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# On Certain Properties and Applications of the Perturbed Meixner–Pollaczek Weight

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**Abstract:** This paper deals with monic orthogonal polynomials orthogonal with a perturbation of classical Meixner–Pollaczek measure. These polynomials, called Perturbed Meixner–Pollaczek polynomials, are described by their weight function emanating from an exponential deformation of the classical Meixner–Pollaczek measure. In this contribution, we investigate certain properties such as moments of finite order, some new recursive relations, concise formulations, differential-recurrence relations, integral representation and some properties of the zeros (quasi-orthogonality, monotonicity and convexity of the extreme zeros) of the corresponding perturbed polynomials. Some auxiliary results for Meixner–Pollaczek polynomials are revisited. Some applications such as Fisher’s information, Toda-type relations associated with these polynomials, Gauss–Meixner–Pollaczek quadrature as well as their role in quantum oscillators are also reproduced.

**Keywords:** orthogonal polynomials; Meixner; perturbed Meixner–Pollaczek; moments; recurrence coefficients; difference equations; differential equations; zeros



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## 1. Introduction

First, let us define some terminologies, notations and conventions that we will use throughout this paper. The set of complex numbers will be denoted by  $\mathbb{C}$  and  $i$  will stand for the imaginary number ( $i^2 = -1$ ); the set of positive integers will be denoted by  $\mathbb{N}$ , and  $\mathbb{N}_0$  will denote the set of non-negative integers. All polynomials considered will be real-valued in one real variable, and  $\mathbb{P}$  will stand for the set of all such polynomials. For each  $n \in \mathbb{N}_0$ , the subset of  $\mathbb{P}$  of all polynomials of degree not greater than  $n$  will be denoted by  $\mathbb{P}_n$ . By a system of monic polynomials, we will mean a sequence  $\{\Phi_n\}_{n=0}^\infty$  of polynomials satisfying  $\Phi_n^{(n)} = n!$  for each  $n \in \mathbb{N}_0$ .

A sequence of real polynomials  $\{\Phi_n\}_{n=0}^\infty$ , where  $\Phi_n$  is of exact degree  $n$ , is orthogonal with respect to a (positive) measure  $\mu$  supported on an interval  $[a, b]$ , if the scalar product

$$\langle \Phi_m, \Phi_n \rangle = \int_a^b \Phi_m(x) \Phi_n(x) d\mu(x) = 0, \quad m \neq n.$$

If  $\mu(x)$  is absolutely continuous, then it can be represented by a real weight function  $w(x) > 0$  so that  $d\mu(x) = w(x) dx$ . If  $\mu(x)$  is discrete with support in  $\mathbb{N}_0$ , then it can be represented by a discrete weight  $w(x) \geq 0$  ( $x \in \mathbb{N}_0$ ), and the scalar product given by

$$\langle \Phi_m, \Phi_n \rangle = \sum_{x=0}^\infty \Phi_m(x) \Phi_n(x) w(x).$$

The orthogonal polynomial families under consideration in this paper are the following ones (see [1]):

- Meixner polynomials ([1], Section 9.10)

$$M_n(x; \beta, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - \frac{1}{c}\right), \tag{1}$$

are orthogonal with respect to the discrete weight  $\rho(x) = \frac{c^x(\beta)_x}{x!}$  on  $(0, \infty)$ , for  $0 < c < 1$  and  $\beta > 0$ , with  $\beta \neq -1, -2, \dots, -n + 1$ . Here,  ${}_2F_1$  is the hypergeometric function defined by

$${}_2F_1\left(\begin{matrix} p, q \\ r \end{matrix} \middle| s\right) = \sum_{k=0}^{\infty} \frac{(p)_k (q)_k}{(r)_k} \frac{s^k}{k!}, \tag{2}$$

where the Pochhammer symbol, or rising factorial,  $(z)_n$ , takes the form

$$(z)_n := (z)(z + 1) \cdots (z + n - 1) = \prod_{i=1}^n (z + i - 1). \tag{3}$$

- Monic Meixner polynomials ([1], Section 9.10) are given by

$$\mathcal{M}_n(x; \beta, c) = (\beta)_n \left(\frac{c}{c-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - \frac{1}{c}\right) = (\beta)_n \left(\frac{c}{c-1}\right)^n M_n(x; \beta, c). \tag{4}$$

- Meixner–Pollaczek polynomials ([1], Section 9.7)

$$p_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} \left(\frac{e^{2i\phi}}{e^{2i\phi} - 1}\right)^n {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - \frac{1}{e^{2i\phi}}\right), \tag{5}$$

are orthogonal with respect to the continuous weight

$$w(x; \phi) = |\Gamma(\lambda + ix)|^2 e^{(2\phi - \pi)x}, \tag{6}$$

on the interval  $(-\infty, \infty)$ , for  $n \in \mathbb{N}$ ,  $\lambda > 0$  and  $0 < \phi < \pi$ . Note that the complex Gamma function in Equation (6) takes the form [2]

$$|\Gamma(\lambda + ix)|^2 = \Gamma(\lambda + ix) \Gamma(\lambda - ix).$$

- Monic Meixner–Pollaczek polynomials ([1], Section 9.7) are given by

$$P_n^{(\lambda)}(x; \phi) = i^n (2\lambda)_n \left(\frac{e^{2i\phi}}{e^{2i\phi} - 1}\right)^n {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - \frac{1}{e^{2i\phi}}\right) = \frac{n! i^n}{e^{in\phi}} p_n^{(\lambda)}(x; \phi). \tag{7}$$

For some properties of Meixner–Pollaczek polynomials including asymptotics, we refer to [3–9].

We recall the following essential facts.

**Definition 1** ([8]). Let  $\{\eta_n\}_{n=0}^{\infty}$  be a sequence of complex numbers and let  $\mathcal{L}$  be a complex valued function on the linear space of all polynomials by

$$\begin{cases} \mathcal{L}[x^n] = \eta_n, & n \in \mathbb{N}_0, \\ \mathcal{L}[\alpha f_1(x) + \beta f_2(x)] = \mathcal{L}[\alpha f_1(x)] + \mathcal{L}[\beta f_2(x)], \end{cases}$$

for  $\alpha, \beta \in \mathbb{C}$  and  $f_i(x)$  ( $i = 1, 2$ ). Then  $\mathcal{L}$  is said to be the moment functional determined by the moments  $\eta_n$  of order  $n$ .

Let  $\mathcal{P}(x; t)$  denote  $R(t)[x]$ , the linear space of all polynomials with rational function (in  $t$ ) coefficients in one variable  $x$ . We call such polynomials, parameterized polynomials. We extend the classical orthogonality results in [3,8,10] to parameterized polynomials. We denote the linear subspace of degree  $m$  parameterized polynomials by  $\mathcal{P}_m[t]$ . The following is an extension of ([8], Theorem 2.1).

**Lemma 1.** Consider a moment functional  $\mathcal{L}$  and a parameterized polynomial sequence  $\{\Psi_n(x; t)\}_{n=0}^\infty$ . Then the following are equivalent (cf. [8], Theorem 2.1):

- (i)  $\{\Psi_n(x; t)\}_{n=0}^\infty$  is an orthogonal polynomial sequence with respect to  $\mathcal{L}$ ,
- (ii)  $\mathcal{L}[\pi(x; t) \Psi_n(x; t)] = 0$  for every polynomial  $\pi(x; t)$  of degree  $m < n$ ; while  $\mathcal{L}[\pi(x; t) \Psi_n(x; t)] \neq 0$  if  $m = n$ ,
- (iii)  $\mathcal{L}[x^m \Psi_n(x; t)] = \zeta_n(t) \delta_{m,n}$  where  $\zeta_n(t) \neq 0$ , for  $0 \leq m \leq n$ .

In [11], Meixner–Pollaczek polynomials are used to explore thermodynamic susceptibilities in the thermodynamic relations of Hermitian Ensembles. One can apply an exponential modification of the measure  $\mu$  and to investigate orthogonal polynomials for the measure  $d\mu_t(x) = e^{-xt} d\mu(x)$ , whenever all the moments of this modified measure exist, and this leads to a new class of semi-classical (non-classical) orthogonal polynomials with respect to the modified measure.

**Definition 2.** Perturbed Meixner–Pollaczek polynomials  $\{Q_n^{(\lambda, \varphi)}(x; t)\}_{n=0}^\infty$  are monic real polynomials which are orthogonal with respect to the weight function

$$w^{(\lambda, \varphi)}(x; t) := \frac{1}{2\pi} e^{(2\varphi - \pi)x} |\Gamma(\lambda + ix)|^2 e^{-axt}, \quad x \in \mathbb{R}, \tag{8}$$

with parameters  $\lambda > 0, a > 0$  and  $0 \leq t < \frac{2\varphi}{a}$ .

Chen and Ismail [11] also discussed Toda lattice equations in the context of Coulomb fluid relations. Perturbed Meixner–Pollaczek polynomials have some applications as shown in ([11], pp. 12–13). In the context of Physics literature, the parameter  $\varphi$  in Equation (8) is the phase of an oscillation,  $t$  is time and  $a$  can be perceived as a positive angular frequency (in Hertz) (angular velocity or angular speed) of a wave, an oscillation (in cycle per second or  $2\pi$  rad per second) or a field (electromagnetic). For example,  $a > 0$  in the mathematical model of (nonlinear) tornado system as the wave speed of frequency of tornadoes is so huge.

The objective of this paper is to unravel some properties of monic orthogonal polynomials with respect to the perturbed Meixner–Pollaczek measure (8) and to explore some of their practical applications.

The structure of the paper is as follows. In Section 1, certain properties and auxiliary results of Meixner–Pollaczek polynomials are given. This section also introduces perturbed Meixner–Pollaczek polynomials with some properties. Section 2 gives the relation between Meixner–Pollaczek and Perturbed Meixner–Pollaczek polynomials. In Section 3, we investigate some results of perturbed Meixner–Pollaczek polynomials with proofs. Certain properties of these polynomials such as orthogonality, concise formulation, new recursive relations and some properties of the zeros (convexity and monotonicity of the extreme zeros) are discussed. Section 4 provides some practical applications; in particular, the applicability of the monic perturbed Meixner–Pollaczek polynomials in the study of Toda lattices in Random Matrix theory, Fisher information, Gaussian quadrature using Meixner–Pollaczek weight and solution to a quantum oscillator in quantum physics [12]. Section 5 ends with conclusions of this work.

1.1. Some Auxiliary Results for the Meixner and Meixner–Pollaczek Weight

In this Subsection, we revisit some properties of Meixner and Meixner–Pollaczek polynomials. The following proposition gives some properties of Meixner polynomials.

**Proposition 1.** For Meixner polynomials, we have

(i) Orthogonality:

$$\langle M_m, M_n \rangle := \sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_m(x; \beta, c) M_n(x; \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{m,n}, \quad m, n \in \mathbb{N}_0;$$

(ii) Forward shift operator identity:

$$\Delta M_n(x; \beta, c) := M_n(x + 1; \beta, c) - M_n(x; \beta, c) = \frac{n}{\beta} \frac{c-1}{c} M_{n-1}(x; \beta + 1, c);$$

(iii) Three-term recursion relation:

$$(n + \beta) M_n(x; \beta + 1, c) = \beta M_n(x; \beta, c) + n M_{n-1}(x; \beta + 1, c);$$

(iv) Expansion formula:

$$M_n(x; \beta + 1, c) = \frac{n!}{(\beta + 1)_n} \sum_{k=0}^n \frac{(\beta)_k}{k!} M_k(x; \beta, c), \quad n \in \mathbb{N}_0. \tag{9}$$

**Proof.** • For the proof of (i) and (ii), we refer to ([1], (1.9.2), (1.9.6)).

- Property (iii) follows, by considering  $z = 1 - \frac{1}{c}$ , from the formulae for  $M_n(x; \beta + 1, c)$  and  $M_n(x; \beta, c)$ :

$$M_n(x; \beta + 1, c) = 1 + \sum_{k=1}^n \binom{n}{k} \frac{x(x-1) \cdots (x-k+1)}{(\beta+1)(\beta+2) \cdots (\beta+k)} z^k \tag{10}$$

$$M_n(x; \beta, c) = 1 + \sum_{k=1}^n \binom{n}{k} \frac{x(x-1) \cdots (x-k+1)}{\beta(\beta+1) \cdots (\beta+k-1)} z^k \tag{11}$$

If we take  $\beta$  multiplied by Equation (11) and then subtract it from Equation (10) multiplied by  $n + \beta$ , the required result immediately follows.

- For the proof of property (iv), we use mathematical induction on  $n$ . One can see easily that Equation (9) holds for  $n = 0$ . We assume it holds true for some  $n \in \mathbb{N}_0$ . By applying induction hypothesis and Equation (9), we have

$$\begin{aligned} M_{n+1}(x; \beta + 1, c) &= \frac{\beta}{n + 1 + \beta} M_{n+1}(x; \beta, c) + \frac{n + 1}{n + 1 + \beta} M_n(x; \beta + 1, c) \\ &= \frac{\beta}{n + 1 + \beta} M_{n+1}(x; \beta, c) + \frac{(n + 1)!}{(\beta + 1)_{n+1}} \sum_{k=0}^n \frac{(\beta)_k}{k!} M_k(x; \beta, c) \\ &= \frac{(n + 1)!}{(\beta + 1)_{n+1}} \sum_{k=0}^{n+1} \frac{(\beta)_k}{k!} M_k(x; \beta, c). \end{aligned}$$

This completes the inductive result.

