Brief paper

A model predictive control approach to the periodic implementation of the solutions of the optimal dynamic resource allocation problem☆

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ABSTRACT

This paper proposes a model predictive control (MPC) approach to the periodic implementation of the optimal solutions of a class of resource allocation problems in which the allocation requirements and conditions repeat periodically over time. This special class of resource allocation problems includes many practical energy optimization problems such as load scheduling and generation dispatch. The convergence and robustness of the MPC algorithm is proved by invoking results from convex optimization. To illustrate the practical applications of the MPC algorithm, the energy optimization of a water pumping system is studied.

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1. Introduction

There is a particular class of problems in resource allocation and energy appliance scheduling pertaining to special periodic constraints in minimizing objective functions. This class of problems includes the scheduling of appliances (pumps, conveyor belts, generators, etc.) on a daily or weekly basis. The demand, operation conditions, and constraints change only periodically during this time. For the purpose of this paper, this class of problems are called *Optimal Dynamic Resource Allocation Problems* (ODRAP). Such problems will usually be solved over one period, e.g. 24 h, and the optimal solution will be repeatedly implemented for other periods without considering interactions between periods (Middelberg, Zhang, & Xia, 2009). The interaction between periods of implementation is what makes the ODRAP different from those resource allocation and scheduling problems generally studied (see Aissi, Bazgan, and Vanderpooten (2009), Belfares, Klibi, Lo, and Guitouni (2007), Biegler and Zavala (2009), Ibaraki and Katoh (1988), Lee, Kumara, and Gautam (2007), Lee and Lee (2006), Munawar and Gudi (2005), Pasadyn, Lee, and Edgar (2008), Patriksson (2008) and Zafra-Cabeza, Ridao, and Camacho (2008) and the references therein). In these papers, the research is focused on developing various algorithmic approaches to resource allocation problems

in which periodic constraints may not exist. Even though periodic constraints are included in the problem, the corresponding backgrounds are either not applicable to ODRAP or not explicitly discussed. A lack of discussion particularly pertaining to the periodic implementations of optimal solutions for periodic resource allocation problems exists. There may, in fact, be unintended interactions between two periods when an optimal solution of an ODRAP for one period is implemented over consequent periods. A ramp rate violation, for example, may occur when the optimal solution of the dynamic economic dispatch (DED) problem over the first day (Xia, Zhang, & Elaiw, 2009) is simply implemented in the second day. This ramp rate violation actually implies that the constraints in an ODRAP formulation for DED are inadequate for periodic implementation. This problem is solved by introducing more constraints and thus formulating a new ODRAP, or a *Perfect Optimal Dynamic Resource Allocation Problem* (PODRAP). Furthermore, Xia et al. (2009) also proposes an algorithm to solve this revised dynamic economic dispatch problem by the model predictive control (MPC) approach. The same approach will be extended to the general PODRAP in this paper.

The ability of the MPC to handle constraints, being able to use simple models, and its closed-loop stability and inherent robustness makes it very practical for use in industrial problems (see Allgöwer and Zheng (2000), Garcia, Prett, and Morari (1989), and Qin and Badgwell (2003)). This MPC approach is also applied in general resource allocation or scheduling problems as done in Ferrari-Trecate et al. (2004), Lee et al. (2007), Lee and Lee (2006), Munawar and Gudi (2005), van Staden, Zhang, and Xia (2009), and Zafra-Cabeza et al. (2008). It is interesting to note that MPC algorithms have a close relationship with optimization since an

MPC algorithm needs to solve an optimization problem in each iteration. Studies on the connections of MPC with optimization were conducted since the 1960's (see Chang and Seborg (1983) and Zadeh and Whalen (1962)). Modern MPC approaches for resource allocation problems do not take the relationship of the MPC solutions and the global optimal resource allocation solutions into consideration.

The aim of this paper is to develop an MPC algorithm for the periodic implementation of the optimal solutions of the PODRAP. Further it will prove the algorithm's convergence and the corresponding robustness against disturbances in controller implementation or state measurement. The convergence result reveals that the optimal solutions in the MPC algorithm converge to the optimal solution of the PODRAP. The formulation of PODRAP and the perfection of ODRAP can be found in Section 2, while the convergence and robust results are in Section 3. An example on the voluntary load shedding problem for a water pumping system is also studied in Section 4 to illustrate the formulation of a PODRAP and the convergence and robustness of the MPC approach. Some concluding remarks are drawn in the last section.

