



Constructing the bulk at the critical point of three-dimensional large N vector theories

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ABSTRACT

In the context of the AdS_4/CFT_3 correspondence between higher spin fields and vector theories, we use the constructive bilocal fields based approach to this correspondence, to demonstrate, at the IR critical point of the interacting vector theory and directly in the bulk, the removal of the $\Delta = 1$ ($s = 0$) state from the higher spins field spectrum, and to exhibit simple Klein-Gordon higher spin Hamiltonians. The bulk variables and higher spin fields are obtained in a simple manner from boundary bilocals, by the change of variables previously derived for the UV critical point (in momentum space), together with a field redefinition.

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1. Introduction

The AdS/CFT correspondence [1–3] has a very interesting application in the context of the higher spin theories/vector model correspondence [4]. Of particular interest to us is the AdS_4/CFT_3 correspondence.¹

Although the higher spin degrees of freedom of Fronsdal and Vasiliev are not those of string theory,² there are several reasons why this correspondence is of importance and deserves further study. These include the absence of supersymmetry and the fact that vector models are “solvable” in the large N limit, allowing for a more concrete and detailed study of the workings of the correspondence, and possibly even providing a definition of (gauge fixed) higher spin theories themselves, through their dual vector valued field theories.

We focus in this communication on the constructive approach of [18–22]. In this approach, the singlet sector of $O(N)$ invariant field theories is described in terms of equal time bilocals, appropriate to an Hamiltonian description of the theory,

$$\psi_{\vec{x}_1 \vec{x}_2} = \sum_{a=1}^N \phi^a(t, \vec{x}_1) \phi^a(t, \vec{x}_2), \quad (1)$$

where \vec{x}_1 and \vec{x}_2 are two dimensional space vectors. For the free theory (the UV fixed point), these 5 degrees of freedom and their canonical conjugates are mapped to $AdS_4 \times S_1$, where the S_1 encodes the spin degrees of freedom. The map is a phase transformation, but is a point transformation in momentum space. In a temporal gauge [22], it is given by:

$$E = E_1 + E_2 = |\vec{p}_1| + |\vec{p}_2| \quad (2)$$

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \quad (3)$$

$$p^z = 2\sqrt{|\vec{p}_1| |\vec{p}_2|} \sin\left(\frac{\varphi_2 - \varphi_1}{2}\right) \quad (4)$$

$$\theta = \arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{(|\vec{p}_1| - |\vec{p}_2|) p^z}\right) \quad (5)$$

with $\varphi_2 - \varphi_1$ being the angle between \vec{p}_1 and \vec{p}_2 , and $\vec{p}_2 \times \vec{p}_1 \equiv p_2^1 p_1^2 - p_2^2 p_1^1$ [22].

The three dimensional $O(N)$ vector theory with a $\frac{\lambda}{N}(\phi^a \phi^a)^2$ interaction has an IR fixed critical point. At this critical point, the theory is expected to contain a state with dimension $\Delta = 2$, a boundary field in the standard AdS/CFT correspondence with the standard positive branch for the expression of the dimension of the operator [4], and no longer the $\Delta = 1$ state present in the UV

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¹ There is a vast literature on the subject; [5–13] are representative of the work on the subject, but they do not form by any means an exhaustive list.

² For attempts to link the two, see for instance [14–17].

critical point. Although general arguments exist relating the two through a Legendre transformation [23], in practice the IR fixed point is described in terms of a non-linear sigma model [24,25]. In this description, the Lagrange multiplier field is naturally identified with the $\Delta = 2$ state, but it is certainly not apparent that the $\Delta = 1$ is no longer present in the theory, or equivalently, that the constraint is enforced beyond the leading large N order.

These issues were discussed and successfully resolved in [26] directly in terms of the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory, using bilocal fields on the field theory boundary. The two point function for bilocals appropriate to the path integral description of the boundary field theory (a Bethe-Sapeter equation [27] in terms of the original field theory variables) was obtained, and shown to take a universal form at the IR critical point.³ It consists of the two free propagators present in the UV limit plus a connected piece with a pole identified with the $\Delta = 2$ state. This bilocal propagator was then shown to be equivalent to the spectrum equation arising in the Hamiltonian bilocal approach as a result of integration of an intermediate energy variable. In both cases, the absence of a boundary $\Delta = 1$ state was demonstrated.

In this communication, we address the question of whether the map (2)–(5) and the construction of bulk fields, established for the UV critical point, is still applicable at the IR fixed point, or if it needs adjusting. It will be shown that the map remains valid, and that by introducing a suitable field redefinition in the definition of bulk higher spin fields, the connected piece of the propagator / spectrum equation precisely removes the $s = 0$ state from the bulk higher spin field.