□

We note from Equation (4) that

$$P_n^{(\lambda)}(x; \phi) = i^n \mathcal{M}_n(-\lambda - ix; 2\lambda, e^{2i\phi}), \tag{12}$$

and Meixner polynomials and Meixner–Pollaczek polynomials are the same polynomials, with a discrete variable in the first case and a continuous variable in the second (cf. [13]). Monic Meixner polynomials satisfy the three-term recurrence relation

$$\mathcal{M}_n(x; \beta, c) = \left(x + \frac{c(\beta + n - 1) + n - 1}{c - 1}\right) \mathcal{M}_{n-1}(x; \beta, c) - \frac{c(n - 1)(\beta + n - 2)}{(c - 1)^2} \mathcal{M}_{n-2}(x; \beta, c)$$

and when we substitute  $x$  with  $-\lambda - ix$ ,  $\beta$  with  $2\lambda$  and  $c$  with  $e^{2i\phi}$ , multiply by  $i^n$  and apply Equation (12), we obtain the three-term recurrence relation for the Meixner–Pollaczek polynomials

$$P_n^{(\lambda)}(x; \phi) = \left(x + a_n^{(\lambda, \phi)}\right) P_{n-1}^{(\lambda)}(x; \phi) - C_n P_{n-2}^{(\lambda)}(x; \phi). \tag{13}$$

where

$$a_n^{(\lambda, \phi)} := \frac{\lambda + n - 1}{\tan \phi}; \quad C_n := C_n^{(\lambda, \phi)} = \frac{(n - 1)(2\lambda + n - 2)}{4 \sin^2 \phi}, \tag{14}$$

with  $P_{-1}^{(\lambda)}(x) = 0$ ,  $P_0^{(\lambda)}(x) = 1$  and  $a_n^{(\lambda, \frac{\pi}{2})} = \lim_{\phi \rightarrow \frac{\pi}{2}} a_n^{(\lambda, \phi)} = 0$ . We note that the coefficient of  $P_{n-2}^{(\lambda)}(x; \phi)$ , behaves like  $O(n^2)$  as  $n \rightarrow \infty$  and using Carleman’s condition [8], the uniqueness of the orthogonality measure holds ([1], Section 9.7).

Let’s recall the following result [14] (see also [6]).

**Proposition 2** ([14]). *For  $\lambda > 0$ , the moments for Meixner–Pollaczek measure are finite; i.e.,*

$$\int_{\mathbb{R}} x^n e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 dx < \infty.$$

**Proof.** The finiteness of the moments follow from [14]

$$\int_{-\infty}^{\infty} e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 dx = \frac{\pi \Gamma(2\lambda)}{(2 \sin \phi)^{2\lambda}}, \tag{15}$$

and by differentiating Equation (15)  $n$ -times with respect to  $\phi$  ([15], Lemma 1); i.e.,

$$\int_{\mathbb{R}} x^n e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 dx = 2^{-n} \pi \Gamma(2\lambda) \frac{d^n}{d\phi^n} (2 \sin \phi)^{-2\lambda}.$$

□

We now consider some results on quasi-orthogonality and interlacing of the zeros of the Meixner–Pollaczek polynomials.

**Definition 3.** *A polynomial  $\Phi_n$  of exact degree  $n \geq r$ , is quasi-orthogonal of order  $r$  on  $[a, b]$  with respect to a weight function  $w(x) > 0$ , if (cf. ([16], p. 159))*

$$\int_a^b x^j \Phi_n(x) w(x) dx \begin{cases} = 0, \text{ for } j = 0, 1, \dots, n - r - 1, \\ \neq 0, \text{ for } j = n - r. \end{cases}$$

For a more general definition of quasi-orthogonality, we refer to [8].

Since the Meixner–Pollaczek polynomials are orthogonal on the real line, zeros departing from the interval of orthogonality will do so in complex conjugate pairs. (This fact is later checked with numerical experiments of the zeros of these polynomials). The quasi-orthogonality of the monic Meixner–Pollaczek polynomials is therefore of even order, as detailed in the next result ([17], Theorem 3.3).

**Theorem 1.** *Let  $n \in \mathbb{N}$ ,  $k \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$  and  $0 < \phi < \pi$ . For  $0 < \lambda < 1$ , the sequence of polynomials  $\{P_n^{\lambda-k}\}_{n=1}^{\infty}$  is quasi-orthogonal of order  $2k$  with respect to the weight function*

$e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2$  on  $(-\infty, \infty)$  and the polynomials have at least  $n - 2k$  real zeros on the real line.

By a change in the variable  $\lambda$ , the result in Theorem 1 can be rephrased by stating that  $P_n^{(\lambda)}(x; \phi)$ , with  $-k < \lambda < -k + 1$ , is quasi-orthogonal of order  $2k$  with respect to  $e^{(2\phi-\pi)x}|\Gamma(\lambda+k+ix)|^2$  on the interval  $(-\infty, \infty)$ .

We can say the following about the interlacing of the zeros of polynomials  $P_n^{(\lambda-1)}$  and  $P_{n-1}^{(\lambda)}$ ,  $\lambda > 0$ .

**Lemma 2.** Let  $n \in \mathbb{N}$ ,  $\lambda > 0$  and  $0 < \phi < \pi$ .

- (a) If  $\lambda > 1$ , the  $n$  zeros of  $P_n^{(\lambda-1)}(x; \phi)$  interlace with the  $(n - 1)$  zeros of  $P_{n-1}^{(\lambda)}(x; \phi)$ .
- (b) If  $0 < \lambda < 1$ , then  $(n - 2)$  zeros of the order two quasi-orthogonal polynomial  $P_n^{(\lambda-1)}(x; \phi)$  interlace with the  $(n - 1)$  zeros of  $P_{n-1}^{(\lambda)}(x; \phi)$ .

**Proof.** The polynomial  $P_n^{(\lambda-1)}(x; \phi)$  can be expressed as follows ([17], Equation (2).2):

$$P_n^{(\lambda-1)}(x; \phi) = P_n^{(\lambda)}(x; \phi) - \frac{n}{\tan \phi} P_{n-1}^{(\lambda)}(x; \phi) + b_n P_{n-2}^{(\lambda)}(x; \phi) \tag{16}$$

with  $b_n = \frac{n(n-1)}{4 \sin^2 \phi}$ .

- (a) Let  $\lambda > 1$ . Then the polynomial  $P_n^{(\lambda-1)}(x; \phi)$  is part of an orthogonal sequence and all its zeros are real. Furthermore,  $b_n < C_n$ , where  $C_n$ , given in (14), is obtained from the three-term recurrence relation satisfied by the Meixner–Pollaczek polynomials and the result follows from (16) and ([18], Theorem 15 (i)).
- (b) Let  $0 < \lambda < 1$ . From Theorem 1 we see that at least  $(n - 2)$  zeros of  $P_n^{(\lambda-1)}(x; \phi)$  are real. Furthermore,  $b_n > C_n$  when  $0 < \lambda < 1$ , and the result follows from (16) and ([18], Theorem 15 (i)).

□

For a detailed discussion on the quasi-orthogonality and location of the zeros of the Meixner polynomials, we refer the reader to [19].

### 1.1.1. Some Numerical Experiment on the Zeros of $P_n^{(\lambda)}(x; \delta)$ , $\delta \in \mathbb{R}$

We now validate the above results related to the zeros of Meixner–Pollaczek polynomials by considering pictorial representations of the first few polynomials. Let  $\delta = \cot \phi \in \mathbb{R}$  and  $\phi \in (0, \pi)$ , the first few polynomials  $P_n^{(\lambda)}(x; \delta)$  are obtained from Equation (13) using symbolic packages (Maple) as follows.

$$\begin{aligned} P_0^{(\lambda)}(x; \delta) &= 1; \\ P_1^{(\lambda)}(x; \delta) &= x + \delta \lambda; \\ P_2^{(\lambda)}(x; \delta) &= x^2 + (\delta \lambda + \lambda + 1)x - 2\delta^2 \lambda + \delta \lambda^2 + \delta \lambda - 2 \lambda; \\ P_3^{(\lambda)}(x; \delta) &= x^3 + (\delta \lambda + 2 \lambda + 3)x^2 + \left( -6 \delta^2 \lambda + 2 \delta \lambda^2 - 2 \delta^2 + 3 \delta \lambda + \lambda^2 - 3 \lambda \right) x \\ &\quad - 4 \delta^3 \lambda^2 - 2 \delta^3 \lambda - 2 \delta^2 \lambda^2 + \delta \lambda^3 - 4 \delta^2 \lambda - \delta \lambda^2 - 2 \lambda^2 - 4 \lambda. \end{aligned}$$

Let's consider the following cases:

**Case I: When  $\phi = \frac{\pi}{6}$ , ( $\delta = \cot(\frac{\pi}{6}) = \sqrt{3} \approx 1.73$ ) and  $\lambda = 0.55$ .**

The first few Meixner–Pollaczek polynomials in this case are given by

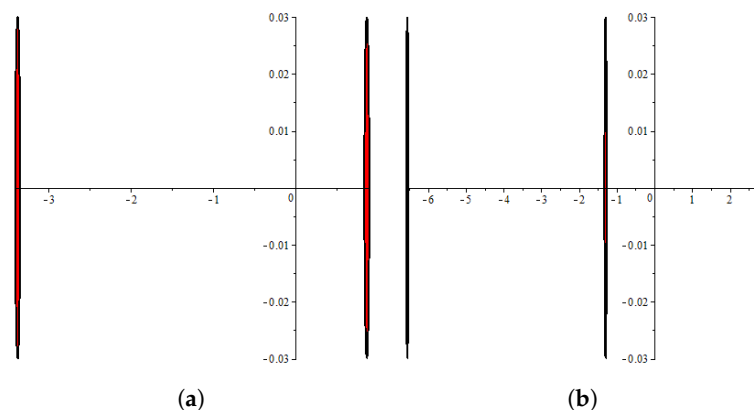
$$\begin{aligned}
 P_1^{(0.55)}(x; \sqrt{3}) &= x + 0.95262, \\
 P_2^{(0.55)}(x; \sqrt{3}) &= x^2 + 2.502627944x - 2.923426687, \\
 P_3^{(0.55)}(x; \sqrt{3}) &= x^3 + 5.052627944x^2 - 13.34172543x, \\
 P_4^{(0.55)}(x; \sqrt{3}) &= x^4 + 8.602627944x^3 - 32.60489624x^2 - 163.9197723x + 25.47242210, \\
 P_5^{(0.55)}(x; \sqrt{3}) &= x^5 + 13.15262794x^4 - 59.06293911x^3 - 643.7244434x^2 + 154.8546466x + 1654.802543,
 \end{aligned}$$

and their corresponding real zeros are tabulated as follows.

Table 1 and Figure 1 show that the zeros of  $\{P_n^{(0.55)}(x; \sqrt{3})\}_{n=1}^5$  are real and simple, which confirms the classical result for  $\phi \in (0, \pi)$  and  $\lambda > 0$  [6].

**Table 1.** Real zeros for  $P_n^{(\lambda)}(x; \delta)$  for  $\lambda = 0.55$  and  $\delta = \sqrt{3}$ .

$P_n^{(0.55)}(x; \sqrt{3})$	Corresponding (Real) Zeros
$P_1^{(0.55)}(x; \sqrt{3})$	-0.95262
$P_2^{(0.55)}(x; \sqrt{3})$	[-3.370090351, 0.8674624068]
$P_3^{(0.55)}(x; \sqrt{3})$	[-6.5436518550, -1.2893767698, 2.7804006808]
$P_4^{(0.55)}(x; \sqrt{3})$	[-10.1996365410, -3.40704030, 0.1510418707, 4.853007027]
$P_5^{(0.55)}(x; \sqrt{3})$	[-14.088331505, -6.1197704981, -1.6483271898, 1.650996725, 7.052804527]



**Figure 1.** Plots for the real zeros of  $P_n^{(\lambda)}(x; \delta)$  (with  $\lambda = 0.55$ , and  $\delta = \sqrt{3}$ ) for  $n = 2, 3$ . (a) Plots for the (real) zeros of  $P_2^{(0.55)}(x; \sqrt{3})$ . (b) Plots for the (real) zeros of  $P_3^{(0.55)}(x; \sqrt{3})$ .

**Case II: When  $\phi = \frac{\pi}{8} \in [0, \frac{\pi}{4}]$ , ( $\delta = \cot(\frac{\pi}{8}) = 1 + \sqrt{2} \approx 2.41$ ) and  $\lambda = -2.75$ .**

For  $\lambda < 0$  and  $\delta > 0$ , we see that real orthogonality fails as complex zeros appear in conjugate pairs for the first few polynomials. For extended orthogonality, see [20] for more details. The first few monic polynomials for case II are given by

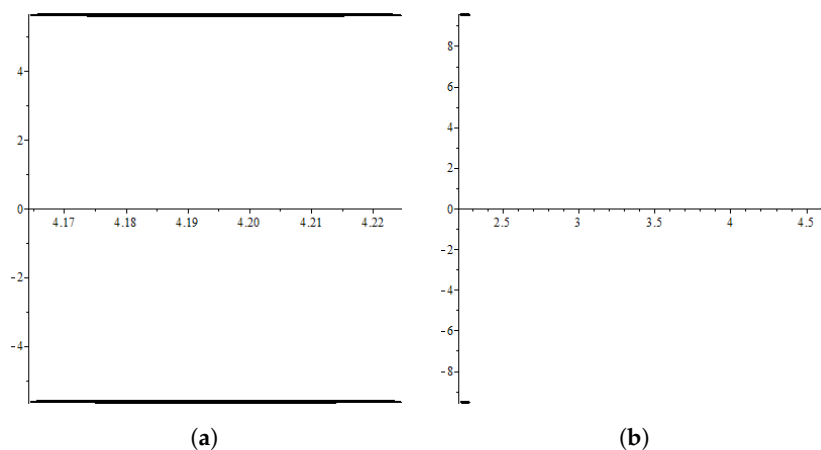
$$\begin{aligned}
 P_1^{(0.55)}(x; 2.41) &= x - 6.6275, \\
 P_2^{(-2.75)}(x; 2.41) &= x^2 - 8.389087296x + 49.17475195, \\
 P_3^{(-2.75)}(x; 2.41) &= x^3 - 9.139087296x^2 + 116.9224115x, \\
 P_4^{(-2.75)}(x; 2.41) &= x^4 - 8.889087296x^3 + 186.3361245x^2 - 1017.146023x + 3414.532260,
 \end{aligned}$$

and their corresponding zeros with plots in the complex plane are given as follows.

Figure 2 demonstrates the pictorial representation of the complex zeros of  $P_n^{(-2.75)}(x; 2.41)$  for  $n = 2, 3$ , whereas Table 2 shows that the zeros of the polynomials  $\{P_n^{(-2.75)}(x; 2.41)\}_{n=1}^4$  exhibit one real and remaining complex zeros in conjugate pairs except  $P_1^{(-2.75)}(x; 2.41)$ . This may likely suggest that for  $\lambda < 0$  and  $\delta > 0$ , complex zeros appear in conjugate pairs for  $n > 1$ .

