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MIXED RECURRENCE RELATIONS AND INTERLACING OF THE ZEROS OF SOME q-ORTHOGONAL POLYNOMIALS FROM DIFFERENT SEQUENCES

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Abstract. We use the generating functions of some q-orthogonal polynomials to obtain mixed recurrence relations involving polynomials with shifted parameter values. These relations are used to prove interlacing results for the zeros of Al-Salam–Chihara, continuous q-ultraspherical, q-Meixner–Pollaczek and q-Laguerre polynomials of the same or adjacent degree as one of the parameters is shifted by integer values or continuously within a certain range. Numerical examples are given to illustrate situations where the zeros do not interlace.

1. Introduction

Let p_n denote a polynomial of degree n and consider $\{p_n(x)\}_{n=0}^{\infty}$, a sequence of polynomials orthogonal on the (finite or infinite) interval (c,d). For each n, p_n has n distinct real zeros on the interval (c,d) and the zeros of p_n and p_{n-1} are interlaced in the sense that if $x_1 < x_2 < \cdots < x_n$ denotes the zeros of p_n and $y_1 < y_2 < \cdots < y_{n-1}$ those of p_{n-1}, then

$$x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n.$$

The manner in which the real zeros of various families of orthogonal polynomials change as the parameter values are shifted is a well studied

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phenomenon. For the so-called classical one and two parameter orthogonal polynomials there are numerous results and papers on properties such as the monotonicity of the zeros, convexity of the zeros, estimates on the distance between the zeros, bounds for the smallest and largest zeros (cf. for example [6], [12], [13], [14], [19], [24], [25]) and, more recently, interlacing of the zeros from different sequences (cf. [7], [8], [15], [22]) following a conjecture by Askey that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leqq \alpha + 2$ (cf. [1]). At the conference Orthogonal Polynomials, Special Functions and Applications 2007, Mourad Ismail posed the question whether existing techniques could be used to prove similar results for the zeros of q-orthogonal polynomials from different sequences.

Interlacing of the zeros of polynomials is important in several contexts. Examples where the interlacing of zeros plays a role include methods for approximating the finite Hilbert transform [17], uniform convergence of derivatives arising in extended Lagrange interpolation [5], approximation of zeros by fixed point iteration techniques [21], stability tests for linear difference forms [16], completeness of the set of eigenfunctions to a self-adjoint Sturm–Liouville eigenvalue problem [2], the Gelfond interpolation problem [11] and positivity of numerical quadrature formulae where zeros are used as nodes [20] and [23].

Interlacing properties of orthogonal polynomials can often be derived from the following simple principle which is a generalization of [3, Theorem 3].

LEMMA 1.1. *Assume that a continuous function $f(x)$ satisfies the equation*

$$f(x) = a(x)p_n(x) + b(x)p_{n-1}(x). \tag{1}$$

(i) *If $a(x)$ is continuous and has constant sign on (c,d), then $f(x)$ has at least one zero between any two consecutive zeros of $p_{n-1}(x)$.*

(ii) *If $b(x)$ is continuous and has constant sign on (c,d), then $f(x)$ has at least one zero between any two consecutive zeros of $p_n(x)$.*

PROOF. We only prove the first part, the proof of the second part is analogous. Let x_k and x_{k+1} be consecutive zeros of p_{n-1}. Then

$$f(x_k)f(x_{k+1}) = a(x_k)a(x_{k+1})p_n(x_k)p_n(x_{k+1}).$$

$a(x)$ is sign-preserving and the zeros of p_n and p_{n-1} interlace, so $p_n(x_k)p_n(x_{k+1}) < 0$. This implies that $f(x_k)$ and $f(x_{k+1})$ must have opposite signs, and it follows from the continuity that f must have at least one zero between x_k and x_{k+1}. $\square$

The following corollary is a slight generalization of Exercise 5.4 in [4].

COROLLARY 1.2. *If f is a polynomial of degree n or $n-1$ satisfying* (1) *and both $a(x)$ and $b(x)$ are continuous and have constant signs on (c,d), then all the zeros of f are real and simple, the zeros of f and p_n interlace and the zeros of f and p_{n-1} interlace.*

Note that there are several possibilities for the arrangement of the zeros. Let $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_{n-1}$ and $t_1 < t_2 < \cdots < t_n$ be the zeros of p_n, p_{n-1} and f, respectively. Then, if f is of degree n, either

$$x_1 < t_1 < y_1 < x_2 < t_2 < \cdots < x_{n-1} < t_{n-1} < y_{n-1} < x_n < t_n \tag{2}$$

or

$$t_1 < x_1 < y_1 < t_2 < x_2 < \cdots < t_{n-1} < x_{n-1} < y_{n-1} < t_n < x_n. \tag{3}$$

One way to determine which of the above cases holds for a specific example is by checking the sign of f at a zero of p_n and a neighboring zero of p_{n-1}. E.g. if the p_n's are monic and f has degree n, then evaluating f at y_{n-1} and x_n gives

$$f(y_{n-1})f(x_n) = a(y_{n-1})b(x_n)p_n(y_{n-1})p_{n-1}(x_n) = (-1)\operatorname{sgn}(a)\operatorname{sgn}(b).$$

Therefore, if a and b have the same sign, then f changes sign between y_{n-1} and x_n and (3) holds, otherwise (2) will hold.

