Witt's theorem in abstract geometric algebra

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Abstract In an earlier paper of the author, a version of the Witt's theorem was obtained within a specific subcategory of the category of A-modules: the *full subcat-egory of convenient A-modules*. A further investigation yields two more versions of the Witt's theorem by revising the notion of convenient A-modules. For the first version, the A-bilinear form involved is either symmetric or antisymmetric, and the two isometric free sub-A-modules, the isometry between which may extend to an isom-etry of the non-isotropic convenient A-module concerned onto itself, are assumed *pre-hyperbolic*. On the other hand, for the second version, the A-bilinear form defined on the non-isotropic convenient A-module involved is set to be symmetric, and the two isometric free sub-A-modules, whose orthogonals are to be proved isometric, are assumed *strongly non-isotropic* and *disjoint*.

Keywords Sheaf of A-radicals · Orthosymmetric A-bilinear forms · Strongly isotropic (non-isotropic) sub-A-modules · Weakly isotropic (non-isotropic) sub-A-modules · Free subpresheaves of modules · Pre-hyperbolic free sub-A-modules

1 Basics of abstract geometric algebra

As has been said by Mallios in many of his works, we would cite here for example [7–12], and [13], Abstract Differential Geometry (ADG in short) offers a new approach to classical Differential Geometry, in the sense that *obstacles*, which do appear when trying to cope with problems of *quantum gravity*, within the standard set-up of the classical Differential Geometry, *do not appear at all* within the context of ADG. The *spirit* of ADG is to perform *differential geometry*, no *space* (viz. smooth manifolds) is virtually required, provided that one has at their disposal a *basic differential*, ∂, alias dx, along with the appropriate *differential-geometric mechanism* that might be afforded thereby.

The major goal of this paper consists, as indicated in the title, in the setting of the *sheaf-theoretic version of the Witt's theorem*. The classical Witt's theorem has several versions, see to this effect for instance [1–5], and [6]. Our main reference as far as abstract geometric algebra is concerned is Mallios[7].

This work is meant to be part of the ongoing project as undertaken in Mallios–Ntumba [14–16], and Ntumba–Orioha [18].

All our $\mathcal{A}$-modules and A-presheaves in this paper are defined on a fixed topological space X and are *torsion-free*. $\mathcal{A}$-modules and A-presheaves with their respective morphisms form categories which we denote $\mathcal{A}\text{-}\mathcal{M}od_X$ and $A\text{-}\mathcal{PS}h_X$, respectively. By virtue of the equivalence $\mathcal{S}h_X \cong \mathcal{C}o\mathcal{PS}h_X$, an $\mathcal{A}$-morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ of $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ may be identified with the A-morphism $\overline{\phi} := (\overline{\phi}_U)_{X\supseteq U,\ open} : E \longrightarrow F$ of the associated A-presheaves. We shall most often denote by just ϕ the corresponding A-morphism associated with the $\mathcal{A}$-morphism ϕ. The meaning of ϕ will always be determined by the situation at hand.

Recall that given an $\mathcal{A}$-module $\mathcal{E}$ and a sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$, the quotient $\mathcal{A}$-module of $\mathcal{E}$ by $\mathcal{F}$ is the $\mathcal{A}$-module generated by the presheaf sending an open $U \subseteq X$ to an $\mathcal{A}(U)$-module $S(U) := \Gamma(U, \mathcal{E})/\Gamma(U, \mathcal{F}) \equiv \mathcal{E}(U)/\mathcal{F}(U)$ such that for every restriction map $\sigma^U_V : \mathcal{E}(U)/\mathcal{F}(U) \longrightarrow \mathcal{E}(V)/\mathcal{F}(V)$, one has $\sigma^U_V(r + \mathcal{F}(U)) := \rho^U_V(r) + \mathcal{F}(V)$ (the ρ^U_V are the restriction maps for the A-presheaf $\Gamma\mathcal{E}$).

For the sake of easy referencing, we also recall some notions, which may be found in our recent papers such as [14–16], and [18]. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules and $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ an $\mathcal{A}$-bilinear morphism. Then, we say that the triple $(\mathcal{E}, \mathcal{F}; \phi)$ forms a *pairing of* $\mathcal{A}$*-modules* or that $\mathcal{E}$ and $\mathcal{F}$ are *paired through* ϕ *into* $\mathcal{A}$. The sub-$\mathcal{A}$-module $\mathcal{F}^\perp$ of $\mathcal{E}$ such that, for every open subset U of X, $\mathcal{F}^\perp(U)$ consists of all $r \in \mathcal{E}(U)$ with $\phi_V(\mathcal{F}(V), r|_V) = 0$ for any open $V \subseteq U$, is called the *left kernel* of the pairing $(\mathcal{E}, \mathcal{F}; \phi)$. In a similar way, one defines the *right kernel* of $(\mathcal{E}, \mathcal{F}; \phi)$ to be the sub-$\mathcal{A}$-module $\mathcal{E}^\perp$ of $\mathcal{F}$ such that, for any open subset U of X, $\mathcal{E}^\perp(U)$ is the set of all (local) sections $r \in \mathcal{F}(U)$ such $\phi_V(r|_V, \mathcal{E}(V)) = 0$ for every open $V \subseteq U$. If $(\mathcal{E}, \mathcal{F}; \phi)$ is a pairing of free $\mathcal{A}$-modules, then, for every open subset U of X, $\mathcal{F}^\perp(U) = \mathcal{F}(U)^\perp := \{r \in \mathcal{E}(U) : \phi_U(\mathcal{F}(U), r) = 0\}$, and similarly $\mathcal{E}^\perp(U) = \mathcal{E}(U)^\perp := \{r \in \mathcal{F}(U) : \phi_U(r, \mathcal{E}(U)) = 0\}$. Let us recall at this stage that the dual of $\mathcal{E}$ is denoted $\mathcal{E}^*$, see [7, p. 298]. A pairing $(\mathcal{E}, \mathcal{F}; \phi)$ of *free* $\mathcal{A}$*-modules* is called a *weakly convenient pairing* if given free sub-$\mathcal{A}$-modules $\mathcal{F}_0$ and $\mathcal{E}_0$ of $\mathcal{F}$ and $\mathcal{E}$, respectively, their *orthogonal* $\mathcal{F}_0^\perp$ and $\mathcal{E}_0^\perp$ are free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Now, let $(\mathcal{E}, \mathcal{E}; \phi)$ be a pairing such that if $r, s \in \mathcal{E}(U)$, where U is an open subset of X, then $\phi_U(r, s) = 0$ if and only if $\phi_U(s, r) = 0$. The left kernel, $\mathcal{E}_l^\perp := \mathcal{E}^\perp$, is the same as the right kernel $\mathcal{E}_r^\perp := \mathcal{E}^\top$. In this instance, we say that the $\mathcal{A}$-bilinear form ϕ is *orthosymmetric* and call $\mathcal{E}^\perp (= \mathcal{E}^\top)$ the *radical sheaf* (or *sheaf of $\mathcal{A}$-radicals,* or simply *$\mathcal{A}$-radical*) of $\mathcal{E}$, and denote it by $\text{rad}_{\mathcal{A}}\mathcal{E} \equiv \text{rad}\, \mathcal{E}$. An $\mathcal{A}$-module $\mathcal{E}$ such that $\text{rad}\, \mathcal{E} \neq 0$ (resp. $\text{rad}\, \mathcal{E} = 0$) is called *isotropic* (resp. *non-isotropic*); $\mathcal{E}$ is *totally isotropic* if ϕ is identically zero, i.e. $\phi_U(r, s) = 0$ for all sections $r, s \in \mathcal{E}(U)$, with U open in X. For any open $U \subseteq X$, a non-zero section $r \in \mathcal{E}(U)$ is called *isotropic* if $\phi_U(r, r) = 0$. For a sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$, the $\mathcal{A}$-radical of $\mathcal{F}$ is defined as rad $\mathcal{F} := \mathcal{F} \cap \mathcal{F}^\perp = \mathcal{F} \cap \mathcal{F}^\top$. If $\text{rad}\, \mathcal{F} = 0$, then $\mathcal{F}$ is said to be *strongly non-isotropic*. In other words, for every section $r \in \mathcal{F}(U)$, there exists a section $s \in \mathcal{F}(U)$ such that $\phi_U(r, s) \neq 0$. If $(\text{rad}\, \mathcal{F})(U) \neq 0$ for every open $U \subseteq X$, then $\mathcal{F}$ is said to be *strongly isotropic*. However, it is possible that a sub-$\mathcal{A}$-module $\mathcal{F}$ is neither strongly isotropic, nor strongly non-isotropic; in such a case, $\mathcal{F}$ is said to either be *weakly isotropic* or *weakly non-isotropic*. Now, let $(\mathcal{E}, \mathcal{F}; \phi)$ be a *pairing of free $\mathcal{A}$-modules,* then for every open subset U of X, $(\text{rad}\, \mathcal{E})(U) = \text{rad}\, \mathcal{E}(U)$ and $(\text{rad}\, \mathcal{F})(U) = \text{rad}\, \mathcal{F}(U)$, where $\text{rad}\, \mathcal{E}(U) = \mathcal{E}(U) \cap \mathcal{E}(U)^\perp$ and $\text{rad}\, \mathcal{F}(U) = \mathcal{F}(U) \cap \mathcal{F}(U)^\perp$. Given a pairing $(\mathcal{E}, \mathcal{E}; \phi)$ with ϕ a *symmetric* $\mathcal{A}$-bilinear morphism, sub-$\mathcal{A}$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$ are said to be *mutually orthogonal* if for every open subset U of X, $\phi_U(r, s) = 0$, for all $r \in \mathcal{E}_1(U)$ and $s \in \mathcal{E}_2(U)$. If $\mathcal{E} = \oplus_{i \in I} \mathcal{E}_i$, where the $\mathcal{E}_i$ are pairwise orthogonal sub-$\mathcal{A}$-modules of $\mathcal{E}$, we say that $\mathcal{E}$ is the direct orthogonal sum of the $\mathcal{E}_i$, and write $\mathcal{E} := \mathcal{E}_1 \perp \cdots \perp \mathcal{E}_i \perp \cdots$.

