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# Zeros of linear combinations of Laguerre polynomials from different sequences

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#### ABSTRACT

We study interlacing properties of the zeros of two types of linear combinations of Laguerre polynomials with different parameters, namely  $R_n = L_n^{\alpha} + a L_n^{\alpha'}$  and  $S_n = L_n^{\alpha} + b L_{n-1}^{\alpha'}$ . Proofs and numerical counterexamples are given in situations where the zeros of  $R_n$ , and  $S_n$ , respectively, interlace (or do not in general) with the zeros of  $L_n^{\alpha}$ ,  $L_n^{\alpha'}$ , k = n or n - 1. The results we prove hold for continuous, as well as integral, shifts of the parameter  $\alpha$ .

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#### 1. Introduction

Let  $\mu$  be a positive Borel measure supported on a finite or infinite interval [a,b] and let  $\{p_n\}_{n=0}^{\infty}$  be the sequence of polynomials, uniquely determined up to normalization, orthogonal with respect to  $\mu$ . Then  $\int_a^b x^k p_n(x) \mathrm{d}\mu(x) = 0$  for  $k = 0, 1, \ldots, n-1$  and it is well known that the zeros of  $p_n$  are real and simple and lie in (a,b). Moreover, if  $a < x_1 < x_2 < \cdots < x_n < b$  and  $a < y_1 < y_2 < \cdots < y_{n-1} < b$  are the zeros of  $p_n$  and  $p_{n-1}$  respectively, then

$$a < x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n < b,$$

a property usually called the interlacing of the zeros of  $p_n$  and  $p_{n-1}$ .

The interlacing of zeros of polynomials is particularly important in numerical quadrature (cf. [1]) and also arises, inter alia, in the context of extremal Zolotarev–Markov problems (cf. [2]), the Korous–Peebles problem (cf. [3]) and the Gelfond interpolation problem (cf. [4]).

In [5], Alfaro, Marcellán, Peña and Rezola derived necessary and sufficient conditions for the orthogonality of  $\{Q_n\}_{n=0}^{\infty}$ , where  $Q_n(x) = p_n(x) + a_1p_{n-1}(x) + \cdots + a_kp_{n-k}(x)$ ,  $a_k \neq 0$ ,  $n \geq k$  and  $\{p_n\}_{n=0}^{\infty}$  is a sequence of monic orthogonal polynomials. Their work extends the results of Peherstorfer (cf. [6]) who established sufficient conditions, when  $\sup(\mu) = (-1, 1)$ , on the real numbers  $\{a_j\}_{j=1}^k$  such that  $p_n + a_1p_{n-1} + \cdots + a_kp_{n-k}$  has n simple zeros in (-1, 1). Marcellán raised a more general question at OPSFA 2007: Given two different orthogonal sequences  $\{p_n\}_{n=0}^{\infty}$  and  $\{q_n\}_{n=0}^{\infty}$ , under what conditions does a linear combination  $r_n = p_n + aq_n$ ,  $a \neq 0$ , form an orthogonal sequence  $\{r_n\}_{n=0}^{\infty}$ ? A related question, relevant for applications, is whether and when the zeros of  $r_n$  interlace with the zeros of  $p_n$ ,  $p_{n-1}$ ,  $q_n$  or  $q_{n-1}$ . One starting point for answering these general questions is to consider linear combinations of classical orthogonal polynomials from the same family but from different sequences.

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In this paper, we consider linear combinations of Laguerre polynomials  $L_n^{\alpha}$  of the form  $R_n^{\alpha,t} = L_n^{\alpha} + a L_n^{\alpha+t}$  and  $S_n^{\alpha,t} = L_n^{\alpha} + b L_{n-1}^{\alpha+t}$  where  $\alpha > -1$ , t > 0 and  $a, b \ne 0$ . We recall that the Laguerre polynomials (cf. [7]) are orthogonal with respect to the weight function  $e^{-x}x^{\alpha}$ ,  $\alpha > -1$ , on the interval  $(0, \infty)$ .

For  $0 < t \le 2$ , we give proofs (or counterexamples) for the interlacing of the zeros of  $R_n^{\alpha,t}$  and  $S_n^{\alpha,t}$  with the zeros of  $L_n^{\alpha}$ ,  $L_{n-1}^{\alpha+t}$ ,  $L_{n-1}^{\alpha}$  and  $L_{n-1}^{\alpha+t}$ .