In this paper we will apply these principles to the Al-Salam–Chihara, continuous q-ultraspherical, q-Meixner–Pollaczek and the q-Laguerre polynomials to show where the zeros of these polynomials interlace with the zeros of polynomials with shifted parameter values. We will make use of mixed recurrence relations that we obtain from the generating functions of each of these polynomials and only consider those q-polynomials that allow use of the generating function method. We give examples in the cases where interlacing fails to take place. For definitions and notations used in this paper see, for example [10] or [14].

2. Interlacing of the zeros of the Al-Salam–Chihara polynomials with different parameters

The Al-Salam–Chihara polynomials are defined by

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\ 0 \end{matrix}\;\middle|\; q;q\right),$$

where $x = \cos\theta$. They are orthogonal on $[-1,1]$ if e.g. a and b are real and $\max\big(|a|,|b|\big) < 1$, with respect to the weight function

$$w(x;a,b|q) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,q^{-1/2})}{h(x,a)h(x,b)\sqrt{1-x^2}},$$

where

$$h(x,t)=\prod_{k=0}^{\infty}(1-2txq^k+t^2q^{2k}). \tag{4}$$

They have the generating function

$$\sum_{n=0}^{\infty}\frac{Q_n(x;a,b|q)}{(q;q)_n}t^n=\frac{(at,bt;q)_\infty}{(e^{i\theta}t,e^{-i\theta}t;q)_\infty}. \tag{5}$$

Set $a=q^\alpha$ with $0<q<1$, $\alpha>0$ and $Q_n^\alpha=Q_n(x;q^\alpha,b|q)$. It follows from (5) that

$$\sum_{n=0}^{\infty}\frac{Q_n^\alpha}{(q;q)_n}t^n=(1-q^\alpha t)\sum_{n=0}^{\infty}\frac{Q_n^{\alpha+1}}{(q;q)_n}t^n.$$

Equating the coefficients we get

$$Q_n^\alpha=Q_n^{\alpha+1}-q^\alpha(1-q^n)Q_{n-1}^{\alpha+1}, \tag{6}$$

and we can immediately apply Corollary 1.2.

THEOREM 2.1. *If $\alpha>0$, $0<q<1$ and $|b|<1$ then the zeros of Q_n^α, $Q_n^{\alpha+1}$ and $Q_{n-1}^{\alpha+1}$ interlace.*

Note that the exact ordering in the interlacing pattern is given in Corollary 2.3.

To consider the interlacing property for a shift of α by 2, we need the following identity resulting from (5):

$$\sum_{n=0}^{\infty}\frac{Q_n^\alpha}{(q;q)_n}t^n=(1-q^\alpha t)\big(1-q^{\alpha+1}t\big)\sum_{n=0}^{\infty}\frac{Q_n^{\alpha+2}}{(q;q)_n}t^n.$$

Equating the coefficients again we have

$$Q_n^\alpha=Q_n^{\alpha+2}-q^\alpha(1+q)(1-q^n)Q_{n-1}^{\alpha+2}+q^{2\alpha+1}\big(1-q^{n-1}\big)(1-q^n)Q_{n-2}^{\alpha+2}. \tag{7}$$

Now we need the three-term recurrence relation to eliminate the last term on the right hand side. The three-term recurrence relation is

$$\begin{aligned}2xQ_n^{\alpha+2}(x)&=Q_{n+1}^{\alpha+2}(x)+q^n\big(b+q^{\alpha+2}\big)Q_n^{\alpha+2}(x)\\&\quad+(1-q^n)\big(1-bq^{\alpha+n+1}\big)Q_{n-1}^{\alpha+2}(x).\end{aligned}$$

Replacing n by $n-1$, expressing $Q_{n-2}^{\alpha+2}$, and substituting into (7) we obtain

$$Q_n^\alpha = \left(1 - \frac{q^{2\alpha+1}(1-q^n)}{1-bq^{\alpha+n}}\right) Q_n^{\alpha+2} + (1-q^n)\left(q^{2\alpha+1}\frac{2x - q^{n-1}(b+q^{\alpha+2})}{1-bq^{\alpha+n}} - q^\alpha(1+q)\right) Q_{n-1}^{\alpha+2}. \tag{8}$$