Lemma 1 *Let ϕ be a non-degenerate $\mathcal{A}$-bilinear form on an $\mathcal{A}$-module $\mathcal{E}$. Then the mappings $\perp \equiv \perp(\phi)$, $\top \equiv \top(\phi)$ have the following properties*:

(1) (a) *If $\mathcal{G} \subseteq \mathcal{H}$, then $\mathcal{G}^\perp \supseteq \mathcal{H}^\perp$*
    (b) *If $\mathcal{G} \subseteq \mathcal{H}$, then $\mathcal{G}^\top \supseteq \mathcal{H}^\top$*
(2) (c) $(\mathcal{G} + \mathcal{H})^\perp = \mathcal{G}^\perp \cap \mathcal{H}^\perp$
    (d) $(\mathcal{G} + \mathcal{H})^\top = \mathcal{G}^\top \cap \mathcal{H}^\top$

for all sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$.

Proof Assertion (1) is clear. However, to prove (2), one has to take care of the very definition of the bi-functor $\mathcal{H}om_{\mathcal{A}}(\cdot, \cdot)$ since it concerns the operation "$\perp$": thus, one should consider the sheaves (: $\mathcal{A}$-modules) involved as (complete) presheaves (*à* la Leray) to handle the corresponding morphisms; see e.g. [7, p. 133, (6.4)/(6.5)].

Now for the sake of what follows, we assume, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an *algebraized space*, where $\mathcal{A}$ is a *unital $\mathbb{C}$-algebra sheaf* such that *every nowhere-zero section of $\mathcal{A}$ is invertible.*

Theorem 1, which has been proved in [15], is pivotal as far as the sheaf-theoretic version of the Witt's theorem is concerned.

Theorem 1 *Let $(\mathcal{E}, \phi)$ be a free $\mathcal{A}$-module of finite rank. Then, every non-isotropic free sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand of $\mathcal{E}$; viz.*

$$\mathcal{E} = \mathcal{F} \perp \mathcal{F}^\perp.$$

Next, let us revise the notion of *convenient $\mathcal{A}$-modules* by altering its format of [15] so as to make some of its hypotheses redundant in case the *coefficient algebra sheaf* $\mathcal{A}$ is a *PID algebra sheaf* (i.e., for every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ is a PID algebra; in other words, given a free $\mathcal{A}$-module and a sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$, one has that $\mathcal{F}$ is *section-wise free*), see [17].

To this end, we need the following

Definition 1 (*A. Mallios*) A subpresheaf F of a presheaf of modules (or more precisely, $A(U)$-modules) E (cf. [7, p. 99, Definition 1.6]) is called a **free subpresheaf** if for every open U in X, $F(U)$ is a free sub-$A(U)$-module of $E(U)$.

Then, we have

Definition 2 A **convenient $\mathcal{A}$-module** is a self pairing $(\mathcal{E}, \mathcal{E}; \phi) \equiv (\mathcal{E}, \phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank and ϕ an orthosymmetric $\mathcal{A}$-bilinear form, such that the following conditions are satisfied: (1) *If $\mathcal{F}$ is a free subpresheaf of $\mathcal{A}(U)$-modules of $\mathcal{E}$, then $\mathcal{F}^{\perp} \equiv \mathcal{F}^{\perp_\phi}$ is a free subpresheaf of $\mathcal{A}(U)$-modules of $\mathcal{E}$;* (2) *Every free subpresheaf $\mathcal{F}$ of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is orthogonally reflexive, i.e., $\mathcal{F}^{\perp\perp} = \mathcal{F}$;* (3) *The intersection of any two free subpresheaves of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is a free subpresheaf of $\mathcal{A}(U)$-modules.*

Note Concerning the above definition of *convenient $\mathcal{A}$-modules*, by supposing that the (*coefficient-*) algebra sheaf $\mathcal{A}$ is a PID-*algebra sheaf*, we obtain that *every subpresheaf of $\mathcal{A}(U)$-modules of a free $\mathcal{A}$-module is free.* So in that context, conditions (1) and (3) in Definition 5 are satisfied. Now, concerning condition (2) of the same definition, the *reflexivity* at hand is a known situation in ordinary Functional Analysis: see, for instance, Hilbert spaces and structures having similar properties; so we do have the so-called *complemented topological algebras, Hilbert algebras* and the likes with the aforementioned property for *ideals* (: *modules*), and also analogous examples in *infinite-dimensional Hamiltonian mechanics.* (I am indebted to A. Mallios for *this comment* on convenient $\mathcal{A}$-modules.)

Orthogonally convenient pairings of $\mathcal{A}$-modules are just as much interesting as convenient $\mathcal{A}$-modules and satisfy in some *restricted way* conditions (1) and (2) of Definition 5.

Definition 3 A pairing $(\mathcal{E}, \mathcal{F}; \phi)$ of free $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ into the $\mathbb{C}$-algebra sheaf $\mathcal{A}$ is called an **orthogonally convenient pairing** if given free sub-$\mathcal{A}$-modules $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp}$ and $\mathcal{F}_0^{\perp}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively.