The following nomenclatures are fixed.

$\sum_{i=j}^{k} a_i$ equals 0 for $j > k$ and $a_j + \cdots + a_k$ if $j \leq k$;

$\mathrm{col}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ equals $(\alpha_1^T, \alpha_2^T, \ldots, \alpha_s^T)^T$ for any column vectors $\alpha_1, \alpha_2, \ldots, \alpha_s$;

s.t.: subject to;

m, p, n: fixed positive integers

k, j: the k-th or j-th sampling time interval;

$(k, k + p)$: the time interval from k to $k + p$ excluding k;

X_i^j: a 1-dimensional real variable for all i, j;

$X[j]$: the m-dimensional real vector variable for all j;

$X[k|j]$: the prediction of $X[k]$ at time j for any $k \geq j$;

$Z[k]$: equals $\mathrm{col}(X[k], X[k + 1], \ldots, X[k + p - 1])$;

$Z[k|j]$: the prediction of $Z[k]$ at time j for any $k \geq j$;

u_k^j: equals $X[k + 1|j] - X[k|j]$;

$u[k|j] := \mathrm{col}(u_k^j, u_{k+1}^j, \ldots, u_{k+p-1}^j) = z[k + 1|j] - z[k|j]$;

$i_1, i_2, \ldots, i_s$: integers with $1 \leq i_1 < i_2 < \cdots < i_s = n$;

$x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$;

$y^\ell := (x_{i_{\ell-1}+1}, x_{i_{\ell-1}}, \ldots, x_{i_\ell})$, $\ell = 1, 2, \ldots, s$, are called *partitions* of x such that $x = \mathrm{col}(y^1, y^2, \ldots, y^s)$.

2. Problem formulation

PODRAP $(k, k + p]$ For any give system dynamics $X[k + 1] = G(X[k])$, a PODRAP over the time interval $(k, k + p]$ is defined as the minimization problem:

$\min \quad J_{k+1}(z[k + 1|k + 1])$

s.t. $\quad H_{k+1}(z[k + 1|k + 1]) \geq 0,$ $\qquad (1)$

where the optimization variable is $z[k + 1|k + 1] = \mathrm{col}(X[k + 1|k + 1], X[k + 2|k + 1], \ldots, X[k + p|k + 1])$, and J_{k+1} and H_{k+1} are smooth, convex functions satisfying the following *periodic invariant* properties for all $k \geq 0$:

- *Periodic invariant objective*:

$J_{k+1}(X[k + 1], X[k + 2], \ldots, X[k + p])$
$\quad = J_{k+2}(X[k + 2], \ldots, X[k + p], X[k + 1]),$ $\qquad (2)$

- *Periodic invariant constraints*:

$\{(X[k + 1], X[k + 2], \ldots, X[k + p]) :$
$\quad H_{k+1}(X[k + 1], X[k + 2], \ldots, X[k + p]) \geq 0\}$
$\quad = \{(X[k + 1], X[k + 2], \ldots, X[k + p]) :$
$\quad H_{k+2}(X[k + 2], X[k + 3], \ldots, X[k + p], X[k + 1]) \geq 0\}.$
$\qquad (3)$

Definition 1. Fix the notations in PODRAP$(k, k + p]$, then the minimization problem (1) is called an *optimal dynamic resource allocation problem* (ODRAP) if the functions J_ℓ and H_ℓ are smooth, convex, and J_ℓ satisfies the periodic invariant property (2) for all $\ell \geq 1$.

Note that the only way in which the ODRAP and PODRAP differ is that an ODRAP may not satisfy the constraints in (3). A practical resource allocation problem may be an ODRAP but not a PODRAP, and the constraints can often be reasonably extended so that the periodic invariant property is satisfied (Xia et al., 2009).

The following two propositions are easy to verify and the proofs are omitted.