Here only the coefficient of $Q_{n-1}^{\alpha+2}$ depends on x and changes sign at

$$x_0 = \frac{1+q-bq^{\alpha+n+1}+q^{2\alpha+n+2}}{2q^{\alpha+1}}.$$

Since $|b|<1$,

$$x_0 > \frac{1+q-q^{\alpha+n+1}+q^{2\alpha+n+2}}{2q^{\alpha+1}} > 1,$$

where the second inequality follows from the fact $(1-q^{\alpha+1})(1-q^{\alpha+n+1}) + q(1-q^\alpha) > 0$. It means that on the interval of orthogonality $[-1,1]$ all the coefficients of (8) have constant sign and we can apply Corollary 1.2 again. To decide between cases (2) and (3), we determine the monotonicity property of the zeros for varying α.

Set $w(x,\alpha) = w(x;q^\alpha,b|q)$ and let $\alpha_1 < \alpha_2$. It is easy to check that

$$\frac{w(x,\alpha_1)}{w(x,\alpha_2)} = \prod_{k=0}^{\infty} \frac{1-2xq^{\alpha_2+k}+q^{2(\alpha_2+k)}}{1-2xq^{\alpha_1+k}+q^{2(\alpha_1+k)}}.$$

Now,

$$\frac{d}{dx}\left(\frac{1-2xq^{\alpha_2+k}+q^{2(\alpha_2+k)}}{1-2xq^{\alpha_1+k}+q^{2(\alpha_1+k)}}\right) = \frac{2q^k(q^{\alpha_1}-q^{\alpha_2})(1-q^{\alpha_1+\alpha_2+2k})}{\left(1-2xq^{\alpha_1+k}+q^{2(\alpha_1+k)}\right)^2} > 0.$$

Therefore $\frac{w(x,\alpha_1)}{w(x,\alpha_2)}$ is an increasing function of x and according to Markov's Theorem (cf. [24, Theorem 6.12.2]) the zeros of Q_n^α decrease with increasing α.

THEOREM 2.2. *Let $\alpha > 0$, $0<q<1$, $|b|<1$ and let $0<x_1<x_2<\cdots<x_n$, $0<y_1<y_2<\cdots<y_{n-1}$ and $0<t_1<t_2<\cdots<t_n$ be the zeros of $Q_n^{\alpha+2}$, $Q_{n-1}^{\alpha+2}$ and Q_n^α, respectively. Then* (2) *holds.*

Although Theorem 2.2 is stated in terms of an integer shift of the parameter, it in fact also holds more generally for a continuous shift in the parameter.

Corollary 2.3. *Let* $\alpha > 0$, $0 < q < 1$, $|b| < 1$ *and let* $0 < p_1 < p_2 < \cdots < p_n$, $0 < q_1 < q_2 < \cdots < q_{n-1}$ *and* $0 < t_1 < t_2 < \cdots < t_n$ *be the zeros of* $Q_n^{\alpha+t}$, $Q_{n-1}^{\alpha+t}$ *and* Q_n^{α}, *respectively, where* $0 < t \leqq 2$. *Then*

$$p_i < t_i < q_i < p_{i+1} < t_{i+1} \quad \text{for} \quad i = 1, 2, \dots, n-1.$$

Proof. Since the zeros of an Al-Salam–Chihara polynomial of fixed degree decrease as the parameter α increases, we have that $p_i < t_i$ and $y_i \leqq q_i$, where y_i is as in Theorem 2.2. In addition, $q_i < p_{i+1}$ since the zeros of two consecutive terms in a sequence of orthogonal polynomials separate each other. By Theorem 2.2, $t_i < y_i$ and the result follows. □

Note that the continuous big q-Hermite polynomials are the limiting case of the Al-Salam–Chihara polynomials and can be handled in exactly the same way (by taking $b = 0$).

Remark. If there exist two measures $w(x)\,dx$ and $x\,w(x)\,dx$, the corresponding orthogonal polynomials, say P_n and Q_n, are related as follows (cf. [4, p. 45])

$$P_n = a_n Q_n + b_n Q_{n-1}, \quad xQ_n = c_n P_{n+1} + d_n P_n,$$

with some constants a_n, b_n, c_n, d_n. Since the coefficients a_n and b_n are independent of x, the triple interlacing of the zeros of P_n, Q_n and Q_{n-1} follows immediately by Corollary 1.2. By an affine change of variables one can also allow multiplication of the weight function $w(x)$ by a linear factor $ax + b$. We note that the result in Theorem 2.1 follows immediately using this method without it being necessary to obtain the mixed recurrence relation (6) explicitly. The weight functions of big q-Laguerre and Al-Salam–Carlitz II polynomials also differ by a linear factor when the parameters β and α respectively are shifted by unity. Hence the interlacing property of the zeros of these polynomials, where the parameter value is shifted by 1, is automatic. It is also possible to obtain the coefficients a_n and b_n in the mixed recurrence relations for these polynomials using the same generating function method as above.