This letter is organised as follows. Overall, Section 2 discusses the bilocal description of the boundary. Subsection 2.1 briefly describes the conformal IR fixed point of the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory at leading order in N . In Subsection 2.2, the bilocal spectrum equation of the $1/N$ quadratic Hamiltonian fluctuations is obtained, and a potential scattering problem ensues. In Subsection 2.3, the most general solution to the spectrum equation is obtained, and is shown to take a universal form at IR criticality. Section 3 describes the construction of the bulk. Using a change of variables from bilocal momenta to bulk momenta (as dictated by the map (2)–(5)), and instituting a field redefinition to define bulk higher spin fields, a bulk Hamiltonian is obtained in Subsection 3.1 for the free case that is simply the sum of Hamiltonians of massless spin s fields in an equal time slice of AdS_4 . In Subsection 3.2, again using the same field redefinition and the map (2)–(5), we are able to obtain the bulk description of the universal boundary eigenstates at the IR critical point. It is then shown directly in the bulk that the $s = 0$ (or $\Delta = 1$ state) is exactly removed from the spectrum. It is remarkable that this direct construction of the bulk is obtained by a simple change of variables (2)–(5) accompanied by a field redefinition in defining bulk higher spin fields from boundary bilocals. Subsection 3.3 exhibits how the bulk $\Delta = 2$ state becomes a boundary state at IR criticality. In addition, we show explicitly that the bulk $AdS_4 \times S_1$ Hamiltonian projects to the boundary Hamiltonian with the correct dispersion relation for a single mode bound state. This was the expected result and serves as a further check of our bulk higher spin field redefinitions. Section 4 is left for a brief discussion and outlook.

2. Bilocal boundary

2.1. Bilocal Hamiltonian and large- N conformal background

Our starting point is the Hamiltonian density of a three (space-time) dimensional scalar vector theory with a quartic interaction

$$\frac{\lambda}{N}(\phi^a\phi^a)^2, \quad a = 1, \dots, N:$$

$$\mathcal{H} = \frac{1}{2}\pi^a\pi^a + \frac{1}{2}\vec{\nabla}\phi^a \cdot \vec{\nabla}\phi^a + \frac{1}{2}m^2\phi^a\phi^a + \frac{\lambda}{4!N}(\phi^a\phi^a)^2,$$

$$\pi^a(\vec{x}) = -i\frac{\partial}{\partial\phi^a(\vec{x})}.$$

We use the collective field theory method [31] to re-express the above Hamiltonian in terms of $O(N)$ invariant equal time bilocals

$$\psi_{\vec{x}_1\vec{x}_2}(t) = \sum_{a=1}^N \phi^a(t, \vec{x}_1)\phi^a(t, \vec{x}_2), \quad (6)$$

and their canonical conjugates, as appropriate to an Hamiltonian approach. This is achieved by a simple change of variables from the original fields of the scalar theory to the invariant bilocals, and by a similarity transformation:

$$\partial_\alpha \rightarrow \partial_\alpha - \frac{1}{2}\partial_\alpha \ln J, \quad \alpha \equiv \psi_{\vec{x}_1\vec{x}_2}.$$

J is the Jacobian induced by the change of variables, and the above transformation ensures that the collective Hamiltonian is explicitly hermitian. For vector models the large N form of the Jacobian is known (see for instance [32], [27]) and its leading large N form is given by:

$$\ln J = \frac{N}{2}\text{Tr} \ln \psi.$$

The trace is in (spatial) functional space. One obtains the form of the collective field theory Hamiltonian sufficient to generate the large N background and spectrum:

$$\begin{aligned} H &= \frac{2}{N}\text{Tr}\Pi\psi\Pi + \frac{N}{8}\text{Tr}(\psi^{-1}) \\ &\quad + N \int d^{d-1}\vec{x} \left(-\frac{1}{2} \lim_{\vec{x} \rightarrow \vec{y}} \partial_{\vec{y}}^2 \psi_{\vec{x}\vec{y}} + \frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{\lambda}{4!}\psi_{\vec{x}\vec{x}}^2 \right) \\ &\equiv \frac{2}{N}\text{Tr}\Pi\psi\Pi + NV_{eff}, \quad \Pi_{\vec{x}\vec{y}} = -i\frac{\partial}{\partial\psi_{\vec{x}\vec{y}}}. \end{aligned} \quad (7)$$

The fields have been rescaled $\psi \rightarrow N\psi$ to make explicit the N dependence. In the large N limit the kinetic term is subleading, and with the large N translationally invariant ansatz:

$$\psi_{\vec{x}\vec{y}}^0 = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \psi_k^0, \quad (8)$$

the standard gap equation

$$s = \frac{1}{2} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{1}{\sqrt{\vec{k}^2 + m^2 + \frac{\lambda}{6}s}}, \quad s = \int \frac{d^2\vec{k}}{(2\pi)^2} \psi_k^0, \quad (9)$$

is obtained. Defining $\alpha \equiv m^2 + \frac{\lambda}{6}s$, one has⁴

$$\begin{aligned} \frac{6}{\lambda}(\alpha - m^2) &= \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{1}{2\sqrt{\vec{k}^2 + \alpha}} = \int \frac{d^3k}{(2\pi)^3} \frac{i}{k^2 - \alpha} \\ &= \int \frac{d^3k_E}{(2\pi)^3} \frac{1}{k_E^2 + \alpha}. \end{aligned}$$

Our regularization is defined as:

³ Path integral bilocal holography was previously discussed in [28], and more recently with the use of conformal group techniques in [29], [30].

