Grade-two fluids on non-smooth domain driven by multiplicative noise: Existence, uniqueness and regularity

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Abstract

In this paper we present a systematic study of a stochastic PDE with multiplicative noise modeling the motion of viscous and inviscid grade-two fluids on a bounded domain $\mathscr{O}$ of $\mathbb{R}^2$. We aim to identify the minimal conditions on the boundary smoothness of the domain for the well-posedness and time regularity of the solution. In particular, we found out that the existence of a $\mathbf{H}^1(\mathscr{O})$ weak martingale solution holds for any bounded Lipschitz domain $\mathscr{O}$. When $\mathscr{O}$ is a convex polygon the solution $\mathbf{u}$ lives in the Sobolev space $\mathbf{W}^{2,r}(\mathscr{O})$ for some $r > 2$ and $\mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ is continuous in $\mathrm{L}^2(\mathscr{O})$ with respect to the time variable. Moreover, pathwise uniqueness of solution holds. The existence result is new for the stochastic inviscid model and improves previous results for the viscous one. The time continuity result is new, even for the deterministic case when the domain $\mathscr{O}$ is a convex polygon.

Keywords: Grade two fluids, Lagrangian Averaged Euler equations, Martingale solution, Space and time regularity, Time discretization, Stochastic transport equations.
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1. Introduction

1.1. General introduction

In general, the constitutive law for a homogeneous incompressible fluid satisfies

$$\mathbb{T} = -\mathfrak{p}\mathbf{1} + \hat{\mathbb{T}}(\mathcal{E}(\mathbf{u})),$$

where $\mathbb{T}$ is the Cauchy stress tensor, $\mathbf{u}$ is the velocity of the fluid, and $\mathfrak{p}$ is the undetermined pressure due to the incompressibility condition, $\mathbf{1}$ is the identity tensor. The argument tensor $\mathcal{E}(\mathbf{u})$ of the symmetric-valued function $\hat{\mathbb{T}}$ is defined through

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right), \quad \mathbf{L} = \nabla\mathbf{u},$$

where the T superscript denotes the matrix transpose. If the extra tensor $\hat{\mathbb{T}}$ is a linear function of $\mathcal{E}(\mathbf{u})$ then we have a Newtonian fluid and the system of Partial Differential Equation obtained for the fluid dynamic is the Navier-Stokes equations. When the extra-tensor $\hat{\mathbb{T}}$ is a nonlinear function of $\mathcal{E}(\mathbf{u})$, then we have a non-Newtonian fluid. In the monograph [1] Noll and Truesdell introduced the theory of fluids of differential type to which belong a grade-two or second grade fluid. The stress tensor of this particular non-Newtonian fluid is given by

$$\hat{\mathbb{T}} = \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{1.1}$$

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Here ν is the kinematic viscosity, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = 2\mathcal{E}(\mathbf{u}) \text{ and } \mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1,$$

where D/Dt denotes the material derivative. The constants α_1 and α_2 represent the normal stress moduli. For the grade-two fluid to be compatible with the theory of thermodynamic, it was shown in [2] that

$$\alpha_1 + \alpha_2 = 0 \quad \text{and} \quad \alpha_1 \geqslant 0, \tag{1.2}$$

must hold. On the basis of the analysis done in Sections 4, 6, 7, 8, and 9 of [2], this condition ensures the unique existence and boundedness of the flow of grade-two fluids. We also refer to [3] and [4] for more recent work concerning these conditions.

Throughout this work we assume that $\alpha_1 = \alpha > 0$ and $\nu \geqslant 0$. Taking these conditions into account and assuming that the fluid is homogeneous with density $\rho = 1$, the system of Partial Differential Equations (PDEs) describing the motion of an incompressible grade-two fluid excited by an external force $\mathbf{f}$ takes the form

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\tilde{\mathfrak{p}} = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.3}$$

where

$$\tilde{\mathfrak{p}} = \mathfrak{p} - \alpha(\mathbf{u}\cdot\Delta\mathbf{u} + \frac{1}{4}|\mathbf{A}_1|) + \frac{1}{2}|\mathbf{u}|^2$$

is the modified pressure and $\operatorname{div} \mathbf{u} = 0$ is considered due to the incompressibility constraint.

The system (1.3) is frequently used to describe fluid models in petroleum industry, polymer technology and suspensions of liquid crystals. It was also used in [5] to study the connection of Turbulence Theory to Non-Newtonian fluids, especially fluids of differential type. When $\nu = 0$, the system (1.3) reduces to what is know the Lagrangian averaged Euler equations (LAEs) which appeared for the first time in the context of averaged fluid models in [6] and [7]. The derivation of LAEs used averaging and asymptotic methods in the variational formulation. The LAEs are also closely related to the following equation

$$u_t - u_{xxt} + 2\kappa u_x - 3uu_x = 2u_xu_{xx} + u_xu_{xxx},$$

where u_x, u_{xy}, etc, denote partial derivatives with respect to the variable x, x and then y, etc. This equation was proposed by Camassa and Holm in [8] to describe a special model of shallow water. As in the case of the grade-two fluid this new model of shallow water also reduces to LAEs when $\kappa = 0$ and in this case it was shown in [9] that it is the geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle. The works [10] and [11] also contain interesting discussions concerning the grade-two fluids and the LAEs.

1.2. Our basic model and results

In this paper we are interested in a stochastic version of the system (1.3). More precisely, we assume that a finite time horizon $[0, T]$, and an initial value $\mathbf{u}_0$ are given. The motion of a grade-two fluid filling a bounded Lipschitz domain $\mathcal{O}$ of $\mathbb{R}^2$ with initial condition $\mathbf{u}_0$ driven by a multiplicative random forcing $G(\mathbf{u})\frac{dW}{dt}$ is governed by the following system of stochastic PDEs

$$d(\mathbf{u} - \alpha\Delta\mathbf{u}) + (-\nu\Delta\mathbf{u} + \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\tilde{\mathfrak{p}})dt = G(\mathbf{u})dW \quad \text{in } \mathscr{O} \times [0, T], \tag{1.4a}$$

$$\operatorname{div} \mathbf{u} = 0 \;\text{ on }\; \mathscr{O} \times [0, T], \tag{1.4b}$$

$$\mathbf{u} = 0 \text{ in } \partial\mathscr{O} \times [0, T], \tag{1.4c}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \;\text{ in }\; \mathscr{O}, \tag{1.4d}$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $\tilde{\mathfrak{p}}$ represent the random velocity, the modified pressure, respectively. The stochastic process $\{W(t);\; t \in [0, T]\}$ is a Wiener process taking values in a given separable Hilbert space $\mathscr{H}$. Hereafter we understand that in $\mathbb{R}^2$ the rotational of a vector $\mathbf{u} = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ is a scalar function defined by

$$\mathbf{rot}\,\mathbf{u} = \frac{\partial \mathbf{u}^{(1)}}{\partial x_2} - \frac{\partial \mathbf{u}^{(2)}}{\partial x_1},$$

and for any vector and scalar functions $\mathbf{v} = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)})$ and z

$$\mathbf{rot}\,\mathbf{u} \times \mathbf{v} = (-\mathbf{v}^{(1)}\,\mathbf{rot}\,\mathbf{u}, \mathbf{v}^{(2)}\,\mathbf{rot}\,\mathbf{u}),$$
$$\mathbf{rot}(z \times \mathbf{v}) = \mathbf{v} \cdot \nabla z := \mathbf{v}^{(1)} \frac{\partial z}{\partial x_1} + \mathbf{v}^{(1)} \frac{\partial z}{\partial x_2}.$$

In order to describe the main results of the paper, let us denote by $\mathbf{V}$ the subspace of the Sobolev space $\mathbf{H}^1(\mathscr{O})$ consisting of divergence free functions that vanish on the boundary of $\mathscr{O}$, and by $\mathbf{W}$ a subspace of $\mathbf{V}$ consisting of functions $\mathbf{v} \in \mathbf{V}$ such $\mathbf{rot}(\mathbf{v} - \alpha\Delta\mathbf{v}) \in \mathrm{L}^2(\mathscr{O})$. Roughly speaking, the main results in this paper can be summarized in the following theorem.

Theorem.

(a) Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$, $Q : \mathscr{H} \to \mathscr{H}$ is a trace class operator and $G : \mathbf{V} \to \mathscr{L}(\mathscr{H}, \mathbf{H})$ is globally Lipschitz with respect to the L^2-norm. Then, for any $\alpha > 0$, $\nu \geqslant 0$, $\mathbf{u}_0 \in \mathbf{W}$ the problem (1.4) has at least a weak martingale solution which consists of a complete filtered probability system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a $\mathscr{H}$-valued Q-Wiener process W and a $\mathbb{F}$-adapted stochastic process $\mathbf{u}$ such that the integral version of (1.4) holds almost surely for any $t \in (0, T)$ in the weak sense.

(b) If, in addition to the above conditions, $\mathscr{O}$ is a convex polygon, then there exists a real number $r_0 > 2$ such that for any $r \in (2, r_0)$,

$$\mathbf{u} \in \mathrm{L}^p \left(\Omega; \mathrm{L}^\infty(0, T; \mathrm{W}^{2,r}(\mathscr{O})) \right).$$

Furthermore, $\mathbf{u} \in \mathrm{L}^p(\Omega; C([0, T]; \mathbf{W}))$ and any two processes $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying (1.4) with the same Wiener process W and starting with the same initial datum $\mathbf{u}_0$ coincide with probability 1.

This theorem improves the existing results, which will be reviewed in the next paragraph, in several respects. First, to the best of our knowledge the existence of weak martingale solution for the Lagrangian Averaged Euler equations (LAEs) driven by multiplicative noise is established for the first time in this paper. Second, while previous results concerning the existence of weak martingale solution of grade-two fluids driven by state-dependent external random perturbation was proved under the assumption the bounded domain $\mathscr{O}$ is simply connected and its boundary is of class C^3, in the present work we only require that $\mathscr{O}$ is a Lipschitz domain. Third, even in the deterministic case, it is not known whether the solution $\mathbf{u}$ is strongly continuous in $\mathbf{W}$ when the domain is a convex polygon. Thus, in the present paper we are able to settle this standing open problem for the stochastic system (1.4) under minimal assumption on $\mathscr{O}$. Although, we proved the time continuity for the stochastic and Lipschitz domain cases, our result is also valid for the deterministic and smooth domain cases. The proofs of all the above results are non-trivial, but the arguments are elementary in that they only need the fine properties of Sobolev spaces, regularity of solutions to elliptic problems on non-smooth domain, some estimates and convergence results from the theory of (semi)martingale.

Before proceeding to the literature review, we should note that while it is difficult to give a particular practical motivation for considering the grade-two fluids on Lipschitz domain, it seems natural, as pointed out in [12], to consider fluid flow in Lipschitz domain as most of partial differential equations which arise in practice are in non-smooth domains with simple geometry. An example of practical motivation we can mention is a fluid past a polygonal obstacle contained in a bounded domain. This produces the vortex shedding phenomenon which finds its application in electrical transmission lines, chimneys, towers, antennae, bridge decks of bluff cross-section. We refer, for instance, to [13] for a detailed and well explained exposition of the vortex shedding phenomenon.

1.3. Literature review

Before we proceed to the outline of the proofs of our results we give a sketchy account of existing mathematical literature related to the grade-two models and the LAEs. Since the beginning of the 80s, the equations for the viscous and non-viscous grade-two fluids have been the object of intensive mathematical studies, but by far the best method for proving the existence of weak solution is due to Cioranescu and Ouazar and can be found in Ouazar's thesis and in [14] and [15]. The method of Cioranescu and Ouazar consists of the blending of compactness method and Galerkin approximation based on a special basis formed by the eigenfunctions of the operator $\mathbf{rot}\,\mathbf{rot}(\mathbf{v}-\alpha\Delta\mathbf{v})$. By using the very method Cioranescu and Girault [16], Bernard [17] proved the global existence of a unique weak solution of three-dimensional grade-two fluids. This result is obtained under some restrictions on the data. In 2002, the author of [18] shows that the NSE can be approximated by the grade-two fluids. More precisely, the author of [18] showed that there exists a subsequence of weak solutions of grade-two fluids which converges weakly in some topology to the weak solution of the NSE. Amongst the important results obtained so far are the existence of global attractor, the regularity of the global attractor and finite-dimensional behavior for the grade-two fluid equations which were proved in [19] and [20]. Most of these results were established under the conditions that the boundary of $\mathscr{O}$ is sufficiently smooth, of class C^3 for example. Unfortunately, for a technical reason that we shall explain at the end of the results review, the method of Cioranescu and Ouazar is not applicable for the grade-two fluids flowing in a Lipschitz domain. Girault and Scott [21] came out with the idea of splitting the equations for the grade-two fluids into a steady Stokes-like and a transport systems to establish the existence of weak solution. This decomposition approach along with a time discretization based on backward Euler scheme was used by Girault and Saadouni in [22] to prove the existence of weak solution of the time-dependent problem in any arbitrary Lipschitz domain. The solution of either the steady or the time-dependent problem is unique as long as $\mathscr{O}$ is a convex polygon. In contrast to the mathematical literature devoted to the study of grade-two models, there are only few mathematical results for the LAEs. Cioranescu-Ouazar's methods was used in [23] to prove simultaneously the existence and uniqueness of solution to the LAEs and the grade-two fluids with Navier-slip boundary conditions. The convergence of the solution of grade-two fluid to the solution of LAEs is studied in [24]. Several local existence and global existence criterion in Besov and Triebel-Lizorkin spaces for the three dimensional LAEs can be found in [25], [26], [27] and [28]. The convergence of the grade two fluids or Lagrangian Averaged Euler to the Euler system has been also the subject of intense investigation and has generated several interesting and important result, see, for instance, [29], [30], [31], [32] and references therein. Of course, there are other results related to the mathematical theory of the deterministic LAEs and the grade-two fluids, and for a detailed of past and recent results related to the deterministic grade-two fluid and the LAEs we refer to [33] and [34]. Despite all these results, the continuity of the solution in $\mathbf{W}$ was left as an open question when $\mathscr{O}$ is a Lipschitz domain.

As far as the stochastic versions of the LAEs and grade-two fluids are concerned, there are only few works related to the problem (1.4). By using Cioranescu-Ouazar's method the global existence of both martingale and strong (in the stochastic calculus sense) solutions were proved in [35] and [36]. When the noise is additive, then the convergence of the solution of (1.4) to the weak martingale solution of the two dimensional stochastic Navier-Stokes equations was established in [37]. Existence of a global weak martingale solution for the grade-two fluids driven by external forcing of Lévy noise type is shown in [38]. Two important results related to the problem (1.4) have been recently posted on Arxiv, see [39] and [40]. The large deviation estimates for the solution to (1.4) was established in [39] by the weak convergence method of Budhiraja and Dupuis [41]. By Odasso's exponential mixing criterion [42] it was shown in [40] that the problem (1.4) has a unique invariant measure which is exponentially mixing. When the viscous term $-\nu\Delta\mathbf{u}$ is replaced with the stronger regularizing term $-\nu\Delta(\mathbf{u}-\alpha\Delta\mathbf{u})$, then the problem (1.4) becomes the Lagrangian-Navier-Stokes-α (LANS-α) which were derived in [43] to describe mathematical model capturing the phenomenon of turbulence at a low computational resolution. In contrast to the system for grade-two fluids, the LANS-α is a parabolic semilinear system and is much easier to solve than the former model. The stochastic LANS-

α has been extensively studied and has generated several important results, see ,for instance, [44], [45] and references therein. Note also that the Lagrangian Averaged Euler equations is different to the inviscid Leray-α models in which the nonlinear term is $\mathbf{u} \cdot \nabla(\mathbf{u} - \alpha\Delta\mathbf{u})$, see, amongst other, [46]. In contrast to the Lagrangian Averaged Euler equations the Leray-α Euler equation, either in two or three dimensional cases, admits a global weak solution. The uniqueness of solution of the Leray-α Euler equations is an open problem for the deterministic case, however when adding a *special* multiplicative noise it was proved in the interesting paper [47] that the solution of the stochastic Euler-α is unique in law.

To end this literature review, we note that the results in [38], [37], [36], [35], [40] and [39] are valid only when the bounded domain $\mathscr{O}$ is simply-connected and its boundary is of class C^3. This regularity of the boundary ensures that the eigenfunctions of $\mathbf{rot}\,\mathbf{rot}(\mathbf{v} - \alpha\Delta\mathbf{v})$ exist and form a subset of $\mathbf{H}^4(\mathscr{O})$. In fact, in the initial proof of Cioranescu and Ouazar it is shown that the eigenfunctions satisfy a steady biharmonic-like system with a $\mathbf{H}^2(\mathscr{O})$-valued external force, and well-known regularity result for elliptic problem in smooth domain yields the desired regularity of the eigenfunctions. This smoothness of the eigenfunctions plays an essential role for the derivation of *a priori* estimates for $\mathbf{rot}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)$ in $\mathrm{L}^2(\mathscr{O})$ where $\mathbf{u}_m$ is the Galerkin solution of (1.4). Since, even with a $\mathbf{H}^2(\mathscr{O})$-valued external forcing, we cannot expect a $\mathbf{H}^4(\mathscr{O})$-regularity of the eigenfunctions of $\mathbf{rot}\,\mathbf{rot}(\mathbf{v} - \alpha\Delta\mathbf{v})$ when the domain $\mathscr{O}$ is only Lipschitz, the method of Cioranescu-Ouazar is no longer applicable to the case of non-smooth domain.

1.4. *Sketch of the approaches and proofs of the main results*

Now, we continue the present introduction with the sketch of the approaches used to derive our main results. We will start with an outline of the proof of the existence of weak martingale solution. Albeit, the existence of solution is a basic question in (stochastic) Partial Differential equations, these turn to be rather challenging for the system (1.4). The structure of the problem is one of the main source of difficulties. In fact, (1.4) is fully nonlinear and behaves as an hyperbolic problem in that while the linear term is only the Laplacian its nonlinear term involves a third-order derivative. Besides this fact, as we have explained above the celebrated method of Cioranescu and Ouazar, thus the approach in [36] and [36], is no longer applicable to our framework. For this reason, we will follow closely the approach used in [22] to establish the existence of a weak martingale solution which, roughly speaking, is consisting of a complete filtered space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ on which is defined a pair $(\mathbf{u}, W)$ such that W is a $\mathscr{H}$-valued Wiener process and with probability 1 $\mathbf{u}$ belongs to $C([0,T]; \mathbf{V}) \cap \mathrm{L}^\infty(0,T; \mathbf{W})$ and satisfies (1.4). The method in [22] consists in splitting (1.4) into a linearized system of stochastic Stokes-like and transport systems and using a time discretization to construct approximating solution of the latter systems. The idea of the decomposition can be briefly described as follows. We set $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$, where $\mathbf{u}$ is a solution to (1.4), and apply the $\mathbf{rot}$ operator to (1.4a) in the sense of distribution to obtain that z solves

$$dz + \left(\frac{\nu}{\alpha} z + \mathbf{u} \cdot \nabla z\right) dt = \frac{\nu}{\alpha}\,\mathbf{rot}\,\mathbf{u}\, dt + \mathbf{rot}\, G(\mathbf{u}) dW.$$

This short discussion motivates us to introduce the following coupled SPDEs with multiplicative noise

$$d(\mathbf{u} - \alpha\Delta\mathbf{u}) + (z \times \mathbf{u} + \nabla\mathcal{P} - \nu\Delta\mathbf{u})\, dt = G(\mathbf{u}) dW, \text{ in } [0,T] \times \mathscr{O}, \tag{1.5a}$$

$$dz + (\frac{\nu}{\alpha} z + \mathbf{u} \cdot \nabla z) dt = \frac{\nu}{\alpha}\,\mathbf{rot}\,\mathbf{u}\; dt + \mathbf{rot}\, G(\mathbf{u}) dW, \text{ in } [0,T] \times \mathscr{O}, \tag{1.5b}$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } [0,T] \times \mathscr{O}, \tag{1.5c}$$

$$\mathbf{u} = 0 \text{ on } [0,T] \times \partial\mathscr{O}, \tag{1.5d}$$

$$z(0) = z_0 := \mathbf{rot}(\mathbf{u}_0 - \alpha\Delta\mathbf{u}_0) \text{ in } \mathscr{O}, \tag{1.5e}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \mathscr{O}. \tag{1.5f}$$

Even though, the relation $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ was discarded, we will, as in the deterministic case, see later on that the two problems (1.4) and (1.5) are equivalent. Thus, in order to prove the existence of weak martingale solution of (1.4) it is sufficient to establish an existence result for the auxiliary stochastic problem (1.5). For this purpose, we will use a combination of time discretization and compactness method. The semi-discrete discretization is based on backward Euler scheme and the resulting equations are basically a coupling of a steady Stokes-like and transport equations. At each time step, the numerical algorithm is shown to have a a unique adapted solution, and the sequence formed by these solutions are unconditionally stable. We exploit this stability result to show the tightness of the laws family of the interpolants of the discrete solutions. This laws tightness combined with the Prokhorov and Skorokhod theorems enables us to construct a new complete filtered probability along with a sequence of processes $(\mathbf{u}_n, z_n, W_n)$ converging in laws and almost surely to a limiting process $(\mathbf{u}, z, W)$ which, upon to passage to the limit in the equation for the interpolants, is shown to solve (1.4). Note that the solution $(\mathbf{u}, z)$ of (1.5) belongs to $C([0,T]; \mathbf{H} \times \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})) \cap \mathrm{L}^\infty(0,T; \mathbf{V} \times \mathrm{L}^2(\mathscr{O}))$ with probability 1. We also note that while the papers [48], [49], [50], [51] and [52] motivated us to use time discretization, our problem does not fit their framework.

Regarding to the other results, the uniqueness mainly relies on the space regularity solution. The former results require that the solution $\mathbf{u}$ belongs at least to $\mathbf{W}^{1,\infty}(\mathscr{O})$ a regularity that cannot be produced by the estimate in $\mathbf{H}^1(\mathscr{O})$ of solution of (1.5a) alone. In order to get a regularity in $\mathbf{W}^{1,\infty}(\mathscr{O})$, we need that the solution $\mathbf{u}$ belongs at least to $\mathbf{W}^{2,r}(\mathscr{O})$, a result that will be obtained by exploiting that $z := \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \in \mathrm{L}^2(\mathscr{O})$. In fact, since $z \in \mathrm{L}^2(\mathscr{O})$ is already the $\mathbf{rot}$ of $\mathbf{u} - \alpha\Delta\mathbf{u}$ and $\mathscr{O}$ is a simply-connected domain, one can construct a vector stream-function $\mathbf{z} \in \mathbf{H}^1(\mathscr{O})$, which depends continuously in z, such that $\mathbf{u} = (\mathrm{Id} + \alpha\mathcal{A})^{-1}\mathbf{z}$, where $\mathcal{A}$ is basically the Stokes operator. The latter identity along with the regularity of the solution of elliptic problems on non-smooth domain implies that there exists a number $r_0 > 2$ depending only on the inner angle of $\mathscr{O}$ such that $\mathbf{u} \in \mathbf{W}^{2,r}(\mathscr{O})$ for any $r \in [2, r_0)$. Thanks to this spatial regularity the uniqueness follows easily from a careful estimate of the nonlinear term, the application of Itô formula and a trick due to Schmalfuß [53].

The idea of the continuity proof of $\mathbf{u}$ in $\mathbf{W}$ is quite simple. In fact, since $(\mathbf{u}, z)$, where $z := \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$, is a solution of (1.5), then $z \in C([0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})) \cap \mathrm{L}^\infty(0,T; \mathrm{L}^2(\mathscr{O}))$ with probability 1, hence it is weakly continuous in $\mathrm{L}^2(\mathscr{O})$. Since the process $\mathbf{u}$ belongs to $C([0,T]; \mathbf{V})$ already, in order to prove the strong continuity in $\mathbf{W}$ it suffices to show that the $\mathrm{L}^2(\mathscr{O})$-norm $|z(\cdot)| : [0,T] \to [0,\infty)$ is continuous. This amounts to show that the process z satisfies an energy equality in $\mathrm{L}^2(\mathscr{O})$. For the deterministic grade-two fluids, this idea appears for the first time in [19] and other proof methods appeared in [20] and [34]. In all these literature, the domain $\mathscr{O}$ was assumed to be either simply-connected and of class C^3 or a two-dimensional torus. In contrast to [19] which used a Galerkin approximation, we will use a spatial regularization argument based on some ideas from [54] and [55]. Observe also that this regularization by convolution was used in [56] and references therein to derive that any L^∞-weak solution of a fairly *general* stochastic transport equations is a renormalized solution. The first step of the proof is to regularize the process z in the space variable by convolution with a special family of mollifiers indexed by a number $k \in \mathbb{N}$ and derive the stochastic equation satisfied by the sequence of the regularized processes z_k, $k \in \mathbb{N}$. The second and final step is the derivation of the energy equation for $|z_k(\cdot)|^2$ from which we will get the energy equation for $|z(\cdot)|^2$ upon passing to the limit. In order to be able to pass to the limit in *good topology* we need the space regularity stated in the above theorem, in particular we need that $\mathbf{u} \in \mathrm{L}^2(\Omega, \mathrm{L}^2(0,T; \mathbf{W}^{1,\infty}(\mathscr{O})))$. The steps we outlined above are crucial, since a crude application of Itô formula to $|z(\cdot)|^2$ or $\|\mathbf{u}(\cdot)\|^2_{\mathbf{W}}$ is doomed to fail. The main reason is that neither the process $\mathbf{u}$ nor z satisfies the general criteria for the application of the Itô formula or for continuity in $\mathbf{W}$ and $\mathrm{L}^2(\mathscr{O})$, see [57, Chapter I, Theorem 3.2] or [58, Chapter 1, Lemma 1.2]. Indeed, $\mathbf{u}$ is $\mathbf{W}$-valued and can be written in a formal way into the form

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t F(s)ds + (\mathrm{Id} + \alpha\mathcal{A})^{-1} \int_0^t G(\mathbf{u}(s))dW(s), \ \forall t \in [0,T],$$

where $F \in \mathrm{L}^2(\Omega, \mathrm{L}^2(0,T; \mathbf{V}))$, but, because of the loss of regularity due to the lack of smoothness of the boundary of $\mathscr{O}$, it is not known whether $(\mathrm{Id} + \alpha\mathcal{A})^{-1} \int_0^t G(\mathbf{u}(s))dW(s)$ is a $\mathbf{W}$-valued martingale. Thus, both the criteria for continuity in $\mathbf{W}$ and the application of the Itô formula to $\|\mathbf{u}(\cdot)\|^2_{\mathbf{W}}$

is not satisfied. For z, owing to the definition of the solution, it can be easily checked that the process $\mathbf{u} \cdot \nabla z \in \mathrm{L}^2(\Omega, \mathrm{L}^2(0, T; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})))$ and that the martingale $\int_0^t \mathbf{rot}\, G(\mathbf{u}(s))dW(s)$ is $\mathrm{L}^2(\mathscr{O})$-valued. However, the process z is not $\mathrm{W}^{1,4}(\mathscr{O})$-valued, hence, as in the case for $\mathbf{u}$, all the requirements for the application of [57, Chapter I, Theorem 3.2] or [58, Chapter 1, Lemma 1.2] are not met.

1.5. Structure of the paper

Let us now close this introduction with the layout of the paper. In Section 2 we introduce several notations and all the assumptions we need in this paper. Amongst the main results that we stated in the next section is the existence of a weak martingale of (1.5); the proof of this results is postponed to Section 4. Thanks to this and the equivalence of (1.4) and (1.5), the existence of weak martingale solution to (1.4) is proved in Section 2. Section 3 is a prelude to the proof of existence of weak martingale solution to the auxiliary problem (1.5). There we introduce and analyze the algorithm used to construct a sequence of discrete random variables which, in turn, will be used to construct the continuous approximating solutions (interpolants) to (1.5). Several key estimates, which will be used to prove the tightness of the interpolants, are also established in section 3. We prove the space regularity and uniqueness results alluded in the description of our main results in Section 5. The continuity in $\mathbf{W}$ of the solution is proved in the last section.