3. Interlacing of the zeros of q-Meixner–Pollaczek polynomials with different parameters

The generating function method also works for the q-Meixner–Pollaczek polynomials. The q-Meixner–Pollaczek polynomials defined by

$$P_n(x; a|q) = \frac{(a^2; q)_n}{a^n e^{in\theta} (q; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n},\ ae^{i(\theta+2\phi)},\ ae^{-i\theta} \\ a^2,\ 0 \end{matrix} \,\middle|\, q; q \right)$$

where $x = \cos(\theta+\phi)$ are orthogonal on $[-1,1]$ when $0 < a < 1$ with respect to

$$w(x;a|q) = \frac{h(x,1)h(x,-1)h\big(x,q^{1/2}\big)\,h\big(x,-q^{1/2}\big)}{h(x,ae^{i\phi})h(x,ae^{-i\phi})},$$

where $h(x,t)$ is given by (4). Let $P_n^\alpha = P_n(x;q^\alpha|q)$ be the nth q-Meixner–Pollaczek polynomial with parameter $a = q^\alpha$, $\alpha > 0$ and $0 < q < 1$. The generating function is

$$\sum_{n=0}^{\infty} P_n^\alpha t^n = \frac{\big(q^\alpha e^{i\phi}t, q^\alpha e^{-i\phi}t; q\big)_\infty}{\big(e^{i(\theta+\phi)}t, e^{-i(\theta+\phi)}t; q\big)_\infty},$$

where $x = \cos(\theta+\phi)$. It is easy to see that

$$\sum_{n=0}^{\infty} P_n^\alpha t^n = \big(1-q^\alpha e^{i\phi}t\big)\big(1-q^\alpha e^{-i\phi}t\big)\sum_{n=0}^{\infty} P_n^{\alpha+1} t^n.$$

Equating the coefficients we have

$$P_n^\alpha = P_n^{\alpha+1} - 2q^\alpha(\cos\phi)P_{n-1}^{\alpha+1} + q^{2\alpha}P_{n-2}^{\alpha+1}. \tag{9}$$

The three term recurrence relation is

$$2xP_n^\alpha = \big(1-q^{n+1}\big)P_{n+1}^\alpha + 2q^{n+\alpha}(\cos\phi)P_n^\alpha + \big(1-q^{n+2\alpha-1}\big)P_{n-1}^\alpha.$$

Replacing α by $\alpha+1$ and n by $n-1$ gives

$$P_{n-2}^{\alpha+1} = \frac{2x-2q^{n+\alpha}\cos\phi}{1-q^{n+2\alpha}}P_{n-1}^{\alpha+1} - \frac{1-q^n}{1-q^{n+2\alpha}}P_n^{\alpha+1}.$$

Substituting this into (9) leads to

$$\big(1-q^{2\alpha+n}\big)P_n^\alpha = \big(1-q^{2\alpha}\big)P_n^{\alpha+1} + 2q^\alpha(xq^\alpha-\cos\phi)P_{n-1}^{\alpha+1}. \tag{10}$$

Now if $|\cos\phi| > q^\alpha$, then all the coefficients have constant sign on $[-1,1]$, so we can again apply Corollary 1.2.

THEOREM 3.1. *Let $0 < q < 1$ and $\alpha > 0$. If $|\cos\phi| > q^\alpha$ then the zeros of P_n^α, $P_n^{\alpha+1}$ and $P_{n-1}^{\alpha+1}$ interlace.*

REMARK. To distinguish between the two types of interlacing (2) and (3) for q-Meixner–Pollaczek polynomials is not straightforward. Using Markov's Monotonicity Theorem (cf. [24, Theorem 6.12.2]) to investigate the monotonicity property of the zeros when the parameter α increases to $\alpha+1$ one obtains

$$\frac{w(x,\alpha)}{w(x,\alpha+1)} = \prod_{k=0}^{\infty} A_n^{(q,\alpha,\phi,k)}(x)$$

where

$$A_n^{(q,\alpha,\phi,k)}(x)$$
$$= \frac{\left(1-2xq^{\alpha+k+1}e^{i\phi}+q^{2(\alpha+k+1)}e^{2i\phi}\right)\left(e^{2i\phi}-2xq^{\alpha+k+1}e^{i\phi}+q^{2(\alpha+k+1)}\right)}{\left(1-2xq^{\alpha+k}e^{i\phi}+q^{2(\alpha+k)}e^{2i\phi}\right)\left(e^{2i\phi}-2xq^{\alpha+k}e^{i\phi}+q^{2(\alpha+k)}\right)}.$$