Proposition 1. *Assume that there exists smooth and convex functions $\alpha_i, \beta_i, i \geq 1$, such that $\alpha_{i+p} \equiv \alpha_i$, $\beta_{i+p} \equiv \beta_i$, and J_{k+1} and H_{k+1} have the following special form:*

$J_{k+1}(X[k + 1|k + 1], \ldots, X[k + p|k + 1])$
$\quad = \sum_{i=k+1}^{k+p} \alpha_i(X[i|k + 1]),$

$H_{k+1}(X[k + 1|k + 1], \ldots, X[k + p|k + 1])$
$\quad = \sum_{i=k+1}^{k+p} \beta_i(X[i|k + 1]).$

Then the following optimization problem is a PODRAP over the time period $(k, k + p]$

$\min \quad J_{k+1}(z[k + 1|k + 1])$
s.t. $\quad H_{k+1}(z[k + 1|k + 1]) \geq 0.$

Proposition 2. *Assume that J and H are symmetric functions in the sense that $J(X[k + 1|k + 1], \ldots, X[k + p|k + 1]) = J(X[\sigma(k + 1)|k + 1], \ldots, X[\sigma(k+p)|k + 1])$ and $H(X[k + 1|k + 1], \ldots, X[k + p|k + 1]) = H(X[\sigma(k + 1)|k + 1], \ldots, X[\sigma(k+p)|k + 1])$ hold for any permutation σ of $(k + 1, k + 2, \ldots, k + p)$, then the following optimization problem is a PODRAP over the time period $(k, k + p]$:*

$\min \quad J(z[k + 1|k + 1])$
s.t. $\quad H(z[k + 1|k + 1]) \geq 0.$

Definition 2. Consider the following optimization problem over the time period $(k, k + p]$

$\min \quad J_{k+1}(z[k + 1|k + 1])$
s.t. $\quad H_{k+1}(z[k + 1|k + 1]) \geq 0,$ $\qquad (4)$

where J_{k+1} and H_{k+1} are convex and smooth, and J_{k+1} satisfies (2) so that (4) is an ODRAP. Denote as Ω_{k+1} the feasible domain $\{z[k + 1|k + 1] : H_{k+1}(z[k + 1|k + 1]) \geq 0\}$, and as Ω'_{k+1} the maximum subset of Ω_{k+1} such that

$\Omega'_{k+1} := \{z[k + 1|k + 1] : H_{k+1}(z[k + 1|k + 1]) \geq 0,$
$\qquad H'_{k+1}(z[k + 1|k + 1]) \geq 0\},$ $\qquad (5)$

with H'_{k+1} a convex and smooth function, and the function $\mathrm{col}(H_{k+1}, H'_{k+1})$ satisfying the periodic invariant property (3). Then the following minimization problem over $(k, k + p]$ is called a *perfection* of the ODRAP (4):

$\min \quad J_{k+1}(z[k + 1|k + 1])$
s.t. $\quad H_{k+1}(z[k + 1|k + 1]) \geq 0, \qquad H'_{k+1}(z[k + 1|k + 1]) \geq 0.$

Obviously the perfection of an ODRAP is a PODRAP. The following result constructs a perfection for an ODRAP. The proof is straightforward and thus omitted.

Proposition 3. *Fix the notations in Definition 2 and consider the ODRAP in (4). Let*

$$\Omega'_{k+1} = \Big\{ z[k+1|k+1] : H_{k+ip+1}(X[k+1], X[k+2], \dots,$$

$$X[k+p]) \geq 0, H_{k+ip+2}(X[k+2], X[k+3], \dots,$$

$$X[k+p], X[k+1]) \geq 0, H_{k+ip+3}(X[k+3], X[k+4],$$

$$\dots, X[k+1], X[k+2]) \geq 0, \dots, H_{k+ip+p}(X[k+p],$$

$$X[k+1], \dots, X[k+p-2], X[k+p-1]) \geq 0,$$

$$\text{for all } i \geq 0 \Big\}, \qquad (6)$$

and suppose that Ω'_{k+1} is nonempty and can be defined by only a finite number of inequalities from (6). Then the following is a perfection of (4):

$$\min \quad J_{k+1}(z[k+1|k+1])$$
$$\text{s.t.} \quad z[k+1|k+1] \in \Omega'_{k+1}.$$

In case Ω'_{k+1} is empty for some k, then the corresponding optimization problem has no solution, and the ODRAP does not have a perfection. The reason for such an ill-conditioned ODRAP can be very complex. Wrong ODRAP formulations, or poor matching between the system dynamics and the constraints can lead to the nonexistence of PODRAP since the latter two are involved in the perfection process in Proposition 3. There do exist many energy problems which have a good matching between system dynamics and constraints and thus are PODRAP. The corresponding examples can be the dynamic economic dispatch problem (Xia et al., 2009), the water pumping system studied in Section 4, or any ODRAP satisfying Proposition 2 or Proposition 3.

