ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Journal of Approximation Theory

www.elsevier.com/locate/jat

Polynomial solutions of differential–difference equations

Diego Dominici[a], Kathy Driver[b], Kerstin Jordaan[c,*]

[a] *Department of Mathematics, State University of New York at New Paltz, 1 Hawk Dr. Suite 9, New Paltz, NY 12561-2443, USA*

[b] *Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa*

[c] *Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa*

Received 12 February 2009; received in revised form 4 May 2009; accepted 25 May 2009

Communicated by Leonid Golinskii

Abstract

We investigate the zeros of polynomial solutions to the differential–difference equation

$$P_{n+1} = A_n P_n' + B_n P_n, \quad n = 0, 1, \ldots$$

where A_n and B_n are polynomials of degree at most 2 and 1 respectively. We address the question of when the zeros are real and simple and whether the zeros of polynomials of adjacent degree are interlacing. Our result holds for general classes of polynomials including sequences of classical orthogonal polynomials as well as Euler–Frobenius, Bell and other polynomials.

Keywords: Interlacing of zeros; Zeros of polynomial solutions of differential–difference equations; Euler–Frobenius polynomials; Bell polynomials

* Corresponding author.

E-mail addresses: dominicd@newpaltz.edu (D. Dominici), kathy.driver@uct.ac.za (K. Driver), kjordaan@up.ac.za (K. Jordaan).

0021-9045/$ - see front matter © 2009 Published by Elsevier Inc.
doi:10.1016/j.jat.2009.05.010

Please cite this article in press as: D. Dominici, et al., Polynomial solutions of differential–difference equations, Journal of Approximation Theory (2009), doi:10.1016/j.jat.2009.05.010

1. Introduction

Let $\{P_n\}_{n=0}^{\infty}$ be the sequence of polynomials defined by

$$P_0 = 1$$
$$P_{n+1} = A_n P_n' + B_n P_n, \quad n = 0, 1, \ldots, \tag{1}$$

where A_n and B_n are polynomials of degree at most 2 and 1 respectively. The best-known families of classical orthogonal polynomials satisfy differential–difference equations of this type:

- Jacobi polynomials

$$P_{n+1}^{(\alpha,\beta)}(x) = \frac{(2n+2+\alpha+\beta)\left(x^2-1\right)}{2(n+1)(n+1+\alpha+\beta)} \frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x} + \frac{(2n+2+\alpha+\beta)x+\alpha-\beta}{2(n+1)} P_n^{(\alpha,\beta)}(x).$$

- Laguerre polynomials

$$L_{n+1}^{\alpha}(x) = \frac{x}{n+1} \frac{\mathrm{d}L_n^{\alpha}}{\mathrm{d}x}(x) + \frac{\alpha+n+1-x}{n+1} L_n^{\alpha}(x).$$

- Hermite polynomials

$$H_{n+1}(x) = -\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} + 2x H_n(x).$$

In these cases, it follows from the theory of orthogonal polynomials (cf. [10]) that the zeros of P_n are real and simple and that the zeros of P_{n+1} and P_n are interlacing. This leads to the question of what can one say about the zeros of a sequence of polynomials satisfying (1) that is not orthogonal. Examples of such sequences include the Bell polynomials (cf. [1]), the Euler–Frobenius polynomials (cf. [6]) and the so-called derivative polynomials (cf. [7]).

A number of authors have investigated the properties of the zeros of sequences of polynomials that are solutions of (1) but are not, in general, orthogonal. In [11], Vertgeim considered polynomials generated by (1) with

$$A_n(x) = a_n x^2 - b_n, \qquad B_n(x) = \alpha a_n x, \quad \alpha, a_n, b_n > 0,$$

that generalize the Euler polynomials. In [4,5], Dubeau and Savoie study interlacing properties of the zeros of polynomial solutions of (1) with

$$A_n(x) = \kappa_n \left(1 - x^2\right), \qquad B_n(x) = -2\kappa_n r_n x, \quad r_n > 0, \kappa_n \neq 0,$$