Now

$$\frac{d}{dx}\left(A_n^{(q,\alpha,\phi,k)}\right)(x)$$
$$= \frac{4e^{4i\phi}(q-1)(q^{\alpha+k})\left(q^{2\alpha+2k+1}-1\right)B_n^{(q,\alpha,\phi,k)}(x)}{\left(1-2xq^{\alpha+k}e^{i\phi}+q^{2(\alpha+k)}e^{2i\phi}\right)^2\left(e^{2i\phi}-2xq^{\alpha+k}e^{i\phi}+q^{2(\alpha+k)}\right)^2}$$

with

$$B_n^{(q,\alpha,\phi,k)}(x) = 4q^{2\alpha+2k+1}\cos\phi\ x^2 - 2q^{\alpha+k}(q+1)\left(q^{1+2\alpha+2k}+1\right)x$$
$$- q^{2\alpha+2k+1}\cos 3\phi + \cos\phi\left(1+q^{2\alpha+2k}\left(1+q+q^2+q^{2(\alpha+k+1)}\right)\right).$$

The zeros of P_n^{α} will decrease as α increases to $\alpha+1$ resulting in type (2) interlacing whenever $\frac{w(x,\alpha)}{w(x,\alpha+1)}$ is an increasing function of x i.e. when $B_n^{(q,\alpha,\phi,k)}$ is positive on the interval of orthogonality. This means that there are four different cases to consider depending on the positivity or negativity of both the discriminant and the leading coefficient of $B_n^{(q,\alpha,\phi,k)}$ as well as the relation of the zeros of the quadratic $B_n^{(q,\alpha,\phi,k)}$ to the interval $[-1,1]$.

There is also another approach to distinguish between case (2) and (3). One can use a modified version of Theorem 7.4.2 in [14], generalized to arbitrary (monic) orthogonal polynomials to determine the monotonicity of the extremal zeros for fixed n and varying α. Since the triple interlacing has already been established, the monotonicity of the largest (or smallest) zeros implies case (2) or (3). The details of this calculation are done in an upcoming paper [9]. The result is that we have case (2) if $\cos\phi > 0$ and case (3) if $\cos\phi < 0$.

4. Interlacing of the zeros of the continuous q-ultraspherical polynomials with different parameters

The continuous q-ultraspherical polynomials are defined by

$$C_n(\cos\phi;\beta|q)=\sum_{k=0}^{n}\frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\phi},$$

where $x=\cos\phi$. They are a special case of the q-Meixner–Pollaczek polynomials with $\theta=0$ and $a=\beta$. They are orthogonal on $[-1,1]$ for $|\beta|<1$ with the weight function

$$w(\cos\phi;\beta|q)=\frac{\left(e^{2i\phi},e^{-2i\phi};q\right)_\infty}{\left(\beta e^{2i\phi},\beta e^{-2i\phi};q\right)_\infty}(\sin\phi)^{-1},$$

and satisfy the three-term recurrence relation

$$2x(1-\beta q^n)C_n(x;\beta|q)$$
$$=\left(1-q^{n+1}\right)C_{n+1}(x;\beta|q)+\left(1-\beta^2q^{n-1}\right)C_{n-1}(x;\beta|q). \tag{11}$$

The generating function is

$$\sum_{n=0}^{\infty}C_n(\cos\phi;\beta|q)t^n=\frac{\left(t\beta e^{i\phi},t\beta e^{-i\phi};q\right)_\infty}{\left(te^{i\phi},te^{-i\phi};q\right)_\infty},\quad |t|<1.$$

In order to get mixed recurrence relations for different parameter values, similar to the q-Meixner–Pollaczek polynomials, we take $\beta=q^\alpha$, $\alpha>0$, $0<q<1$. To ease notation from now on we write C_n^α for $C_n(\cos\phi;q^\alpha|q)$.

Since now $x=\cos\phi$, (10) gives

$$\frac{1-q^{2\alpha+n}}{1-q^\alpha}C_n^\alpha=(1+q^\alpha)C_n^{\alpha+1}-2xq^\alpha C_{n-1}^{\alpha+1},$$

and we can use the results from Section 1. Of course, the coefficient of $C_{n-1}^{\alpha+1}$ changes sign at 0, but it is natural, since the zeros of C_n^α are symmetric in the origin, having a zero at the origin for odd n, so it is sufficient to investigate the interlacing property for the positive zeros.

So as to determine the exact order of the zeros (case (2) or (3)), one needs the monotonicity of the zeros of q-ultraspherical polynomials with respect to α. This can be done similarly to the proof of monotonicity of the Gegenbauer polynomials (c.f. [24, Section 6.21]). It can be seen, by using

Markov's Theorem just like in Section 2, that the zeros of the continuous q-Jacobi polynomials decrease with α and increase with β. Then it follows from the quadratic transformations between the continuous q-ultraspherical and continuous q-Jacobi polynomials that the positive zeros of the continuous q-ultraspherical polynomials decrease with α.