⁴ Our notations are as follows: $k = (E, \vec{k})$ with Minkowski signature $(+, -, -)$ and k_E is the euclidean momentum 3-vector.

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + \alpha} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \alpha^{\frac{d-2}{2}} \rightarrow -\frac{1}{4\pi} \sqrt{\alpha},$$

for $d = 3$. (10)

Thus one obtains the equation $\alpha + \frac{\lambda}{24\pi} \sqrt{\alpha} - m^2 = 0$. The IR fixed point is associated with the root:

$$\sqrt{\alpha} = \frac{24\pi m^2}{\lambda} + O\left(\frac{m^4}{\lambda^3}\right) \quad (11)$$

and is approached by keeping m^2 finite and taking $|\lambda| \rightarrow \infty$. At the critical point then, the background propagator takes the conformal form:

$$\psi_k^0 = \frac{1}{2|k|}, \quad (12)$$

and is the $O(N)$ invariant two point function of the underlying scalar fields.

2.2. Quadratic Hamiltonian and spectrum equation

$1/N$ corrections yield the spectrum, which is obtained from small fluctuations about the large- N conformal background. One shifts,

$$\psi_{\vec{x}_1 \vec{x}_2} = \psi_{\vec{x}_1 \vec{x}_2}^0 + \frac{1}{\sqrt{N}} \eta_{\vec{x}_1 \vec{x}_2}; \quad \Pi_{\vec{x}_1 \vec{x}_2} = \sqrt{N} \pi_{\vec{x}_1 \vec{x}_2},$$

from which the quadratic Hamiltonian follows:

$$H^{(2)} = 2\text{Tr}(\pi \psi^0 \pi) + \frac{1}{8} \text{Tr}((\psi^0)^{-1} \eta (\psi^0)^{-1} \eta (\psi^0)^{-1}) + \frac{\lambda}{4!} \int d^2 \vec{x} \eta_{\vec{x}\vec{x}}^2. \quad (13)$$

The equations of motion for η are then:

$$\ddot{\eta}_{\vec{x}_1 \vec{x}_2} = -\frac{1}{4} [(\psi^0)^{-1} \eta (\psi^0)^{-1} + \eta (\psi^0)^{-2} + (\psi^0)^{-2} \eta + (\psi^0)^{-1} \eta (\psi^0)^{-1}]_{\vec{x}_1 \vec{x}_2} - \frac{\lambda}{6} (\psi_{\vec{x}_1 \vec{x}_2}^0 (\eta_{\vec{x}_1 \vec{x}_1} + \eta_{\vec{x}_2 \vec{x}_2})).$$

Looking for eigen-frequencies, and Fourier transforming:

$$\eta_{\vec{x}_1 \vec{x}_2}(t) = e^{-iEt} \eta_{\vec{x}_1 \vec{x}_2},$$

$$\eta_{\vec{x}_1 \vec{x}_2} = \int \frac{d^2 \vec{k}_1}{2\pi} \int \frac{d^2 \vec{k}_2}{2\pi} e^{i\vec{k}_1 \vec{x}_1 + i\vec{k}_2 \vec{x}_2} \eta_{\vec{k}_1 \vec{k}_2},$$

one obtains the spectrum equation:

$$E^2 \eta_{\vec{k}_1 \vec{k}_2} = \frac{1}{4} \left((\psi_{\vec{k}_1}^0)^{-1} + (\psi_{\vec{k}_2}^0)^{-1} \right)^2 \eta_{\vec{k}_1 \vec{k}_2} + \frac{\lambda}{6} (\psi_{\vec{k}_1}^0 + \psi_{\vec{k}_2}^0) \int \frac{d^2 \vec{l}}{(2\pi)^2} \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}. \quad (14)$$

At the UV point ($\lambda = 0$), the large N background is also conformal, and

$$E_{\vec{k}_1 \vec{k}_2}^2 = \frac{1}{4} \left((\psi_{\vec{k}_1}^0)^{-1} + (\psi_{\vec{k}_2}^0)^{-1} \right)^2 = \left(|\vec{k}_1| + |\vec{k}_2| \right)^2, \quad (15)$$

a result known for some time [33] and at the root of the AdS_4/CFT_3 constructive map [19,22] at the free UV fixed point. At the IR fixed point, the spectrum is to be understood as that of a quantum mechanical (relativistic) potential scattering problem for the set of continuum states with $E_{\vec{k}_1 \vec{k}_2}^2 = \left(|\vec{k}_1| + |\vec{k}_2| \right)^2$. It can then be expected that the AdS_4/CFT_3 constructive map of [22] remains valid.

