

# Endogenous Fluctuations in an Endogenous Growth Model: An analysis of Inflation Targeting as a Policy

## Abstract

We implement two different monetary policies – an inflation targeting policy as well as a cash reserve requirement – in a monetary endogenous growth overlapping generations model characterized by production lags and analyse the growth dynamics that emerge from this framework. The growth process is endogenized by allowing productive government expenditure on infrastructure, complementing the lagged private capital input. In the presence of these monetary policies, we show that multiple equilibria emerge along different growth paths, with the low-growth (high-growth) equilibrium being unstable (stable) and locally determinate (locally indeterminate). In addition, we highlight the possibility of convergent or divergent endogenous fluctuations and even topological chaos around the high-growth equilibrium in the growth path where the monetary authority follows a high inflation targeting regime. Conversely, when the monetary authority follows a low inflation targeting regime, oscillations do not occur around either the low-growth or high-growth equilibrium. Moreover, a strictly non-negative cash reserve requirement is a necessary and sufficient condition to initiate the growth process.

*Keywords:* endogenous fluctuations, inflation targeting, production lags, chaos, indeterminacy.

*JEL Classification:* C62, E32, O42

# 1 Introduction

“Economic progress, in capitalist society, means turmoil.” –  
Joseph A. Schumpeter (1942)<sup>1</sup>

Since the seminal work of Kalecki (1935) on production lags and business cycles, the study of the impact of different types of lags (including production lags) has been the intense focus of a vast literature<sup>2</sup>. Noteworthy contributions analysing the impact of various forms of lags on theoretical issues relating to prices, markets, investments, and cycles, among others include those of Goodwin (1947), Mayer (1960), May (1976), Kydland and Prescott (1982), Day (1983), Grandmont (1985) and again, Goodwin (1990).

More closely related to the discussion in this paper are the studies of El-Hodiri, Loehman and Whinston (1972) who are the first to introduce lags into a model of optimal growth; Benhabib and Day (1982) who characterize wide classes of utility functions which generate erratic dynamics in an overlapping generations OLG model; Reichlin (1986) who uses the Hopf bifurcation theorem to detect stable or unstable equilibrium trajectories where the determining parameter of the bifurcation is the technological externality to the production process; Galor and Ryder (1991) who study the dynamic efficiency of equilibria in an OLG model; Michel (1993) who shows that the growth rate of the economy oscillates on a transitional path if capital used in production is *not* contemporaneous; Matsuyama (1999) who finds that economic fluctuations are driven by periods of high investment or periods of high innovation; Kitagawa and Shibata (2001) who shows that investment gestation lags causes permanent cyclical movements in the *level* of national income without a production technology; and lastly Kitagawa and Shibata (2005) who develop a simple OLG model with investment gestation lags and find that if the production technology is of the AK-type, the existence of investment lags causes permanent cyclical fluctuations in the growth rate.

The motivation for the discussion in this paper stems largely from the initial findings in Michel (1993), the subsequent analysis detailed in Chetty and Ratha (1996) and finally, Gupta (2011) who studies growth dynamics in a similar OLG model with no money or productive government expenditure, but rather using firm-specific and per capita economy-wide capital input. In this setting, endogenous convergent fluctuations emerge due to the lagged production input, where the speed of the convergence is determined by the marginal product of labour given the initial value of the gross growth rate. The Chetty and Ratha (1996) findings, that growth is feasible in an

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<sup>1</sup>Capitalism, Socialism and Democracy.

<sup>2</sup>Although Jevons (1835-1882) has been credited with opening the discussion on production lags and cycles in output whilst it was more likely Hearn (1826-1888) who first broached the concept, it was in fact Kalecki who formally analysed this first in his 1935 paper, “A Macrodynamic Theory of Business Cycles” published in *Econometrica*.

OLG model with production lags if the productivity of capital is sufficiently high and borrowing is for capital services only, and not to finance wages in advance as well, is not only contradictory to the well-known Jones and Manuelli (1990) result – that growth rates in an OLG model with convex production is bounded above by zero, and hence, sustained growth is not feasible – but it also highlights the crucial impact of the time-structure of production on the economic growth rate.

Due to the technology used in Chetty and Ratha (1996) being of the Solow (1956)-type, and hence, the results pertaining only to exogenous growth processes determined by the population growth, there was no implication for endogenous growth processes in a setting where the time-structure of the production process is altered. In this sense, the Chetty and Ratha (1996) findings suffer from the same drawback as any Solow (1956)-type model, in that it lacks the ability to explain the non-zero growth in the per-capita standard of living in steady-state observed in the data. The Gupta (2011) findings do extend the impact of lagged production inputs to endogenous growth processes, however the economic environment presented therein included a Romer (1986)-type production technology and did not include a role for money, and hence, a role for monetary policy.

Complimentary to the theoretical motivation, Day (1983) provides a more philosophical reminder:

...concerned with the emergence of erratic fluctuations in economic growth processes, fluctuations of a highly irregular or unstable nature termed “chaotic” in the mathematical literature, that emerge *endogenously* [own emphasis] through the interplay of technology, preferences, and behavioural rules alone, with *no exogenous interference* [own emphasis] from stochastic shocks.

Against this backdrop, we develop a monetary endogenous growth overlapping generations model with inflation targeting, characterised by production lags, to analyse growth dynamics. The growth process is endogenized by allowing for productive government expenditure on infrastructure<sup>3</sup> in the vein of Barro (1990). Money is introduced through an obligatory reserve requirement, set by government and imposed on the banking system, which otherwise operate in a perfectly competitive environment<sup>4</sup>. Moreover, instead of the standard money growth rule set by the monetary authority,

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<sup>3</sup>The infrastructure referred to here, comprises typical government activities that stimulate firm activities, such as highways, railways, water systems, power systems, police and fire services, and courts. For simplicity, we will assume that there is no congestion as in Barro and Sala-i-Martin (1992) and the infrastructure provided is a non-rival public good for firms.