We will make use of two well known identities (cf. [8], 22.7.30 and 22.7.29)

$$L_n^{\alpha} = L_n^{\alpha+1} - L_{n-1}^{\alpha+1} \tag{1}$$

and 
$$xL_n^{\alpha+1}(x) = (x-n)L_n^{\alpha}(x) + (\alpha+n)L_{n-1}^{\alpha}(x)$$
. (2)

#### 2. Linear combinations of Laguerre polynomials of the same degree

Ι<sub>Δ</sub>t

$$R_n^{\alpha,t} = L_n^{\alpha} + aL_n^{\alpha+t}, \quad a \neq 0, \ \alpha > -1. \tag{3}$$

**Theorem 2.1.** For  $0 < t \le 2$ , the zeros of  $R_n^{\alpha,t}$  interlace with the zeros of (i)  $L_n^{\alpha}$ , (ii)  $L_n^{\alpha+t}$ .

**Proof.** We know from [9, Theorem 2.3] that the zeros of  $L_n^{\alpha}$  interlace with the zeros of  $L_n^{\alpha+t}$  for  $0 < t \le 2$  which implies that  $L_n^{\alpha}$  has a different sign at successive zeros of  $L_n^{\alpha+t}$  and vice versa. Evaluating (3) at successive zeros  $x_i$  and  $x_{i+1}$  of  $L_n^{\alpha}$  we obtain

$$R_n^{\alpha,t}(x_i)R_n^{\alpha,t}(x_{i+1}) = a^2 L_n^{\alpha+t}(x_i) L_n^{\alpha+t}(x_{i+1}), \quad 1 = 1, 2, \dots, n-1$$
  
< 0 for all  $a \neq 0$ .

Therefore  $R_n^{\alpha,t}$  has a different sign at successive zeros of  $L_n^{\alpha}$  and so the zeros interlace. The same argument shows that the zeros of  $R_n^{\alpha,t}$  interlace with those of  $L_n^{\alpha+t}$  by evaluating (3) at successive zeros of  $L_n^{\alpha+t}$ .  $\square$ 

**Remark:** For the integer values t=1 and t=2,  $R_n^{\alpha,1}$  and  $R_n^{\alpha,2}$  are in fact each a linear combination of orthogonal polynomials from the same sequence. Indeed, using the identity (1), we see that

$$R_n^{\alpha,1} = (a+1)L_n^{\alpha+1} - L_{n-1}^{\alpha+1},\tag{4}$$

and the restrictions on a to ensure that  $\{R_n^{\alpha,1}\}_{n=0}^{\infty}$  has all its zeros in  $(0,\infty)$  can be deduced from ([10], Theorem 3(v)). Similarly, applying (1) iteratively, we obtain

$$R_n^{\alpha,2} = (a+1)L_n^{\alpha+2} - 2L_{n-1}^{\alpha+2} + L_{n-2}^{\alpha+2}$$
(5)

and the zeros of this type of linear combination are discussed in ([10], Theorem 5, and [1]).

Evaluating (4) at successive zeros of  $L_{n-1}^{\alpha+1}$ , one can also prove that the zeros of  $R_n^{\alpha,1}$  interlace with the zeros of  $L_{n-1}^{\alpha+1}$ . However, the zeros of  $R_n^{\alpha,t}$  do not interlace with the zeros of  $L_{n-1}^{\alpha}$  even in the simple special case when t=1, as illustrated by the following example: For n=5, a=2.33,  $\alpha=1.45$  and t=1, the zeros of  $L_a^{1.45}$  are

while those of  $R_5^{1.45,1}$  are

{1.17057, 3.01797, 5.80288, 9.83574, 15.9213}.

#### 3. Linear combinations of Laguerre polynomials of different degree

Next we consider linear combinations of the type

$$S_n^{\alpha,t} = L_n^{\alpha} + b L_{n-1}^{\alpha+t}, \quad b \neq 0, \quad \alpha > -1.$$
 (6)

We will need information on the interlacing properties of the two polynomials  $L_n^{\alpha}$  and  $L_{n-1}^{\alpha+1}$  in the linear combination.