**Table 2.** The zeros for  $P_n^{(\lambda)}(x; \delta)$  when  $\lambda = -2.75$  and  $\delta = 2.41$ .

$P_n^{(-2.75)}(x; 2.41)$	Corresponding (Real/Complex) Zeros
$P_1^{(-2.75)}(x; 2.41)$	6.6275
$P_2^{(-2.75)}(x; 2.41)$	[4.194543648 – 5.619657955I, 4.194543648 + 5.619657955I]
$P_3^{(-2.75)}(x; 2.41)$	[2.253498955 – 9.537679842I, 2.2534989556 + 9.5376798424I, 4.6320893848]
$P_4^{(-2.75)}(x; 2.41)$	[1.188253595 – 12.0969643912I, 1.1882535946 + 12.096964391I, 3.2562900534 – 3.5365254217I, 3.25629005341 + 3.5365254217I]



**Figure 2.** Plots for the complex zeros of  $P_n^{(\lambda)}(x; \delta)$  (with  $\lambda = -2.75$  and  $\delta \approx 2.41$ ) for  $n = 2, 3$ . (a) Plots for the complex zeros of  $P_2^{(-2.75)}(x; 2.41)$ . (b) Plots for the complex zeros of  $P_3^{(-2.75)}(x; 2.41)$ .

**Case III:** When  $\phi = -\frac{\pi}{3} \in [-\frac{\pi}{2}, 0]$ , ( $\delta = \cot(-\frac{\pi}{3}) = 0.577$ ),  $\lambda = -3.67$ .

The first few polynomials in this case are

$$P_1^{(-3.67)}(x; -0.577) = x + 3.8412,$$

$$P_2^{(-3.67)}(x; -0.577) = x^2 - 0.551124512x + 4.129269119,$$

$$P_3^{(-3.67)}(x; -0.577) = x^3 - 2.221124512x^2 + 21.95631373x + 28.92724218,$$

$$P_4^{(-3.67)}(x; -0.577) = x^4 - 2.891124512x^3 + 44.80446716x^2 + 2.44449240x + 68.81993616,$$

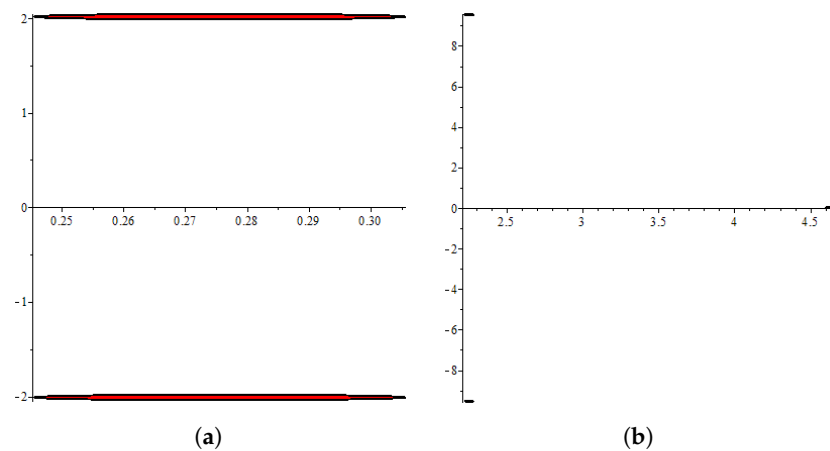
and their corresponding zeros are given as follows.

Table 3 shows that all the zeros of  $\{P_n^{(-3.67)}(x; -0.577)\}_{n=1}^4$  are complex in conjugate pairs, and plots for the complex zeros of  $P_n^{(-3.67)}(x; -0.577)$  for  $n = 2, 3$  are given below in Figure 3.



**Table 3.** Complex zeros for  $P_n^{(\lambda)}(x; \delta)$  for  $\lambda = -3.67$  and  $\delta = -0.577$ .

$P_n^{(-3.67)}(x; -0.577)$	Corresponding Zeros
$P_2^{(-3.67)}(x; -0.577)$	$[4.194543648 - 5.619657955I, 4.194543648 + 5.619657955I]$
$P_3^{(-3.67)}(x; -0.577)$	$[2.25349895559073 - 9.53767984236043I, 2.25349895559073 + 9.53767984236043I, 4.63208938481853]$
$P_4^{(-3.67)}(x; -0.577)$	$[1.188253590 - 12.09696440I, 1.1882536 + 12.0969644I, 3.256290053 - 3.53652542I, 3.25629005 + 3.5365254I]$



**Figure 3.** Plots for complex zeros of  $P_n^{(\lambda)}(x; \delta)$  (with  $\lambda = -3.67$  and  $\delta \approx -0.577$ ) for  $n = 2, 3$ . **(a)** Plots for the complex zeros of  $P_2^{(-3.67)}(x; -0.577)$ . **(b)** Plots for the complex zeros of  $P_3^{(-3.67)}(x; -0.577)$ .

**Remark 1.** The above numerical experiments elaborate how the restriction of parameter values influence real orthogonality and these numerical findings also likely verify the results given in Lemma 1 and Theorem 1.

**2. Relation between the Monic Polynomials  $P_n^{(\lambda)}$  and  $Q_n^{(\lambda, \varphi)}$**

It is known that classical orthogonal polynomials, namely the polynomials of Jacobi, Laguerre, and Hermite, obey numerous well-known properties corresponding to their several explicit relations [3]; nevertheless, when the conditions on such relations are less restricted, semi-classical (non-classical) orthogonal polynomials [21] are obtained. For mathematical completeness and applications of polynomials in numerous fields, one requires polynomials that are orthogonal with respect to shifting of the weight function in transcendental forms. For semi-classical measure modification from classical weights, we refer to some works [21–24].

It is known that the classical polynomial  $P_n^{(\lambda)}$  is orthogonal with respect to the weight [1,3]

$$w(x; \lambda, \phi) = \frac{1}{2\pi} e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2, \quad x \in \mathbb{R}, \quad \lambda > 0, \quad \phi \in (0, \pi). \tag{17}$$

However, the polynomial  $Q_n^{(\lambda, \varphi)}$  is orthogonal with the weight in Equation (17) perturbed by  $e^{-ax}$ . This perturbation leads to the phase shift from phase  $\varphi$  to  $(\varphi - \frac{at}{2})$ , which likely turns out to guarantee certain shared properties such as orthogonality, three-term recurrence relation, generating functions, etc. In this sense,  $P_n^{(\lambda)}$  and  $Q_n^{(\lambda, \varphi)}$  behave like the same polynomials with different parameters involved in their respective weight function as parameter restrictions in the weight greatly affect some properties of the corresponding

polynomials; for e.g., certain properties of the zeros (such as monotonicity, convexity, quasi-orthogonality, etc.), concise formulation of the recurrence coefficients, etc are some that may deviate as shown in literature [22,25–27]. This work also signifies the need for time-dependent orthogonal polynomials, mainly in terms of their practical applications. We believe that there are few works in related literature that treated certain properties and applications of perturbed classical weights and we hope this work would then contribute to filling this gap.

### 3. Main Results of the Perturbed Meixner–Pollaczek Weight

#### 3.1. Finite Moments

It is shown in Proposition 2 that the moments of the Meixner–Pollaczek measure are finite. We now present a result proving the finiteness of moments of the perturbed Meixner–Pollaczek measure.

**Theorem 2.** *Suppose  $a > 0, t > 0, \varphi \in (0, \pi)$  and  $x \in \mathbb{R}$ . The moments  $\eta_j(t; \varphi)$  associated with the weight Equation (8) are finite of all orders.*

**Proof.** For the weight given in Equation (8), the moments  $\eta_j(t; \varphi)$  take the form

$$\eta_k(w^{(\lambda, \varphi)}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^k e^{(2\varphi - \pi)x} |\Gamma(\lambda + ix)|^2 e^{-axt} dx, \quad k \in \mathbb{N}_0. \tag{18}$$

Now, using the fact that  $\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} [f(x) + f(-x)] dx$ , Equation (18) gives

$$\eta_k(w^{(\lambda, \varphi)}) = \frac{1}{2\pi} \int_0^{\infty} x^k |\Gamma(\lambda + ix)|^2 \left( e^{(2\varphi - \pi - at)x} + (-1)^k e^{-(2\varphi - \pi - at)x} \right) dx, \quad k \in \mathbb{N}_0. \tag{19}$$

From Stirling’s approximation (cf. [28]) for the complex Gamma function, we have

$$\Gamma(z) \approx \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z,$$

and from the fact that  $\Gamma(z)$  is a holomorphic function for  $\Re(z) > 0, \Gamma(\bar{z}) = \overline{\Gamma(z)}$ , we obtain

$$|\Gamma(z)|^2 = \Gamma(z)\Gamma(\bar{z}) \approx \frac{2\pi}{\sqrt{z\bar{z}}} \left(\frac{z}{e}\right)^z \left(\frac{\bar{z}}{e}\right)^{\bar{z}} = \frac{2\pi}{|z|} \left(\frac{z}{e}\right)^z \left(\frac{\bar{z}}{e}\right)^{\bar{z}}.$$

By employing  $z = re^{i\theta} = \lambda + ix$ , we have

$$|\Gamma(z)|^2 \approx \frac{2\pi}{r} (re^{-1+i\theta})^z (re^{-1-i\theta})^{\bar{z}} = 2\pi r^{z+\bar{z}-1} \exp(z(-1+i\theta) - \bar{z}(1+i\theta)).$$

Using  $z + \bar{z} = 2\lambda, z - \bar{z} = 2ix$  and  $r^2 = \lambda^2 + x^2$ , we obtain

$$|\Gamma(z)|^2 \approx 2\pi (\lambda^2 + x^2)^{\frac{(2\lambda-1)}{2}} \exp(-2\lambda - 2x\theta).$$

By assuming that  $x \gg \lambda$  &  $x \gg 1$ , we have  $\lambda^2 + x^2 \approx x^2$  and using  $\theta \approx \frac{\pi}{2}$  gives

$$|\Gamma(z)|^2 = |\Gamma(\lambda + ix)|^2 \approx 2\pi x^{2\lambda-1} \exp(-\pi x),$$

in which the term  $2\lambda$  in the argument of the exponential vanishes since  $2\lambda$  is negligible compared to  $\pi x$ . Since  $2 \cosh x = e^x + e^{-x} \approx e^x$  for large  $x$ , we finally attain that

$$|\Gamma(\lambda + ix)|^2 \approx \frac{\pi x^{2\lambda-1}}{\cosh(\pi x)}. \tag{20}$$

Substituting Equation (20) into Equation (19) yields

$$\eta_k(w^{(\lambda,\varphi)}) \approx \frac{1}{2\pi} \int_0^\infty x^k \frac{\pi x^{2\lambda-1}}{\cosh(\pi x)} \left( e^{(2\varphi-\pi-at)x} + (-1)^k e^{-(2\varphi-\pi-at)x} \right) dx \approx \frac{1}{2} \int_0^\infty \frac{x^{k+2\lambda-1}}{\cosh(\pi x)} \left( e^{Mx} + (-1)^k e^{-Mx} \right) dx, \tag{21}$$

where  $M := 2\varphi - \pi - at$ , with  $M < 0$  for the weight to be defined.

By using Equation (21), the even and odd moments are given as follows.

(i) The even moments ( $\eta_{2n}$ ):

$$\eta_{2n}(w^{(\lambda,\varphi)}) \approx \frac{1}{2} \int_0^\infty \frac{x^{2n+2\lambda-1}}{\cosh(\pi x)} \left( e^{Mx} + e^{-Mx} \right) dx \approx \frac{1}{2} \int_0^\infty x^{2n+2\lambda-1} \frac{2 \cosh(Mx)}{\cosh(\pi x)} dx, \tag{22}$$

By employing the following cosh inequality:  $\frac{e^x - e^{-x}}{2} = \cosh x \leq e^{\frac{x^2}{2}}, \forall x \in \mathbb{R}$ , we write

$$\frac{\cosh(Mx)}{\cosh(\pi x)} \leq \frac{e^{M^2 x^2 / 2}}{e^{\pi^2 x^2 / 2}} = e^{(M^2 - \pi^2)x^2 / 2},$$

so that Equation (22) reduces to

$$\eta_{2n}(w^{(\lambda,\varphi)}) \approx \frac{1}{2} \int_0^\infty x^{2n+2\lambda-1} \frac{2 \cosh(Mx)}{\cosh(\pi x)} dx \leq \int_0^\infty x^{2n+2\lambda-1} e^{(M^2 - \pi^2)x^2 / 2} dx = \frac{1}{(M^2 - \pi^2)^{(n+\lambda)}} \Gamma(n + \lambda) < \infty. \tag{23}$$

(ii) Similarly, for the odd moments ( $\eta_{2n+1}$ ), we use the following sinh inequality:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \leq \frac{e^x}{2}, \text{ as } e^{-x} > 0, \forall x \in \mathbb{R},$$

to obtain

$$\begin{aligned} \eta_{2n+1}(w^{(\lambda,\varphi)}) &\approx \frac{1}{2} \int_0^\infty \frac{x^{2n+2\lambda}}{\cosh(\pi x)} \left( e^{Mx} - e^{-Mx} \right) dx = \int_0^\infty x^{2n+2\lambda} \frac{\sinh(Mx)}{\cosh(\pi x)} dx, \\ &\leq \frac{1}{2} \int_0^\infty x^{2n+2\lambda} e^{Mx - \frac{\pi^2 x^2}{2}} dx = \frac{1}{2} \int_0^\infty x^{2n+2\lambda} e^{-\frac{\pi^2}{2} \left( -\frac{M}{\pi^2} + \left(x - \frac{M}{\pi^2}\right)^2 \right)} dx < \infty. \end{aligned} \tag{24}$$

Thus, from Equations (22)–(24), we see that the moments associated with the weight in Equation (8) are finite of all orders.  $\square$

### 3.2. Orthogonality and Generating Function

We now present some result related to orthogonality of the perturbed Meixner–Pollaczek weight given in (8).

