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Full Length Research Paper

A new derivative-free method for solving nonlinear equations

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This paper presents a new method for solving nonlinear equations. The method is free of derivatives and easy to implement. Although its convergence is linear, numerical experiments conducted indicate that its performance compares well not only with Newton-based methods of higher order but with other derivative-free methods as well.

Key words: Derivative-free iterative method, nonlinear equations.

INTRODUCTION

Newton's method is one of the most popular techniques for solving the nonlinear equation (Abbasbandy et al., 2007).

$$\phi(x) = 0. \quad (1)$$

As it is well known, the method is quadratic in convergence but its convergence is slow or it fails to converge at all when either $f'(x)$ is too small at any stage of the computation, or when the initial values are too far removed from the true solution.

This paper presents a new method for solving nonlinear equations. The method is derivative-free and its convergence is linear. However, as numerical experiments conducted indicate, although its convergence is of order one, for certain problems, the method converges faster than some Newton-based methods of higher order; in particular, when Newton-based methods fail to converge, the method can be used to find the solution of the problem. Numerical experiments conducted also show that the method compares well with other iterative methods that do not depend on derivatives.

DEVELOPMENT OF THE METHOD

Consider the solution of the equation

$$x^{\psi_1+\psi_2} = 1, \quad (2)$$

where $\psi_1 + \psi_2$ is an index, by using the iterative scheme of steps 1 - 6 as thus shown.

Step 1

$x^{(0)} := x_0$, x_0 given;

Step 2

$i = 0$;

Step 3

Set $y^{(i)} := 1/\left(x^{\psi_2}\right)$ (3)

Step 4

Set $x^{(i+1)} := \left(y^{(i)}\right)^{1/\psi_1}$ (4)

Step 5:

If $\left|x^{(i+1)} - x^{(i)}\right| < \varepsilon$ where ε is a predetermined accuracy limit, go to Step 6, otherwise increase i by 1 and go to Step 3;

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Table 1. Tableau to the solution of the equation $x^{\psi_1+\psi_2}=1.$

i	$x^{(i)}$	$y^{(i)}$
0	x_0	$(x_0)^{-\psi_2}$
1	$(x_0)^{-\psi_2/\psi_1}$	$(x_0)^{\psi_2^2/\psi_1}$
2	$(x_0)^{\psi_2^2/\psi_1^2}$	$(x_0)^{-\psi_2^3/\psi_1^2}$
3	$(x_0)^{-\psi_2^3/\psi_1^3}$	$(x_0)^{\psi_2^4/\psi_1^3}$
4	$(x_0)^{\psi_2^4/\psi_1^4}$	$(x_0)^{-\psi_2^5/\psi_1^4}$

Step 6 (End of iteration)

This iterative scheme is essentially a rearrangement of (2) into two constituent equations that define the algorithm. These equations are:

$$y = 1/x^{\psi_2} \quad \text{and} \quad x = y^{1/\psi_1}.$$

CONVERGENCE OF ALGORITHM

Table 1 shows the construction of the sequence of trial solutions for the solution of (2), where the sequence is generated by the terms:

$$\left\{ (x_0)^{\left(-\psi_2/\psi_1\right)^i} \right\}_{i=0}^{\infty}$$

Theorem 1

Let

$$x_{i+1} = \phi(x_i) = (x_i)^{-\psi_2/\psi_1}. \tag{5}$$

The algorithm (5) converges when:

$$\left|\frac{-\psi_2}{\psi_1}\right| < 1. \tag{6}$$

Proof

Let

$$\{S_i\}_{i=0}^{\infty} = \{x_i\}_{i=0}^{\infty}, \quad = x_0, x_1, x_2, \cdots.$$

The result follows since, if we let

$r = \dfrac{-\psi_2}{\psi_1}$, it follows that

$$\frac{\ln\{S_i\}_{i=0}^{\infty}}{\ln x_0} = \{r^i\}_{i=0}^{\infty} \text{ so that } \lim_{i\to\infty}\{r^i\} \to 0 \text{ when } |r|<1.$$

Theorem 2

The convergence of iterative process (5) is of order one.

Proof

Following the definition of Traub [1964], an iterative process ϕ is of order p if and only if

$$\phi(x_0) = x_0, \phi^{(j)}(x_0) = 0, j = 1, 2, \cdots, p-1; \phi^{(p)} = 0. \tag{7}$$

From (5), $\phi(x_0) = x_0$ since

$$\begin{aligned}\phi(x_0) &= (x_0)^{\frac{-\psi_2}{\psi_1}} \\ &= \left(x_0^{-\psi_2}\right)^{\frac{1}{\psi_1}} \\ &= x_0\end{aligned}$$

using (2). Also the formal differentiation of (5), and using (2) again, gives

$$\begin{aligned}\phi'(x_i) &= \frac{-\psi_2}{\psi_1}(x_i)^{\left(\frac{-\psi_2}{\psi_1}-1\right)} \\ &= \frac{\psi_2}{\psi_1}(x_i)^{\left((-\psi_2)\left(\frac{1}{\psi_1}\right)-1\right)} \\ &= \frac{-\psi_2}{\psi_1}\end{aligned}$$

Thus $p=1$, and hence algorithm (5) is of order one.

Theorem 3

Let $e_{i+1} = x_{i+1} - x_0$ and $e_i = x_i - x_0$.

Table 2. Comparison with methods of Abbasbandy, Tan and Liao [2007].

Eg	NM	NT	ADM	HPM	HAM (-2.0)	HAM (-3.0)	HAM (-4.0)
1	11	103	53	53	38	39	24

Table 3. Comparison with the method of Kogan, Sapir and Sapir (KSS) [2007] and Newton's method.