<sup>4</sup>This is a standard treatment of money in the literature. See Bittencourt, Gupta and Stander (2014) as well as Gupta and Stander (2013) and the references cited therein, for a detailed account of the related literature and the motivation for this treatment of money. Alternatively, we could have introduced money through a cash-in-advance constraint as in

we assume that the monetary authority implements an inflation-targeting (IT) regime across both generations. This assumption introduces growth dynamics into the model that yield results much richer than those presented by Michel (1993), Chetty and Ratha (1996) or Gupta (2011).

The model developed herein, albeit in an OLG framework, could be compared to the “time-to-build” model of Kydland and Prescott (1982), and the “vintage capital” infinitely-lived, representative agent model of Benhabib and Rustichini (1991). The first comprehensive survey on indeterminacy of equilibria in OLG models, was provided by Woodford (1984). Quite clearly, the endeavour to understand the impact of production lags and growth dynamics across two different strands of the literature is not new. Though, the use of lagged inputs is not as prevalent in analysing growth processes as perhaps it should be, especially in the theoretical<sup>5</sup> growth literature where the use of contemporaneous capital and labour inputs are almost the standard treatment. This could perhaps largely be attributed to a concern first shared in Fabbri and Gozzi (2008), that “...the introduction of vintage capital allows to explain some growth facts, but strongly increases the mathematical difficulties”. Fabbri and Gozzi (2008) then proceed to analyse fluctuations using the Maximum Principle as well as Dynamic Programming.

In contrast, we follow a simpler tract in analysing growth fluctuations that are generated endogenously. The theoretical OLG model presented here is developed along the lines of Gupta (2011). However, we extend this analysis by allowing for the role of money and hence, monetary policy as well as productive government expenditure and the include a role for the banking sector. Furthermore, introducing an inflation targeting regime ties the optimal growth rate to the inflation target, which allows for a deeper understanding of the role of monetary policy on the equilibrium growth path and endogenous fluctuations in the characterised economy.

The rest of this paper is organised as follows: Section 2-4, respectively, defines the economic environment and solves the specific model, characterises equilibrium along a balanced growth path (BGP) and then examines the growth dynamics of the characterised economy. Section 5 offers some concluding remarks.

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Kudoh (2007). However, our results would remain unchanged, as long as we either assume the parameterization of the utility function is such that either savings is positively related to the interest rate or is inelastic with respect to the interest rate. However, in the former case, where the savings function is positively related to the interest rate, no closed-form solutions are obtained and we need to linearize equations. Introducing money through the cash-reserve requirement allows us to avoid these complications without any loss of generality.

<sup>5</sup>In the empirical literature, however, the standard inclusion of lags of the growth process in the modelling of growth dynamics could be seen as a *de facto* attempt to account for the impact of lagged inputs on the growth process.

## 2 The economic setting

Time is divided into discrete segments and indexed by  $t = 1, 2, \dots$ . The principal economic activities are: (i) every possible two-period lived overlapping generation consumer/labourer, receives a positive young-age labour endowment of unit one, but retires and consumes only when old<sup>6</sup>. Thus, at time point  $t$ , there are two coexisting generations of young-age and old-age consumers.  $N$  people are born at each time point  $t \geq 1$ . At time point  $t = 1$ , there exist  $N$  people in the economy called the initial old, who live for only one period. The young-age consumer supply their one unit of labour inelastically to earn a wage income. The entire wage income is deposited into banks for future consumption; (ii) each infinitely-lived producer uses the same production technology to produce a single final good, using the inelastically supplied labour, physical capital which is borrowed from the banks and public capital supplied by the government; (iii) the banks operate in a competitive environment and perform a simple pooling function<sup>7</sup> by collecting the deposits from the consumers and lending it out to the firms after meeting an obligatory cash reserve requirements. We assume that banks do not spend any resources in performing this intermediary function<sup>8</sup>; and (iv) there is an infinitely-lived government which meets its productive expenditure on infrastructure by generating seigniorage income. The government follows an inflation targeting (IT) regime and also controls the reserve requirement. The government balances its budget on a period-by-period basis. There is a continuum of each type of economic agent with unit mass.

### 2.1 Consumers

All consumers have the same preferences, so there is a representative agent in each period. When young, consumers inelastically supply their unit of time endowment,  $n_t$  to earn a real wage of  $w_t$  that is saved as a bank deposit,  $d_t$ . The consumer retires when old and consumes  $c_{t+1}$  from the

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<sup>6</sup>This assumption ensures tractability and makes the analysis independent of the consumers utility function, as it abstracts from the consumption-savings decision. See Woodford (1984) for initial discussion on this, although this assumption is frequently used in the OLG literature. See among others, Cazzavillan (1996) and Sodini (2011) for more details. In fact, Bhattacharya and Qiao (2007), Gupta and Vermeulen (2010) and Gupta (2011) stress that the interest-inelastic nature of the savings function is a realistic representation of the true world.

<sup>7</sup>This is an implicit assumption that individual consumers cannot finance the firm's demand for investment, as the firm requires a minimum level of supply of capital that the consumer can not meet.

<sup>8</sup>This is a simplifying assumption, but the profit function of the bank could easily be adapted to account for a fraction of the deposits spent as resource cost. See footnote 13 for more details.

investment of young-age savings<sup>9</sup>. Formally, the young-age consumer wants to maximize<sup>10</sup>:

$$\max_{c_{t+1}} U = u(c_{t+1}) \quad (1)$$

subject to:

$$p_t d_t = p_t w_t \quad (2)$$

$$p_{t+1} c_{t+1} = (1 + i_{dt+1}) p_t d_t \quad (3)$$

where  $U$  is assumed to be twice-differentiable, such that  $U'(c) > 0$  and  $U''(c) < 0$ .  $c_{t+1}$  is old-age consumption,  $d_t = \frac{D_t}{p_t}$  measures real deposits held by consumers at the banks (where  $D_t$  is the amount of nominal deposits held by consumers),  $w_t$  is the real wage earned by the young-age consumer,  $i_{dt+1}$  is the nominal interest rate received on deposits at  $t + 1$ ,  $p_t$  is the price level in  $t$  and  $p_{t+1}$  is the price level in  $t + 1$ . Equation (2) is the feasibility (first-period budget) constraint for the young-age consumer and (3) is the budget constraint of the old-age consumer.