3. MPC approach to PODRAP

Substituting the relations $X[k+1|j] = X[k|j] + u_k^j$ into the PODRAP$(k, k+p)$, one has:

$$\min \quad \widetilde{J_{k+1}}(X[k+1|k+1], u_{k+1}^{k+1}, \dots, u_{k+p-1}^{k+1})$$
$$\text{s.t.} \quad \widetilde{H_{k+1}}(X[k+1|k+1], u_{k+1}^{k+1}, \dots, u_{k+p-1}^{k+1}) \geq 0. \qquad (7)$$

If $X[k+1|k+1]$ in (7) is substituted by a constant vector, then the optimization problem (7) has been reduced into a new optimization problem which has only $(u_{k+1}^{k+1}, \dots, u_{k+p-1}^{k+1})$ as its optimization variable. Denote this new optimization problem by PODRAP$_u(k, k+p)$. Similarly, the substitution of $X[k+1|k+1]$ by a constant vector in (1) leads to a new optimization problem with optimization variables $(X[k+2|k+1], \dots, X[k+p|k+1])$, denoted by PODRAP$_X(k, k+p)$. Now the following MPC algorithm in control variable form is obtained.

Algorithm 1. *Initialization* Input $X[1|1]$, and let $k = 0$.
(1) Measure $X[k+1|k+1]$ and solve PODRAP$_u(k, k+p)$ to find its optimal solution $u_{opt}^k = col(u_{k+1}^{k+1}|_{opt}, \dots, u_{k+p-1}^{k+1}|_{opt})$;
(2) Implement $X[k+2|k+2] = X[k+1|k+1] + u_{k+1}^{k+1}|_{opt}$ in the system, let $k = k + 1$ and go to Step (1).

This algorithm is equivalent to the next algorithm in state variable form:

Algorithm 1′. *Initialization* Input $X[1|1]$, and let $k = 0$.
(1) Measure $X[k+1|k+1]$ and solve PODRAP$_X(k, k+p)$ to find its optimal solution $z_{opt}^k = col(X[k+2|k+1]|_{opt}, \dots, X[k+p|k+1]|_{opt})$;
(2) Implement $X[k+2|k+2] = X[k+2|k+1]|_{opt}$, let $k = k + 1$ and go to Step (1).

The convergence of the MPC algorithm can be obtained by studying the following convex optimization problem:

$$\min \quad f(x)$$
$$\text{s.t.} \quad x \in \Omega := \{ x : g(x) \geq 0, h(x) = 0 \}, \qquad (8)$$

where f is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^1$, g and h are vector-valued smooth functions, f is convex over the feasible domain Ω which is nonempty, bounded and convex.

The above convex optimization problem can be solved by the gradient method. The optimal solution x_{grad}^* and the global minimum value $f(x_{grad}^*)$ are obtained. If the function is strictly convex, then this optimal solution x_{grad}^* is also unique. In the following we will show that the same problem can also be solved by Algorithm 2.

Algorithm 2. Take a partition $(y^1, y^2, \dots, y^s)$ for x with $y^j \in \mathbb{R}^{i_j - i_{j-1}}$ and $i_0 = 0$. Choose any initial value $x^0 = col(y_0^1, y_0^2, \dots, y_0^s) \in \Omega \subseteq \mathbb{R}^n$ and an error bound $\epsilon > 0$, let Initial$^a = y_0^1$, Initial$^b = col(y_0^2, \dots, y_0^s)$, $k = 1, f_{old}^* = f(y_0^1, y_0^2, \dots, y_0^s)$.

(i) Let $\tilde{y} = col(y^1, \dots, y^{k-1}, y^{k+1}, \dots, y^s)$, $F_k(\tilde{y}) := f(y^1, y^2, \dots, y^s)|_{y^k = \text{Initial}^a}$, and solve the following problem

$$\min \quad F_k(\tilde{y})$$
$$\text{s.t.} \quad g(y^1, y^2, \dots, y^s)|_{y^k = \text{Initial}^a} \geq 0, \qquad (9)$$
$$h(y^1, y^2, \dots, y^s)|_{y^k = \text{Initial}^a} = 0,$$

by the initial value $\tilde{y} = \text{Initial}^b$. Denote the optimal solution by $\tilde{y}_{new}^* = col(y_*^1, \dots, y_*^{k-1}, y_*^{k+1}, \dots, y_*^s)$.