**Proposition 3.** Let  $\lambda > 0, t > 0$  and  $\varphi > 0$ . The orthogonality relation of the monic perturbed Meixner–Pollaczek polynomials is given by

$$\mathcal{L}[x^m \mathcal{Q}_n^{(\lambda,\varphi)}(x;t)] = \int_{-\infty}^\infty \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) x^m w^{(\lambda,\varphi)}(x;t) dx = \zeta_n^{(\lambda,\varphi)}(t) \delta_{n,m}, \quad n, m \in \mathbb{N}_0, \tag{25}$$

where the weight  $w^{(\lambda,\varphi)}(x;t)$  is as given in Equation (6) with

$$\zeta_n^{(\lambda,\varphi)}(t) = \mathcal{L}\left[x^n \mathcal{Q}_n^{(\lambda,\varphi)}(x;t)\right] \neq 0, \quad n \geq 0. \tag{26}$$

**Proof.** The result immediately follows from Lemma 1 together with fact that the parameter  $t$ , which likely leads to shifting the phase  $\varphi$  to  $\varphi - \frac{at}{2}$ . Equation (26) also follows from the positivity condition of the coefficient  $\beta_n^{(\lambda,\varphi)}(t) > 0$  of the recurrence relation for orthogonality to occur [3]. The constant  $\zeta_n^{(\lambda,\varphi)}(t), n \geq m \geq 0$ , takes the form

$$\zeta_n^{(\lambda,\varphi)}(t) = \int_{-\infty}^\infty \left(\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)\right)^2 w^{(\lambda,\varphi)}(x;t) dx = \zeta_m^{(\lambda,\varphi)}(t) \prod_{j=m+1}^n \beta_j(t) = \zeta_0^{(\lambda,\varphi)}(t) \prod_{j=1}^n \beta_j(t), \tag{27}$$

where  $\zeta_0^{(\lambda,\varphi)} > 0$ ; in particular,  $\zeta_0^{(\lambda,\varphi)} = 1$ . We see, for  $\lambda > 0$ , that

$$\zeta_n^{(\lambda,\varphi)}(t) = \prod_{j=1}^n \beta_j(t) = \prod_{j=1}^n \frac{1}{4} j(j+2\lambda-1) \csc^2\left(\varphi - \frac{at}{2}\right) = (n!) \left(\frac{1}{2} \csc\left(\varphi - \frac{at}{2}\right)\right)^{2n} \prod_{j=1}^n (j+k) > 0,$$

with  $k = 2\lambda - 1$ , and using the fact that  $\prod_{j=1}^n j = n!$  and  $\prod_{j=1}^n j^2 = (n!)^2$  and hence the result holds.  $\square$

It now follows that the sequence of monic polynomials  $\{\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)\}_{n=0}^\infty$  obey the three-term recurrence relation

$$x\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) = \mathcal{Q}_{n+1}^{(\lambda,\varphi)}(x;t) + \alpha_n^{(\lambda,\varphi)}(t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) + \beta_n^{(\lambda,\varphi)}(t) \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t), \quad n \geq 1, \quad (28)$$

with initial conditions  $\mathcal{Q}_{-1}^{(\lambda,\varphi)} = 0$ ;  $\mathcal{Q}_0^{(\lambda,\varphi)} = 1$ , where the recurrence coefficients are given by

$$\begin{cases} \alpha_n(t) := \alpha_n^{(\lambda,\varphi)}(t) = -(\lambda + n) \cot\left(\varphi - \frac{at}{2}\right), \\ \beta_n(t) := \beta_n^{(\lambda,\varphi)}(t) = \frac{1}{4} n(n + 2\lambda - 1) \csc^2\left(\varphi - \frac{at}{2}\right). \end{cases} \quad (29)$$

**Lemma 3.** Let  $\lambda > 0$ ,  $a > 0$ ,  $0 \leq t < \frac{2\varphi}{a}$ , fixed. The following holds for the monic perturbed Meixner-Pollaczek polynomials  $\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)$ :

(i) The generating function

$$\sum_{n=0}^\infty \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) s^n = (1 - e^{i(\varphi - \frac{at}{2})} s)^{-\lambda + ix} (1 - e^{-i(\varphi - \frac{at}{2})} s)^{-\lambda - ix}, \quad |e^{\pm i(\varphi - \frac{at}{2})} s| < 1, \quad (30)$$

(ii) The hypergeometric representation

$$\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) = \frac{e^{in(\varphi - \frac{at}{2})} (2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i(\varphi - \frac{at}{2})}\right). \quad (31)$$

**Proof.** (i) This result follows from the modification of the weight

$$W^{(\lambda,\varphi)}(x) := e^{(2\varphi - \pi)x} |\Gamma(\lambda + ix)|^2 \longrightarrow W^{(\lambda,\varphi)}(x) e^{-axt} := w^{(\lambda,\varphi)}(x;t),$$

which leads to the modification  $P_n^{(\lambda)}(x;\varphi) \rightarrow P_n^{(\lambda)}(x;\varphi - \frac{at}{2}) := \mathcal{Q}_n^{(\lambda,\varphi)}(x;t)$ , and hence Equation (30) is immediate from the hypergeometric formulation of Meixner–Pollaczek polynomials ([1], Section 9.7).

(ii) In order to prove the result in Equation (31), we employ the generating function (30) together with the identity ([29], p. 82)

$$(1 - u)^{a-b} (1 - u + uz)^{-a} = \sum_{n=0}^\infty \frac{(b)_n}{n!} {}_2F_1\left(\begin{matrix} -n, a \\ b \end{matrix} \middle| z\right) u^n,$$

with  $u = se^{i(\varphi - \frac{at}{2})}$ ,  $a = \lambda + ix$ ,  $b = 2\lambda$ ,  $z = 1 - e^{-2i(\varphi - \frac{at}{2})}$  to obtain

$$\left(1 - e^{i(\varphi - \frac{at}{2})} s\right)^{-\lambda + ix} \left(1 - e^{-i(\varphi - \frac{at}{2})} s\right)^{-\lambda - ix} = \sum_{n=0}^\infty \frac{e^{i(n+1)(\varphi - \frac{at}{2})} (2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i(\varphi - \frac{at}{2})}\right) s^n,$$

and later comparing the coefficients of the power series of both sides to arrive at the desired result.

$\square$

### 3.3. Concise Formulation

In the sequel, we use Lemma 3 to obtain concise formulations of the perturbed Meixner–Pollaczek polynomials.

**Theorem 3.** Let  $\lambda > 0, a > 0, 0 \leq t < \frac{2\varphi}{a}$ , fixed. The following formulations hold for the monic perturbed Meixner–Pollaczek polynomials  $\mathcal{Q}_n^{(\lambda, \varphi)}(x; t)$ :

$$\begin{aligned}
 (i) \quad \mathcal{Q}_n^{(\lambda, \varphi)}(x; t) &= (-1)^n e^{in(\varphi - \frac{at}{2})} \sum_{k=0}^n \binom{-\lambda + ix}{n-k} \binom{-\lambda - ix}{k} e^{-2ik(\varphi - \frac{at}{2})} \\
 &= \sum_{k=0}^n \binom{n}{k} \frac{(\lambda - ix)_{n-k} (\lambda + ix)_k}{n!} e^{i(n-2k)(\varphi - \frac{at}{2})}, \\
 (ii) \quad \mathcal{Q}_n^{(\lambda, \varphi)}(x; t) &= \sum_{\ell=0}^n \frac{1}{n!} \binom{n}{\ell} (\lambda + ix)_{n-\ell} (2\lambda + n - \ell)_\ell e^{i(n-\ell)\varphi} e^{-\frac{i(n-\ell)at}{2}} \left[ e^{-2i(\varphi - \frac{at}{2})} - 1 \right]^{n-\ell} e^{i(\varphi - \frac{at}{2})\ell}.
 \end{aligned}$$

**Proof.** (i) The proof for (i) uses generalized binomial Theorem

$$(1 + x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n$$

on the generating function in Equation (30) and applying Cauchy’s product of the series by using the identity

$$\binom{-a}{n} = \frac{(-1)^n (a)_n}{n!}, \quad a \in \mathbb{C},$$

where  $(a)_n$  is the Pochhammer symbol given in Equation (3).

(ii) By considering ([1], Equation (1.7.11)) and upon some rearrangement as in ([10], p. 172), the generating function takes the form

$$\sum_{n=0}^\infty P_n^{(\lambda)}(x; \varphi) s^n = \sum_{n=0}^\infty \frac{(\lambda + ix)_k}{k!} e^{ik\varphi} (e^{-2i\varphi} - 1)^k (1 - se^{i\varphi})^{-2\lambda-k} s^k, \quad (32)$$

where  $P_n^{(\lambda)}(x; \varphi)$  is the Meixner–Pollaczek polynomial. Since  $\mathcal{Q}_n^{(\lambda, \varphi)}(x; t) := P_n^{(\lambda)}(x, \varphi - \frac{at}{2})$ , it follows that

$$\begin{aligned}
 \sum_{n=0}^\infty \mathcal{Q}_n^{(\lambda, \varphi)}(x; t) s^n &= \sum_{n=0}^\infty \frac{(\lambda + ix)_k}{k!} e^{ik(\varphi - \frac{at}{2})} (e^{-2i(\varphi - \frac{at}{2})} - 1)^k (1 - se^{i(\varphi - \frac{at}{2})})^{-2\lambda-k} s^k \\
 &= \sum_{n=0}^\infty \frac{(\lambda + ix)_k}{k!} e^{ik\varphi} e^{-\frac{ikat}{2}} (e^{-2i(\varphi - \frac{at}{2})} - 1)^k (1 - se^{i(\varphi - \frac{at}{2})})^{-2\lambda-k} s^k. \quad (33)
 \end{aligned}$$

Expanding  $(1 - se^{i(\varphi - \frac{at}{2})})^{-2\lambda-k}$  using Pochhammer’s identity  $(-\lambda)_k = (-1)^k \frac{(\lambda)_k}{k!}$  gives

$$\begin{aligned}
 (1 - se^{i(\varphi - \frac{at}{2})})^{-2\lambda-k} &= \sum_{\ell=0}^\infty \binom{-2\lambda-k}{\ell} (-se^{i(\varphi - \frac{at}{2})})^\ell \\
 &= \sum_{\ell=0}^\infty \frac{(-1)^\ell (2\lambda + k)_\ell}{\ell!} (-1)^\ell e^{i(\varphi - \frac{at}{2})\ell} s^\ell = \sum_{\ell=0}^\infty \frac{(2\lambda + k)_\ell}{\ell!} e^{i\varphi\ell} e^{-\frac{iata\ell}{2}} s^\ell. \quad (34)
 \end{aligned}$$

By substituting Equations (34) into (33) and using the summation identity

$$\sum_{n=0}^\infty \sum_{k=0}^\infty f(k, n) = \sum_{n=0}^\infty \sum_{k=0}^n f(k, n - k),$$

we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{(\lambda, \varphi)}(x; t) s^n &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(\lambda + ix)_k (2\lambda + k)_\ell}{k! \ell!} e^{ik\varphi} e^{-\frac{ikat}{2}} \left[ e^{-2i(\varphi - \frac{at}{2})} - 1 \right]^k e^{i\varphi\ell} e^{-\frac{i\ell t}{2}} s^\ell s^k \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \frac{(\lambda + ix)_{k-\ell} (2\lambda + k - \ell)_\ell}{(k - \ell)! \ell!} e^{i(k-\ell)\varphi} e^{-\frac{i(k-\ell)at}{2}} \left[ e^{-2i(\varphi - \frac{at}{2})} - 1 \right]^{(k-\ell)} e^{i\varphi\ell} e^{-\frac{i\ell t}{2}} s^k. \end{aligned} \tag{35}$$

By writing  $n$  instead of  $k$  in Equation (35), we may write

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{(\lambda, \varphi)}(x; t) s^n &= \sum_{n=0}^{\infty} \sum_{\ell=0}^n \frac{(\lambda + ix)_{n-\ell} (2\lambda + n - \ell)_\ell}{(n - \ell)! \ell!} e^{i(n-\ell)\varphi} e^{-\frac{i(n-\ell)at}{2}} \left[ e^{-2i(\varphi - \frac{at}{2})} - 1 \right]^{n-\ell} e^{i\varphi\ell} e^{-\frac{i\ell t}{2}} s^n \\ &= \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^n \frac{1}{n!} \binom{n}{\ell} (\lambda + ix)_{n-\ell} (2\lambda + n - \ell)_\ell e^{i(n-\ell)\varphi} e^{-\frac{i(n-\ell)at}{2}} \left[ e^{-2i(\varphi - \frac{at}{2})} - 1 \right]^{n-\ell} e^{i\varphi\ell} e^{-\frac{i\ell t}{2}} \right) s^n. \end{aligned} \tag{36}$$

Thus, the required result follows by comparing the coefficients of  $s$  on both sides of the last equality.

□

### 3.4. Some New Recursive Relations

In this Subsection, let's now denote, for notational convenience, the perturbed Meixner–Pollaczek polynomials by  $Q_n^{(\lambda, a)}(x; \varphi, t)$  in order to show the role of the parameters in Equation (8). We may also sometimes omit some parameters for simplicity. We can now state one of our main results giving new recursive relations fulfilled by the perturbed polynomials using hypergeometric identities.