2. Notations, hypotheses and the main results

2.1. Notations

We introduce necessary definitions of functional spaces frequently used in this work. Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$. We denote by $\mathrm{L}^p(\mathscr{O})$ and $\mathrm{W}^{m,p}(\mathscr{O})$, $p \in [1, \infty]$, $m \in \mathbb{N}$, the well-known Lebesgue and Sobolev spaces. In particular, $\mathrm{W}_0^{1,p}(\mathscr{O})$ is the Sobolev spaces of functions vanishing (in the sense of trace) on the boundary $\partial\mathscr{O}$ of $\mathscr{O}$. We simply write $\mathrm{H}^m(\mathscr{O})$ when $p = 2$. We refer to the monograph [59] for more detailed information about Sobolev spaces.

In what follows we denote by $\mathbf{X}$ the space of $\mathbb{R}^2$-valued functions such that each component belongs to X. We introduce the spaces

$$\begin{aligned} \mathcal{V} &= \left\{ \mathbf{u} \in [\mathcal{C}_c^\infty(\mathscr{O})]^2 \text{ such that } \operatorname{div} \mathbf{u} = 0 \right\} \\ \mathbf{V} &= \text{ closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\mathscr{O}) \\ \mathbf{H} &= \text{ closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\mathscr{O}), \end{aligned}$$

where $[\mathcal{C}_c^\infty(\mathscr{O})]^2 := \mathcal{C}_c^\infty(\mathscr{O}, \mathbb{R}^2)$ denotes the spaces of all infinitely differentiable functions with compact support in $\mathscr{O}$. We denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbf{L}^2(\mathscr{O})$ on $\mathbf{H}$, respectively. The inner product and the norm induced by that of $\mathbf{H}_0^1(\mathscr{O})$ on $\mathbf{V}$ are denoted respectively by $((\cdot, \cdot))$ and $||\cdot||$. Let $\Pi : \mathrm{L}^2(\mathscr{O}) \to \mathbf{H}$ be the Helmholtz-Leray projection, and $\mathcal{A} = -\Pi\Delta$ be the Stokes operator with the domain $D(\mathcal{A}) = \mathbf{H}^2(\mathscr{O})^2 \cap \mathbf{H}$. It is well-known that $\mathcal{A}$ is a self-adjoint positive operator with compact inverse, see for instance [60, Chapter 1, Section 2.6]. Hence, it has an orthonormal sequence of eigenvectors $\{e_j;\ j \in \mathbb{N}\}$ with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 < ....$

Observe that in the space $\mathbf{V}$, the norm $\|\cdot\|$ is equivalent to the norm generated by the following scalar product

$$((\mathbf{u}, \mathbf{w}))_\alpha = (\mathbf{u}, \mathbf{w}) + \alpha((\mathbf{u}, \mathbf{w})), \text{ for any } \mathbf{u}\ \mathbf{w} \in \mathbf{V}, \text{and} \alpha > 0. \tag{2.1}$$

More precisely, we have

$$\alpha||\mathbf{u}||^2 \leqslant \|\mathbf{u}\|_\alpha^2 \leqslant \left(\frac{1}{\lambda_1} + \alpha\right) ||\mathbf{u}||^2, \forall \mathbf{u} \in \mathbf{V}, \tag{2.2}$$

where λ_1 is the least of the eigenvalues of the Stokes operator $\mathcal{A}$. From now on, we will equip $\mathbf{V}$ with the norm $\|\mathbf{u}\|_\alpha$ generated by the inner product defined in (2.1).

We also introduce the following space

$$\mathbf{W}=\left\{\mathbf{u}\in\mathbf{V};\ \mathbf{rot}(\mathbf{u}-\alpha\Delta\mathbf{u})\in \mathrm{L}^2(\mathscr{O})\right\},$$

which is a Hilbert space equipped with the norm generated by the following scalar product

$$((\mathbf{u},\mathbf{v}))_{\mathbf{W}}=((\mathbf{u},\mathbf{v}))_\alpha+(\mathbf{rot}(\mathbf{u}-\alpha\Delta\mathbf{u}),\mathbf{rot}(\mathbf{v}-\alpha\Delta\mathbf{v})),\ \ \forall\mathbf{u},\ \mathbf{v}\in\mathbf{W}.$$

Note that for $\mathbf{v}\in\mathbf{V}$, $\alpha\,\mathbf{rot}\,\Delta\mathbf{v}\in\mathrm{L}^2(\mathscr{O})$ is understood in its weak sense.

For any Banach space $\mathbf{B}$ we denote its dual by $\mathbf{B}^*$ and by $\langle\mathbf{f},\mathbf{v}\rangle$ the action of any element $\mathbf{f}$ of $\mathbf{B}^*$ on an element $\mathbf{v}\in\mathbf{B}$. By identifying $\mathbf{H}$ with its dual space $\mathbf{H}^*$ via the Riesz representation, we have the Gelfand-Lions triple

$$\mathbf{V}\subset\mathbf{H}\subset\mathbf{V}^\star,$$

where each space is dense in the next one and the inclusions are continuous. It follows from the above identification that we can write

$$((\mathbf{v},\mathbf{w}))_\alpha=\langle\mathbf{v},\mathbf{w}\rangle, \tag{2.3}$$

for any $\mathbf{v}\in\mathbf{H},\mathbf{w}\in\mathbf{V}$.

For a fixed $\mathbf{v}\in\mathbf{H}^1(\mathscr{O})$, we set

$$\mathrm{L}_{\mathbf{v}}:=\{f\in\mathrm{L}^2(\mathscr{O});\ \mathbf{v}\cdot\nabla f\in\mathrm{L}^2(\mathscr{O})\},$$

which defines a Hilbert space when endowed with the graph norm

$$\|f\|_{\mathrm{L}_{\mathbf{v}}}:=|f|+|\mathbf{v}\cdot\nabla f|,\ \forall f\in\mathrm{L}_{\mathbf{v}}.$$

As in the definition of $\mathbf{W}$, for $f\in\mathrm{L}^2(\mathscr{O})$ and $\mathbf{v}\in\mathbf{H}^1(\mathscr{O})$ $\mathbf{v}\cdot\nabla f\in\mathrm{L}^2(\mathscr{O})$ is understood in the weak sense.

Now, we will fix the assumption on the noise entering the system. Let $\mathscr{U}:=(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space where the filtration $\mathbb{F}=\{\mathscr{F}_t;\ \ t\in[0,T]\}$ satisfies the usual condition. Let $\{\beta_j;\ j\in\mathbb{N}\}$ be a sequence of mutually independent and identically distributed standard Brownian motions on $\mathscr{U}$. Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{L}_1(\mathscr{H})$ be the space of all trace class operators on $\mathscr{H}$. Let $Q\in\mathscr{L}_1(\mathscr{H})$ be a symmetric, nonnegative operator and $\{h_j; j\in\mathbb{N}\}$ be an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of Q. Let $\{q_j;\ j\in\mathbb{N}\}$ be the eigenvalues of Q and W the process defined by

$$W(t)=\sum_{j=1}^{\infty}\sqrt{q_j}\beta_j(t)h_j,\quad t\in[0,T].$$

It is well-known, see [61, Theorem 4.5], that the above series converges in $\mathrm{L}^2(\Omega;C([0,T];\mathscr{H}))$ and it defines a $\mathscr{H}$-valued Wiener process with covariance operator Q. Furthermore, for any positive integer $\ell>0$ there exists a constant $C_\ell>0$ such that

$$\mathbb{E}\|W(t)-W(s)\|_{\mathscr{H}}^{2\ell}\leqslant C_\ell|t-s|^\ell\,(\mathrm{Tr}\,Q)^\ell, \tag{2.4}$$

for any $t,s\geqslant 0$ with $t\neq 0$.

Let K be a separable Banach space, $\mathscr{L}(\mathscr{H},\mathrm{K})$ be the space of all bounded linear K-valued operators defined on $\mathscr{H}$, $\mathscr{M}_T^2(\mathrm{K}):=\mathscr{M}^2(\Omega\times[0,T];\mathrm{K})$ be the space of all equivalence classes of $\mathbb{F}$-progressively measurable processes $\Psi:\Omega\times[0,T]\to\mathrm{K}$ satisfying

$$\mathbb{E}\int_0^T\|\Psi(s)\|_{\mathrm{K}}^2ds<\infty.$$

If $Q\in\mathscr{L}_1(\mathscr{H})$ is a symmetric, nonnegative and trace class operator then $Q^{\frac12}\in\mathscr{L}_2(\mathscr{H})$ and for any $\Psi\in\mathscr{L}(\mathscr{H},\mathrm{K})$ we have $\Psi\circ Q^{\frac12}\in\mathscr{L}_2(\mathscr{H},\mathrm{K})$, where $\mathscr{L}_2(\mathscr{H},\mathrm{K})$ is the Hilbert space of all operators $\Psi\in\mathscr{L}(\mathscr{H},\mathrm{K})$ satisfying

$$\|\Psi\|_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}^2=\sum_{j=1}^{\infty}\|\Psi h_j\|_{\mathrm{K}}^2<\infty.$$

Furthermore, from the theory of stochastic integration on infinite dimensional Hilbert space, see [62, Chapter 5, Section 26] and [61, Chapter 4], for any $\Psi \in \mathscr{M}^2_T(\mathscr{L}(\mathscr{H}, \mathrm{K}))$ the process M defined by

$$M(t) = \int_0^t \Psi(s)dW(s), t \in [0,T],$$

is a K-valued martingale. Moreover, we have the following Itô isometry

$$\mathbb{E}\left(\left\|\int_0^t \Psi(s)dW(s)\right\|^2_{\mathrm{K}}\right) = \mathbb{E}\left(\int_0^t \|\Psi(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}ds\right), \forall t \in [0,T], \tag{2.5}$$

and the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}\left\|\int_0^s \Psi(s)dW(s)\right\|^q\right) \leqslant C_q\mathbb{E}\left(\int_0^t \|\Psi(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}ds\right)^{\frac{q}{2}}, \forall t \in [0,T], \forall q \in (1,\infty). \tag{2.6}$$

2.2. The standing hypotheses and main results

Now, we impose the following set of conditions on the nonlinear term $G(\cdot)$ and the Wiener process W.

(**N**) Let $\mathscr{H}$ be a separable Hilbert space. We assume that we are given a nonnegative and symmetric covariance operator $Q \in \mathscr{L}_1(\mathscr{H})$.

(**G**) We assume that we are given a nonlinear function G from $\mathbf{V}$ into $\mathscr{L}(\mathscr{H},\mathbf{V})$ such that there exists a constant $C_1 > 0$ for which the following hold

$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathscr{L}(\mathscr{H},\mathbf{V})} \leqslant C_1|\mathbf{u}-\mathbf{v}|,$$
$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathscr{L}(\mathscr{H},\mathbf{V})} \leqslant C_1\|\mathbf{u}-\mathbf{v}\|_\alpha,$$

for any $\mathbf{u} \in \mathbf{V}$, $\mathbf{v} \in \mathbf{V}$.

Remark 2.1.

(a) Note that the above assumption implies that there exists a constant $C_2 > 0$ such that

$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathscr{L}(\mathscr{H},\mathbf{H})} \leqslant C_2\|\mathbf{u}-\mathbf{v}\|_\alpha,$$
$$\|G(\mathbf{u})\|_{\mathscr{L}(\mathscr{H},\mathbf{H})} \leqslant C_2(1+\|\mathbf{u}\|_\alpha),$$

for any $\mathbf{u},\ \mathbf{v} \in \mathbf{V}$.

(b) There exists also a number $C_3 > 0$ such that

$$\|\,\mathbf{rot}[G(\mathbf{u}) - G(\mathbf{v})]\|_{\mathscr{L}(\mathscr{H},\mathrm{L}^2(\mathscr{O}))} \leqslant C_3\|\mathbf{u}-\mathbf{v}\|_\alpha,$$
$$\|\,\mathbf{rot}\,G(\mathbf{u})\|_{\mathscr{L}(\mathscr{H},\mathrm{L}^2(\mathscr{O}))} \leqslant C_3(1+\|\mathbf{u}\|_\alpha),$$

for any $\mathbf{u},\ \mathbf{v} \in \mathbf{V}$.

(c) Owing to item (b) of the present remark, if $\mathbf{u} \in M^2_T(\mathbf{V})$, then $\mathbf{rot}\,G(\mathbf{u})$ belongs to $M^2_T(\mathscr{L}(\mathscr{H},\mathrm{L}^2(\mathscr{O})))$ and the stochastic integral $\int_0^t \mathbf{rot}\,G(\mathbf{u}(s))dW(s)$ is a well defined $\mathrm{L}^2(\mathscr{O})$-valued martingale.

To alleviate notation we introduce the concept of stochastic basis.

Definition 2.2. A *stochastic basis* $\mathscr{U} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$ consists of

(a) a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the filtration $\mathbb{F} = \{\mathscr{F}_t;\quad t \in [0,T]\}$ satisfies the usual condition.

(b) a $\mathscr{H}$-valued Q-Wiener process W defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We now formulate several definitions.

Definition 2.3. By a solution of the system (1.4), we mean a pair $(\mathscr{U}, \mathbf{u})$ such that

(a) $\mathscr{U} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$ is a stochastic basis,

(b) $\mathbf{u}$ is an $\mathbb{F}$-adapted stochastic process and

$$\mathbf{u} \in \mathrm{L}^8(\Omega; C([0,T]; \mathbf{V}) \cap \mathrm{L}^\infty(0,T; \mathbf{W})),$$

(c) the following integral equation of Itô type holds with probability 1

$$\begin{aligned}((\mathbf{u}(t), \mathbf{v}))_\alpha &+ \int_0^t [\nu((\mathbf{u}(s), \mathbf{v})) + (\mathbf{rot}(\mathbf{u}(s) - \alpha\Delta\mathbf{u}(s)) \times \mathbf{u}(s), \mathbf{v})]\, ds \\ &= ((\mathbf{u}_0, \mathbf{v}))_\alpha + \int_0^t (G(\mathbf{u}(s)), \mathbf{v}) dW(s)\end{aligned} \tag{2.7}$$

for any $t \in (0,T]$ and $\mathbf{v} \in \mathbf{V}$.

In the next proposition we will show that the systems (1.5) and (1.4) are equivalent, but for now let us proceed to the definition of weak martingale to the former system.

Definition 2.4. By a solution of the system (1.5), we mean a system

$$(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F}, W, \mathbf{u}, z),$$

where

(a) $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$ is a stochastic basis,

(b) the process $(\mathbf{u}, z)$ is $\mathbb{F}$-adapted and

$$(\mathbf{u}, z) \in \mathrm{L}^8(\Omega; C([0,T]; \mathbf{V})) \times \mathrm{L}^p(\Omega; C([0,T]; W^{-1,\frac{4}{3}}(\mathscr{O})) \cap \mathrm{L}^\infty(0,T; \mathrm{L}^2(\mathscr{O}))).$$

(c) the following integral equation of Itô type holds with probability 1

$$\begin{aligned}((\mathbf{u}(t), \mathbf{v}))_\alpha + \int_0^t [\nu((\mathbf{u}(s), \mathbf{v})) + (z(s) \times \mathbf{u}(s), \mathbf{v})]\, ds &= ((\mathbf{u}_0, \mathbf{v}))_\alpha + \int_0^t (G(\mathbf{u}(s)), \mathbf{v}) dW(s) \\ (z(t), \phi) + \int_0^t \left(\frac{\nu}{\alpha}[z(s) - \mathbf{rot}\,\mathbf{u}(s)] + \mathbf{u}(s) \cdot \nabla z(s), \phi\right) ds &= (z_0, \phi) + \int_0^t (\mathbf{rot}\, G(\mathbf{u}(s)), \phi) dW(s),\end{aligned} \tag{2.8}$$

for any $t \in (0,T]$, $\mathbf{v} \in \mathbf{V}$ and $\phi \in W_0^{1,4}(\mathscr{O})$.

We shall prove the following proposition which will play an important role in our analysis.

Proposition 2.5. *If $(\mathscr{U}, \mathbf{u})$, where $\mathscr{U}$ is a stochastic basis, is a weak martingale solution of (1.4) then $(\mathscr{U}, \mathbf{u}, z)$, with $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$, is a weak martingale solution to (1.5). Conversely, if $(\mathscr{U}, \mathbf{u}, z)$ is a weak martingale solution to (1.5), then $(\mathscr{U}, \mathbf{u})$ is a weak martingale solution to (1.4) and $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$.*

Proof. It is not difficult to show that if $(\mathscr{U}, \mathbf{u})$ is a solution to (1.4) in the sense of Definition 2.3, then $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ solves (1.5a) on the same stochastic basis $\mathscr{U}$ (see also the discussion in the introduction). That is, $(\mathscr{U}, \mathbf{u}, \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}))$ is a weak martingale solution to (1.5). This proves the first part of the proposition. Now, assume that we have found a weak martingale solution $(\mathscr{U}, \mathbf{u}, z)$ to (1.5). Then taking the **rot** of (1.4a) (in the sense of distribution) we obtain that

$$d\tilde{z} + (\frac{\nu}{\alpha}\tilde{z} + \mathbf{u} \cdot \nabla\tilde{z})dt = \frac{\nu}{\alpha}\mathbf{rot}\,\mathbf{u}\, dt + \mathbf{rot}\, G(\mathbf{u})dW,$$

where $\tilde{z} := \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$. Setting $y = \tilde{z} - z$ and subtracting the last identity and (1.4b) yields

$$\frac{\partial y}{\partial t} + \frac{\nu}{\alpha} y = 0.$$

This means that y solves the following random ordinary differential equations in the Hilbert $\mathrm{H}^{-2}(\mathscr{O})$

$$\frac{d\varphi}{dt} + \frac{\nu}{\alpha}\varphi = 0, \quad \varphi(0) = y_0.$$

The above ODEs admits a unique solution φ with

$$\varphi(\cdot) = e^{-\frac{\nu}{\alpha}\mathrm{Id}(\cdot)} y_0 \in \mathrm{C}([0,T]; \mathrm{H}^{-2}), \text{ a.s. },$$

where $\{e^{-\frac{\nu}{\alpha}\mathrm{Id}t}; t \in [0,T]\}$ is the semigroup generated by the identity operator Id on $\mathrm{H}^{-2}(\mathscr{O})$. Since $y_0 = 0$, we have $y = 0$ and

$$z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \in C([0,T]; \mathrm{H}^{-2}(\mathscr{O})), \text{ a.s. },$$

from which we easily conclude the proof of the second part. □

Theorem 2.6. *Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$ and assume that $Q \in \mathscr{L}_1(\mathscr{H})$ and G satisfy* ***(N)*** *and* ***(G)****, respectively. Then, for any $\alpha > 0$, $\nu \geqslant 0$, $\mathbf{u}_0 \in \mathbf{W}$ the problem* (1.5) *has a solution in the sense of Definition* 2.4.

Proof. The proof of this theorem will be given in Section 4. □

Theorem 2.7. *Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$ and assume that $Q \in \mathscr{L}_1(\mathscr{H})$ and G satisfy* ***(N)*** *and* ***(G)****, respectively. Then, for any $\alpha > 0$, $\nu \geqslant 0$, $\mathbf{u}_0 \in \mathbf{W}$ the problem* (1.4) *has at least a weak martingale solution.*

Proof. The proof of this theorem follows from Proposition 2.5 and Theorem 2.6 given above. □

Before proceeding further we make the following remark.

Remark 2.8. In the framework of this paper we are not given a priori a probability space, thus we are not allowed to take stochastic or random initial data. In fact the filtered probability space along with the Wiener process is a part of our solution. However, it is possible to take the initial data as a probability distribution μ_0 on $\mathbf{W}$. In this case we have to modify the definition of our solution by requiring that the initial value $\mathbf{u}(0)$ of the solution process $\mathbf{u}$ has a probability distribution equal to μ_0. Some steps of the proofs also need to be modified, but this is too complicated to be described in this remark. We instead refer, for instance, to [52] for the possible modifications (either in the concept of solution or proofs steps) that need to be carried out.

Now we turn our attention to the space and time regularities of the solution. We first prove the space regularity by using tools from the theory of deterministic elliptic differential equations on non-smooth domain. We then use this space regularity result to prove the time smoothness and the uniqueness of solution. The space-time regularity is stated in the following theorem.

Theorem 2.9. *In addition to the assumptions of Theorem 2.7, suppose that $\mathscr{O}$ is a convex polygon. Let $(\mathbf{u}, \mathscr{U})$ be a weak martingale solution of* (1.4) *given by Theorem 2.7.*

(a) Then, there exist a real number $r_0 > 2$ such that for any $r \in (2, r_0)$,

$$\mathbf{u} \in \mathrm{L}^8\left(\Omega; \mathrm{L}^\infty(0,T; \mathrm{W}^{2,r}(\mathscr{O}))\right).$$

(b) Furthermore,

$$\mathbf{u} \in \mathrm{L}^8\left(\Omega; C([0,T]; \mathbf{W})\right).$$

Proof. The proof of item (a) of this theorem is given in Subsection 5.1. That of item (b) will be carried out in Subsection 6.2. □

As mentioned above we use the result from the last theorem to prove the pathwise uniqueness of solution to (1.4).

Theorem 2.10. *In addition to the assumptions of Theorem 2.7, suppose that $\mathscr{O}$ is a convex polygon. Then, for any $\alpha > 0$, $\nu \geqslant 0$, $\mathbf{u}_0 \in \mathbf{W}$ the weak martingale solution to problem (1.4) is pathwise unique, i.e., any two processes $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying (1.4) on the same stochastic basis $\mathscr{U} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$ and starting with the same initial datum $\mathbf{u}_0$ are equal with probability 1.*

Proof. The proof of this theorem will be carried out in Subsection 5.2. □

3. Description of the algorithm and Energy estimates

This section 3 serves as a prelude to the proof of existence of weak martingale solution to the auxiliary problem (1.5). As we mentioned earlier in the introduction we will use a time discretization and compactness method to establish Theorem 2.6. Thus, in this section we introduce and analyze the algorithm used to construct a sequence of discrete random variables which, in turn, will be used to construct the continuous approximating solutions (interpolants) to (1.5). Several key estimates, which will be used to prove the tightness of the interpolants, are also established.

3.1. *Description of the algorithm*

We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and for any real numbers a and b with $a \leqslant b$ we put $[\![a, b]\!] := [a, b] \cap \mathbb{N}_0$. We fix an integer $n \geqslant 0$, set $k = T/n$ as the time step, and $\Pi_n := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of $[0, T]$ where the grid points are $t_\ell = \ell k$, $k \in [\![0, n]\!]$. We first consider the system (1.5) on a fixed stochastic basis $\mathbf{U} := (\mathbf{\Omega}, \mathfrak{F}, \mathbf{F}, \mathbf{P}, \boldsymbol{\eta})$, i.e, we study (1.5) with the Wiener noise W replaced by $\boldsymbol{\eta}$ on $(\mathbf{\Omega}, \mathfrak{F}, \mathbf{F}, \mathbf{P})$. We assume that $\boldsymbol{\eta}$ is $\mathscr{H}$-valued Wiener process with covariance Q satisfying Assumption (N). For any $i \in [\![0, n-1]\!]$, we define a $\mathscr{H}$-valued Gaussian random variable $\Delta_i \boldsymbol{\eta}$ by

$$\Delta_i \boldsymbol{\eta} = \boldsymbol{\eta}(t_{i+1}) - \boldsymbol{\eta}(t_i).$$

With all these in mind, the time-discrete problem associated to (1.5) is given in the following algorithm.

Algorithm 1

Let $n \in \mathbb{N}$, $\mathbf{u}^0 := \mathbf{u}_0 \in \mathbf{W}$, $z^0 = \mathbf{rot}(\mathbf{u}^0 - \alpha \Delta \mathbf{u}^0)$ and a $\mathscr{H}$-valued Wiener process $\boldsymbol{\eta}$ with covariance Q satisfying (N) be given.

Then, construct two sequences $\{\mathbf{u}^\ell;\ \ell \in [\![1, n]\!]\} \subset \mathbf{V}$ and $\{z^\ell;\ \ell \in [\![1, n]\!]\} \subset \mathrm{L}^2(\mathscr{O})$ such that for each $\ell \in [\![0, n-1]\!]$ and for all $\mathbf{v} \in \mathbf{V}$,

$$((\mathbf{u}^{\ell+1} - \mathbf{u}^\ell, \mathbf{v}))_\alpha + \nu k((\mathbf{u}^{\ell+1}, \mathbf{v})) + k(z^\ell \times \mathbf{u}^{\ell+1}, \mathbf{v}) = (G(\mathbf{u}^\ell)\Delta_\ell \boldsymbol{\eta}, \mathbf{v}), \tag{3.1}$$

$$z^{\ell+1} - z^\ell + k(\frac{\nu}{\alpha} z^{\ell+1} + \mathbf{u}^{\ell+1} \cdot \nabla z^{\ell+1} - \frac{\nu}{\alpha} \mathbf{rot}\, \mathbf{u}^{\ell+1}) = \tilde{G}(\mathbf{u}^\ell)\Delta_\ell \boldsymbol{\eta}, \tag{3.2}$$

where $\tilde{G}(\cdot) := \mathbf{rot}\, G(\cdot)$.

The formula (3.1) and (3.2) are respectively the weak formulation of a time-discrete version of a generalized Stokes equation and a transport equation. We will show that for each $\ell \in [\![1, n]\!]$ Algorithm (1) admits a unique weak solution $(\mathbf{u}^\ell, z^\ell) \in \mathbf{V} \times \mathrm{L}^2(\mathscr{O})$. To this end, we will first state and prove the following two lemmata.

Lemma 3.1.

(a) Let $\psi \in L^2(\mathscr{O})$, and $\gamma > 0$ and $\delta \geqslant 0$ be two real numbers. Then, for any $\mathbf{f} \in \mathbf{V}^*$, there exists a unique $\mathbf{u} \in \mathbf{V}$ such that for all $\mathbf{v} \in \mathbf{V}$,

$$(\mathbf{u}, \mathbf{v}) + \gamma((\mathbf{u}, \mathbf{v})) + \delta(\psi \times \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle. \tag{3.3}$$

(b) Moreover, the map

$$\begin{aligned} \mathscr{S} : \mathbf{V} \times \mathscr{H} \times L^2(\mathscr{O}) &\to \mathbf{V} \\ (\mathbf{w}, \boldsymbol{\beta}, \psi) &\mapsto \mathbf{u}, \end{aligned}$$

where $\mathbf{u}$ is the weak unique solution of (3.3) with right hand side $\mathbf{f} = G(\mathbf{w})\boldsymbol{\beta}$, is continuous.

