Geometric Steady States of Nonlinear Systems

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Abstract—**The analytic concept of steady states for nonlinear systems was introduced by Isidori and Byrnes, and its geometric properties were also given implicitly mixed with the solvability of the output regulation problem for nonlinear systems with neutrally stable exogenous signals. In this technical note, a geometric definition of steady states for nonlinear systems, which is named as geometric steady state, is formulated independent of the output regulation problem so that it can be applied to many problems other than output regulation and the exogenous system can be unstable too. Some sufficient conditions for the existence of geometric steady states and a practical application in robotics are also provided.**

Index Terms—**Attractiveness, controlled invariance, output regulation, steady state, Sylvester equation.**

I. Introduction

STEADY state analysis of linear systems comes from linear circuit theory. The concept was soon found very useful in general linear and nonlinear systems. A formal analytic definition of steady states for nonlinear systems was introduced by Isidori and Byrnes in 1990 in their work on output regulation problems [7]. Although this analytic definition of nonlinear steady states is quite general, the main attention of [7] is the output regulation problem and therefore the results of [7] are limited to nonlinear systems with neutrally stable exogenous systems. Geometric characterizations on the solvability of this kind of output regulation problem are given in [7] and [6], and the inherent geometric properties of steady states are implied and given mixed together with the solvability of output regulation problem. It is also noted by Chen and Huang [2] that the neutral stability of the exogenous system is restricted sometimes even for the output regulation problem. Therefore this technical note devotes to give a general geometric definition of steady states so that it is able to be applied in more problems other than output regulation and this new definition can cover the case of unstable exogenous signals too. This geometric definition is named as geometric steady states to indicate the difference with usual understanding of steady states. In fact, a geometric steady state may not be a constant state, and it can be a manifold with dimension greater than 0.

For a general nonlinear system $\dot{x} = f(x, u)$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, assume that $f(0,0) = 0$ and $x(t, x^0, u(\cdot))$ is the value of the state x at time t under initial value x^0 and input $u(\cdot)$. Then [6] defines a (local) *steady state* of this nonlinear system to be a state $x(t, x^*, u^*(\cdot))$ where x^* is an initial state, $u^*(\cdot)$ is a specific input, and $x(t, x^*, u^*(\cdot))$ satisfies

$$\lim_{t \longrightarrow \infty} \|x(t, x^0, u^*(\cdot) - x(t, x^*, u^*(\cdot))\| = 0 \tag{1}$$

for every x^0 in some neighborhood U^* of x^*.

Note that the above concept of steady state is independent of the output regulation problem.

Manuscript received July 25, 2008; revised January 27, 2009. First published March 01, 2010; current version published June 09, 2010. This technical note was presented in part at the 6th IEEE International Conference on Control and Automation, Guangzhou, China, May 30–June 1, 2007. This work was supported by the National Research Foundation. Recommended by Associate Editor Z. Qu.

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Digital Object Identifier 10.1109/TAC.2010.2044261

In an attempt to solve the output regulation problem, Isidori and Byrnes [7] considered the (local) steady state for nonlinear control systems

$$\dot{x}(t) = f(x(t), u(t), \omega(t)) \tag{2}$$

$$\dot{\omega}(t) = S(\omega(t)) \tag{3}$$

under the assumption that the exogenous signal $\omega(t)$ comes from a system (3) satisfying that all eigenvalues of $\partial S(0)/\partial\omega$ have zero real parts. By invoking the Centre Manifold Theorem [1], it is proven [7], [6] that a local steady state of (2) with exponential convergence in the corresponding relation (1) exists if and only if: i) (A, B) is a stabilizable pair, where $A = \partial f(0,0,0)/\partial x$, $B = \partial f(0,0,0)/\partial u$, and, ii) *the regulator equation* is solvable for smooth functions $\pi(\omega)$ and $c(\omega)$

$$\frac{\partial\pi(\omega)}{\partial\omega} S(\omega) = f(\pi(\omega), c(\omega), \omega). \tag{4}$$

In differential geometric terms, the solvability of (4) is equivalent to the existence of $\pi(\omega)$ such that $N = \{(x, \omega) \mid x - \pi(\omega) = 0\}$ is a locally controlled invariant submanifold [6] of (2) and (3). Reference [7] further showed, among other things, that the output regulation problem of the system (2) and (3) with an output $y = h(x(t), \omega(t))$ is locally solvable via a static state feedback if and only if a steady state of the form $x_{ss}(t) = \pi(\omega(t))$ exists for the function π, and the controlled invariant submanifold N is output-zeroing, i.e., $h(\pi(\omega(t)), \omega(t)) = 0$.

Note that in the above nonlinear definition of steady states, it is required that a steady state solutions exist for *each and all* $\omega(t)$ generated by (3) through assigning freely the initial conditions. The nonlinear steady state concept is meaningful if steady state solutions exist for the whole class of exogenous signals. This requirement probably stems from the output regulation problem in which the system output should emulate all (not just one) signals from a whole class.

In this technical note we direct our attention to the definition and existence of geometric steady states of nonlinear systems with exogenous signals. This geometric formulation is obtained by a thorough study on the mixed properties of steady states and solvability of output regulation problem obtained in [7] and [6]. The new definition of geometric steady states covers the case of unstable exogenous systems too. We start with some easy characterizations of the linear cases. Sufficient conditions are given for the existence of geometric steady states of nonlinear systems. Applications of the geometric steady states theory can be made to nonlinear observer design and the output regulation of nonlinear systems with unstable exogenous signals. The new definition of nonlinear geometric steady states is based on the following observation.

For neutrally stable exogenous signals, the center manifold theory can be applied and a geometric description on the existence of steady states can be formulated as the controlled invariance and attractiveness of the solution manifold of the center manifold equation. Such a formulation tells more geometric information about steady states than the analytic definition which uses the existence of a suitable input function and an initial state.

For unstable exogenous signals where the center manifold theory is not applicable, the above geometric description is adopted as a definition for nonlinear geometric steady states in this technical note. As the same as Isidori and Byrnes' approach, the exogenous system has not been assigned any initial value so that a class of exogenous signals are considered simultaneously. Furthermore, the obtained definition gives both the local and global version of geometric steady states in a unified way which enables one to give the existence criteria for local and global geometric steady states in some unified way too.