which contain the generalized Euler–Frobenius and the ultraspherical polynomials as special cases. They also consider the Hermite-like polynomials defined by (1) with

$$A_n(x) = \kappa_n, \qquad B_n(x) = -2\kappa_n x, \quad \kappa_n \neq 0.$$

In [9], Liu and Wang analyze polynomial solutions of the equation

$$P_{n+1} = A_n P_n' + B_n P_n + C_n P_{n-1}, \quad n = 0, 1, \ldots. \tag{2}$$

By assuming that P_n has strictly nonnegative coefficients (resp. alternating in sign) and $A_n < 0$ or $C_n < 0$ for $x \leq 0$ (resp. $x \geq 0$), they show that $\{P_n\}_{n=0}^{\infty}$ forms a Sturm sequence.

Please cite this article in press as: D. Dominici, et al., Polynomial solutions of differential–difference equations, Journal of Approximation Theory (2009), doi:10.1016/j.jat.2009.05.010

Although (2) is more general than (1), the condition on the sign of the coefficients of P_n is *a priori* very difficult to check, except in rather isolated situations.

In this paper, we shall take an approach similar to that used in [11]. We establish criteria that ensure, for a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ satisfying (1), either that all the zeros of P_n are real and simple, or the zeros of P_n and P_{n+1} are interlacing, or both, based on conditions that can be checked directly from A_n and B_n. We present several interesting examples and consider possible extensions.

2. Preliminary results

It is obvious that any sequence of polynomials is a solution of (1), if we allow A_n and B_n to be (non-unique) rational functions. The question of characterizing which sequences of polynomials $\{P_n\}_{n=0}^{\infty}$ are solutions of (1), when A_n and B_n are polynomials, is addressed in the following theorem.

Theorem 1. *Suppose that P_n is a polynomial with n simple zeros. Let*

$$P_n(x_{n,k}) = 0, \qquad y_{n,k} = \frac{P_{n+1}(x_{n,k})}{P_n'(x_{n,k})}, \quad k = 1, \dots, n. \tag{3}$$

Then the following are equivalent:

(i) *There exist polynomials A_n and B_n of degree at most 2 and 1 respectively such that $\{P_n\}_{n=0}^{\infty}$ satisfies (1). For every $n \geq 3$, A_n and B_n are unique.*

(ii) *For every $n \geq 2$, there exists a quadratic polynomial A_n such that*

$$A_n(x_{n,k}) = y_{n,k}, \quad k = 1, \dots, n.$$

Proof. Evaluating (1) at the zeros of P_n, we clearly see that (i) $\Longrightarrow$ (ii). Assume now that (ii) holds. For $n = 0$, we have

$$P_1 = A_0 P_0' + B_0 P_0 = B_0$$

since $P_0 = 1$. Thus, $A_0(x)$ can be any polynomial of degree at most 2 and $B_0(x) = P_1(x)$. For $n = 1$, let $B_1(x)$ be an arbitrary polynomial of degree at most 1 and define $A_1(x)$ by

$$A_1 = \frac{P_2 - B_1 P_1}{P_1'}.$$

Since P_1' is a non-zero constant, we see that (i) holds for $n = 1$. For $n \geq 2$, let A_n be a quadratic polynomial such that

$$y_{n,k} = A_n(x_{n,k}), \quad k = 1, \dots, n.$$

Then $P_{n+1} - A_n P_n'(x)$ is a polynomial of degree at most $n + 1$ that is zero at each $x_{n,k}$ and therefore divisible by $P_n(x)$. This yields the existence of a B_n that is at most linear. When $n \geq 3$, A_n is uniquely determined so (i) holds. ■

Remark 2. The assumption that P_n has n simple zeros is not very restrictive in the sense that the statement and proof of Theorem 1 can be modified to cater for other possibilities.