Proof. The identity (3.3) is the weak formulation of a generalized Stokes problem with Dirichlet boundary condition. The existence of a weak solution of its version with a tangential boundary condition was established in [21, Proposition 2.2] and the argument therein can be easily adapted to our framework, thus we omit the proof of the part (a) of the lemma. Now let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{V}$, $\psi_1, \psi_2 \in L^2(\mathscr{O})$, $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathscr{H}$, and set

$$\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2, \ \boldsymbol{\beta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2, \ \psi = \psi_1 - \psi_2.$$

We also put

$$\begin{aligned} \mathbf{u} =& \mathscr{S}(\mathbf{w}_1, \boldsymbol{\beta}_1, \psi_1) - \mathscr{S}(\mathbf{w}_2, \boldsymbol{\beta}_2, \psi_2) \\ =& \mathbf{u}_1 - \mathbf{u}_2. \end{aligned}$$

The function $\mathbf{u}$ satisfies

$$a(\mathbf{u}, \mathbf{v}) + \delta\left(c(\psi, \mathbf{u}_1, \mathbf{v}) + c(\psi_2, \mathbf{u}, \mathbf{v})\right) = \left([G(\mathbf{w}_1) - G(\mathbf{w}_2)]\boldsymbol{\beta}_1 + G(\mathbf{w}_2)\boldsymbol{\beta}, \mathbf{v}\right),$$

where

$$a(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v}) + \gamma((\mathbf{u}, \mathbf{v})) \text{ and } c(\psi, \mathbf{u}, \mathbf{v}) := \delta(\psi \times \mathbf{u}, \mathbf{v}),$$

are bilinear and trilinear forms defined on $\mathbf{V} \times \mathbf{V}$ and $L^2(\mathscr{O}) \times \mathbf{V} \times \mathbf{V}$, respectively. By taking $\mathbf{v} = \mathbf{u}$ in the above equation, using the Hölder and Poincaré inequalities, the Sobolev embedding $\mathbf{V} \subset \mathbf{L}^4(\mathscr{O})$, and the Assumption (G) in the resulting equation we infer that there exists a constant $\theta > 0$, such that

$$\|\mathbf{u}\|_\alpha^2 \leqslant \theta\left(|\psi|\|\mathbf{u}_1\|_\alpha\|\mathbf{u}\|_\alpha + \|\mathbf{w}\|_\alpha\|\boldsymbol{\beta}_1\|_{\mathscr{H}} + \|\mathbf{w}_2\|_\alpha\|\boldsymbol{\beta}\|_{\mathscr{H}}\right).$$

Owing to the Cauchy inequality with $\epsilon = \frac{1}{2\theta}$, there exists $C > 0$ such that

$$\frac{1}{2}\|\mathbf{u}\|_\alpha^2 \leqslant C\left(|\psi|^2\|\mathbf{u}_1\|_\alpha^2 + \|\mathbf{w}\|_\alpha\|\boldsymbol{\beta}_1\|_{\mathscr{H}} + \|\mathbf{w}_2\|_\alpha\|\boldsymbol{\beta}\|_{\mathscr{H}}\right),$$

from which we readily conclude the continuity of the map $\mathscr{S}$ and the proof of part (b). ☐

Before proceeding further we state the following remark.

Remark 3.2. It follows from the Hölder inequality and the Sobolev embedding $\mathbf{V} \subset \mathbf{L}^4(\mathscr{O})$ that

$$\begin{aligned} |(\psi \times \mathbf{u}, \mathbf{v})| \leqslant& |\psi|\|\mathbf{v}\|_{\mathbf{L}^4}\|\mathbf{v}\|_{\mathbf{L}^4} \\ \leqslant& C|\psi|\|\mathbf{v}\|_\alpha\|\mathbf{v}\|_\alpha, \ \forall \psi \in L^2(\mathscr{O}), \ \mathbf{u}, \ \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{3.4}$$

Lemma 3.3.

(a) Let $\lambda > 0$, $\mathbf{u} \in \mathbf{V}$ and $f \in L^2(\mathscr{O})$. Then, the transport equation

$$\lambda z + \mathbf{u} \cdot \nabla z = f, \tag{3.5}$$

has a unique solution $z \in L_{\mathbf{u}}$ such that

$$|z| \leqslant |f|.$$

Moreover, the following Green's formula hold

$$(\mathbf{u} \cdot \nabla z, y) = -(\mathbf{u} \cdot \nabla y, z), \tag{3.6}$$

for any z, $y \in L_{\mathbf{u}}$.

(b) The map

$$\begin{aligned}\mathscr{T}: \mathbf{V}\times\mathbf{V}\times\mathscr{H}\times \mathrm{L}^2(\mathscr{O}) &\to \mathbf{V}\\ (\mathbf{u},\mathbf{v},\boldsymbol{\beta},\psi) &\mapsto z,\end{aligned}$$

where z is the unique solution of (3.5) *with right hand side $f = \psi + \mathbf{rot}\, G(\mathbf{v})\boldsymbol{\beta}$ has a closed graph.*

Proof. We refer to [21, Theorem 2.5] for the proof of part (a) .

Let $\{(z_\ell,\mathbf{u}_\ell,f_\ell);\ \ell\in\mathbb{N}\}\subset \mathrm{L}^2\times\mathbf{V}\times\mathrm{L}^2$ be a sequence such that for each $\ell\geqslant 1$ z_ℓ denotes the unique solution of (3.5) with $\mathbf{u}$ and f replaced by $\mathbf{u}_\ell$ and f_ℓ , respectively. If $\{\mathbf{v}_\ell,\beta_\ell,\psi_\ell;\ell\in\mathbb{N}\}\subset \mathbf{V}\times\mathscr{H}\times\mathrm{L}^2(\mathscr{O})$ is a sequence converging to $(\mathbf{v},\beta,\psi)$ in $\mathbf{V}\times\mathscr{H}\times\mathrm{L}^2(\mathscr{O})$, then it is not difficult to check that as $\ell\to\infty$

$$\psi_\ell + \mathbf{rot}\, G(\mathbf{v}_\ell)\beta_\ell \to \psi + \mathbf{rot}\, G(\mathbf{v})\beta \text{ in } \mathrm{L}^2(\mathscr{O}).$$

Thus, in order to prove part (b) , it is sufficient to show that if the sequence $\{(z_\ell,\mathbf{u}_\ell,f_\ell);\ \ell\in\mathbb{N}\}$ converge in $\mathrm{L}^2\times\mathbf{V}\times\mathrm{L}^2$ to $(z,\mathbf{u},f)$ then z is a solution to (3.5). To this end, we first notice that for each ℓ we have

$$\lambda(z_\ell - z,\phi) + \lambda(z,\phi) - (\mathbf{u}_\ell\cdot\nabla\phi, z_\ell - z) - (\mathbf{u}_\ell\cdot\nabla\phi, z) - (f_\ell,\phi) = 0,$$

for any $\phi\in \mathrm{W}^{1,4}_0(\mathscr{O})$. Letting $\ell\to\infty$ in the above identity implies that

$$\lambda(z,\phi) - (\mathbf{u}\cdot\nabla\phi, z) - (f,\phi) = 0,$$

i.e., we have proved that z is a solution of (3.5). This completes the proof of the lemma. □

Proposition 3.4. *Let $n\in\mathbb{N}$, $\mathbf{u}^0 := \mathbf{u}_0\in\mathbf{W}$, $z^0 = \mathbf{rot}(\mathbf{u}^0 - \alpha\Delta\mathbf{u}^0)$ and a $\mathscr{H}$-valued Wiener process $\boldsymbol{\eta}$ with covariance Q satisfying* (N) *be given. Then, with probability one we can find two sequences $\{\mathbf{u}^\ell;\ \ell\in[\![1,n]\!]\}\subset \mathbf{V}$ and $\{z^\ell;\ \ell\in[\![1,n]\!]\}\subset\mathrm{L}^2(\mathscr{O})$ such that for each $\ell\in[\![0,n-1]\!]$ and for all $\mathbf{v}\in\mathbf{V}$ and $\phi\in\mathrm{W}^{1,4}_0(\mathscr{O})$, $\mathbf{u}^\ell$ and z^ℓ satisfies* (3.1) *and* (3.2)*, respectively. Moreover, if all spaces are equipped with their respective Borel σ-algebra, then for each $\ell\in[\![1,n]\!]$ $\mathbf{u}^\ell$ and z^ℓ are $\mathfrak{F}_{t_\ell}$-measurable.*

Proof. We prove the proposition by induction and we start with the proof of the existence of solution. In what follows, we consider a sequence of events $\{\boldsymbol{\Omega}_\ell;\ \ell\in[\![0,n]\!]\}\subset\boldsymbol{\Omega}$ defined by

$$\boldsymbol{\Omega}_\ell = \{\omega; \|\boldsymbol{\eta}(\omega,t_\ell)\|_{\mathscr{H}} < \infty\}.$$

Since $\boldsymbol{\eta}$ is a $\mathscr{H}$-valued Wiener process, we have $\mathbf{P}(\boldsymbol{\Omega}_\ell) = 1$ and $\mathbf{P}(\bigcap_{\ell=1}^n\boldsymbol{\Omega}_\ell) = 1$. Throughout this proof, the arguments below will hold on $\boldsymbol{\Omega}_n = \bigcap_{\ell=1}^n\boldsymbol{\Omega}_\ell$. Since $\mathbf{u}^0\in\mathbf{W}$, $z^0\in\mathrm{L}^2(\mathscr{O})$ and $G(\mathbf{u}^0)\Delta_0\eta\in\mathscr{H}$ are given, then using Lemma 3.1 we can find $\mathbf{u}^1\in\mathbf{V}$ such that (3.1) holds for any $\mathbf{v}\in\mathbf{V}$. Having found $\mathbf{u}^1\in\mathbf{V}$ we invoke Lemma 3.3 to infer the existence of z^1 satisfying (3.3). Now, assuming that $\mathbf{u}^\ell\in\mathbf{V}$ and $z^\ell\in\mathrm{L}^2(\mathscr{O})$ are given, we can argue as above to infer the existence of $\mathbf{u}^{\ell+1}\in\mathbf{V}$ and $z^{\ell+1}\in\mathrm{L}^2(\mathscr{O})$. This completes the proof of the existence.

Observe that for any $\ell\in[\![0,n-1]\!]$, $\mathbf{u}^{\ell+1} = \mathscr{S}(\mathbf{u}^\ell,\Delta_\ell,z^\ell)$ and $z^{\ell+1} = \mathscr{T}(\mathbf{u}^{\ell+1},\mathbf{u}^\ell,\Delta_\ell\boldsymbol{\eta},z^\ell)$. Thus, arguing by induction and using the continuity of $\mathscr{S}$ and the closedness of the graph of $\mathscr{T}$ one can show easily that for each $\ell\in[\![1,n]\!]$ $\mathbf{u}^\ell$ and z^ℓ are $\mathfrak{F}_{t_\ell}$-measurable. This completes the proof of our proposition. □

3.2. *Energy estimates*

In this subsection we will derive several energy estimates for the solution of the Algorithm 1. These estimates are of the essence in the remaining part of the proof of our main result. Before we embark on the statements and proofs of these results let us recall identities and inequalities that are relevant for our analysis. First recall that for any Hilbert space K we have

$$((\psi_1-\psi_2, 2\psi_1))_{\mathrm{K}} = \|\psi_1\|^2_{\mathrm{K}} - \|\psi_2\|^2_{\mathrm{K}} + \|\psi_1-\psi_2\|^2_{\mathrm{K}}, \forall\psi_1,\ \psi_2\in\mathrm{K}. \tag{3.7}$$

We also need the following inequalities: for any $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that

$$\sum_{i=1}^{3} a_i^p \leqslant C_p \left(\sum_{i=1}^{3} a_i\right)^p \leqslant C_p \sum_{i=1}^{3} a_i^p, \tag{3.8}$$

for any non-negative numbers a_i, $i \in [\![1,3]\!]$.
In what follows we will use the following lemma, which is taken from [33] and known as the (discrete) Gronwall lemma, without further notice.

Lemma 3.5. *Let $\{a_n;\ n \in \mathbb{N}_0\}$, $\{b_n;\ n \in \mathbb{N}_0\}$ and $\{c_n;\ n \in \mathbb{N}_0\}$ be three sequences of non-negative real numbers such that $\{c_n;\ n \in \mathbb{N}\}$ is monotonic increasing, $a_0 + b_0 \leqslant c_0$, and there exists a real number $\kappa > 0$ such that*

$$a_n + b_n \leqslant c_n + \kappa \sum_{j=0}^{n-1} a_n,$$

for any $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}_0$

$$a_n + b_n \leqslant c_n e^{n\kappa}.$$

We also need the following lemma.

Lemma 3.6. *Let $\ell \in \mathbb{N}$, $R \in \{G, \tilde{G}\}$, and x^ℓ be a $\mathscr{F}_{t_\ell}$-measurable L^2-valued random variable. Then, for any integer $r \geqslant 1$, and real number $q \geqslant 0$, there exists a constants $C > 0$ such that*

$$\mathbf{E}\Big[|R(\mathbf{u}^\ell)\Delta_\ell \boldsymbol{\eta}|^{2r} |x^\ell|^{2q}\Big] \leqslant C k^r \mathbf{E}\Big[(1 + \|\mathbf{u}^\ell\|_\alpha^{2r})|x^\ell|^{2q}\Big], \tag{3.9}$$

provided that the term in the right-hand side is finite. With the above conditions, we also have

$$\mathbf{E}\left(|x^\ell|^{2r}\left(R(\mathbf{u}^\ell)\Delta_\ell \boldsymbol{\eta}, x^\ell\right)\right) = 0. \tag{3.10}$$

Proof. From the $\mathscr{F}_{t_\ell}$-measurability of x^ℓ, the tower property of the conditional mathematical expectation, the independence of the increments of the Wiener process $\boldsymbol{\eta}$, the inequality (2.4) we derive the following chain of equalities/inequalities

$$\begin{aligned}
\mathbf{E}\Big[|R(\mathbf{u}^\ell)\Delta_\ell \boldsymbol{\eta}|^{2r} |x^\ell|^{2q}\Big] &= \mathbf{E}\Big[\mathbf{E}\Big(\|R(\mathbf{u}^\ell)\|^{2r}_{\mathscr{L}(\mathscr{H},\mathbf{H})} \|\Delta_\ell \boldsymbol{\eta}\|^{2r}_{\mathscr{H}} |x^\ell|^{2q} \| \mathscr{F}_{t_\ell}\Big)\Big], \\
&= \mathbf{E}\Big[\|R(\mathbf{u}^\ell)\|^{2r}_{\mathscr{L}(\mathscr{H},\mathbf{H})} |x^\ell|^{2q}\Big] \mathbf{E}\left(\|\Delta_\ell \boldsymbol{\eta}\|^{2r}_{\mathscr{H}}\right) \\
&\leqslant C (\operatorname{Tr} Q)^r k^r \mathbf{E}\Big[\|R(\mathbf{u}^\ell)\|^{2r}_{\mathscr{L}(\mathscr{H},\mathbf{H})} |x^\ell|^{2q}\Big].
\end{aligned}$$

From the last line and Remark (2.1) we easily derive the sought estimate in Lemma 3.6.

Thanks to the $\mathscr{F}_{t_\ell}$-measurability of x^ℓ and $\mathbf{u}^\ell$ the second part of the lemma easily follows from the fact that $\Delta_\ell \boldsymbol{\eta}$ is a Gaussian random variable with zero mean. □

Now, we proceed to one of the main topics of this section.

Proposition 3.7. *Let $\mathbf{u}^0 \in \mathbf{W}$ and $\boldsymbol{\eta}$ be an $\mathscr{H}$-valued Wiener process with covariance satisfying Assumption (N). Then, for any $\alpha > 0$, $T > 0$ and $p \in [\![1,3]\!]$ there exists a constant $C > 0$ such that for any fixed $n \in \mathbb{N}$ and $\nu \geqslant 0$*

$$\mathbf{E}\left(\max_{0 \leqslant \leqslant n} \|\mathbf{u}^\ell\|_\alpha^{2^p}\right) + \mathbf{E}\sum_{\ell=0}^{n-1} \|\mathbf{u}^{\ell+1} - \mathbf{u}^\ell\|_\alpha^2 + k\nu \mathbf{E}\sum_{\ell=1}^{n} \|\mathbf{u}^\ell\|^2 \leqslant C(1 + \|\mathbf{u}^0\|_\alpha^{2^p}) \tag{3.11}$$

$$\mathbf{E}\left(\max_{0 \leqslant \leqslant n} |z^\ell|^{2^p}\right) + \mathbf{E}\sum_{\ell=0}^{n-1} |z^{\ell+1} - z^\ell|^2 + \frac{\nu}{\alpha} k \mathbf{E}\sum_{\ell=1}^{n} |z^\ell|^2 \leqslant C(1 + |z^0|^{2^p} + \|\mathbf{u}^0\|_\alpha^{2^p}). \tag{3.12}$$

Proof. Since the proofs of (3.11) and (3.12) are very similar, we will only give the proof of (3.12). We will closely follow [48, Proof of Lemma 3.1(iii)].

In order to prove (3.12) for $p = 1$ we fix $\ell \in [\![0, n]\!]$. Given $z^i \in L^2(\mathscr{O})$ and $\mathbf{u}^{i+1} \in \mathbf{V}$, we infer from Lemma 3.3(a) that the random variable z^{i+1} solving Algorithm 1 satisfies $z^{i+1} \in L_{\mathbf{u}^{i+1}}$ for any $i \in [\![0, n-1]\!]$. Thus for any $i \in [\![0, \ell]\!]$ and $\phi \in L_{\mathbf{u}^{i+1}}$, we have

$$(z^{i+1} - z^i, \phi) + \frac{k\nu}{\alpha}(z^{i+1}, \phi) - (z^{i+1}, \mathbf{u}^{i+1} \cdot \nabla\phi) = \frac{k\nu}{\alpha}(\mathbf{rot}\,\mathbf{u}^{i+1}, \phi) + (\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, \phi), \tag{3.13}$$

where, here and throughout, we set $\tilde{G}(\cdot) = \mathbf{rot}\, G(\cdot)$. Thus, every term in the identity (3.13) makes sense when taking $\phi = 2z^{i+1}$. By doing so and invoking (3.6) and (3.7) we derive that

$$\begin{aligned}|z^{i+1}|^2 - |z^i|^2 + |z^{i+1} - z^i|^2 + \frac{2k\nu}{\alpha}|z^{i+1}|^2 = \frac{2k\nu}{\alpha}(\mathbf{rot}\,\mathbf{u}^{i+1}, z^{i+1} - z^i) + \frac{2k\nu}{\alpha}(\mathbf{rot}\,\mathbf{u}^{i+1}, z^i)\\ +2(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, z^{i+1} - z^i) + 2(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, z^i).\end{aligned} \tag{3.14}$$

By using the Cauchy-Young inequality and summing from $i = 0$ to $i = \ell - 1$, it is not difficult to show that

$$\begin{aligned}|z^\ell|^2 + \frac{1}{2}\sum_{i=0}^{\ell-1}|z^{i+1} - z^i|^2 + \frac{2k\nu}{\alpha}\sum_{i=1}^{\ell}|z^i|^2 \leqslant |z^0|^2 + \frac{4\nu^2(k^2+k)}{\alpha^2}\sum_{i=0}^{\ell-1}|\,\mathbf{rot}\,\mathbf{u}^{i+1}|^2 + \frac{k}{4}\sum_{i=0}^{\ell-1}|z^i|^2\\ +4\sum_{i=0}^{\ell-1}|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^2 + 2\sum_{i=0}^{\ell-1}(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, z^i).\end{aligned} \tag{3.15}$$

After taking the mathematical expectation, using the second part of Lemma (3.6) to get rid of the last term and taking the max over $\ell \in [\![0, n]\!]$ in the last estimate we derive that

$$\begin{aligned}\max_{\ell\in[\![0,n]\!]}\mathbf{E}|z^\ell|^2 + \frac{1}{2}\mathbf{E}\sum_{i=0}^{n-1}|z^{i+1} - z^i|^2 + \frac{2k\nu}{\alpha}\mathbf{E}\sum_{i=1}^{n}|z^i|^2 \leqslant |z^0|^2 + \frac{4\nu^2(k+1)T}{\alpha^2}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|\,\mathbf{rot}\,\mathbf{u}^{i+1}|^2\\ +\frac{k}{4}\sum_{\ell=0}^{n-1}\max_{i\in[\![0,\ell]\!]}\mathbf{E}|z^i|^2 + 4\,\mathrm{Tr}\,Qk\mathbf{E}\sum_{i=0}^{n-1}\|\tilde{G}(\mathbf{u}^i)\|^2_{\mathscr{L}(\mathscr{H},\mathbf{H})},\end{aligned} \tag{3.16}$$

from which along with the application of Assumption (G) and Remark 2.1, the fact that $|\,\mathbf{rot}\cdot|$ and $\|\cdot\|_\alpha$ are equivalent on $\mathbf{V}$, the estimate (3.11) and finally the discrete Gronwall lemma we infer that there exists a constant $C > 0$ depending only on T, $\mathrm{Tr}\, Q$ and α such that

$$\max_{\ell\in[\![0,n]\!]}\mathbf{E}|z^\ell|^2 \leqslant C(1 + |z^0|^2 + |\mathbf{u}^0|^2), \tag{3.17}$$

which altogether with (3.16) implies that

$$\mathbf{E}\sum_{i=0}^{n-1}|z^{i+1} - z^i|^2 + \frac{2k\nu}{\alpha}\mathbf{E}\sum_{i=1}^{n}|z^i|^2 \leqslant C(1 + |z^0|^2 + \|\mathbf{u}^0\|^2_\alpha). \tag{3.18}$$

Now dropping out all but $|z^\ell|^2$ positive terms in the LHS of (3.15), taking the maximum over $\ell \in [\![0, n]\!]$ and the mathematical expectation yields

$$\begin{aligned}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^2 \leqslant |z^0|^2 + \frac{4\nu^2(k+1)T}{\alpha^2}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|\,\mathbf{rot}\,\mathbf{u}^{i+1}|^2 + \frac{k}{4}\sum_{\ell=0}^{n-1}\mathbf{E}|z^\ell|^2\\ + 4\,\mathrm{Tr}\,Qk\mathbf{E}\sum_{i=0}^{n-1}\|\tilde{G}(\mathbf{u}^i)\|^2_{\mathscr{L}(\mathscr{H},\mathbf{H})} + 2\mathbf{E}\max_{\ell[\![0,n]\!]}\sum_{i=0}^{\ell-1}(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, z^i).\end{aligned} \tag{3.19}$$

Note that by using the equivalence of the norms $|\mathbf{rot}(\cdot)|$ and $\|\cdot\|_\alpha$ on $\mathbf{V}$, Lemma 3.6 and the estimates (3.11) and (3.17) the sum of the first four terms of the above inequality can be bounded from above by $C(1+\|\mathbf{u}^0\|_\alpha^2+|z^0|^2)$. Thus, we derive by considering the last term as a stochastic integral with piecewise constant integrand, applying the Burkholder-Davis-Gundy inequality, the Cauchy inequality and applying Lemma 3.6 and the estimate (3.7) that

$$\begin{aligned}
\mathrm{I} &\leqslant C(1+\|\mathbf{u}^0\|_\alpha^2+|z^0|^2)+C\mathbf{E}\left(\sum_{\ell=0}^{n-1}|\tilde{G}(\mathbf{u}^\ell)\Delta_\ell||z^\ell|^2\right)^{\frac{1}{2}} \\
&\leqslant C(1+\|\mathbf{u}^0\|_\alpha^2+|z^0|^2)+\frac{1}{2}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^2+Ck\mathbf{E}\sum_{\ell=0}^{n-1}(1+\|\mathbf{u}^\ell\|^2) \\
&\leqslant C(1+\|\mathbf{u}^0\|_\alpha^2+|z^0|^2)+\frac{1}{2}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^2,
\end{aligned}$$

where we denoted by I the RHS of (3.19). Now, absorbing the term $\frac{1}{2}\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^2$ in the LHS of (3.19) implies

$$\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^2\leqslant C(1+|\mathbf{u}^0|^2+|z^0|^2),$$

from which along with (3.18) follows (3.12) for $p=1$.

To treat the case $p=2$ we first multiply (3.14) by $2|z^{i+1}|^2$, then apply (3.7) to obtain

$$|z^{i+1}|^4-|z^i|^4+\left||z^{i+1}|^2-|z^i|^2\right|^2+2|z^{i+1}|^2|z^{i+1}-z^i|^2+4\frac{\nu}{\alpha}k|z^{i+1}|^4=\sum_{m=1}^{4}\mathrm{J}_{m,i}, \tag{3.20}$$

where the summands $\mathrm{J}_{m,i}$ are defined as follows

$$\begin{aligned}
\mathrm{J}_{1,i} &:= \frac{4k\nu}{\alpha}|z^{i+1}|^2\left(\mathbf{rot}\,\mathbf{u}^{i+1},z^{i+1}-z^i\right) \\
\mathrm{J}_{2,i} &:= \frac{4k\nu}{\alpha}|z^{i+1}|^2\left(\mathbf{rot}\,\mathbf{u}^{i+1},z^i\right) \\
\mathrm{J}_{3,i} &:= 2|z^{i+1}|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},z^{i+1}-z^i\right) \\
\mathrm{J}_{4,i} &:= 2|z^{i+1}|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},z^i\right).
\end{aligned}$$

using well-known and elementary inequalities such as the Cauchy-Schwarz and Cauchy inequalities we can show that they satisfy

$$\begin{aligned}
\mathrm{J}_{1,i} &\leqslant \frac{1}{2}|z^{i+1}|^2|z^{i+1}-z^i|^2+\frac{1}{8}\left||z^{i+1}|^2-|z^i|^2\right|^2+C(k^4+k^2)|\mathbf{rot}\,\mathbf{u}^{i+1}|^4+ck^2|z^i|^4, \\
\mathrm{J}_{2,i} &\leqslant \frac{1}{8}\left||z^{i+1}|^2-|z^i|^2\right|^2+C(k^2+k)\left(|\mathbf{u}^{i+1}|^4+|z^i|^4\right), \\
\mathrm{J}_{3,i} &\leqslant \frac{1}{2}|z^{i+1}|^2|z^{i+1}-z^i|^2+\frac{1}{8}\left||z^{i+1}|^2-|z^i|^2\right|^2+C|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^4+|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^2|z^i|^2, \\
\mathrm{J}_{4,i} &\leqslant \frac{1}{8}\left||z^{i+1}|^2-|z^i|^2\right|^2+C|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^2|z^i|^2+C|z^i|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},z^i\right).
\end{aligned}$$

After plugging these inequalities in (3.20), absorbing some terms in LHS, and summing from $i=0$ to $i=\ell-1$ we deduce that

$$\begin{aligned}
&|z^\ell|^4+\frac{1}{2}\sum_{i=0}^{\ell-1}\left||z^{i+1}|^2-|z^i|^2\right|^2+\sum_{i=0}^{\ell-1}|z^{i+1}|^2|z^{i+1}-z^i|^2+4\frac{\nu}{\alpha}k\sum_{i=1}^{\ell}|z^i|^4 \\
&\leqslant |z^0|^4+C(k^2+k)\left(\sum_{i=1}^{\ell}\|\mathbf{u}^i\|^4+\sum_{i=0}^{\ell-1}|z^i|^4\right)+C\sum_{i=0}^{\ell-1}|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^4 \\
&\quad+C\sum_{i=0}^{\ell-1}|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|^2|z^i|^2+C\sum_{i=0}^{\ell-1}|z^i|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},z^i\right).
\end{aligned} \tag{3.21}$$

Now, by taking the mathematical expectation, using Lemma (3.6) to get rid of the last term in the above inequality and to estimate the terms containing $\tilde{G}(\mathbf{u}^i)$, and taking the maximum over $\ell \in [\![0,n]\!]$ we infer that

$$\begin{aligned}
&\max_{\ell\in[\![0,n]\!]} \mathbf{E}|z^\ell|^4 + \frac{1}{2}\mathbf{E}\sum_{\ell=0}^{n-1}\left||z^{\ell+1}|^2-|z^\ell|^2\right|^2 + \mathbf{E}\sum_{\ell=0}^{n-1}|z^{\ell+1}|^2|z^{\ell+1}-z^\ell|^2 + 4\frac{\nu}{\alpha}k\mathbf{E}\sum_{\ell=1}^{n}|z^\ell|^4\\
&\leqslant |z^0|^4 + C(k^2+k)\sum_{\ell=1}^{n}\mathbf{E}\|\mathbf{u}^\ell\|_\alpha^4 + C(k^2+k)\mathbf{E}\sum_{\ell=0}^{n-1}|z^\ell|^4 + Ck\sum_{\ell=0}^{n-1}\mathbf{E}\left([1+\|\mathbf{u}^\ell\|_\alpha^2]|z^\ell|^2\right), \qquad (3.22)\\
&\leqslant |z^0|^4 + C(k^2+k)\sum_{\ell=1}^{n}\mathbf{E}\|\mathbf{u}^\ell\|_\alpha^4 + C(k^2+k)\mathbf{E}\sum_{\ell=0}^{n-1}|z^\ell|^4 + CT\mathbf{E}\max_{\ell\in[\![0,n]\!]}[1+\|z^\ell\|_\alpha^4],
\end{aligned}$$

which, after applying (3.11) and the discrete Gronwall inequality, implies

$$\max_{\ell\in[\![0,n]\!]}\mathbf{E}|z^\ell|^4 \leqslant C(1+\|\mathbf{u}^0\|_\alpha^4+|z^0|^4). \qquad (3.23)$$

As in the case $p=1$, after dropping all, except the first term, positive terms in the LHS of (3.21), taking the maximum over $\ell \in [\![0,n]\!]$ and the mathematical expectation, and utilizing Lemma 3.6 to estimate the term containing $|\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}|$ we obtain

$$\begin{aligned}
\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^4 &\leqslant |z^0|^4 + C(k^2+k)\sum_{\ell=1}^{n}\mathbf{E}\|\mathbf{u}^\ell\|_\alpha^4 + C(k^2+k)\mathbf{E}\sum_{\ell=0}^{n-1}|z^\ell|^4\\
&\quad + \mathbf{E}\max_{\ell\in[\![0,n]\!]}\sum_{i=0}^{\ell-1}|z^i|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},\mathbf{u}^i\right) \qquad (3.24)\\
&\leqslant C(1+\|\mathbf{u}^o\|^4+|z^0|^4) + \mathbf{E}\max_{\ell\in[\![0,n]\!]}\sum_{i=0}^{\ell-1}|z^i|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},\mathbf{u}^i\right)
\end{aligned}$$

Here we used (3.11) and (3.23) to derive the last line. The last term in the last line is estimated by means of the Burkholder-Davis-Gundy inequality after considering the sum as a stochastic integral with piecewise constant integrand:

$$\begin{aligned}
\mathbf{E}\max_{\ell\in[\![0,n]\!]}\sum_{i=0}^{\ell-1}|z^i|^2\left(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},\mathbf{u}^i\right) &\leqslant C\mathbf{E}\left(\sum_{\ell=0}^{n-1}|z^\ell|^4|\tilde{G}(\mathbf{u}^\ell)\Delta_l\boldsymbol{\eta}|^2|z^\ell|^2\right)^{\frac{1}{2}} \qquad (3.25)\\
&\leqslant \frac{1}{2}\mathbf{E}\max_{\ell\in[\![0n]\!]}|z^\ell|^4 + Ck\mathbf{E}\sum_{\ell=0}^{n-1}\left(\mathbf{E}\|\mathbf{u}^\ell\|_\alpha^4+|z^\ell|^4\right) \qquad (3.26)\\
&\leqslant \frac{1}{2}\mathbf{E}\max_{\ell\in[\![0n]\!]}|z^\ell|^4 + C(1+\|\mathbf{u}^0\|_\alpha^4+|z^0|^4), \qquad (3.27)
\end{aligned}$$

where the Cauchy inequality, Lemma 3.6 and the estimates (3.11) and (3.23) was used to obtain the second and third line of the above chain of inequalities. With these last two chains of estimates we derive that

$$\mathbf{E}\max_{\ell\in[\![0,n]\!]}|z^\ell|^4 \leqslant C(1+\|\mathbf{u}^0|_\alpha^4+|z^0|^4), \qquad (3.28)$$

which ends the proof of (3.12) for the case $p=2$.

