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A Characterization of Procyclic Groups via Complete Exterior Degree

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Abstract: We describe the nonabelian exterior square $G\widehat{\wedge}G$ of a pro-p-group G (with p arbitrary prime) in terms of quotients of free pro-p-groups, providing a new method of construction of $G\widehat{\wedge}G$ and new structural results for $G\widehat{\wedge}G$. Then, we investigate a generalization of the probability that two randomly chosen elements of G commute: this notion is known as the "complete exterior degree" of a pro-p-group and we will use it to characterize procyclic groups. Among other things, we present a new formula, which simplifies the numerical aspects which are connected with the evaluation of the complete exterior degree.

Keywords: nonabelian exterior square; pro-p-groups; Schur multiplier; free profinite groups

MSC: 20P05; 20J05; 20E10; 20J06

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1. Introduction and Formulation of the Main Results

In the present paper we deal only with topological Hausdorff groups. Topological groups with a discrete topology are called *discrete groups* or *abstract groups*. Topological groups with a compact topology are called *compact groups*. Of course, finite groups are examples of discrete compact groups but the additive group $\mathbb{Z}_p$ of the p-adic integers (p prime) is an example of an infinite nondiscrete compact group, see Example 1.28 (i) and Exercise E1.10 in [1]. The usual notion of an *abelian tensor product* of two abstract abelian groups, which is described by Propositions A1.44, A1.45 and A1.46 of [1], has been adapted to the context of profinite groups in §5.4 and §5.5 of [2], introducing the *complete abelian tensor product* of two profinite abelian groups. Usually, one formulates a universal property and then provides an explicit construction, as indicated in Lemma 5.5.1, Lemma 5.5.2 and Proposition 5.5.4 of [2]. Brown and Loday [3,4] introduced the *nonabelian tensor product* of two abstract groups, which are not necessarily abelian. Adapting the notion of Brown and Loday to topological groups, the presence of a topology should be compatible with the algebraic structure of the tensor product and some difficulties can appear even if we consider compact groups.

Let us focus on a special class of compact groups. First of all, we mention from §1.1 of [2] that a topological space X, which arises as projective limit on a given (directed) set of indices J, can be written as

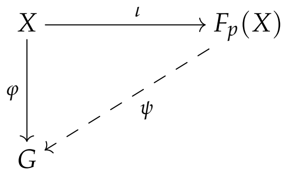
$$X = \varprojlim_{j \in J} X_j \quad \text{with} \quad X_j \ \text{finite space endowed with the discrete topology} \ \ \forall j \in J \qquad (1)$$

and X is called *profinite space*. Secondly, we may look at finite groups and a projective limit

$$G = \varprojlim_{j \in J} G_j \quad \text{with } \; G_j \; \text{ finite } p-\text{group } \; \forall j \in J \tag{2}$$

is called a *pro-p-group*, as indicated by Definition 1.27 of [1]. These are special types of *profinite groups*, that is, totally disconnected compact groups, which are described by Theorem 1.34 of [1]. Several results of classification of compact groups involve maximal closed subgroups which are pro-p-groups; just to give an idea, Corollaries 8.5, 8.6 and 8.8 of [1] classify compact abelian groups via their pro-p-subgroups; in particular Corollary 8.8 of [1] shows that any compact abelian group is totally disconnected if and only if it is the direct product of pro-p-groups.

Following §3.3 of [2], if X is a profinite space, $F_p(X)$ a pro-p-group and $\iota : X \to F_p(X)$ a continuous map such that $F_p(X) = \overline{\langle \iota(X) \rangle}$, we say that the pair $(F_p(X), \iota)$, or briefly $F_p(X)$, is a *free pro-p-group* on X, if the following universal property is satisfied



(3)

where $\varphi : X \to G$ is a continuous map into a pro-p-group G, $\varphi(X)$ *topologically generates* G (i.e., $\overline{\langle \varphi(X) \rangle} = G$) and $\psi : F_p(X) \to G$ is a continuous homomorphism such that (3) commutes (i.e., $\psi \circ \iota = \varphi$). Diagram (3) describes a universal property defining $F_p(X)$. The reader can look at Theorem A3.28 of [1] for a more general perspective on the definition of objects in a prescribed category by a universal property. Here, we assume that X is nonempty and $|X| \geq 2$ in order to avoid trivial examples. We also note that ι is an embedding by Lemma 3.3.1 of [2] and that for every profinite space X there exists a unique free pro-p-group $F_p(X)$ on X by Proposition 3.3.2 of [2].

We recall briefly the details of the construction of $F_p(X)$ here. For instance, if X is a profinite space, F a free abstract group on X and $N \triangleleft F$ (i.e., N is a normal subgroup of F), then one observes that

$$\mathcal{N}(F) = \{ N \triangleleft F \mid F/N \text{ finite } p\text{-group and } X \cap fN \text{ open in } X,\ \forall f \in F \} \tag{4}$$

is a filter basis which allows us to form the projective limit

$$\lim_{N \in \mathcal{N}(F)} F/N =: F_p(X). \tag{5}$$

This is a concrete construction of the free pro-p-group on X and we may check that $F_p(X)$ satisfies the universal property expressed by (3), as illustrated by Proposition 3.3.2 and Exercise 3.3.3 of [2]. Note also that $F_p(X)$ possesses the topology induced by (4) and is compatible with the structure of the projective limit. The same logic applies to any filter basis and works in fact for free compact groups, but also for a free pro-$\mathcal{C}$-group with $\mathcal{C}$ an arbitrary class of finite groups closed under taking quotients and finite subdirect products (and containing groups of order two). The reader may refer to Chapter 11 of [1], to Chapter 3 of [2] and to [5] for more information on topological free groups.

We recall now that a *Cantor cube* is a topological space which is homeomorphic to a product space $\{0, 1\}^{\aleph}$ for some infinite cardinal $\aleph$, see Definition A4.30 in [1]. A *dyadic space* is a continuous image of a Cantor cube. Our first main result provides a new approach to the notion of a *complete nonabelian tensor square* $G \widehat{\otimes} G$ of a pro-p-group G, originally studied in [6,7] for pro-p-groups, but introduced by Brown and Loday [3,4] for finite groups.