## 2.2 Financial intermediaries

There exist a finite number of banks in this economy, which we assume to behave competitively and who are all subject to an obligatory cash reserve requirement  $\gamma_t$ , set by the government. Two simplifying assumptions – that no resources are used to operate the banking system and bank deposits are essentially one period contracts – guarantee that all competitive banks levy the same cost on their loans, the nominal loan rate  $i_{lt}$  and guarantee the depositor a nominal deposit rate,  $i_{dt}$ . Banks accept and pool deposits<sup>11</sup>, choose their allocation portfolio of loans and required cash reserves and then extend loans to firms, subject to  $\gamma_t$ , with the goal of maximising their profits. Subsequently, banks receive interest income from loans to firms and meet their interest obligations to depositors. The bank's balance sheet is constrained by the reserve requirement, and is represented by  $(1 - \gamma)d_t = l_t$ . Hence, all banks attempt to:

$$\max \Pi_{Bt} = i_{lt} L_t - i_{dt} D_t \quad (4)$$

subject to:

$$M_t + L_t \leq D_t \quad (5)$$

$$M_t \geq \gamma_t D_t \quad (6)$$

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<sup>9</sup>We omit tax transfers from the consumer's budget constraint for simplicity. See Gupta and Vermeulen (2010) and the references cited therein, for further details.

<sup>10</sup>Optimisation solutions for the different economic agents are fully set out in the Appendix.

<sup>11</sup>Pooling occurs along the lines described in Bryant and Wallace (1980), since we assume that capital is illiquid and is only created in large minimum denominations.

where  $\Pi_{Bt}$  is the bank's net profit function<sup>12</sup>;  $M_t$  is the cash reserves held by the banks to meet the reserve requirement,  $\gamma_t$  is the reserve requirement ratio, and  $L_t$  are the amount of nominal loans extended to the firms. Equation (5) is the feasibility constraint resulting from optimal financing contracts<sup>13</sup> and (6) is the bank's reserve requirement constraint. A competitive banking sector is characterised by free entry, which drives profits to zero. Thus, given that (5) and (6) binds, the solution to the bank's problem yields:

$$i_{lt} = \frac{i_{dt}}{1 - \gamma_t} \quad (7)$$

Equation (7) clearly shows that cash reserve requirements lead to a distortion in financial intermediation, in the sense that it induces a wedge between the interest rate on deposits and the lending rates<sup>14</sup>. Since the cash reserves held by banks,  $M_t$  is rate-of-return dominated by loans, (6) will be binding as banks will hold just enough real money balances to satisfy the legal reserve requirements.

## 2.3 Firms

Firms have access to the same Barro (1990)-type production technology to produce a single final good, using physical capital  $k_t$ , labour,  $n_t$  and public capital,  $g_t$  with the technology specified by:<sup>15,16</sup>

$$y_t = A k_{t-1}^\alpha (n_t g_t)^{1-\alpha} \quad (8)$$

where  $A > 0$ ,  $0 < \alpha(1 - \alpha) < 1$  is the elasticity of output with respect to capital and labour/publicly-provided infrastructure, respectively.  $k_{t-1}$ ,  $n_t$  and  $g_t$  denote the lagged capital, labour and government infrastructure

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<sup>12</sup>Note that although the reserve requirement  $M_t$  is part of the bank's resources, it only forms part of the bank's gross profit function as in Chari et al. (1995), Haslag and Young (1998) and Basu (2001).

<sup>13</sup>See Myerson (1979) for more detail.

<sup>14</sup>The simplifying assumption that banks operate without spending resources, could easily be dropped in favour of the following environment: if we assume that banks spend a portion of the deposits as resource cost in operating the bank system, the bank's optimisation problem is based on the net profit function represented by  $\max \Pi_{Bt} = i_{lt}L_t - i_{dt}D_t - cD_t$ , with  $c$  being the fraction of deposits banks spend on its operations. This would lead to the optimisation solution (with the same constraints) where  $i_{lt} = \frac{i_{dt}}{1 - \gamma_t - c}$ . Redefining  $\gamma_t^c = (\gamma_t + c)$  will not affect our results.

<sup>15</sup>Note also that the form of the production function implies that the public services are complementary with the private inputs in the sense that an increase in  $g_t$  raises the marginal products of  $n_t$  and  $k_t$ . This setting parallels the production function for the learning-by-doing/spillovers model of Romer (1986), except that the per-capita aggregate capital stock,  $\bar{k}_t$ , has been replaced by the quantity of public goods,  $g_t$ .

<sup>16</sup>One could also allow the public capital input in the production function to be lagged by one period, which would yield a technology of the form:  $y_t = A k_{t-1}^\alpha (n_t g_{t-1})^{1-\alpha}$ . This specification, however, does not affect our results in any way, as can be seen from the brief discussion in the Appendix.

expenditure inputs at time  $t$ , respectively. Investment in physical capital,  $i_{kt}$  is limited by the availability of funding to the firms since we assume that firms are able to convert loans,  $L_{t-1}$  into fixed capital such that  $p_{t-1}i_{kt-1} = L_{t-1}$ . Following Diamond and Yellin (1990) and Chen, Chiang and Wang (2008), we assume that the goods producer uses up the unsold consumption good in a way which is consistent with lifetime value maximization of the firms. The representative firm therefore maximises its discounted stream of net profit flows subject to the capital evolution constraint and the loan constraint. Formally:

$$\max_{k_t, n_t} \sum_{i=0}^{\infty} \rho^i [p_t y_t - p_t w_t n_t - (1 + i_{Lt}) L_{t-1}] \quad (9)$$

subject to:

$$k_t \leq (1 - \delta_k) k_{t-1} + i_{kt-1} \quad (10)$$

$$p_{t-1} i_{kt-1} \leq L_{t-1} \quad (11)$$

$$L_{t-1} \leq (1 - \gamma_t) D_{t-1} \quad (12)$$

where  $\rho$  is the firm's (constant) discount rate and  $\delta_k$  is the (constant) rate of capital depreciation. The firm solves the following recursive problem to solve for the demand of labour and investment:

$$V(k_{t-1}) = \max_{k_t, n_t} [p_t A k_{t-1}^\alpha (n_t g_t)^{1-\alpha} - p_t w_t n_t - p_t (1 + i_{Lt}) (k_t - (1 - \delta_k) k_{t-1})] + \rho V(k_t) \quad (13)$$