Examples from forced and unforced systems are also included to show that the above geometric definition is reasonable for the corresponding systems.

After the derivation of the definition of nonlinear geometric steady states, the attention of the technical note is then to find a good existence criterion for nonlinear geometric steady states. Note that the newly formed Definition 4 requires both the controlled invariance and the attractiveness of some manifold. The controlled invariance can be characterized as the solvability of Sylvester equation $\partial\pi/\partial\omega S(\omega) = f(\pi(\omega), u(\omega), \omega)$, which can be simplified by our recent approach for the solvability of regulator equation and the parametrization of friend sets (see [10]). Therefore the main theorems of this technical note focusses on the conditions that ensure the attractiveness.

The layout of the technical note is as follows. Section II gives the definition and existence of geometric steady states. Both unforced and forced linear and nonlinear systems are discussed. A practical application on a single-link flexible joint robot is also presented. Section III is some concluding remarks. The Appendix lists some frequently cited results from [3] and [1].

II. Existence of Geometric Steady States

A. Unforced Systems

1) Linear Systems: Consider first a unforced linear system

$$\begin{cases} \dot{x} = Ax + P\omega, \\ \dot{\omega} = S\omega \end{cases} \tag{5}$$

where $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}^s$, and $A,\ P$ and S are matrices of proper sizes.

Different concepts of geometric steady states can be defined for (5).

Note that in (5), the ω part is autonomous. An initial condition ω_0 corresponds to a solution $\omega(t, \omega_0)$, or an exogenous signal. A solution $x_{ss}(t)$ to

$$\dot{x} = Ax + P\omega(t, \omega_0) \tag{6}$$

is called a *geometric steady state* of (5) with respect the exogenous signal $\omega(t, \omega_0)$, if it is an attracting solution to (6).

It is easy to see and left to the reader to verify that for a linear system such as (5), if a geometric steady state exists for one initial condition ω_0, then geometric steady states exist for all initial conditions, and a necessary and sufficient condition for the existence is the stability of the matrix A, or $\dot{x} = Ax$ is a stable system.

However, this is not the concept needed to deal with the output regulation problem.

Definition 1: A linear subspace of the form $N = \{(x, \omega) \mid x = \Pi\omega\}$, for an $n \times s$ matrix is called a geometric steady state to (5) if

i) it is invariant with respect to (5);
ii) it is attracting: denote as $(x(t, x_0, \omega_0), \omega(t, \omega_0))$ the solution of (5) initialized at (x_0, ω_0), then for all initial conditions x_0 and ω_0

$$\lim_{t\to\infty} (x(t, x_0, \omega_0) - \Pi\omega(t, \omega_0)) = 0. \tag{7}$$

The system (5) is said to have a unique geometric steady state if the corresponding Π is unique.

For simplicity, we sometimes say that $x = \Pi\omega$ is a geometric steady state.

Lemma 1: Let Π be an $n \times s$ matrix, then

i) $x = \Pi\omega$ is a geometric steady state of (5) if and only if A is a Hurwitz matrix, and

$$\Pi S = A\Pi + P. \tag{8}$$

ii) $x = \Pi\omega$ is a unique geometric steady state of (5) if and only if the conditions in i) hold and, furthermore, S and A have no eigenvalues in common.

Proof:

(i) Let $x = \Pi\omega$ be a geometric steady state with initial value $(\Pi\omega_0, \omega_0)$, then the subspace $N = \{(x,\omega)|\ x = \Pi\omega\}$ is invariant, which is equivalent to the condition that $\mathrm{d}(x - \Pi\omega)/\mathrm{d}t = 0$ along the dynamics of the system (5). However $\mathrm{d}(x - \Pi\omega)/\mathrm{d}t = (A\Pi + P - \Pi S)\omega$, therefore N is a invariant if and only if $A\Pi + P - \Pi S = 0$.
Now let x be a solution of (5) with initial condition (x_0, ω_0), and $e = x - \Pi\omega$, then $\dot{e} = Ae$, therefore N is attractive if and only if A is Hurwitz.

(ii) This follows from i) and Lemma 2.7 of [11]. □

Example 1: Consider the following four systems:

$$\Sigma_1 : \begin{cases} \dot{x} = -x + \omega, \\ \dot{\omega} = -\omega, \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x} = x - 2\omega, \\ \dot{\omega} = -\omega, \end{cases}$$
$$\Sigma_3 : \begin{cases} \dot{x} = -x, \\ \dot{\omega} = -\omega, \end{cases} \quad \Sigma_4 : \begin{cases} \dot{x} = -x + 3\omega, \\ \dot{\omega} = 2\omega. \end{cases}$$

It is easily seen that a geometric steady state exists for any exogenous signal for Σ_1. For example, $x_{ss}(t) = 0$ is a geometric steady state with respect to $\omega(t) = 0$; $x_{ss}(t) = t\exp(-t)$ is a geometric steady state with respect to $\omega(t) = \exp(-t)$. However, no geometric steady states exist in the sense of Definition 1, because there is no solution to the (8). Σ_2 has no geometric steady states, even though $x = \omega$ is invariant or $\Pi = 1$ satisfies the (8). Subspaces defined by $x = \omega$ and $x = 2\omega$ respectively are two geometric steady states for Σ_3. While Σ_4 has a unique geometric steady state defined by $x = \omega$.

2) Nonlinear Systems: Consider a nonlinear, unforced system

$$\begin{cases} \dot{x} = f(x,\omega) = Ax + P\omega + \phi(x,\omega), \\ \dot{\omega} = S(\omega) = S_0\omega + \psi(\omega), \end{cases} \tag{9}$$

in which, it is assumed that $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}^s$, $f(0,0) = 0$, $S(0) = 0$, and

$$A = \frac{\partial f(0,0)}{\partial x}, \quad P = \frac{\partial f(0,0)}{\partial \omega}, \quad S_0 = \frac{\partial S(0)}{\partial \omega}$$

and $\phi(x,\omega)$ and $\psi(x,\omega)$ are higher order terms. Let M be a given open neighborhood of 0 in $\mathbb{R}^n$, and W a given open neighborhood of 0 in $\mathbb{R}^s$. The notations M and W will be fixed throughout the technical note, and the notations M_0 and W_0 are used to denote some smaller open neighborhoods of the origins contained in M and W respectively. A result holding on $M \times W$ is said to be held globally on $M \times W$, while a result holding on $M_0 \times W_0$ for some $M_0 \subseteq M$ and $W_0 \subseteq W$ is referred to be held locally in $M \times W$.

