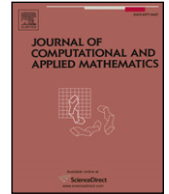




Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Zeros of linear combinations of Laguerre polynomials from different sequences

Kathy Driver^{a,*}, Kerstin Jordaan^b^a Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa^b Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

ARTICLE INFO

Article history:

Received 27 October 2007

MSC:

33C45

42C05

Keywords:

Laguerre polynomials

Zeros

Interlacing properties

Linear combinations

ABSTRACT

We study interlacing properties of the zeros of two types of linear combinations of Laguerre polynomials with different parameters, namely $R_n = L_n^\alpha + aL_n^{\alpha'}$ and $S_n = L_n^\alpha + bL_{n-1}^{\alpha'}$. Proofs and numerical counterexamples are given in situations where the zeros of R_n , and S_n , respectively, interlace (or do not in general) with the zeros of $L_k^\alpha, L_k^{\alpha'}$, $k = n$ or $n - 1$. The results we prove hold for continuous, as well as integral, shifts of the parameter α .

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Let μ be a positive Borel measure supported on a finite or infinite interval $[a, b]$ and let $\{p_n\}_{n=0}^\infty$ be the sequence of polynomials, uniquely determined up to normalization, orthogonal with respect to μ . Then $\int_a^b x^k p_n(x) d\mu(x) = 0$ for $k = 0, 1, \dots, n - 1$ and it is well known that the zeros of p_n are real and simple and lie in (a, b) . Moreover, if $a < x_1 < x_2 < \dots < x_n < b$ and $a < y_1 < y_2 < \dots < y_{n-1} < b$ are the zeros of p_n and p_{n-1} respectively, then

$$a < x_1 < y_1 < x_2 < y_2 < \dots < x_{n-1} < y_{n-1} < x_n < b,$$

a property usually called the interlacing of the zeros of p_n and p_{n-1} .

The interlacing of zeros of polynomials is particularly important in numerical quadrature (cf. [1]) and also arises, inter alia, in the context of extremal Zolotarev–Markov problems (cf. [2]), the Koros–Peebles problem (cf. [3]) and the Gelfond interpolation problem (cf. [4]).

In [5], Alfaro, Marcellán, Peña and Rezola derived necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n=0}^\infty$, where $Q_n(x) = p_n(x) + a_1 p_{n-1}(x) + \dots + a_k p_{n-k}(x)$, $a_k \neq 0$, $n \geq k$ and $\{p_n\}_{n=0}^\infty$ is a sequence of monic orthogonal polynomials. Their work extends the results of Peherstorfer (cf. [6]) who established sufficient conditions, when $\text{supp}(\mu) = (-1, 1)$, on the real numbers $\{a_j\}_{j=1}^k$ such that $p_n + a_1 p_{n-1} + \dots + a_k p_{n-k}$ has n simple zeros in $(-1, 1)$. Marcellán raised a more general question at OPSFA 2007: Given two different orthogonal sequences $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$, under what conditions does a linear combination $r_n = p_n + a q_n$, $a \neq 0$, form an orthogonal sequence $\{r_n\}_{n=0}^\infty$? A related question, relevant for applications, is whether and when the zeros of r_n interlace with the zeros of p_n , p_{n-1} , q_n or q_{n-1} . One starting point for answering these general questions is to consider linear combinations of classical orthogonal polynomials from the same family but from different sequences.

* Corresponding author.

E-mail addresses: Kathy.Driver@uct.ac.za (K. Driver), kjordaan@up.ac.za (K. Jordaan).

Proof. A simple computation using (1) and (2) leads to

$$(\alpha + 1)L_n^{\alpha+1}(x) = (\alpha + n + 1)L_n^\alpha(x) + xL_{n-1}^{\alpha+2}(x). \tag{7}$$

Evaluating (7) at successive zeros x_k and x_{k+1} of $L_n^\alpha(x)$, we obtain

$$x_k x_{k+1} L_{n-1}^{\alpha+2}(x_k) L_{n-1}^{\alpha+2}(x_{k+1}) = (\alpha + 1)^2 L_n^{\alpha+1}(x_k) L_n^{\alpha+1}(x_{k+1}).$$

The expression on the right is negative since the zeros of L_n^α and $L_n^{\alpha+1}$ interlace (cf. [9, Theorem 2.3]) and therefore

$$x_k < X_k < x_{k+1} \quad \text{for each fixed } k, k = 1, \dots, n - 1.$$

The zeros of L_{n-1}^α increase as α increases (cf. [7], p. 122), hence

$$y_k < t_k < X_k \quad \text{for each fixed } k, k = 1, \dots, n.$$

Finally, since the zeros of L_n^α and L_{n-1}^α separate each other, we know that

$$x_k < y_k < x_{k+1} \quad \text{for each fixed } k, k = 1, \dots, n - 1$$

and this completes the proof. \square

Note that this result extends Theorem 2.4 in [9] to the case of polynomials of different degree with continuously varying parameters.