Please cite this article in press as: D. Dominici, et al., Polynomial solutions of differential–difference equations, Journal of Approximation Theory (2009), doi:10.1016/j.jat.2009.05.010

Example 3. Let P_n be the family of orthonormal polynomials with respect to the Freud-type weight

$$w(x) = \sqrt{\frac{2}{t}} \frac{1}{\mathrm{K}_{\frac{1}{4}}\left(\frac{t^2}{2}\right)} \exp\left(-x^4 + 2tx^2 - \frac{t^2}{4}\right),$$

where $\mathrm{K}_\nu(z)$ is the Bessel function of the second kind. In this case, we have (cf. [8])

$$A_n(x) = -\frac{1}{4a_{n+1}\left(x^2 + a_{n+1}^2 + a_n^2 - t\right)}, \qquad B_n(x) = \frac{x\left(x^2 + a_{n+1}^2 - t\right)}{a_{n+1}\left(x^2 + a_{n+1}^2 + a_n^2 - t\right)},$$

where the numbers a_n are the coefficients in the three-term recurrence relation

$$xP_n(x) = a_{n+1}P_{n+1}(x) + a_nP_{n-1}(x)$$

satisfying the string equation

$$n = 4a_n^2\left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 - t\right), \quad n \geq 0.$$

The initial values for a_n are

$$a_0 = 0, \qquad a_1(t) = \sqrt{\frac{t}{2}} \sqrt{\frac{\mathrm{K}_{\frac{1}{4}}\left(\frac{t^2}{4}\right)}{\mathrm{K}_{\frac{3}{4}}\left(\frac{t^2}{4}\right)} - 1}, \qquad a_1(0) = \frac{\Gamma\left(\frac{3}{4}\right)}{2^{\frac{1}{4}}\sqrt{\pi}}, \tag{4}$$

where $a_1 > 0$ is chosen so that $P_1(x) = \frac{x}{a_1}$ has unit norm.

For $n = 5$, we have

$$P_5(x) = \frac{x^5 - \alpha x^3 + \beta x}{a_1a_2a_3a_4a_5},$$

with

$$\alpha = a_1^2 + a_2^2 + a_3^2 + a_4^2, \qquad \beta = a_1^2a_3^2 + a_1^2a_4^2 + a_2^2a_4^2$$

and therefore, with the notation of (3),

$$x_{5,1} = 0, \qquad x_{5,2} = \sqrt{\zeta_+}, \qquad x_{5,3} = -\sqrt{\zeta_+}, \qquad x_{5,4} = \sqrt{\zeta_-}, \qquad x_{5,5} = -\sqrt{\zeta_-}$$

and

$$\zeta_\pm = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\beta}\right).$$

One can show, using a simple algebraic argument, that the polynomial interpolating the points $\left(x_{5,k}, y_{5,k}\right)$, $k = 1, \ldots, 5$, has degree 4, unless $a_1 = 0$, which contradicts (4). We deduce from Theorem 1 that there are no polynomials A_n and B_n such that P_n satisfies (1).

3. Main result

In our proofs, we will find it convenient to rewrite the differential–difference Eq. (1) in the form

$$P_{n+1}(x) = \frac{A_n(x)}{K_n(x)} \frac{\mathrm{d}[K_n(x)P_n(x)]}{\mathrm{d}x}, \tag{5}$$

where K_n is an integrating factor, defined by

$$K_n(x) = \exp\left(\int_0^x \frac{B_n(t)}{A_n(t)} \mathrm{d}t\right). \quad (6)$$

Since A_n and B_n are polynomials, we can obtain all possible functions K_n by considering the location of the zeros of A_n and B_n. This leads to the following classification, where we define the extended real line by $(-\infty, \infty) \cup \{-\infty, \infty\}$ and the notation $f(\pm\infty) = 0$ means $\lim_{x\to\pm\infty} f(x) = 0$.

Theorem 4. *Let $K_n(x)$ be defined by (6). Then, $K_n(x)$ has at most two zeros on the extended real line.*

Proof. Since K_n depends on the ratio B_n/A_n, without loss of generality, we can choose A_n to be monic. We consider the possible cases in turn.