Definition 2: The system (9) is said to admit a (local) geometric steady state if

i) there is a C^r ($r \geqslant 1$) function $\pi(\omega)$ such that a submanifold defined by $N = \{(x,\omega) \mid x - \pi(\omega) = 0\}$ is (locally) invariant with respect to the closed-loop system (9): for all $\omega \in W$ ($\omega \in W_0$, W_0 is some open subset of W)

$$\frac{\partial \pi}{\partial \omega} S(\omega) = f(\pi(\omega), \omega). \tag{10}$$

That is, the Sylvester (10) is solvable.

ii) N is (locally) attracting: for each $\omega_0 \in W$ ($\omega_0 \in W_0$, W_0 is some open subset of W) and $x_0 \in M$ ($x_0 \in M_0$, M_0 is some open subset of M), there exist a $\mathcal{KL}$ function $\rho(\cdot,\cdot)$ and a trajectory of (9) starting from (x_0, ω_0) which satisfy the following inequality for all $t \in [0, \infty)$:

$$\|x(t, x_0, \omega_0) - \pi(\omega(t, \omega_0))\| < \rho(\|x_0 - \pi(\omega_0)\|, t). \tag{11}$$

The following example illustrates the difference between a local geometric steady state and a global geometric steady state.

Example 2: Consider the system

$$\dot{x} = -x + x\omega,$$
$$\dot{\omega} = 0.$$

The solution can be found as $x(t) = x(0)\exp((-1 + \omega(0))t)$. The solution $x = 0$ is a local geometric steady state, but not a global one, because $x(t)$ tends to infinity when $\omega(0) > 1$.

Another example was given in [2] when $\dot{\omega} = S(\omega)$ is unstable.

Implicitly implied in the definition of geometric steady states are the following three conditions:

A1) the Sylvester (10) is (locally) solvable for a C^r function $\pi(\omega)$;

A2) the system (9) admits global solutions on $t \in [0, \infty)$ for all initial conditions (in $M_0 \times W_0$);

A3) the attracting property (11).

Let $\mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^m)$ denote the Banach space of all C^r functions whose derivatives up to the r-th order are uniformedly bounded on $\mathbb{R}^n$ and $\|\cdot\|_r$ denotes the norm in $\mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^m)$. Let $\epsilon > 0$ and $\mathcal{C}^r_\epsilon(\mathbb{R}^n, \mathbb{R}^m) = \{w \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^m) : \|w\|_r < \epsilon\}$. In case of no confusion, we also use $\mathcal{C}^r_\epsilon$ when the domains and ranges are distinct, and clear from the context.

There are some situations when A1) is guaranteed.

Proposition 1: Suppose that all eigenvalues of A have nonzero real parts, and all eigenvalues of S_0 have zero real parts.

If for the system (9)

$$\phi(x,\omega) = o((\|x\| + \|\omega\|)), \text{ as } \|x\| + \|\omega\| \to 0,$$
$$\psi(\omega) = o(\|\omega\|), \quad \text{as } \|\omega\| \to 0 \tag{12}$$

then there is a δ and a C^1 function $\pi(\omega)$ defined for $\|\omega\| < \delta$ such that $\pi(0) = 0$, $\partial\pi(0)/\partial\omega = 0$, and

$$\frac{\partial \pi(\omega)}{\partial \omega} S(\omega) = f(\pi(\omega), \omega). \tag{13}$$

Proof: When $P = 0$ in (9), the result is a special form of Theorem A in the Appendix (see also [3]). When $P \neq 0$, note that one needs only to let $\lambda = \mu = 0$ in the corresponding Theorem A from the Appendix , and substitute (x, y) in Theorem A by (ω, x) for our problem. Therefore the cut-off function $\chi(x)$ in the proof of Theorem A is replaced by $\chi(\omega)$, where $\chi : \mathbb{R}^s \to [0,1]$ is C^∞ with

$$\chi(\omega) = \begin{cases} 1, & \|\omega\| \leqslant 1; \\ 0, & \|\omega\| \geqslant 2. \end{cases}$$

Let $\tilde{u}(\omega, x) = P\omega + \phi(x,\omega)$, and $u(\omega, x) = \tilde{u}(\omega\chi(\omega/\epsilon), x) = P\omega\chi(\omega/\epsilon) + \phi(x, \omega\chi(\omega/\epsilon))$. Then ϵ can be chosen sufficiently small so that $u(\omega, x) \in \mathcal{C}^1_\epsilon \cap C^r_\alpha$ for some $\alpha > 0$. And one can perform exactly the same as the rest of the proof of Theorem A to obtain the result. This ends the proof.

Proposition 1 is obviously different from Theorem A although its proof relies largely on Theorem A (see the Appendix for a comparison).

Proposition 2: Suppose that all the eigenvalues of A have negative real parts, and all the eigenvalues of S_0 have nonnegative real parts. If for the system (9), (12) holds, then there is a δ and a C^1 function $\pi(\omega)$ defined for $\|\omega\| < \delta$ such that $\pi(0) = 0$, and (13) is satisfied.

Proof: By the proof of Proposition 1, the assumptions of Theorem B in the Appendix (see also [3]) are fulfilled, therefore the result follows from Theorem B. □

Proposition 2 is actually a combination of Proposition 1 and Theorem B, which is tailored to the need of geometric steady state.

Obviously, these two propositions do not provide a complete solution to the solvability of (13). The following example is not covered by the two propositions.

Example 3: Consider the system

$$\dot{x} = x + \omega^2, \ \ \dot{\omega} = \omega.$$

Then $x = \omega^2$ is invariant with respect to the dynamics of the system, and solves the Sylvester (10) for the system.