Theorem 1. *Let $G = \varprojlim_{j \in J} G_j$ be a pro-p-group with G_j finite p-groups for all $j \in J$.*

Then, there exists a profinite space Y such that the complete nonabelian tensor square $G\widehat{\otimes}G$ is the pro-p-group which is topologically isomorphic to the quotient group $F_p(Y)/K$ of the free pro-p-group $F_p(Y)$ on Y by the smallest closed normal subgroup K of $F_p(Y)$ containing the set

$$\{\iota(gz,h)\iota(g,h)^{-1}\iota(z^g,h^g)^{-1}, \iota(g,ht)\iota(g^h,t^h)^{-1}\iota(g,h)^{-1} \mid g,z,h,t \in G\}.$$

Moreover, ι embeds Y into $F_p(Y)$ and, if in addition $G\widehat{\otimes}G$ is metrizable, then $G\widehat{\otimes}G$ is a dyadic space.

We should mention that Theorem 1 has been recently proved for arbitrary compact groups in Theorem 1.4 of [8], involving the representation theory of compact groups. Here we do not use the representation theory and offer an argument, which involves only constructions via projective limits. The concrete description of Theorem 1 allows us to illustrate our second main result.

First of all, we consider $\widehat{\nabla}(G) = \overline{\langle x\widehat{\otimes}x \mid x \in G\rangle}$, which turns out to be a closed normal subgroup of $G\widehat{\otimes}G$, called a *diagonal subgroup* of $G\widehat{\otimes}G$, and then we form the quotient

$$(G\widehat{\otimes}G)/\widehat{\nabla}(G) = G\widehat{\wedge}G \tag{6}$$

which is called a *complete nonabelian exterior square* of the pro-p-group G. We mention that the set

$$\widehat{C}_G(x) = \{a \in G \mid a\widehat{\wedge}x = 1\} \tag{7}$$

is a closed subgroup of G, called a *complete exterior centralizer* of x in G (see [7,9]), and the Haar measure μ on G, whose properties are illustraed by Theorem 2.8 and Exercise E2.3 in [1], allows us to introduce the *complete exterior degree*

$$\widehat{\mathrm{d}}(G) := \int_G \mu(\widehat{C}_G(x))d\mu \tag{8}$$

of the pro-p-group G. In particular, if G is finite and we consider the counting measure on G, then we find *the exterior degree of finite groups* in [10–12]. The *complete exterior center*

$$\widehat{Z}(G) = \bigcap_{x \in G} \widehat{C}_G(x) = \{a \in G \mid a\widehat{\wedge}x = 1, \forall x \in G\}, \tag{9}$$

plays a relevant role in [7,9] and is always a subgroup of the usual center $Z(G)$ of G.

Secondly, we note that the notion of the *FC-center* is well known (i.e., it is the set of elements with finite conjugacy classes) and investigated by Baer in 14.5.6 of [13] and by Neumann in 14.5.9 and 14.5.11 of [13]. In case of a pro-p-group G the *complete exterior FC-center* is more recent:

$$\widehat{FC}(G) := \{x \in G \mid |G : \widehat{C}_G(x)| \text{ is finite}\}. \tag{10}$$

This set turns out to be a closed normal subgroup of G by Lemma 3 of [9].

Thirdly, we recall that a pro-p-group P is *procyclic*, if it is topologically generated by a single element. As indicated by Proposition 2.7.1 of [2], a procyclic group P is either isomorphic to $\mathbb{Z}_p$ or to the cyclic group $\mathbb{Z}(p^n)$ of p-power order (with $n \geq 1$). Detecting procyclic groups among totally disconnected compact groups turns out to be relevant for several results of classification (for instance, they are involved in the theory of *near abelian compact groups*, whose structure is described by Theorems 6, 15, 26, 35 in Overview of [14]. In particular, some homological notions such as the notion of the *Schur multiplier* should be involved (see [2,14] and Definition 3 below). Our second main result is devoted to recognize procyclic groups through the complete exterior degree.

Theorem 2. *For a pro-p-group G, the following conditions are satisfied:*

(i). $\widehat{\mathrm{d}}(G) = 0$ *if and only if* $\widehat{FC}(G)$ *is not open in G;*

(ii). $\widehat{\mathrm{d}}(G) = 1$ *if and only if G is procyclic.*

In particular, if G is a nonprocyclic pro-p-group with $\widehat{Z}(G)$ *open in G, then there exists a finite p-group E with number of conjugacy classes* $k(E)$ *such that*

$$\widehat{\mathrm{d}}(G) = \widehat{\mathrm{d}}(E) = \sum_{i=1}^{k(E)} \frac{|\widehat{C}_E(x_i)|}{|C_E(x_i)|}. \tag{11}$$

Note that Theorem 2 was inspired by a similar property, which was shown by Abdollahi and others [15] for *the probability* $\mathrm{d}(G)$ *that two randomly picked elements* x, y *commute* in a pro-p-group G. In fact, our third main result connects $\widehat{\mathrm{d}}(G)$ with $\mathrm{d}(G)$.

Theorem 3. *If G is a pro-p-group with a trivial Schur multiplier, then there exists a finite p-group H such that* $\mathrm{d}(G) = \widehat{\mathrm{d}}(G) = \widehat{\mathrm{d}}(H)/|G : \widehat{FC}(G)|^2$.

Section 2 proves Theorem 1 and gives a formal description of complete nonabelian tensor squares and complete nonabelian exterior squares in terms of quotients of free pro-p-groups. Section 3 recalls some facts of homological algebra and previous bounds on the exterior degree, setting the ground for the proofs of the remaining main theorems which are given in Section 4. Examples appear at the end, in order to support the main results. Notations and terminology are standard and follow [1,2,14,16].

2. The First Main Theorem and Its Proof

We say that a pro-p-group G acts *compatibly and continuously on itself by conjugation*, if the action $(a, b) \in G \times G \mapsto a^b \in G$ is continuous and the compatibility relations $x^{(y^z)} = x^{z^{-1}yz}$ and $t^{(z^y)} = t^{y^{-1}zy}$ are satisfied for all $x, y, z, t \in G$.

Definition 1 (Continuous Crossed Pairings of Pro-p-Groups)**.** *Let A be a pro-p-group and G another pro-p-group acting compatibly and continuously on itself by conjugation. A map* $\varphi : G \times G \to A$ *is called a continuous crossed pairing if for all* $g, h, t, z \in G$ *we have*

$$\varphi(gz, h) = \varphi(z^g, h^g)\ \varphi(g, h) \ \text{ and } \ \varphi(g, ht) = \varphi(g, h)\ \varphi\left(g^h, t^h\right) \tag{12}$$

If G and A are profinite abelian groups, Definition 1 gives the notion of a *bilinear continuous map*, or *middle linear continuous map*, according to §5.5 of [2]. It is possible to introduce categorically the complete nonabelian tensor square via an appropriate universal property.