1. Let $A_n(x) = (x - \lambda_n)(x - \xi_n)$, $\deg(B_0) = 1$ and $B_n(x) = \mu_n$, $\mu_n \neq 0$ for $n \geq 1$. If $\lambda_n \neq \xi_n$, we have

$$K_n(x) = \exp\left[\frac{\mu_n}{\lambda_n - \xi_n} \ln\left(\frac{x - \lambda_n}{x - \xi_n}\right)\right].$$

If $\xi_n = \lambda_n$, we have

$$K_n(x) = \exp\left(-\frac{\mu_n}{x - \lambda_n}\right).$$

2. Let $A_n(x) = (x - \lambda_n)(x - \xi_n)$, and $B_n(x) = \kappa_n(x - \mu_n)$, $\kappa_n \neq 0$. If $\xi_n \neq \lambda_n \neq \mu_n$, we have

$$K_n(x) = \exp\left[\kappa_n \frac{\lambda_n - \mu_n}{\lambda_n - \xi_n} \ln(x - \lambda_n) + \kappa_n \frac{\mu_n - \xi_n}{\lambda_n - \xi_n} \ln(x - \xi_n)\right].$$

If $\xi_n = \lambda_n \neq \mu_n$, we have

$$K_n(x) = (x - \lambda_n)^{\kappa_n} \exp\left(\kappa_n \frac{\mu_n - \lambda_n}{x - \lambda_n}\right).$$

If $\xi_n \neq \lambda_n = \mu_n$ or $\xi_n = \lambda_n = \mu_n$, we have

$$K_n(x) = (x - \xi_n)^{\kappa_n}.$$

3. Let $A_n(x) = x - \lambda_n$, and $B_n(x) = \kappa_n(x - \mu_n)$, $\kappa_n \neq 0$. If $\lambda_n \neq \mu_n$, we have

$$K_n(x) = \exp[\kappa_n x + \kappa_n(\lambda_n - \mu_n)\ln(x - \lambda_n)].$$

If $\lambda_n = \mu_n$, we have

$$K_n(x) = \exp(\kappa_n x).$$

4. Let $A_n(x) = 1$, and $B_n(x) = \kappa_n(x - \mu_n)$, $\kappa_n \neq 0$. We have

$$K_n(x) = \exp\left[\kappa_n x\left(\frac{x}{2} - \mu_n\right)\right]. \quad \blacksquare$$

Note that in some of the above cases K_n is multi-valued because of the logarithmic function. However, K_n can be made single-valued by changing the lower bound in (6) and doing so will not affect the use of K_n in the next theorem.

Please cite this article in press as: D. Dominici, et al., Polynomial solutions of differential–difference equations, Journal of Approximation Theory (2009), doi:10.1016/j.jat.2009.05.010

Theorem 5. *We denote the zeros of P_n in increasing order by $\gamma_{i,n}$, $i = 1, \ldots, n$. Let*

$$-\infty \leq \alpha_{n+1} \leq \alpha_n < \beta_n \leq \beta_{n+1} \leq \infty$$

and K_n be continuous on $[\alpha_n, \beta_n]$ and differentiable on (α_n, β_n). Then:

(a) *If $K_n = 0$ only at $x = \alpha_n$ and $x = \beta_n$ for $1 \leq n \leq N$ and $\alpha_1 < \gamma_{1,1} < \beta_1$, then the zeros of P_n and P_{n+1} interlace and are in the interval (α_n, β_n) for $n = 1, \ldots, N$.*

(b) *If $K_n = 0$ only when $x = \alpha_n$ (resp. β_n), $\frac{A_n}{K_n} = 0$ when $x = \beta_n$ (resp. α_n), $\beta_n < \beta_{n+1}$ for $1 \leq n \leq N$ (resp. $\alpha_{n+1} < \alpha_n$ for $1 \leq n \leq N$) and $\alpha_1 < \gamma_{1,1} < \beta_1$, then the zeros of P_n and P_{n+1} interlace and are in the interval (α_n, β_n) for $n = 1, \ldots, N$.*