**Theorem 4.** *Let  $a > 0, \varphi > 0$  and  $t > 0$ . Then the following recursive relations hold for monic perturbed Meixner–Pollaczek polynomials  $Q_n^{(\lambda, a)}(x; \varphi, t)$ :*

(i)

$$2i(\lambda + ix) \sin \varphi Q_n^{(\lambda + \frac{1}{2}, a)}\left(x - \frac{1}{2}i; \varphi, t\right) = e^{i\varphi}(n + 2\lambda) Q_n^{(\lambda, a)}(x; \varphi, t) - 2 \sin \varphi Q_{n+1}^{(\lambda, a)}(x; \varphi, t), \tag{37}$$

(ii)

$$e^{i\varphi} Q_n^{(\lambda + \frac{1}{2}, a)}\left(x - \frac{1}{2}i; \varphi, t\right) - e^{i\varphi} Q_n^{(\lambda, a)}(x; \varphi, t) = \frac{n}{2 \sin \varphi} Q_{n-1}^{(\lambda + \frac{1}{2}, a)}\left(x - \frac{1}{2}i; \varphi, t\right). \tag{38}$$

**Proof.** (i) In order to prove the result in (37), let's rewrite the monic perturbed Meixner–Pollaczek polynomials

$$Q_n^{(\lambda, a)}(x; \varphi, t) = \frac{n!}{(2 \sin \varphi)^n} P_n^{(\lambda)}\left(x; \varphi - \frac{at}{2}\right) = \frac{(2\lambda)_n}{(2 \sin \varphi)^n} e^{in\varphi} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi}\right). \tag{39}$$

Now, by using the  ${}_2F_1$ -hypergeometric formulation given in Equation (39), we rewrite Equation (37) as

$$\begin{aligned} &2i(\lambda + ix) \sin \varphi \left\{ \frac{(2\lambda + 1)_n}{(2 \sin \varphi)^n} e^{in\varphi} {}_2F_1\left(\begin{matrix} -n, \lambda + ix + 1 \\ 2\lambda + 1 \end{matrix} \middle| 1 - e^{-2i\varphi}\right) \right\} \\ &= e^{i\varphi}(n + 2\lambda) \left\{ \frac{(2\lambda)_n}{(2 \sin \varphi)^n} e^{in\varphi} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi}\right) \right\} \\ &\quad - 2 \sin \varphi \left\{ \frac{(2\lambda)_{n+1}}{(2 \sin \varphi)^{n+1}} e^{in\varphi} e^{i\varphi} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi}\right) \right\}. \end{aligned} \tag{40}$$

Simplifying Equation (40) and using the identities in ([30], Theorem 9.2)

$$\left. \begin{aligned} (2\lambda + 1)_n &= \frac{n+2\lambda}{2\lambda} (2\lambda)_n, \\ (2\lambda)_n &= \frac{(2\lambda)_{n+1}}{n+2\lambda}, \\ \sin \varphi &= \frac{1}{2i} (e^{i\varphi} - e^{-i\varphi}), \end{aligned} \right\} \tag{41}$$

we obtain the following relations:

$$\begin{aligned} &(\lambda + ix) (1 - e^{-2i\varphi}) (2\lambda + 1)_n {}_2F_1 \left( \begin{matrix} -n, \lambda + ix + 1 \\ 2\lambda + 1 \end{matrix} \middle| 1 - e^{-2i\varphi} \right) \\ &= (n + 2\lambda) (2\lambda)_n {}_2F_1 \left( \begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi} \right) - (2\lambda)_{n+1} {}_2F_1 \left( \begin{matrix} -n - 1, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi} \right). \end{aligned} \tag{42}$$

Thus, the result in Equation (37) immediately follows from Equation (42) together with the  ${}_2F_1$ -contagious hypergeometric identity (cf. [31], Equation (2.11))

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = {}_2F_1 \left( \begin{matrix} a - 1, b \\ c \end{matrix} \middle| z \right) + \frac{bz}{c} {}_2F_1 \left( \begin{matrix} a, b + 1 \\ c + 1 \end{matrix} \middle| z \right), \tag{43}$$

where  $a = -n, b = \lambda + ix$ , and  $c = 2\lambda$ .

- (ii) To prove the second, we rewrite the left hand side of Equation (38), using Equation (39), to obtain

$$\begin{aligned} &e^{i\varphi} \mathcal{Q}_n^{(\lambda + \frac{1}{2}, a)} \left( x - \frac{1}{2}i; \varphi, t \right) - e^{i\varphi} \mathcal{Q}_n^{(\lambda, a)} (x; \varphi, t) \\ &= \frac{e^{i(n+1)\varphi}}{(2 \sin \varphi)^n} \left\{ (2\lambda + 1)_n {}_2F_1 \left( \begin{matrix} -n, \lambda + ix + 1 \\ 2\lambda + 1 \end{matrix} \middle| 1 - e^{-2i\varphi} \right) - (2\lambda)_{n+1} {}_2F_1 \left( \begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\varphi} \right) \right\}. \end{aligned} \tag{44}$$

Besides, the right hand side of Equation (38) also takes the form

$$\begin{aligned} \frac{n}{2 \sin \varphi} \mathcal{Q}_{n-1}^{(\lambda + \frac{1}{2}, a)} \left( x - \frac{1}{2}i; \varphi, t \right) &= \frac{n}{2 \sin \varphi} \left\{ \frac{(2\lambda + 1)_{n-1}}{(2 \sin \varphi)^{n-1}} e^{i(n-1)\varphi} {}_2F_1 \left( \begin{matrix} -n, \lambda + ix + 1 \\ 2\lambda + 1 \end{matrix} \middle| 1 - e^{-2i\varphi} \right) \right\} \\ &= \frac{n}{(2 \sin \varphi)^n} (2\lambda + 1)_{n-1} e^{i(n-1)\varphi} {}_2F_1 \left( \begin{matrix} -n, \lambda + ix + 1 \\ 2\lambda + 1 \end{matrix} \middle| 1 - e^{-2i\varphi} \right). \end{aligned} \tag{45}$$

We now see that the result in Equation (38) follows by combining Equations (44) and (45) together with the  ${}_2F_1$  hypergeometric contagious identity (cf. ([31], Equation (2.6))

$$(a - c + 1) {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right) + (c - 1) {}_2F_1 \left( \begin{matrix} a, b - 1 \\ c - 1 \end{matrix} \middle| z \right) - a(1 - z) {}_2F_1 \left( \begin{matrix} a + 1, b \\ c \end{matrix} \middle| z \right) = 0, \tag{46}$$

where  $a = -n, b = \lambda + ix + 1, c = 2\lambda + 1$  and  $z = 1 - e^{-2i\varphi}$ .

□

Our next proposition gives some properties of the perturbed Meixner–Pollaczek polynomials.

### 3.5. Addition Formulation and Integral Representation

**Proposition 4.** Let  $\lambda > 0, a > 0$  and  $0 \leq t < \frac{2\varphi}{a}$ . The following properties hold for  $\mathcal{Q}_n^{(\lambda, \varphi)} (x; t)$ :

(i) Addition formulation

$$Q_n^{(\alpha+\beta,\varphi)}(x+y;t) = \sum_{k=0}^n Q_{n-k}^{(\alpha,\varphi)}(x;t) Q_k^{(\beta,\varphi)}(y;t). \tag{47}$$

(ii) Integral representation

$$Q_n^{(\lambda,\varphi)}(x;t) = \frac{1}{n!} \frac{e^{in(\varphi-\frac{at}{2})}}{|\Gamma(\lambda+ix)|^2} \int_0^\infty \int_0^\infty e^{-(s+r)} (sr)^{\lambda-1} \left(s + e^{-2i(\varphi-\frac{at}{2})}r\right)^n \left(\frac{r}{s}\right)^{ix} ds dr. \tag{48}$$

**Proof.** (i) By replacing  $\lambda \rightarrow \alpha + \beta$ , ( $\alpha > 0$ ,  $\beta > 0$ ) and  $x \rightarrow x + y$  in Equation (30) and then applying Cauchy’s product, we obtain

$$\begin{aligned} \sum_{n=0}^\infty Q_n^{(\alpha+\beta,\varphi)}(x+y;t) s^n &= \left[1 - e^{i(\varphi-\frac{at}{2})}s\right]^{-(\alpha+\beta-ix-iy)} \left[1 - e^{-i(\varphi-\frac{at}{2})}s\right]^{-(\alpha+\beta+ix+iy)} \\ &= \left(\sum_{n=0}^\infty Q_n^{(\alpha,\varphi)}(x;t) s^n\right) \left(\sum_{n=0}^\infty Q_n^{(\beta,\varphi)}(y;t) s^n\right) \\ &= \sum_{n=0}^\infty \sum_{k=0}^\infty Q_n^{(\alpha,\varphi)}(x;t) Q_k^{(\beta,\varphi)}(y;t) s^{k+n} \\ &= \sum_{n=0}^\infty \sum_{k=0}^n Q_{n-k}^{(\alpha,\varphi)}(x;t) Q_k^{(\beta,\varphi)}(y;t) s^n. \end{aligned} \tag{49}$$

Thus, Equation (47) follows by comparing the coefficients of  $s$  on both sides of the last equality.

(ii) In order to prove (48), we use the generating function in ([1], Equation (9.7.13)) (by setting  $\gamma = 2\lambda$ ) and by applying the definition of Gamma function

$$\Gamma(z) b^{-z} = \int_0^\infty e^{-bt} t^{z-1} dt, \quad [\Re(z) > 0],$$

to obtain

$$\begin{aligned} \sum_{n=0}^\infty e^{-in(\varphi-\frac{at}{2})} Q_n^{(\lambda,\varphi)}(x;t) u^n &= (1-u)^{-(\lambda-ix)} \left(1 - ue^{-2i(\varphi-\frac{at}{2})}\right)^{-(\lambda+ix)} \\ &= \frac{1}{\Gamma(\lambda-ix)} \frac{1}{\Gamma(\lambda+ix)} \int_0^\infty \int_0^\infty e^{-(1-u)s} s^{\lambda-1-ix} ds e^{-(1-ue^{-2i(\varphi-\frac{at}{2})})r} r^{\lambda-1+ix} dr \\ &= \frac{1}{|\Gamma(\lambda+ix)|^2} \int_0^\infty \int_0^\infty e^{-(s+r)} (sr)^{\lambda-1} e^{iu(s+e^{-2i(\varphi-\frac{at}{2})}r)} (sr)^{\lambda-1} \left(\frac{r}{s}\right)^{ix} ds dr \\ &= \sum_{n=0}^\infty \frac{1}{n!} \left[ \frac{1}{|\Gamma(\lambda+ix)|^2} \int_0^\infty \int_0^\infty e^{-(s+r)} (sr)^{\lambda-1} \left(s + e^{-2i(\varphi-\frac{at}{2})}r\right)^n \left(\frac{r}{s}\right)^{ix} ds dr \right] u^n. \end{aligned} \tag{50}$$

Thus, (48) follows by comparing the coefficients of  $u^n$  on both sides of Equation (50).  $\square$

### 3.6. Some Properties of the Zeros Associated with the Perturbed Weight in (8)

For  $\varphi \in (0, \pi)$  and  $\lambda > 0$ , the zeros of the Meixner–Pollaczek polynomials,  $\{P_n^{(\lambda)}(x; \varphi)\}_{n=0}^\infty$ , are simple and real, and consequently, the zeros interlace [3]. The monotonicity properties of all the zeros with respect to a parameter of orthogonal polynomials associated with an even weight function, specifically, the symmetric Meixner–Pollaczek case, are given in [26] (see also [32]). In what follows, we state some fresh results related to certain properties of the zeros of the perturbed Meixner–Pollaczek polynomials.

#### 3.6.1. Monotonicity of the Zeros

**Proposition 5.** Let  $\lambda > 0$ ,  $a > 0$  and  $x \in \mathbb{R}$ . The zeros  $\left\{x_{nk}^{(\lambda,\varphi)}(t)\right\}_{k=1}^n$  of the monic perturbed Meixner–Pollaczek polynomials  $Q_n^{(\lambda,a)}(x; \varphi, t)$  are

(i) monotone decreasing functions of  $t$  on the interval  $2\varphi - 2\pi < at < 2\varphi$ ,  $t > 0$ .



(ii) monotone increasing functions of  $\varphi$  for  $0 < \varphi - \frac{at}{2} < \pi$  and fixed  $t > 0$ .

**Proof.** (i) By applying Markov’s monotonicity Theorem (cf. [10], Theorem 7.1.1), it is easy to check that for the weight in Equation (8), we have

$$\ln w^{(\lambda,\varphi)}(x;t) = (2\varphi - \pi)x + \ln\left(|\Gamma(\lambda + ix)|^2\right) - ax, \quad a > 0, \quad x \in \mathbb{R}. \quad (51)$$

Differentiating Equation (51) with respect to  $t$  gives  $G(x;t) = \frac{\partial \ln w^{(\lambda,\varphi)}(x;t)}{\partial t} = -ax$ , and hence  $G(x;t)$  is decreasing function of  $x$  for  $x \in \mathbb{R}$  since  $\frac{\partial G}{\partial x} = -a < 0$  for  $a > 0$  and  $x \in \mathbb{R}$ . We can easily infer from (cf. [10], Theorem 7.1.1) that the zeros of  $\mathcal{Q}_n^{(\lambda,a)}(x; \varphi, t)$  decrease as a function of  $t$ , for  $t \in (\frac{2\varphi-2\pi}{a}, \frac{2\varphi}{a})$ .

(ii) It is easy to check that for the perturbed weight in Equation (8), we have

$$\ln w^{(\lambda,\varphi)}(x;t) = (2\varphi - \pi)x + \ln\left(|\Gamma(\lambda + ix)|^2\right) - ax, \quad a > 0, \quad x \in \mathbb{R}. \quad (52)$$

Differentiating Equation (52) with respect to  $t$  gives  $H(x;t) = \frac{\partial \ln w^{(\lambda,\varphi)}(x;t)}{\partial \varphi} = 2x$ .

Since  $H(x;t)$  is monotone increasing of  $x$  for  $x \in \mathbb{R}$ , as  $\frac{\partial H}{\partial x} > 0$  for  $a > 0$  and  $x \in \mathbb{R}$ , it is easy to deduce from ([10], Theorem 7.1.1) that the zeros of  $\mathcal{Q}_n^{(\lambda,a)}(x; \varphi, t)$  increase as a function of  $\varphi$ , for  $\varphi \in (\frac{at}{2}, \pi + \frac{at}{2})$ ,  $a > 0$ , with  $t \in (\frac{2\varphi-2\pi}{a}, \frac{2\varphi}{a})$ .

□

Our next result gives the connection between Hellmann–Feynman Theorem [33] and the monotonicity of the zeros associated with the perturbed weight given in Equation (8).

**Theorem 5.** Let  $\lambda > \frac{1}{2}$ ,  $\{x_{n,k}(\varphi, t)\}_{k=1}^n$  be the zeros of  $\mathcal{Q}_n^{(\lambda,a)}(x; \varphi, t)$  in such a way that

$$x_{n,1}(\varphi, t) > x_{n,2}(\varphi, t) > \dots > x_{n,n}(\varphi, t).$$

The following monotone properties of the zeros hold true for  $t \in [0, \frac{2\varphi}{a})$ :

- (i)  $\frac{\partial x_{n,1}(\varphi,t)}{\partial \varphi} > 0$  for  $\frac{\pi}{2} < \varphi - \frac{at}{2} < \pi$ ,  $a > 0$ .
- (ii)  $\frac{\partial x_{n,n}(\varphi,t)}{\partial \varphi} > 0$  for  $0 < \varphi - \frac{at}{2} < \frac{\pi}{2}$ ,  $a > 0$ .
- (iii)  $\frac{\partial x_{n,1}(\varphi,t)}{\partial t} < 0$  for  $\frac{\pi}{2} < \varphi - \frac{at}{2} < \pi$ ,  $a > 0$ .
- (iv)  $\frac{\partial x_{n,n}(\varphi,t)}{\partial t} < 0$  for  $\varphi - \frac{at}{2} \in (0, \frac{\pi}{2})$ ,  $a > 0$ .