The corollary of this dynamic formulation is the following efficiency conditions for the choice variables:

$$n_t : \quad w_t = (1 - \alpha) A \left( \frac{k_{t-1}}{n_t} \right)^\alpha g_t^{1-\alpha} \quad (14)$$

$$k_t : \quad (1 + i_{Lt}) = \rho \left( \frac{p_{t+1}}{p_t} \right) \left[ \alpha A \left( \frac{n_{t+1} g_{t+1}}{k_t} \right)^{1-\alpha} + (1 + i_{Lt+1}) (1 - \delta_k) \right] \quad (15)$$

(15) represents the efficiency condition for the optimal investment decision of the firm. Intuitively, the firm weighs the cost of increasing investment in the current period with the future stream of benefits generated from the additional capital invested in the current period. Furthermore, assuming that capital depreciates fully between periods, or  $\delta_k = 1$  without any loss of generality<sup>17</sup>, simplifies (15) to  $(1 + i_{Lt}) = \rho \left( \frac{p_{t+1}}{p_t} \right) \left[ \alpha A \left( \frac{n_{t+1} g_{t+1}}{k_t} \right)^{1-\alpha} \right]$ . Equation (14) represents the optimal hiring decision for a firm, which results in

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<sup>17</sup>This assumption provides analytical tractability in the same vein as Barnett, Bhattacharya and Bunzel (2013), and should be viewed against the generational structure of the OLG model, where a typical generation might span 20 or 30 years. Also see Cazzavillan (1996), Chen (2006) and Dávila (2012).



the firm hiring labour up to the point where the marginal product of labour is equal to the real wage.

## 2.4 Government

An infinitely-lived consolidated government – comprising a monetary as well as a treasury (or fiscal) wing – purchases  $g_t$  units of consumption goods, and government expenditure is assumed to be a productive factor in the firm’s production function. The government finances its productive consumption expenditure solely through the collection of seigniorage income. Formally, the government’s budget constraint is:

$$g_t = \frac{M_t - M_{t-1}}{p_t} \quad (16)$$

and furthermore the monetary authority implements an inflation targeting regime to bring about macroeconomic stability. These interventions could be directed at either price stability or output stability, or a combination of both, given that government also provides productivity-enhancing infrastructure that is used as a factor in the production of the consumption good. Hence, we have that  $\Pi_t = \hat{\Pi}$  for all  $t$ . Note, now with  $(1 + i_{Lt}) = \rho \hat{\Pi} [\alpha A (\frac{n_{t+1} g_{t+1}}{k_t})^{1-\alpha}]$  from (15),  $n_{t+1} = 1$  and from the government budget constraint  $\frac{g_{t+1}}{k_t} = (\gamma_t(1 - \alpha)A[1 - \frac{1}{\Omega_{t+1}\hat{\Pi}}])^{\frac{1}{\alpha}}$ , this leads to an interest rate rule where the monetary authority not only responds to inflation, but also the growth rate of the economy. Given this policy rule for the rate of inflation, the nominal quantity of money adjusts endogenously to satisfy the demand for money. Therefore, using  $M_t = \gamma_t D_t$  from (6), the government budget constraint in real terms can be rewritten as:

$$g_t = \gamma_t d_t \left(1 - \frac{1}{\Omega_t \hat{\Pi}}\right) \quad (17)$$

where  $\Omega_t$  is the gross growth rate at time  $t$  and  $\hat{\Pi}$  is the gross inflation rate across all  $t$ .<sup>18</sup> Note that the actual monetary policy rule is stated by (17), which as mentioned earlier, implies that the government responds to both inflation and growth under an inflation targeting regime.<sup>19</sup>

<sup>18</sup>It should be noted that because of this IT-regime, it implies that inflation is constant across both periods. Hence, to contrast a low versus high inflation environment entails simply changing the value of  $\hat{\Pi}$ . However, most countries that have implemented IT, have actually implemented an inflation targeting band. In the case of South Africa, for example, the inflation targeting band is 3 – 6%. Therefore, any discussion around a high versus low inflation target is intended to also include the sensitivity, or accuracy, of these IT bands.

<sup>19</sup>Although not a traditional Taylor rule, we are of the opinion that it does resemble a “Taylor-type” rule, in the sense that it incorporates information from both the growth rate (as an approximation of deviations from the output gap) as well as from the inflation

### 3 Equilibrium

A valid, perfect-foresight equilibrium for the characterised economy is defined as a sequence of prices  $\{p_t, i_{Lt}, i_{Dt}\}_{t=0}^{\infty}$ , allocations  $\{c_{t+1}, n_t, i_{kt}\}_{t=0}^{\infty}$ , stock of financial assets  $\{m_t, d_t\}_{t=0}^{\infty}$  as well as policy variables  $\{g_t, \gamma_t\}_{t=0}^{\infty}$  such that:

- Taking  $g_t, \gamma_t, p_t, p_{t+1}, i_{Dt}$  and  $w_t$ , the consumer maximises utility in (1) such that both (2) and (3) holds;
- Banks maximise profits subject to  $i_{Lt}, i_{Dt}$  and  $\gamma_t$  such that (7) holds;
- The real allocations solve the firm's date  $t$  profit maximization problem, given prices and policy variables, such that (14) and (15) holds;
- The equilibrium money market condition,  $m_t = \gamma_t d_t$  holds for all  $t \geq 0$ ;
- The loanable funds market equilibrium condition,  $p_{t-1} i_{kt-1} = L_{t-1}$  given the total supply of loans  $L_{t-1} = (1 - \gamma_t) D_{t-1}$ , holds for all  $t \geq 0$ ;
- The equilibrium goods market resource constraint,  $c_t + i_{kt-1} + g_t = A k_{t-1}^{\alpha} (n_t g_t)^{1-\alpha}$  holds, where  $i_{kt-1} = (1 - \gamma_t) d_{t-1}$  ;
- The government budget constraint in (17) is balanced on a period-by-period basis; and
- $d_t, p_t, i_{Lt}, i_{Dt}$  and  $A$  are positive for all  $t \geq 0$ .