(c) *If $K_n = 0$ only when $x = \alpha_n$ (resp. β_n), $\frac{A_n}{K_n} = 0$ when $x = \beta_n$ (resp. α_n) for $1 \leq n \leq N$ and $\alpha_1 < \gamma_{1,1} \leq \beta_1$ (resp. $\alpha_1 \leq \gamma_{1,1} < \beta_1$), then all the zeros of P_n are real, simple and in the interval $(\alpha_n, \beta_n]$ (resp. $[\alpha_n, \beta_n)$) for $n = 1, \ldots, N$.*

(d) *If $K_n \neq 0$, $\frac{A_n}{K_n} = 0$ when $x = \alpha_n, \beta_n$ and $\alpha_{n+1} < \alpha_n < \beta_n < \beta_{n+1}$ for $1 \leq n \leq N$ with $\alpha_1 < \gamma_{1,1} < \beta_1$, then the zeros of P_n and P_{n+1} interlace for $n = 1, \ldots, N$.*

Proof. We use the familiar extension of Rolle's theorem to an infinite open interval.

(a) When $n = 1$ we have $P_2(x) = \frac{A_1(x)}{K_1(x)} \frac{\mathrm{d}[K_1(x)P_1(x)]}{\mathrm{d}x}$ and $K_1(x)P_1(x) = 0$ at α_1, β_1 and $\gamma_{1,1}$ with $\alpha_1 < \gamma_{1,1} < \beta$. It follows from Rolle's theorem that $P_2(x) = 0$ at $\gamma_{1,2}$, $\gamma_{2,2}$ with

$$\alpha_2 \leq \alpha_1 < \gamma_{1,2} < \gamma_{1,1} < \gamma_{2,2} < \beta_1 \leq \beta_2.$$

Now, let $n \in \mathbb{N}$ satisfy $2 \leq n \leq N$ and assume that we have proved the result for P_n. Since $K_n(x)P_n(x)$ vanishes at α_n, β_n and $\gamma_{i,n}$ for $i = 1, \ldots, n$ with $\alpha_n < \gamma_{1,n} < \cdots < \gamma_{n,n} < \beta_n$, Rolle's theorem applied to (5) yields

$$\alpha_{n+1} \leq \alpha_n < \gamma_{1,n+1} < \gamma_{1,n} < \cdots < \gamma_{n,n} < \gamma_{n+1,n+1} < \beta_n \leq \beta_{n+1}$$

and the result follows.

(b) We prove the result for $\frac{A_n}{K_n}(\beta_n) = 0$ for $n \in \mathbb{N}$, $1 \leq n \leq N$, the other case being analogous. When $n = 1$ we have $P_2(x) = \frac{A_1(x)}{K_1(x)} \frac{\mathrm{d}[K_1(x)P_1(x)]}{\mathrm{d}x}$ and $K_1(x)P_1(x) = 0$ at α_1 and $\gamma_{1,1}$ with $\alpha_1 < \gamma_{1,1} < \beta_1$ and it follows from Rolle's theorem that $\alpha_1 < \gamma_{1,2} < \gamma_{1,1}$. The second zero of $P_2(x)$ coincides with β_1, so we have

$$\alpha_2 \leq \alpha_1 < \gamma_{1,2} < \gamma_{1,1} < \gamma_{2,2} = \beta_1 < \beta_2.$$

Now, let $n \in \mathbb{N}$ satisfy $2 \leq n \leq N$ and assume that we have proved the result for P_n. Since $K_n(x)P_n(x)$ vanishes at α_n and $\gamma_{i,n}$ for $i = 1, \ldots, n$ with $\alpha_n < \gamma_{1,n} < \cdots < \gamma_{n,n} < \beta_n$, Rolle's theorem applied to (5) yields

$$\alpha_{n+1} \leq \alpha_n < \gamma_{1,n+1} < \gamma_{1,n} < \cdots < \gamma_{n,n}.$$

Since the largest zero of $P_{n+1}(x)$ coincides with β_n we have

$$\alpha_{n+1} \leq \alpha_n < \gamma_{1,n+1} < \gamma_{1,n} < \cdots < \gamma_{n,n} < \gamma_{n+1,n+1} = \beta_n < \beta_{n+1}$$

and the result follows.