The condition (12) does not imply the existence of global solutions on $[0, \infty)$ to the system (9). In fact, Example 2 is such an example by noting the fact $\lim_{\|x\|+\|\omega\| \to 0} x\omega/\|x\| + \|\omega\| = 0$.

The maximal interval of existence of solutions, and global solution in particular, of ordinary differential equations have been classical problems [4]. The following result from Theorem C in the Appendix (see also [3]) gives conditions for both the global existence of solutions A2) and the attracting property A3).

Proposition 3: ([3]): Suppose that all the eigenvalues of A have negative real parts, and all the eigenvalues of S_0 have nonnegative real parts. If for an $\epsilon > 0$ such that $\phi, \psi \in \mathcal{C}_\epsilon^1$, then:

i) there exists a $\delta > 0$ such that solutions to (9) exist globally on $[0, \infty)$ for all initial conditions in $\|x(0)\| + \|\omega(0)\| < \delta$;
ii) there exist a unique C^1 function $\pi(\omega)$, positive constant $\beta > 0$ and $L > 0$ with L depending on $(x(0), \omega(0))$ such that

$$\|x(t) - \pi(\omega(t))\| \leqslant Le^{-\beta t}, \quad t \geqslant 0. \tag{14}$$

Though the conditions of Proposition 3 seem to be strong, there are examples which do not satisfy the attracting property when some of the conditions of the proposition are violated. The following is an example in which the condition $\phi \in \mathcal{C}_\epsilon^1$ is violated and thus (14) does not hold.

Example 4:

$$\dot{x} = -x + x\omega, \quad \dot{\omega} = \omega.$$

It is easily seen that $x = \pi(\omega) = 0$ is a solution to the Sylvester (10). Let $e(t) = x(t) - \pi(\omega) = x(t)$, we have $\dot{e}(t) = -e(t) + \omega(t)e(t)$, $e(t) = C_0 \exp(-t + \omega(0)\exp(t))$, and $e(t)$ diverges for any $C_0 \neq 0$ and $\omega(0) > 0$. Therefore the system has no a geometric steady state in this case.

Proposition 4: (Centre Manifold Theorem, [1], [6]): Suppose that all the eigenvalues of A have negative real parts, and all the eigenvalues of S_0 have zero real parts. If for an $\epsilon > 0$ such that $\phi, \psi \in \mathcal{C}_\epsilon^2$, then

i) there exists a $\delta > 0$ such that solutions to (9) exist globally on $[0, \infty)$ for all initial conditions in $\|x(0)\| + \|\omega(0)\| < \delta$;
ii) there exist a unique C^2 function $\pi(\omega)$ defined for $\|\omega\| < \delta$, positive constant $\beta > 0$ and $L > 0$ with L depending on $(x(0), \omega(0))$ such that (14) holds as long as $\|\omega(t)\| < \delta$ for $t \geqslant 0$.

Proposition 4 is a combination of the Centre Manifold Theorems from [1] and [6] (c.f Theorem D in the Appendix). The boundedness of $\omega(t)$ is crucial for the attracting property in this proposition.

Example 5:

$$\dot{x} = -x + \omega_1 x, \quad \dot{\omega}_1 = \omega_2, \quad \dot{\omega}_2 = 0.$$

It can be found that $x = \pi(\omega) = 0$ is a solution to the Sylvester (10), and $\omega_2(t) = \omega_2(0)$, $\omega_1(t) = w_2(0)t + w_1(0)$. Then $e(t) = x(t) - \pi(\omega(t)) = C\exp\left((w_1(0) - 1)t + (\omega_2(0)/2)t^2\right)$, and it tends to infinity for any $C \neq 0$ and $\omega_2(0) > 0$. Therefore the system has no geometric steady states when $C \neq 0$ and $\omega_2(0) > 0$. Note that this example satisfies all the requirements in Proposition 4 except for the boundedness of $\omega(t)$.

Example 6:

$$\dot{x} = \omega_1 x, \quad \dot{\omega}_1 = \omega_2, \quad \dot{\omega}_2 = -\omega_1.$$

Again $x = \pi(\omega) = 0$ is a solution to the Sylvester (10), $\omega_1(t) = \omega_1(0)\cos(t) + \omega_2(0)\sin(t)$, $\omega_2(t) = -\omega_1(0)\sin(t) + \omega_2(0)\cos(t)$, and $e(t) = x(t) - \pi(\omega(t)) = C\exp(\omega_1(0)\cos(t) + \omega_2(0)\sin(t))$ which has no limit when t tends to infinity for $C \neq 0$ and $\omega_1(0)^2 + \omega_2(0)^2 \neq 0$. Thus the system has no geometric steady states. In this example, all the conditions in Proposition 4 are satisfied except that the matrix $A = 0$ violates the requirement.

Example 7:

$$\dot{x} = -x^3, \quad \dot{\omega} = -\omega.$$

This system has a geometric steady state $x = 0$ by Definition 2. However this system does not satisfy the conditions in any of Proposition 1, 2, 3, or 4, nor does it satisfy the stabilizability of (A, B) in (2)–(4) in the definition of Isidori and Byrnes. Note that the stabilizability of (A, B) as obtained in [7] and [6] is a typical first order approximation property superimposed by the requirement of exponential convergence in the output regulation problem.

Now consider the following assumptions which are essentially the conditions A1), A2) and A3). The above Examples 4, 5 and 6 show that even though both the system (9) and the Sylvester (10) admit global solutions, that is, conditions A1) and A2) are fulfilled, the attracting property A3) may still not hold. One needs to give more conditions to guarantee attractiveness. Therefore the following hypothesis A3') is added.

A1') The Sylvester (10) admits solution $\pi(\omega)$ for all $\omega \in W_0 \subset W$.

A2') The system (9) admits global solutions on $t \in [0, \infty)$ for all initial conditions $(x_0, \omega_0) \in M_0 \times W_0 \subset M \times W$.

