Synthetic Phase II Shewhart-type Attributes Control Charts When Process Parameters are Estimated

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The performance of attributes control charts (such as c and np charts) is usually evaluated under the assumption of known process parameters (i.e., the nominal proportion of nonconforming units or the nominal number of nonconformities). However, in practice, these process parameters are rarely known and have to be estimated from an in-control phase I data set. In this paper, we derive the run length properties of the phase II synthetic *c* and *np* charts with *estimated* parameters, and we investigate the number *m* of phase I samples that would be necessary for these charts in order to obtain similar in-control average run lengths as in the *known* parameters case. We also propose new specific chart parameters that allow these charts to have approximately the same in-control average run lengths as the ones obtained in the *known* parameter case.

Keywords: phases I and II; attribute control chart; synthetic chart; binomial; Poisson; unknown parameter; average run length; Markov chain

1. Introduction

Because of the global market that considers customer satisfaction as a primary objective, quality has been established as a key competitive priority in the world of business. Statistical Process Control (SPC) is a collection of statistical techniques providing a rational management of a manufacturing process, which allows high-quality final products to be produced. Among the statistical process control tools, control charts are undeniably the most widely used for identifying changes in processes. When the considered quality characteristic consists of *qualitative* information, like the nominal proportion of nonconforming units or the nominal number of nonconformities, traditional variable control charts like the $\overline{X}$, S, and R charts cannot be used and must be replaced by attributes control charts like the *np* or the *c* charts (see, for instance, Montgomery[1]). Control charts have important applications, and they are particularly useful in service industries and in non-manufacturing quality-improvement efforts because many of the quality characteristics found in these environments are not easily measured on a numerical scale.

Shewhart-type control charts (for variables or attributes) are known to be rather inefficient in detecting small or moderate changes in a process compared with more advanced approaches such as the run rules charts, the adaptative charts (i.e., variable sampling interval and variable sampling size), and the exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) charts. Among these more advanced approaches, it is worth to mention the synthetic chart, introduced by Wu and Spedding,[2] that combines the Shewhart $\overline{X}$ chart and the conforming run length (CRL) chart. Under the normal distribution, the synthetic chart provides significantly better detection power than the basic $\overline{X}$ chart for all levels of mean shift and the EWMA and joint $\overline{X}$-EWMA charts for a mean shift δ larger than 0.8. Wu and Spedding[3] also provided a program in C for an optimal design of a synthetic chart. Calzada and Scariano[4] studied the robustness of the synthetic chart to non-normality. Davis and Woodall[5] presented a Markov chain model of the synthetic chart and used it to evaluate the zero-state and steady-state average run length (ARL) performances. Scariano and Calzada[6] worked on a synthetic control chart for exponential data. Sim[7] discussed combined $\overline{X}$ and CRL charts assuming that the quality characteristic follows a gamma distribution. Synthetic charts for process dispersion were proposed by Chen and Huang[8] by merging the sample range, R chart with the CRL chart, and Huang and Chen,[9] by combining the sample standard deviation, S with

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the *CRL* chart. Chen and Huang[10] suggested a synthetic chart as well as its variable sampling interval schemes by integrating the Max and CRL charts. Costa and Rahim[11] proposed a synthetic chart based on the non-central chi-square statistics, where the chart can detect a shift more effectively than the joint $\overline{X}$ and R chart. Bourke[12] re-evaluated the performance of a synthetic chart for detecting increases in fraction nonconforming by comparing it with the *np* chart, CRL-CUSUM chart and RL_2 chart, that is based on the moving sum of two successive CRLs. Khoo *et al*[13] and Castagliola and Khoo[14] proposed synthetic charts for skewed populations based on the weighted variance and scaled weighted variance methods, respectively. Aparisi and de Luna[15] developed a synthetic chart, which does not detect shifts in a region of admissible shifts but detects shifts that are considered important. (Admissible shifts are shifts smaller than a given magnitude that are not desired to be detected or, at least, to delay as far as possible this detection. Contrarily, important shifts are shifts larger than a given magnitude that are desired to be detected.) Scariano and Calzada[16] suggested generalized synthetic charts by considering the EWMA and CUSUM charts. Very recently, Calzada and Scariano[17] suggested a synthetic control chart for monitoring the coefficient of variation, and Zhang *et al*[18] investigated the synthetic $\overline{X}$ chart with estimated parameters. The growing interest in synthetic charts may be explained by the fact that many practitioners prefer waiting until the occurrence of a second point beyond the control limits before looking for an assignable cause.