As above, the beginning of the proof for the case $p=3$ consists in establishing an identity for $|z^{i+1}|^8$. For this aim we infer from multiplying (3.20) by $2|z^{i+1}|^4$, then applying (3.7) that

$$\begin{aligned}
&|z^{i+1}|^8 - |z^{i+1}|^8 + \left||z^{i+1}|^4-|z^i|^4\right|^2 + 2|z^{i+1}|^4\left||z^{i+1}|^2-|z^i|^2\right|^2\\
&= -4|z^{i+1}|^6|z^{i+1}-z^i|^2 - 8\frac{\nu}{\alpha}k|z^{i+1}|^8 + 2\sum_{m=1}^{4}\mathrm{J}_{m,i}|z^{i+1}|^4.
\end{aligned} \qquad (3.29)$$

With this identity at hand we can use similar arguments as in the proof of the case $p = 2$ to first derive a uniform estimate for $\max_{\ell \in [\![0,n]\!]} \mathbf{E}|z^\ell|^8$, and then a uniform estimate for $\mathbf{E} \max_{\ell \in [\![0,n]\!]} |z^\ell|^8$ which with (3.18) yields (3.12). We omit the detail and remaining part of the proof since the calculations, although long and tedious, are quite similar to the case $p = 2$. □

We also need the following results.

Proposition 3.8. *There exists a constant $C > 0$ such that for any $\ell \in [\![0, n-1]\!]$, we have*

$$k\mathbf{E} \sum_{j=0}^{n-\ell} \|\mathbf{u}^{j+\ell} - \mathbf{u}^j\|_\alpha^4 \leqslant C t_\ell^2 \left(\frac{\nu^4}{\alpha^4} + 1 \right), \tag{3.30}$$

$$k\mathbf{E} \sum_{j=0}^{n-\ell} \|z^{j+\ell} - z^j\|^4_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})} \leqslant C t_\ell^2 \left(\frac{\nu^4}{\alpha^4} + 1 \right) k^2. \tag{3.31}$$

Proof. As in the statement of the proposition, the proof will be divided in two parts.

(i) To start with the proof of (3.30) we recall that for any $\mathbf{v} \in \mathbf{V}$, $\mathbf{u}^{i+1}$ satisfies

$$((\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{v}))_\alpha + \nu k((\mathbf{u}^{i+1}, \mathbf{v})) + k(z^i \times \mathbf{u}^{i+1}, \mathbf{v}) = (G(\mathbf{u}^i)\Delta_i \boldsymbol{\eta}, \mathbf{v}). \tag{3.32}$$

Summing (3.32) from $i = j$ to $i = j + \ell - 1$ gives

$$\frac{1}{k}((\mathbf{u}^{j+\ell} - \mathbf{u}^j, \mathbf{v}))_\alpha - \nu \sum_{i=j}^{j+\ell-1} \Big[((\mathbf{u}^{i+1}, \mathbf{v})) + (z^i \times \mathbf{u}^{i+1}, \mathbf{v}) \Big] = \frac{1}{k} \sum_{i=j}^{j+\ell-1} (G(\mathbf{u}^i)\Delta_i \boldsymbol{\eta}, \mathbf{v}).$$

By taking $\mathbf{v} = \mathbf{u}^{j+\ell} - \mathbf{u}^j$ in the above identity, then raising to the power 2 the resulting equation and summing from $j = 0$ to $j = n - \ell$, and using (3.8) we obtain

$$\begin{aligned} k \sum_{j=0}^{n-\ell} \|\mathbf{u}^{j+\ell} - \mathbf{u}^j\|_\alpha^4 \leqslant Ck^3 \sum_{j=0}^{n-\ell} \left[\left| \sum_{i=j}^{j+\ell-1} \nu((\mathbf{u}^{i+1}, \mathbf{u}^{j+\ell} - \mathbf{u}^j)) \right|^2 + \left| \sum_{i=j}^{j+\ell-1} (z^i \times \mathbf{u}^{i+1}, \mathbf{u}^{j+\ell} - \mathbf{u}^j) \right|^2 \right] \\ + Ck \sum_{j=0}^{n-\ell} \left| \sum_{i=j}^{j+\ell-1} (G(\mathbf{u}^i)\Delta_i \boldsymbol{\eta}, \mathbf{u}^{j+\ell} - \mathbf{u}^j) \right|^2 \\ \leqslant C_p k^3 \sum_{j=0}^{n-\ell} (I_{j,1} + I_{j,2}) + C_p k \sum_{j=0}^{n-\ell} I_{j,3}. \end{aligned} \tag{3.33}$$

Thus, in order to prove (3.30) we will successively estimate the terms in the right hand side of (3.33). From the Cauchy-Schwarz, the inequality (2.2), the Hölder inequalities along with the Cauchy-Young inequality with $\varepsilon > 0$ we obtain

$$\begin{aligned} k^3 \mathbf{E} \sum_{j=0}^{n-\ell} I_{j,1} \leqslant & k^3 \ell \frac{\nu^2}{\alpha^2} \mathbf{E} \sum_{j=0}^{n-\ell} \sum_{i=j}^{j+\ell-1} \|\mathbf{u}^{i+1}\|_\alpha^2 \|\mathbf{u}^{j+\ell} - \mathbf{u}^j\|_\alpha^2 \end{aligned} \tag{3.34}$$

$$\begin{aligned} \leqslant & k^4 \ell^2 C_\varepsilon \frac{\nu^4}{\alpha^4} \mathbf{E} \sum_{j=0}^{n-\ell} \sum_{i=j}^{j+\ell-1} \|\mathbf{u}^{i+1}\|_\alpha^4 + k^2 \ell \varepsilon \mathbf{E} \sum_{j=0}^{n-\ell} \|\mathbf{u}^{j+\ell} - \mathbf{u}^j\|_\alpha^4, \\ \leqslant & t_\ell^3 C_\varepsilon T \frac{\nu^4}{\alpha^4} \mathbf{E} \max_{j \in [\![0,n]\!]} \|\mathbf{u}^j\|_\alpha^4 + \varepsilon T k \mathbf{E} \sum_{j=0}^{n-\ell} \|\mathbf{u}^{j+\ell} - \mathbf{u}^j\|_\alpha^4. \end{aligned} \tag{3.35}$$

To estimate the second term involving the cross-product of z^i and $\mathbf{u}^{i+1}$, we use the Hölder inequality, the Sobolev embedding $\mathbf{V} \subset \mathrm{L}^4(\mathscr{O})$, and the Cauchy inequality with $\varepsilon > 0$, and we obtain

$$\begin{aligned} k^3\mathbf{E}\sum_{j=0}^{n-\ell} I_{j,2} \leqslant & k^3\ell\mathbf{E}\left(\sum_{j=0}^{n-\ell}\|\mathbf{u}^{i+1}\|^2_{\mathrm{L}^4(\mathscr{O})}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^2_{\mathrm{L}^4(\mathscr{O})}|z^i|^2\right) \\ \leqslant & Ck^3\ell\mathbf{E}\left(\sum_{j=0}^{n-\ell}\|\mathbf{u}^{i+1}\|^2_{\alpha}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^2_{\alpha}|z^i|^2\right) \\ \leqslant & C_\varepsilon k^4\ell^2\mathbf{E}\sum_{j=0}^{n-\ell}\sum_{i=j}^{j+\ell-1}\|\mathbf{u}^{i+1}\|^4_{\alpha}|z^i|^4+\varepsilon k^2\ell\sum_{j=0}^{n-\ell}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^4_{\alpha} \qquad (3.36) \\ \leqslant & C_\varepsilon t_\ell^3 T\mathbf{E}\left(\max_{i\in[\![0,n]\!]}[|z^i|^4\|\mathbf{u}^i\|^4_{\alpha}]\right)+\varepsilon Tk\mathbf{E}\sum_{j=0}^{n-\ell}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^4_{\alpha}. \qquad (3.37) \end{aligned}$$

The following chain of inequalities can be checked using the Cauchy-Schwarz and the Hölder inequalities along with Remark 2.1(a), (2.4) and the Cauchy-Young inequality with $\varepsilon > 0$:

$$\begin{aligned} k\mathbf{E}\sum_{j=0}^{n-\ell} I_{j,3} \leqslant & Ck\sum_{j=0}^{n-\ell}\left(\mathbf{E}|\mathbf{u}^{j+\ell}-\mathbf{u}^j|^4\mathbf{E}\left[\sum_{i=j}^{j+\ell-1}\|G(\mathbf{u}^i)\|_{\mathscr{L}(\mathscr{H},\mathbf{H})}\|\Delta_i\boldsymbol{\eta}\|_{\mathscr{H}}\right]^4\right)^{\frac12} \\ \leqslant & Ck\sum_{j=0}^{n-\ell}\left(\mathbf{E}|\mathbf{u}^{j+\ell}-\mathbf{u}^j|^4\right)^{\frac12}\left(\ell\mathbf{E}\sum_{i=j}^{j+\ell-1}(1+\|\mathbf{u}^i\|^4_{\alpha})\|\Delta_i\boldsymbol{\eta}\|^4_{\mathscr{H}}\right)^{\frac12} \\ \leqslant & \left(C\ell^2\max_{i\in[\![0,n]\!]}\left[\mathbf{E}(1+\|\mathbf{u}^i\|^4_{\alpha})^2\mathbf{E}\|\Delta_i\boldsymbol{\eta}\|^8_{\mathscr{H}}\right]^{\frac12}\right)^{\frac12}k\sum_{j=0}^{n-\ell}\left(\mathbf{E}|\mathbf{u}^{j+\ell}-\mathbf{u}^j|^4\right)^{\frac12} \\ \leqslant & C_\varepsilon Ck^2\ell^2\max_{i\in[\![0,n]\!]}\left[\mathbf{E}(1+\|\mathbf{u}^i\|^4_{\alpha})^2\right]^{\frac12}+\varepsilon kT\mathbf{E}\sum_{j=0}^{n-\ell}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^4_{\alpha}. \end{aligned}$$

From the last line we readily derive that

$$k\mathbf{E}\sum_{j=0}^{n-\ell} I_{j,3} \leqslant C_\varepsilon t_\ell^2(1+\mathbf{E}\max_{i\in[\![0,n]\!]}\|\mathbf{u}^i\|^8_{\alpha})+\varepsilon kT\mathbf{E}\sum_{j=0}^{n-\ell}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^4_{\alpha} \qquad (3.38)$$

Summing up, we can derive from the estimates (3.33), (3.35), (3.37) and (3.38) that

$$(1-4\varepsilon T)k\mathbf{E}\sum_{j=0}^{n-\ell}\|\mathbf{u}^{j+\ell}-\mathbf{u}^j\|^4_{\alpha} \leqslant Ct_\ell^2\left(\frac{\nu^4}{\alpha^4}+1\right)\left(\mathbf{E}\max_{i\in[\![0,n]\!]}\|\mathbf{u}^i\|^8+\mathbf{E}\max_{i\in[\![0,n]\!]}|z^i|^8\right). \qquad (3.39)$$

Substituting $\varepsilon = \frac{1}{8T}$ in the above inequality and plugging (3.11) and (3.12) in the resulting equations yields (3.30).

(ii) Now, we proceed to the proof of (3.31). Summing (3.13) from $i = j$ to $i = j + \ell - 1$ implies that for any $\phi \in \mathrm{W}^{1,4}_0(\mathscr{O})$

$$(z^{\ell+j}-z^j,\phi) = -k\sum_{i=j}^{j+\ell-1}\left[\frac{\nu}{\alpha}(z^{i+1}-\mathbf{rot}\,\mathbf{u}^{i+1},\phi)-(\mathbf{u}^{i+1}\cdot\nabla\phi,z^{i+1})\right]+\sum_{i=j}^{j+\ell-1}(\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta},\phi),$$

where, as in the proof of (3.12), we have set $\tilde{G}(\cdot) = \mathbf{rot}\, G(\cdot)$. From this identity and (3.8) we readily infer that

$$k\mathbf{E}\sum_{j=0}^{n-\ell} \|z^{\ell+j} - z^j\|_{X^*}^4 \leqslant C_p k^5 \mathbf{E}\sum_{j=0}^{n-\ell} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \left| \sum_{i=j}^{j+\ell-1} \left[\frac{\nu}{\alpha}(z^{i+1} - \mathbf{rot}\,\mathbf{u}^{i+1}, \phi) + (\mathbf{u}^{i+1}\cdot\nabla z^{i+1}, \phi)\right]\right|^4 + k\mathbf{E}\sum_{j=0}^{n-\ell} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \left|\sum_{i=j}^{j+\ell-1} (\tilde{G}(\mathbf{u}^i)\Delta_i \boldsymbol{\eta}, \phi)\right|^4, \tag{3.40}$$

where $X := W_0^{1,4}(\mathscr{O})$. We easily infer from the Cauchy-Schwarz inequality, the Hölder inequality for counting measure and the Sobolev embedding $W_0^{1,4}(\mathscr{O}) \subset L^2(\mathscr{O})$ that

$$\begin{aligned}
&k^5\mathbf{E} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \sum_{j=0}^{n-\ell} \left|\sum_{i=j}^{j+\ell-1} \left[\frac{\nu}{\alpha}(z^{i+1} - \mathbf{rot}\,\mathbf{u}^{i+1}, \phi)\right]\right|^4 \\
&\leqslant \frac{\nu^4}{\alpha^4} k^5 \ell \mathbf{E} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \sum_{j=0}^{n-\ell}\sum_{i=j}^{j+\ell-1} (|z^{i+1}|^4 + |\mathbf{rot}\,\mathbf{u}^{i+1}|^4)|\phi|_X^4 \\
&\leqslant \frac{\nu^4}{\alpha^4} t_\ell^2 T k^2 \mathbf{E}[\max_{i\in[\![0,n]\!]} (|z^i|^4 + \|\mathbf{u}^i\|_\alpha^4)].
\end{aligned} \tag{3.41}$$

A successive applications of Hölder's inequality yields

$$\begin{aligned}
&k^5\mathbf{E}\sum_{j=0}^{n-\ell} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \left|\sum_{i=j}^{j+\ell-1} (\mathbf{u}^{i+1}\cdot\nabla z^{i+1}, \phi)\right|^4 \\
&\leqslant Ck^4 t_\ell \mathbf{E} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \sum_{j=0}^{n-\ell}\sum_{i=j}^{j+\ell-1} \|\mathbf{u}^{i+1}\|_{L^4(\mathscr{O})}^4 \|\nabla\phi\|_{L^4(\mathscr{O})}^4 |z^{i+1}|^4 \\
&\leqslant CTk^2 t_\ell^2 \left(\mathbf{E}\max_{i\in[\![0,n]\!]} |z^i|^8 \mathbf{E}\max_{i\in[\![0,n]\!]} \|\mathbf{u}^i\|_\alpha^8\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.42}$$

For the stochastic perturbation we have

$$k\mathbf{E}\sum_{j=0}^{n-\ell} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \left|\sum_{i=j}^{j+\ell-1} (\tilde{G}(\mathbf{u}^i)\Delta_i\boldsymbol{\eta}, \phi)\right|^4 \tag{3.43}$$

$$\begin{aligned}
&\leqslant Ct_\ell \mathbf{E} \sup_{\phi\in X; \|\phi\|_X \leqslant 1} \sum_{j=0}^{n-\ell}\sum_{i=j}^{j+\ell-1} \|\tilde{G}(\mathbf{u}^i)\|_{\mathscr{L}(\mathscr{H}, L^2(\mathscr{O}))}^4 \|\Delta_i\boldsymbol{\eta}\|_{\mathscr{H}}^4 |\phi|^4 \\
&\leqslant Ct_\ell \sum_{j=0}^{n-\ell}\sum_{i=j}^{j+\ell-1} \left(\mathbf{E}\|\tilde{G}(\mathbf{u}^i)\|_{\mathscr{L}(\mathscr{H}, L^2(\mathscr{O}))}^8 \mathbf{E}\|\Delta_i\boldsymbol{\eta}\|_{\mathscr{H}}^8\right)^{\frac{1}{2}} \\
&\leqslant Ct_\ell^2 T \left(\mathbf{E}\max_{i\in[\![0,n]\!]} \|\mathbf{u}^i\|_\alpha^8\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.44}$$

Thus, by substituting (3.41), (3.42) and (3.44) into (3.40) we obtain

$$k\mathbf{E}\sum_{j=0}^{n-\ell} \|z^{\ell+j} - z^j\|_{X^*}^4 \leqslant Ct_\ell^2 k^2 \left(\frac{\nu^4}{\alpha^4} + 1\right)\left(\mathbf{E}\max_{i\in[\![0,n]\!]} |z^i|^{16} + \mathbf{E}\max_{i\in[\![0,n]\!]} \|\mathbf{u}^i\|_\alpha^{16}\right),$$

which along with (3.11) and (3.12) implies (3.31).

□

3.3. Construction of the approximating solution and tightness

In this subsection we will study the compactness of some interpolants of the sequences $\{\mathbf{u}^\ell;\ \ell \in [\![0,n]\!]\}$ and $\{z^\ell;\ \ell \in [\![0,n]\!]\}$. More precisely, associated to $\{\mathbf{u}^\ell;\ \ell \in [\![0,n]\!]\}$ we define the piecewise affine, globally continuous $\mathrm{u}_n : [0,T] \to \mathbf{V}$ by

$$\mathrm{u}_n(t) = \sum_{\ell=0}^{n-1}\left(\mathbf{u}^\ell + \frac{\mathbf{u}^{\ell+1}-\mathbf{u}^\ell}{k}(t-t_\ell)\right)1_{[t_\ell,t_{\ell+1}]}(t),\ t\in[0,T].$$

We also introduce

$$\check{\mathrm{u}}_n(t) = \sum_{\ell=0}^{n-1}\mathbf{u}^\ell 1_{[t_\ell,t_{\ell+1})}(t),\quad t\in[0,T],$$

and

$$\hat{\mathrm{u}}_n(t) = \sum_{\ell=0}^{n-1}\mathbf{u}^{\ell+1} 1_{(t_\ell,t_{\ell+1}]}(t),\quad t\in[0,T].$$

Analogously, we define

$$y_n(t) = \sum_{\ell=0}^{n-1}\left(z^\ell + \frac{z^{\ell+1}-z^\ell}{k}(t-t_\ell)\right)1_{[t_\ell,t_{\ell+1}]}(t),\ t\in[0,T],$$

$$\check{y}_n(t) = \sum_{\ell=0}^{n-1} z^\ell 1_{[t_\ell,t_{\ell+1})}(t),\quad t\in[0,T],$$

$$\hat{y}_n(t) = \sum_{\ell=0}^{n-1} z^{\ell+1} 1_{(t_\ell,t_{\ell+1}]}(t),\quad t\in[0,T],$$

where $z_0 := z^0 = \mathbf{rot}(\mathbf{u}_0 - \alpha\Delta\mathbf{u}_0)$. Observe that u_n, $\hat{\mathrm{u}}_n$, y_n and $\hat{y}_n$ are not $\mathfrak{F}$-adapted, but $\check{\mathbf{u}}_n$, $\check{y}_n$ are. We formulate several estimates for u_n, $\hat{\mathrm{u}}_n$, y_n and $\hat{y}_n$ in the following proposition.

Proposition 3.9. *For any $p \in [\![1,3]\!]$ and $\alpha > 0$ there exists a constant $C > 0$ such that for any fixed $\nu \geqslant 0$,*

$$\sup_{n\in\mathbb{N}}\mathbf{E}\sup_{t\in[0,T]}[\|\mathrm{u}_n(t)\|_\alpha^{2^p} + \|\hat{\mathrm{u}}_n(t)\|_\alpha^{2^p} + \|\check{\mathrm{u}}_n(t)\|_\alpha^{2^p}] \leqslant C(\|\mathbf{u}_0\|^{2^p}+1), \tag{3.45}$$

$$\sup_{n\in\mathbb{N}}\mathbf{E}\sup_{t\in[0,T]}[\|y_n(t)\|_\alpha^{2^p} + \|\hat{y}_n(t)\|_\alpha^{2^p} + \|\check{y}_n(t)\|_\alpha^{2^p}] \leqslant C(|z^0|^{2^p} + \|\mathbf{u}^0\|_\alpha^{2^p}+1). \tag{3.46}$$

Proof. The present proposition is a corollary of Proposition 3.7 □

Proposition 3.10. *In this proposition we extend the functions u_n and y_n, $n\in\mathbb{N}$, by zero outside $[0,T]$. Then, for any $\alpha > 0$ there exists a constant $C > 0$ such that for any $\nu \geqslant 0$ and $\delta > 0$,*

$$\sup_{n\in\mathbb{N}}\mathbf{E}\int_0^{T-\delta}\|\mathrm{u}_n(t+\delta)-\mathrm{u}_n(t)\|_{\mathbf{V}}^4 \leqslant C\delta^2, \tag{3.47}$$

$$\sup_{n\in\mathbb{N}}\mathbf{E}\int_0^{T-\delta}\|y_n(t+\delta)-y_n(t)\|_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})}^4 \leqslant C\delta^2. \tag{3.48}$$

Proof. Noticing that, for any $\theta > 0$, $k^\theta < T^\theta$, the estimates in the present proposition follows from (3.30) and (3.31) and [49, Lemma 3.2]. □

The following convergences will also play a central role in the remaining part of our paper.

Proposition 3.11. *We have*

$$\lim_{n\to\infty}\mathbf{E}\int_0^T\|\mathrm{u}_n(t)-\check{\mathrm{u}}_n(t)\|_\alpha^2 dt + \lim_{n\to\infty}\mathbf{E}\int_0^T\|\mathrm{u}_n(t)-\hat{\mathrm{u}}_n(t)\|_\alpha^2 dt = 0, \tag{3.49}$$

$$\lim_{n\to\infty}\mathbf{E}\int_0^T|y_n(t)-\check{y}_n(t)|^2 dt + \lim_{n\to\infty}\mathbf{E}\int_0^T\|y_n(t)-\hat{y}_n(t)\|_\alpha^2 dt = 0. \tag{3.50}$$

Proof. The convergences (3.49) and (3.50) can be proved in the same fashion and we only prove the first one. Also, notice that the arguments of the proof for the convergences of the two terms in (3.49) are very similar. Hence, to fix the idea we will only establish

$$\lim_{n\to\infty} \mathbf{E}\int_0^T \|\mathrm{u}_n(t)-\hat{\mathrm{u}}(t)\|_\alpha^2 dt = 0. \tag{3.51}$$

To this end, we use the definition of u_n and $\hat{\mathrm{u}}_n$ to derive that

$$\begin{aligned}\mathbf{E}\int_0^T \|\mathrm{u}_n(t)-\hat{\mathrm{u}}_n(t)\|_\alpha^2 dt =& \mathbf{E}\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}} \|\mathrm{u}_n(t)-\hat{\mathrm{u}}_n(t)\|_\alpha^2 dt \\ \leqslant& \frac{C}{k^2}\mathbf{E}\sum_{\ell=1}^{n-1}\int_{t_\ell}^{t_{\ell+1}} (t-t_\ell)^2\|\mathbf{u}^\ell-\mathbf{u}^{\ell+1}\|_\alpha^2 \\ &+ C\mathbf{E}\sum_{\ell=1}^{n-1}\int_{t_\ell}^{t_{\ell+1}} \|\mathbf{u}^\ell-\mathbf{u}^{\ell+1}\|_\alpha^2 dt.\end{aligned}$$

From the last line and (3.11) we infer that

$$\mathbf{E}\int_0^T \|\mathrm{u}_n(t)-\hat{\mathrm{u}}(t)\|_\alpha^2 dt \leqslant C\frac{T}{n},$$

which, upon passing to the limit, implies (3.51). □

We close this subsection by showing that u_n and y_n is in fact a solution of the integral form of the system (1.5) up to some small error terms. We mainly prove the following proposition.