**Proof.** To apply Hellmann–Feynman’s Theorem in terms of the three-term recurrence relation (cf. [34], Theorem 1.1), we have to consider recurrence coefficients of the monic perturbed Meixner–Pollaczek polynomials in Equation (28),

$$\alpha_n^{(\lambda,\varphi)}(t) = -(\lambda + n) \cot\left(\varphi - \frac{at}{2}\right); \quad \beta_n^{(\lambda,\varphi)}(t) = \frac{n(n + 2\lambda - 1)}{4} \csc^2\left(\varphi - \frac{at}{2}\right).$$

(i) We now first consider the derivative of the coefficient  $\alpha_n(\lambda, \varphi)$ ; i.e.,  $\alpha'_n(\lambda, \varphi)$  as

$$\alpha'_n(\varphi) := \frac{\partial \alpha_n^{(\lambda,\varphi)}(t)}{\partial \varphi} = -(\lambda + n) \csc^2\left(\varphi - \frac{at}{2}\right), \quad n \geq 0, \quad (53)$$

and we see from (53) that  $\alpha'_n(\varphi) < 0$  for  $\varphi - \frac{at}{2} \in (k\pi, \pi + k\pi)$ ,  $k \in \mathbb{Z}$ . Hence the coefficient  $\alpha_n(\lambda, \varphi)$ ,  $n \geq 0$  is a monotone decreasing function of  $\varphi$  in the interval  $\varphi - \frac{at}{2} \in (0, \pi) + k\pi$  for  $k \in \mathbb{Z}$  and fixed  $t > 0$ .

Next, we examine the derivative of  $\beta'_n(\lambda, \varphi)$ . For  $n \geq 1$ ,

$$\beta'_n(\varphi) := \frac{\partial \beta_n^{(\lambda, \varphi)}(t)}{\partial \varphi} = -\frac{1}{2}(n)(\lambda + n - 1) \csc\left(\varphi - \frac{at}{2}\right) \cot\left(\varphi - \frac{at}{2}\right). \tag{54}$$

From Equation (54), we see that  $\beta'_n(\varphi) < 0$  for  $\varphi - \frac{at}{2} \in (k\pi, \frac{\pi}{2} + k\pi)$ ,  $k \in \mathbb{Z}$  with fixed parameter  $t > 0$ ; and  $\beta'_n(\varphi) > 0$  for  $\varphi - \frac{at}{2} \in (\frac{\pi}{2} + k\pi, k\pi)$ ,  $k \in \mathbb{Z}$ . In particular, for  $k = 0$ , we see that  $\beta'_n(\varphi) < 0$  if  $\varphi - \frac{at}{2} \in (0, \frac{\pi}{2})$  and  $\beta'_n(\varphi) > 0$  if  $\varphi - \frac{at}{2} \in (\frac{\pi}{2}, \pi)$  for fixed positive  $t$ . Thus, the coefficient  $\beta_n(\lambda, \varphi)$ ,  $n \geq 0$  is a monotone decreasing function of the parameter  $\varphi$  in the interval  $\varphi \in (at, \frac{\pi}{2} + at)$  for  $k \in \mathbb{Z}$  and for fixed positive  $t$ ; and  $\beta_n(\lambda, \varphi)$ ,  $n \geq 0$  is a monotone increasing function of  $\varphi$  in the interval  $\varphi \in (\frac{\pi}{2} + at, \pi + at)$ ,  $k \in \mathbb{Z}$  and fixed  $t > 0$ . Thus, the assumptions of Hellman-Feynman Theorem are fulfilled, and so is Theorem 5.

(ii) The proofs for (ii), (iii) and (iv) share similar approach.

□

### 3.6.2. Convexity of the Extreme Zeros

In the following, we shall now prove the convexity of zeros related to the perturbed Meixner–Pollaczek weight (8).

**Theorem 6.** Let  $\lambda > \frac{1}{2}$ ,  $\{x_{n,k}(\varphi, t)\}_{k=1}^n$  be the zeros of  $Q_n^{(\lambda, a)}(x; \varphi, t)$  in such a way that  $x_{n,1}(\varphi, t) > x_{n,2}(\varphi, t) > \dots > x_{n,n}(\varphi, t)$ . The following convexity results of the extreme zeros hold true:

- (i)  $\frac{\partial^2 x_{n,1}(\varphi, t)}{\partial \varphi^2} > 0$  and  $\frac{\partial^2 x_{n,1}(\varphi, t)}{\partial t^2} > 0$  for  $\varphi - \frac{at}{2} \in (\frac{\pi}{2}, \pi)$ ,  $a, t > 0$ .
- (ii)  $\frac{\partial^2 x_{n,1}(\varphi, t)}{\partial \varphi^2} < 0$  and  $\frac{\partial^2 x_{n,1}(\varphi, t)}{\partial t^2} < 0$  for  $\varphi - \frac{at}{2} \in (0, \frac{\pi}{2})$ ,  $a, t > 0$ .

**Proof.** (i) By following the idea of Dimitrov (cf. [35], Lemma 1), the convexity of the extreme zeros follows from the derivatives

$$\begin{aligned} \frac{d^2 \alpha_n^{(\lambda, \varphi)}}{dt^2} &= -(n + \lambda) \frac{d^2}{d\varphi^2} \left( \cot\left(\varphi - \frac{at}{2}\right) \right) \\ &= -2(n + \lambda)a^2 \csc^2\left(\varphi - \frac{at}{2}\right) \cot\left(\varphi - \frac{at}{2}\right) \\ &= \begin{cases} < 0, & \text{if } 0 < \varphi - \frac{at}{2} < \frac{\pi}{2}, \\ > 0, & \text{if } \frac{\pi}{2} < \varphi - \frac{at}{2} < \pi, \end{cases} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \frac{d^2 \beta_n^{(\lambda, \varphi)}}{d\varphi^2} &= \frac{d^2}{d\varphi^2} \left( \csc^2\left(\varphi - \frac{at}{2}\right) \right) \\ &= \frac{n(n + 2\lambda - 1)a^2}{4} \left[ 4 \csc^2\left(\varphi - \frac{at}{2}\right) \cot^2\left(\varphi - \frac{at}{2}\right) + 2 \csc^4\left(\varphi - \frac{at}{2}\right) \right] \\ &= n(n + 2\lambda - 1) a^2 \left[ \csc^2\left(\varphi - \frac{at}{2}\right) \cot^2\left(\varphi - \frac{at}{2}\right) + \frac{1}{2} \csc^4\left(\varphi - \frac{at}{2}\right) \right] > 0, \end{aligned} \tag{56}$$

for all values of  $\varphi - \frac{at}{2}$ , and in particular,  $0 < \varphi - \frac{at}{2} \in \pi$  for fixed  $t > 0$  and  $a > 0$ . By combining Equations (55) and (56) and applying ([35], Lemma 1), the above convexity result of the largest zero of the perturbed Meixner–Pollaczek polynomials follows immediately.

(ii) The concavity of the smallest zero of the perturbed Meixner–Pollaczek polynomials also follows from Equations (55) and (56), in a similar manner, by applying ([35], Lemma 1).

□

**Remark 2.** A similar numerical experimentation of the zeros of the perturbed Meixner–Pollaczek polynomials can be done to give analog results to the ones in Section 1.1.1 with careful restriction of involved parameters.

#### 4. Some Applications of the Polynomial $Q_n^{(\lambda,a)}(x; \varphi, t)$

In this Section, certain applications of the perturbed Meixner–Pollaczek polynomials are explored. These polynomials have wider applicability in the Random matrix theory of level statistics using partition functions (via Toda molecule equation) [11], wave functions in Quantum Mechanics, the Fisher information theory and in the study of Gaussian quadrature (cf. [36]), to mention a few.

##### 4.1. Exposition of Toda-Type Lattice/Molecule Equation

Toda lattice is a system of particles on the line with exponential interaction of nearest neighbours [37]. Toda was the first to study such a system for infinitely many particles on the line [38]. The Toda lattice equations are investigated from the Newtonian equations of motion (see, for example, [37])

$$\dot{x}_n = e^{x_{n-1}-x_n} - e^{x_n-x_{n+1}}, \quad n \geq 1,$$

when one takes  $\alpha_n = \dot{x}_n$  and  $\beta_n = e^{x_{n-1}-x_n}$  for  $n \in \mathbb{N}$ . (Note that  $\alpha_n$  and  $\beta_n$  are the recurrence coefficients for corresponding monic orthogonal polynomials on the real line [3,8]).

The fact that perturbed Meixner–Pollaczek polynomials are time-dependent orthogonal polynomials, allows us to study the time-evolution equation related to Toda lattices. The Perturbed Meixner–Pollaczek weight in (8) is obtained from deformation of classical Meixner–Pollaczek weight by  $\exp(-axt)$ . For similar measure deformation, we refer to [21,23,27] (See also [39]). We now mention in the following result of the perturbed Meixner–Pollaczek polynomials satisfying a similar scaled Toda lattice/molecule equation.

**Proposition 6.** The recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in (29) associated with the monic perturbed Meixner–Pollaczek polynomials  $Q_n^{(\lambda,\varphi)}(x; t)$  for  $\varphi \in (\frac{at}{2}, \pi + \frac{at}{2})$ , obey a scaled Toda molecule equation

$$\begin{cases} \frac{\partial \alpha_n}{\partial t} = a(\beta_n - \beta_{n+1}), \\ \frac{\partial \beta_n}{\partial t} = a\beta_n(\alpha_{n-1} - \alpha_n), \quad a > 0. \end{cases} \tag{57}$$

**Proof.** This result immediately follows from orthogonality and iterated recurrences, see [21,39].  $\square$

The proof of this result is given in Appendix A.1 of Appendix A just for the reader’s convenience.

**Remark 3.** We now see that Equation (29) solves the differential-recurrence (Toda) equation in Equation (57) associated with the monic perturbed Meixner–Pollaczek polynomials.

##### 4.2. Fisher Information of the Monic Polynomial $Q_n^{(\lambda,a)}(x; \varphi, t)$

Following the approach given in [36], the Fisher information of the Meixner–Pollaczek polynomials is computed using the concept introduced for general orthogonal polynomials by Sánchez-Ruiz and Dehesa in [40]. They considered a sequence of real polynomials orthogonal with respect to the weight function  $\rho(x)$  on the interval  $[a, b]$

$$\int_a^b P_n(x) P_m(x) \rho(x) dx = \zeta_n \delta_{n,m}, \quad n, m = 0, 1, \dots,$$

with  $\deg(P_n) = n$ . Introducing the normalized density functions

$$\rho_n(x) = \frac{[P_n(x)]^2 \rho(x)}{\zeta_n}, \tag{58}$$

they in fact defined the Fisher information corresponding to the densities in Equation (58)

$$\mathcal{I}(n) = \int_a^b \frac{[\rho'_n(x)]^2}{\rho_n(x)} dx. \tag{59}$$

Applying the formula in Equation (59) to the classical hypergeometric polynomials, the authors in [41] evaluated  $\mathcal{I}(n)$  for Jacobi, Laguerre and Hermite polynomials. We quote the following result by Dominici from [36]:

**Theorem 7 ([36]).** *The Fisher information of the Meixner–Pollaczek polynomials is given by*

$$I_\varphi(P_n^{(\lambda)}) = \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial \varphi} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)} dx = \frac{2[n^2 + (2n + 1)\lambda]}{\sin^2(\varphi)}, \quad n \in \mathbb{N}_0,$$

where the normalized function  $\rho_n(x)$  is as defined in Equation (58).

Based on the above discussion, we shall now reproduce the following application of the monic perturbed Meixner–Pollaczek polynomials.

**Theorem 8.** *The Fisher information of the monic perturbed Meixner–Pollaczek polynomials with respect to the parameter  $\varphi$  is given, in terms of the recurrence coefficients, by*

$$I_\varphi(Q_n^{(\lambda, \varphi)}(x; t)) = \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial \varphi} \rho_n(x; t) \right]^2 \frac{1}{\rho_n(x; t)} dx = 4(\beta_n^{(\lambda, \varphi)} + \beta_n^{(\lambda, \varphi)} + [\alpha_n^{(\lambda, \varphi)}]^2) + 4(2n + 2\lambda - 1)\alpha_n^{(\lambda, \varphi)} + (2n + 2\lambda - 1)^2 \left( \cot\left(\varphi - \frac{at}{2}\right) \right)^2, \tag{60}$$

where the normalized function  $\rho_n(x; t)$  is as given in Equation (58).

**Proof.** By employing the three-term recurrence relation in Equation (28) associated with the weight in (8) and using the orthogonality relation in Equation (25) with its (monic) normalization constant, we have normalized function

$$\rho_n(x; t) = \frac{[Q_n^{(\lambda, \varphi)}(x; t)]^2 w^{(\lambda, \varphi)}(x; t)}{\zeta_n^{(\lambda, \varphi)}} = \frac{e^{(2\varphi - \pi)x} |\Gamma(\lambda + ix)|^2 e^{-axt} [2 \sin(\varphi - \frac{at}{2})]^{2n+2\lambda} [Q_n^{(\lambda, \varphi)}(x; t)]^2}{2\pi\Gamma(n + 2\lambda)\Gamma(n + 1)} \tag{61}$$

and we note that  $\int_{\mathbb{R}} \rho_n(x; t) = 1$  for  $n \in \mathbb{N}_0$ .