### 4 Growth Dynamics

In this section, we analyse the growth dynamics derived from the model. Using (2), (10), (11), (11), (12), (14) and (17) we obtain the following relation between the gross growth rate in  $t + 1$ ,  $\Omega_{t+1}$  and the gross growth rate in  $t$ ,  $\Omega_t$ . More clearly, we have  $\Omega_{t+1} = f(\Omega_t)$ <sup>20</sup> which is given by:

$$\Omega_{t+1} = (1 - \gamma_t)(1 - \alpha)A \left[ \gamma_t(1 - \alpha)A \left(1 - \frac{1}{\Omega_t \hat{\Pi}}\right) \right]^{\frac{1-\alpha}{\alpha}} \frac{1}{\Omega_t} \quad (18)$$

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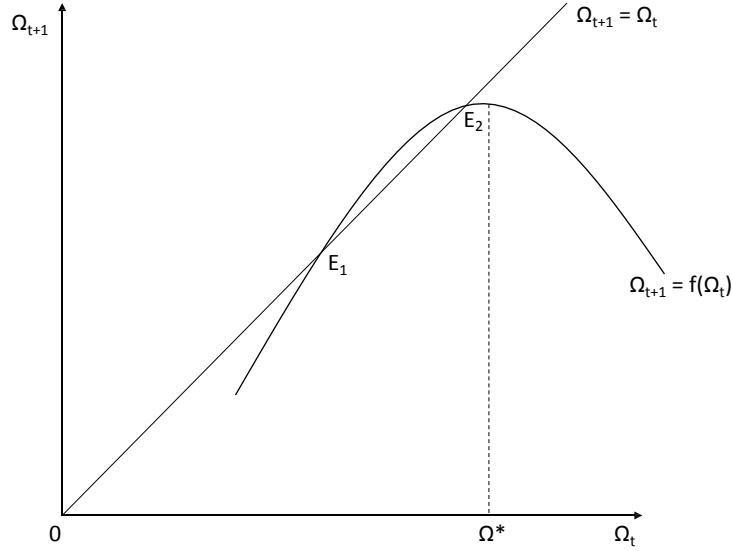
target itself (under IT, expected future inflation is just inflation in the current period, and hence there are no deviations). See Taylor (2012), Taylor (2013) and Hatcher and Minford (2014) for a detailed discussion on this point.

<sup>20</sup>Note, if we relaxed the assumption that  $\delta_k = 1$  and we allow for any non-negative range of depreciation, say  $0 < \delta_k < 1$ , the growth equation in (18) would just be amended with an additional term on the RHS, equal to  $+(1 - \delta_k)$ .

The function  $f(\Omega_t)$  satisfies the following conditions: (a)  $f'(\Omega_t) > 0$  for  $\Omega < \Omega^*$  [ $= \frac{1}{\hat{\Pi}\alpha}$ ]<sup>21</sup>; (b)  $f'(\Omega_t) = 0$  for  $\Omega = \Omega^*$ ; (c)  $f'(\Omega_t) < 0$  for  $\Omega > \Omega^*$ ; (d)  $\lim_{\Omega \rightarrow 0} f'(\Omega_t) = \infty$ ; (e)  $\lim_{\Omega \rightarrow \infty} f'(\Omega_t) = -\infty$ ; and (f)  $\lim_{\Omega \rightarrow \infty} f(\Omega_t) = 0$ .

Depending upon the values of  $A$ ,  $\alpha$ ,  $\gamma_t$ <sup>22</sup> and  $\hat{\Pi}$  and based on the given properties of  $f(\Omega)$  described above, there are different types of equilibrium growth paths, yielding two discrete versions of multiple equilibria – one low-growth and one high-growth – as clearly shown in Figure 1 and Figure 2.

Figure 1: Multiple Equilibria *without* endogenous fluctuations



The inverted U-shape nature of the function,  $f(\Omega)$  should be intuitively

<sup>21</sup>When  $g_{t-1}$ , along with  $k_{t-1}$ , is included in the production structure instead of  $g_t$ , the optimal value for  $\Omega^* = \frac{2-\alpha}{\hat{\Pi}}$ , which shifts the turning point of the unimodal  $f(\Omega)$  to the left, relative to the current case. This does not, however, change our results in terms of analysing indeterminacy, stability, endogenous fluctuations or chaos in any possible way.

<sup>22</sup>It is easily verified that in the case of  $\gamma_t = 0$ , or when the cash reserve requirement is zero, the equilibrium growth rate falls to 0, or in the case of non-negative depreciation ( $0 < \delta_k < 1$ ) it collapses to  $(1 - \delta_k)$ , the undepreciated portion of the capital stock carried over from the previous period. This equilibrium is not economically interesting, as it turns out that for the growth process to initiate, a strictly non-negative (small)  $\gamma_t$  is required. Absent the cash reserve requirement, the uni-modal  $f(\Omega)$  becomes a horizontal line, regardless of the parameter values of  $A$ ,  $\alpha$  or  $\hat{\Pi}$ . We will offer a brief discussion on the different alternative rules we could have used to introduce money in this economy, in the concluding remarks beside those contained in footnote (4).

understood given there exists both a positive and negative effect of an increase in  $\Omega_t$ , the gross growth rate at  $t$ . An increase in  $\Omega_t$  results in higher seigniorage revenue for government, raising the ratio of real government expenditure to real wage. This implies a higher growth rate via the higher available resources for productive expenditure on publicly-provided infrastructure relative to real wage. Due to the complementarity between private and public capital investment in output, private capital investment in output now also increases. The negative effect stems purely from the time-structure of the production function, where  $f(\Omega)$  decreases as  $k_{t-1}$  decreases.