Theorem 3.2. For $0 < t \leq 2$, the zeros of $S_n^{\alpha,t}$ interlace with the zeros of (i) L_n^α , (ii) $L_{n-1}^{\alpha+t}$.

Proof. We know from Theorem 3.1 that the zeros of L_n^α interlace with the zeros of $L_{n-1}^{\alpha+t}$ for $0 < t \leq 2$ which implies that $L_n^{\alpha+t}$ has a different sign at successive zeros of L_n^α and vice versa. Evaluating (6) at successive zeros x_i and x_{i+1} of L_n^α we obtain

$$\begin{aligned} S_n^{\alpha,t}(x_i) S_n^{\alpha,t}(x_{i+1}) &= b^2 L_{n-1}^{\alpha+t}(x_i) L_{n-1}^{\alpha+t}(x_{i+1}), \quad i = 1, 2, \dots, n - 1 \\ &< 0 \quad \text{for all } b \neq 0. \end{aligned}$$

Therefore $S_n^{\alpha,t}$ has a different sign at successive zeros of L_n^α and so the zeros interlace. The same argument shows that the zeros of $S_n^{\alpha,t}$ interlace with those of $L_{n-1}^{\alpha+t}$ by evaluating (6) at successive zeros of $L_{n-1}^{\alpha+t}$. \square

It is interesting to note that in the case of linear combinations of Laguerre polynomials of different degree, the zeros of $S_n^{\alpha,t}$ do not interlace with the zeros of L_{n-1}^α . Indeed, even in the simplest case when $t = 1$ and $n = 5, b = 2.33, \alpha = 1.45$ in (6), the zeros of $S_5^{1.45,1}$ are

$$\{1.34638, 3.48132, 6.74108, 11.6384, 20.6928\}$$

while those of $L_4^{1.45}$ are

$$\{0.954365, 2.94834, 6.26071, 11.6366\},$$

and interlacing does not occur. The zeros of $S_n^{\alpha,t}$ and $L_n^{\alpha+t}$ are interlacing when $t = 1$ since $S_n^{\alpha,1} = L_n^{\alpha+1} + (b - 1)L_{n-1}^{\alpha+1}$. However, when $t = 2$, the zeros of $S_5^{1.45,2}$ are

$$\{1.94417, 4.47751, 8.08954, 12.6085, 16.7802\}$$

while those of $L_5^{1.45+2}$ are

$$\{1.70945, 3.92167, 7.07942, 11.5061, 18.0334\},$$

and interlacing fails in this case.

Acknowledgements

Research by the first author was supported by the National Research Foundation of South Africa under grant number 2053730 and the research by the second author was supported by the National Research Foundation of South Africa under grant number 2054423.

References

[1] J.A. Shohat, On mechanical quadratures, in particular, with positive coefficients, Trans. Amer. Math. Soc. 42 (1937) 491–496.
 [2] Z. Grinshpun, Special linear combinations of orthogonal polynomials, J. Math. Anal. Appl. 299 (1) (2004) 1–18.

- [3] V.B. Uvarov, On the connection between polynomials, orthogonal with different weights, Dokl. Akad. Nauk USSR 126 (1959) 32–36.
- [4] A. Gelfond, On the polynomials least deviating from zero with their derivatives, Dokl. Akad. Nauk USSR 96 (1954).
- [5] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? Available in ArXiv [math.CA/07111740](https://arxiv.org/abs/math.CA/07111740).
- [6] F. Peherstorfer, Linear combination of orthogonal polynomials generating positive quadrature formulas, Math. Comput. 55 (191) (1990) 231–241.
- [7] G. Szegő, Orthogonal Polynomials, American Mathematical Society, New York, 1959.
- [8] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, in: Nat. Bur. Standards Appl. Series, vol. 55, US Government Printing Office, Washington, DC, 1964.
- [9] K. Driver, K. Jordaan, Interlacing of zeros of shifted sequences of one-parameter orthogonal polynomials, Numer. Math. 107 (4) (2007) 615–624.
- [10] C. Brezinski, K.A. Driver, M. Redivo-Zaglia, Quasi-orthogonality with applications to some families of classical orthogonal polynomials, Appl. Numer. Math. 48 (2) (2004) 157–168.