Definition 2 (Universal Property of Complete nonabelian Tensor Squares)**.** *Consider a pro-p-group A and pro-p-group G acting compatibly and continuously on itself by conjugation. The complete nonabelian tensor square of G is the pro-p-group* $G\widehat{\otimes}G$ *together with a continuous crossed pairing* $\widehat{\otimes} : (g, h) \in G \times G \mapsto g\widehat{\otimes}h \in G\widehat{\otimes}G$ *such that for any continuous crossed pairing* $\varphi : G \times G \to A$ *there is a unique homomorphism* $\widehat{\varphi} : G\widehat{\otimes}G \to A$ *of pro-p-groups making commutative the following diagram (i.e.,* $\widehat{\varphi} \circ \widehat{\otimes} = \varphi$*)*

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \widehat{\otimes}\ } & G\widehat{\otimes}G \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle\widehat{\varphi}} & \\ A & & \end{array} \tag{13}$$

Note the universal property of complete abelian tensor products in §5.5 of [2]. It is also useful to compare Lemma 5.5.1, Lemma 5.5.2 and Proposition 5.4 of [2] with Theorems

2.1 and 3.1 of [6], in order to understand how we generalize the results on complete abelian tensor squares of profinite groups to complete nonabelian tensor squares of profinite groups. We now begin with the proof of our first main result.

Proof of Theorem 1. First of all, we note that $Y = G \times G$ is a profinite space since we have

$$Y = G \times G = \varprojlim_{i \in J} G_i \times \varprojlim_{j \in J} G_j = \varprojlim_{i,j \in J} (G_i \times G_j). \tag{14}$$

If $\varphi : (g,h) \in Y \mapsto \varphi(g,h) \in A$ is a continuous crossed pairing of pro-p-groups, then the universal property defining $F_p(Y)$ implies that there is a continuous homomorphism $\widehat{\varphi} : F_p(Y) \to A$, which is unique in making commutative the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & F_p(Y) \\ {\scriptstyle \varphi}\downarrow & \swarrow {\scriptstyle \widehat{\varphi}} & \\ A & & \end{array} \tag{15}$$

Here, ι is the embedding of Y into $F_p(Y)$. Let K be the smallest closed normal subgroup of $F_p(Y)$ that is topologically generated by the elements

$$\iota(gz,h)\iota(g,h)^{-1}\iota(z^g,h^g)^{-1} \text{ and } \iota(g,ht)\iota(g^h,t^h)^{-1}\iota(g,h)^{-1} \tag{16}$$

for all $g,z,h,t \in G$. Since φ is a crossed pairing,

$$\widehat{\varphi}\Big(\iota(gz,h)\iota(g,h)^{-1}\iota(z^g,h^g)^{-1}\Big) = \widehat{\varphi}(\iota(gz,h))\widehat{\varphi}\Big(\iota(g,h)^{-1}\Big)\widehat{\varphi}\Big(\iota(z^g,h^g)^{-1}\Big) \tag{17}$$

$$= \varphi(gz,h)\varphi(g,h)^{-1}\varphi(z^g,h^g)^{-1} = \varphi(z^g,h^g)\varphi(g,h)\varphi(g,h)^{-1}\varphi(z^g,h^g)^{-1} = 1.$$

We also have for the same reason

$$\widehat{\varphi}\Big(\iota(g,ht)\iota(g^h,t^h)^{-1}\iota(g,h)^{-1}\Big) = \widehat{\varphi}(\iota(g,ht)\widehat{\varphi}(\iota(g^h,t^h)^{-1})\widehat{\varphi}(\iota(g,h)^{-1}) \tag{18}$$

$$= \varphi(g,ht)\varphi(g^h,t^h)^{-1}\varphi(g,h)^{-1} = \varphi(g,h)\varphi(g^h,t^h)\varphi(g^h,t^h)^{-1}\varphi(g,h)^{-1} = 1.$$

Therefore, $K \subseteq \ker \widehat{\varphi}$ and $\widehat{\varphi}$ is a continuous homomorphism of pro-p-groups vanishing on the topological generators of K. Now, $\pi : F_p(Y) \to F_p(Y)/K$ is the quotient homomorphism, hence a surjective continuous homomorphism of pro-p-groups, and we may consider the composition $\pi \circ \iota : Y \to F_p(Y)/K$. In this situation, there is a continuous homomorphism of pro-p-groups $\widehat{\varphi}_K : F_p(Y)/K \to A$, which is unique in making commutative the following diagram

$$\begin{array}{ccccc} & & \overset{\pi}{\curvearrowleft} & & \\ F_p(Y)/K & \xleftarrow{\pi\circ\iota} & Y & \xrightarrow{\iota} & F_p(Y) \\ & {\scriptstyle \widehat{\varphi}_K}\searrow & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle \widehat{\varphi}} & \\ & & A & & \end{array} \tag{19}$$

Setting $G \widehat{\otimes} G = F_p(Y)/K$ and $g \widehat{\otimes} h = \pi(\iota(g,h))$, Definition 2 is satisfied by the left portion of the diagram above. Of course, we may repeat the proof taking any pro-p-group which is topologically isomorphic to $F_p(Y)/K$ and we reach the same conclusions. The first part of Theorem 1 follows.

Concerning the remaining part of the theorem, we shall note that $G\widehat{\otimes}G$ is a pro-p-group and by a result of Alexandroff in Lemma A4.31 of [1], so it is a dyadic space, whenever it is metrizable. □

From Theorem A4.16 of [1], it can be useful to mention that first countable pro-p-groups are always metrizable. This happens for instance when pro-p-groups are topologically finitely generated. Therefore, one could also note from the proof above that if G is topologically finitely generated, then so is $G\widehat{\otimes}G$ and in this situation automatically $G\widehat{\otimes}G$ is metrizable, hence a dyadic space.

3. Some Observations on the Schur Multipliers

In the present section we report some results on the exterior degree in [2,7,9] but also some results of homological algebra in [1,2,16]. Here, $\overline{[G,G]}$ denotes the closure of the *commutator subgroup*

$$[G,G] = \langle [x,y] \mid x,y \in G\rangle = \langle x^{-1}y^{-1}xy \mid x,y \in G\rangle. \tag{20}$$

While in general $Z(G)$ is a closed subgroup of a pro-p-group G, it can easily be seen that this is not the case for $[G,G]$, and so we need to consider $\overline{[G,G]}$ if we want to preserve both its algebraic and topological structure in a pro-p-group, see [1].