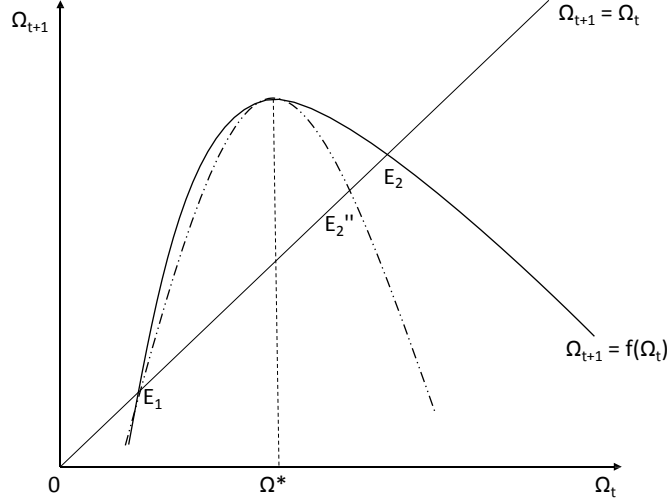
The positive effect is prevalent for values of  $\Omega_t$  less than the optimal  $\Omega^* = \frac{1}{\hat{\Pi}\alpha}$ , hence  $f(\Omega)$  is positively-sloped until  $\Omega^*$ . Beyond the optimal  $\Omega^*$ , the negative effect is prevalent, hence a negatively-sloped  $f(\Omega)$  is observed in this region. It should be clear that the different positional  $f$  loci in Figure 1 and Figure 2, is determined by different values of the parameter set consisting of  $\{A, \alpha, \gamma_t, \hat{\Pi}\}$ . It obtains then, that the position and existence of the different equilibria hinges critically on the values of these parameters, and these values have to be such that the function  $f(\Omega)$  intersects the 45 degree line, which is representative of the loci of steady states of this OLG economy. Based on this, the particular inference is:

- The low-growth (high-growth) equilibrium depicted in  $E_1$  ( $E_2$ ) is unstable (stable) and locally determinate (locally indeterminate)<sup>23</sup>. The low-growth (high-growth) equilibrium is unstable (stable) under perfect foresight because the  $f$  loci intersects the 45 degree line from below (above). Furthermore, although  $k_{t-1}$  is a state variable and cannot jump,  $\Omega_t = \frac{k_t}{k_{t-1}}$  is not a state variable and, hence, can jump<sup>24</sup>. This resultant jump then implies that there is infinitely many rational expectations (RE) paths to the high-growth and stable equilibrium from any initial given value for  $k_1$ . Hence, the stable equilibrium in this economy at  $E_2$  suffers from local indeterminacy, as there is still asymptotic convergence to the BGP;
- As depicted in Figure 2, on a different growth path the high-growth and stable equilibrium can also be characterized by endogenous fluctuations, given that the slope of the  $f$  locus is negative at point  $E_2$ . At point  $E_2$ , however, the growth path might still display convergent fluctuations, if the slope of the function  $f$  is not “too steep”, or formally, if the slope of  $f < 1$  in absolute terms. Recall, the position of

<sup>23</sup>See Eggoh and Villieu (2014) for a detailed discussion and technical graphical analysis of indeterminacy. Also, Mitra and Nishimura (2001) provides an extensive survey on stability and indeterminacy.

<sup>24</sup>Vind (1967) first formulated the idea that if time as a variable can be controlled, and it should be clear that  $\Omega_t = \frac{k_t}{k_{t-1}}$  is a mechanism for controlling, or eliminating time, then jumps with respect to time is possible.

Figure 2: Multiple equilibria *with* endogenous fluctuations



the  $f$  locus depends on the values of the structural parameters of the model,  $A$ ,  $\alpha$ ,  $\gamma_t$  and  $\hat{\Pi}$ ;

- As in Gupta and Vermeulen (2010), our framework too can yield chaotic behaviour of the growth rate around the high-growth equilibrium, given that the slope of the function  $f$  is “steep enough”, or if the slope of  $f > 1$  in absolute terms. This “steepness requirement” holds if and only if the following conditions are satisfied<sup>25</sup>: Mitra (2001) describes the conditions required for chaos to emerge from growth cycles, namely the map  $f$  is required to be a continuous function from  $X$  to  $X$ , where the state space  $X$  is an interval on the non-negative part of the real line, with  $(X, f)$  defining the dynamical system. Further, the map  $f$  must have a unimodal distribution with a maximum at  $\Omega^*$  with  $f(\Omega^*) > \Omega^*$  and the high-growth equilibrium at point  $E_2''$  must ensure that the steady-state level of growth rate corresponding to point  $E_2''$ , exceeds  $\Omega^*$ . Plainly, for a non-linear function with a single maximum, the value of  $\Omega^*$  must be less than the value (say,  $\Omega_H$ ) corresponding to point  $E_2''$  – or the intersection point of  $f(\Omega)$  with the steady

<sup>25</sup>See Bhattacharya and Qiao (2007) as well as Boldrin, Nishimura, Shigoka and Yano (2001) for full details.

state locus (the 45 degree line). Furthermore, the value of  $\Omega^*$  must be smaller than the value of its first iterate and bigger than the value of its second iterate. Evidently, the high-growth equilibrium point  $E_2''$  on the equilibrium path depicted in Figure 2 is exactly such point, and therefore exhibits topological chaos at that equilibrium. Economically, a change in the IT regime could lead to such chaotic behaviour.