(ii) If $F_k(\tilde{y}_{new}^*) < f_{old}^* - \epsilon$, then let $f_{old}^* = F_k(\tilde{y}_{new}^*), k = (k+1) \mod s$, Initial$^a = y_*^k$, Initial$^b = col(y_*^1, \dots, y_*^{k-1}, y_*^{k+1}, \dots, y_*^s)$ for the case $2 \leq k \leq s$ and Initial$^b = col(y_*^2, y_*^3, \dots, y_*^s)$ for $k = 1, x^k = col(y_*^1, \dots, y_*^{k-1}, \text{Initial}^a, y_*^{k+1}, \dots, y_*^s)$, and go to Step (i); otherwise stop the algorithm and return k, Initial^a, Initial^b, f_{old}^* and x^k.

Remark 1. Note that the above algorithm solves iteratively subproblems (9) to approach the solution of (8), each subproblem (9) has a lower dimension than the original problem (8). This algorithm does not specify any solution algorithm to solve the optimization problem in each iteration. Generally gradient based algorithms are good enough to compute these convex optimization problems. However, when uncertainties are considered, the optimization problems may not be convex and alternative algorithms such as genetic algorithms, particle swarm optimization, ant colony optimization, etc., from intelligent computing (Schrijver, 1998) can be very helpful.

Theorem 1. *Fix the notations in problem (8) and a partition $x = col(y^1, y^2, \dots, y^s)$, then Algorithm 2 solves problem (8) in the sense that its output x^k satisfies $\lim_{k \to \infty} x^k = x_0^*$ for some x_0^* and $f(x_0^*) = f(x_{grad}^*)$. If f is also strictly convex, then $x_0^* = x_{grad}^*$.*

Proof. Note that Ω is bounded and the value of f strictly decreases in Algorithm 2, this algorithm must converge to some point x_0^*, and it suffices to show that $f(x_0^*)$ equals $f(x_{grad}^*)$. Let $x_0^* = col(y_*^1, y_*^2, \dots, y_*^s)$ and $x_{grad}^* = col(y_{grad}^1, y_{grad}^2, \dots, y_{grad}^s)$. Since Algorithm 2 stops at x_0^*, the point $col(y_*^1, \dots, y_*^{k-1}, y_*^{k+1}, \dots, y_*^s)$ is a global optimal solution of the subproblem (9) for any $k = 1, 2, \dots, s$. By Theorem 3.4.3 of Bazaraa, Sherali, and Shetty (1993), we have $\nabla F_k(\tilde{y})|_{x_0^*} \tilde{y} \geq 0$ for all $\tilde{y}$. In other words, we have

$$\frac{\partial f}{\partial y^1}\Big|_{x_0^*} y^1 + \cdots + \frac{\partial f}{\partial y^{k-1}}\Big|_{x_0^*} y^{k-1} + \frac{\partial f}{\partial y^{k+1}}\Big|_{x_0^*} y^{k+1} + \cdots + \frac{\partial f}{\partial y^s}\Big|_{x_0^*} y^s \geq 0.$$

Note that the above inequality holds for all $k = 1, \dots, s$, and all $y^1, y^2, \dots, y^s$, therefore, $\nabla f|_{x_0^*} x = \frac{\partial f}{\partial y^1}\Big|_{x_0^*} y^1 + \frac{\partial f}{\partial y^2}\Big|_{x_0^*} y^2 + \cdots + \frac{\partial f}{\partial y^s}\Big|_{x_0^*} y^s \geq 0$, where $x = col(y^1, y^2, \dots, y^s)$ is arbitrary. Again by Theorem 3.4.3 of Bazaraa et al. (1993), one has that x_0^* is a global optimal solution of (8), and $f(x_0^*) = f(x_{grad}^*)$. When f is strictly convex, the solution is unique and thus $x_0^* = x_{grad}^*$. □

The proof of the convergence of Algorithm 1′ follows from the observation that there is a one-to-one correspondence between the loops in Algorithm 1′ and the loops in Algorithm 2. For instance, consider the first loop of Algorithm 1′ for $k = 0$. Given $X[1|1] = X_0$, the algorithm aims to find the minimum value of $J_1(X_0, X[2|1], X[3|1], \ldots, X[p + 1|1])$ under the constraints $H_1(X_0, X[2|1], X[3|1], \ldots, X[p + 1|1]) \geq 0$. Now consider the first loop of applying Algorithm 2 to the PODRAP. Fix the partition $z[1|1] = \text{col}(y^1, y^2, \ldots, y^p)$ with $y^i := X[i|1]$, $i = 1, \ldots, p$. Take Initial$^a = X_0$, the optimization problem in Step (i) of Algorithm 2 is the same as that in Algorithm 1′, which establishes the one-to-one correspondence.