Definition 3 (See [6,16])**.** *The Schur multiplier $M(G)$ of a pro-p-group G is defined to be the second homology group $H_2(G,\mathbb{Z}_p)$ with coefficients in the ring $\mathbb{Z}_p$ of the p-adic integers.*

The above notion is largely used in [7,16–18] but it is useful to recall how Definition 3 should be interpreted in case of a finite p-group. For instance, we may consider a pro-p-group on a countable set of indices, that is, $G = \varprojlim_{m\in\mathbb{N}} G_m$ with each G_m finite p-group. The situation does not change if $G = \varprojlim_{j\in J} G_j$ and J is an arbitrary set of indices, but take J being countable as a temporary assumption.

From Proposition 6.5.7 of [2], there is a continuous homomorphism of pro-p-groups such that

$$H_2(G,\mathbb{Z}_p) = H_2\left(\varprojlim_{m\in\mathbb{N}} G_m, \varprojlim_{m\in\mathbb{N}} \frac{\mathbb{Z}}{p^m\mathbb{Z}}\right) \simeq \varprojlim_{m\in\mathbb{N}} H_2\left(G_m, \frac{\mathbb{Z}}{p^m\mathbb{Z}}\right), \tag{21}$$

where $\mathbb{Z}/p^m\mathbb{Z} = \mathbb{Z}(p^m)$ denotes the cyclic group of order p^m as per Example 1.28 (i) of [1]. Let us carefully examine the construction of the above homology groups with coefficients in $\mathbb{Z}_p$. Consider a *free homogeneous Bar resolution* (with each L_n free profinite $\mathbb{Z}_p$-modules on the profinite space $\{(1,x_1,\ldots,x_n) \mid x_i \in G\}$) according to §6.2 in [2]

$$\ldots \longrightarrow L_n \xrightarrow{\partial_n} L_{n-1} \longrightarrow \ldots L_1 \longrightarrow L_0 \xrightarrow{\epsilon} \mathbb{Z}_p \longrightarrow 0, \tag{22}$$

where ∂_n is the *boundary map* defined by

$$\partial_n(x_0,x_1,\ldots,x_n) = \sum_{i=0}^{n} (-1)^i\,(x_0,x_1,\ldots x_{i-1},x_{i+i},\ldots,x_n), \tag{23}$$

and ϵ is the *augmentation map* defined by

$$\epsilon : x \in L_0 \mapsto \epsilon(x) = 1 \in \mathbb{Z}_p. \tag{24}$$

Both (23) and (24) are continuous homomorphisms of pro-p-groups. Now, G is a pro-p-group, $\mathbb{Z}_p$ is a commutative pro-p-ring and we may consider B which is a pro-p right $[[\mathbb{Z}_pG]]$-module, that is, a pro-p-module on the complete group algebra $[[\mathbb{Z}_pG]]$. See §5.1, §5.2 and §5.3 of [2] for definitions and details. In this situation, $\mathrm{Tor}_n^{[[\mathbb{Z}_pG]]}(B,\mathbb{Z}_p)$ is the n-th left derived functor of the complete abelian tensor product, $\widehat{\otimes}_{[[\mathbb{Z}_pG]]}\,\mathbb{Z}_p$, as noted in [2] (§6.3), and so we have

$$\ldots \longrightarrow B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_n \xrightarrow{\partial_n} B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_{n-1} \longrightarrow \ldots \xrightarrow{\partial_2} B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_1 \xrightarrow{\partial_1} B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_0 \xrightarrow{\epsilon} \mathbb{Z}_p \longrightarrow 0. \quad (25)$$

Since $u\widehat{\otimes}(x_0, x_1, \ldots, x_n) \in B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_n \mapsto u\widehat{\otimes}\partial_n(x_0, x_1, \ldots, x_n) \in B\widehat{\otimes}_{[[\mathbb{Z}_pG]]}L_{n-1}$, we may use the symbol ∂_n in (23) also in (25), because it is induced by (23). In particular, $B = \mathbb{Z}_p$ can be regarded as a pro-p-module on $[[\mathbb{Z}_pG]]$ and so we have

$$H_2(G, \mathbb{Z}_p) = \mathrm{Tor}_2^{[[\mathbb{Z}_pG]]}(\mathbb{Z}_p, \mathbb{Z}_p) = \frac{\ker \partial_2}{\mathrm{im}\, \partial_3}. \quad (26)$$

A careful examination of §6.8 of [2] suggests that we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{q} \frac{\mathbb{Z}}{p\mathbb{Z}} \longrightarrow 0 \quad (27)$$

where p denotes the multiplication by p in $\mathbb{Z}_p$ and q the limit map from $\mathbb{Z}_p$ to $\mathbb{Z}/p\mathbb{Z}$ arising from the structure of the projective limit of $\mathbb{Z}_p$, and so there is a long exact sequence of abelian pro-p-groups

$$\begin{aligned} \ldots &\longrightarrow H_2(G,\mathbb{Z}_p) \xrightarrow{p_2} H_2(G,\mathbb{Z}_p) \xrightarrow{q_2} H_2(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_2} \\ &\xrightarrow{\delta_2} H_1(G,\mathbb{Z}_p) \xrightarrow{p_1} H_1(G,\mathbb{Z}_p) \xrightarrow{q_1} H_1(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_1} \\ &\xrightarrow{\delta_1} H_0(G,\mathbb{Z}_p) \xrightarrow{p_0} H_0(G,\mathbb{Z}_p) \xrightarrow{q_0} H_0(G,\mathbb{Z}/p\mathbb{Z}) \end{aligned} \quad (28)$$

where p_1 and p_2 are induced by p, q_1 and q_2 by q, and δ_1 and δ_2 are connecting continuous homomorphisms. Since $H_0(G,\mathbb{Z}_p) \xrightarrow{p_0} H_0(G,\mathbb{Z}_p) = \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p$ is a monomorphism, we have

$$H_1(G,\mathbb{Z}_p) \xrightarrow{p_1} H_1(G,\mathbb{Z}_p) \longrightarrow H_1(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \quad (29)$$

and so (28) becomes

$$\begin{aligned} \ldots &\longrightarrow H_2(G,\mathbb{Z}_p) \xrightarrow{p_2} H_2(G,\mathbb{Z}_p) \xrightarrow{q_2} H_2(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_2} \\ &\xrightarrow{\delta_2} H_1(G,\mathbb{Z}_p) \xrightarrow{p_1} H_1(G,\mathbb{Z}_p) \xrightarrow{q_1} H_1(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \end{aligned} \quad (30)$$