Based on [14, pp. 399–401], if $\mathcal{E}$ is an $\mathcal{A}$-module, $\mathcal{F}$ and $\mathcal{G}$ are sub-$\mathcal{A}$-modules of $\mathcal{E}$ such $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$, then

$$\mathcal{E}/\mathcal{F} = \mathcal{G} \tag{1}$$

within an $\mathcal{A}$-isomorphism. Furthermore, if $\mathcal{E}$ is a free $\mathcal{A}$-module and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then, for every open $U \subseteq X$,

$$(\mathcal{E}/\mathcal{F})(U) \simeq \mathcal{E}(U)/\mathcal{F}(U). \tag{2}$$

On another side, given an orthogonally convenient $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, $\mathcal{E}_0$ and $\mathcal{F}_0$ free sub-$\mathcal{A}$-modules of (free $\mathcal{A}$-modules) $\mathcal{E}$ and $\mathcal{F}$, respectively, one has, for every open $U \subseteq X$,

$$(\mathcal{E}/\mathcal{F}_0^{\perp})(U) = \mathcal{E}(U)/\mathcal{F}_0^{\perp}(U) \tag{3}$$

and

$$(\mathcal{F}/\mathcal{E}_0^{\perp})(U) = \mathcal{F}(U)/\mathcal{E}_0^{\perp}(U). \tag{4}$$

From [14], we also single out the following result.

Theorem 2 *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules paired into a $\mathbb{C}$-algebra sheaf $\mathcal{A}$, and assume that $\mathcal{E}^{\perp} = 0$. Moreover, let $\mathcal{F}_0$ be a sub-$\mathcal{A}$-module of $\mathcal{F}$ and $\mathcal{E}_0$ a sub-$\mathcal{A}$-module of $\mathcal{E}$. There exist natural $\mathcal{A}$-isomorphisms into*:

$$\mathcal{E}/\mathcal{F}_0^{\perp} \longrightarrow \mathcal{F}_0^{*}, \tag{5}$$

and

$$\mathcal{E}_0^{\perp} \longrightarrow (\mathcal{E}/\mathcal{E}_0)^{*}. \tag{6}$$

Definition 4 The pairing $(\mathcal{E}, \mathcal{E}^*; \phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module and such that for every open $U \subseteq X$,

$$\phi_U(r, \psi) := \psi_U(r)$$

with $\psi \in \mathcal{E}^*(U) := \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$ and $r \in \mathcal{E}(U)$, is called the **canonical pairing** of $\mathcal{E}$ and $\mathcal{E}^*$.

Theorem 3 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite rank. The canonical pairing $(\mathcal{E}, \mathcal{E}^*; \phi)$ is orthogonally convenient.*

Proof First, we notice by [14, Theorem 2.2] that both kernels, i.e. $\mathcal{E}^{\perp}$ and $(\mathcal{E}^*)^{\perp}$, are 0. Let $\mathcal{E}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and consider the map (6) of Theorem 2: $\mathcal{E}_0^{\perp} \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*$. It is an $\mathcal{A}$-isomorphism into, and we shall show that it is onto. Fix an open set U in X, and let $\psi \in (\mathcal{E}/\mathcal{E}_0)^*(U) := \mathcal{H}om_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_0)|_U, \mathcal{A}|_U)$. Let us consider a family $\overline{\psi} \equiv (\overline{\psi}_V)_{U \supseteq V,\ open}$ where if V, W are open in U with $W \subseteq V$, then

$$\tau_W^V \circ \overline{\psi}_V = \overline{\psi}_W \circ \rho_W^V$$

(the $\{\rho_V^U\}$ and $\{\tau_V^U\}$ are the restriction maps for the (complete) presheaf of sections of $\mathcal{E}$ and $\mathcal{A}$, respectively) and

$$\overline{\psi}_V(r) := \psi_V(r + \mathcal{E}_0(V)),\ r \in \mathcal{E}(V). \tag{7}$$

It is easy to see that $\overline{\psi}_V$ is $\mathcal{A}(V)$-linear for any open $V \subseteq U$. Thus,

$$\overline{\psi} \in \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) =: \mathcal{E}^*(U).$$

Suppose $r \in \mathcal{E}_0(V)$, where V is open in U. Then

$$\overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V)) = \psi_V(\mathcal{E}_0(V)) = 0,$$

therefore

$$\phi_V(\mathcal{E}_0(V), \overline{\psi}|_V) = \overline{\psi}_V(\mathcal{E}_0(V)) = 0,$$

i.e. $\overline{\psi} \in \mathcal{E}_0^{\perp}(U)$. We contend that $\overline{\psi}$ has the given ψ as image under the map (6), and this will show the ontoness of (6) and that $\mathcal{E}_0^{\perp}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$.

Let us find the image of $\overline{\psi}$. Consider the pairing $(\mathcal{E}/\mathcal{E}_0, \mathcal{E}_0^{\perp}; \Theta)$ such that for any open $V \subseteq X$, we have

$$\Theta_V(r + \mathcal{E}_0(V), \alpha) := \phi_V(r, \alpha) = \alpha_V(r),$$

where $r \in \mathcal{E}(V)$,$\alpha \in \mathcal{E}_0^{\perp}(V) \subseteq \mathcal{E}^*(V)$. Clearly, the *right kernel* of this new pairing is 0. For $\alpha = \overline{\psi} \in \mathcal{E}_0^{\perp}(U) \subseteq \mathcal{E}^*(U)$, we have

$$\Theta_U(r + \mathcal{E}_0(U), \overline{\psi}) = \overline{\psi}_U(r)$$

where $r \in \mathcal{E}(U)$, and the map

$$\overline{\Theta}_U : \mathcal{E}_0^{\perp}(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$$

given by

$$\overline{\psi} \longmapsto \overline{\Theta}_{U,\overline{\psi}} \equiv \left((\overline{\Theta}_{U,\overline{\psi}})_V\right)_{U \supseteq V,\ \text{open}}$$

and such that for any $r \in \mathcal{E}(V)$

$$(\overline{\Theta}_{U,\overline{\psi}})_V(r + \mathcal{E}_0(V)) := \overline{\Theta}_V(r + \mathcal{E}_0(V), \overline{\psi}|_V) = \overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V))$$

is the image. Thus the image of $\overline{\psi}$ is ψ, hence the map $\mathcal{E}_0^{\perp}(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$, derived from (6), is onto, and therefore an $\mathcal{A}(U)$-isomorphism. Since $\mathcal{E}/\mathcal{E}_0$ is free by Corollary 1, so are $(\mathcal{E}/\mathcal{E}_0)^*$ and $\mathcal{E}_0^{\perp}$ free.

Now, let $\mathcal{F}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}^* \cong \mathcal{E}$ (cf. Mallios [7, p.298, (5.2)]); on considering $\mathcal{F}_0$ as a free sub-$\mathcal{A}$-module of $\mathcal{E}$, according to all that precedes above $\mathcal{F}_0^{\perp}$ is free in $\mathcal{E}^* \cong \mathcal{E}$, and so the proof is finished.