A3') There is a $\gamma > 0$ such that for all $(x, \omega) \in M_0 \times W_0$

$$\|\phi(x + \pi(\omega), \omega) - \phi(\pi(\omega), \omega)\| < \gamma\|x\|.$$

Theorem 1: Assume that A1'), A2'), and A3') hold for some open subsets $M_0 \subseteq M$ and $W_0 \subseteq W$. Let A be Hurwitz, and R and Q are symmetric and positive definite matrices satisfying the following Lyapunov equation

$$A^T R + RA = -Q.$$

If $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(R)$, where $\lambda_{\min}(Q)$ is the minimal eigenvalue of Q, and $\lambda_{\max}(R)$ is the maximal eigenvalue of R, then a local geometric steady state exists for the system (9). If $M_0 = M$ and $W_0 = W$, then a global geometric steady state exists.

Proof: It suffices to show the attractiveness of the solution $\pi(\omega)$ of (9), which corresponds to an initial condition $(\pi(\omega_0), \omega_0)$, for the sets (W_0, M_0) (or (W, M) for the global version). Let $x(t, x_0, \omega_0)$ be a solution of (9) for the initial condition (x_0, ω_0), $e = x - \pi(\omega)$, and $\xi = \phi(x, \omega) - \phi(\pi, \omega)$, then $\dot{e} = Ae + \xi$. Let $V = e^T Re$, it suffices to show V is a Lyapunov function, or equivalently, $\dot{V} < 0$. By the assumption A3') and Theorem 1 of [8] (see also [9]), $\dot{V} < 0$, $\lim_{t \to \infty} e = 0$. The result follows. □

Theorem 2: Assume that A1'), A2'), and A3') hold for some open subsets $M_0 \subseteq M$ and $W_0 \subseteq W$, and A is Hurwitz. If there is an $\epsilon > 0$ such that the following Algebraic Riccati Equation (ARE) has a symmetric, positive definite matrix solution R

$$AR + RA^T + \gamma^2 R^2 + (1 + \epsilon)I = 0 \tag{15}$$

then the system (9) has a local geometric steady state. If $M_0 = M$ and $W_0 = W$, then a global geometric steady state exists.

Proof: The proof is similar as that of Theorem 2 in [8]. Take $\pi(w)$ and x as that in the proof of Theorem 1. Since $\epsilon > 0$, one has $AR + RA^T + \gamma^2 R^2 + I < 0$. By Lemma 1 of [8], there exists a positive definite, symmetric matrix R_1 such that $A^T R_1 + R_1 A + \gamma^2 R_1^2 + I < 0$. Let $V = e^T R_1 e$, where $e = x - \pi(\omega)$. Then $\dot{e} = Ae + \xi$, $\xi = \phi(x, \omega) - \phi(\pi, \omega)$, and

$$\begin{aligned}\dot{V} &= (e^T A^T + \xi^T)R_1 e + e^T R_1(Ae + \xi) \\ &= e^T(A^T R_1 + R_1 A)e + 2\xi^T R_1 e \\ &\leqslant e^T(A^T R_1 + R_1 A)e + \gamma^2 e^T R_1 e + e^T e \\ &= e^T(A^T R_1 + P_1 A + \gamma^2 R_1 + I)e < 0\end{aligned}$$

where the first inequality follows from Lemma 2 of [8]. Therefore $\lim_{t\to\infty} e = 0$, and $\pi(w)$ is a geometric steady state. □

The condition in A3') looks like the Lipschitz condition in M_0, and in fact, it is implied by such a Lipschitz condition. That is, if $\phi(x,\omega)$ is Lipschitz on M_0, then there exists a constant $\gamma > 0$ such that for all $x',\ x'' \in M_0,\ \omega \in W_0$, the inequality $\|\phi(x',\omega) - \phi(x'',\omega)\| < \gamma\|x' - x''\|$ holds; and thus A3') holds which can also be illustrated by the following example.

Example 8: Consider

$$\dot{x} = -2x + \sin(\omega)x, \quad \dot{\omega} = \omega.$$

Then $\phi(x,\omega) = \sin(\omega)x$, $\|\phi(x,\omega) - \phi(\tilde{x},\omega)\| \leqslant \|x - \tilde{x}\|$, and A3') is satisfied. Obviously $x = \pi(\omega) = 0$ solves the Sylvester (10). Let $e = x - \pi = x$, then $\dot{e} = -2e + \sin(\omega)e$, where $\omega(t) = \omega(0)\exp(t)$. Let $V = e^2$, then $\dot{V} = 2e^2(-2 + \sin(\omega)) < 0$ for any nonzero e, therefore V is a Lyapunov function and $\lim_{t\to\infty} e(t) = 0$. Thus $x = \pi(\omega) = 0$ is a geometric steady state.

The condition in A3') is in fact weaker than the above Lipschitz condition on M_0 as shown by the following example.

Example 9:

$$\dot{x} = -2x + (x - \omega^2)\sin x + 4\omega^2, \quad \dot{\omega} = \omega.$$

It is easy to see that this system does not satisfy the Lipschitz condition on M_0. However, it can be found that the Sylvester (10) has a solution $\pi(\omega) = \omega^2$. And $\phi(x,\omega) = (x - \omega^2)\sin x + 4\omega^2$ satisfies $\phi(x + \pi(\omega),\omega) - \phi(\pi(\omega),\omega) = x\sin(x + \omega^2)$. That is, A3') holds for $\gamma = 1$. Let $e = x - \pi(\omega) = x - \omega^2$ and $V = e^2$, then $\dot{e} = e(-2 + \sin(e + \omega^2))$, and $\dot{V} = 2e^2(-2 + \sin(e + \omega^2)) < 0$ for $e \neq 0$. Therefore $\lim_{t\to\infty} e(t) = 0$, and $x = \pi(\omega) = \omega^2$ is a geometric steady state.

B. Forced Systems

Now consider a forced linear system

$$\begin{cases} \dot{x} = Ax + Bu + P\omega, \\ \dot{\omega} = S\omega, \end{cases} \tag{16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\omega \in \mathbb{R}^s$, and $A,\ B,\ P$ and S are matrices of proper sizes.

Definition 3: A subspace $N = \{(x,\omega) \mid x = \Pi\omega\}$ for an $n \times s$ matrix Π, or simply $x = \Pi\omega$, is called a controlled geometric steady state of the system (16) if there is a state feedback

$$u = Kx + G\omega \tag{17}$$

for matrices K and G of proper sizes, such that the closed-loop system

$$\begin{cases} \dot{x} = (A + BK)x + (BG + P)\omega, \\ \dot{\omega} = S\omega \end{cases} \tag{18}$$

has the geometric steady state $x = \Pi\omega$.