Theorem 2. *Let z^* be the globally optimal solution of the PODRAP $(0, p)$, $z_{\text{opt}}^k = \text{col}(X[k + 2|k + 1]|_{\text{opt}}, \ldots, X[k + p|k + 1]|_{\text{opt}})$, the optimal solution of the k-th loop in Algorithm 1′ which is obtained under the initial value $X[k + 1|k + 1] = X_{\text{initial}}^k$, and $X_{\text{initial}}^k = X[k + 1|k]|_{\text{opt}}$ for $k \geq 1$. Suppose $k = pk_1 + r_1$ with $0 \leq r_1 < p$. Denote $\bar{z}^k = \text{col}(X[pk_1 + p + 1|k + 1]|_{\text{opt}}, \ldots, X[pk_1 + p + r_1|k + 1]|_{\text{opt}}, X_{\text{initial}}^k, X[pk_1 + r_1 + 2|k + 1]|_{\text{opt}}, \ldots, X[pk_1 + p|k + 1]|_{\text{opt}})$ for the case $r_1 \geq 1$, and $\bar{z}^k = \text{col}(X_{\text{initial}}^k, X[k + 2|k + 1]|_{\text{opt}}, \ldots, X[k + p|k + 1]|_{\text{opt}})$ for $r_1 = 0$; then $\lim_{k \to \infty} J_k(\bar{z}^k) = J_1(z^*)$. If, furthermore, the functions J_1 and H_1 are strictly convex, then the optimal solution of each loop in Algorithm 1′ is unique, and therefore $\lim_{k \to \infty} \bar{z}^k = z^*$.*

Now consider the robustness of Algorithm 1′. For convenience, we consider only the case that uncertainty happens in the execution of the controller or the measurement of states, that is, consider $X[k + 2|k + 2] = X[k + 1|k + 1] + u_{k+1}^{k+1}|_{\text{opt}} + w_{k+1}$ or equivalently $X[k + 2|k + 2] = X[k + 2|k + 1]|_{\text{opt}} + w_{k+1}$, where w_{k+1} is a disturbance vector satisfying $\|w_{k+1}\| < e$ and e is a constant. By a similar reason as the proof of convergence of the MPC algorithm, the robustness can be shown by proving the robustness of Algorithm 2. To this end, let Initial$^a = y_*^k + w_*^k$ in Step (ii) of Algorithm 2, where w_*^k is a disturbance vector satisfying $\|w_*^k\| < e$.

Theorem 3. *Fix the notations in problem (8) and a partition $x = \text{col}(y^1, y^2, \ldots, y^s)$, and assume that $\|\nabla f(x)\| \leq L$ for a constant L for all $x \in \Omega$. Let x_{grad}^* be the optimal solution of problem (8) obtained by gradient method; ϵ the error bound defined in Algorithm 2; c a positive constant which is less than ϵ, the constant disturbance w_*^k satisfying $\|w_*^k\| < e$; Initial$^a = y_*^k + w_*^k$ executed in Step(ii) of Algorithm 2; and e small enough so that $e < \min\{c/L, (\epsilon - c)/L\}$; then there exists an integer N_0 such that for any $k > N_0$, the output x^k of the k-th loop in Algorithm 2 belongs to the domain $\Omega' := \{x : \|x - x_{\text{grad}}^*\| < c\}$.*

Proof. Consider the k-th loop in Algorithm 2. Step (i) gives an optimal solution $\tilde{y}_{\text{new}}^* = \text{col}(y_*^1, \ldots, y_*^{k-1}, y_*^{k+1}, \ldots, y_*^s)$ for Initial^a equals some vector ξ^k. Denote $z_k = \text{col}(y_*^1, \ldots, y_*^{k-1}, \xi^k, y_*^{k+1}, \ldots, y_*^s)$, and suppose $F_k(\tilde{y}_{\text{new}}^*) < f_{\text{old}}^* - \epsilon$. Define $\bar{z}_k = \text{col}(y_*^1, \ldots, y_*^{k-1}, \xi^k, y_*^{k+1} + w_*^k, y_*^{k+2}, \ldots, y_*^s)$. In the $(k + 1)$-th loop, denote the optimal solution of step (i) by $\text{col}(\tilde{y}_*^1, \ldots, \tilde{y}_*^k, \tilde{y}_*^{k+2}, \ldots, \tilde{y}_*^s)$, and define similarly $z_{k+1} = \text{col}(\tilde{y}_*^1, \ldots, \tilde{y}_*^k, y_*^{k+1} + w_*^k, \tilde{y}_*^{k+2}, \ldots, \tilde{y}_*^s)$ and the corresponding $\bar{z}_{k+1}$ under the disturbance w_*^{k+1}.