Now, if $(\mathcal{E}, \mathcal{F}; \phi)$ is an orthogonally convenient pairing, $\mathcal{E}_0$ and $\mathcal{F}_0$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively, by (1), $\mathcal{E}/\mathcal{F}_0^{\perp}$ and $\mathcal{E}/\mathcal{E}_0$ are free $\mathcal{A}$-modules. Since the maps in Theorem 2 are $\mathcal{A}$-isomorphisms into,

$$\text{rank}(\mathcal{E}/\mathcal{F}_0^{\perp}) \leq \text{rank } \mathcal{F}_0^{*} = \text{rank } \mathcal{F}_0 \tag{8}$$

and

$$\text{rank } \mathcal{E}_0^{\perp} \leq \text{rank}(\mathcal{E}/\mathcal{E}_0)^{*} = \text{rank}(\mathcal{E}/\mathcal{E}_0). \tag{9}$$

Inequalities (8) and (9) can also be written in the form

$$\text{corank } \mathcal{F}_0^{\perp} \leq \text{rank } \mathcal{F}_0$$

and

$$\text{rank } \mathcal{E}_0^{\perp} \leq \text{corank } \mathcal{E}_0.$$

If we put $\mathcal{E}_0 = \mathcal{F}_0^{\perp}$ in the last inequality and combine it with the first one, we get

$$\text{rank } \mathcal{F}_0^{\perp\perp} \leq \text{corank } \mathcal{F}_0^{\perp} \leq \text{rank } \mathcal{F}_0. \tag{10}$$

But $\mathcal{F}_0$ is a free sub-$\mathcal{A}$-module of $\mathcal{F}_0^{\perp\perp}$, so that rank $\mathcal{F}_0 \leq$ rank $\mathcal{F}_0^{\perp\perp}$, and (10) becomes

$$\text{rank } \mathcal{F}_0^{\perp\perp} = \text{corank } \mathcal{F}_0^{\perp} = \text{rank } \mathcal{F}_0. \tag{11}$$

Let us consider the formula (11) in the case where rank $\mathcal{F}_0$ is finite. We clearly have $\mathcal{F}_0^{\perp\perp} = \mathcal{F}_0$ within an $\mathcal{A}$-isomorphism. The $\mathcal{A}$-module $\mathcal{F}_0$ is said to be *orthogonally reflexive*. In the $\mathcal{A}$-morphism (5), both free $\mathcal{A}$-modules have the same finite rank, the $\mathcal{A}$-isomorphism into is, therefore, onto and thus

$$\mathcal{E}/\mathcal{F}_0^{\perp} = \mathcal{F}_0^{*}$$

within an $\mathcal{A}$-isomorphism. Hence, $\mathcal{E}/\mathcal{F}_0^{\perp}$ may be regarded naturally as the dual $\mathcal{A}$-module of $\mathcal{F}_0$. For the $\mathcal{A}$-morphism (6), put $\mathcal{E}_0 = \mathcal{F}_0^{\perp}$; thus (6) becomes an $\mathcal{A}$-isomorphism

$$\mathcal{F}_0^{\perp\perp} \cong (\mathcal{E}/\mathcal{F}_0^{\perp})^{*}.$$

Putting $\mathcal{F}_0 = \mathcal{F}$ in (11), we obtain

$$\text{corank } \mathcal{F}^{\perp} = \text{rank } \mathcal{F}. \tag{12}$$

Now, assume in our orthogonally convenient pairing $(\mathcal{E}, \mathcal{F}; \phi)$ that the right kernel $\mathcal{E}^{\perp}$ is not 0. Let $\Psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E} \oplus (\mathcal{F}/\mathcal{E}^{\perp}), \mathcal{A})$ such that

$$\Psi_U(s, t + \mathcal{E}^{\perp}(U)) := \phi_U(s, t),$$

where U is an open subset of X, $t + \mathcal{E}^{\perp}(U) \in (\mathcal{F}/\mathcal{E}^{\perp})(U) \cong \mathcal{F}(U)/\mathcal{E}^{\perp}(U)$, (cf. (4)) and $s \in \mathcal{E}(U)$.

The element $t + \mathcal{E}^{\perp}(U)$ lies in the right kernel $\mathcal{E}^{\perp}(U) \cong \mathcal{E}(U)^{\perp}$ if $\phi_U(s, t) = 0$ for all $s \in \mathcal{E}(U)$. But this means $t \in \mathcal{E}^{\perp}(U)$, so that $t + \mathcal{E}^{\perp}(U) = \mathcal{E}^{\perp}(U)$. It follows that the right kernel of the pairing $(\mathcal{E}, \mathcal{F}/\mathcal{E}^{\perp}; \Psi)$ is 0. The left kernel is obviously the old $\mathcal{F}^{\perp}$. Applying (12), we have

$$\operatorname{rank}(\mathcal{E}/\mathcal{F}^{\perp}) = \operatorname{rank}(\mathcal{F}/\mathcal{E}^{\perp}). \tag{13}$$

Suppose now that both kernels $\mathcal{E}^{\perp}$ and $\mathcal{F}^{\perp}$ are zero, and that rank $\mathcal{F}$ is finite. (13) shows that rank $\mathcal{E}$ is also finite and rank $\mathcal{E}$ = rank $\mathcal{F}$. So whenever $\mathcal{E}^{\perp} = 0 = \mathcal{F}^{\perp}$, by [14, Theorem 2.3], we see that each of the free $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{E}$ is naturally the dual of the other.

Now, still under the condition $\mathcal{E}^{\perp} = 0 = \mathcal{F}^{\perp}$ for the orthogonally convenient pairing $(\mathcal{E}, \mathcal{F}; \phi)$, let us look at the correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\perp}$ of a free sub-$\mathcal{A}$-module $\mathcal{F}_0$ of $\mathcal{F}$ and the free sub-$\mathcal{A}$-module $\mathcal{F}_0^{\perp}$ of $\mathcal{E}$. Any free sub-$\mathcal{A}$-module $\mathcal{E}_0$ of $\mathcal{E}$ is obtainable from an $\mathcal{F}_0$; indeed we merely have to put $\mathcal{F}_0 = \mathcal{E}_0^{\perp}$. And if $\mathcal{F}_0 \ncong \mathcal{F}_1$, then $\mathcal{F}_0^{\perp} \ncong \mathcal{F}_1^{\perp}$. The correspondence $\mathcal{F}_0 \longleftrightarrow \mathcal{F}_0^{\perp}$, where $\mathcal{F}_0$ is any free sub-$\mathcal{A}$-module of $\mathcal{F}$, is one-to-one, and also if $\mathcal{F}_0 \subseteq \mathcal{F}_1$ then $\mathcal{F}_0^{\perp} \supseteq \mathcal{F}_1^{\perp}$.

Let us collect all our results.

Theorem 4 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be an orthogonally convenient pairing. Then,*

(a) *rank $(\mathcal{F}/\mathcal{E}^{\perp})$ = rank $(\mathcal{E}/\mathcal{F}^{\perp})$; in particular if one of the free $\mathcal{A}$-modules $\mathcal{F}/\mathcal{E}^{\perp}$ and $\mathcal{E}/\mathcal{F}^{\perp}$ has finite rank, so has the other one, and the ranks are equal.*

(b) *If the right kernel $\mathcal{E}^{\perp}$ is zero, and $\mathcal{F}_0 \subseteq \mathcal{F}$ is a free sub-$\mathcal{A}$-module, then*

$$\textit{rank } \mathcal{F}_0 = \textit{corank } \mathcal{F}_0^{\perp} = \textit{rank } \mathcal{F}_0^{\perp\perp}. \tag{14}$$

If rank $\mathcal{F}_0$ is finite, then $\mathcal{F}_0^{\perp\perp} = \mathcal{F}_0$ and $\mathcal{E}/\mathcal{F}_0^{\perp} = \mathcal{F}_0^*$ within an $\mathcal{A}$-isomorphism, i.e. each of the free $\mathcal{A}$-modules $\mathcal{F}_0$ and $\mathcal{E}/\mathcal{F}_0^{\perp}$ is naturally the dual of the other.