On the other hand, Lemma 6.8.6 of [2] allows us to conclude that $H_1(G,\mathbb{Z}/p\mathbb{Z}) \simeq G/G^p\overline{[G,G]}$ and $H_1(G,\mathbb{Z}_p) \simeq G/\overline{[G,G]}$; hence, (23) becomes (up to isomorphisms of abelian pro-p-groups) the following long exact sequence of abelian pro-p-groups

$$\begin{aligned} \ldots &\longrightarrow H_2(G,\mathbb{Z}_p) \xrightarrow{p_2} H_2(G,\mathbb{Z}_p) \xrightarrow{q_2} H_2(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_2} \\ &\xrightarrow{\delta_2} G/\overline{[G,G]} \xrightarrow{p_1} G/\overline{[G,G]} \xrightarrow{q_1} G/G^p\overline{[G,G]} \longrightarrow 0 \end{aligned} \quad (31)$$

We make two observations on the basis of the homological algebra, which was used.

Remark 1. Assume we start with $G = G_1$ finite p-group, that is, $G = G_1 = G_2 = G_3 = \ldots$ in the projective limit describing G. Then, there is a presentation $1 \to R \to F \to G \to 1$ for $G = F/R$ with F a free abstract group and R a normal subgroup of F. Applying Theorem 9.5.10 of [16], we obtain the isomorphism of finite abelian p-groups

$$H_2(G,\mathbb{Z}/p\mathbb{Z}) \simeq \frac{R \cap (F'F^p)}{[R,F]F^p}, \quad (32)$$

where $F^p = \langle a^p \mid a \in F \rangle$ denotes the subgroup of p-powers of F, $[F,F] = F'$ the commutator subgroup of F and $[R,F] = \langle [a,b] \mid a \in R, b \in F \rangle \subseteq F'$. In particular, the long exact sequence (31) becomes in this situation

$$\ldots \xrightarrow{q_2} R \cap (F'F^p)/[R,F]F^p \xrightarrow{\delta_2} G/[G,G] \xrightarrow{p_1} \tag{33}$$
$$G/[G,G] \xrightarrow{q_1} G/G^p[G,G] \longrightarrow 0$$

so we may concretely visualize Definition 3 in the case of finite p-groups.

Note that (32) modifies the *Hopf's Formula* for the Schur multiplier, which is available in Theorem 9.5.6 of [16] and usually formulated as

$$H_2(G,\mathbb{Z}) \simeq \frac{R \cap F'}{[R,F]}, \tag{34}$$

when $G = F/R$ is an arbitrary abstract group with presentation $1 \to R \to F \to G \to 1$, so not necessarily a finite p-group. Now, we make our second observation as a further description of Definition 3.

Remark 2. Comparing (34) with (32), we note that the terms F^p and $p\mathbb{Z}$ are significant in the case of finite p-groups and this justifies the construction of $H_2(G,\mathbb{Z}_p)$, which is designed for infinite pro-p-groups as large projective limits of finite p-groups. The temporary assumption of working with J countable facilitates the understanding of the functorial behavior in (21), where $H_2(G,\mathbb{Z}_p)$ preserves the structure of projective limit. This relevant observation and general versions of Hopf's Formula such as (32) allow us to consider $H_2(G,\mathbb{Z}_p)$ as a projective limit of smaller homology groups $H_2(G_m,\mathbb{Z}/p^m\mathbb{Z})$ when m tends to infinity. The reader can find in [19] details of a categorical nature on Hopf Formulas.

Now we remove the temporary assumption to have a countable J and consider a pro-p-group G which is a projective limit of G_j with an arbitrary set J of indices. Note that Theorem 1 involves the nonabelian tensor square $G\widehat{\otimes}G$ of an arbitrary pro-p-group G and one has the following maps

$$\widehat{\kappa} : x\widehat{\otimes}y \in G\widehat{\otimes}G \mapsto [x,y] \in \overline{[G,G]} \text{ and } \widehat{\kappa'} : x\widehat{\wedge}y \in G\widehat{\wedge}G \mapsto [x,y] \in \overline{[G,G]}, \tag{35}$$

which are continuous surjective homomorphisms of pro-p-groups such that

$$\widehat{J}_2(G) = \ker \widehat{\kappa} \supseteq \widehat{\nabla}(G) \text{ and } \ker \widehat{\kappa'} \simeq M(G). \tag{36}$$

We give a proof of the following result for convenience of the reader.

Lemma 1. *In a pro-p-group G, the maps $\widehat{\kappa}$ and $\widehat{\kappa'}$ in (35) are continuous surjective homomorphisms of pro-p-groups and the following diagram has rows which are central extensions of pro-p-groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{J}_2(G) & \longrightarrow & G\widehat{\otimes}G & \xrightarrow{\widehat{\kappa}} & \overline{[G,G]} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \widehat{\varepsilon}_|} & & \downarrow{\scriptstyle \widehat{\varepsilon}} & & \| & & \\ 1 & \longrightarrow & M(G) & \longrightarrow & G\widehat{\wedge}G & \xrightarrow{\widehat{\kappa'}} & \overline{[G,G]} & \longrightarrow & 1. \end{array} \tag{37}$$

Moreover, we have $\ker \widehat{\kappa} \supseteq \widehat{\nabla}(G)$ *and there is a continuous isomorphism of pro-p-groups such that* $\ker \widehat{\kappa'} \simeq M(G)$.