Proposition 3.12. *Let $n\in\mathbb{N}$, $t\in[0,T]$, $\ell_n=\min\{\ell\in[\![0,n]\!];\ t\in[t_\ell,t_{\ell+1}]\}$ and $\tau_n=\ell_n k$. For each $n\in\mathbb{N}$, the functions u_n and y_n satisfies*

$$\begin{aligned}((\mathrm{u}_n(t),\mathbf{v}))_\alpha + \int_0^t [((\hat{\mathrm{u}}_n(s),\mathbf{v})) + (\check{y}_n(s)\times\hat{\mathrm{u}}_n(s),\mathbf{v})]ds =& \int_0^t (G(\check{\mathrm{u}}_n(s))dW(s),\mathbf{v}) \\ &+((\mathbf{u}_0,\mathbf{v}))_\alpha + (\mathscr{E}_n(t),\mathbf{v}),\end{aligned} \tag{3.52}$$

$$\begin{aligned}(y_n(t),\phi) + \frac{\nu}{\alpha}\int_0^t (\hat{y}_n(s)-\hat{\mathrm{u}}_n(s)\cdot\nabla\hat{y}_n(s),\phi)ds =& (z_0,\phi) + \frac{\nu}{\alpha}\int_0^t (\mathbf{rot}\,\hat{\mathrm{u}}_n(s),\phi)ds \\ &+ \int_0^t (\tilde{G}(\check{\mathrm{u}}_n(s))dW(s),\phi) + (\tilde{\mathscr{E}}_n(t),\phi),\end{aligned} \tag{3.53}$$

for any $t\in[0,T]$, $\mathbf{v}\in\mathbf{V}$ and $\phi\in \mathrm{W}_0^{1,4}(\mathscr{O})$. Here, we have put $\tilde{G}(\cdot) := \mathbf{rot}\, G(\cdot)$ and

$$\begin{aligned}\mathscr{E}_n(\cdot) :=& \int_t^{\tau_n+1} G(\check{\mathrm{u}}_n(s))dW(s) - \frac{k\wedge t}{k}\int_0^k G(\check{\mathrm{u}}_n(s))dW(s) \\ &- \int_0^t 1_{[0,k]}(s)\,[\Delta\hat{\mathrm{u}}_n(s) + \check{y}_n(s)\times\hat{\mathrm{u}}_n(s)]\,ds, \\ \tilde{\mathscr{E}}_n(\cdot) :=& -\int_0^t 1_{[0,k]}(s)[\frac{\nu}{\alpha}(\mathbf{rot}\,\hat{\mathrm{u}}_n(s)-\hat{y}_n(s)) - \hat{\mathrm{u}}_n(s)\cdot\nabla\hat{y}_n(s)]ds \\ &+ \int_t^{\tau_n+1}\tilde{G}(\check{\mathrm{u}}_n(s))dW(s) - \frac{k\wedge t}{k}\int_0^k \tilde{G}(\check{\mathrm{u}}_n(s))dW(s).\end{aligned}$$

Proof. The proof of this proposition is quite easy and in following the spirit of [52] will just give a rather sketchy proof of it. Using the definition of $\mathfrak{u}_n$ and (3.1) we have

$$\begin{aligned}((\mathfrak{u}_n(t),\mathbf{v}))_\alpha =&((\mathbf{u}_0,\mathbf{v}))_\alpha+\int_0^t((\frac{\partial \mathfrak{u}_n}{\partial t}(s),\mathbf{v}))_\alpha ds,\\ =&((\mathbf{u}_0,\mathbf{v}))-\sum_{\ell=0}^{n-1}\int_0^t[\nu((\mathbf{u}^{\ell+1},\mathbf{v}))+(z^\ell\times\mathbf{u}^{\ell+1},\mathbf{v})]1_{[t_\ell,t_{\ell+1}]}(s)ds\\ &+\frac{1}{k}\sum_{\ell=1}^{n-1}\int_0^t(G(\mathbf{u}^\ell)\Delta_\ell\boldsymbol{\eta}1_{[t_\ell,t_{\ell+1}]},\mathbf{v})ds+\int_0^t[\nu((\mathbf{u}^1,\mathbf{v}))+(z^0\times\mathbf{u}^1,\mathbf{v})]1_{[0,k]}(s)ds.\end{aligned}$$

Thanks to the definition of $\hat{\mathfrak{u}}_n$, $\check{y}_n$, we obtain

$$\begin{aligned}((\mathfrak{u}_n(t),\mathbf{v}))_\alpha =&-\int_0^t[((\hat{\mathfrak{u}}_n(s),\mathbf{v}))+(\check{y}_n(s)\times\hat{\mathfrak{u}}_n(s),\mathbf{v})]ds+\frac{1}{k}\sum_{\ell=1}^{n-1}\int_0^t(G(\mathbf{u}^\ell)\Delta_\ell\boldsymbol{\eta}1_{[t_\ell,t_{\ell+1}]},\mathbf{v})ds\\ &+\int_0^t[((\hat{\mathfrak{u}}_n(s),\mathbf{v}))+(\check{y}_n(s)\times\hat{\mathfrak{u}}_n(s),\mathbf{v})]1_{[0,k]}(s)ds+((\mathbf{u}_0,\mathbf{v})).\end{aligned}\tag{3.54}$$

Now, observe that

$$\begin{aligned}\frac{1}{k}\sum_{\ell=1}^{n-1}\int_0^t(G(\mathbf{u}^\ell)\Delta_\ell\boldsymbol{\eta}1_{[t_\ell,t_{\ell+1}]},\mathbf{v})ds =&\frac{1}{k}\sum_{\ell=0}^{\ell_n}\int_0^t\left(\int_{t_\ell}^{t_{\ell+1}\wedge t}G(\mathbf{u}^\ell)dW(r),\mathbf{v}\right)1_{[t_\ell,t_{\ell+1}]}(s)ds\\ &+\frac{1}{k}\int_0^t\left(\int_t^{\tau_n+1}G(\mathbf{u}^{\ell_n})dW(r),\mathbf{v}\right)1_{[t_\ell,t_{\ell+1}]}(s)ds\\ &-\frac{1}{k}\int_0^t\left(\int_0^k G(\mathbf{u}_0)dW(r),\mathbf{v}\right)1_{[0,k]}(s)ds,\end{aligned}$$

from which and the definition of $\check{\mathfrak{u}}_n$ we infer that

$$\begin{aligned}\frac{1}{k}\sum_{\ell=1}^{n-1}\int_0^t(G(\mathbf{u}^\ell)\Delta_\ell\boldsymbol{\eta}1_{[t_\ell,t_{\ell+1}]},\mathbf{v})ds =\sum_{\ell=0}^{\ell_n}&\left(\int_{t_\ell}^{t_{\ell+1}\wedge t}G(\check{\mathfrak{u}}_n(r))dW(r),\mathbf{v}\right)+\left(\int_t^{\tau_n}G(\check{\mathfrak{u}}_n(r))dW(r),\mathbf{v}\right)\\ &-\frac{t\wedge k}{k}\left(\int_0^k G(\check{\mathfrak{u}}_n(r))dW(r),\mathbf{v}\right).\end{aligned}$$

Substituting the last identity into (3.54) yields (3.52).

The proof of (3.53) is very similar to the argument above, thus we omit it. □

Now we proceed to the tightness of the functions we defined above. To this end we define additional functional spaces which are very important for our study.
Let Y be a Banach space, $\gamma\in(0,1)$ and $p\in[1,\infty)$. The Nikolskii space $\mathrm{N}_T^{\gamma,p}:=\mathrm{N}^{\gamma,p}(0,T;\mathrm{Y})$ is the space of functions $f\in\mathrm{L}^p(0,T;\mathrm{Y})$ such that

$$\|f\|_{\mathrm{N}_T^{\gamma,p}}:=\sup_{\delta>0}\delta^{-\gamma}\|f(\cdot+\delta)-f(\cdot)\|_{\mathrm{L}^p(0,T-\delta;\mathrm{Y})}<\infty.$$

The fractional Sobolev space $\mathrm{W}_T^{\gamma,p}:=\mathrm{W}^{\gamma,p}(0,T;\mathrm{Y})$ is the space of functions $f\in\mathrm{L}^p(0,T;\mathrm{Y})$ such that

$$\|f\|_{\mathrm{W}_T^{\gamma,p}}:=\left(\int_0^T\int_0^T\left(\frac{\|f(r)-f(s)\|_{\mathrm{Y}}}{|r-s|^\gamma}\right)^p\frac{drds}{|r-s|}\right)^{\frac{1}{p}}.$$

From [63, Section 13, Corollary 24] we derive the following embedding, which plays a important role in the sequel,

$$\mathrm{N}_T^{\gamma,p}\subset\mathrm{W}_T^{\beta,p},\text{ for all }\gamma>\beta.\tag{3.55}$$

We now recall the following lemma.

Lemma 3.13. *Let* $\mathrm{Y}_1, \mathrm{Y}_2$ *be two Banach spaces such that the embedding* $\mathrm{Y}_1 \subset \mathrm{Y}_2$ *is compact. Let* $\gamma \in (0,1)$ *and* $p \in [1,\infty)$*. Then, the space* $\mathrm{L}^1(0,T;\mathrm{Y}_1) \cap \mathrm{N}^{\gamma,p}(0,T;\mathrm{Y}_2)$ *is relatively compact in* $\mathrm{L}^p(0,T;\mathrm{Y}_2)$*.*

Proof. Note that one has, uniformly in f from the unit ball of $\mathrm{N}^{\gamma,p}(0,T;\mathrm{Y}_2)$,

$$\lim_{\delta\to 0} \|f(\cdot+\delta) - f(\cdot)\|_{\mathrm{L}^p(0,T-\delta;\mathrm{Y}_2)} = 0.$$

Thus, the conclusion of the lemma follows from this observation and the applicability of [64, Section 6, Theorem 3]. □

We also recall the following result which is taken from [65, Theorem 2.2].

Lemma 3.14. *Let* $\mathrm{Y}_1, \mathrm{Y}_2$ *be two Banach spaces satisfying the assumptions of Lemma 3.13, and* $\gamma \in (0,1)$, $p > 1$ *such that* $\gamma p > 1$*. Then, the embedding* $\mathrm{W}^{\gamma,p}(0,T;\mathrm{Y}_1) \subset C([0,T];\mathrm{Y}_2)$ *is compact.*

Now, we can proceed to the heart of the subject in this subsection. For this aim, for a Polish space K we denote by $\mathcal{M}(\mathrm{K})$ the space of probability measures on $(\mathrm{K}, \mathscr{B}(\mathrm{K}))$ where $\mathscr{B}(\mathrm{K})$ is the Borel σ-algebra of K. We also set

$$\begin{aligned}
\mathbf{U}_T &:= C([0,T];\mathbf{L}^4(\mathscr{O})),\\
\tilde{\mathbf{U}}_T &:= \mathrm{L}^2(0,T;\mathbf{L}^4(\mathscr{O})),\\
Z_T &:= C([0,T];\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})),\\
\tilde{Z}_T &:= \mathrm{L}^2(0,T;\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})),\\
\mathscr{W}_T &:= C([0,T];\mathscr{H}).
\end{aligned}$$

Finally, we define a sequence of $\mathscr{H}$-valued Wiener processes $\{\boldsymbol{\eta}_n;\ n \in \mathbb{N}\}$ defined by

$$\boldsymbol{\eta}_n = \boldsymbol{\eta},\ \forall n \in \mathbb{N}.$$

The family of laws of $\{\boldsymbol{\eta}_n;\ n \in \mathbb{N}\}$ on $\mathscr{W}_T$ is denoted by $\{\gamma_n;\ n \in \mathbb{N}\}$. The following result is of the essence for the existence result in Proposition 2.6.

Proposition 3.15. *Let us denote by* $\{\mu_n;\ n \in \mathbb{N}\}$ *(reps.* $\{\rho_n;\ n \in \mathbb{N}\}$*) the family of laws of* $\{\mathrm{u}_n;\ n \in \mathbb{N}\}$ *(resp.* $\{y_n;\ n \in \mathbb{N}\}$*) on* $\mathbf{U}_T$ *(resp. on* Z_T*). Then, the family* $\{(\mu_n, \rho_n, \gamma_n); n \in \mathbb{N}\}$ *is tight on* $\mathbf{U}_T \times Z_t \times \mathscr{W}_T$*.*

Proof. Because a cartesian product of finite compact sets is compact, it is sufficient to consider the tightness of each component of $(\mu_n, \rho_n, \gamma_n)$. Hence, we firstly prove that the family $\{\mu_n;\ n \in \mathbb{N}\}$ is tight on $\mathbf{U}_T$. From (3.45) and (3.47) and the Sobolev embedding $\mathbf{V} \subset \mathbf{L}^4(\mathscr{O})$ we infer that the family $\{\mathrm{u}_n;\ n \in \mathbb{N}\}$ forms an uniformly bounded subset of $\mathrm{N}^{\frac{1}{2},4}(0,T;\mathbf{L}^4(\mathscr{O})) \cap \mathrm{L}^\infty(0,T;\mathbf{V})$. Thanks to (3.55), the family $\{\mathrm{u}_n;\ n \in \mathbb{N}\}$ also forms an uniformly bounded subset of $\mathrm{W}^{\beta,4}(0,T;\mathbf{L}^4(\mathscr{O}))$ for any $\beta \in (\frac{1}{4}, \frac{1}{2})$. Due to these remarks and the compact embeddings $\mathbf{V} \subset \mathbf{L}^4(\mathscr{O})$, the desired tightness of the family $\{\mu_n;\ n \in \mathbb{N}\}$ on $\mathbf{U}_T$ follows from Lemma 3.14.

Secondly, thanks to (3.46), (3.48) and the compact embedding $\mathrm{L}^2(\mathscr{O}) \subset \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})$, we can use the same argument as above to establish the tightness of $\{\rho_n;\ n \in \mathbb{N}\}$ on Z_T.

Finally, endowed with the uniform convergence, $C([0,T];\mathscr{H})$ is a Polish space, then it follows from [66, Theorem 6.8] that the space of probability measure on $C([0,T];\mathscr{H})$ endowed with the Prohorov's metric is a separable and complete metric space. By construction the family of probability laws $\{\gamma_n;\ n \in \mathbb{N}\}$ is reduced to one element which is the law of $\boldsymbol{\eta}$ and belongs to $\mathcal{M}(C([0,T];\mathscr{H}))$. Therefore, invoking [67, Chapter II, Theorem 3.2] we easily deduce that the family $\{\gamma_n;\ n \in \mathbb{N}\}$ is tight on $\mathcal{M}(C(0,T;\mathscr{H}))$. □

Remark 3.16. Due to the continuous embeddings $\mathbf{U}_T \times Z_T \times \mathscr{W}_T \subset \tilde{\mathbf{U}}_T \times \tilde{Z}_T \times \mathscr{W}_T$ and $\mathbf{U}_T \times Z_T \times \mathscr{W}_T \subset \mathrm{L}^2(0,T;\mathbf{V}^*) \times \tilde{Z}_T \times \mathscr{W}_T$, the family $\{(\mu_n, \rho_n, \gamma_n);\ n \in \mathbb{N}\}$ is also tight on $\tilde{\mathbf{U}}_T \times \tilde{Z}_T \times \mathscr{W}_T$ and on $\mathrm{L}^2(0,T;\mathbf{V}^*) \times \tilde{Z}_T \times \mathscr{W}_T$. One can also use Lemma 3.13 to prove these claims.

4. Passage to the Limit and the Proof of Theorem 2.6

This section contains the proof of the existence of a weak martingale solution to the problem (1.5).

By Proposition 3.15 and the Prokhorov Theorem in the version given in [66, Chapter 1, Theorem 3.1], we can find a subsequence of n, still denoted by n, such that the family of laws $\{(\mu_n, \rho_n, \gamma_n);\ n \in \mathbb{N}\}$ weakly converge to a probability measure (μ, ρ, γ) on $\mathbf{U}_T \times Z_T \times \mathscr{W}_T$. Thanks to Remark 3.16, [66, Chapter 1, Theorem 3.1], (3.49) and (3.50), we also infer that the family of laws of $\{(\hat{u}_n, \hat{y}_n);\ n \in \mathbb{N}\}$ and $\{(\check{u}_n, \check{y}_n);\ n \in \mathbb{N}\}$, denoted respectively by $\{(\hat{\mu}_n, \hat{\rho}_n);\ n \in \mathbb{N}\}$ and $\{(\check{\mu}_n, \check{\rho}_n);\ n \in \mathbb{N}\}$, converge to (μ, ρ) on $\tilde{\mathbf{U}}_T \times \tilde{Z}_T$.

Proposition 4.1. *(i) There exist a new probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which one can find a sequence of $\mathbf{U}_T \times Z_T \times \mathscr{W}_T$-valued random variables (r.v.) $\{(\mathbf{u}_n, z_n, W_n);\ n \in \mathbb{N}\}$ such that its family of laws on $\mathbf{U}_T \times Z_T \times \mathscr{W}_T$ is equal to $\{(\mu_n, \rho_n, \gamma_n);\ n \in \mathbb{N}\}$. On $(\Omega, \mathscr{F}, \mathbb{P})$ one can also find a $\mathbf{U}_T \times Z_T \times \mathscr{W}_T$-valued r.v. $(\mathbf{u}, z, W)$ such that*

$$(\mathbf{u}_n, z_n, W_n) \to (\mathbf{u}, z, W) \text{ in } \mathbf{U}_T \times Z_T \times \mathscr{W}_T \ \mathbb{P}\ a.s.. \tag{4.1}$$

(ii) There exists two sequences of $\tilde{\mathbf{U}}_T \times \tilde{Z}_T$-valued r.v. $\{(\hat{\mathbf{u}}_n, \hat{z}_n);\ n \in \mathbb{N}\}$, $\{(\check{\mathbf{u}}_n, \check{z}_n);\ n \in \mathbb{N}\}$, and two $\tilde{\mathbf{U}}_T \times \tilde{Z}_T$-valued r.v. $(\hat{\mathbf{u}}, \hat{z})$, $(\check{\mathbf{u}}, \check{z})$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$ such that we have the following equalities of laws and convergences

$$(\hat{\mathbf{u}}_n, \hat{z}_n) \overset{law}{=} (\hat{u}_n, \hat{y}_n) \text{ on } \tilde{\mathbf{U}}_T \times \tilde{Z}_T, \tag{4.2}$$

$$(\check{\mathbf{u}}_n, \check{z}_n) \overset{law}{=} (\check{u}_n, \check{y}_n) \text{ on } \tilde{\mathbf{U}}_T \times \tilde{Z}_T, \tag{4.3}$$

$$(\hat{\mathbf{u}}_n, \hat{z}_n) \to (\hat{\mathbf{u}}, \hat{z}) \text{ in } \tilde{\mathbf{U}}_T \times \tilde{Z}_T \ \mathbb{P}\ a.s., \tag{4.4}$$

$$(\check{\mathbf{u}}_n, \check{z}_n) \to (\check{\mathbf{u}}, \check{z}) \text{ in } \tilde{\mathbf{U}}_T \times \tilde{Z}_T \ \mathbb{P}\ a.s.. \tag{4.5}$$

Proof. This result follows from Skorokhod's representation theorem, see, for instance, [66, Chapter 1, Theorem 6.7]. □

Remark 4.2. Because of Remark 3.16 we can assume that the equalities of laws above also hold with $\mathbf{U}_T$ and $\tilde{\mathbf{U}}_T$ replaced by $L^2(0, T; \mathbf{V}^*)$. Since, by [68, Theorem 1.1 of Chapter I], the Borel subsets of $C([0, T]; \mathbf{V} \times L^2(\mathscr{O}))$ are Borel subsets of $\mathbf{U}_T$ and, by construction,

$$\mathbf{P}\left((u_n, y_n) \in C([0, T]; \mathbf{V} \times L^2(\mathscr{O})), \forall n \in \mathbb{N}\right) = 1,$$

we can and will assume that $\{(\mathbf{u}_n, z_n);\ n \in \mathbb{N}\} \subset C([0, T]; \mathbf{V} \times L^2(\mathscr{O}))$ and that its family of laws on $C([0, T]; \mathbf{V} \times L^2(\mathscr{O}))$ is equal to that of $\{(u_n, y_n);\ n \in \mathbb{N}\}$. Analogously, the same assumption will be imposed for the sequences $\{(\hat{\mathbf{u}}_n, \hat{z}_n);\ n \in \mathbb{N}\}$ and $\{(\check{\mathbf{u}}_n, \check{z}_n);\ n \in \mathbb{N}\}$.

The above remark and proposition will be used to derive the following estimates.

Proposition 4.3. *For any $p \in [\![1, 3]\!]$ and $\alpha > 0$ there exists a constant $C > 0$ such that for any fixed $\nu \geqslant 0$,*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t \in [0,T]} [\|\mathbf{u}_n(t)\|_\alpha^{2^p} + \|\hat{\mathbf{u}}_n(t)\|_\alpha^{2^p} + \|\check{\mathbf{u}}_n(t)\|_\alpha^{2^p}] \leqslant C(\|\mathbf{u}_0\|^{2^p} + 1), \tag{4.6}$$

$$\sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t \in [0,T]} [|z_n(t)|^{2^p} + |\hat{z}_n(t)|^{2^p} + |\check{z}_n(t)|^{2^p}] \leqslant C(|z_0|^{2^p} + \|\mathbf{u}_0\|_\alpha^{2^p} + 1), \tag{4.7}$$

$$\lim_{n \to \infty} \mathbb{E} \int_0^T \|\mathbf{u}_n(t) - \check{\mathbf{u}}_n(t)\|_\alpha^2 dt + \lim_{n \to \infty} \mathbb{E} \int_0^T \|\mathbf{u}_n(t) - \hat{\mathbf{u}}_n(t)\|_\alpha^2 dt = 0, \tag{4.8}$$

$$\lim_{n \to \infty} \mathbb{E} \int_0^T |z_n(t) - \check{z}_n(t)|^2 dt + \lim_{n \to \infty} \mathbb{E} \int_0^T |z_n(t) - \hat{z}_n(t)|^2 dt = 0. \tag{4.9}$$

Furthermore, for any $p \in [\![1, 3]\!]$ there exists constant $C > 0$ such that

$$\mathbb{E} \sup_{t \in [0,T]} [\|\mathbf{u}(t)\|_\alpha^{2^p} + \|\hat{\mathbf{u}}(t)\|_\alpha^{2^p} + \|\check{\mathbf{u}}(t)\|_\alpha^{2^p}] \leqslant C(\|\mathbf{u}_0\|_\alpha^{2^p} + 1), \tag{4.10}$$

$$\mathbb{E} \sup_{t \in [0,T]} [|z(t)|^{2^p} + |\hat{z}(t)|^{2^p} + |\check{z}(t)|^{2^p}] \leqslant C(|z_0|^{2^p} + \|\mathbf{u}_0\|_\alpha^{2^p} + 1). \tag{4.11}$$

Proof. The estimates and convergences (4.6)-(4.9) follows from the equality of laws stated in Remark 4.2 and the estimate (3.45).

Thanks to (4.6) and (4.7), the estimates (4.10) and (4.11) can be proved by arguing exactly as in [69, Proof of (4.12), page 20]. □

We will also exploit the results in Proposition 4.1 and Remark 4.2 to derive the following two important propositions. In the first one, we will show that the limit process W defines a $\mathscr{H}$-valued Wiener process with covariance Q. In the second one, we will prove that for each $n \in \mathbb{N}$ the stochastic processes $(\mathbf{u}_n, z_n)$, $(\hat{\mathbf{u}}_n, \hat{z}_n)$ and $(\check{\mathbf{u}}_n, \check{z}_n)$ satisfy a system of equations very similar to the original problem (1.5) up to small errors which converge to zero when the time step k approaches zero.

Proposition 4.4. *The stochastic process $\{W(t);\ t \in [0,T]\}$ is a $\mathscr{H}$-valued Wiener process on $(\Omega, \mathscr{F}, \mathbb{P})$ with covariance Q. Furthermore, if $0 \leqslant s < t \leqslant T$ then the increments $W(t) - W(s)$ are independent of the σ-algebra generated by $(\mathbf{u}(r), z(r), W(r))$ for $r \in [0,s]$.*

Proof. We closely follow [69, Lemma 5.2] and [70, Proposition 3.11]. By Proposition 4.1 the family of laws of $\{(\check{\mathbf{u}}_n, \check{z}_n, W_n);\ n \in \mathbb{N}\}$ are equal to those of $\{(\check{u}_n, \check{y}_n, \boldsymbol{\eta});\ n \in \mathbb{N}\}$ on $\tilde{\mathbf{U}}_T \times \tilde{Z}_T \times \mathscr{W}_T$ and, by construction, $\boldsymbol{\eta}$ is a $\mathscr{H}$-valued Wiener process with covariance Q. Hence it is easy to check that $\{W_n;\ n \in \mathbb{N}\}$ is a sequence of Wiener processes taking values in $\mathscr{H}$. Moreover, for any $s, t \in [0,T]$ such that $0 \leqslant s < t \leqslant T$, the increments $W_n(t) - W_n(s)$ are independent of the σ-algebra generated by $(\check{\mathbf{u}}_n(r), \check{z}_n(r), W_n(r))$, for $r \in [0,s]$. Now, by arguing exactly as in [70, Proposition 3.11] we can show that W satisfies the Lévy characterization of the finite dimensional distribution of a $\mathscr{H}$-valued Wiener process with covariance Q; that is, for any partition $\Pi_N = \{0 = s_0 < s_1 < \cdots < s_N = T\}$ of $[0,T]$ and $h \in \mathscr{H}$, we have

$$\mathbb{E}\left[e^{i \sum_{\ell=1}^N t_j \langle W(s_\ell) - W(s_{\ell-1}), h \rangle_{\mathscr{H}}} \right] = e^{-\frac{1}{2} \sum_{\ell=1}^N t_\ell^2 (s_\ell - s_{\ell-1}) \langle Qh, h \rangle_{\mathscr{H}}},$$

where here i denotes the complex number satisfying $i^2 = -1$.