(c) *If both kernels are zero, and rank $\mathcal{F}$ is finite, then $\mathcal{F} = \mathcal{E}$ within an $\mathcal{A}$-isomorphism. The correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\perp}$ is a bijection between the free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and the free sub-$\mathcal{A}$-modules of $\mathcal{E}$, and it reverses any inclusion relation.*

As a corollary of the preceding theorem, we have

Theorem 5 *Let $(\mathcal{E}, \mathcal{E}^*; \phi)$ be the canonical pairing between (free $\mathcal{A}$-modules) $\mathcal{E}$ and $\mathcal{E}^*$, and let $\mathcal{E}_0$ be a free $\mathcal{A}$-module of $\mathcal{E}$. Then $\mathcal{E}_0^{\perp\perp} = \mathcal{E}_0$ and $\mathcal{E}_0^{\perp} = (\mathcal{E}/\mathcal{E}_0)^*$ within an $\mathcal{A}$-isomorphism, and rank $\mathcal{E}_0^{\perp}$ = corank $\mathcal{E}_0$. The correspondence $\mathcal{F}_0 \longmapsto \mathcal{F}_0^{\perp}$ is a bijection between free sub-$\mathcal{A}$-modules $\mathcal{F}_0 \subseteq \mathcal{E}^*$ of finite rank and all the free sub-$\mathcal{A}$-modules of $\mathcal{E}$ with finite corank.*

2 Witt's theorem first version

Before stating the theorem, let us recall the result, see [18], that given an $\mathcal{A}$-module $\mathcal{E}$, equipped with an orthosymmetric $\mathcal{A}$-bilinear form $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$, then for every open subset U of X, ϕ_U is either symmetric or skew-symmetric. When ϕ_U is symmetric, the geometry on the $\mathcal{A}(U)$-module $\mathcal{E}(U)$ is called *orthogonal*; on the other hand, if ϕ_U is skew-symmetric, the geometry is called *symplectic*. No other case can occur if ϕ must be orthosymmetric.

Lemmas 2 and 3, below, are needed for the proof of the Witt's theorem. Proofs of Lemmas 2 and 3 are found in [15].

Lemma 2 *Let $(\mathcal{E}, \phi)$ be a free $\mathcal{A}$-module of rank 2, endowed with a non-degenerate symmetric or antisymmetric $\mathcal{A}$-bilinear form ϕ. For an open subset $U \subseteq X$, the non-isotropic $\mathcal{A}(U)$-plane $\mathcal{E}(U)$ is hyperbolic if it contains a nowhere-zero isotropic section r.*

Lemma 3 *Let $(\mathcal{E}, \phi)$ be a non-isotropic convenient $\mathcal{A}$-module, where $\mathcal{A}$ is a PID algebra sheaf, and F any free sub-$\mathcal{A}(U)$-module of $\mathcal{E}(U)$. Moreover, let sections $s_1, s_2, \ldots, s_k \in F$ form a basis of rad F and G a free sub-$\mathcal{A}(U)$-module of $\mathcal{E}(U)$ such that $F = \text{rad } F \perp G$. Then, there are isotropic sections $t_1, t_2, \ldots, t_k \in \mathcal{E}(U)$ such that the planes $P_i := [s_i, t_i]$ are hyperbolic, pairwise orthogonal and also orthogonal to $\mathcal{G}(U)$. The $\mathcal{A}(U)$-module*

$$P_1 \perp P_2 \perp \ldots \perp P_k \perp G$$

contains F.

On the basis of Lemma 3, above, we introduce the following notion.

Definition 5 *Let $(\mathcal{E}, \phi)$ be a non-isotropic convenient $\mathcal{A}$-module, and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$ of rank k. The free sub-$\mathcal{A}$-module $\mathcal{F}$ is called **pre-hyperbolic** if there are pairwise orthogonal hyperbolic $\mathcal{A}$-planes*

$$\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_l \subseteq \mathcal{E}$$

such that if $\mathcal{F} = \text{rad } \mathcal{F} \perp \mathcal{G}$ with rad $\mathcal{F} \cong \mathcal{A}^l$ and $\mathcal{G} \cong \mathcal{A}^{k-l}$, then

$$\mathcal{H}_1 \perp \mathcal{H}_2 \perp \ldots \perp \mathcal{H}_l \perp \mathcal{G}$$

is non-isotropic and contains $\mathcal{F}$.

Theorem 6 (Witt's Theorem) *Let $(\mathcal{E}, \phi)$ be a non-isotropic convenient $\mathcal{A}$-module, with the $\mathcal{A}$-bilinear form ϕ symmetric or antisymmetric. Let $\mathcal{F}$ and $\mathcal{F}'$ be **pre-hyperbolic free sub-$\mathcal{A}$-modules** of $\mathcal{E}$, and let $\sigma \in Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{F}')$ be an isomtery. Then, σ **extends to an isometry of $\mathcal{E}$ onto itself.***

Proof For every open set $U \subseteq X$, $\sigma_U : \mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$ is an $\mathcal{A}(U)$-isometry. Suppose that σ_U extends to an isometry $\widehat{\sigma}_U : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$ such that $\rho_V^U \circ \widehat{\sigma}_U = \widehat{\sigma}_V \circ \rho_V^U$, where the ρ_V^U are restriction maps for $\mathcal{E}$, then the $\mathcal{A}$-morphism $\beta : \mathcal{E} \longrightarrow \mathcal{E}$ such that $\beta_U := \widehat{\sigma}_U$ is an $\mathcal{A}$-isometry of $\mathcal{E}$ onto itself and extends σ.

We shall see that the proof reduces to the case when $\mathcal{F}$ is non-isotropic. For, suppose $\mathcal{F}$ is isotropic, and $\mathcal{F} = \text{rad } \mathcal{F} \perp \mathcal{G}$, where rad $\mathcal{F} \cong \mathcal{A}^l$ and $\mathcal{G} \cong \mathcal{A}^{k-l}$. Since $\sigma \in Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{F}')$ is an isometry, for every open $U \subseteq X$, $\mathcal{F}'(U) = \sigma_U(\mathcal{F}(U)) = \sigma_U((\text{rad } \mathcal{F})(U) \perp \sigma_U(\mathcal{G}(U)) = \sigma_U(\text{rad } \mathcal{F}(U)) \perp \sigma_U(\mathcal{G}(U))$. Clearly, rad $\mathcal{F}'(U) = \sigma_U(\text{rad } \mathcal{F}(U))$. Now, based on the fact that both $\mathcal{F}$ and $\mathcal{F}'$ are pre-hyperbolic, we can *enlarge* them to orthogonal sums

$$\mathcal{H}_1 \perp \cdots \perp \mathcal{H}_l \perp \mathcal{G} \quad \text{and} \quad \mathcal{H}'_1 \perp \cdots \perp \mathcal{H}'_l \perp \sigma(\mathcal{G}),$$

where, for every open subset $U \subseteq X$, and $1 \leq i \leq l$,

$$\mathcal{H}_i(U) = [e_{i,U}, f_{i,U}]$$

in keeping with the notations of Definition 5, and if $\sigma_U(e_{i,U}) = e'_{i,U}$, we can find $f'_{i,U} \in \mathcal{F}'(U)$ such that,

$$\mathcal{H}'_i(U) = [e'_{i,U}, f'_{i,U}]$$

is an hyperbolic $\mathcal{A}(U)$-plane, orthogonal to $\mathcal{G}'(U) := \sigma_U(\mathcal{G}(U)) \cong \mathcal{A}^{k-l}(U) \cong \mathcal{A}(U)^{k-l}$. We extend every σ_U to an $\mathcal{A}$-isometry

$$\overline{\sigma}_U : \mathcal{H}_1(U) \perp \cdots \perp \mathcal{H}_l(U) \perp \mathcal{G}(U) \longrightarrow \mathcal{H}'_1(U) \perp \cdots \perp \mathcal{H}'_l(U) \perp \mathcal{G}'(U)$$

by requiring that $\overline{\sigma}_U(f_{i,U}) = f'_{i,U}$. The $\mathcal{A}$-morphism $\overline{\sigma} \equiv (\overline{\sigma}_U)$ thus obtained is an $\mathcal{A}$-isometry

$$\mathcal{H}_1 \perp \cdots \perp \mathcal{H}_l \perp \mathcal{G} \longrightarrow \mathcal{H}'_1 \perp \cdots \perp \mathcal{H}'_l \perp \mathcal{G}'.$$