Without loss of generality suppose the three points $\bar{z}_k$, z_k, and z_{k+1} are different from each other. Obviously $f(\bar{z}_k) < f(z_{k+1})$, and $\delta_k := f(\bar{z}_k) - f(z_{k+1}) > 0$, $|f(z_k) - f(\bar{z}_k)| = |\nabla f|_\xi (z_k - \bar{z}_k)| \leq \|w_*^k\| \|\nabla f|_\xi\| \leq eL$. Note that if $\delta_k < c$, then $|f(z_k) - f(z_{k+1})| \leq |f(z_k) - f(\bar{z}_k)| + |f(\bar{z}_k) - f(z_{k+1})| < c + eL < \epsilon$, and Algorithm 2 stops. This implies that z_k is treated as x_{grad}^*, thus $|f(\bar{z}_k) - f(x_{\text{grad}}^*)| = |f(\bar{z}_k) - f(z_k)| \leq eL < c$, and $\bar{z}_k$ enters the domain Ω'. If $\delta_k \geq c$, then $f(z_k) - f(z_{k+1}) = f(z_k) - f(\bar{z}_k) + f(\bar{z}_k) - f(z_k) > \delta_k - eL \geq c - eL > 0$.

Repeat the above steps for all the k, then either $f(\bar{z}_k)$ enters the domain Ω', or $f(z_k)$ continues decreasing. Note that the problem

is over a bounded convex domain, therefore in the latter case $f(z_k)$ converges to $f(x_{\text{grad}}^*)$. Then by a similar analysis as done for the case $\delta_k < c$, $f(\bar{z}_k)$ will eventually enter the domain Ω'. □

A similar procedure as the proof of the convergence of the MPC algorithm shows the following robustness result.

Corollary 1. *Suppose $\Omega_{\text{PODRAP}} := \{\text{col}(X[1], \ldots, X[p + 1]) : H_1(X[1], \ldots, X[p + 1]) \geq 0\}$ is the feasible domain of the problem PODRAP$(0, p)$, z^* is the globally optimal solution of PODRAP$(0, p)$, the norm of the gradient of the cost function J_1 of PODRAP$(0, p)$ has the upper bound L on Ω_{PODRAP}, $z_{\text{opt}}^k = \text{col}(X[k + 2|k + 1]|_{\text{opt}}, \ldots, X[k + p|k + 1]|_{\text{opt}})$ is the optimal solution of the k-th loop in Algorithm 1′ which is obtained under the initial value $X[k + 1|k + 1] = X_{\text{initial}}^k$, and $X_{\text{initial}}^k = X[k + 1|k]|_{\text{opt}} + w_k$ for $k \geq 1$. Assume also that ϵ is a small enough positive constant, c is a positive constant which is less than ϵ, the disturbance w_k satisfies $\|w_k\| < e$, e is small enough so that $e < \min\{c/L, (\epsilon - c)/L\}$, $k = pk_1 + r_1$ with $0 \leq r_1 < p$, and define $\bar{z}^k$ as Theorem 2, then $\|\bar{z}^k - z^*\| < c$.*

Remark 2. In a practical energy system, the end-user demand will often be approximately periodic which corresponds to the case that disturbances happen in some coefficients in the function H_1. Since the optimal solution of the convex optimization problem depends smoothly on the corresponding coefficients in H_1, the MPC algorithm is also robust against the disturbances in the demand.