Proof. From Definition 2, there are continuous crossed pairings $\widehat{\otimes} : (x,y) \in G \times G \mapsto x\widehat{\otimes}y \in G\widehat{\otimes}G$ and $\kappa : (x,y) \in G \times G \to [x,y] \in \overline{[G,G]}$ of pro-p-groups, inducing the map $\widehat{\kappa} : x\widehat{\otimes}y \in G\widehat{\otimes}G \mapsto [x,y] \in \overline{[G,G]}$, which is a continuous crossed pairing such that $\widehat{\kappa} \circ \widehat{\otimes} = \kappa$ and the upper part of the following diagram is commutative

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \widehat{\otimes}\ } & G\widehat{\otimes}G \\ \kappa \downarrow & \overset{\widehat{\kappa}}{\swarrow} & \downarrow \widehat{\varepsilon} \\ \overline{[G,G]} & \xleftarrow{\ \widehat{\kappa'}\ } & G\widehat{\wedge}G \end{array} \tag{38}$$

This shows that $\widehat{\kappa}$ is a continuous homomorphism of pro-p-groups. Note that $\widehat{\kappa}$ is surjective, because $\widehat{\kappa} \circ \widehat{\otimes} = \kappa$ and κ is surjective by construction. Concerning $\widehat{\kappa'}$, we look at the lower part of the same diagram, where $\widehat{\varepsilon} : G\widehat{\otimes}G \to G\widehat{\wedge}G$ is the natural projection of $G\widehat{\otimes}G$ onto $G\widehat{\wedge}G$ with $\ker \widehat{\varepsilon} = \widehat{\nabla}(G)$. Then, $\widehat{\kappa'}$ is induced by $\widehat{\kappa}$ modulo $\widehat{\nabla}(G)$ and is a continuous homomorphism of pro-p-groups such that $\widehat{\kappa'} \circ \widehat{\varepsilon} = \widehat{\kappa}$. Of course, $\ker \widehat{\varepsilon} \subseteq \ker \widehat{\kappa}$ and, if $x\widehat{\otimes}x \in \ker \widehat{\varepsilon} = \widehat{\nabla}(G)$, then for all $y\widehat{\otimes}z \in G\widehat{\otimes}G$ we have

$$(x\widehat{\otimes}x)\,(y\widehat{\otimes}z)\,(x\widehat{\otimes}x)^{-1} = (y\widehat{\otimes}z)^{[x,x]} \implies (x\widehat{\otimes}x)\,(y\widehat{\otimes}z) = (y\widehat{\otimes}z)\,(x\widehat{\otimes}x) \tag{39}$$

showing that $\ker \widehat{\varepsilon} \subseteq Z(G\widehat{\otimes}G)$. Note that more generally the same argument applies to any element $a\widehat{\otimes}b \in \ker \widehat{\kappa}$, in fact

$$(a\widehat{\otimes}b)\,(y\widehat{\otimes}z)\,(a\widehat{\otimes}b)^{-1} = (y\widehat{\otimes}z)^{[a,b]} \implies (a\widehat{\otimes}b)\,(y\widehat{\otimes}z) = (y\widehat{\otimes}z)\,(a\widehat{\otimes}b) \tag{40}$$

hence, $\ker \widehat{\kappa} \subseteq Z(G\widehat{\otimes}G)$. We may conclude that both the sequence

$$1 \longrightarrow \widehat{J}_2(G) \longrightarrow G\widehat{\otimes}G \xrightarrow{\ \widehat{\kappa}\ } \overline{[G,G]} \longrightarrow 1 \tag{41}$$

and the sequence

$$1 \longrightarrow \ker \widehat{\kappa'} \longrightarrow G\widehat{\wedge}G \xrightarrow{\ \widehat{\kappa'}\ } \overline{[G,G]} \longrightarrow 1. \tag{42}$$

are short exact sequences, which describe central extensions of pro-p-groups. Considering the restriction $\widehat{\varepsilon}_|$ of $\widehat{\varepsilon}$ to $\widehat{J}_2(G)$ we may conclude that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{J}_2(G) & \longrightarrow & G\widehat{\otimes}G & \xrightarrow{\ \widehat{\kappa}\ } & \overline{[G,G]} & \longrightarrow & 1 \\ & & \widehat{\varepsilon}_| \downarrow & & \widehat{\varepsilon} \downarrow & & \| & & \\ 1 & \longrightarrow & \ker \widehat{\kappa'} & \longrightarrow & G\widehat{\wedge}G & \xrightarrow{\ \widehat{\kappa'}\ } & \overline{[G,G]} & \longrightarrow & 1. \end{array} \tag{43}$$

is a commutative diagram whose rows are central extensions of pro-p-groups. It remains to show that $\ker \widehat{\kappa'}$ is isomorphic as a pro-p-group to $M(G)$. This is proved in [6] (Proposition 2.2), so we omit the details here. The result follows. □

Even if in principle $\widehat{\nabla}(G)$ may be a proper subgroup of $\widehat{J}_2(G)$, the computations show that most of the time $\widehat{J}_2(G) = \widehat{\nabla}(G)$ in case of finite groups (see [3,4]) and so (37) becomes most of the time

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\nabla}(G) & \xrightarrow{\ \alpha\ } & G\widehat{\otimes}G & \xrightarrow{\ \widehat{\kappa}\ } & \overline{[G,G]} & \longrightarrow & 1 \\ & & \widehat{\varepsilon}_| \downarrow & & \widehat{\varepsilon} \downarrow & & \| & & \\ 1 & \longrightarrow & M(G) & \xrightarrow{\ \beta\ } & G\widehat{\wedge}G & \xrightarrow{\ \widehat{\kappa'}\ } & \overline{[G,G]} & \longrightarrow & 1 \end{array} \tag{44}$$

where α embeds $\widehat{\nabla}(G)$ into $G\widehat{\otimes}G$ and β is induced by α and by $\widehat{\varepsilon}_| : x\widehat{\otimes}x \in \widehat{\nabla}(G) \mapsto x\widehat{\wedge}x \in M(G)$. Of course, if $\widehat{J}_2(G) \neq \widehat{\nabla}(G)$, then $\operatorname{Im} \alpha \subseteq \ker \widehat{\kappa}$ and $\operatorname{Im} \beta \subseteq \ker \widehat{\kappa'}$, so we have inclusions and lose the exactness of the sequences, that is, the diagram (44) is no longer formed by central extensions as rows but it is still (44) a commutative diagram.

Remark 3. In a finite p-group G, the quotient $C_G(x)/\widehat{C}_G(x)$ is isomorphic to a subgroup of the abelian group $H_2(G,\mathbb{Z})$ by the results in [7,10]. In particular, groups with $H_2(G,\mathbb{Z}) = 1$ have $C_G(x) = \widehat{C}_G(x)$ for all $x \in G$ and $Z(G) = \widehat{Z}(G)$ as well.