(c) We prove the result for $\frac{A_n}{K_n}(\beta_n) = 0$ for $n \in \mathbb{N}$, $1 \leq n \leq N$, the other case being analogous. When $n = 1$ we have $P_2(x) = \frac{A_1(x)}{K_1(x)} \frac{\mathrm{d}[K_1(x)P_1(x)]}{\mathrm{d}x}$ and $K_1(x)P_1(x) = 0$ at α_1 and $\gamma_{1,1}$ with $\alpha_1 < \gamma_{1,1} \leq \beta_1$ and it follows from Rolle's theorem that $\alpha_1 < \gamma_{1,2} < \gamma_{1,1}$. The second zero of $P_2(x)$ coincides with β_1, so we have

$$\alpha_2 \leq \alpha_1 < \gamma_{1,2} < \gamma_{1,1} \leq \gamma_{2,2} = \beta_1 \leq \beta_2.$$

Now, let $n \in \mathbb{N}$ where $2 \leq n \leq N$ and assume that we have proved the result for P_n. Since $K_n(x)P_n(x)$ vanishes at α_n and $\gamma_{i,n}$ for $i = 1, \ldots, n$ with $\alpha_n < \gamma_{1,n} < \cdots < \gamma_{n,n} \leq \beta_n$, Rolle's theorem applied to (5) yields

$$\alpha_n < \gamma_{1,n+1} < \gamma_{1,n} < \cdots < \gamma_{n-1,n} < \gamma_{n,n+1} < \gamma_{n,n}.$$

The largest zero of $P_{n+1}(x)$ coincides with β_n, so we have

$$\alpha_{n+1} \leq \alpha_n < \gamma_{1,n+1} < \gamma_{1,n} < \cdots < \gamma_{n,n} \leq \gamma_{n+1,n+1} = \beta_n \leq \beta_{n+1}$$

and the result follows.

(d) When $n = 1$ we have $P_2(x) = \frac{A_1(x)}{K_1(x)} \frac{\mathrm{d}[K_1(x)P_1(x)]}{\mathrm{d}x}$ and $P_2(x) = 0$ at α_1 and β_1 with $\alpha_1 = \gamma_{1,2} < \gamma_{2,2} = \beta_1$. When $n = 2$, $\gamma_{1,3} = \beta_2$ and $\gamma_{3,3} = \beta_2$. Furthermore, $P_3(x) = \frac{A_2(x)}{K_2(x)} \frac{\mathrm{d}}{\mathrm{d}x} [K_2(x)P_2(x)]$ and Rolle's theorem implies that $\alpha_1 = \gamma_{1,2} < \gamma_{2,2} = \beta_1$. Hence

$$\alpha_2 = \gamma_{1,3} < \alpha_1 = \gamma_{1,2} < \gamma_{2,3} < \gamma_{2,2} = \beta_1 < \gamma_{3,3} = \beta_2.$$

Now, let $n \in \mathbb{N}$ with $3 \leq n \leq N$ and assume that we have proved the result for P_n. The smallest and largest zero of $P_{n+1}(x)$ are $\gamma_{1,n+1} = \alpha_n$ and $\gamma_{n+1,n+1} = \beta_n$. The remaining $n - 1$ zeros are obtained by applying Rolle's theorem to the function inside the square brackets in (5) and we obtain

$$\alpha_n = \gamma_{1,n+1} < \gamma_{1,n} < \gamma_{2,n+1} < \cdots < \gamma_{n,n+1} < \gamma_{n,n} < \gamma_{n+1,n+1} = \beta_n. \quad \blacksquare$$

4. Examples

We conclude by giving some examples where our results apply, highlighting particular choices of A_n and B_n that give rise to known families of polynomials.