- The economic intuition for the scenario's described above, flows from the parameter set of (18). For the dynamics in the growth process to initiate, a “small” strictly non-negative  $\gamma_t$  is required, without which the dynamics in the growth process never takes off. It is evident that without  $\gamma_t$ , there is also no productive government expenditure, and hence output is produced by  $y = Ak_{t-1}^\alpha$ . Recall  $0 < \alpha(1 - \alpha) < 1$ , implies decreasing returns to scale in capital which would result in no feasible sustained growth. The initial dynamics is then kick-start by say  $\gamma_t = 0.01$ . In fact, in this setting  $\gamma_t > 0$  is a sufficient and necessary condition for growth dynamics, and hence fluctuations to occur. The shape of the map (or curve) is determined solely by  $\hat{\Pi}$ , which controls the position of the optimal  $\Omega^*$ , given a fixed value of  $\alpha$ , and  $\alpha$ , which controls the curvature (or “steepness”) of the map, especially for values of  $\Omega_t > \Omega^*$ . The technology parameter  $A$  shifts the map up (down) for bigger (smaller) values of  $A$ , along the vertical. So, for observed growth rates to the left of the optimum, higher growth results in higher seigniorage revenue and hence increased production.

#### 4.1 Policy Implications

From the perspective of inflation targeting monetary policy, our theoretical results have very important implications. While gestation lags are part of any production process, irrespective of whether we consider either developed or developing economies, they are of greater importance and more likely to occur in the latter set of countries. Hence, our model is perhaps more evident of developing nations pursuing inflation targeting monetary policy, contingent on production lags in the investment process.

Consequently, we show that central banks in developing countries should be exceptionally careful in deciding on their inflation targets. Inflation targets should be chosen based on the structural parameters of the model, and independent of any pressure from groups, such as the labour unions. Under external pressures, it is more likely that the inflation target will be relatively less stringent, and likely to be higher. As we show, in such a scenario, uncertain growth dynamics like fluctuations and even chaos are likely to emerge. What is of even greater concern is that these uncertain fluctuations will occur around the high-growth path; which, though indeterminate, is the stable equilibrium.

In other words, setting a high inflation target as a result of extraneous pressure instead of being theoretically motivated is likely to lead to tremendous uncertainty in the economy, at the extreme even making it chaotic, and nullifying the possible positive growth-effects of inflation targeting. Recall that the whole idea of inflation targeting was to ensure low stable rates of inflation to allow stability in the growth process, which in turn, is likely to ensure the achievement of other goals like addressing unemployment, eradicating poverty and reducing income inequality.

Therefore the message is quite clear, in the presence of production lags, monetary authorities should be pursuing low inflation targets to avoid uncertainty in the behaviour of the growth process.

## 5 Concluding remarks

Although recent studies in an OLG model environment also provide evidence of fluctuations, and some point to chaotic behaviour of the equilibrium growth rate, most of the recent findings hinges critically on non-linearities from the technology or preference set (Grandmont (1985); Michel (1993); Cazzavillan (1996) and Matsuyama (1999)), the returns-to-scale characteristics of the production function (Boldrin and Rustichini (1994); Benhabib and Farmer (1996); Mitra and Nishimura (2001) and Mino (2001)) and the specific characteristics of the generations contained in the model, specifically the bequest motive (Constantinides, Donaldson and Mehra (2007) and Barnett, et al. (2013)). Moreover, to the best of our knowledge, only Barnett, Bhattacharya and Bunzel (2010) as well as Gupta and Vermeulen (2010) specifically model the role of money in an OLG environment to analyse the existence of cycles.

Here, we develop a monetary endogenous growth overlapping generations models with inflation targeting and characterised by lagged capital in the production structure, and analyse growth dynamics that arise endogenously. The growth process is endogenized through productive government expenditure that is necessary for producing the consumption good. Money is introduced into the model through a banking sector that intermediates between depositors (consumers) and borrowers (firms) in a competitive environment, but are subjected to a cash reserve requirement on deposits. This obligation leads to a distortion in financial intermediation, driving a wedge between the real loan rate and the real deposit rate in equilibrium. If we introduced a simple money growth rule, with no lagged inputs there would be no growth dynamics. If we account for lagged inputs, but do not introduce an IT-regime, we could still admit endogenous fluctuations. An IT regime with no lagged inputs would produce convergent growth dynamics, without any oscillations. Growth dynamics are introduced through an inflation targeting monetary authority regime.

Within this framework, different equilibrium growth paths emerge once we have a strictly non-negative cash reserve requirement ratio. We detail the existence of multiple equilibria, both high-growth and low-growth where the low-growth equilibria is unstable and locally determinate. The opposite emerge for the high-growth equilibria. In addition, we find that endogenous fluctuations and topological chaos is possible around the high-growth equilibrium, due to the monetary authority following an inflation targeting regime and output depending on lagged capital inputs. Specifically, when the monetary authority sets a high inflation target, we can observe possible convergent or divergent endogenous fluctuations – and even chaotic behaviour – around the high-growth equilibrium on this growth path, if the value of  $\alpha$  is sufficient. Additionally, if the monetary authority sets a low inflation target, the resultant equilibria do not display any oscillations or chaotic behaviour, but the low-growth, low-welfare equilibrium for this growth path is unstable.

Most importantly, we highlight the critical role of a strictly non-zero cash reserve requirement in starting the growth process, and the subsequent but equally critical role that a low-inflation inflation targeting policy plays in ensuring certainty in the growth process.



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## A Appendix

### Optimisation solutions for all economic agents

Note that from the solution to the consumer's problem, we have:

$$d_t = w_t \quad (\text{A.1})$$

$$c_{t+1} = (1 + r_{dt+1})d_t \quad (\text{A.2})$$

from (2) and (3). The bank's solution follows directly from its net profit function, and the fact that (5) and (6) holds. We also obtain, from putting (6) into (5) that:

$$l_t = (1 - \gamma_t)d_t \quad (\text{A.3})$$

Recall the firm's optimisation problem, in recursive form:

$$V(k_{t-1}) = \max_{k_t, n_t} [p_t A k_{t-1}^\alpha (n_t g_t)^{1-\alpha} - p_t w_t n_t - p_t(1 + i_{Lt})(k_t - (1 - \delta_k)k_{t-1})] + \rho V(k_t) \quad (\text{A.4})$$

which yields the following first order conditions (FOC):

$$k_t : \quad p_t(1 + i_{Lt}) = \rho V'(k_t) \quad (\text{A.5})$$

$$n_t : \quad w_t = (1 - \alpha)A\left(\frac{k_{t-1}}{n_t}\right)^\alpha g_t^{1-\alpha} \quad (\text{A.6})$$