By taking the derivatives of  $\rho_n$  with respect to  $\varphi$  and using the perturbed weight (8)

$$\frac{\partial w^{(\lambda, \varphi)}}{\partial \varphi} = (2x) w^{(\lambda, \varphi)}, \tag{62}$$

together with the result in ([36], Equation (12)) gives

$$\frac{\partial Q_n^{(\lambda, \varphi)}(x; t)}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left( \frac{n!}{(2 \sin \varphi)^n} P_n^{(\lambda)}\left(x; \varphi - \frac{at}{2}\right) \right) = \frac{-n(n + 2\lambda - 1)}{2 \sin^2(\varphi - \frac{at}{2})} Q_{n-1}^{(\lambda, \varphi)}(x; t). \tag{63}$$

Using Equation (61), it follows that

$$\frac{\partial \rho_n(x;t)}{\partial \varphi} = \frac{\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)w^{(\lambda,\varphi)}(x;t)}{\zeta_n^{(\lambda,\varphi)}} \left\{ \frac{-n(n+2\lambda-1)}{\sin^2\left(\varphi-\frac{at}{2}\right)} \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) + 2x\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) - \left(\frac{1}{\zeta_n^{(\lambda,\varphi)}} \frac{\partial \zeta_n^{(\lambda,\varphi)}}{\partial \varphi}\right) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \right\} \quad (64)$$

From the orthogonality of Meixner-Pollaczek polynomials [1], we note here that

$$\zeta_n^{(\lambda,\varphi)}(t) = \frac{2\pi(n!)\Gamma(n+2\lambda)}{[2\sin(\varphi-\frac{at}{2})]^{2n+2\lambda}} = \frac{2\pi\Gamma(n+1)\Gamma(n+2\lambda)}{[2\sin(\varphi-\frac{at}{2})]^{2n+2\lambda}}, \quad (65)$$

for the perturbed Meixner-Pollaczek polynomials. It then follows from Equation (65) that

$$\begin{aligned} \frac{1}{\zeta_n^{(\lambda,\varphi)}} \frac{\partial \zeta_n^{(\lambda,\varphi)}}{\partial \varphi} &= \frac{(-2\pi)(2n+2\lambda-1)\Gamma(n+1)\Gamma(n+2\lambda)\left(\frac{1}{2}\right)^{2n+2\lambda} [2\csc(\varphi-\frac{at}{2})]^{2n+2\lambda} \cot(\varphi-\frac{at}{2})}{\Gamma(n+1)\Gamma(n+2\lambda)\left(\frac{1}{2}\right)^{2n+2\lambda} [2\csc(\varphi-\frac{at}{2})]^{2n+2\lambda}} \\ &= -(2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right). \end{aligned} \quad (66)$$

By using Equations (66) and (61), Equation (64) becomes

$$\begin{aligned} \frac{\partial \rho_n(x;t)}{\partial \varphi} &= \frac{1}{\zeta_n^{(\lambda,\varphi)}} \mathcal{Q}_n^{(\lambda,\varphi)}(x;t)w^{(\lambda,\varphi)}(x;t) \left\{ \frac{-n(n+2\lambda-1)}{\sin^2\left(\varphi-\frac{at}{2}\right)} \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) + 2x\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) - \left(\frac{1}{\zeta_n^{(\lambda,\varphi)}} \frac{\partial \zeta_n^{(\lambda,\varphi)}}{\partial \varphi}\right) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \right\} \\ &= \frac{\rho_n(x;t)}{\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)} \left\{ \frac{-n(n+2\lambda-1)}{\sin^2\left(\varphi-\frac{at}{2}\right)} \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) + 2x\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) + (2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \right\} \\ &= \frac{\rho_n(x;t)}{\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)} \left\{ \left[ 2x + (2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right) \right] \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) - 4\beta_n^{(\lambda,\varphi)} \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \right\}. \end{aligned} \quad (67)$$

Thus, using Equation (67), we attain that

$$\begin{aligned} \left(\frac{\partial \rho_n}{\partial \varphi}\right)^2 \frac{1}{\rho_n} &= \frac{\rho_n(x;t)}{[\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2} \left\{ \left[ 2x + (2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right) \right]^2 [\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2 \right. \\ &\quad \left. - 8\beta_n^{(\lambda,\varphi)} [2x + (2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right)] \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) + 16[\beta_n^{(\lambda,\varphi)}]^2 [\mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t)]^2 \right\} \\ &= \frac{w^{(\lambda,\varphi)}(x;t)}{\zeta_n^{(\lambda,\varphi)}} \left\{ \left[ 4x^2 + 4(2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right)x + (2n+2\lambda-1)^2 \left(\cot\left(\varphi-\frac{at}{2}\right)\right)^2 \right] [\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2 \right. \\ &\quad \left. - 16\beta_n^{(\lambda,\varphi)} x \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) - 8\beta_n^{(\lambda,\varphi)} (2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right) \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \right. \\ &\quad \left. + 16[\beta_n^{(\lambda,\varphi)}]^2 [\mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t)]^2 \right\}. \end{aligned} \quad (68)$$

By integrating Equation (68) and using the orthogonality relation in Equation (25) and iterating the recurrence (28)

$$\begin{aligned} x\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) &= \mathcal{Q}_{n+1}^{(\lambda,\varphi)}(x;t) + \alpha_n(t)\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) + \beta_n(t)\mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t), \\ x^2\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) &= \mathcal{Q}_{n+2}^{(\lambda,\varphi)}(x;t)(\alpha_n + \alpha_{n+1})\mathcal{Q}_{n+1}^{(\lambda,\varphi)}(x;t) + (\beta_{n+1} + \beta_n + \alpha_n^2)\mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \\ &\quad + \beta_n\alpha_{n-1}\mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) + \beta_n\beta_{n-1}\mathcal{Q}_{n-2}^{(\lambda,\varphi)}(x;t), \end{aligned} \quad (69)$$

we obtain

$$\begin{aligned} \left(\frac{\partial \rho_n}{\partial \varphi}\right)^2 \frac{1}{\rho_n} &= \frac{1}{\zeta_n^{(\lambda,\varphi)}} \int 4x^2 [\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2 w^{(\lambda,\varphi)}(x;t) dx \\ &\quad + \frac{1}{\zeta_n^{(\lambda,\varphi)}} \int 4(2n+2\lambda-1) \cot\left(\varphi-\frac{at}{2}\right) x [\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2 w^{(\lambda,\varphi)}(x;t) dx \\ &\quad + \frac{1}{\zeta_n^{(\lambda,\varphi)}} \int (2n+2\lambda-1)^2 \left(\cot\left(\varphi-\frac{at}{2}\right)\right)^2 [\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)]^2 w^{(\lambda,\varphi)}(x;t) dx \\ &\quad - 16\beta_n^{(\lambda,\varphi)} \frac{1}{\zeta_n^{(\lambda,\varphi)}} \int x \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx + 16[\beta_n^{(\lambda,\varphi)}]^2 \frac{1}{\zeta_n^{(\lambda,\varphi)}} \int [\mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t)]^2 w^{(\lambda,\varphi)}(x;t) dx \\ &= 4\left(\beta_n^{(\lambda,\varphi)} + \beta_n^{(\lambda,\varphi)} + [\alpha_n^{(\lambda,\varphi)}]^2\right) + 4(2n+2\lambda-1)\alpha_n^{(\lambda,\varphi)} + (2n+2\lambda-1)^2 \left(\cot\left(\varphi-\frac{at}{2}\right)\right)^2, \end{aligned} \quad (70)$$

and this completes the proof.  $\square$

**Remark 4.** The Fisher information of the classical orthogonal polynomials with respect to a parameter is given in [41]. In our case, the Fisher information of the perturbed Meixner–Pollaczek polynomials with respect to the parameter  $a > 0$  can also be obtained in a similar procedure, using the fact that

$$\frac{\partial w^{(\lambda, \varphi)}}{\partial a} = (-tx) w^{(\lambda, \varphi)}.$$

### 4.3. Gauss–Meixner–Pollaczek Quadrature

Let’s first recall a quadrature rule,

$$\int_{\mathbb{R}} f(x) d\mu(x) \approx \sum_{j=1}^n \omega_j f(x_j),$$

where the integral of a function  $f$  relative to some (in general positive) measure  $d\mu$  is approximated by a finite sum involving  $n$  values of  $f$  at suitably selected distinct nodes  $x_j$ , where these nodes are obtained from the zeros of orthogonal polynomials  $\Phi_n(x; w)$  and the quadrature weights  $\omega_j$ ,  $j = 1, 2, \dots, n$  can also be given by [42]

$$\omega_j = \frac{\langle \Phi_{n-1}, \Phi_{n-1} \rangle_w}{\Phi_{n-1}(x_j) \Phi'_n(x_j)}. \tag{71}$$

where the prime denotes differentiation with respect to  $x$ .

Just for simplicity, this Subsection emphasizes to explore Gaussian quadrature rule related to symmetric monic Meixner–Pollaczek polynomials, which are special cases of the perturbed Meixner–Pollaczek polynomials  $\mathcal{Q}_n^{(\lambda, \varphi)}(x; t)$  when  $t = 0$  and  $\varphi = \frac{\pi}{2}$ . As given in ([1], Section 9.7), symmetric monic Meixner–Pollaczek polynomials, are defined by

$$\mathcal{S}_n^{(\lambda)}(x) := P_n^{(\lambda)}\left(x; \frac{\pi}{2}\right) = \frac{(2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 2\right), \tag{72}$$

and are orthogonal on  $\mathbb{R}$  for  $\lambda > 0$  with respect to the continuous weight

$$W(x; \lambda) = \frac{1}{2\pi} |\Gamma(\lambda + ix)|^2, \quad \lambda > 0, \quad x \in \mathbb{R}. \tag{73}$$

Since the sequence of monic polynomials  $\{\mathcal{S}_n^{(\lambda)}\}_{n=0}^\infty$  defined in Equation (72) are symmetric with respect to the origin, it follows from orthogonality that they obey symmetric recurrence relation [6]

$$\begin{cases} x\mathcal{S}_n^{(\lambda)}(x) = \mathcal{S}_{n+1}^{(\lambda)}(x) + \beta_n(\lambda) \mathcal{S}_{n-1}^{(\lambda)}(x), & n \in \mathbb{N}, \\ \mathcal{S}_0^{(\lambda)}(x) \equiv 1, \quad \mathcal{S}_{-1}^{(\lambda)}(x) \equiv 0, \end{cases} \tag{74}$$

where the coefficient  $\beta_n(\lambda)$  from Equation (74) is given by [6]

$$\beta_n(\lambda) = \frac{(n)(2\lambda + n - 1)}{4 \sin^2(\frac{\pi}{2})} = \frac{(n)(n + 2\lambda - 1)}{4}, \quad n \geq 1. \tag{75}$$

It now follows from [1] that the normalization constant associated with the weight in (73) is given by

$$\zeta_n^{(\lambda)} = \frac{2\pi \Gamma(n + 1) \Gamma(n + 2\lambda)}{2^{2n+2\lambda}}. \tag{76}$$

We note that, for  $\lambda = \frac{1}{2}$ , taking into account of Euler’s duplication formula ([2], Equation (5.5.5)), we have from (73)

$$W\left(x; \frac{1}{2}\right) = \frac{1}{2\pi} \left| \Gamma\left(\frac{1}{2} + ix\right) \right|^2 = \frac{1}{2\pi} \left( \frac{\pi}{\cosh(\pi x)} \right) = \frac{1}{2 \cosh(\pi x)}.$$

Similarly, if we take  $\lambda = 1$ , again using Euler’s duplication formula [2], we obtain

$$W(x; 1) = \frac{1}{2\pi} \left| \Gamma(1 + ix) \right|^2 = \frac{1}{2\pi} \Gamma(1 + ix) \Gamma(1 - ix) = \frac{1}{2\pi} \left( \frac{\pi x}{\sin(\pi x)} \right) = \frac{x}{2 \sin(\pi x)}.$$

We now establish the following result, which is an application of Gaussian quadrature formula based on (symmetric) Meixner-Pollaczek weight (74).

**Proposition 7.** *The Gauss quadrature rule for a continuous function  $f(x)$  associated with symmetric Meixner-Pollaczek weight in (73) is given by*

$$\int_{-\infty}^{\infty} f(x) W(x; \lambda) dx \approx \sum_{k=1}^j \omega_{j,k}^{(\lambda)} f(x_{j,k}^{(\lambda)}), \tag{77}$$

where  $f(x)$  can be a polynomial, and the quadrature weights are given by

$$\omega_{j,n}^{(\lambda)} = \int_{\mathbb{R}} \left| \Gamma(\lambda + ix) \right|^2 \left[ \prod_{\substack{n \neq k \\ k=1}}^j \left( \frac{x - x_{j,k}^{(\lambda)}}{x_{j,n}^{(\lambda)} - x_{j,k}^{(\lambda)}} \right) \right] dx,$$

for  $n = 1, 2, 3, \dots, j$  and  $\{x_{j,1}^{(\lambda)}, x_{j,2}^{(\lambda)}, \dots, x_{j,j}^{(\lambda)}\}$  are the zeros of Meixner-Pollaczek polynomials  $\mathcal{S}_j^{(\lambda)}$ .

**Proof.** Suppose  $f \in \mathbb{P}_{2j-1}$ . Then, by using division algorithm, we have

$$f = \mathcal{S}_j^{(\lambda)}(x) V(x) + R_j(x), \tag{78}$$

where the degree of  $R_j(x)$  is  $(j - 1)$  and  $\mathcal{S}_j^{(\lambda)}(x)$  is orthogonal to any polynomials of degree  $< j$ , and  $V(x)$  is of degree  $(j - 1)$  and then we have  $\langle \mathcal{S}_j^{(\lambda)}, V(x) \rangle = 0$ . Now, by using orthogonality property and Equation (78), we have

$$\begin{aligned} \int_{\mathbb{R}} f(x) W(x; \lambda) dx &= \int_{\mathbb{R}} W(x; \lambda) \left[ \mathcal{S}_j^{(\lambda)}(x) V(x) + R_j(x) \right] dx \\ &= \int_{\mathbb{R}} W(x; \lambda) \mathcal{S}_j^{(\lambda)}(x) V(x) dx + \int_{\mathbb{R}} W(x; \lambda) R_j(x) dx \\ &= \int_{\mathbb{R}} W(x; \lambda) R_j(x) dx. \end{aligned}$$

However, by orthogonality, and since  $R_j(x)$ , a polynomial of degree  $(j - 1)$ , is approximated by using Lagrange interpolating polynomial,  $L_j(x)$ , and it is given as,

$$R_j(x) \approx L_j(x) = \sum_{k=1}^j \ell_{j,k}(x) R_j(x), \quad \text{where} \quad \ell_{j,k}(x) = \prod_{\substack{\ell=1 \\ k \neq \ell}}^j \left( \frac{x - x_{j,\ell}^{(\lambda)}}{x_{j,k}^{(\lambda)} - x_{j,\ell}^{(\lambda)}} \right).$$

Now,

$$\begin{aligned} \int_{\mathbb{R}} W(x; \lambda) f(x) dx &= \int_{\mathbb{R}} W(x; \lambda) R_j(x) dx = \int_{\mathbb{R}} W(x; \lambda) \left[ \sum_{k=1}^j \ell_{j,k}(x) R_j(x_k) \right] dx \\ &= \sum_{k=1}^j \int_{\mathbb{R}} W(x; \lambda) \ell_{j,k}(x) R_j(x_k) dx = \sum_{k=1}^j R_j(x_k) \int_{\mathbb{R}} W(x; \lambda) \ell_{j,k}(x) dx \\ &= \sum_{k=1}^j R_j(x_k) \omega_{j,k}^{(\lambda)} \quad \left( \text{where } \omega_{j,k}^{(\lambda)} = \int_{\mathbb{R}} W(x; \lambda) \ell_{j,k}(x) dx \right) \\ &= \sum_{k=1}^j \omega_{j,k}^{(\lambda)} f(x_{j,k}^{(\lambda)}) \quad \left( \text{since } f(x_{j,k}^{(\lambda)}) = R_j(x_k) \right). \end{aligned}$$

Therefore,  $\int_{\mathbb{R}} f(x) W(x; \lambda) dx = \sum_{k=1}^j \omega_{j,k}^{(\lambda)} f(x_{j,k}^{(\lambda)})$  where  $\omega_{j,k}^{(\lambda)} = \int_{\mathbb{R}} \ell_{j,k}(x) W(x; \lambda) dx$ .  
 □

In order to implement Proposition 7, the first few monic polynomials  $\mathcal{S}_n^{(\lambda)}(x)$ , for some values of  $\lambda$ , are shown in the following Table 4.