Lemma 2: The system (16) admits a geometric steady state if and only if (A,B) is stabilizable and there exist matrices Π and C of proper sizes such that

$$\Pi S = A\Pi + BC + P. \tag{19}$$

Proof: The manifold $N = \{(x,\omega) | \ x = \Pi\omega\}$ is a geometric steady state if and only if it is both controlled invariant and attractive. Similar as the proof of Lemma 1, N is attractive if and only if the solution of the differential $\mathrm{d}e/\mathrm{d}t = (A + BK)e$ tends to zero when t tends to infinity, where $e = x - \Pi\omega$, x is a solution of (16), K is determined by a controller $u = Kx + G\omega$. Therefore N is attractive if and only if $A + BK$ is Hurwitz, or equivalently, (A,B) is stabilizable.

By Lemma 1, N is controlled invariant with respect to the system (18) if and only if there exists a controller $u = Kx + G\omega$ such that the equation $(A + BK)\Pi + (BG + P) - \Pi S = 0$ has a solution Π for the given (K,G).

Since $(A + BK)\Pi + (BG + P) - \Pi S = A\Pi + BC + P - \Pi S$, where $C = K\Pi + G$, it is obvious that N is a geometric steady state if and only if there exists K, C and Π such that $A + BK$ is Hurwitz, and $A\Pi + BC + P - \Pi S = 0$. This ends the proof. □

Remark 1: For a unforced linear system, the geometric steady state is unique if condition (ii) of Lemma 1 holds. For a forced system, it is meaningless to discuss the uniqueness of geometric steady state because the admissible controller $u = Kx + G\omega$ may not be unique, and the resulting matrix C may not be unique. Therefore the geometric steady state $x = \Pi w$ will be dependent on (K,G) and may not be unique.

Consider a nonlinear control system

$$\begin{cases} \dot{x} = f(x,u,\omega), \\ \dot{\omega} = S(\omega) \end{cases} \tag{20}$$

where $x \in M = \mathbb{R}^n$, $\omega \in W = \mathbb{R}^s$, and f and S are smooth.

Recall [6] that a (local) regular submanifold $N = \{(x,\omega) \mid \Phi(x,\omega) = 0\}$, or simply $\Phi(x,\omega) = 0$, is controlled invariant if there is a (locally defined) feedback $u = \alpha(x,\omega)$ such that for all $(x,\omega) \in N$

$$\frac{\partial\Phi}{\partial x} f(x,\alpha(x,\omega),\omega) + \frac{\partial\Phi}{\partial\omega} S(\omega) = 0. \tag{21}$$

By the definition of controlled invariant manifold it is obvious that system (20) admits a controlled invariant submanifold of the form $x - \pi(\omega) = 0$ if and only if the following Sylvester equation holds for some $C^r(r \geqslant 0)$ function $c(\omega)$:

$$\frac{\partial\pi}{\partial\omega} S(\omega) = f(\pi(\omega), c(\omega), \omega). \tag{22}$$

Definition 4: The system (20) is said to admit a (local) geometric steady state if

i) there is a $C^r(r \geqslant 1)$ function $\pi(\omega)$ such that a submanifold defined by $N = \{(x,\omega) \mid x - \pi(\omega) = 0\}$ is (locally) controlled invariant, that is, there is a feedback $u = \alpha(x,\omega)$ such that N is invariant with respect to the closed-loop system

$$\begin{cases} \dot{x} = f(x,\alpha(x,\omega),\omega), \\ \dot{\omega} = S(\omega). \end{cases} \tag{23}$$

ii) N is (locally) attracting: for each $\omega_0 \in W$ ($\omega_0 \in W_0$, W_0 is some open subset of W) and $x_0 \in M$ ($x_0 \in M_0$, M_0 is some open subset of M), there exist a $\mathcal{KL}$ function $\rho(\cdot,\cdot)$ and a trajectory of (23) starting from (x_0,ω_0) which satisfy the following inequality for all $t \in [0,\infty)$:

$$\|x(t,x_0,\omega_0) - \pi(\omega(t,\omega_0))\| < \rho(\|x_0 - \pi(\omega_0)\|, t). \tag{24}$$

Denote $A = \partial f(0,0,0)/\partial x$, $B = \partial f(0,0,0)/\partial u$, $P = \partial f(0,0,0)/\partial\omega$, $S_0 = \partial S(0)/\partial\omega$, and rewrite the system (20) as

$$\begin{cases} \dot{x} = Ax + Bu + P\omega + \phi(x,u,\omega), \\ \dot{\omega} = S_0\omega + \psi(\omega) \end{cases} \tag{25}$$

in which $\phi(x,u,\omega)$ and $\psi(\omega)$ are higher order terms.

Similar to the unforced case, the following hypotheses are made to give some sufficient results on the existence of geometric steady state for forced nonlinear systems.

A1'') There exists a controller $u = \alpha(x,\omega) = K_1x + K_2\omega + \alpha_1(x,\omega)$ such that $K_1 = \partial\alpha(0,0)/\partial x$, $K_2 = \partial\alpha(0,0)/\partial\omega$, and the Sylvester (22) admits solution $\pi(\omega)$ for all $\omega \in W_0$, where $c(\omega) = \alpha(\pi(\omega), \omega)$.

A2'') For the same controller $u = \alpha(x, \omega)$ in A1'') the system (20) admits global solutions on $t \in [0, \infty)$ for all initial conditions $(x_0, \omega_0) \in M_0 \times W_0 \subset M \times W$.