2.3. States on the bilocal boundary

As is well known, the most general solution of the spectrum equation (14) for potential scattering with (squared) energy $E_{\vec{p}_1 \vec{p}_2}^2 = (|\vec{p}_1| + |\vec{p}_2|)^2$ can be written as:

$$\eta_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} = \rho_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} + \frac{\lambda}{12} \frac{1}{E_{\vec{p}_1 \vec{p}_2}^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \times \int \frac{d^2 \vec{l}}{(2\pi)^2} \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}^{\vec{p}_1, \vec{p}_2},$$

where $\rho_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2}$ is a solution of the free equation, which we normalize to $\rho_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} = \delta^2(\vec{p}_1 - \vec{k}_1) \delta^2(\vec{p}_2 - \vec{k}_2)$. In the above, (\vec{p}_1, \vec{p}_2) labels the states and (\vec{k}_1, \vec{k}_2) are momentum coordinates.

Integration of both sides of the full scattering solution results in ([26])

$$\int \frac{d^2 l}{(2\pi)^2} \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}^{\vec{p}_1, \vec{p}_2} = \frac{\delta^2(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{(2\pi)^2 \left(1 + \frac{\lambda}{6i} \int \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2(p_1 + p_2 - l)^2} \right)}, \quad (16)$$

where we have used the result [26]

$$\frac{1}{E_p^2 - (|\vec{l}| + |\vec{p} - \vec{l}|)^2} \left(\frac{1}{|\vec{l}|} + \frac{1}{|\vec{p} - \vec{l}|} \right) = 2i \int \frac{dE_l}{(2\pi)} \frac{1}{l^2(p-l)^2}. \quad (17)$$

Since

$$\int \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2(p-l)^2} = \frac{i}{8|p_E|},$$

the form of the scattering solution for finite λ is:

$$\eta_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} = \delta^2(\vec{p}_1 - \vec{k}_1) \delta^2(\vec{p}_2 - \vec{k}_2) + \frac{\delta^2(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{E_{\vec{p}_1 \vec{p}_2}^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \left(\frac{\lambda}{48\pi^2} \right) \frac{1}{1 + \frac{\lambda}{48|p_E|}}, \quad (18)$$

with $|p_E| = \sqrt{-(|p_1| + |p_2|)^2 + (\vec{p}_1 + \vec{p}_2)^2}$. At the IR critical point ($|\lambda| \rightarrow \infty$), the scattering states take a universal critical form:

$$\eta_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} = \delta^2(\vec{p}_1 - \vec{k}_1) \delta^2(\vec{p}_2 - \vec{k}_2) + \frac{|p_E|}{\pi^2} \frac{\delta^2(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{E_{\vec{p}_1 \vec{p}_2}^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \quad (19)$$

where $E_{\vec{p}_1 \vec{p}_2}^2 = (|\vec{p}_1| + |\vec{p}_2|)^2$. On the boundary, that the $\Delta = 1$ state is no longer in the spectrum is more simply shown by taking the limit $|\lambda| \rightarrow \infty$ in equation (16).⁵

Alternatively it can also be confirmed directly from the critical form (19), by integration with $\vec{k}_1 + \vec{k}_2$ fixed, and using the integral results stated in the above. This agrees with results obtained with path integral bilocal correlators at criticality [26].

Bound states are well known to correspond to eigenspectrum solutions in the absence of an incident wave, or equivalently as particular solutions of (14):

$$\eta_{\vec{k}_1 \vec{k}_2}^B = \frac{\lambda}{12} \frac{1}{E^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \int \frac{d^2 l}{(2\pi)^2} \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}^B$$

Use of the integral results stated above determines its energy to be

⁵ $\eta_{\vec{x}\vec{x}} = \int d^2 p e^{i\vec{p}\vec{x}} \int \frac{d^2 l}{(2\pi)^2} \eta_{\vec{p}-\vec{l}, \vec{l}}$ is a boundary field, as $z \sim (\vec{x}_1 - \vec{x}_2) \cdot \vec{f}(\vec{p}_1, \vec{p}_2)$.

$$E^2 = (\vec{k}_1 + \vec{k}_2)^2 - \left(\frac{\lambda}{48}\right)^2$$

As expected, bound states can also be identified as poles in the connected piece (transmission amplitude) of (18) occurring at $48|p_E| = -\lambda$.

This is a state of infinite (tachyon) squared mass present as $\lambda \rightarrow -\infty$. It appears as an infinite pole $|p_E| \rightarrow \infty$ in the universal connected piece of (19). This is the $\Delta = 2$ state [26].⁶

3. Constructing the bulk

3.1. Higher spin fields in the bulk - field redefinition and quadratic Hamiltonian

We now wish to use the map (2)–(5) to explicitly construct the higher spin fields in the bulk. We first discuss the free case.