But $\mathcal{H}_1 \perp \cdots \perp \mathcal{H}_l \perp \mathcal{G}$ and $\mathcal{H}'_1 \perp \cdots \perp \mathcal{H}'_l \perp \mathcal{G}'$ are non-isotropic, so the proof is reduced to the anticipated case, viz. the case when $\mathcal{F}$ is non-isotropic.

(A) **Symplectic geometry** By Theorem 1, since $\mathcal{F}$ and $\mathcal{F}'$ are non-isotropic

$$\mathcal{E} = \mathcal{F} \perp \mathcal{F}^{\perp} \text{ and } \mathcal{E} = \mathcal{F}' \perp \mathcal{F}'^{\perp}.$$

We need only show that $\mathcal{F}^{\perp}$ and $\mathcal{F}'^{\perp}$ are $\mathcal{A}$-isometric; in fact $\mathcal{F}^{\perp}$ and $\mathcal{F}'^{\perp}$ are free non-isotropic sub-$\mathcal{A}$-modules of $\mathcal{E}$ of the same rank. Moreover, by Theorem 1 , since restrictions $\phi|_{\mathcal{F}^{\perp}}$ and $\phi|_{\mathcal{F}'^{\perp}}$ (of ϕ to $\mathcal{F}^{\perp}$ and $\mathcal{F}'^{\perp}$, respectively) are non-degenerate, $\mathcal{F}^{\perp}$ and $\mathcal{F}'^{\perp}$ are direct orthogonal sums of hyperbolic $\mathcal{A}$-planes. Based on the latter observations, $\mathcal{F}^{\perp}$ and $\mathcal{F}'^{\perp}$ are $\mathcal{A}$-isometric.

(B) **Orthogonal geometry** We will proceed stepwise. ***Case*** (1) Suppose that $\mathcal{F} = \mathcal{F}'$, i.e. σ is an $\mathcal{A}$-isometry of $\mathcal{F}$ onto itself. We extend σ by keeping, for every open $U \subseteq X$, every section in $\mathcal{F}^{\perp}(U)$ fixed. ***Case*** (2) Assume rank $\mathcal{F} =$ rank $\mathcal{F}' = 1$, i.e. $\mathcal{F} \cong \mathcal{F}' \cong \mathcal{A}$, and $\mathcal{F} \neq \mathcal{F}'$. Thus, for some open subset $U \subseteq X$, $\mathcal{F}(U) \neq \mathcal{F}'(U)$. Say $\mathcal{F}(U) = [e_U]$ and $\mathcal{F}'(U) = [e'_U]$ for every open $U \subseteq X$. $\mathcal{F}$ and $\mathcal{F}'$ being $\mathcal{A}$-modules of rank 1, it is clear that if U and V are open sets in X such that $V \subseteq U$, then $e_V = e_U|_V$ and $e'_V = e'_U|_V$. Next, since $\mathcal{F} = \mathcal{F}'$ within an $\mathcal{A}$-isometry, and $\mathcal{F}$ and $\mathcal{F}'$ are non-isotropic, we have that $\phi_U(e_U, e_U) = \phi_U(e'_U, e'_U) \neq 0$, for every open $U \subseteq X$. Furthermore, the correspondence

$$U \longmapsto \mathcal{J}(U) := [e_U, e'_U]$$

along with the obvious restriction maps (in fact, the restriction maps δ^U_V are given by the prescription $\delta^U_V(ae_U + be'_U) = \lambda^U_V(a)\rho^U_V(e_U) + \lambda^U_V(b)\rho^U_V(e'_U)$ for every $a, b \in \mathcal{A}(U)$) yields a presheaf of $\mathcal{A}(U)$-modules. For an open subset $U \subseteq X$ such that $\mathcal{F}(U) = \mathcal{F}'(U)$, it follows that $\mathcal{J}(U)$ is of rank 1 and non-isotropic. In this case, since $\mathcal{E}(U) = \mathcal{J}(U) \perp \mathcal{J}(U)^{\perp}$, σ_U is extended by keeping the sections in $\mathcal{J}(U)^{\perp}$ fixed. On the other hand, for an open U such that $\mathcal{J}(U) \equiv [e_U, e'_U]$ has rank 2, we distinguish two situations.

If $\mathcal{J}(U)$ is non-isotropic, the map, which sends e_U to e'_U and e'_U to e_U, is an $\mathcal{A}(U)$-isometry. Then, we apply ***Case*** (1) to get an $\mathcal{A}(U)$-isometry β_U of $\mathcal{E}(U)$ onto itself. Clearly for each open $U \subseteq X$,

$$\beta_V \circ \rho^U_V = \rho^U_V \circ \beta_U.$$

Hence, $\beta \equiv (\beta_U)$ is an $\mathcal{A}$-isometry extending σ.

Next, ***if*** $\mathcal{J}(U)$ is isotropic, then rad $\mathcal{J}(U)$ has rank 1. Let s_U be a generator of rad $\mathcal{J}(U)$. There exist nowhere-zero sections $a, b \in \mathcal{A}(U)$ such that $e'_U = ae_U + bs_U$. Then, $\phi_U(e'_U, e'_U) = a^2\phi_U(e_U, e_U)$ and because $\phi_U(e'_U, e'_U)$ and $\phi_U(e_U, e_U)$ are nowhere-zero sections of $\mathcal{A}$ on U, it follows that $a = \pm 1$. Assume that $a = 1$, and let us replace bs_U by s_U, then

$$s_U = e'_U - e_U.$$

Let $t_U = e_U + e'_U$. It is obvious that $[s_U, t_U] = [s_U] \oplus [t_U]$ and rad $([s_U, t_U]) = [s_U]$; by Lemma 3, we can find a section $z_U \in \mathcal{E}(U)$ such that

$$\phi_U(z_U, t_U) = 0, \quad \phi_U(z_U, z_U) = 0 \quad \text{and} \quad \phi_U(s_U, z_U) = 1.$$

The $\mathcal{A}(U)$-module $[t_U] \perp [s_U, z_U]$ is non-isotropic, being an orthogonal sum of $[t_U]$ and the hyperbolic plane $[s_U, z_U]$. There exists an $\mathcal{A}(U)$-isometry such that

$$t_U \longleftrightarrow t_U, \quad s_U \longleftrightarrow -s_U, \quad z_U \longleftrightarrow -z_U.$$

But $e_U = \frac{1}{2}(t_U - s_U)$ is mapped on $e'_U = \frac{1}{2}(t_U + s_U)$ by this isometry. Resorting to ***Case*** (1), we obtain an $\mathcal{A}(U)$-isometry $\mathcal{E}(U) \longrightarrow \mathcal{E}(U)$ which extends σ_U. This part of the proof is therefore finished.