A3'') For the same controller $u = \alpha(x, \omega)$ in A1''), there is a $\gamma > 0$ such that for all $(x, \omega) \in M_0 \times W_0$

$$\begin{aligned}&\|B\left(\alpha_1(x+\pi(\omega),\omega) - \alpha_1(\pi(\omega),\omega)\right) + \\ &(\phi(x+\pi(\omega),\alpha(\pi(\omega),\omega),\omega) - \phi(\pi(\omega),\alpha(\pi(\omega),\omega),\omega))\,\| \\ &< \gamma\|x\|.\end{aligned}$$

Remark 2: The hypothesis A1'') guarantees the controlled invariance of some manifold. By the analytic method in [5] or the geometric approach for the solvability of regulator equation and the parametrization of friend sets in [10], one can simplify the computation of the feasible controllers (i.e., friends) which ensure the controlled invariance. Note that such feasible controllers may not be unique, therefore the results in [10] will be helpful to find all the possible friends. Now one can try to find, among all the possible friends, a controller u which satisfies furthermore hypotheses A2'') and A3''). Since A2'') can often be satisfied in a lot of cases, this technical note aims to find, under hypotheses A1'') and A2''), conditions for which the attractiveness condition A3'') holds.

Theorem 4: Assume that A1''), A2'') and A3'') hold for some open subsets $M_0 \subseteq M$ and $W_0 \subseteq W$. Let $\tilde{A} := A + BK_1$ be Hurwitz, and R and Q are symmetric and positive definite matrices satisfying the following Lyapunov equation:

$$\tilde{A}^T R + R\tilde{A} = -Q.$$

If $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(R)$, where $\lambda_{\min}(Q)$ is the minimal eigenvalue of Q, and $\lambda_{\max}(R)$ is the maximal eigenvalue of R, then a local geometric steady state exists for the system (20). If $M_0 = M$ and $W_0 = W$, then a global geometric steady state exists.

Proof: Fix the controller $u = \alpha(x, \omega)$ in A1''), and let $\pi(\omega)$ be a geometric steady state corresponding to the initial condition $(\pi(\omega_0), \omega_0)$, and x the solution of (20) corresponding to the initial condition (x_0, ω_0). Then by letting $e = x - \pi(w)$, one has the following equation:

$$\begin{aligned}\dot{e} = \tilde{A}e + B(\alpha_1(x,\omega) - \alpha_1(\pi,\omega)) \\ +\phi(x,\alpha(\pi,\omega),\omega) - \phi(\pi,\alpha(\pi,\omega),\omega).\end{aligned} \quad (26)$$

Then by the proof of Theorem 1, the result follows. □

Theorem 4: Assume that A1''), A2'') and A3'') hold for some open subsets $M_0 \subseteq M$ and $W_0 \subseteq W$, and $\tilde{A} := A + BK_1$ is Hurwitz. If there is an $\epsilon > 0$ such that the following ARE has a symmetric, positive definite matrix solution R

$$\tilde{A}R + R\tilde{A}^T + \gamma^2 R^2 + (1+\epsilon)I = 0 \quad (27)$$

then the system (20) has a local geometric steady state. If $M_0 = M$ and $W_0 = W$, then a global geometric steady state exists.

Proof: Since (26) still holds, the result follows from a similar procedure in the proof of Theorem 2. □

Example 10: Consider the system

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\end{pmatrix} \\ + \begin{pmatrix}13\omega^2 - x_1^3\\ x_2^2\end{pmatrix}, \\ \dot{\omega} = \omega.\end{aligned}$$

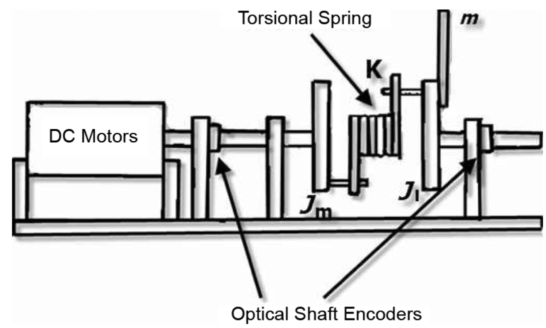

Fig. 1. Schematic of an elastic robot.

Let $u_1 = -2x_1 - 2x_2 + x_1^3, u_2 = -2x_2 - x_2^2$, then

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \tilde{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}13\omega^2\\ 0\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}-1 & -2\\ 2 & -1\end{pmatrix}$$

and $\tilde{A}$ is Hurwitz since its eigenvalues are $-1+2i$ and $-1-2i$. Clearly, $\pi_1 = 3\omega^2$ and $\pi_2 = 2\omega^2$ solve the Sylvester (22). The above equation is an ordinary differential system with constant coefficients and has a solution on $t \in [0, \infty)$ for any given domain of initial values.

Let $R = I,\ Q = 2I$, where I is the 2×2 identity matrix, then $\tilde{A}^T R + R\tilde{A} = -Q$. It is obvious that γ in A3'') equals 0 for this example, and therefore the conditions in Theorem 4 hold and the system has a geometric steady state.

Example 11: Fig. 1 is taken from [8] to show the schematic of a laboratory model of a single-link flexible joint robot, in which J_m is the inertial of the DC motor, J_l is the inertia of the controlled link. Let x_1 and x_3 be the angular rotations of the motor and the link respectively, and x_2 and x_4 are their angular velocities. Assume that k is the torsional compliance, B is the viscous friction coefficient, K_τ is the amplifier gain, m is the pointer mass, and $2b$ is the link length. Then the system is modeled as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \frac{k}{J_m}(x_3 - x_1) - \frac{B}{J_m}x_2 + \frac{K_\tau}{J_m}u,\\ \dot{x}_3 &= x_4,\\ \dot{x}_4 &= -\frac{k}{J_l}(x_3 - x_1) - \frac{mgh}{J_l}\sin(x_3).\end{aligned} \quad (28)$$

Take the same system parameter as [8], then the above system is in the form

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 48.6(x_3 - x_1) - 1.25x_2 + 21.6u,\\ \dot{x}_3 &= x_4,\\ \dot{x}_4 &= -19.5(x_3 - x_1) - 3.33\sin(x_3).\end{aligned} \quad (29)$$

Since a geometric steady state is determined by the corresponding input, the following two subsections consider two kinds of input functions respectively.

(i) Constant Input

Let the input be a constant determined by $u = u_0 = \omega$ and $\dot{\omega} = 0$, where ω is an exogenous signal. Now consider the following two problems.