**Table 4.**  $\mathcal{S}_n^{(\lambda)}(x)$  for  $0 \leq n \leq 4$  and  $\lambda = 1, \frac{1}{2}, \frac{1}{4}$ .

	$\lambda = 1$	$\lambda = \frac{1}{2}$	$\lambda = \frac{1}{4}$
$n = 0$	1	1	1
$n = 1$	$x$	$x$	$x$
$n = 2$	$x^2 - \frac{1}{2}$	$x^2 - \frac{1}{4}$	$x^2 - \frac{1}{8}$
$n = 3$	$x^3 - 2x$	$x^3 - \frac{5}{4}x$	$x^3 - \frac{7}{8}x$
$n = 4$	$x^4 - 5x^2 + \frac{3}{2}$	$x^4 - \frac{7}{2}x^2 + \frac{9}{16}$	$x^4 - \frac{11}{4}x^2 + \frac{15}{64}$
$n = 5$	$x^5 - 10x^3 + \frac{23}{2}x$	$x^5 - \frac{15}{2}x^3 + \frac{89x}{16}$	$x^5 - \frac{25x^3}{4} + \frac{211x}{64}$

The following example elaborates the applicability of Proposition 7.

**Example 1.** Construct a two-point Gauss quadrature rule for the symmetric Meixner-Pollaczek weight  $W(x; \lambda) = |\Gamma(1 + ix)|^2$  with parameters ( $\lambda = \frac{1}{2}$  and  $\lambda = 1$ ) and also compute the specific zeros  $x_{j,k}^{(\lambda)}$  and the quadrature weights  $\omega_{j,k}^{(\lambda)}$  of this quadrature rule.

**Solution:** By considering the zeros of symmetric Meixner-Pollaczek polynomials,  $\mathcal{S}_n^{(\lambda)}(x)$  with parameter ( $\lambda = \frac{1}{2}$  and  $\lambda = 1$ ) and by recalling that  $f \in \mathbb{P}_{2j-1}$ , we compute the Gauss quadrature rule, as given in (77), as follows.

**The case when  $\lambda = 1$ :**

The zeros of  $\mathcal{S}_2^{(1)}(x) = x^2 - \frac{1}{2}$  are  $x_{2,1}^{(1)} = \frac{1}{\sqrt{2}}$  and  $x_{2,2}^{(1)} = -\frac{1}{\sqrt{2}}$  and the corresponding quadrature weights are given by

$$\omega_{2,1}^{(1)} = \int_{\mathbb{R}} |\Gamma(1 + ix)|^2 \left[ \prod_{k=1}^2 \left( \frac{x - x_{2,k}^{(1)}}{(x_{2,1}^{(1)} - x_{2,k}^{(1)})} \right) \right] dx = \int_{-\infty}^{\infty} \frac{\pi x}{\sin \pi x} \left( \frac{x - x_{2,2}^{(1)}}{(x_{2,1}^{(1)} - x_{2,2}^{(1)})} \right) dx = \int_{-\infty}^{\infty} \frac{\pi x}{\sin \pi x} \left( \frac{x + \frac{1}{\sqrt{2}}}{\sqrt{2}} \right) dx, \tag{79}$$

and

$$\omega_{2,2}^{(1)} = \int_{\mathbb{R}} |\Gamma(1 + ix)|^2 \left[ \prod_{k=1}^2 \left( \frac{x - x_{2,k}^{(1)}}{(x_{2,2}^{(1)} - x_{2,k}^{(1)})} \right) \right] dx = \int_{-\infty}^{\infty} \frac{\pi x}{\sin \pi x} \left( \frac{x - x_{2,1}^{(1)}}{(x_{2,2}^{(1)} - x_{2,1}^{(1)})} \right) dx = \int_{-\infty}^{\infty} \frac{\pi x}{\sin \pi x} \left( \frac{x - \frac{1}{\sqrt{2}}}{-\sqrt{2}} \right) dx. \tag{80}$$



In order to determine the quadrature weights in Equations (79) and (80), we use Equations (71) and (76) and Table 4 together with orthogonality, to obtain

$$\begin{cases} \omega_{2,1}^{(1)} = \frac{\langle P_1, P_1 \rangle_W}{2\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{2\pi \Gamma(n+1) \Gamma(n+2\lambda)}{2^{2n+2\lambda}}, \\ \omega_{2,2}^{(1)} = \frac{\langle P_1, P_1 \rangle_W}{2\left(-\frac{1}{\sqrt{2}}\right)^2} = \frac{2\pi \Gamma(n+1) \Gamma(n+2\lambda)}{2^{2n+2\lambda}}. \end{cases} \tag{81}$$

Hence,

$$\int_{\mathbb{R}} f(x) W(x; \lambda) dx = \left(\frac{2\pi \Gamma(n+1) \Gamma(n+2\lambda)}{2^{2n+2\lambda}}\right) f(x_{2,1}^{(1)}) + f(x_{2,2}^{(1)}) \left(\frac{2\pi \Gamma(n+1) \Gamma(n+2\lambda)}{2^{2n+2\lambda}}\right).$$

The case when  $\lambda = \frac{1}{2}$ : The computation of Gaussian nodes and weights can be done in a similar manner.

**Remark 5.** For numerical computation of Gauss weights and nodes for an arbitrary weight function using Matlab, see [43] and also [42].

#### 4.4. Meixner-Pollaczek Polynomials as Solution for Cauchy Problem

It is shown in [12] that the Cauchy problem for the  $n$ -dimensional Schrödinger equation for a free particle

$$i\psi_t + \Delta\psi = 0$$

with

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}\psi, \quad \mathcal{H} = -\Delta = \frac{1}{2} \sum_{s=1}^n (a_s + a_s^\dagger)^2, \tag{82}$$

and the Hamiltonian  $\mathcal{H}$  takes the form

$$\mathcal{H}\psi = \left[ \frac{1}{2} \sum_{s=1}^n \left( -(1 + \cos 2t) \frac{\partial^2}{\partial x_s^2} + (1 - \cos 2t) x_s^2 \right) - \frac{i}{2} \sin 2t \sum_{s=1}^n \left( 2x_s \frac{\partial}{\partial x_s} + 1 \right) \right] \psi,$$

the particular solution for Equation (82) is explained in terms of Meixner-Pollaczek polynomials, which satisfies conditions in quantum mechanics (orthogonality and normalizability). The result in [12] generalizes time-dependent simple harmonic motion oscillator and angular momentum problem oscillator of quantum mechanics in a Cartesian and spherical coordinate system.

### 5. Conclusions

By introducing a time variable to the Meixner–Pollaczek measure, we have found certain interesting properties such as some recursive relations, moments of finite order, concise hypergeometric formulae and orthogonality relation, certain analytic properties of the zeros of the corresponding monic perturbed Meixner–Pollaczek polynomials. As practical applications, we have reproduced the scaled Toda molecule equation in Random matrix theory, Fisher’s information with respect to some new parameter, and Gaussian-type quadrature related to the perturbed Meixner–Pollaczek polynomials and also their role as a solution to quantum oscillators.

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### Appendix A

In this appendix, we provide the proof to Proposition 6 to aid the reader.

#### Appendix A.1. Proof for Proposition 6

**Proof.** The proof follows from orthogonality and the corresponding recurrence coefficients for monic polynomials that are orthogonal with respect to the weight  $w^{(\lambda,\varphi)}(x;t)$  given in (8). Now, by considering the three-term recurrence relation in (29) and taking derivatives of the coefficients in Equation (29) with respect to  $t$ , we have that

$$x \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} = \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} + \alpha_n^{(\lambda,\varphi)}(t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} + \frac{d\alpha_n}{dt} Q_n^{(\lambda,\varphi)}(x;t) + \frac{d\beta_n}{dt} Q_{n-1}^{(\lambda,\varphi)} + \beta_n^{(\lambda,\varphi)} \frac{dQ_{n-1}^{(\lambda,\varphi)}(x;t)}{dt}. \tag{A1}$$

Multiplying Equation (A1) by  $Q_n^{(\lambda,\varphi)}(x;t)$  and integrating with respect to the measure  $w^{(\lambda,\varphi)}(x;t)$  yields

$$\begin{aligned} \frac{d\alpha_n^{(\lambda,\varphi)}}{dt} \zeta_n^{(\lambda,\varphi)} &= \int \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} (x Q_n(x) w^{(\lambda,\varphi)}(x;t) dx - \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx \\ &\quad - \alpha_n^{(\lambda,\varphi)} \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx - \beta_n^{(\lambda,\varphi)} \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n-1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx, \end{aligned} \tag{A2}$$

where we have used the orthogonality of  $\frac{dQ_{n-1}^{(\lambda,\varphi)}}{dt}$  and  $Q_n^{(\lambda,\varphi)}$ .

Again, employing the recurrence relation (29) and the orthogonality relation, (A2) is equivalently given as

$$\begin{aligned} \frac{d\alpha_n^{(\lambda,\varphi)}}{dt} \zeta_n^{(\lambda,\varphi)} &= \int \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} (Q_{n+1}^{(\lambda,\varphi)} + \alpha_n^{(\lambda,\varphi)} Q_n^{(\lambda,\varphi)} + \beta_n^{(\lambda,\varphi)} Q_{n-1}^{(\lambda,\varphi)}) w^{(\lambda,\varphi)}(x;t) dx \\ &\quad - \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx - \alpha_n^{(\lambda,\varphi)} \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx \\ &= \beta_n^{(\lambda,\varphi)} \int Q_{n-1}^{(\lambda,\varphi)}(x;t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx - \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx. \end{aligned} \tag{A3}$$

Now, if we consider the weight given in Equation (8) and if we differentiate the orthogonality condition

$$\int Q_n^{(\lambda,\varphi)}(x;t) Q_{n\pm 1}^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx = 0$$

with respect to  $t$ , we obtain the following relations respectively:

$$\int Q_{n-1}^{(\lambda,\varphi)}(x;t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx + \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx - \int x Q_n^{(\lambda,\varphi)}(x;t) Q_{n-1}^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx = 0, \tag{A4}$$

$$\int Q_{n+1}^{(\lambda,\varphi)}(x;t) \frac{dQ_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx + \int Q_n^{(\lambda,\varphi)}(x;t) \frac{dQ_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx - \int x Q_{n+1}^{(\lambda,\varphi)}(x;t) Q_n^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx = 0. \tag{A5}$$

Now using iterated three-term recurrences, Equations (A4) and (A5) lead to

$$\int \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \frac{d\mathcal{Q}_n^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx = \int x \mathcal{Q}_{n-1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx = \beta_n^{(\lambda,\varphi)} \zeta_{n-1}^{(\lambda,\varphi)}. \tag{A6}$$

and

$$\int \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) \frac{d\mathcal{Q}_{n+1}^{(\lambda,\varphi)}(x;t)}{dt} w^{(\lambda,\varphi)}(x;t) dx = \int x \mathcal{Q}_{n+1}^{(\lambda,\varphi)}(x;t) \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx = \beta_{n+1}^{(\lambda,\varphi)} \zeta_n^{(\lambda,\varphi)}. \tag{A7}$$

Thus, using orthogonality, (A6) and (A7) into Equation (A3), we obtain the first equation in (57).

Similarly, if we differentiate the normalization constant

$$\zeta_n^{(\lambda,\varphi)} = \int \mathcal{Q}_n^{(\lambda,\varphi)}(x;t) w^{(\lambda,\varphi)}(x;t) dx, \text{ where}$$

$$(w^{(\lambda,\varphi)}(x;t))' = -xw^{(\lambda,\varphi)}(x;t)$$

with the prime denoting differentiation with respect to  $x$  and if we use the orthogonality relation and the recurrence relation, we find that

$$\frac{d\zeta_n^{(\lambda,\varphi)}}{dt} = \alpha_n^{(\lambda,\varphi)} \zeta_n^{(\lambda,\varphi)}. \tag{A8}$$

Now using (A8) and considering the derivative of  $\beta_n^{(\lambda,\varphi)}(t)$  with respect to  $t$ , we have that

$$\begin{aligned} \frac{d\beta_n^{(\lambda,\varphi)}}{dt} &= \frac{d}{dt} \left( \frac{\zeta_n^{(\lambda,\varphi)}}{\zeta_{n-1}^{(\lambda,\varphi)}} \right) = \frac{1}{\zeta_{n-1}^2} \left( \zeta_{n-1}^{(\lambda,\varphi)} \frac{d\zeta_n^{(\lambda,\varphi)}}{dt} - \zeta_n^{(\lambda,\varphi)} \frac{d\zeta_{n-1}^{(\lambda,\varphi)}}{dt} \right) \\ &= \frac{1}{\zeta_{n-1}^2} \left( \zeta_{n-1}^{(\lambda,\varphi)} \alpha_n^{(\lambda,\varphi)} \zeta_n^{(\lambda,\varphi)} - \zeta_n^{(\lambda,\varphi)} \alpha_{n-1}^{(\lambda,\varphi)} \zeta_{n-1}^{(\lambda,\varphi)} \right) = \alpha_n^{(\lambda,\varphi)} \beta_n^{(\lambda,\varphi)} - \alpha_{n-1}^{(\lambda,\varphi)} \beta_n^{(\lambda,\varphi)}, \end{aligned}$$

which yields the second equation in Equation (57) and this completes the proof.  $\square$

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