We also have to distinguish according to the parity of the degree.

THEOREM 4.1. *Let $\alpha > 0$, $0 < q < 1$ and let $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_{n-1}$ and $t_1 < t_2 < \cdots < t_n$ be the positive zeros of $C_{2n}^{\alpha+1}$ $(C_{2n+1}^{\alpha+1})$, $C_{2n-1}^{\alpha+1}$ $(C_{2n}^{\alpha+1})$ and C_{2n}^{α} (C_{2n+1}^{α}), respectively. Then*

$$0 < x_1 < t_1 < y_1 < x_2 < t_2 < \cdots < x_{n-1} < t_{n-1} < y_{n-1} < x_n < t_n < 1$$

$$(0 < y_1 < x_1 < t_1 < y_2 < x_2 < \cdots < x_{n-1} < t_{n-1} < y_n < x_n < t_n < 1).$$

One can formulate the interlacing property for the continuous shift, too, similar to Corollary 2.3.

COROLLARY 4.2. *Let $\alpha > 0$, $0 < q < 1$ and let $p_1 < p_2 < \cdots < p_n$, $q_1 < q_2 < \cdots < q_{n-1}$ and $t_1 < t_2 < \cdots < t_n$ be the positive zeros of $C_{2n}^{\alpha+t}$ $(C_{2n+1}^{\alpha+t})$, $C_{2n-1}^{\alpha+t}$ $(C_{2n}^{\alpha+t})$ and C_{2n}^{α} (C_{2n+1}^{α}), respectively, where $0 < t \leqq 1$. Then*

$$0 < p_1 < t_1 < q_1 < p_2 < t_2 < \cdots < p_{n-1} < t_{n-1} < q_{n-1} < p_n < t_n < 1$$

$$(0 < q_1 < p_1 < t_1 < q_2 < p_2 < \cdots < p_{n-1} < t_{n-1} < q_n < p_n < t_n < 1).$$

To get the interlacing property for a shift by 2, more tedious work is required. The generating function gives

$$\left(1 - 2xq^{\alpha-2}t + q^{2\alpha-4}t^2\right)\left(1 - 2xq^{\alpha-1}t + q^{2\alpha-2}t^2\right)\sum_{n=0}^{\infty} C_n^{\alpha} t^n = \sum_{n=0}^{\infty} C_n^{\alpha-2} t^n.$$

Equating like powers of t:

$$C_n^{\alpha-2} = C_n^{\alpha} - 2x\left(q^{\alpha-1} + q^{\alpha-2}\right)C_{n-1}^{\alpha} + \left(q^{2\alpha-2} + q^{2\alpha-4} + 4x^2q^{2\alpha-3}\right)C_{n-2}^{\alpha}$$
$$- 2x\left(q^{3\alpha-4} + q^{3\alpha-5}\right)C_{n-3}^{\alpha} + q^{4\alpha-6}C_{n-4}^{\alpha}.$$

Now we have to use the recurrence relation (11) three times (with shifted indices) to eliminate C_{n-4}^{α}, C_{n-3}^{α}, and C_{n-2}^{α}. In the end we have

$$\frac{\left(1 - q^{2\alpha+n-2}\right)\left(1 - q^{2\alpha+n-3}\right)\left(1 - q^{2\alpha+n-4}\right)}{\left(1 - q^{\alpha-1}\right)\left(1 - q^{\alpha-2}\right)} C_n^{\alpha-2} = A(\alpha, n; x)C_n^{\alpha} + 2xB(\alpha, n; x)C_{n-1}^{\alpha}, \tag{12}$$

where

$$A(\alpha,n;x)=\left(1+q^{\alpha-1}\right)\left(1+q^{\alpha-2}\right)\left(1-q^{2\alpha+n-3}\right)-4q^{2\alpha-3}(1-q^n)x^2$$

and

$$B(\alpha,n;x)=4q^{2\alpha-3}\left(1-q^{\alpha+n-1}\right)x^2$$
$$+q^{3\alpha+n-4}\left(2+q^{\alpha-2}+q^{\alpha-1}\right)-q^{\alpha-2}-q^{\alpha-1}-2q^{2\alpha-3}.$$

Now $A(\alpha,n;x)$ changes sign at $x=\pm\sqrt{D}$, where

$$D=\frac{\left(1+q^{\alpha-1}\right)\left(1+q^{\alpha-2}\right)\left(1-q^{2\alpha+n-3}\right)}{4q^{2\alpha-3}(1-q^n)}$$
$$=\left(\frac{1+q^{\alpha-1}+q^{\alpha-2}+q^{2\alpha-3}}{4q^{2\alpha-3}}\right)\left(q^{2\alpha-3}+\frac{1-q^{2\alpha-3}}{1-q^n}\right).$$