with the solution to the FOC for  $k_t$  found in the derivative of the value function with respect to  $k_{t-1}$ , updated for one period. Formally:

$$V'(k_t) = p_{t+1}\alpha A\left(\frac{n_{t+1}g_{t+1}}{k_t}\right)^{1-\alpha} + p_{t+1}(1 + i_{Lt+1})(1 - \delta_k) \quad (\text{A.7})$$

which results in (15). Simply substituting  $\delta_k = 1$  and  $n_{t+1}$  into (15), yields:

$$(1 + r_{Lt}) = \rho\left[\alpha A\left(\frac{g_{t+1}}{k_t}\right)^{1-\alpha}\right] \quad (\text{A.8})$$

and because  $(1 + r_{Lt})$  is constant, it implies that  $\left(\frac{g_{t+1}}{k_t}\right)^{1-\alpha}$  must be constant as well. From (7), we can show that  $(1 + r_{dt}) = (1 + r_{lt})(1 - \gamma_t) + \frac{\gamma_t}{\Pi}$ . From (15) it is also clear that  $(1 + r_{lt}) = \rho\left[\alpha A\left(\frac{g_{t+1}}{k_t}\right)^{1-\alpha}\right]$ , hence a higher inflation target would lead to a lower  $(1 + r_{lt})$  and due to the real rate on loans and the real rate on deposits being tied, in equilibrium, would also lead to lower  $(1 + r_{dt})$ .

From the government's budget constraint in (16), we have:

$$g_t = \frac{M_t - M_{t-1}}{p_t} \quad (\text{A.9})$$

$$= m_t - \frac{M_{t-1}}{M_t} \frac{M_t}{p_t} \quad (\text{A.10})$$

$$= m_t - \frac{1}{\Omega_t \hat{\Pi}} m_t \quad (\text{A.11})$$

$$= \gamma_t d_t \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \quad (\text{A.12})$$

with  $m_t = \gamma_t d_t$  and  $\Omega_t$  defined as the gross growth rate in period  $t$ , and because the government targets inflation in this setting, i.e.  $\Pi_t = \hat{\Pi}$ , we can define  $\hat{\Pi}$  as the gross inflation rate  $\forall t$ . Further, from (16), we also have  $\frac{g_{t+1}}{k_t} = (\gamma_t(1 - \alpha)A[1 - \frac{1}{\Omega_{t+1}\hat{\Pi}}])^{\frac{1}{\alpha}}$ , which implies the government responds to the growth rate as well, in addition to its inflation targeting goal.

### Derivation of the BGP gross growth rate

From the government's budget constrain in (17), and  $\delta_k = 1$  and  $n_{t+1} = 1$  we have:

$$g_t = \gamma_t d_t \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \quad (\text{A.13})$$

and using the fact that  $d_t = w_t$  and using a rearranged (14) twice, we first have:

$$g_t = \gamma_t(1 - \alpha)Ak_{t-1}^\alpha g_t^{1-\alpha} \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \quad (\text{A.14})$$

which simplifies to:

$$g_t = \left( \gamma_t(1 - \alpha)Ak_{t-1}^\alpha \left[ 1 - \frac{1}{\Omega_t \hat{\Pi}} \right] \right)^{\frac{1}{\alpha}} \quad (\text{A.15})$$

Plugging this expression for  $g_t$  back into the rearranged (14), along with realising that  $w_t = \frac{k_{t+1} - (1 - \delta)k_t}{(1 - \gamma_t)}$ , we have:

$$\frac{k_{t+1} - (1 - \delta)k_t}{(1 - \gamma_t)} = (1 - \alpha)A \left[ \gamma_t(1 - \alpha)A \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \right]^{\frac{1-\alpha}{\alpha}} k_{t-1} \quad (\text{A.16})$$

Recall that  $\delta_k = 1$  and dividing both sides with  $k_t$ , yields the expression for the gross growth rate found in (18):

$$\Omega_{t+1} = (1 - \gamma_t)(1 - \alpha)A \left[ \gamma_t(1 - \alpha)A \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \right]^{\frac{1-\alpha}{\alpha}} \frac{1}{\Omega_t} \quad (\text{A.17})$$

which clearly makes  $\Omega_{t+1} = f(\Omega_t)$ .

The derivative of  $\Omega_{t+1}$  is given as:

$$\frac{\partial \Omega_{t+1}}{\partial \Omega_t} = \frac{(\gamma_t - 1)[A(1 - \alpha)\gamma_t(1 - \frac{1}{\Omega_t \hat{\Pi}})]^{\frac{1}{\alpha}}(\alpha \Omega_t \hat{\Pi} - 1)}{(\hat{\Pi} \Omega_t - 1)\alpha \gamma_t \Omega_t} \quad (\text{A.18})$$

with the optimal solution for  $\Omega_t^* = \frac{1}{\hat{\Pi}\alpha}$ .

### **The BGP gross growth rate with $g_{t-1}$**

When lagged public capital is included, together with lagged private capital, the resultant ratio of public and private capital inputs from the government's budget constraint, is:

$$\frac{g_t}{k_t} = \left( \gamma_t(1 - \alpha)A \left[ 1 - \frac{1}{\Omega_{t+1} \hat{\Pi}} \right] \frac{1}{\Omega_{t+1}} \right)^{\frac{1}{\alpha}} \quad (\text{A.19})$$

The expression for the gross growth rate becomes:

$$\Omega_{t+1} = (1 - \gamma_t)(1 - \alpha)A \left[ \gamma_t(1 - \alpha)A \left( 1 - \frac{1}{\Omega_t \hat{\Pi}} \right) \right]^{\frac{1-\alpha}{\alpha}} \frac{1}{\Omega_t^{\frac{1}{\alpha}}} \quad (\text{A.20})$$

which still implies that  $\Omega_{t+1} = f(\Omega_t)$ . The related optimal solution for  $\Omega_t^* = \frac{2-\alpha}{\hat{\Pi}}$ , with the gross growth rate again inversely related to the gross inflation rate.