Next we prove that the increments $W(t) - W(s)$, $0 \leqslant s < t \leqslant T$, are independent of the σ-algebra generated by $(\mathbf{u}(r), z(r), W(r))$ for any $r \in [0,s]$. To this end, let us consider $\{\Phi_\ell;\ \ell = 1, \ldots, N\} \subset C_b(\mathrm{L}^4(\mathscr{O}) \times \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))$ and $\{\Psi_\ell;\ \ell = 1, \ldots, N\} \subset C_b(\mathscr{H})$, where

$$C_b(\mathbf{B}) = \{\Phi : \mathbf{B} \to \mathbb{R}, \Phi \text{ is continuous and bounded}\},$$

for any Banach space $\mathbf{B}$. Let also $0 \leqslant r_1 < \cdots < r_N \leqslant s < t \leqslant T$, $\Psi \in C_b(\mathscr{H})$. For each $n \in \mathbb{N}$, there holds

$$\begin{aligned}
\mathbb{E}\Bigg[\Bigg(\prod_{\ell=1}^N \Phi_\ell(\check{\mathbf{u}}_n(r_\ell), \check{z}_n(r_\ell)) \prod_{\ell=1}^N \Psi_\ell(W_n(r_\ell)) \Bigg) \times \Psi(W_n(t) - W_n(s)) \Bigg] \\
= \mathbb{E}\Bigg[\prod_{\ell=1}^N \Phi_\ell(\check{\mathbf{u}}_n(r_\ell), \check{z}_n(r_\ell)) \prod_{\ell=1}^N \Psi_\ell(W_n(r_\ell)) \Bigg] \\
\times \mathbb{E}\left(\Psi(W_m(t) - W_m(s)) \right).
\end{aligned}$$

Thanks to (4.1), (4.5), the Lebesgue Dominated Convergence Theorem and Remark 4.7, the same identity is true with $(\mathbf{u}, z, W)$ in place of $(\check{\mathbf{u}}_n, \check{z}_n, W_n)$. □

To rigorously deal with all the stochastic integrals below we define the following filtrations: let $\mathscr{N}$ be the set of null sets of $\mathscr{F}$ and for any $t \geqslant 0$ and $n \in \mathbb{N}$, let

$$\begin{aligned}
\check{\mathscr{F}}_t^n &:= \sigma\Big(\sigma\Big((\check{\mathbf{u}}_n(s), \check{z}_n(s), W_n(s)); s \leqslant t \Big) \cup \mathscr{N} \Big), \\
\mathscr{F}_t &:= \sigma\Big(\sigma\Big((\mathbf{u}(s), z(s), W(s)); s \leqslant t \Big) \cup \mathscr{N} \Big),
\end{aligned}$$

be the completion of the natural filtration generated by $(\check{\mathbf{u}}_n, \check{z}_n, W_n)$ and $(\mathbf{u}, z, W)$, respectively. Note that from the proof of Proposition 4.4 we see that W (resp. W_n) is a $\mathscr{H}$-valued Wiener process adapted to the filtration $\mathbb{F} := \{\mathcal{F}_t : t \in [0,T]\}$ (resp. $\check{\mathbb{F}}^n := \{\check{\mathscr{F}}^n_t : t \in [0,T]\}$). The $\mathbf{H}$-valued stochastic processes $\check{\mathbf{u}}^n$ and $\mathbf{u}$ are adapted with respect to $\mathbb{F}$ and $\check{\mathbb{F}}^n$ as well. Thus, they are also predictable in $\mathbf{H}$ because their sample paths are left-continuous in $\mathbf{H}$. Hence, the existence of the following stochastic integrals is justified:

$$
\begin{aligned}
\tilde{\mathfrak{M}}_n(\cdot) &:= \int_0^{\cdot} G(\check{\mathbf{u}}_n(s))dW(s),\\
\tilde{\mathfrak{N}}_n(\cdot) &:= \int_0^{\cdot} \mathbf{rot}\, G(\check{\mathbf{u}}_n(s))dW(s),\\
\tilde{\mathfrak{M}}(\cdot) &:= \int_0^{\cdot} G(\mathbf{u}(s))dW(s),\\
\tilde{\mathfrak{N}}(\cdot) &:= \int_0^{\cdot} \mathbf{rot}\, G(\mathbf{u}(s))dW(s).
\end{aligned}
$$

For fixed $n \in \mathbb{N}$, let also $\mathfrak{M}_n$, $\mathfrak{M} \in \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*)$ and $\mathfrak{N}_n$, $\mathfrak{N} \in \mathrm{L}^2(\Omega \times [0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))$ be four stochastic processes defined by

$$
\begin{aligned}
(\mathfrak{M}_n(t), \mathbf{v}) &:= ((\mathbf{u}_n(t), \mathbf{v}))_\alpha - ((\mathbf{u}_0, \mathbf{v}))_\alpha + \int_0^t [\nu((\hat{\mathbf{u}}_n(s), \mathbf{v})) + (\check{z}_n(s) \times \hat{\mathbf{u}}_n(s), \mathbf{v})]\, ds,\\
(\mathfrak{N}_n(t), \phi) &:= (z_n(t), \phi) - (z_0, \phi) + \int_0^t \left[\frac{\nu}{\alpha}(\hat{z}_n(s) - \mathbf{rot}\, \hat{\mathbf{u}}_n(s), \phi) - (\hat{\mathbf{u}}_n(s) \cdot \nabla\phi, \hat{z}_n(s))\right] ds,\\
(\mathfrak{M}(t), \mathbf{v}) &:= ((\mathbf{u}(t), \mathbf{v}))_\alpha - ((\mathbf{u}_0, \mathbf{v}))_\alpha + \int_0^t [\nu((\mathbf{u}(s), \mathbf{v})) + (z(s) \times \mathbf{u}(s), \mathbf{v})]\, ds,\\
(\mathfrak{N}(t), \phi) &:= (z(t), \phi) - (z_0, \phi) + \int_0^t \left[\frac{\nu}{\alpha}(z(s) - \mathbf{rot}\, \mathbf{u}(s), \phi) - (\mathbf{u}(s) \cdot \nabla\phi, z(s))\right] ds,
\end{aligned}
$$

for any $t \in [0,T]$, $\mathbf{v} \in \mathbf{V}$ and $\phi \in \mathrm{W}^{1,4}_0(\mathscr{O})$.

In the next two lemma we will show that on $(\Omega, \mathscr{F}, \mathbb{P})$ the stochastic processes $(\mathbf{u}_n, z_n)$, $(\hat{\mathbf{u}}_n, \hat{z}_n)$ and $(\check{\mathbf{u}}_n, \check{z}_n)$ satisfies the integral and weak form of (1.5) up to small error terms $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$.

Proposition 4.5. *The following identities holds $\mathbb{P}$-a.s*

$$
\langle \mathfrak{M}_n(t), \mathbf{v}\rangle - (\mathcal{E}_n(t), \mathbf{v}) = (\tilde{\mathfrak{M}}_n(t), \mathbf{v}) \tag{4.12}
$$

$$
\langle \mathfrak{N}_n(t), \phi\rangle - (\tilde{\mathcal{E}}_n(t), \phi) = (\tilde{\mathfrak{N}}_n(t), \phi), \tag{4.13}
$$

for any $t \in [0,T]$, $\mathbf{v} \in \mathbf{V}$ and $\phi \in \mathrm{W}^{1,4}_0(\mathscr{O})$. Here we put $\tilde{G}(\cdot) := \mathbf{rot}\, G(\cdot)$ and

$$
\begin{aligned}
\mathcal{E}_n(\cdot) &:= \int_{\cdot}^{\tau_n + 1} G(\check{\mathbf{u}}_n(s))dW(s) - \frac{k \wedge \cdot}{k} \int_0^k G(\check{\mathbf{u}}_n(s))dW(s)\\
&\quad + \int_0^{\cdot} 1_{[0,k]}(s)\, [-\nu\Delta\hat{\mathbf{u}}_n(s) + \check{z}_n(s) \times \hat{\mathbf{u}}_n(s)]\, ds,\\
\tilde{\mathcal{E}}_n(\cdot) &:= -\int_0^{\cdot} 1_{[0,k]}(s) \left[\frac{\nu}{\alpha}(\mathbf{rot}\, \hat{\mathbf{u}}_n(s) - \hat{z}_n(s)) - \hat{\mathbf{u}}_n(s) \cdot \nabla\hat{z}_n(s)\right] ds\\
&\quad + \int_{\cdot}^{\tau_n + 1} \tilde{G}(\check{\mathbf{u}}_n(s))dW(s) - \frac{k \wedge \cdot}{k} \int_0^k \tilde{G}(\check{\mathbf{u}}_n(s))dW(s).
\end{aligned}
$$

Proof. The proof of the proposition follows the exact same lines as the proof of [35, Theorem 4.9], thus we omit it. □

We will justify in the next lemma the alluded term *small error terms* by showing that $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ become very small when performing a time grid refinement, that is, when n takes large values.

Lemma 4.6. *We have*

$$\lim_{n\to\infty} \mathbb{E}\|\mathcal{E}_n\|^2_{\mathrm{L}^2(0,T;\mathbf{V}*)} = 0, \tag{4.14}$$

$$\lim_{n\to\infty} \mathbb{E}\|\tilde{\mathcal{E}}_n\|^2_{\mathrm{L}^2(0,T;\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))} = 0. \tag{4.15}$$

Proof. We will only prove the lemma for $\nu = 0$ because the treatment of the case $\nu > 0$ differs to the former case up to the study of linear terms which are easy to deal with.

First, notice that thanks to (4.6) and (4.7) it is a straightforward task to check that

$$\mathbb{E}\int_0^T \|\check{z}_n(t)\times\hat{\mathbf{u}}_n(t)\|^2_{\mathbf{V}*}dt \leqslant \left(\mathbb{E}\sup_{t\in[0,T]}|\check{z}_n(t)|^4\right)^{\frac{1}{2}}\left(\mathbb{E}\sup_{t\in[0,T]}\|\hat{\mathbf{u}}\|^4_\alpha\right)^{\frac{1}{2}} < C, \tag{4.16}$$

$$\mathbb{E}\int_0^T \|\hat{\mathbf{u}}_n(t)\cdot\nabla\hat{z}_n(t)\|^2_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})}dt \leqslant \mathbb{E}\sup_{t\in[0,T]}|\check{z}_n(t)|^4\mathbb{E}\sup_{t\in[0,T]}\|\hat{\mathbf{u}}\|^4_\alpha < C. \tag{4.17}$$

Thanks to (4.16), we easily infer that

$$\begin{aligned}\mathbb{E}\int_0^T\left\|\int_0^{t\wedge k}[\check{z}_n(s)\times\hat{\mathbf{u}}_n(s)]\,ds\right\|^2_{\mathbf{V}*}dt &\leqslant \mathbb{E}\sup_{t\in[0,T]}|\check{z}_n(t)|^4\mathbb{E}\sup_{t\in[0,T]}\|\hat{\mathbf{u}}\|^4_\alpha\int_0^T(t\wedge k)^2dt\\ &\leqslant C\left(\int_0^k(t\wedge k)^2dt+\int_k^T(t\wedge k)^2dt\right)\\ &\leqslant C(k^3+k^2).\end{aligned}$$

Now we proceed to the derivation of some estimates for the stochastic integrals. Making use of the Fubini theorem, the Itô isometry we obtain

$$\mathbb{E}\int_0^T\left\|\int_t^{\tau_n+1}G(\check{\mathbf{u}}_n(s))dW_n(s)\right\|^2_\alpha dt = \operatorname{Tr}Q\int_0^T\mathbb{E}\int_t^{\tau_n+1}\|G(\check{\mathbf{u}}_n(s))\|^2_{\mathscr{L}(\mathscr{H},\mathbf{V})}dt.$$

Since, by definition, amongst the subdivision intervals of $[0,T]$, $[\tau_n,\tau_n+1]$ is the first interval containing t, we infer, from Assumption (G) and the estimate (4.6), that

$$\begin{aligned}\mathbb{E}\int_0^T\left\|\int_t^{\tau_n+1}G(\check{\mathbf{u}}_n(s))dW_n(s)\right\|^2_\alpha dt &\leqslant \int_0^T\mathbb{E}\int_{\tau_n}^{\tau_n+1}\|G(\check{\mathbf{u}}_n(s))\|^2_{\mathscr{L}(\mathscr{H},\mathbf{V})}dt\\ &\leqslant CTk\mathbb{E}\sup_{t\in[0,T]}(1+\|\check{\mathbf{u}}_n\|^2)\leqslant Ck.\end{aligned}$$

Analogously,

$$\begin{aligned}\mathbb{E}\int_0^T\left\|\frac{t\wedge k}{k}\int_0^k G(\check{\mathbf{u}}_n(s))dW_n(s)\right\|^2_\alpha &= \mathbb{E}\int_0^T\frac{(t\wedge k)^2}{k^2}\left(\mathbb{E}\int_0^k\|G(\check{\mathbf{u}}_n(s))\|^2_{\mathscr{L}(\mathscr{H},\mathbf{V})}ds\right)dt\\ &\leqslant Ck\mathbb{E}\sup_{t\in[0,T]}(1+\|\check{\mathbf{u}}_n\|^2)\left[\int_0^k\frac{(t\wedge k)^2}{k^2}dt+\int_k^T\frac{(t\wedge k)^2}{k^2}dt\right]\\ &\leqslant C(k^2+k).\end{aligned}$$

Summing up, we have shown that

$$\mathbb{E}\|\mathcal{E}_n\|^2_{\mathrm{L}^2(0,T;\mathbf{V}*)}\leqslant c(k^3+k^2+k),$$

from which we easily derive (4.14). Because of the similarity of the estimates (4.16), (4.17) and the terms in the definition of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$, the convergence (4.15) can be proved exactly by the same argument given above. Thus, we omit the proof of (4.15). □

To complete the proof of the existence of solution we need to pass to the limit in the other terms of (4.12) and (4.13). To this end, we will derive several convergences which are consequences of the facts stated in Proposition 4.3. By (4.6) and (4.7) we can find a subsequence of n, still denoted by n, such that for any $p \in [1,8]$ we have the following weak convergences

$$(\mathbf{u}_n, z_n), (\hat{\mathbf{u}}_n, \hat{z}_n), (\check{\mathbf{u}}_n, \check{z}_n) \rightharpoonup (\mathbf{u}, z), (\hat{\mathbf{u}}, \hat{z}), (\check{\mathbf{u}}, \check{z}) \text{ in } \mathrm{L}^{2p}(\Omega \times [0,T]; \mathbf{V} \times \mathrm{L}^2(\mathscr{O})). \tag{4.18}$$

Due to (4.10) and (4.11) it is easy to see that the norm of $(\mathbf{u}_n, z_n)$ is uniformly integrable in $\mathrm{L}^4(\Omega, \mathbf{U}_T \times Z_T \cap \tilde{Z}_T)$. Thus, it follows from the almost sure convergences stated in Proposition 4.1 and Vitali's Convergence Theorem that

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|\mathbf{u}_n(t) - \mathbf{u}(t)\|_{\mathrm{L}^4(\mathscr{O})} = 0, \tag{4.19}$$

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|z_n(t) - z(t)\|_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})} = 0. \tag{4.20}$$

In the same way, we can show that

$$\lim_{n\to\infty} \mathbb{E} \int_0^t \|\hat{\mathbf{u}}_n(t) - \hat{\mathbf{u}}(t)\|^2_{\mathrm{L}^4(\mathscr{O})} dt + \lim_{n\to\infty} \mathbb{E} \int_0^t \|\check{\mathbf{u}}_n(t) - \check{\mathbf{u}}(t)\|^2_{\mathrm{L}^4(\mathscr{O})} dt = 0, \tag{4.21}$$

$$\lim_{n\to\infty} \mathbb{E} \int_0^t \|\hat{z}_n(t) - \hat{z}(t)\|^2_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})} dt + \lim_{n\to\infty} \mathbb{E} \int_0^t \|\check{z}_n(t) - \check{z}(t)\|^2_{\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})} dt = 0. \tag{4.22}$$

Remark 4.7. Note that (4.21) and (4.22) along with (4.8) and (4.9) enable us to make the following identification

$$\mathbf{u} = \hat{\mathbf{u}} = \check{\mathbf{u}} \text{ in } \mathrm{L}^4(\mathscr{O}) \ \mathbb{P} \otimes \lambda \ a.e., \tag{4.23}$$

$$z = \hat{z} = \check{z} \text{ in } \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}) \ \mathbb{P} \otimes \lambda \ a.e.. \tag{4.24}$$

where λ denotes the Lebesgue measure on $[0,T]$.

For the nonlinear terms, we can find a subsequence of n, still denoted by n, such that

$$\check{z}_n \times \hat{\mathbf{u}}_n \rightharpoonup z \times \mathbf{u} \text{ in } \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*), \tag{4.25}$$

$$\hat{\mathbf{u}}_n \cdot \nabla \hat{z}_n \rightharpoonup \mathbf{u} \cdot \nabla z \text{ in } \mathrm{L}^2(\Omega \times [0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})). \tag{4.26}$$

In fact, as a result of (4.16) and (4.17), the sequences $\{\check{z}_n \times \hat{\mathbf{u}}_n;\ n \in \mathbb{N}\}$ and $\{\hat{\mathbf{u}}_n \times \hat{z}_n;\ n \in \mathbb{N}\}$ are bounded in $\mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*)$ and $\mathrm{L}^2(\Omega \times [0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))$, respectively. Therefore, by Eberlein-Smulyan Theorem, see [71, Chapter 21, Proposition 21.23-(h)], there exists a subsequence of n, denoted in the same way as the original sequence, and two stochastic processes $\Gamma \in \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*)$ and $\Theta \in \mathrm{L}^2(\Omega \times [0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))$ such that

$$\check{z}_n \times \hat{\mathbf{u}}_n \rightharpoonup \Gamma \text{ in } \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*),$$
$$\hat{\mathbf{u}}_n \cdot \nabla \hat{z}_n \rightharpoonup \Theta \text{ in } \mathrm{L}^2(\Omega \times [0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})).$$

Thus, we need to identify Γ (resp. Θ) with $z \times \mathbf{u}$ (resp. $\mathbf{u} \cdot \nabla z$). To this end, let $\mathbb{D} \subset \mathrm{L}^\infty(\Omega \times [0,T]; \mathbf{V})$ be a dense subset of $\mathrm{L}^2(\Omega \times [0,T]; \mathbf{V})$. For any $\Phi \in \mathbb{D}$, we have

$$\left| \mathbb{E} \int_0^T (\check{z}_n(t) \times \hat{\mathbf{u}}_n(t) - z(t) \times \mathbf{u}(t), \Phi(t)) dt \right| \leqslant \mathrm{CI}_n + \mathrm{CII}_n$$

where

$$\mathrm{I}_n := \left(\mathbb{E}\sup_{t\in[0,T]}|\check{z}_n(t)|^4\right)^{\frac{1}{4}}\left(\mathbb{E}\sup_{t\in[0,T]}\|\Phi(t)\|^4_{\mathbf{L}^4}\right)^{\frac{1}{4}}\left(\mathbb{E}\int_0^T\|\hat{\mathbf{u}}_n(t)-\mathbf{u}_n(t)\|^2_{\mathbf{L}^4}+\|\mathbf{u}_n(t)-\mathbf{u}(t)\|^2_{\mathbf{L}^4}\right),$$

$$\mathrm{II}_n := \left|\mathbb{E}\int_0^T([\check{z}_n(t)-z(t)]\times\mathbf{u}(t),\Phi(t))dt\right|,$$

and, for the sake of simplicity, we have set $\mathbf{L}^4 := \mathbf{L}^4(\mathscr{O})$. Thanks to (4.7) and the strong convergence (4.21), from a successive application of the Cauchy-Schwarz and Hölder inequalities we infer that

$$\lim_{n\to\infty}\mathrm{I}_n = 0.$$

Let $\mathrm{K} := \mathrm{L}^2(\Omega\times[0,T];\mathrm{L}^2(\mathscr{O}))\times\mathrm{L}^4(\Omega;\mathrm{L}^\infty(0,T;\mathbf{V}))\times\mathrm{L}^4(\Omega;\mathrm{L}^\infty(0,T;\mathbf{V}))$. Then, we can argue as in the proof of (3.4) to show that the trilinear form $\mathfrak{c}(\cdot,\cdot,\cdot)$ defined on K by

$$\mathfrak{c}(\psi\times\mathbf{v},\mathbf{w}) := (\psi\times\mathbf{v},\mathbf{w}),\ \forall(\psi,\mathbf{v},\mathbf{w})\in\mathrm{K},$$

is continuous. Thus, thanks to the weak convergences (4.18) we easily infer that

$$\lim_{n\to\infty}\mathrm{II}_n = 0.$$

Summing up, we have shown that for any $\Phi\in\mathbb{D}$

$$\lim_{n\to\infty}\left|\mathbb{E}\int_0^T(\check{z}(t)\times\hat{\mathbf{u}}_n(t)-z(t)\times\mathbf{u}(t),\Phi(t))dt\right| = 0.$$

By [71, Proposition 21.23], we readily infer that $\varGamma = z\times\mathbf{u}$ which also concludes the proof of (4.25) from the last identity. Since the procedure for identifications of $\varTheta$ with $\mathbf{u}\cdot\nabla z$ is very similar to the argument above, we omit the proof of (4.26).

With the convergences in (4.18), Proposition (6.9) we see that

$$\mathfrak{M}_n\to\mathfrak{M}\text{ weakly in }L^2(\Omega\times[0,T];\mathbf{V}^*),\tag{4.27}$$

$$\mathfrak{N}_n\to\mathfrak{N}\text{ weakly in }L^2(\Omega\times[0,T];\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})).\tag{4.28}$$

Due to the convergences (4.14), (4.15), (4.27) and (4.28), in order to complete the proof of the existence of solution we need to identify $\mathfrak{M}$ and $\mathfrak{N}$ respectively with the stochastic integrals $\tilde{\mathfrak{M}}(\cdot)$, and $\tilde{\mathfrak{N}}(\cdot)$ in appropriate topologies. These identification will be the object of the sequence of lemmata below.

Lemma 4.8. *We have the following weak convergences*

$$\tilde{\mathfrak{M}}_n(\cdot)\rightharpoonup\tilde{\mathfrak{M}}(\cdot)\text{ in }\mathrm{L}^2(\Omega\times[0,T];\mathbf{V}^*),\tag{4.29}$$

$$\tilde{\mathfrak{N}}_n(\cdot)\rightharpoonup\tilde{\mathfrak{N}}(\cdot)\text{ in }\mathrm{L}^2(\Omega\times[0,T];\mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})).\tag{4.30}$$

Proof. Firstly, we will show that $\tilde{\mathfrak{M}}_n$ converges strongly in $\mathrm{L}^2(\Omega\times[0,T];\mathbf{H})$ to $\tilde{\mathfrak{M}}$. For this aim, we observe that because of the Itô isometry, the Assumption **(G)** and the estimate (4.6), the family of maps $[0,T]\ni t\mapsto\mathbb{E}\left|\int_0^t G(\mathbf{u}_n(s))dW_n(s)\right|^2\in\mathrm{L}^2(0,T)$ is uniformly integrable in $\mathrm{L}^2(0,T)$. Hence it is sufficient to show that $\tilde{\mathfrak{M}}_n(t)$ converges strongly in $\mathrm{L}^2(\Omega;\mathbf{H})$ to $\tilde{\mathfrak{M}}(t)$ for any $t\in[0,T]$ which will follow from (4.8), (4.21) and [58, Part I, Theorem 3.3]. Indeed, we can write

$$\mathbb{E}\left|\tilde{\mathfrak{M}}_n(t)-\tilde{\mathfrak{M}}(t)\right|^2\leqslant 2\mathbb{E}\left|\int_0^t[G(\check{\mathbf{u}}_n(s))-G(\mathbf{u}_n(s))]dW_n(s)\right|^2$$
$$+2\mathbb{E}\left|\int_0^t G(\mathbf{u}_n(s))dW_n(s)-\int_0^t G(\mathbf{u}(s))dW(s)\right|^2,$$

and because of the Itô's isometry, Assumption (G) and (4.8) we see that the first term of the right hand side of the above inequality converges to zero as $n \to \infty$. To show that the second term converges to zero as $n \to \infty$, it is sufficient to use [58, Part I, Theorem 3.3] which is permissible because of the following two reasons:

(i) by the Lipschitz continuity of G in $\mathscr{L}(\mathscr{H}, \mathbf{H})$ w.r.t to the $\mathbf{H}-$norm and (4.21), we see that $G(\mathbf{u}_n)$ converges to $G(\mathbf{u})$ in $\mathrm{L}^2(\Omega, C(0,T;\mathscr{L}(\mathscr{H}, \mathbf{H})))$.

(ii) Since the Wiener process W_n with covariance Q converges in $C([0,T];\mathscr{H})$ to W with probability 1 and $\|W_n\|^2_{C([0,T];\mathscr{H})}$ is uniformly integrable, a fact which follows from (2.4), we see that W_n converges to W in $\mathrm{L}^2(\Omega, C([0,T];\mathscr{H}))$.

Secondly, let $\Phi \in \mathrm{X} := \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V})$, $\mathrm{Y} := \mathrm{L}^2(\Omega \times [0,T]; \mathbf{H})$, and $\langle \cdot, \cdot \rangle$ be the duality pairing of X and its dual X^*. Since $\tilde{\mathfrak{M}}_n, \tilde{\mathfrak{M}} \in \mathrm{Y}$ for all $n \in \mathbb{N}$, it readily follows from the identification (2.3) that

$$|\langle \tilde{\mathfrak{M}}_n - \tilde{\mathfrak{M}}, \Phi \rangle| = |((\tilde{\mathfrak{M}}_n - \tilde{\mathfrak{M}}, \Phi))_{\mathrm{Y}}|, \forall n \in \mathbb{N},$$

which along with the strong convergence of $\{\tilde{\mathfrak{M}}_n;\ n \in \mathbb{N}\}$ to $\tilde{\mathfrak{M}}$ in Y we infer that for any $\Phi \in \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V})$

$$\lim_{n\to\infty} |\langle \tilde{\mathfrak{M}}_n - \tilde{\mathfrak{M}}, \Phi \rangle| = 0.$$

This completes the proof of (4.29).

Finally, the convergence (4.30) easily follows from the strong convergence of $\{\tilde{\mathfrak{M}}_n;\ n \in \mathbb{N}\}$ to $\tilde{\mathfrak{M}}$ in $\mathrm{L}^2(\Omega \times [0,T]; \mathbf{H})$ and the boundedness of the linear map $\mathbf{rot} : \mathbf{L}^2(\mathscr{O}) \to \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O})$. This completes the proof of the lemma. □

Now we state and prove the following important proposition.

Lemma 4.9. *The following identities holds* $\mathbb{P}$*-a.s.*

$$\langle \mathfrak{M}(t), \mathbf{v} \rangle = \langle \tilde{\mathfrak{M}}(t), \mathbf{v} \rangle,$$
$$\langle \mathfrak{N}(t), \phi \rangle = \langle \tilde{\mathfrak{N}}(t), \phi \rangle,$$

for any $t \in [0,T]$, $\mathbf{v} \in \mathbf{V}$ *and* $\phi \in \mathrm{W}^{1,4}_0(\mathscr{O})$.

Proof. It follows from (4.29) that $\tilde{\mathfrak{M}}_n$ weakly converges to $\tilde{\mathfrak{M}}$ in $\mathrm{L}^2(\Omega, \mathrm{L}^2(0,T,\mathbf{V}^*))$, and from Proposition 4.5 we derive that $\mathfrak{M}_n - \mathcal{E}_n = \tilde{\mathfrak{M}}_n$ in $\mathrm{L}^2(\Omega, \mathrm{L}^2(0,T,\mathbf{V}^*))$. Hence we derive from (4.14), (4.27) and the uniqueness of the weak limit that $\mathfrak{M} = \tilde{\mathfrak{M}}$ in $\mathrm{L}^2(\Omega, \mathrm{L}^2(0,T;\mathbf{V}^*))$. This fact implies that $\mathbb{P}$-a.s. $\mathfrak{M}(t) = \tilde{\mathfrak{M}}(t)$ for almost all $t \in [0,T]$. Since $\mathfrak{M}$ and $\tilde{\mathfrak{M}}(t)$ are $\mathbf{V}^*$-valued continuous functions which agree for almost all $t \in [0,T]$, they must be equal for all $t \in [0,T]$. This ends the first part of the proposition. Thanks to (4.14), (4.27) and (4.30) the previous argument can be carried out to establish the second identity of the proposition. □

Now we are ready to give the proof of the existence of weak martingale solution formulate in Theorem2.6.

Proof of Theorem 2.6. Let $\mathcal{N}$ be the set of null sets of $\mathcal{F}$. Let $\mathbb{F} = \{\mathscr{F}_t : t \in [0,T]\}$, where the σ-algebra $\mathcal{F}_s$ is defined by

$$\mathscr{F}_t := \sigma\Big(\sigma\Big((\mathbf{u}(s), z(s), W(s)); s \leqslant t\Big) \cup \mathcal{N}\Big).$$

We will check that $(\mathscr{U}, \mathbf{u}, z)$, where $\mathscr{U} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$, is a weak martingale solution to the problem (1.5). To establish this claim we need to check the items (a)-(c) of Definition 2.4.