The Jacobian for the change of variables from bilocals to AdS coordinates [22] is given by

$$\left| \frac{\partial \vec{k}_{AdS \times S^1}}{\partial \vec{k}_{biloca}} \right| = \frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}. \quad (20)$$

We use the notation $\vec{k}_{AdS \times S^1} = (\vec{k}, k^z, \theta) \equiv \vec{k}$ and $\vec{k}_{biloca} = (\vec{k}_1, \vec{k}_2)$. The equal time slice is the same.

We wish to preserve the canonical structure under this change of variables:

$$\begin{aligned} \left[\pi_{\vec{k}_1 \vec{k}_2}, \eta_{\vec{k}'_1 \vec{k}'_2} \right] &= -i \delta^2(\vec{k}_1 - \vec{k}'_1) \delta^2(\vec{k}_2 - \vec{k}'_2) \\ &= -i \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \delta(\vec{k}_{AdS \times S^1} - \vec{k}'_{AdS \times S^1}) \\ \Rightarrow \left[\frac{\pi_{\vec{k}_1 \vec{k}_2}, \eta_{\vec{k}'_1 \vec{k}'_2}}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}} \right] &= -i \delta(\vec{k}_{AdS \times S^1} - \vec{k}'_{AdS \times S^1}) \end{aligned}$$

This requires a redefinition of at least one of the fields.

Let us now consider the form of the (free) quadratic Hamiltonian (13) in momentum space:

$$\begin{aligned} H_2 &= \int d^2 k_1 \int d^2 k_2 \left(\pi_{\vec{k}_1 \vec{k}_2} (\psi_{\vec{k}_1}^0 + \psi_{\vec{k}_2}^0) \pi_{-\vec{k}_2 - \vec{k}_1} \right) \\ &\quad + \frac{1}{16} \int d^2 k_1 \int d^2 k_2 \eta_{\vec{k}_1 \vec{k}_2} \\ &\quad \left((\psi_{\vec{k}_1}^0)^{-2} (\psi_{\vec{k}_2}^0)^{-1} + (\psi_{\vec{k}_2}^0)^{-2} (\psi_{\vec{k}_1}^0)^{-1} \right) \eta_{-\vec{k}_2 - \vec{k}_1} \\ &= \int d^2 k_1 \int d^2 k_2 \frac{1}{2} \left(\pi_{\vec{k}_1 \vec{k}_2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \pi_{-\vec{k}_2 - \vec{k}_1} \right) \\ &\quad + \frac{1}{2} \eta_{\vec{k}_1 \vec{k}_2} (|\vec{k}_1|^2 |\vec{k}_2| + |\vec{k}_2|^2 |\vec{k}_1|) \eta_{-\vec{k}_2 - \vec{k}_1}. \end{aligned}$$

By factorizing the Jacobian, this can be re-written as

$$\begin{aligned} H_2 &= \int d^2 k_1 \int d^2 k_2 \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \frac{1}{2} \left[\left(\pi_{\vec{k}_1 \vec{k}_2} \pi_{-\vec{k}_2 - \vec{k}_1} \right) \right. \\ &\quad \left. + \frac{\eta_{\vec{k}_1 \vec{k}_2} (|\vec{k}_1| + |\vec{k}_2|)^2 \eta_{-\vec{k}_2 - \vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right)} \right]. \quad (21) \end{aligned}$$

This then suggests that we define the bulk higher spin field and its conjugate field as [34,35]⁷

$$\mathcal{H}(\vec{k}) = \mathcal{H}(\vec{k}, k^z, \theta) \equiv \frac{\eta_{\vec{k}_1 \vec{k}_2}}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}} \Big|_{\vec{k}_1, \vec{k}_2(\vec{k}, k^z, \theta)} \quad (22)$$

$$\Pi(\vec{k}) = \Pi_{\mathcal{H}}(\vec{k}, k^z, \theta) = \pi_{\vec{k}_1 \vec{k}_2} \Big|_{\vec{k}_1, \vec{k}_2(\vec{k}, k^z, \theta)}. \quad (23)$$

Since by construction (2), $|\vec{k}_1| + |\vec{k}_2| = E = \sqrt{\vec{k}^2 + (k^z)^2} \equiv (P^0)_{AdS}$, the Hamiltonian can be written directly as an integral over $AdS \times S^1$:

$$\begin{aligned} H_2 &= \frac{1}{2} \int d\vec{k}_{AdS \times S^1} \left[\left(\Pi_{\mathcal{H}}(\vec{k}, k^z, \theta) \Pi_{\mathcal{H}}(-\vec{k}, -k^z, -\theta) \right) \right. \\ &\quad \left. + (P^0)^2 \mathcal{H}(\vec{k}, k^z, \theta) \mathcal{H}(-\vec{k}, -k^z, -\theta) \right]. \end{aligned}$$