We recall that the *commutativity degree* of a pro-p-group G is defined by the formula

$$\mathrm{d}(G) := \int_G \mu(C_G(x))d\mu, \tag{45}$$

where μ is the Haar measure on G. This notion has been studied extensively in [20–23] both in the finite case and in the infinite case. Of course, (45) represents for finite groups the probability that a randomly picked pair (x,y) of elements of $G \times G$ commutes, that is, satisfies the condition $[x,y] = 1$. Following the same notion at the level of elements for the operator of exterior degree, instead of commutator, we find the probability that the same randomly picked pair (x,y) of elements of $G \times G$ satisfies $x \widehat{\wedge} y = 1$ (instead of $[x,y] = 1$). The two notions are related by the following result:

Lemma 2 (See [7], Theorem 1.1)**.** *A pro-p-group G satisfies the following inequality*

$$\widehat{\mathrm{d}}(G) \leq \mathrm{d}(G) - \left(\frac{p-1}{p}\right)\Big(\mu(Z(G)) - \mu(\widehat{Z}(G))\Big).$$

Furthermore, if $M(G)$ is finite, then

$$\widehat{\mathrm{d}}(G) \geq \mu(\widehat{Z}(G)) + \frac{1}{|M(G)|}\Big(\mathrm{d}(G) - \mu(\widehat{Z}(G))\Big).$$

It is also useful to mention that the abelian pro-p-group $Z(G)/\widehat{Z}(G)$ can be embedded in the abelian pro-p-group $M(G)$, involving some numerical invariants such as *the rank* of a pro-p-group G

$$\mathrm{rk}(G) = \sup\{m(H) \mid H = \overline{H} \text{ subgroup of } G\}, \tag{46}$$

where $m(H)$ is the minimal number of elements which topologically generate H. If G is a torsion-free pro-p-group, $\mathrm{rk}(G) = \mathrm{tf}(G)$ is called the *torsion-free rank*, see [17].

Lemma 3 (See [7], Theorem 1.2)**.** *Let G be a pro-p-group such that* $\mathrm{rk}(G/\widehat{Z}(G)) = a$, $\mathrm{rk}(M(G)) = b$ *and* $\mathrm{tf}(M(G)) = c$.

(i). *If $M(G)$ is finite, then $|Z(G)/\widehat{Z}(G)|$ divides $|M(G)|^a$.*
(ii). *If $M(G)$ is infinite, then $\mathrm{rk}(Z(G)/\widehat{Z}(G)) \leq b^a$. In particular, if $M(G)$ is torsion-free, then $\mathrm{tf}(Z(G)/\widehat{Z}(G)) \leq c^a$.*

Now, we recall a characterization for the extremal cases of exterior degree equal to zero, or equal to one, via the notions of the complete exterior center and complete exterior centralizer.

Lemma 4 (See [7], Proposition 3.3)**.** *A pro-p-group G has $\widehat{\mathrm{d}}(G) = 1$ if and only if $\widehat{Z}(G) = G$.*

We report a result similar to that of Abdollahi and others [15] for compact groups. This was at the origin of our investigations.

Lemma 5 (See [15], Theorem 1.1)**.** *In a pro-p-group G of $\mathrm{d}(G) > 0$ there exists a finite p-group H such that $\mathrm{d}(G)$ is proportional to $\mathrm{d}(H)$ via a constant $\alpha = |G : FC(G)|^{-2}$ depending only on the FC-center $FC(G) = \{g \in \mid |G : C_G(g)| \text{ is finite}\}$ of G. In particular, $\mathrm{d}(G) = \alpha\, \mathrm{d}(H)$.*

We note explicitly that the formulation above is designed for our present context. It is also useful to collect some bounds, which can be obtained in terms of subgroups and quotients.

Lemma 6 (See [7], Proposition 3.6 and Corollary 5.3)**.** *Assume that G is a pro-p-group.*

(i). *If N is a closed normal subgroup of G, then* $\widehat{\mathrm{d}}(G) \le \widehat{\mathrm{d}}(G/N)$ *and the equality holds if* $N \le \widehat{Z}(G)$*;*

(ii). *If G is abelian nonprocyclic, then*

$$\widehat{\mathrm{d}}(G) \le \frac{p^2+p-1}{p^3}$$

and the equality holds if and only if $G/\widehat{Z}(G)$ *is p-elementary abelian of rank two;*

(iii). *If G is nonabelian and* $\widehat{Z}(G)$ *is a proper subgroup of* $Z(G)$*, then*

$$\widehat{\mathrm{d}}(G) \le \frac{p^3+p-1}{p^4}.$$

While the upper bounds on $\widehat{\mathrm{d}}(G)$ are useful to measure how far we are from the extremal case $\widehat{\mathrm{d}}(G) = 1$ in $[0,1]$, the lower bounds on $\widehat{\mathrm{d}}(G)$ may reveal the presence of quotients, which are small enough.

Remark 4. From Lemma 2, a pro-p-group G always has $\widehat{\mathrm{d}}(G) \le \mathrm{d}(G)$, and, if $M(G)$ is finite, $\mu(\widehat{Z}(G))$ is finite and $\mu(\widehat{Z}(G)) \ne \mathrm{d}(G)$, then $\mathrm{d}(G)$ is nontrivially bounded from below. Note that nontrivial lower bounds for $\mathrm{d}(G)$ imply that G is virtually abelian by [22].

4. Proofs of the Main Theorems

With the results of the previous section at hand, we show Theorem 2.

Proof of Theorem 2. (i). The normalized Haar measure μ on the pro-p-group G is a left invariant Borel probability measure which respects the closed subgroups of G (see [22] for terminology); hence, for any closed subgroup M of G and $k \ge 1$, we have

$$\mu(M) = \begin{cases} \frac{1}{p^k}, & \text{if} \quad |G:M| = p^k \\ 0, & \text{if} \quad |G:M| = \infty. \end{cases} \tag{47}$$

Consider

$$\widehat{\mathrm{d}}(G) = \int_G \mu(\widehat{C}_G(x))d\mu(x). \tag{48}$$

We have from (47) that $\mu(\widehat{C}_G(x)) > 0$ iff $\widehat{C}_G(x)$ has p-power index in G, that is, $\mu(\widehat{C}_G(x)) = 0$ iff $\widehat{C}_G(x)$ has infinite index in G iff $x \notin \widehat{FC}(G)$. Since μ is a nonnegative normalized Haar measure on G, we have

$$0 = \widehat{\mathrm{d}}(G) = \int_G \mu(\widehat{C}_G(x))d\mu(x) \iff \mu(\widehat{C}_G(x)) = 0, \ \forall x \in G \tag{49}$$

iff $x \notin \widehat{FC}(G)$ for all $x \in G$ iff there are no elements in the interior $\widehat{FC}(G)$, i.e., $\widehat{FC}(G)^\circ = \varnothing$ but we know that (any set so in particular) $\widehat{FC}(G)$ is open iff $\widehat{FC}(G)^\circ = \widehat{FC}(G)$. This cannot happen since $1 \in \widehat{FC}(G)$ and $\widehat{FC}(G) \ne \varnothing$. Therefore, $\widehat{\mathrm{d}}(G) = 0$ happens iff $\widehat{FC}(G)^\circ = \varnothing$ iff $\widehat{FC}(G)$ is not open.