It follows from Proposition 4.28 that the stochastic process W, defined on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, is a $\mathscr{H}$-valued Wiener process with covariance Q. By construction the filtration $\mathbb{F}$ satisfies the usual condition. Therefore, $\mathscr{U} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$ is a stochastic basis. This proves the item (a) of Definition

2.4. From the construction of the filtration $\mathbb{F}$ it is clear that $(\mathbf{u}, z)$ is $\mathbb{F}$-adapted. Because of (4.11) and (4.22), we have

$$z \in \mathrm{L}^p(\Omega, C([0,T]; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathscr{O})),\ p \in [2,16].$$

We also observe from (4.10) and (4.21) that

$$\mathbf{u} \in \mathrm{L}^p(\Omega, C([0,T];\mathbf{H}) \cap \mathrm{L}^\infty(0,T;\mathbf{V})),\ p \in [2,16].$$

Hence in order to prove Definition 2.4(b) we need to show the continuity of $\mathbf{u}$ in $\mathbf{V}$. To this end, observe that by using the Lax-Milgram lemma as in the proof of Lemma 3.1, we can find a linear isomorphism $\mathcal{A} : \mathbf{V} \to \mathbf{V}^*$ such that for all $\mathbf{v}$, $\mathbf{w} \in \mathbf{V}$,

$$((\mathbf{v}, \mathbf{w})) = \langle \mathcal{A}\mathbf{v}, \mathbf{w}\rangle.$$

From (3.4) one can find a bilinear map $\mathcal{C} : \mathrm{L}^2(\mathscr{O}) \times \mathbf{V} \to \mathbf{V}^*$ such that

$$\langle \mathcal{C}(y, \mathbf{v}), \mathbf{w}\rangle = (y \times \mathbf{v}, \mathbf{w}). \tag{4.31}$$

With this observation and by denoting the identity map on $\mathbf{V}$ by Id, we can rewrite the first identity in Lemma 4.9, *i.e.*, $\mathfrak{M} = \tilde{\mathfrak{M}}$, in the following form

$$(\mathrm{Id} + \alpha\mathcal{A})\mathbf{u}(t) + \int_0^t [\nu\mathcal{A}\mathbf{u}(s) + \mathcal{C}(z(s), \mathbf{u}(s))]\, ds = \mathbf{u}_0 + \int_0^t G(\mathbf{u}(s))dW(s), \tag{4.32}$$

for any $t \in [0, T]$. For the sake of simplicity, we will set

$$\begin{aligned}\mathscr{A} &:= (\mathrm{Id} + \alpha\mathcal{A})^{-1} \circ \mathcal{A},\\ \mathscr{C} &:= (\mathrm{Id} + \alpha\mathcal{A})^{-1} \circ \mathcal{C},\\ \mathscr{G} &= (\mathrm{Id} + \alpha\mathcal{A})^{-1} \circ G.\end{aligned}$$

With this in mind, we derive from (4.32) that

$$\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t [\nu\mathscr{A}\mathbf{u}(s) + \mathscr{C}(z(s), \mathbf{u}(s))]\, ds + \int_0^t \mathscr{G}(\mathbf{u}(s))dW(s), \tag{4.33}$$

for all $t \in [0, T]$. As a result of (4.16) and (4.10) we see that $\mathcal{A}\mathbf{u} + \mathcal{C}(z, \mathbf{u}) \in \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V}^*)$ which implies that $\mathscr{A}\mathbf{u}(s) + \mathscr{C}(z(s), \mathbf{u}(s)) \in \mathrm{L}^2(\Omega \times [0,T]; \mathbf{V})$. We also have that $\mathscr{G}(\mathbf{u}) \in \mathscr{M}^2_T(\mathscr{L}(K, \mathbf{V}))$, thus the stochastic integral defines a martingale which is continuous in $\mathbf{V}$. From these observations we readily infer that there exists $\Omega^* \in \mathscr{F}$ such that $\mathbb{P}(\Omega^*) = 1$ and for each $\omega \in \Omega^*$ the function $\mathbf{u}(\omega, \cdot) : [0,T] \to \mathbf{V}$ is continuous. Thus, with (4.10) we readily see that $\mathbf{u} \in \mathrm{L}^p(\Omega, C([0,T]; \mathbf{V}))$. Hence, we have finished the proof Definition 2.4(b).
The last item, *i.e.*, Definition 2.4(c), readily follows from the identities in Lemma 4.9. Thus, the proof of Theorem 2.6 is completed. □

5. Proofs of Theorems 2.9(a) and 2.10: space regularity and uniqueness of solution

This section, which is divided in 2 subsections, is devoted to the proof of the space- regularity and the uniqueness of solution stated in Theorems 2.9 and 2.10.

5.1. *Space regularity of the solution: proof of Theorem 2.9(a)*

The smoothness in spatial variable that we stated in Theorem 2.9 and prove in this subsection plays a crucial role in the remaining part of the paper. Its proof is quite elementary and relies only on the theory of deterministic elliptic differential equations on non-smooth domains.

Proof of Theorem 2.9(a). Let $(\mathbf{u}, \mathscr{U})$ be a weak martingale solution of (1.4). From Definition (2.3) we have $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ in $L^2(\mathscr{O})$ almost all $(\omega, t) \in \Omega \times [0, T]$. The calculation in what follows hold *for almost all* $(\omega, t) \in \Omega \times [0, T]$ a term that, for the sake of simplicity, we will omit for the remaining part of this proof. Since $\operatorname{div} z = 0$, extending z by zero outside $\mathscr{O}$ and using the Fourier transform as in [72, Theorem 3.1] we can find an element $\mathbf{z} \in \mathbf{H}^1(\mathscr{O})$ such that

$$z = \mathbf{rot}\,\mathbf{z}, \text{ that is, } \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u} - \mathbf{z}) = 0. \tag{5.1}$$

Moreover, arguing as in [72, Proof Proposition 3.1] we can show that there exist a constant $C > 0$ depending only on $\mathscr{O}$ such that

$$\|\mathbf{z}\|_{\mathbf{H}^1(\mathscr{O})} \leqslant C|z|. \tag{5.2}$$

Since $\mathbf{u} \in \mathbf{V}$ and any convex polygon is a simply connected domain, we infer from (5.1) and [72, Theorem 2.9] that there exists a class of functions $\bar{\mathfrak{q}} \in H^1(\mathscr{O})/\mathbb{R}$ such that $(\mathbf{u}, \bar{\mathfrak{q}})$ is the solution of the generalized Stokes problem

$$\mathbf{u} - \alpha\Delta\mathbf{u} + \nabla\bar{\mathfrak{q}} = \mathbf{z} \text{ in } \mathscr{O}, \tag{5.3a}$$

$$\operatorname{div}\mathbf{u} = 0 \text{ in } \mathscr{O}, \tag{5.3b}$$

$$\mathbf{u} = 0 \text{ on } \partial\mathscr{O}. \tag{5.3c}$$

Since $\mathbf{z} \in \mathbf{H}^1(\mathscr{O}) \subset L^r(\mathscr{O})$, $r \in [2, \infty)$, we can conclude as in [21, Proof of Proposition 5.3], see also [12, Theorem 7.3.3.1], that there exists a constant $r_0 > 2$ depending on the largest interior angle of $\mathscr{O}$ such that $\mathbf{u} \in \mathbf{W}^{2,r}(\mathscr{O})$ for any $r \in (2, r_0)$. Moreover, there exists a constant $C > 0$ depending only on r (hence on $\mathscr{O}$) such that

$$\|\mathbf{u}\|_{\mathbf{W}^{2,r}(\mathscr{O})} \leqslant C\|\mathbf{z}\|_{\mathbf{H}^1(\mathscr{O})}.$$

Plugging (5.2) into the last estimate and invoking (4.11) we infer that

$$\mathbb{E}\|\mathbf{u}\|^p_{L^\infty(0,T;\mathbf{W}^{2,r}(\mathscr{O}))} \leqslant \mathbb{E}\|z\|^p_{L^\infty(0,T;L^2(\mathscr{O}))} < \infty.$$

This completes the proof of the Theorem 2.9(a). □

5.2. *Proof of the uniqueness of solution*

In this subsection we will prove the uniqueness stated in Theorem 2.10. To achieve this goal we first establish few preparatory results which are mainly some estimates on the nonlinear term $\mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u}$. To this aim, we introduce the well known trilinear form b used in the study of the Navier-Stokes equation by setting

$$b(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \sum_{i,j=1}^{2} \int_{\mathscr{O}} \mathfrak{u}^{(i)} \frac{\partial \mathfrak{v}^{(j)}}{\partial x_i} \mathfrak{w}^{(j)} dx,$$

for any $\mathfrak{u} \in L^{r_1}(O)$, $\mathfrak{v} \in \mathbf{W}^{1,r_2}(\mathscr{O})$, $\mathfrak{w} \in L^{r_3}(\mathscr{O})$ with r_i, $i \in [\![1, 3]\!]$ satisfying $\sum_{i=1}^3 \frac{1}{r_i} = 1$. In the above formula $\mathfrak{u}^{(i)}$ is the i-th component of the vector $\mathfrak{u} = (\mathfrak{u}^{(1)}, \mathfrak{u}^{(2)})$.

We recall the following formula which was established in [21, Proof of Proposition 5.6]: for any $\mathfrak{u}, \mathfrak{v} \in \mathbf{W}^{1,4}(\mathscr{O})$ there holds

$$\begin{aligned}(\mathbf{rot}(\mathfrak{u} - \alpha\Delta\mathfrak{u}) \times \mathfrak{v}, \mathfrak{u}) =& b(\mathfrak{u}, \mathfrak{v}, \mathfrak{u}) + \alpha b(\mathfrak{u}, \mathbf{rot}\,\mathfrak{v}, \mathbf{rot}\,\mathfrak{u}) \\ & - 2\alpha \int_{\mathscr{O}} \mathbf{rot}\,\mathfrak{u}(x) \left(\nabla\mathfrak{v}^{(1)}(x) \cdot \nabla\mathfrak{u}^{(2)}(x) - \nabla\mathfrak{v}^{(2)}(x) \cdot \nabla\mathfrak{u}^{(1)}\right) dx.\end{aligned} \tag{5.4}$$

In the following lemma we state an important property satisfied by the bilinear map $\mathcal{C}(\cdot, \cdot)$ defined in (4.31).

Lemma 5.1. *Let r_0 be the positive number from Theorem 2.9(a). Then, there exists a constant $\kappa > 0$ such that for any $\nu \geqslant 0$ and $\mathfrak{u}, \mathfrak{v} \in \mathbf{W}$ we have*

$$|\langle \mathcal{C}(\mathbf{rot}(\mathfrak{u} - \alpha\Delta\mathfrak{u}), \mathfrak{v}), \mathfrak{u}\rangle| \leqslant \kappa\|\mathfrak{u}\|_\alpha |\,\mathbf{rot}(\mathfrak{v} - \alpha\Delta\mathfrak{v})|.$$

Proof. Throughout this proof all the constants are independent of $\nu \geqslant 0$. Let $r \in (2, r_0)$ and $s = \frac{2r}{r-1}$. Since $\frac{1}{2} + \frac{1}{s} + \frac{1}{r} = 1$, we infer from the Hölder inequality and the Sobolev embeddings $\mathbf{H}^1_0(\mathscr{O}) \subset \mathbf{L}^s(\mathscr{O})$ that

$$\begin{aligned} |b(\mathfrak{u}, \mathbf{rot}\,\mathfrak{v}, \mathbf{rot}\,\mathfrak{u})| &\leqslant C\|\mathfrak{u}\|_{\mathbf{L}^s(\mathscr{O})}|\,\mathbf{rot}\,\mathfrak{u}|\|\nabla(\mathbf{rot}\,\mathfrak{v})\|_{\mathbf{L}^r(\mathscr{O})}, \\ &\leqslant C|\nabla\mathfrak{u}|^2\|\mathfrak{v}\|_{\mathbf{W}^{2,r}(\mathscr{O})} \\ &\leqslant \frac{C}{\alpha}\|\mathfrak{u}\|^2_\alpha\|\mathfrak{v}\|_{\mathbf{W}^{2,r}(\mathscr{O})}, \end{aligned}$$

for any $\mathfrak{u} \in \mathbf{V},\ \mathfrak{v} \in \mathbf{W}^{2,r}(\mathscr{O})$. In a similar manner, we can prove that

$$\begin{aligned} \left|2\alpha\int_{\mathscr{O}} \mathbf{rot}\,\mathfrak{u}(x)\left(\nabla\mathfrak{v}^{(1)}(x)\cdot\nabla\mathfrak{u}^{(2)}(x) - \nabla\mathfrak{v}^{(2)}(x)\cdot\nabla\mathfrak{u}^{(1)}\right)dx\right| &\leqslant C\|\mathfrak{u}\|\ \|\nabla\mathfrak{v}\|_{\mathbf{L}^\infty(\mathscr{O})}|\,\mathbf{rot}\,\mathfrak{u}| \\ &\leqslant \frac{C}{\alpha}\|\mathfrak{u}\|^2_\alpha\|\mathfrak{v}\|_{\mathbf{W}^{1,\infty}(\mathscr{O})}, \end{aligned}$$

for any $\mathfrak{u} \in \mathbf{V},\ \mathfrak{v} \in \mathbf{W}^{1,\infty}(\mathscr{O})$. Also, there exists a constant $C > 0$ such that for any $\mathfrak{u} \in \mathbf{V},\ \mathfrak{v} \in \mathbf{H}^1(\mathscr{O})$ the following chain of inequalities holds

$$\begin{aligned} |b(\mathfrak{u}, \mathfrak{v}, \mathfrak{u})| &\leqslant C\|\mathfrak{u}\|^2_{\mathbf{L}^4(\mathscr{O})}\|\mathfrak{v}\| \\ &\leqslant \frac{C}{\alpha}\|\mathfrak{u}\|^2_\alpha\|\mathfrak{v}\|. \end{aligned}$$

Now notice that from the Sobolev embedding $\mathbf{W}^{2,r}(\mathscr{O}) \subset \mathbf{W}^{1,\infty}(\mathscr{O})$ and the proof of Theorem 2.9(a) we see that $\mathbf{W} \subset \mathbf{W}^{2,r}(\mathscr{O}) \subset \mathbf{W}^{1,\infty}(\mathscr{O})$ and for any $\mathfrak{v} \in \mathbf{W}$ we have

$$\|\mathfrak{v}\|_{\mathbf{W}^{1,\infty}(\mathscr{O})} \leqslant C\|\mathfrak{v}\|_{\mathbf{W}^{2,2}(\mathscr{O})} \leqslant C|\,\mathbf{rot}(\mathfrak{v} - \alpha\Delta\mathfrak{v})|.$$

Thus, from the definition of $\mathcal{C}(\cdot,\cdot)$, see 4.31, (5.4) and all the estimates above, we infer that there exists a constant $\kappa > 0$ such that

$$|\langle\mathcal{C}(\mathbf{rot}(\mathfrak{u} - \alpha\Delta\mathfrak{u}), \mathfrak{v}), \mathfrak{u}\rangle| = |(\mathbf{rot}(\mathfrak{u} - \alpha\Delta\mathfrak{u}) \times \mathfrak{v}, \mathfrak{u})| \leqslant \kappa\|\mathfrak{u}\|_\alpha|\,\mathbf{rot}(\mathfrak{v} - \alpha\Delta\mathfrak{v})|,$$

for any $\nu \geqslant 0$, $\mathfrak{u}$, $\mathfrak{v} \in \mathbf{W}$. This completes the proof of the lemma. ☐

We are now ready to give the promised proof of Theorem 2.10.

Proof. We recall that any solution $\mathfrak{u}$ of the problem (1.4) with the initial condition $\mathfrak{u}_0 \in \mathbf{W}$ satisfies, $\mathbb{P}$ a.s.

$$\mathfrak{u}(t) + \int_0^t [\nu\mathscr{A}\mathfrak{u}(s) + \mathscr{C}(\mathbf{rot}(\mathfrak{u}(s) - \alpha\Delta\mathfrak{u}(s), \mathfrak{u}(s)]\,ds = \mathfrak{u}_0 + \int_0^t \mathcal{G}(\mathfrak{u}(s))dW(s),$$

for any $t \in [0, T]$. Thus, if $\mathfrak{u}$ and $\mathfrak{v}$ are two solutions to (1.4) on the same stochastic basis $\mathscr{U}$ and respectively starting with the initial conditions $\mathfrak{u}_0$, $\mathfrak{v}_0 \in \mathbf{W}$, then the difference $\mathbf{u} = \mathfrak{v} - \mathfrak{u}$ satisfies $\mathbb{P}$ a.s.

$$\begin{aligned} \mathbf{u}(t) + \int_0^t [\nu\mathscr{A}\mathbf{u}(s) + \mathscr{C}(z(s), \mathfrak{u}(s)) + \mathscr{C}(\mathbf{rot}(\mathfrak{u}(s) - \alpha\Delta\mathfrak{u}(s)), \mathbf{u}(s)]\,ds \\ = \mathbf{u}(0) + \int_0^t [\mathcal{G}(\mathfrak{v}(s)) - \mathcal{G}(\mathfrak{u}(s))]\,dW(s), \end{aligned}$$

for any $t \in [0, T]$. In the above formula, we put $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$, $\mathbf{u}(0) = \mathfrak{v}_0 - \mathfrak{u}_0$. By the application of Itô formula, see [62, Theorem 26.5], to $\|\mathbf{u}(t)\|^2_\alpha$ and the identity

$$(((\mathrm{Id} + \alpha\mathcal{A})^{-1}\mathbf{f}, \mathbf{v})) = \langle\mathbf{f}, \mathbf{v}\rangle \text{ for any } \mathbf{f} \in \mathbf{V}^*,$$

we obtain

$$\|\mathbf{u}(t)\|_\alpha^2 + 2\int_0^t \Big[\nu\|\mathbf{u}(s)\|^2 + \langle \mathcal{C}(z(s),\mathfrak{u}(s)),\mathbf{u}(s)\rangle\Big]\,ds$$
$$= \|\mathbf{u}(0)\|_\alpha^2 + \int_0^t \|[G(\mathfrak{v}(s)) - G(\mathfrak{u}(s))]Q^{\frac12}\|^2_{\mathscr{L}_2(\mathscr{H},\mathbf{V})}ds + 2\int_0^t (\mathbf{u}(s), G(\mathfrak{v}(s)) - G(\mathfrak{u}(s))dW(s)).$$

We should notice that in the proof of the last identity we also used the equality

$$\langle \mathcal{C}(\mathbf{rot}(\mathfrak{u}-\alpha\Delta\mathfrak{u}),\mathbf{u}),\mathbf{u}\rangle = (\mathbf{rot}(\mathfrak{u}-\alpha\Delta\mathfrak{u})\times\mathbf{u},\mathbf{u}) = 0,$$

which is valid because $\mathfrak{u}$, $\mathbf{u}\in\mathbf{W}$ for almost all $(\omega,t)\in\Omega\times[0,T]$. Now, let $\kappa>0$ be the constant in Lemma 5.1, θ and ϑ be two stochastic processes defined by

$$\theta(t) = 2\kappa|\,\mathbf{rot}(\mathfrak{v}(t)-\alpha\Delta\mathfrak{v}(t))| \text{ and } \vartheta(t) = e^{-\int_0^t\theta(s)ds},\ t\in[0,T].$$

Applying the Itô formula to $\vartheta(t)\|\mathbf{u}(t)\|_\alpha^2$ and using the inequality in Lemma 5.1 yields

$$\begin{aligned}\vartheta(t)\|\mathbf{u}(t)\|_\alpha^2 =&\|\mathbf{u}(0)\|_\alpha^2 + \int_0^t\vartheta(s)d(\|\mathbf{u}(s)\|_\alpha^2) - \kappa\int_0^t[\|\mathbf{u}(s)\|_\alpha^2|\,\mathbf{rot}(\mathfrak{v}(s)-\alpha\Delta\mathfrak{v}(s))|]\vartheta(s)ds\\ \leqslant& C\,\mathrm{Tr}(Q)\int_0^t\|G(\mathfrak{v}(s))-G(\mathfrak{u}(s))\|^2_{\mathscr{L}(\mathscr{H},\mathbf{V})}\vartheta(s)ds - 2\nu\int_0^t\|\mathbf{u}(s)\|^2\vartheta(s)ds\\ &+2\int_0^t\vartheta(s)(\mathbf{u}(s),G(\mathfrak{v}(s))-G(\mathfrak{u}(s))dW(s)) + \|\mathbf{u}(0)\|_\alpha^2.\end{aligned}$$

By invoking Assumption (G), taking the mathematical expectation and noticing that the term with the stochastic integral is a martingale with zero mean, we obtain

$$\mathbb{E}\Big[\vartheta(t)\|\mathbf{u}(t)\|_\alpha^2\Big] + 2\nu\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^2\vartheta(s)ds \leqslant \|\mathbf{u}(0)\|_\alpha^2 C\,\mathrm{Tr}(Q)\int_0^t\mathbb{E}[\|\mathbf{u}(s)\|_\alpha^2\vartheta(s)]ds.$$

This estimate along with the Gronwall lemma implies

$$\mathbb{E}\Big[\vartheta(t)\|\mathbf{u}(t)\|_\alpha^2\Big] + 2\nu\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^2\vartheta(s)ds \leqslant \|\mathbf{u}(0)\|_\alpha^2 e^{C\,\mathrm{Tr}\,(Q)T},$$

from which we readily conclude the proof of Theorem 2.10. ☐

6. Proof of the time regularity stated in Theorem 2.9(b)

This section is devoted to the proof of the time-continuity in the Hilbert space $\mathbf{W}$ of the solution to (1.4). This aim will be achieved in observing that $z := \mathbf{u}-\alpha\Delta\mathbf{u}$ is weakly continuous in $\mathrm{L}^2(\mathscr{O})$ and showing that the norm of $z=\mathbf{u}-\alpha\Delta\mathbf{u}$ in $\mathrm{L}^2(\mathscr{O})$ is continuous. This part of the paper will be divided into two subsections.

6.1. *Regularization technique and convergences of (semi)martingale*

The continuity of $|z(\cdot)|^2$ will follow from an energy equation for the norm of z in $\mathrm{L}^2(\mathscr{O})$. Due to the lack of regularity of z, the derivation of this energy inequality is non-trivial and require the use of a regularization technique. Hence, we start recalling the following property of Lipschitz-continuous domain.

Lemma 6.1. *Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$; then $\mathscr{O}$ has a finite open covering,*

$$\bar{\mathscr{O}}\subset\bigcup_{r=1}^m\mathcal{O}_r,$$

with the following property. For each r with $1 \leqslant r \leqslant m$, there exists a nonzero vector y_r of $\mathbb{R}^2$ and a number $\delta_r > 0$ such that for all $0 < \epsilon \leqslant 1$ and for all $x \in \bar{\mathscr{O}} \cap \mathcal{O}_r$,

$$B(x;\epsilon\delta_r) + \epsilon y_r \subset \mathscr{O}, \tag{6.1}$$

where $B(x;\delta)$ denotes the ball with center x and radius δ.

Proof. The lemma is exactly the same as [54, Lemma 2.1]. □

Following the idea in [54], we will construct a special mollifier which does not use values outside $\mathscr{O}$. To this end, we consider a standard mollifier ϱ_r with support in $B(0;\delta_r)$, i.e, $\varrho_r \in [C_c^\infty(\mathbb{R}^2)]^2$, $0 \leqslant \varrho_r \leqslant 1$ in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} \varrho_r(x)dx = \int_{B(0;\delta_r)} \varrho_r(x)dx = 1.$$

Furthermore, for any index $1 \leqslant r \leqslant m$ we set $\mathscr{O}_r = \bar{\mathscr{O}} \cap \mathcal{O}_r$ and for any $\varepsilon \in (0,1]$ we put

$$\varrho_{\varepsilon,r}(x) = \frac{1}{\varepsilon^2}\varrho_r\left(\frac{x}{\varepsilon} + y_r\right).$$

For a function $f \in \mathrm{L}^p(0,T;\mathrm{L}^q(\mathscr{O}))$ (its extension by zero outside $\mathscr{O}$ is still denoted by f) we define its convolution with $\varrho_{\varepsilon,r}$ by

$$f * \varrho_{\varepsilon,r}(x,t) := \int_{B(0;\delta_r)} f(x-\varepsilon(y-y_r),t)\varrho_r(y)dy, \quad \text{a.e. in } \mathscr{O}_r \times [0,T].$$

We see from this last formula and (6.1) that the convolution with $\varrho_{\varepsilon,r}$ regularizes f without using its extension outside $\mathscr{O}_r$. We proceed now to the statement and proof of the following proposition which can be viewed as a generalization of the results in [54, Proposition 2.2 & Corollary 2.3].

Proposition 6.2. *Let $\mathscr{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$ and $f \in \mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathscr{O}))$ with $p \in [1,\infty]$. Assume also that we are given a function $\mathbf{v}$ such that $\mathbf{v} \in \mathrm{L}^\gamma(0,T;\mathbf{W}^{1,q}(\mathscr{O}))$ for some $q \geqslant \frac{p}{p-1}$. Let $s > 0$ be the real number defined by*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

Then, there exists a constant $C > 0$ independent of f and $\mathbf{v}$ such that

$$\|\mathbf{v}\cdot\nabla(f*\varrho_{\varepsilon,r}) - (\mathbf{v}\cdot\nabla f)*\varrho_{\varepsilon,r}\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}_r))} \leqslant C\|f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathscr{O}))}\|\mathbf{v}\|_{\mathrm{L}^\gamma(0,T;\mathbf{W}^{1,q}(\mathscr{O}))}, \tag{6.2}$$

for any index $1 \leqslant r \leqslant m$, $\varepsilon(0,1]$.

Furthermore, for all $1 \leqslant r \leqslant m$ we have

$$\lim_{\varepsilon\to 0}\|\mathbf{v}\cdot\nabla(f*\varrho_{\varepsilon,r}) - (\mathbf{v}\cdot\nabla f)*\varrho_{\varepsilon,r}\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}_r))} = 0 \tag{6.3}$$

Proof. There are several way to prove this lemma, but the easiest and shortest way is to use some results from [73].
Proof of (6.2). It is proved in [73, Lemma 1.2] that there exists a constant $C > 0$ independent of f and $\mathbf{v}$ such that for any index $1 \leqslant r \leqslant m$, $\varepsilon(0,1]$, we have

$$\|\mathbf{v}\cdot\nabla(f*\varrho_{\varepsilon,r}) - (\mathbf{v}\cdot\nabla f)*\varrho_{\varepsilon,r}\|_{\mathrm{L}^s(\mathscr{O}_r)} \leqslant C\|f\|_{\mathrm{L}^p(\mathscr{O})}\|\mathbf{v}\|_{\mathbf{W}^{1,q}(\mathscr{O})}. \tag{6.4}$$

From this estimate we easily conclude the proof of (6.2).
Proof of (6.3). From [73, Corollary 1.1] we derive that for almost all $t \in [0,T]$

$$\lim_{\varepsilon\to 0}\|[\mathbf{v}\cdot\nabla(f*\varrho_{\varepsilon,r})](\cdot,t) - [(\mathbf{v}\cdot\nabla f)*\varrho_{\varepsilon,r}](\cdot,t)\|_{\mathrm{L}^s(\mathscr{O}_r)} = 0, \tag{6.5}$$

which along with (6.2) and the Lebesgue Dominated Convergence Theorem yields (6.3). □

Now we will regularize the stochastic process $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$, where $\mathbf{u}$ is the weak martingale solution to (1.4). For this purpose, let us still denote by z the extension of z by zero outside $\mathscr{O}$, let $\{\psi_r; 1 \leqslant r \leqslant m\}$ be a partition of unity in $\bar{\mathscr{O}}$, subordinated to the finite covering $\{\mathcal{O}_r\ 1 \leqslant r \leqslant m\}$, and set $z^r = z\psi_r$. For any integer $k \geqslant 1$ we set

$$z_k = \sum_{r=1}^{m} z^r * \varrho_{\frac{1}{k},r}. \tag{6.6}$$

In what follows, we extend $\mathbf{u}$ outside $\mathscr{O}$ via an extension operator that is linear and continuous (bounded) as a map $\mathbf{W}^{1,r}(\mathscr{O}) \to \mathbf{W}^{1,r}(\mathbb{R}^2)$ so that the extended function, still denoted by $\mathbf{u}$, belongs to $L^2(0,T;\mathbf{W}^{1,r}(\mathbb{R}^2))$.

We have the following result.