In other words, we have obtained a bulk quadratic Hamiltonian in $AdS_4 \times S^1$ by field redefinition and a simple change of variables:

$$H_2 = \frac{1}{2} \int d\vec{k} \left[\Pi(\vec{k}) \Pi(-\vec{k}) + (P^0)^2 \mathcal{H}(\vec{k}) \mathcal{H}(-\vec{k}) \right]. \quad (24)$$

We expand in spin fields [22]

$$h(k^z, \vec{k}, \theta) = \sum_{s=0, \pm 2, \dots}^{\infty} \frac{e^{is\theta}}{\sqrt{\pi}} h_s(k^z, \vec{k}). \quad (25)$$

The spin field $h_s(k^z, \vec{k})$ can be further expanded [22], but for the purposes of this communication it is sufficient to observe the important property that $\theta \in [0, \pi]$. Indeed, from (5), $\theta \sim \theta + \pi$, and the fact that s is even follows. This corresponds to the spectrum of the minimal type A Vasiliev higher spin theory with a $\Delta = 1$ scalar.

We then recognise (24) as a sum of Hamiltonians of massless spin s fields in AdS_4 ⁸:

$$\begin{aligned} H_2 &= \frac{1}{2} \sum_{s=0, \pm 2, \dots}^{\infty} \int d\vec{k}_{AdS} \left[\pi_s(\vec{k}, k^z) \pi_s(-\vec{k}, -k^z) + (P^0)^2 h_s(\vec{k}, k^z) \right. \\ &\quad \left. \times h_s(-\vec{k}, -k^z) \right] \quad (26) \end{aligned}$$

3.2. Bulk higher spin fields at the IR critical point

At the IR critical point, recall that the universal form (19) of the energy eigenstates was found to be:

$$\begin{aligned} \eta_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} &= \delta^2(\vec{p}_1 - \vec{k}_1) \delta^2(\vec{p}_2 - \vec{k}_2) \\ &\quad + \frac{|p_E|}{\pi^2} \frac{\delta^2(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{E_{\vec{p}_1 \vec{p}_2}^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right). \end{aligned}$$

Further recall that (\vec{k}_1, \vec{k}_2) are momentum coordinates and that (\vec{p}_1, \vec{p}_2) label the states with (squared) energy $E_{\vec{p}_1 \vec{p}_2}^2 = (|\vec{p}_1| + |\vec{p}_2|)^2$. We observe that

$$\begin{aligned} |p_E| &= \sqrt{-(|p_1| + |p_2|)^2 + (\vec{p}_1 + \vec{p}_2)^2} \\ &= i \sqrt{(|p_1| + |p_2|)^2 - (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2)} \\ &= i|p^z|, \end{aligned}$$

using the map (2)–(5).

⁶ In 3 euclidean dimensions $\int \frac{d^3 x}{x^2} e^{ik_E x} \sim |k_E|$.

⁷ This is opposite to the $c = 1$ case [36] where it is the conjugate momentum that is rescaled in the change of variables to (asymptotic) Liouville coordinates.

⁸ π_s and h_s are canonically conjugate fields.

The most general solution to the spectrum equations can then be written as an arbitrary linear combination of the universal energy eigenstates:

$$\begin{aligned} \eta_{\vec{k}_1 \vec{k}_2} &= \int d\vec{p}_1 \int d\vec{p}_2 \eta_{\vec{k}_1, \vec{k}_2}^{\vec{p}_1, \vec{p}_2} \psi_{\vec{p}_1 \vec{p}_2} \\ &= \psi_{\vec{k}_1 \vec{k}_2} + \frac{i}{\pi^2} \int d\vec{p}_1 \int d\vec{p}_2 |p^z| \\ &\quad \frac{\delta^{d-1}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2) \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right)}{E_{\vec{p}_1 \vec{p}_2}^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \psi_{\vec{p}_1 \vec{p}_2}. \end{aligned}$$

Following the prescription of the previous subsection, to change to bulk higher spin variables, we make the identification

$$\begin{aligned} H(\vec{k}) &\equiv \frac{\eta_{\vec{k}_1 \vec{k}_2}}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right)} \\ h(\vec{k}) &\equiv \frac{\psi_{\vec{k}_1 \vec{k}_2}}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right)}, \end{aligned}$$

and change variables to $AdS_4 \times S_1$ coordinates. Hence we obtain, in the bulk:

$$\begin{aligned} H(\vec{k}) &= h(\vec{k}) + \frac{i}{\pi^2} \int d\vec{p} \int dp^z \int d\theta |p^z| \\ &\quad \frac{\delta^2(\vec{p} - \vec{k})}{(p^z)^2 + (\vec{p})^2 - (k^z)^2 - (\vec{k})^2} h(p^z, \vec{p}, \theta) \\ &= h(k^z, \vec{k}, \theta_{\vec{k}}) \\ &\quad + \frac{i}{\pi^2} \int dp^z \int d\theta \frac{|p^z|}{(p^z)^2 - (k^z)^2} h(p^z, \vec{k}, \theta). \end{aligned}$$