(ii). Assume that G is procyclic. If $G \simeq \mathbb{Z}(p^n)$ or $G \simeq \mathbb{Z}_p$, then $M(G)$ is trivial. Hence, $\widehat{Z}(G) = Z(G)$ by Lemma 3 and so $\widehat{Z}(G) = Z(G) = G$ is abelian. The bounds of Lemma 2 imply $\widehat{\mathrm{d}}(G) = 1$. Conversely, assume that G is a pro-p-group with $\widehat{\mathrm{d}}(G) = 1$.

From Lemma 4, $\widehat{\mathrm{d}}(G) = 1$ if and only if $\widehat{Z}(G) = G$. Hence, G is abelian. Therefore, we are assuming that G is an abelian pro-p-group of $\widehat{\mathrm{d}}(G) = 1$. Either G is procyclic or G is nonprocyclic. In the first case, the result follows. In the second case, Lemma 6 (ii) implies $p^3 \leq p^2 + p - 1$, which is a contradiction. Then, G must be necessarily procyclic.

(iii). Of course, $\widehat{\mathrm{d}}(G) \in [0,1]$. From (ii) above $\widehat{\mathrm{d}}(G) \neq 1$ iff G is nonprocyclic. On the other hand, (i) above shows that $\widehat{\mathrm{d}}(G) = 0$ iff $\widehat{FC}(G)^\circ = \emptyset$. Since $\widehat{Z}(G) \subseteq \widehat{FC}(G)$ and $\widehat{Z}(G)$ is open in G, we have $\widehat{Z}(G)^\circ \subseteq \widehat{FC}(G)^\circ$ hence $\widehat{FC}(G)^\circ \neq \emptyset$. This implies that $\widehat{\mathrm{d}}(G) > 0$. Therefore, a nonprocyclic pro-p-group G with $\widehat{Z}(G)$ open in G automatically has $\widehat{\mathrm{d}}(G) \in (0,1)$ and we may proceed with the proof of the formula for the computation of the complete exterior degree. Consider Lemma 6 (i) and that $\widehat{Z}(G)$ is also closed in G by Proposition A4.25 (ii) of [1] (in fact any open subgroup is closed). It follows that $G/\widehat{Z}(G) \simeq E$ is a finite p-group but also that

$$\widehat{\mathrm{d}}(G) = \widehat{\mathrm{d}}\left(\frac{G}{\widehat{Z}(G)}\right) = \widehat{\mathrm{d}}(E) = \sum_{i=1}^{k(E)} \frac{|\widehat{C}_E(x_i)|}{|C_E(x_i)|}, \tag{50}$$

where the last equality is due to [10] (Lemma 2.2). □

Now, we proceed to prove another main result.

Proof of Theorem 3. From Lemmas 2 and 3 we have $\mu(\widehat{Z}(G)) = \mu(Z(G))$ and $\mathrm{d}(G) = \widehat{\mathrm{d}}(G)$. Moreover, $\widehat{C}_G(x) = C_G(x)$ for all $x \in G$ in this situation; hence, $\widehat{FC}(G) = FC(G)$. From Lemma 5, we have a finite p-group H such that $\mathrm{d}(G) = \mathrm{d}(H)/|G : FC(G)|^2$, that is, $\widehat{\mathrm{d}}(G) = \widehat{\mathrm{d}}(H)/|G : \widehat{FC}(G)|^2$. □

As evidence of Theorem 2, we present the following construction.

Example 1. The present example appears in [9], so we report the main information only and a few new computations. Consider the elementary abelian p-group

$$A = \mathbb{Z}(p)^{(\mathbb{N})} \tag{51}$$

of countable rank, where $A_i = \langle a_i \rangle = \mathbb{Z}(p)$ is cyclic of order p and $i \in \mathbb{N}$. Then, consider

$$B = \mathbb{Z}(p)^n = A_1 \times \ldots \times A_n \tag{52}$$

elementary abelian p-subgroup of rank n of A. We have that

$$1 = \mathrm{d}(B) > \widehat{\mathrm{d}}(B) = \frac{p^n + p^{n-1} - 1}{p^{2n-1}} \quad \text{and} \quad 1 = \mathrm{d}(A) \geq \frac{p^2 + p - 1}{p^3} > \widehat{\mathrm{d}}(A). \tag{53}$$

Note that the complete exterior degree of abelian pro-p-groups is also described by Lemma 6 (ii). In fact, Theorem 2 shows that computations such as those in Lemma 6 are in general tedious, so that formulas of reduction are very useful. In addition to Example 1, we mention below a pro-p-group, whose structure is described in [17].

Example 2. Consider the infinite pro-2-group (with $r \geq 1$ arbitrary)

$$G = \overline{\langle a, t \mid a^{2^r} = 1, a^{-1}ta = t^{-1} \rangle} = \mathbb{Z}_2 \rtimes \mathbb{Z}(2^r), \tag{54}$$

which appears also in §1 of [18]. We have $M(G) = 1$ and so $Z(G) = \widehat{Z}(G) = 1$, but also $\widehat{\mathrm{d}}(G) = \mathrm{d}(G)$ and $\widehat{C}_G(x) = C_G(x)$ for all $x \in G$.

The following computations were carried out in Example 5.2 of [7] and are presented here for the convenience of the reader. First of all, we note that for $i = 0$ we have $\mu(\widehat{C}_G(t^i)) = 1$ but, for all $i \neq 0$, $\mu(\widehat{C}_G(t^i)) = 1/2^r$ and for all i and $1 \leq j \leq 2^r - 1$ instead $\mu(\widehat{C}_G(a^j t^i)) = 0$.

If $T = \overline{\langle t \rangle} = \mathbb{Z}_2$, then

$$\widehat{\mathrm{d}}(G) = \mu(\widehat{Z}(G)) + \int_{T-\widehat{Z}(G)} \mu(\widehat{C}_G(x))d\mu(x) + \int_{G-T} \mu(\widehat{C}_G(x))d\mu(x) = \frac{1}{2^r}\mu(T-\{1\}) = \frac{1}{2^r}\mu(T) = \frac{1}{4^r}. \tag{55}$$

Theorem 2 cannot be used here and $FC(G) = \widehat{FC}(G) = T$ is closed and open in G.

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