Example 6. Let

$$A_n(x) = \kappa_n \left(1 - x^2\right), \qquad B_n(x) = -2\kappa_n r_n x, \qquad r_n > 0, \kappa_n \neq 0.$$

Then

$$K_n(x) = \left(x^2 - 1\right)^{r_n}$$

and Theorem 5(a) applies. In [4,5], the authors obtain the same result using a different approach.

Example 7. The Bell polynomials $\mathfrak{B}_n(x)$ are defined by

$$\mathfrak{B}_n(x) = \sum_{k=0}^{n} S_k^n x^k, \quad n = 0, 1, \ldots,$$

where S_k^n is the Stirling number of the second kind. They satisfy the differential–difference equation

$$\mathfrak{B}_{n+1}(x) = x \left[\mathfrak{B}_n'(x) + \mathfrak{B}_n(x)\right],$$

from which we obtain

$$A_n(x) = x, \qquad K_n(x) = \mathrm{e}^x .$$

In this case, we have $\alpha_n = -\infty$ and $\beta_n = 0$ and from Theorem 5(c) it follows that the zeros of $\mathfrak{B}_n(x)$ are real and simple and lie in the interval $(-\infty, 0]$. This result has been obtained by different methods, for example in [2, p. 271].

Please cite this article in press as: D. Dominici, et al., Polynomial solutions of differential–difference equations, Journal of Approximation Theory (2009), doi:10.1016/j.jat.2009.05.010

Example 8. Let P_n be the family of polynomials defined by

$$P_n(x) = (c)_n \; {}_2F_1\left(\begin{matrix} -n, & b \\ c & \end{matrix} \middle| \, x \right).$$

In this case, we have

$$A_n(x) = x(1-x), \qquad B_n(x) = n + c - bx$$

and thus

$$K_n(x) = x^{n+c}(x-1)^{b-c-n}.$$

Provided that $n \in (-c, b-c)$, it follows from Theorem 5(a) that the zeros of P_n and P_{n+1} are interlacing and lie in the interval $(0, 1)$. The same result is obtained in [3] using a different technique.

Acknowledgment

The work of D. Dominici was supported by a Humboldt Research Fellowship for experienced researchers provided by the Alexander von Humboldt Foundation. The work of K. Driver and K. Jordaan was supported by the National Research Foundation of South Africa under grant numbers 2053730 and 2054423 respectively.

References

[1] L. Carlitz, Single variable Bell polynomials, Collect. Math. 14 (1962) 13–25.
[2] L. Comtet, Advanced Combinatorics, The Art of Finite and Infinite Expansions, D. Reidel Publishing Co., Dordrecht, 1974.
[3] K. Driver, K. Jordaan, Separation theorems for the zeros of certain hypergeometric polynomials, J. Comput. Appl. Math. 199 (1) (2007) 48–55.
[4] F. Dubeau, J. Savoie, On interlacing properties of the roots of orthogonal and Euler–Frobenius polynomials, in: Approximation Theory VIII, vol. 1 (College Station, TX, 1995), in: Ser. Approx. Decompos., vol. 6, World Sci. Publ., River Edge, NJ, 1995, pp. 185–191.
[5] F. Dubeau, J. Savoie, On the roots of orthogonal polynomials and Euler–Frobenius polynomials, J. Math. Anal. Appl. 196 (1) (1995) 84–98.
[6] F.G. Frobenius, Über die Bernoullischen Zahlen und die Eulerischen Polynome, Sitz. ber. Preuss. Akad. Wiss. (1910) 809–847.
[7] M.E. Hoffman, Derivative polynomials for tangent and secant, Amer. Math. Monthly 102 (1) (1995) 23–30.
[8] M.E.H. Ismail, Classical and quantum orthogonal polynomials in one variable, in: Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, 2005.
[9] L.L. Liu, Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. Appl. Math. 38 (4) (2007) 542–560.
[10] G. Szegő, Orthogonal Polynomials, American Mathematical Society, Providence, R.I., 1975.
[11] B.A. Vertgeĭm, Euler polynomials and their generalization, Sibirsk. Mat. Zh. 33 (2) (1992) 164–167.