Under mild assumptions on the behaviour of $h(p^z, \vec{k}, \theta)$ as $|p^z| \rightarrow \infty$ (also requiring that $h(-k^z) = h(k^z)$), one has

$$\int dp^z \frac{|p^z| h(p^z, \vec{k}, \theta)}{(p^z)^2 - (k^z)^2 - i\epsilon} = i\pi h(k^z, \vec{k}, \theta),$$

so that

$$H(k^z, \vec{k}, \theta) = h(k^z, \vec{k}, \theta) - \frac{1}{\pi} \int_0^\pi d\theta h(k^z, \vec{k}, \theta). \quad (27)$$

Expanding $h(k^z, \vec{k}, \theta)$ in spin s fields as in (25):

$$h(k^z, \vec{k}, \theta) = \sum_{s=0, \pm 2, \dots}^{\infty} \frac{e^{is\theta}}{\sqrt{\pi}} h_s(k^z, \vec{k})$$

with $0 < \theta < \pi$, we see that the latter term in equation (27) precisely removes the $s = 0$ field ($\Delta = s + 1 = 1$) in the bulk, and thus

$$H(k^z, \vec{k}, \theta) = \sum_{s \neq 0, s = \pm 2, \dots}^{\infty} \frac{e^{is\theta}}{\sqrt{\pi}} h_s(k^z, \vec{k}),$$

in agreement with [37]. More precisely, our result corresponds, in terms of higher spin representations without $\Delta = 1$, to the spectra of the (antisymmetric) direct product of two 3d free $O(N)$ Majorana Di singletons.⁹ That is,

$$[Di \otimes Di]_A = (2, 0) \oplus \bigoplus_{s=1}^{\infty} (2s + 1, 2s) \quad (28)$$

The 3d free $O(N)$ Majorana fermion theory is dual to the minimal type B Vasiliev higher spin theory which, in $d = 3$, has the same spectra, up to boundary conditions for the scalar field, as the minimal type A Vasiliev higher spin theory with a $\Delta = 2$ scalar [7,8]. This provides conclusive evidence of the appropriateness of the identification of the bulk higher spin fields as in (22) and (23).

It is left to observe that the interaction term $\frac{\lambda}{4!} \int d^{d-2} \vec{x} (\eta_{\vec{x}\vec{x}})^2$ of the Hamiltonian does not contribute at the critical point, since

$$\eta_{\vec{x}\vec{x}} \sim 1/\lambda + \mathcal{O}(1/\lambda^2).$$

As such, the expression for the Hamiltonian (26) simply changes to exclude the $s = 0$ term:

$$\begin{aligned} H_2 &= \frac{1}{2} \sum_{s \neq 0, s = \pm 2, \dots} \int d\vec{k}_{AdS} [\pi_s(\vec{k}, k^z) \pi_s(-\vec{k}, -k^z) \\ &\quad + (P^0)^2 h_s(\vec{k}, k^z) h_s(-\vec{k}, -k^z)] \end{aligned}$$

3.3. The $\Delta = 2$ state and its Hamiltonian

Returning to the $\Delta = 2$ state, we recall that the particular solution wave function of (14) is given by

$$\begin{aligned} \eta_{\vec{k}_1 \vec{k}_2}^B &= \frac{\lambda}{48\pi^2} \frac{1}{E^2 - (|\vec{k}_1| + |\vec{k}_2|)^2} \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right) J(\vec{k}_1 + \vec{k}_2), \\ J(\vec{k}_1 + \vec{k}_2) &= \int d^2 l \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}^B, \end{aligned}$$

provided

$$E^2 = E_{\vec{k}_1 + \vec{k}_2}^2 = (\vec{k}_1 + \vec{k}_2)^2 - \left(\frac{\lambda}{48} \right)^2. \quad (29)$$

In other words,

$$\frac{\eta_{\vec{k}_1 \vec{k}_2}^B}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|} \right)} = \frac{\lambda}{48\pi^2} \frac{J(\vec{k}_1 + \vec{k}_2)}{(\vec{k}_1 + \vec{k}_2)^2 - (|\vec{k}_1| + |\vec{k}_2|)^2 - \left(\frac{\lambda}{48} \right)^2}$$

We can now implement the map (2)-(5) and the field redefinition (22) to obtain the state directly in the bulk:

$$H^B(\vec{k}) = -\frac{\lambda}{48\pi^2} \frac{J(\vec{k})}{(k^z)^2 + \left(\frac{\lambda}{48} \right)^2}, \quad (30)$$

$$\int dk^z e^{ik^z z} H^B(\vec{k}) \sim e^{-|\lambda z|/48}. \quad (31)$$

The above bulk description of the $\Delta = 2$ state establishes it as a spin 0 state with an exponential decay into the bulk. At criticality, the state is then a boundary state, in agreement with [26].