Eg	x_0	Solution	NM	NT	KSS
2	3	2.236067978	12	Fails	5

Then

$$\frac{e_{i+1}}{e_i} = \frac{-\psi_2}{\psi_1}. \tag{8}$$

Proof

Since, from Theorem 2,

$\phi'(x_i) = \frac{-\psi_2}{\psi_1},$ it follows that

$$\phi^{(j)}(x_i) = 0, \qquad j = 2, 3, \cdots.$$

Now, formally making a Taylor expansion of $\phi(x_i)$ about x_0 gives

$$x_{i+1} = \phi(x_i) = x_0 - \frac{\psi_2}{\psi_1}(x_i - x_0),$$

from which the required result follows.

NUMERICAL EXPERIMENTS

Example 1

This is from Abbasbandy et al. (2007).

$$\phi(x) = x^2 - e^x - 3x + 2$$

Equations (3) and (4) above are taken in the form

$$y^{(m)} = e^{x^{(m)}} - \left(x^{(m)}\right)^2; \tag{9}$$

$$x^{(m+1)} = \frac{1}{2}\left(2 - y^{(m)}\right). \tag{10}$$

It is clear from Table 2 that, for this example, the new method (NM) takes the least number of iterations to converge to the solution compared with Newton's method (NT), Adomian's decomposition method (ADM), the homotopy perturbation method (HPM) and the homotopy analysis method (HAM) with the auxiliary parameter equal to 2.0, -3.0 and -4.0. This supports the claim that although the convergence of the new method is linear, for certain problems, it converges faster than Newton-based method of higher order.

Example 2

This is from Kogan et al. (2007).

$$\phi(x) = (x^2 - 3)\ln\left(x^2 + \sqrt{x^2 - 1}\right) - \frac{x^2}{x^2 + 5} - 2\ln 7 + \frac{1}{2};$$

Here we let

$$y^{(m)} = \frac{A}{B} \tag{11}$$

where,

$A = \frac{\left(x^{(m)}\right)^2}{\left(x^{(m)}\right)^2 + 5} + 2\ln 7 - \frac{1}{2}$ and

$$B = \ln\left(\left(x^{(m)}\right)^2 + \sqrt{\left(x^{(m)}\right)^2 - 1}\right); \; x^{(m+1)} = \sqrt{y^{(m)} + 3}. \tag{12}$$

Table 3 supports the assertion that the new method (NM) can be used to solve certain nonlinear equations when Newton's method (NT) fails to converge. Furthermore, although the non-stationary second-order method takes fewer iterations than the new method to converge to the solution, it has the general disadvantage that, if the solution is required to a higher accuracy, more complicated formulas need to be employed to obtain successive approximations.

Examples 3, 4 and 5 are from Wu and Han (2000).

Table 4. Comparison of the new method (NM) with Newton's method (NT), the Trapezoidal rule (TR) and the Higher Order method (HO).

Eg	x_0	Solution	NM	NT	TR	HO
3	-0.5	1.36523001341410	53	132	7	4
4	3.5	3	225	13	9	6
5	2	1	1	7	5	3

Example 3.

$$\phi(x) = x^3 + 4x^2 - 10$$

In this example we let

$$y^{(m)} = 10 - \left(x^{(m)}\right)^3; \quad (13)$$

$$x^{(m+1)} = \frac{1}{2}\sqrt{y^{(m)}}. \quad (14)$$

Example 4

$$\phi(x) = e^{x^2+7x-30} - 1$$

Here we set

$$y^{(m)} = 30 - \left(x^{(m)}\right)^2; \quad (15)$$

$$x^{(m+1)} = \frac{1}{7}y^{(m)} \quad (16)$$

Example 5

$$\phi(x) = \ln(x)$$

For this example let

$$y^{(m)} = 1; \quad (17)$$

$$x^{(m+1)} = e^{y^{(m)}-1} \quad (18)$$

Table 4 supports the claim that, for certain problems, the new method converges faster than Newton's method.

Examples 6, 7, 8 and 9 are from Wu and Wu (2006) and in Steffenson (1933).

Example 6

$$\phi(x) = \arctan(x)$$

$$y^{(m)} = 1; \quad (19)$$

$$x^{(m+1)} = \tan(y^{(m)} - 1). \quad (20)$$

Example 7

$$\phi(x) = xe^x - 0.1$$

$$y^{(m)} = 0.1e^{x^{(m)}}; \quad (21)$$

$$x^{(m+1)} = y^{(m)}. \quad (22)$$

Example 8

$$\phi(x) = \ln(x)$$

$$y^{(m)} = 1; \quad (23)$$

$$x^{(m+1)} = e^{y^{(m)}-1}. \quad (24)$$

Example 9

$$\phi(x) = x + 1 - e^{\sin x}$$

$$y^{(m)} = e^{\sin x^{(m)}} - 1; \quad (25)$$

$$x^{(m+1)} = y^{(m)}. \quad (26)$$

Table 5 supports the claim that the new method fares well in comparison with other derivative-free methods.

CONCLUSION

In this paper a new derivative-free iterative method has been presented for solving nonlinear equations. The convergence of the method is linear but numerical experiments conducted support the claim that the method can be used to solve nonlinear equations when Newton-based methods of higher order fail to converge. The

Table 5. Comparison with the methods in Wu and Wu 2000] and in Steffenson [1933].

Eg	x_0	Solution	$\lvert f(x) \rvert$	NM	Wu	Steffenson
6	3	0	0	1	7	Diverges
7	1	0.11183255915896296498	0.1166×10^{-17}	20	6	Fails
8	5	1	0	1	8	Diverges
9	1	1.69681238680975152905	0.6967×10^{-17}	37	9	Diverges

method also compares well with other derivative-free methods.

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