Proposition 6.3. *There exists an integer k_0 such that for all $f \in \mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}))$, $\gamma \in [1,\infty)$ and $s \in [1,\infty]$, extended by zero outside $\mathscr{O}$, for all $k \geqslant k_0$ and for all $1 \leqslant r \leqslant m$, the support of the function $(f\psi_r) * \varrho_{\frac{1}{k},r}$ is contained in $\mathscr{O}_r$. Moreover,*

$$\lim_{k\to\infty} \left\| \sum_{r=1}^{m} (f\psi_r) * \varrho_{\frac{1}{k},r} - f \right\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}))} = 0. \tag{6.7}$$

Proof. The first part of the proposition is exactly the first part of [54, Lemma 2.4]. The argument of the proof of the second part is quite similar to the idea of the proof of the second part of [54, Lemma 2.4], but for the sake of completeness we give a rather sketchy proof of it. Observe that, since $\{\psi_r; 1 \leqslant r \leqslant m\}$ is a partition of unity in $\bar{\mathscr{O}}$,

$$\begin{aligned}\left\| \sum_{r=1}^{m} [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - f(\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O})} &= \left\| \sum_{r=1}^{m} \Big([(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - [(f\psi_r)](\cdot,t) \Big) \right\|_{\mathrm{L}^s(\mathscr{O})} \\ &\leqslant \sum_{r=1}^{m} \left\| [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - [(f\psi_r)](\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O})},\end{aligned}$$

for almost all $t \in [0,T]$. Owing to the property of $\varrho_{\frac{1}{k},r}$ we have for almost all $t \in [0,T]$ and for all $1 \leqslant r \leqslant m$

$$\lim_{k\to\infty} \left\| [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - [f\psi_r](\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O}_r)} = 0, \tag{6.8}$$

from which and the first part of the proposition we derive that for almost all $t \in [0,T]$ and for all $1 \leqslant r \leqslant m$

$$\lim_{k\to\infty} \left\| [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - [f\psi_r](\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O})} = \lim_{k\to\infty} \left\| [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - [f\psi_r](\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O}_r)} = 0.$$

Thus, for almost all $t \in [0,T]$

$$\lim_{k\to\infty} \left\| \sum_{r=1}^{m} [(f\psi_r) * \varrho_{\frac{1}{k},r}](\cdot,t) - f(\cdot,t) \right\|_{\mathrm{L}^s(\mathscr{O})} = 0. \tag{6.9}$$

Notice also that

$$\begin{aligned}\left\| \sum_{r=1}^{m} (f\psi_r) * \varrho_{\frac{1}{k},r} - f \right\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}))} &\leqslant \sum_{r=1}^{m} \left\| (f\psi_r) * \varrho_{\frac{1}{k},r} - f \right\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}))} \\ &\leqslant C_m \|f\|_{\mathrm{L}^\gamma(0,T;\mathrm{L}^s(\mathscr{O}))},\end{aligned}$$

from which altogether with (6.9) and the Lebesgue Dominated Convergence Theorem we easily conclude the proof of (6.7) and the proposition. ∎

We will also need the following result.

Proposition 6.4. *Let $s \in (1,\infty)$ and X be a Banach space such that the injection $\mathrm{L}^s(\mathscr{O}) \subset \mathrm{X}$ is continuous. Then, the first part of Proposition 6.3 remains valid for any $f \in \mathrm{L}^\infty(0,T;\mathrm{L}^s(\mathscr{O}))$ such that $f:[0,T] \to \mathrm{X}$ is weakly continuous. Furthermore, for all $t \in [0,T]$*

$$\lim_{k\to\infty} \left\| \sum_{r=1}^m (f\psi_r) * \varrho_{\frac{1}{k},r} - f \right\|_{\mathrm{L}^s(\mathscr{O})} = 0. \tag{6.10}$$

Proof. The proof of the first part follows the same lines of the proof of the first part of Proposition 6.3. Investigating closely the proof of [74, Theorem 2.1], we infer from [74, Eq. (2.1), page 544] that $f(t) \in \mathrm{L}^s(\mathscr{O})$ for all $t \in [0,T]$. With this observation, we can repeat, mutatis mutandis, the proof of (6.9) to complete the proof of (6.10). □

Hereafter, for the sake of simplicity we set $\mathrm{L}^q := \mathrm{L}^q(\mathscr{O})$ for any $q \geqslant 1$. To close the paragraph about the mollifier $\varrho_{\frac{1}{k},r}$ we formulate the following remarks.

Remark 6.5. It is not difficult to see that the map $\Lambda_k : \mathrm{L}^q \to \mathrm{L}^q$, $q \in [1,\infty)$ defined by

$$\Lambda_k v = \sum_{r=1}^m (\psi_r v) * \varrho_{\frac{1}{k},r}, \forall v \in \mathrm{L}^q,$$

is linear, continuous and closed. It can act on $\mathcal{D}'(\mathscr{O})$ which is the dual of $C_c^\infty(\mathscr{O})$. Furthermore, the convergences 6.7 and (6.10) can be reformulated using Λ_k.

Now, we will state and prove several results related to the theory of (semi)martingales. To this end, let $\mathbb{H}$ be a separable Hilbert space with norm $|||\cdot|||$, Z be a $\mathbb{H}$-valued semimartingale with quadratic variation $[Z] := \{[Z]_t;\ t \in [0,T]\}$. Here we closely follow the notation of [62], in particular, we refer to [62, Theorem 26.5] for the definition of $[Z]$.

Lemma 6.6. *Let $\{Z_k;\ k \in \mathbb{N}\}$ be sequence of $\mathbb{H}$-valued semimartingales and Z a $\mathbb{H}$-valued semimartingale such that*

(S) the sequence $\{[Z_k + Z]_T;\ k \in \mathbb{N}\}$ is uniformly bounded and

$$\lim_{k\to\infty} \mathbb{E}([Z_k - Z]_T) = 0.$$

Then,

$$\lim_{k\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} |[Z_k]_t - [Z]_t| \right) = 0. \tag{6.11}$$

Proof. For the sake of simplicity we will write $X^t := X(t)$ where X denotes either Z_k or Z. For any partition $J = \{0 < t_1 < \ldots < t_n = T\}$ of $[0,T]$ we also set

$$\Delta_i Z^t = Z^{t\wedge t_{i+1}} - Z^{t\wedge t_i}.$$

Using the definition of the quadratic variation given in [62, Theorem 26.5], we obtain

$$\begin{aligned}
|[Z_k]_t - [Z]_t| &= \lim_n (\mathbb{P}) \left| \sum_i |||\Delta_i \Lambda_k Z^t\,|||^2 - |||\,\Delta_i Z^t|||^2 \right| \\
&= \lim_n (\mathbb{P}) \left| \sum_{J_i} \left(\Delta_i (Z_k^t - Z^t), \Delta_i (Z_k^t + Z^t) \right) \right| \\
&\leqslant \lim_n (\mathbb{P}) \left[\sum_i |||\Delta_i (Z_k^t - Z^t)\,|||^2 \sum_{J_i} |||\Delta_i (Z_k^t + Z)|||^2 \right]^{\frac{1}{2}} \\
&\leqslant ([Z_k - Z]_t [Z_k + Z]_t)^{\frac{1}{2}},
\end{aligned}$$

where $\lim(\mathbb{P})$ means *limit in probability*. Since $[X]$ is an increasing function w.r.t. the time variable, we easily obtain from the last line of the above chain inequalities that

$$\mathbb{E} \sup_{t\in[0,T]} |[Z_k]_t - [Z]_t| \leqslant \Big(\mathbb{E}\left([Z_k - Z]_T\right) \mathbb{E}\left([Z_k + Z]_T\right)\Big)^{\frac{1}{2}}, \tag{6.12}$$

which altogether with Assumption (S) yield the desired convergence. □

We will also need the following result.

Lemma 6.7. *Let $f \in L^2(\Omega, L^\infty(0,T;\mathbb{H}))$ be a predictable process and $\{f_k;\ k\in\mathbb{N}\}$ a sequence of predictable stochastic processes in $L^2(\Omega, L^\infty(0,T;\mathbb{H}))$ such that*

$$\lim_{k\to\infty} \mathbb{E}\int_0^T |||f_k(s) - f(s)|||^2\, ds = 0.$$

Let $\{Z_k;\ k\in\mathbb{N}\}$ (resp. Z) be a sequence of càdlàg martingales (resp. a martingale) satisfying Assumption (S) of Lemma 6.6. We assume further that there exists a real-valued process $\Phi \geqslant 0$ and a sequence of non-negative real-valued processes $\{\Phi_k; k\in\mathbb{N}\}$ such that

$$[Z]_\cdot = \int_0^\cdot \Phi(s)ds,\quad [Z_k]_\cdot = \int_0^\cdot \Phi_k(s)ds,$$

and there exists a constant $C>0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left(\Phi_k^2(s) + \Phi^2(s)\right)\right) < C,$$

for any $k\in\mathbb{N}$. Then,

$$\lim_{k\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t (f_k(s-), dZ_k(s))_{\mathbb{H}} - \int_0^t (f(s-), dZ(s))_{\mathbb{H}}\right|\right) = 0. \tag{6.13}$$

Proof. From an application of the Burkholder-Davis-Gundy inequality we infer that there is a constant $C>0$ such that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t (f_k(s-), dZ_k(s))_{\mathbb{H}} - \int_0^t (f(s-), dZ(s))_{\mathbb{H}}\right|\right) &\leqslant C\mathbb{E}\left(\int_0^T |||f_k(s) - f(s)|||^2\, d[Z_k]_s\right)^{\frac{1}{2}} \\ &+ C\mathbb{E}\left(\int_0^T |||f(s)|||^2\, d[Z_k - Z]_s\right)^{\frac{1}{2}},\end{aligned} \tag{6.14}$$

which along with the assumption of the lemma implies

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t (f_k(s-), dZ_k(s))_{\mathbb{H}} - \int_0^t (f(s-), dZ(s))_{\mathbb{H}}\right|\right) \\ &\quad\leqslant C\mathbb{E}\left(\sup_{s\in[0,T]}\Phi_k^2(s)\int_0^T |||f_k(s) - f(s)|||^2\, ds\right)^{\frac{1}{2}} + C\mathbb{E}\left(\sup_{s\in[0,T]} |||f(s)|||\sqrt{[Z_k - Z]_T}\right).\end{aligned} \tag{6.15}$$

Owing to the Cauchy-Schwarz inequality, we readily infer that

$$\begin{aligned}&\lim_{k\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t (f_k(s-), dZ_k(s))_{\mathbb{H}} - \int_0^t (f(s-), dZ(s))_{\mathbb{H}}\right|\right) \\ &\quad\leqslant C\lim_{k\to\infty}\left[\mathbb{E}\left(\sup_{s\in[0,T]}\Phi_k(s)\right)\mathbb{E}\int_0^T |||f_k(s) - f(s)|||^2\, ds\right]^{\frac{1}{2}} \\ &\quad+ C\lim_{k\to\infty}\left[\mathbb{E}\left(\sup_{s\in[0,T]} |||f(s)|||^2\right)\mathbb{E}[Z_k - Z]_T\right]^{\frac{1}{2}},\end{aligned} \tag{6.16}$$

The desired result follows easily by passing to the limit in the last line of the above estimate. □

6.2. The actual proof of Theorem 2.9(b)

In this subsection we will give the promised proof of the time regularity of the weak martingale solution to problem (1.4). To this aim, let $\mathbf{u}$ be the weak martingale solution of (1.4), $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ and $\{z_k;\ k \in \mathbb{N}\}$ be the sequence defined by (6.6). For each $k \in \mathbb{N}$, we set

$$\begin{aligned} L_k(\mathbf{u}) &= \frac{\nu}{\alpha}\sum_{r=1}^{m}(\psi_r\,\mathbf{rot}\,\mathbf{u}) * \varrho_{\frac{1}{k},r}, \\ A_k &= \sum_{r=1}^{m}\left[\mathbf{u}\cdot\nabla(z^r * \varrho_{\frac{1}{k},r}) - (\mathbf{u}\cdot\nabla z^r) * \varrho_{\frac{1}{k},r}\right], \\ B_k &= \sum_{r=1}^{m}\Big([(\mathbf{u}\cdot\nabla\psi_r)z] * \varrho_{\frac{1}{k},r} - (\mathbf{u}\cdot\nabla\psi_r)z\Big). \end{aligned}$$

We also put

$$M^{\mathbf{u}}(\cdot) := \int_0^{\cdot}\mathbf{rot}\,G(\mathbf{u}(s))dW(s), \text{ and } M_k^{\mathbf{u}}(\cdot) := \Lambda_k M^{\mathbf{u}}(\cdot),\ \forall k \in \mathbb{N}.$$

Remark 6.8. By the Remark 6.5 and [61, Proposition 4.30], we have

$$M_k^{\mathbf{u}}(\cdot) = \int_0^{\cdot}\Lambda_k\,\mathbf{rot}\,G(\mathbf{u}(s))dW(s).$$

Thus, thanks to Assumption (G) and the fact that $\mathbf{u} \in \mathrm{L}^2(\Omega; C(0,T;\mathbf{V}))$, for each integer $k \geqslant 1$, $M_k^{\mathbf{u}}$ and $M^{\mathbf{u}}$ are L^2-valued martingales and $M^{\mathbf{u}}, M_k^{\mathbf{u}} \in \mathrm{L}^2(\Omega, C(0,T;\mathrm{L}^2))$. Note also that

$$\begin{aligned} [M]_{\cdot} &= \int_0^{\cdot}\mathrm{Tr}\left((\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}})^*(\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}})\right)ds \\ &= \int_0^{\cdot}\left(\sum_{j=1}^{\infty}|\,\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}}h_j|^2\right)ds, \end{aligned}$$

and for each $k \geqslant 1$

$$\begin{aligned} [M_k]_{\cdot} &= \int_0^{\cdot}\mathrm{Tr}\left((\Lambda_k\,\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}})^*(\Lambda_k\,\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}})\right)ds, \\ &= \int_0^{\cdot}\left(\sum_{j=1}^{\infty}|\Lambda_k\,\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}}h_j|^2\right)ds, \end{aligned} \tag{6.17}$$

which along with Remark 6.5 implies that there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$

$$[M_k]_{\cdot} \leqslant C\int_0^{\cdot}\left(\sum_{j=1}^{\infty}|\,\mathbf{rot}\,G(\mathbf{u}(s))Q^{\frac{1}{2}}|^2\right)ds \leqslant C[M]_{\cdot}. \tag{6.18}$$

Hence, owing to Assumption (G) and the fact that $\mathbf{u} \in \mathrm{L}^2(\Omega; C(0,T;\mathbf{V}))$

$$\sup_{k\in\mathbb{N}}\mathbb{E}\sup_{t\in[0,T]}[M_k]_t \leqslant C\mathbb{E}\sup_{t\in[0,T]}[M]_t \leqslant C.$$

The following proposition plays an important role in the proof of the time-continuity in $\mathbf{W}$ of $\mathbf{u}$.

Proposition 6.9. *There exists a subset $\Omega_4 \subset \Omega$ with $\mathbb{P}(\Omega_4) = 1$ and a subsequence of integers k' such that the following limit hold for any $\omega \in \Omega_4$ and $t \in [0, T]$*

$$\lim_{k'\to\infty} \int_0^t (A_{k'}(s) + B_{k'}(s), z_{k'}(s))ds = 0, \tag{6.19}$$

$$\lim_{k'\to\infty} \int_0^t (L_{k'}(\mathbf{u}(s)), z_{k'}(s))ds = \int_0^t (\mathbf{rot}\,\mathbf{u}(s), z(s))ds, \tag{6.20}$$

$$\lim_{k'\to\infty} \int_0^t (z_{k'}(s), dM^{\mathbf{u}}_{k'}(s)) = \int_0^t (z(s), dM^{\mathbf{u}}(s)), \tag{6.21}$$

$$\lim_{k'\to\infty} [M^{\mathbf{u}}_{k'}]_t = [M]_t. \tag{6.22}$$

Proof. The proof will be divided in several steps.

(**Step 1**) In this step we will show that (6.19) and (6.20) hold with probability 1 for any $t \in [0, T]$ for the whole sequence k. Indeed, by Theorem 2.9(a) we have $\mathbf{u} \in \mathrm{L}^\infty(0, T; \mathbf{W}^{2,r}(\mathscr{O}))$ for some $r > 0$. Since the space dimension is 2 and the embedding $\mathbf{W}^{1,r}(\mathscr{O}) \subset \mathbf{L}^\infty(\mathscr{O})$, $r > 2$, is continuous, we also have $\mathbf{u} \in \mathrm{L}^2(0, T; \mathbf{W}^{1,\infty}(\mathscr{O}))$. Thus, $\mathbb{P}(\Omega_1 \cap \Omega_0) = 1$ where

$$\Omega_0 = \{\omega \in \Omega;\ \ z \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\mathscr{O})) \cap \mathrm{L}^2(0, T; \mathrm{W}^{-1,\frac{4}{3}}(\mathscr{O}))\},$$
$$\Omega_1 = \{\omega \in \Omega;\ \ \mathbf{u} \in \mathrm{L}^\infty(0, T; \mathbf{W}^{1,\infty}(\mathscr{O}))\}.$$

It follows from (6.3) and (6.7) that on $\Omega_1 \cap \Omega_0$

$$\lim_{k\to\infty} \int_0^t (A_k(s) + B_k(s), z_k(s))ds = 0 \tag{6.23}$$

for all $t \in [0, T]$. From (6.23), (6.10), (6.7) we readily deduce that the convergences (6.19) and (6.20) hold on $\Omega_0 \cap \Omega_1$ for any $t \in [0, T]$ for the whole sequence k.

(**Step 2**) Now, we prove that

$$\lim_{k\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} |[M^{\mathbf{u}}_k]_t - [M^{\mathbf{u}}]_t|\right) = 0. \tag{6.24}$$

Since $M^{\mathbf{u}}_k$ and $M^{\mathbf{u}}$, $k \geqslant 1$, are both continuous and square integrable L^2-valued martingales, we can argue as in the proof of (6.7) and (6.10) to show that for all $t \in [0, T]$

$$\lim_{k\to\infty} \mathbb{E}|M^{\mathbf{u}}_k(t) - M^{\mathbf{u}}(t)|^2 = 0.$$

Now, observe that because of the definition of $M^{\mathbf{u}}_k - M^{\mathbf{u}}$, the Itô isometry and (6.25) we have

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}[M^{\mathbf{u}}_k - M^{\mathbf{u}}]_T &= \lim_{k\to\infty} \mathbb{E}\left|\int_0^T [\Lambda_k \tilde{G}(\mathbf{v}(t)) - \tilde{G}(\mathbf{v}(t))]dW(t)\right|^2, \\ &= \lim_{k\to\infty} \mathbb{E}|M^{\mathbf{u}}_k(T) - M^{\mathbf{u}}(T)|^2 = 0.\end{aligned} \tag{6.25}$$

(**Step 3**) In this step we shall show that

$$\lim_{k\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} \left|\int_0^t (z_k(s), dM_k(s)) - \int_0^t (z(s), dM(s))\right|\right) = 0. \tag{6.26}$$

Because $\mathbf{u}$ is weak martingale solution to (1.4) we have $\mathbf{u} \in \mathrm{L}^2(\Omega, \mathrm{L}^\infty(0, T; \mathbf{W}))$ and we can infer from Remark 6.5 that there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} |z_k(t)|^2 \leqslant C\mathbb{E} \sup_{t\in[0,T]} |z(t)|^2 < C. \tag{6.27}$$

Owing to (6.7) and the above estimate we can infer, with the help of the Lebesgue Dominated Convergence Theorem, that

$$\lim_{k\in[0,T]} \mathbb{E}\int_0^T |z_k(s)-z(s)|^2 ds = 0. \tag{6.28}$$

Because of Remark 6.8, (6.27) and (6.28), all the assumptions of Lemma 6.7 are verified by $z_k, z, M_k^{\mathbf{u}}$ and $M^{\mathbf{u}}$. Therefore, (6.26) holds.

(**Step 4**) This is the final step of our proof. Thanks to (6.24), (6.26) and an application of Egorov's theorem one can find a subsequence k' and a subset $\tilde{\Omega}_4$ with $\mathbb{P}(\tilde{\Omega}_4) = 1$ such that (6.21) and (6.22) holds on $\tilde{\Omega}_4$ for any $t \in [0,T]$. We conclude the proof by taking $\Omega_4 = \Omega_0 \cap \Omega_1 \cap \Omega_4$ and k'.

□

As our final preliminary result, we will show in the next proposition that the regularized process z_k given in (6.6) satisfies a tic PDES very similar to (1.4b).

Proposition 6.10. *Let $\Omega_4 \subset \Omega$ and k' be respectively the subset and subsequence of integers given by Proposition 6.9. For the sake of simplicity we still denote k' by k. Let $\mathbf{u}$ be the weak martingale solution of (1.4) given by Theorem 2.7, $z = \mathbf{rot}(\mathbf{u} - \alpha\Delta\mathbf{u})$ and z_k be the function defined in (6.6). Then, z_k solves*

$$dz_k + \left(\frac{\nu}{\alpha} z_k + \mathbf{u}\cdot\nabla z_k\right) dt = (L_k(\mathbf{u}) + A_k + B_k)dt + \Lambda_k \,\mathbf{rot}\, G(\mathbf{u})dW, \tag{6.29a}$$

$$z_k(0) = \sum_{r=1}^m (\psi_r z_0) * \varrho_{\frac{1}{k},r}, \tag{6.29b}$$

where $z_0 = \mathbf{rot}(\mathbf{u}_0 - \alpha\Delta\mathbf{u}_0)$.

Proof. First, note that z solves (1.4a) with the initial condition z_0, *i.e.*, with probability 1

$$z(t) + \int_0^t \left(\frac{\nu}{\alpha} z(s) + \mathbf{u}(s)\cdot\nabla z(s)\right) ds = z_0 + \frac{\nu}{\alpha}\int_0^t \mathbf{rot}\,\mathbf{u}(s)ds + \int_0^t \mathbf{rot}\, G(\mathbf{u}(s))dW(s), \tag{6.30}$$

for all $t \in [0,T]$. Multiplying this identity by ψ_r, regularizing both sides of the resulting equation by convolution with $\varrho_{\frac{1}{k},r}$ and summing from $r=1$ to $r=m$ yields

$$z_k(t) + \int_0^t \left[\frac{\nu}{\alpha} z_k(s) + \sum_{r=1}^m \left([(\mathbf{u}\cdot\nabla z)\psi_r] * \varrho_{\frac{1}{k},r}\right)(s)\right] ds = z_k(0) + \int_0^t L_k(\mathbf{u}(s))ds \\ + \Lambda_k N^{\mathbf{u}}(t). \tag{6.31}$$

Second, observe that

$$(\mathbf{u}\cdot\nabla z)\psi_r = -(\mathbf{u}\cdot\nabla\psi_r)z + \mathbf{u}\cdot\nabla z^r,$$

and since $\{\psi_r; 1 \leqslant r \leqslant m\}$ is a partition of unity in $\bar{\mathscr{O}}$, we also have

$$\sum_{r=1}^m (\mathbf{u}\cdot\nabla\psi_r)z = (\mathbf{u}\cdot\nabla[\sum_{r=1}^m \psi_r])z = 0.$$

Therefore,

$$\begin{aligned}\sum_{r=1}^m [(\mathbf{u}\cdot\nabla z)\psi_r] * \varrho_{\frac{1}{k},r} =& \mathbf{u}\cdot\nabla z_k + \sum_{r=1}^m (\mathbf{u}\cdot\nabla z^r) * \varrho_{\frac{1}{k},r} - \mathbf{u}\cdot\nabla(z^r * \varrho_{\frac{1}{k},r}) \\ &+ (\mathbf{u}\cdot\nabla\psi_r) - [(\mathbf{u}\cdot\nabla\psi_r)z] * \varrho_{\frac{1}{k},r}, \\ =& \mathbf{u}\cdot\nabla z_k - A_k - B_k.\end{aligned}$$

Then (6.29) follows by substituting the last line into (6.31). □

After all these preparatory results we are now ready to give the promised proof of Theorem 2.9(b).

Proof of Theorem 2.9(b). First, define $\Omega_5 = \bigcap_{j=1}^4 \Omega_j$ where Ω_4 is given by Proposition 6.9, Ω_0 and Ω_1 are defined in the proof of Proposition 6.9, and

$$\Omega_2 = \{\omega \in \Omega;\ \ z : [0,T] \to \mathrm{L}^2 \text{ is weakly continuous}\},$$
$$\Omega_3 = \{\omega \in \Omega;\ \ (6.30) \text{ holds }\}.$$

Owing to Theorem 2.9(a) we have $\mathbf{P}(\Omega_1) = 1$. Thanks to [74, Theorem 2.1] and Proposition 2.6 we have $\mathbb{P}(\Omega_0) = \mathbb{P}(\Omega_2) = \mathbb{P}(\Omega_3) = 1$. Hence, $\mathbb{P}(\Omega_5) = 1$. Set also

$$\tilde{G}_k(\cdot) := \Lambda_k \,\mathbf{rot}\, G(\cdot).$$

Thanks to the Itô formula [62, Theorem 26.5], the identity $(\mathbf{u}\cdot\nabla z_k, z_k) = 0$ and Remark 6.8, we have

$$\begin{aligned} |z_k(t)|^2 + \frac{2\nu}{\alpha}\int_0^t |z_k(s)|^2 ds = 2\int_0^t (L_k(\mathbf{u}(s)) + A_k(s) + B_k(s), z_k(s))ds + [M_k^{\mathbf{u}}]_t \\ +2\int_0^t (z_k(s), dM_k^{\mathbf{u}}(s)), \end{aligned} \tag{6.32}$$

for all $\omega \in \Omega_5$, $t \in [0,T]$. From (6.10) and (6.7) respectively, we infer that for all $\omega \in \Omega_5$ and for all $t \in [0,T]$,

$$\lim_{k\to\infty} |z_k(t)|^2 = |z(t)|^2,$$
$$\lim_{k\to\infty} \int_0^t |z_k(s)|^2 ds = \int_0^t |z(s)|^2 ds.$$

Setting $\tilde{G}(\cdot) := \mathbf{rot}\, G(\cdot)$, taking the last two convergences and those in Proposition 6.9 into account and passing to the limit in (6.32) imply

$$\begin{aligned} |z(t)|^2 + \frac{2\nu}{\alpha}\int_0^t |z(s)|^2 ds =& 2\int_0^t (\mathbf{rot}(\mathbf{u}(s)), z(s))ds \\ &+ \int_0^t \mathrm{Tr}\left((\tilde{G}(\mathbf{u}(s))Q^{\frac{1}{2}})(\tilde{G}(\mathbf{u}(s))Q^{\frac{1}{2}})^*\right) ds \\ &+ 2(z, \int_0^t \tilde{G}(\mathbf{u}(s))dW(s)), \end{aligned}$$

for any $\omega \in \Omega_5$ and for any $t \in [0,T]$. The last identity implies that $|z(\cdot)|^2$ is continuous on Ω_5. This fact along with the weak continuity of $z : [0,T] \to \mathrm{L}^2$ implies that z is continuous in L^2 on Ω_5. Now, recalling that from Theorem 2.7 we can find a subset Ω_6 with $\mathbb{P}(\Omega_6) = 1$ such that $\mathbf{u}(\cdot) : [0,T] \to \mathbf{V}$ is continuous on Ω_6. Hence, once can readily show that on $\Omega' = \Omega_5 \cap \Omega_6$, which clearly satisfies $\mathbb{P}(\Omega') = 1$, the function $\mathbf{u}$ is continuous in $\mathbf{W}$. Since $\mathbf{u} \in \mathrm{L}^p(\Omega, \mathrm{L}^\infty(0,T;\mathbf{W}))$ and is continuous in $\mathbf{W}$, we infer that $\mathbf{u} \in \mathrm{L}^p(\Omega; C([0,T];\mathbf{W}))$. This completes the proof of Theorem 2.9(b). □

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