Case (3) We finish the proof by induction. Let $\mathcal{F} = \mathcal{F}_1 \perp \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are free sub-$\mathcal{A}$-modules of $\mathcal{E}$ of rank greater than or equal to 1. Then, $\mathcal{F}_2 \subseteq \mathcal{F}_1^\perp$, consequently by applying Lemma 1, we have that $\mathcal{F}_2 \cap \mathcal{F}_2^\perp \subseteq \mathcal{F}_1^\perp \cap \mathcal{F}_2^\perp = \mathcal{F}^\perp$. It easily follows that

$$\mathcal{F} \cap \mathcal{F}_2 \cap \mathcal{F}_2^\perp = \mathcal{F}_2 \cap \mathcal{F}_2^\perp \subseteq \mathcal{F} \cap \mathcal{F}^\perp = 0,$$

i.e. $\mathcal{F}_2$ is non-isotropic. One applies a similar argument to show that $\mathcal{F}_1$ is also non-isotropic. Now, let us fix an open subset $U \subseteq X$. We have $\mathcal{F}(U) = \mathcal{F}_1(U) \perp \mathcal{F}_2(U)$ and

$$\sigma_U(\mathcal{F}(U)) = \sigma_U(\mathcal{F}_1(U)) \perp \sigma_U(\mathcal{F}_2(U)).$$

Let $\sigma_{1,U} = \sigma_U|_{\mathcal{F}_1(U)}$ be the restriction of σ_U to $\mathcal{F}_1(U)$. By induction, we can extend $\sigma_{1,U}$ to an $\mathcal{A}(U)$-isometry

$$\overline{\sigma}_{1,U} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U).$$

Then, $\overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U)) = (\sigma_{1,U}(\mathcal{F}_1(U)))^\perp$. Indeed, let $r \in \mathcal{F}_1^\perp(U)$ and $s \in \mathcal{F}_1(U)$. Then

$$\phi_U(\overline{\sigma}_{1,U}(r), \sigma_{1,U}(s)) = \phi_U(\overline{\sigma}_{1,U}(r), \overline{\sigma}_{1,U}(s)) = \phi_U(r, s) = 0.$$

Thus, $\overline{\sigma}_{1,U}(r) \in (\sigma_{1,U}(\mathcal{F}_1(U)))^\perp$ and hence

$$\overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U)) \subseteq (\sigma_{1,U}(\mathcal{F}_1(U)))^\perp.$$

Conversely, let $r \in (\sigma_{1,U}(\mathcal{F}_1(U)))^\perp$. Then $\phi_U(r, \sigma_{1,U}(s)) = 0$, $s \in \mathcal{F}_1(U)$. Since $\overline{\sigma}_{1,U} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$ is an isometry and $\sigma_{1,U} = \overline{\sigma}_{1,U}|_{\mathcal{F}_1(U)}$, one has, given that $r = \overline{\sigma}_{1,U}(t)$ for some $t \in \mathcal{E}(U)$,

$$\phi_U(\overline{\sigma}_{1,U}(t), \overline{\sigma}_{1,U}(s)) = 0,$$

which, in turn, implies

$$\phi_U(t, s) = 0.$$

Consequently, $t \in (\mathcal{F}_1(U))^\perp = \mathcal{F}_1^\perp(U)$. But $r := \overline{\sigma}_{1,U}(t) \in \overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U))$, therefore

$$(\sigma_{1,U}(\mathcal{F}_1(U)))^\perp \subseteq \overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U)).$$

Since $\sigma_U(\mathcal{F}_2(U))$ is orthogonal to $\sigma_U(\mathcal{F}_1(U)) := \sigma_{1,U}(\mathcal{F}_1(U))$, it follows that $\sigma_U(\mathcal{F}_2(U)) \subseteq \overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U))$. Let $\sigma_{2,U} = \sigma_U|_{\mathcal{F}_2(U)}$. Then, the $\mathcal{A}(U)$-isometry

$$\sigma_{2,U} : \mathcal{F}_2(U) \longrightarrow \sigma_{2,U}(\mathcal{F}_2(U)) := \sigma_U(\mathcal{F}_2(U))$$

extends by induction to an $\mathcal{A}(U)$-isometry

$$\overline{\sigma}_{2,U} : \mathcal{F}_1^\perp(U) \longrightarrow \overline{\sigma}_{1,U}(\mathcal{F}_1^\perp(U)).$$

The pair $(\sigma_{1,U}, \overline{\sigma}_{2,U})$ applies isometrically $\mathcal{F}_1(U) \perp \mathcal{F}_1^\perp(U) = \mathcal{E}(U)$ onto itself, as desired. Since $\mathcal{F}_1$ and $\mathcal{F}_1^\perp$ are sub-$\mathcal{A}$-modules of $\mathcal{E}$ and each diagram

$$\begin{array}{ccc} \mathcal{F}_1(U)\perp \mathcal{F}_1^\perp(U) & \longrightarrow & \mathcal{E}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V)\perp \mathcal{F}_1^\perp(V) & \longrightarrow & \mathcal{E}(V) \end{array}$$

where U and V are open subsets of X such that $V \subseteq U$, is commutative, $(\sigma_1, \overline{\sigma}_2) \equiv (\sigma_{1,U}, \overline{\sigma}_{2,U})$ is an $\mathcal{A}$-isometry of $\mathcal{E}$ onto $\mathcal{E}$, and the proof is finished.

3 Witt's theorem second version

Unlike Theorem 6, in which the $\mathcal{A}$-bilinear morphism ϕ may be symmetric or anti-symmetric, the $\mathcal{A}$-bilinear morphism of Theorem 7, below, is assumed to be *symmetric case* and free sub-$\mathcal{A}$-modules $\mathcal{F}_1$ and $\mathcal{F}_2$ of the convenient $\mathcal{A}$-module $\mathcal{E}$ are *disjoint* and need not be pre-hyperbolic.

Theorem 7 *Let $\phi \equiv (\phi_U)$ be a symmetric $\mathcal{A}$-bilinear form on a non-isotropic convenient $\mathcal{A}$-module $\mathcal{E}$ of rank $m \geq 2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be strongly non-isotropic free sub-$\mathcal{A}$-modules of $\mathcal{E}$ such that $\mathcal{F}_1 \cap \mathcal{F}_2 = 0$, and let $\sigma : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be an $\mathcal{A}$-isometry. Then $\mathcal{F}_1^\perp$ and $\mathcal{F}_2^\perp$ are isometric.*

Proof We prove the theorem by induction on $n = \text{rank}\ (\mathcal{F}_1) = \text{rank}\ (\mathcal{F}_2)$. First, let $\Gamma(\mathcal{E}) \equiv (\mathcal{E}(U), \rho_V^U)$ and $\Gamma(\mathcal{A}) \equiv (\mathcal{A}(U), \lambda_V^U)$ denote, as usual, the (*complete*)*presheaf of sections* of $\mathcal{E}$ and $\mathcal{A}$, respectively.