**Theorem 3.1.** Let  $\alpha > -1$  and let

$$0 < x_1 < x_2 < \cdots < x_n$$
 be the zeros of  $L_n^{\alpha}$ ,

$$0 < y_1 < y_2 < \cdots < y_{n-1}$$
 be the zeros of  $L_{n-1}^{\alpha}$ ,

$$0 < t_1 < t_2 < \cdots < t_{n-1}$$
 be the zeros of  $L_{n-1}^{\alpha+t}$  and

$$0 < X_1 < X_2 < \cdots < X_{n-1}$$
 be the zeros of  $L_{n-1}^{\alpha+2}$ 

where 0 < t < 2. Then

$$0 < x_1 < y_1 < t_1 < X_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < X_{n-1} < x_n$$

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**Proof.** A simple computation using (1) and (2) leads to

$$(\alpha + 1)L_n^{\alpha + 1}(x) = (\alpha + n + 1)L_n^{\alpha}(x) + xL_{n-1}^{\alpha + 2}(x). \tag{7}$$

Evaluating (7) at successive zeros  $x_k$  and  $x_{k+1}$  of  $L_n^{\alpha}(x)$ , we obtain

$$x_k x_{k+1} L_{n-1}^{\alpha+2}(x_k) L_{n-1}^{\alpha+2}(x_{k+1}) = (\alpha+1)^2 L_n^{\alpha+1}(x_k) L_n^{\alpha+1}(x_{k+1}).$$

The expression on the right is negative since the zeros of  $L_n^{\alpha}$  and  $L_n^{\alpha+1}$  interlace (cf. [9, Theorem 2.3]) and therefore

$$x_k < X_k < x_{k+1}$$
 for each fixed  $k, k = 1, ..., n - 1$ .

The zeros of  $L_{n-1}^{\alpha}$  increase as  $\alpha$  increases (cf. [7], p. 122), hence

$$y_k < t_k < X_k$$
 for each fixed  $k, k = 1, ..., n$ .

Finally, since the zeros of  $L_n^{\alpha}$  and  $L_{n-1}^{\alpha}$  separate each other, we know that

$$x_k < y_k < x_{k+1}$$
 for each fixed  $k, k = 1, ..., n-1$ 

and this completes the proof.

Note that this result extends Theorem 2.4 in [9] to the case of polynomials of different degree with continuously varying parameters.

**Theorem 3.2.** For  $0 < t \le 2$ , the zeros of  $S_n^{\alpha,t}$  interlace with the zeros of (i)  $L_n^{\alpha}$ , (ii)  $L_{n-1}^{\alpha+t}$ .

**Proof.** We know from Theorem 3.1 that the zeros of  $L_n^{\alpha}$  interlace with the zeros of  $L_{n-1}^{\alpha+t}$  for  $0 < t \le 2$  which implies that  $L_{n-1}^{\alpha+t}$  has a different sign at successive zeros of  $L_n^{\alpha}$  and vice versa. Evaluating (6) at successive zeros  $x_i$  and  $x_{i+1}$  of  $L_n^{\alpha}$  we obtain

$$S_n^{\alpha,t}(x_i)S_n^{\alpha,t}(x_{i+1}) = b^2 L_{n-1}^{\alpha+t}(x_i)L_{n-1}^{\alpha+t}(x_{i+1}), \quad i = 1, 2, \dots, n-1$$
  
< 0 for all  $b \neq 0$ .

Therefore  $S_n^{\alpha,t}$  has a different sign at successive zeros of  $L_n^\alpha$  and so the zeros interlace. The same argument shows that the zeros of  $S_n^{\alpha,t}$  interlace with those of  $L_{n-1}^{\alpha+t}$  by evaluating (6) at successive zeros of  $L_{n-1}^{\alpha+t}$ .

It is interesting to note that in the case of linear combinations of Laguerre polynomials of different degree, the zeros of  $S_n^{\alpha,t}$  do not interlace with the zeros of  $L_{n-1}^{\alpha}$ . Indeed, even in the simplest case when t=1 and n=5, b=2.33,  $\alpha=1.45$  in (6), the zeros of  $S_5^{1.45,1}$  are

while those of  $L_4^{1.45}$  are

and interlacing does not occur. The zeros of  $S_n^{\alpha,t}$  and  $L_n^{\alpha+t}$  are interlacing when t=1 since  $S_n^{\alpha,1}=L_n^{\alpha+1}+(b-1)L_{n-1}^{\alpha+1}$ . However, when t=2, the zeros of  $S_5^{1.45,2}$  are

while those of  $L_5^{1.45+2}$  are

and interlacing fails in this case.

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