It is easy to see that in the last expression both terms are greater than 1 if $\alpha>\frac{3}{2}$, which holds true in this case, since we need $\alpha>2$ for (12). Similarly, the zeros of $B(\alpha,n;x)$ are $x=\pm\sqrt{F}$, where

$$F=\frac{q^{\alpha-2}+q^{\alpha-1}+2q^{2\alpha-3}-q^{3\alpha+n-4}\left(2+q^{\alpha-2}+q^{\alpha-1}\right)}{4q^{2\alpha-3}\left(1-q^{\alpha+n-1}\right)}$$
$$=\frac{2+q^{\alpha-2}+q^{\alpha-1}}{4}+\frac{(1+q)\left(1-q^{2\alpha-3}\right)}{4q^{\alpha-1}\left(1-q^{\alpha+n-1}\right)}.$$

This is a decreasing function of n, therefore with $n\to\infty$ we get

$$F>\frac{1+q+2q^{\alpha-1}}{4q^{\alpha-1}}>1.$$

Therefore the coefficients of (12) have no zero inside $(-1,1)$ (except for the origin). Replacing α by $\alpha+2$ we get

THEOREM 4.3. *Let $\alpha>0$, $0<q<1$ and let $x_1<x_2<\dots<x_n$, $y_1<y_2<\dots<y_{n-1}$ and $t_1<t_2<\dots<t_n$ be the positive zeros of $C_{2n}^{\alpha+1}$ $(C_{2n+1}^{\alpha+2})$, $C_{2n-1}^{\alpha+2}$ $(C_{2n}^{\alpha+1})$ and C_{2n}^{α} (C_{2n+1}^{α}), respectively. Then*

$$0<x_1<t_1<y_1<x_2<t_2<\dots<x_{n-1}<t_{n-1}<y_{n-1}<x_n<t_n<1$$

$$(0<y_1<x_1<t_1<y_2<x_2<\dots<x_{n-1}<t_{n-1}<y_n<x_n<t_n<1).$$

Obviously one can formulate the corresponding corollary for the continuous shift as well.

Remark. The interlacing property cannot hold for larger integer shift of the parameter, since the q-ultraspherical polynomials tend to the Gegenbauer polynomials when $q \to 1$ and for the latter the interlacing with a shift of 3 fails (c.f. [8]). And, indeed, the zeros of C_n^α and $C_n^{\alpha+3}$ do not interlace for $q = 0.99$, $\alpha = 0.01$ and $n = 14$. The largest two zeros of C_n^α are:

$$0.943569, \quad 0.993573,$$

and those of $C_n^{\alpha+3}$:

$$0.859068, \quad 0.942525.$$

The zeros of C_n^α and $C_{n-1}^{\alpha+3}$ do not interlace either, since for the same parameter values as above, the largest two zeros of the latter are:

$$0.841258, \quad 0.935139.$$

Moreover, the zeros of C_n^α and $C_{n+1}^{\alpha+1}$ do not interlace either, since for the same parameter values the 4th and 5th positive zeros of C_n^α are:

$$0.706692, \quad 0.846319,$$

while those of $C_{n+1}^{\alpha+1}$ are:

$$0.706774, \quad 0.831145.$$

5. Interlacing of the zeros of the q-Laguerre polynomials with different parameters

Let L_n^α denote the n-th q-Laguerre polynomial. We know that its zeros are positive and simple. We make use of the mixed recurrence relations (cf. [18, formulas (4.12) and (4.15)])

$$L_n^\alpha = q^{-n}\big(L_n^{\alpha+1} - L_{n-1}^{\alpha+1}\big) \tag{13}$$

$$xL_n^{\alpha+2} = \left(xq^n - \frac{1-q^n}{1-q}q^{-\alpha-n-1}\right)L_n^{\alpha+1} + \frac{1-q^{\alpha+n+1}}{1-q}q^{-\alpha-n-1}L_{n-1}^{\alpha+1}. \tag{14}$$

Theorem 5.1. *Let $\alpha > -1$, $0 < q < 1$ and let $0 < x_1 < x_2 < \cdots < x_n$, $0 < y_1 < y_2 < \cdots < y_{n-1}$, $0 < t_1 < t_2 < \cdots < t_{n-1}$ and $0 < X_1 < X_2 < \cdots < X_{n-1}$ be the zeros of L_n^α, L_{n-1}^α, $L_{n-1}^{\alpha+t}$ and $L_{n-1}^{\alpha+2}$, respectively, where $0 < t < 2$. Then*

$$0 < x_1 < y_1 < t_1 < X_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < X_{n-1} < x_n.$$