Let $n = 1, m = 2, r_1 \equiv r_{1,X}$ generate $\mathcal{F}_1(X)$, and $r_2 \equiv r_{2,X} := \sigma_X(r_1)$ generate $\mathcal{F}_2(X)$; since $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-isotropic and isometric,

$$a := \phi_X(r_1, r_1) = \phi_X(r_2, r_2) \neq 0$$

in the sense that $\lambda_U^X(a) \equiv a|_U \neq 0$ for any open subset U of X. On using Theorem 1, $\mathcal{E} = \mathcal{F}_i \perp \mathcal{F}_i^\perp$; so $\mathcal{E}(X)$ has bases $B_{i,X} = \{r_i, r_i^\perp\}$ where $r_i^\perp$ spans $\mathcal{F}_i^\perp(X)$. Suppose that $\phi_X(r_1, r_2) = \phi_X(r_1, r_1) = \phi_X(r_2, r_2)$. Clearly, $r_1 - r_2 \in \mathcal{F}_1^\perp(X)$ and $\mathcal{F}_1^\perp(X) = \mathcal{F}_2^\perp(X)$. Since $\mathcal{E}$ is convenient, we have $\mathcal{F}_1(X) = \mathcal{F}_1^{\perp\perp}(X) = \mathcal{F}_2^{\perp\perp}(X) = \mathcal{F}_2(X)$, which is impossible according to our hypothesis. Thus the matrix representing ϕ_X

with respect to the basis $\{r_1, r_2\}$ is non-singular. By Adkins-Weintraub [1, Theorem 2.21, p. 3.57], the matrix $[\phi_X]_{B_{i,X}}$ representing ϕ_X with respect to the basis $B_{i,X}$ is non-singular, and consequently $\mathcal{F}_i^{\perp}(X)$ is non-isotropic. Hence,

$$[\phi_X]_{B_{i,X}} = \operatorname{diag}(a,\ b_i) \quad \text{for } i = 1, 2,$$

with $b_i := \phi_X(r_i^{\perp}, r_i^{\perp})$ a nowhere-zero section in $\mathcal{A}(X)$. By Adkins–Weintraub [1, Theorem 2.13, p. 354], there is an invertible matrix P with

$$[\phi_X]_{B_{2,X}} = P^t\, [\phi_X]_{B_{1,X}}\, P,$$

and taking determinants shows that $ab_2 = c^2 ab_1$ (where $c := \det P$ is a nowhere-zero section in $\mathcal{A}(X)$); so $f \equiv f_X : \mathcal{F}_1^{\perp}(X) \longrightarrow \mathcal{F}_2^{\perp}(X)$ defined by $f_X(r_1^{\perp}) = c^{-1} r_2^{\perp}$ yields an isometry.

Let $B_{i,U} := \{r_i|_U \equiv r_{i,U}\}$ $(i = 1, 2)$; $B_{i,U}$ is a basis of $\mathcal{F}_i(U)$. Likewise, let $(B_{i,U}^{\perp}) = \{r_i^{\perp}|_U \equiv r_{i,U}^{\perp}\}$; $(B_{i,U}^{\perp})$ is a basis of $\mathcal{F}_i^{\perp}(U)$. Fix an open set U in X, the matrices $[\phi_U]_{B_{1,U}}$ and $[\phi_U]_{B_{2,U}}$ relate to each other as follows:

$$[\phi_U]_{B_{2,U}} = M_n(\lambda_U^X)(P^t)\, [\phi_U]_{B_{1,U}}\, M_n(\lambda_U^X)(P),$$

where if $P = (p_{ij})_{1 \leq i,j \leq 2}$, then

$$M_n(\lambda_U^X)(P) = (\lambda_U^X(p_{ij})),$$

cf. Mallios [7, (1.7), p. 281]. Clearly, $\det M_n(\lambda_U^X)(P) = \lambda_U^X(c) \equiv c|_U$; so $f_U : \mathcal{F}_1^{\perp}(U) \longrightarrow \mathcal{F}_2^{\perp}(U)$, defined by setting that $f_U(r_{1,U}^{\perp}) = (c|_U)^{-1} r_{2,U}^{\perp}$ gives rise to an isometry between $\mathcal{F}_1^{\perp}(U)$ and $\mathcal{F}_2^{\perp}(U)$. But

$$\rho_V^U \circ f_U = f_V \circ \rho_V^U,$$

therefore $f \equiv (f_U) : \mathcal{F}_1^{\perp} \longrightarrow \mathcal{F}_2^{\perp}$ is an $\mathcal{A}$-morphism, from which one derives an isometry between $\mathcal{F}_1^{\perp}$ and $\mathcal{F}_2^{\perp}$; hence this part of the proof is finished.

Now, we apply induction on n. Assume that the theorem is true for

$$\operatorname{rank}(\mathcal{F}_1) = \operatorname{rank}(\mathcal{F}_2) < n,$$

and let

$$\operatorname{rank}(\mathcal{F}_1) = \operatorname{rank}(\mathcal{F}_2) = n.$$

First, we claim that there is an $r \in \mathcal{F}_1(X)$ with $\phi_X(r, r) \neq 0$ for every open $U \subseteq X$. Indeed, let $s \in \mathcal{F}_1(X)$. If $\phi_X(s, s) \neq 0$, set $r = s$. If $\phi_X(s, s) = 0$, pick $t \in \mathcal{F}_1(X)$ with $\phi_X(s, t) \neq 0$. Such a section t exists because $\mathcal{F}_1$ is assumed to be non-isotropic. If $\phi_X(t, t) \neq 0$, set $r = t$. Otherwise, note that for any $a \in \mathcal{A}(X)$,

$$\phi_X(as + t, as + t) = 2a\phi_X(s, t).$$

Setting $r := as + t$, with $a = (\phi_X(s, t))^{-1}$, we have $\phi_X(r, r) \neq 0$.

On the basis of the preceding argument, let $r_1 \in \mathcal{F}_1(X)$ with $\phi_X(r_1, r_1) \neq 0$, and let $r_2 := \sigma_X(r_1) \in \mathcal{F}_2(X)$; so

$$\phi_X(r_2, r_2) = \phi_X(r_1, r_1) \neq 0.$$

Let $\mathcal{F}_{11}$ and $\mathcal{F}_{21}$ be sub-$\mathcal{A}$-modules of $\mathcal{E}$ generated by r_1 and r_2, respectively. Then

$$\mathcal{E} = \mathcal{F}_{11} \perp \mathcal{F}_{11}^{\perp} = \mathcal{F}_{21} \perp \mathcal{F}_{21}^{\perp}$$

or

$$\mathcal{E} = \mathcal{F}_{11} \perp (\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1) \perp \mathcal{F}_1^{\perp} = \mathcal{F}_{21} \perp (\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2) \perp \mathcal{F}_2^{\perp}.$$

But

$$\mathcal{F}_1 = \mathcal{F}_{11} \perp (\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1)$$

and

$$\mathcal{F}_2 = \mathcal{F}_{21} \perp (\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2),$$

since $\sigma : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ and $\sigma|_{\mathcal{F}_{11}} : \mathcal{F}_{11} \longrightarrow \mathcal{F}_{21}$ are $\mathcal{A}$-isometries, it follows that $\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1$ and $\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2$ are isometric to each other. Moreover, it is easy to see that $\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1$ and $\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2$ are non-isotropic and of rank smaller than n. (in fact, $r_1 \notin \mathcal{F}_{11}^{\perp}(X)$ and $r_2 \notin \mathcal{F}_{21}^{\perp}(X)$.) Next, observe that

$$(\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1)^{\perp} = \mathcal{F}_{11} \perp \mathcal{F}_1^{\perp}$$

and

$$(\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2)^{\perp} = \mathcal{F}_{21} \perp \mathcal{F}_2^{\perp}.$$

So, by applying the inductive hypothesis on both $\mathcal{F}_{11}^{\perp} \cap \mathcal{F}_1$ and $\mathcal{F}_{21}^{\perp} \cap \mathcal{F}_2$ we note that $\mathcal{F}_{11} \perp \mathcal{F}_1^{\perp}$ and $\mathcal{F}_{21} \perp \mathcal{F}_2^{\perp}$ are isometric to each other, and consequently $\mathcal{F}_1^{\perp}$ and $\mathcal{F}_2^{\perp}$ are isometric; the theorem is proved.

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