4. A case study on a water pumping system

Voluntary load shedding (VLS), or the so-called strategic offer, is a scheme proposed by a South African electricity supplier to solve the serious electricity shortage problem (Zhang, Xia, & Alexander, 2008). In Zhang et al. (2008), the optimal scheduling of a pumping system with 21 pumps in terms of VLS is solved with an open loop control approach. In this case study, this pumping system is reconsidered to illustrate the MPC approach. In order to simplify the model, it is assumed that all the water from the 21 pumps flows to a big reservoir instead of the 4 reservoirs in Zhang et al. (2008). The VLS model over a 24 h period with $24 \times 21 = 504$ number of variables and 101 constraints can be summarized as follows.

$$\min \sum_{i=1}^{21} \left[\sum_{j=1}^{24} c_j P_i u_{ij} - q \sum_{peak} P_i (1 - u_{i\ell}) \right]$$

$$\text{s.t.} \quad \sum_{i=1}^{21} V_i u_{ij} \geq D_j, \qquad R^{\min} \leq R_k \leq R^{\max},$$

$$0 \leq u_{ij} \leq 1, \quad \sum_{i=1}^{21} P_i u_{ij} \leq P_{\max}, \tag{10}$$

$$\sum_{i=1}^{21} V_i u_{ij} \leq V_{\text{supply}},$$

$$\sum_{\ell \in peak} \sum_{i=1}^{21} P_i u_{i\ell} \leq r P_{\max},$$

$$1 \leq j \leq 24, \qquad 2 \leq k \leq 24,$$

where $u_{ij} \in [0, 1]$ is the switching status of the i-th pump at the j-th time interval. *Peak* means the index ℓ in the summation varies in the peak tariff time intervals. The electricity price at time j is c_j, q is the incentive constant and V_i denotes the volume of water that the i-th pump can pump within one hour when the pump is working at its maximum power P_i. $R^{\min}$ and $R^{\max}$ are constants representing the capacity of the reservoir, R_k is the water level of the reservoir at time k, $R_1 = R^{\min}$, $P_{\max}$ is the specified maximum electricity demand. V_{supply} is a constant denoting the incoming water supply rate (Mega-liter per hour, or Ml/h), and r is the percentage so that a percentage $(1 - r)$ of the maximal possible total load during peak hours is shed.
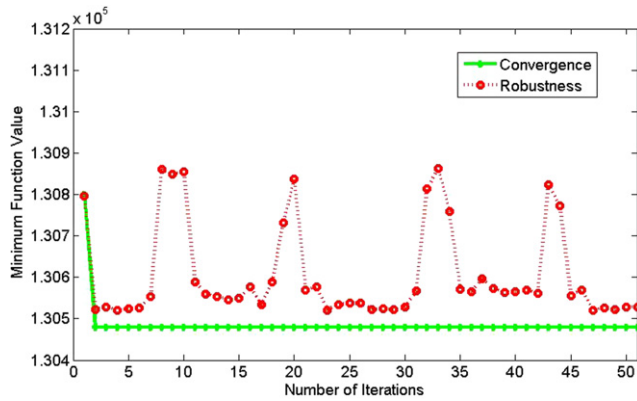
Fig. 1. Convergence and robustness.

This is an ODRAP. By Proposition 3, a PODRAP is obtained by adding the missing constraints $\sum_{i=1}^{21} V_i(u_{i\,24} + \cdots + u_{i2} + u_{i1}) \leq R^{\max} - R^1 + (D_1 + D_2 + \cdots + D_{24})$. For this PODRAP, MPC Algorithm 1 converges to the global minimum 1.3048×10^5 of VLS(0, 24] at the second loop (see the solid line in Fig. 1). As for the robustness of the MPC algorithm, assume that $u_{\mathrm{opt}}[k|k]$ in Step (2) of Algorithm 1 is replaced by $u_{\mathrm{opt}}[k|k] + w[k]$, where $w[k]$ is a noise vector and each of its components is generated randomly by the Matlab uniform distribution function rand$(1)/50$. The minimum objective function value varies within a small neighborhood of the global minimum value 1.3048×10^5 as is clear from the dotted line in Fig. 1. If the first iteration step is excluded, then Fig. 1 shows that the maximum error happens at the 8-th step. The relative error of 0.29% is quite satisfactory.

5. Conclusions

This paper introduces the perfect optimal dynamic resource allocation problem and provides a corresponding MPC algorithm to solve the periodic implementation problem for the optimal solution. The convergence and robustness of the MPC algorithm are proved. This establishes a close connection of an MPC solution and a solution of a global optimization problem. An application of the MPC approach to the voluntary load shedding problem for a water pumping system illustrates the convergence and robustness of this MPC algorithm.

Acknowledgements

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