Let $x = (x_1, x_2, x_3, x_4)^T$, then (29) is rewritten as

$$\begin{aligned}\dot{x} &= Ax + P\omega + \phi(x,\omega),\\ \dot{\omega} &= 0\end{aligned} \quad (30)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -22.83 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{pmatrix},$$

$$\phi(x, \omega) = (0, 0, 0, 3.33(x_3 - \sin(x_3)))^T. \quad (31)$$

Then all the conditions of Proposition 2 are satisfied, therefore there exists a local geometric steady state. The geometric steady state can be further computed by the solution of the Sylvester equation $0 = \partial\pi/\partial\omega 0 = A\pi + P\omega + \phi(\pi, \omega)$, which gives $\pi_2 = \pi_4 = 0, \pi_3 = \arcsin(2600/999\omega), \pi_1 = \pi_3 + 4/9\omega = \arcsin(2600/999\omega) + 4/9\omega$.

(ii) Sinusoidal Input

Assume that the input is some sinusoidal function determined by $u = \omega_1, \dot{\omega}_1 = \omega_2$, and $\dot{\omega}_2 = -\omega_1$, where ω_1 and ω_2 denote exogenous signals. Now take the same x as that of case (i) in this example, then

$$\dot{x} = Ax + \widetilde{P}\omega + \phi(x, \omega), \quad \dot{\omega}_1 = \omega_2, \quad \dot{\omega}_2 = -\omega_1 \quad (32)$$

where A and ϕ are the same as (31), and

$$\widetilde{P} = \begin{pmatrix} 0 & 0 \\ 21.6 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the conditions of Proposition 2 are satisfied, and there exists a local geometric steady state.

III. Conclusion

This technical note gives a geometric definition of nonlinear steady states which is consistent with the linear definition and furthermore applicable to nonlinear exogenous systems which are not neutrally stable. Sufficient results are given for the existence of nonlinear geometric steady states. Note that the theorems obtained in this technical note are limited to the attractiveness condition in the definition. Fine results could be obtained when considering results for the existence of global solution for $t \in [0, \infty)$, the conditions of controlled invariance, and the existence of friends. As a further work, the application of this geometric steady states definition to observer design and output regulation will be conducted.

Appendix

For the readers' convenience, some results from [3] and [1] which are frequently cited in this technical note are recalled.

Theorem A: Theorem 2.2 of Chapter 9 in [3]: Consider the system

$$\dot{x} = Ax + \tilde{u}(x, y, \lambda), \quad \dot{y} = By + \tilde{v}(x, y, \mu) \quad (33)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, all eigenvalues of A have zero real parts, and all eigenvalues of B have nonzero real parts. Suppose that $\tilde{u}$, $\tilde{v}$ are C^r, $\lambda, \mu \in \wedge$, $\wedge$ is a Banach space, and $\tilde{u}(x, y, 0), \tilde{v}(x, y, 0) = o(\|x\| + \|y\|)$ as $\|x\| + \|y\| \to 0$. If $r \geqslant 1$ is given, then there exist $\delta = \delta(r) > 0$ and a C^r function $y = h(x, \lambda, \mu), \|x\| + \|\lambda\| + \|\mu\| < \delta$, such that

(i) $h(0, 0, 0) = 0, D_x h(0, 0, 0) = 0$;

(ii) for fixed λ and μ, $\tilde{M}_c = \{(x, y) : y = h(x, \lambda, \mu), \|x\| + \|\lambda\| + \|\mu\| < \delta\}$, called the (local) centre manifold, is a C^r manifold and is locally invariant with respect to (33); i.e., for every $(x_0, y_0) \in \tilde{M}_c$ there exists $t_0 = t_0(x_0, y_0) > 0$ such that the unique solution $(x(t), y(t))$ of (33) with $(x(0), y(0)) = (x_0, y_0)$ satisfies $(x(t), y(t)) \in \tilde{M}_c, |t| < t_0$.

Theorem B: (Theorem 2.11 of Chapter 9, [3]): Consider the system

$$\dot{x} = Ax + u(x, y), \quad \dot{y} = By + v(x, y) \quad (34)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Assume that all the eigenvalues of A have nonnegative real parts; all the eigenvalues of B have negative real parts. For any integer $r > 0$ and $\alpha > 0$, there exists $\epsilon_0 = \epsilon_0(r, \alpha, A, B) > 0$ such that, for every $0 < \epsilon < \epsilon_0$ and $u, v \in \mathcal{C}^1_\epsilon \cap \mathcal{C}^r_\alpha$, there exists a unique function $g(u, v) \in \mathcal{C}^r$ with the following properties:

(i) $g(0, 0) = 0$;

(ii) for fixed u and v, the set, $M_{cu} = \{(x, y) : y = g(u, v)(x)\}$, called the centre unstable manifold, is an invariant manifold of (34);

(iii) M_{cu} contains exactly those solutions $(x(t), y(t))$ of (34) with $\sup_{t \leqslant 0} \|y(t)\| < \infty$. Moreover, M_{cu} is unique with respect to properties (i), (ii) and (iii).

Theorem C: (Theorem 2.13 of Chapter 9, [3]): Let the assumptions of Theorem B be fulfilled. If $(x(t), y(t))$ is a solution of (34) and g defines the centre unstable manifold M_{cu} as in Theorem B, then there exists $\beta_1 > 0$ and $L_1 > 0$ with L_1 depending on $(x(0), y(0))$ such that $\|y(t) - g(x(t))\| \leqslant L_1 e^{-\beta_1 t}, \ t \geqslant 0$.

Now the Centre Manifold Theorem from [1] is recalled here.

Consider the following system:

$$\dot{x} = Ax + f(x, y), \quad \dot{y} = By + +g(x, y) \quad (35)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, the eigenvalues of A have zero real parts, the eigenvalues of B have negative real parts and f and g are C^2 functions which vanish together with their derivatives at the origin.

Theorem D: (Theorem 1, Section 2.3 of [1]): Equation (35) has a local centre manifold $y = h(x), \|x\| < \delta$, where h is C^2.

Acknowledgment

The authors would like to thank Dr. C. Moog, IRCCyN, Ecole Centrale de Nantes, France, who constructed Example 7 for us, and also the anonymous reviewers and the associate editor for their many helpful comments.

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