In order to obtain the Hamiltonian description of the state, we investigate the bulk properties of $J(\vec{k}_1 + \vec{k}_2)$:

$$\begin{aligned} J(\vec{k}_1 + \vec{k}_2) &= \int d^2 l \eta_{\vec{k}_1 + \vec{k}_2 - \vec{l}, \vec{l}}^B \\ &= \int d^2 l_1 d^2 l_2 \delta^2(\vec{k}_1 + \vec{k}_2 - \vec{l}_1 + \vec{l}_2) \eta_{\vec{l}_1 \vec{l}_2}^B \\ &= \int d\vec{l}_{AdS} \delta^2(\vec{k} - \vec{l}) H^B(l^z, \vec{l}, \theta_l) \\ &= \int d l^z \int d\theta_l H^B(l^z, \vec{k}, \theta_l) = J(\vec{k}) \end{aligned}$$

$J(\vec{k})$ is then a spin 0 boundary ($z = 0$) state. For finite λ , consistency of the solution (30) can be established directly in the bulk, requiring $\lambda < 0$.

⁹ The $SO(3, 2)$ representations are labelled by (Δ, s) and $Di = (1, \frac{1}{2})$.

The interaction term in equation (13) now takes the form:

$$\frac{\lambda}{4!} \int d^2\vec{x} \eta_{xx}^2 = \frac{\lambda}{96\pi^2} \int d^2k J(\vec{k}) J(-\vec{k})$$

From the quadratic Hamiltonian (21) and the field redefinitions (22) and (23), it follows that $\Pi_H(\kappa) = \dot{H}(-\kappa)$, so that

$$\Pi_H^B(\kappa) = -\frac{\lambda}{48\pi^2} \frac{\Pi_J(-\vec{k})}{(k^2)^2 + (\frac{\lambda}{48})^2}, \quad (32)$$

where $\Pi_J(\vec{k})$ is the canonical conjugate to the boundary field $J(\vec{k})$. Substituting (30) and (32) into the Hamiltonian (24), performing the integrals over k^2 and (trivially) over θ and finally adding the above interaction contribution results in the Hamiltonian:

$$H_2^B = \frac{24}{\pi^2|\lambda|} \left\{ \frac{1}{2} \int d^2k \Pi_J(\vec{k}) \Pi_J(-\vec{k}) + \frac{1}{2} \int d^2k \left(k^2 - \left(\frac{\lambda}{48} \right)^2 \right) J(\vec{k}) J(-\vec{k}) \right\}$$

This is, up to a factor, the expected Hamiltonian for a single mode with dispersion relation (29).

4. Discussion and outlook

In this paper, we built on the constructive approach which was developed in Refs. [19,22] in both the light-cone gauge and the temporal gauge for the free theory, in which an explicit map between the conformal field theory in $d=2+1$ dimensions and the higher spin theory in $AdS_4 \times S_1$ was established. In the Hamiltonian approach, the $1+2+2=5$ coordinates of the equal time bilocals, map (in phase space) to the coordinates of $AdS_4 \times S_1$. We made use of the Hamiltonian approach in a time like gauge [22], and for the IR critical point, we considered an $O(N)$ vector theory with a quartic interaction [26]. The quartic interaction contributes linearly in the bilocal field fluctuation equations, and the spectrum problem is then that of a potential scattering problem. The eigenstate solutions take a universal form at the critical point [26].

The bulk description of these boundary eigenstates was obtained by developing a remarkably simple first principles approach, consisting of a simple change of variables from bilocal momenta to bulk momenta (2)–(5), but requiring a field redefinition in defining the bulk higher spin field. In this way, simple quadratic Klein-Gordon bulk Hamiltonians are derived for the higher spin fields in both UV and IR critical points, and, at the IR critical point, the absence of an AdS_4 spin 0 field is established directly in the bulk. The $\Delta=2$ state is shown to be a boundary state at IR criticality. Moreover, after integrating over k^2 and θ , the boundary quadratic Hamiltonian was obtained and has the expected dispersion relation for the bound state. In future, it will be interesting to look also at the non-decaying states which corresponds to mass deformations in the bulk dual theory [30].

The higher spin fields considered in this communication are different from those in [22], but the approach should be equivalent, at quadratic level, to the oscillator expressions obtained in that article for the conformal generators. An extensive and comprehensive study of the conformal algebra using the approach described in this communication was carried out in [35] and shown to be indeed equivalent to the bulk oscillator expressions obtained in [22]. This will be reported elsewhere [38].

It is of great interest to apply the approach developed in this communication to generate interactions. The $1/N$ expansion of the collective field Hamiltonian is well established (e.g. [39], [21]). Work in this direction is currently underway.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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