PROOF. Expressing $L_{n-1}^{\alpha+1}$ from (14) we have

$$\begin{aligned}\frac{1-q^{\alpha+n+1}}{1-q}q^{-\alpha-n-1}L_{n-1}^{\alpha+1} &= \frac{1-q^n}{1-q}q^{-\alpha-n-1}L_n^{\alpha+1} + x\big(L_n^{\alpha+2} - q^n L_n^{\alpha+1}\big)\\ &= \frac{1-q^n}{1-q}q^{-\alpha-n-1}L_n^{\alpha+1} + xL_{n-1}^{\alpha+2}\end{aligned}$$

From this follows

$$\begin{aligned}&\left(\frac{1-q^{\alpha+n+1}}{1-q}q^{-\alpha-n-1} - \frac{1-q^n}{1-q}q^{-\alpha-n-1}\right)L_{n-1}^{\alpha+1}\\ &= \frac{1-q^n}{1-q}q^{-\alpha-n-1}\big(L_n^{\alpha+1} - L_{n-1}^{\alpha+1}\big) + xL_{n-1}^{\alpha+2}.\end{aligned}$$

Applying (13) on both sides:

$$\frac{1-q^{\alpha+1}}{1-q}q^{-\alpha-1}\big(L_n^{\alpha+1} - q^n L_n^{\alpha}\big) = \frac{1-q^n}{1-q}q^{-\alpha-1}\big(L_n^{\alpha} + xL_{n-1}^{\alpha+2}\big).$$

This finally gives

$$\frac{1-q^{\alpha+1}}{1-q}q^{-\alpha-1}L_n^{\alpha+1} = \frac{1-q^{\alpha+n+1}}{1-q}q^{-\alpha-1}L_n^{\alpha} + xL_{n-1}^{\alpha+2}. \tag{15}$$

This is not exactly like equation (1), but we can use the same principle as in Section 1. Evaluating (15) at successive zeros x_k and x_{k+1} of $L_n^{\alpha}(x)$, we obtain

$$x_k x_{k+1} L_{n-1}^{\alpha+2}(x_k) L_{n-1}^{\alpha+2}(x_{k+1}) = \left(\frac{1-q^{\alpha+1}}{1-q}q^{-\alpha-1}\right)^2 L_n^{\alpha+1}(x_k) L_n^{\alpha+1}(x_{k+1}).$$

The expression on the right is negative since the zeros of L_n^{α} and $L_n^{\alpha+1}$ interlace (cf. [18], Theorem 3) and therefore

$$x_k < X_k < x_{k+1} \quad \text{for each fixed } k, \quad k = 1, \dots, n-1.$$

The zeros of L_{n-1}^{α} increase as α increases (cf. [18], Corollary of Theorem C), hence

$$y_k < t_k < X_k \quad \text{for each fixed } k, \quad k = 1, \dots, n.$$

Finally, since the zeros of L_n^{α} and L_{n-1}^{α} separate each other, we know that

$$x_k < y_k < x_{k+1} \text{ for each fixed } k, \quad k = 1, \dots, n-1$$

and this completes the proof. □

Remark. This result matches well with the one for the interlacing property of the zeros of the (common, not q-) Laguerre polynomials (cf. [8]). The Laguerre polynomials have the interlacing property for a fixed degree up to a shift of parameter value by 2, but the interlacing property in general fails at a shift by 3. Since the zeros of the Laguerre polynomials are the limits of the zeros of the q-Laguerre as $q \to 1$, we can expect the same phenomenon for the q-Laguerre. And, indeed the zeros of L_n^α and $L_n^{\alpha+3}$ do not interlace for $n = 4$, $\alpha = -0.8533$, and $q = 0.9$ (cf. [8, Remark after Theorem 2.3]). For these values the zeros of L_n^α are

$$0.00475276, \quad 0.133847, \quad 0.476193, \quad 1.23967,$$

and those of $L_n^{\alpha+3}$ are

$$0.17967, \quad 0.523333, \quad 1.16211, \quad 2.43239.$$

Moreover, the zeros of $L_{n-1}^{\alpha+3}$ are

$$0.212982, \quad 0.643396, \quad 1.55213,$$

which shows that the zeros of L_n^α and $L_{n-1}^{\alpha+3}$ do not interlace in general. Furthermore, the zeros of L_n^α and $L_{n+1}^{\alpha+1}$ do not interlace in general, either. The zeros of L_n^α for $n = 4$, $\alpha = 1$, and $q = 0.5$ are

$$1.88953, \quad 12.7552, \quad 67.7788, \quad 382.576,$$

while those of $L_{n+1}^{\alpha+1}$ are:

$$4.38644, \quad 25.7767, \quad 124.761, \quad 592.76, \quad 3189.32.$$

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