An indispensable assumption for the development of control charts is that the process parameters are assumed known. In practice, the process parameters are rarely known. The process parameters are usually estimated from an in-control historical data set (phase I). As explained in Chakraborti *et al*.,[19] (…) *obtaining the reference sample is accomplished using an iterative procedure in Phase I in which trial limits are first constructed from the data* (*several subgroups or a number of observations*), *and then some charting statistics are placed on a control chart* (…) *subgroups corresponding to the charting statistics which are located outside the control limits are generally considered OOC* (*Out-of-Control*) *and removed from the analysis*, *and new limits are estimated from the remaining data*. *This step is repeated until no more charting statistics appear OOC so that the remaining data is taken to be IC* (*In-Control*), *at which point the final control limits may be constructed for Phase II monitoring*. In our case, the control chart used during the phase I is either a Shewhart *c* or an *np* chart.

When the parameters are estimated, the performance of the control charts differs from the known parameters case because of the variability of the estimators used during the phase I. Recently, some authors have studied the impact of the estimation of the in-control process parameters on the properties of the corresponding control charts. For a literature review on control charts with estimated parameters, see Jensen *et al*.[20] Most of these authors have focused on location-type (i.e., $\overline{X}$) control charts: Quesenberry,[21] Del Castillo,[22] Chen,[23] Ghosh *et al*.,[24] Jones *et al*.,[25] Nedumaran and Pignatiello, Jr.,[26] Jones,[27] Yang *et al*.,[28] Champ and Jones,[29] Chakraborti,[30] Jensen *et al*.,[31] Zhang and Castagliola,[32] and Zhang *et al*.[18] Some other authors have focused on dispersion-type (i.e., S, S^2, and R) control charts: Hillier,[33] Hawkins,[34] Quesenberry,[35] Chen,[36] Maravelakis *et al*.,[37] Maravelakis and Castagliola,[38] Castagliola *et al*.,[39] and Castagliola and Maravelakis.[40]

Very little work has been done concerning attributes control charts with estimated process parameters, as far as we know, and the only significant contributions were proposed by Braun,[41] Chakraborti and Human[42,43] and Castagliola and Wu.[42] In these papers, the run length properties (i.e. *ARL*, standard deviation of the run length (SDRL), and cumulative distribution function (c.d.f.)) are derived for the *c* and *p* or *np* charts in the case where the parameters are estimated, and they are compared with that obtained in the case where these parameters are known. One of the main conclusions stated in Braun[41] is '… as for variables charts, the effect of estimation on quantities, such as the average run length (*ARL*) can be quite dramatic …'

The goal of this paper is thus to extend the contributions of Braun[41]; Chakraborti and Human,[43,44] and Castagliola and Wu[42] to the case of the *synthetic c* and *np* control charts as follows:

- By deriving the run length properties of the investigated synthetic *c* and *np* charts,
- By providing an analysis concerning the required number *m* of phase I samples in order to have similar in-control ARLs in both the estimated and known parameters cases, and
- By proposing new chart parameters especially dedicated to the number of phase I samples used in practice.

It is worth mentioning that in the case where there is no available phase I sample, an alternative approach is to use the so-called *Q*-charts proposed by Quesenberry,[35] which can be used to monitor processes from start-up and to overcome the problem of having to wait to collect an in-control reference sample. Although this approach seems to be appealing, it nevertheless has some drawbacks: (i) as pointed by Del Castillo,[45] the assumption of a stable process at start-up in the absence of prior process knowledge is unrealistic; (ii) sample points are no longer plotted in the original scale of measurement, and shop floor personnel may prefer to work with plots made in the original familiar scale of measurement; and (iii) by transforming the data into independent and identically distributed $N(0,1)$ Q statistics, the ability to detect process shifts in the transformed data decreases because the transformation hides the shifts. Moreover, the synthetic *c* and *np* charts are not meant to be used in short runs (or low-volume manufacturing environment) but are rather suitable to be implemented in production runs where a reliable in-control phase I dataset is available.

The rest of the paper is structured as follows: the synthetic *c* and *np* charts will be defined in Section 2, and their corresponding run length properties will be presented for the *known* parameters case. In Section 3, the run length properties of the synthetic *c* and *np* charts with *estimated* parameters will be derived. A subsection will be devoted to show how the run length properties in the estimated parameters case are computed in practice. In Section 4, a numerical comparison between the known and the estimated parameters case will be conducted for both the synthetic *c* and *np* charts, providing an insight into the practical number *m* of phase I samples to be used and also providing new chart parameters to be used in the estimated parameters case that give similar in-control ARLs to that of the known parameters case. Then the final conclusions will be summarized in Section 5.

2. The synthetic *c* and *np* charts with known parameters

2.1. *The synthetic c* chart with known parameter c_0

Let $\{Y_{i,1},\ldots,Y_{i,n}\}$, $i=1,2,\ldots$, be a sample of $n\geq 1$ independent random variables $Y_{i,j}$ such that $Y_i=\sum_{j=1}^{n}Y_{i,j}\sim P(c_1)$, that is, a Poisson distribution with parameter c_1, where c_1 is an out-of-control average number of nonconformities. Similar to the synthetic $\overline{X}$ chart developed by Wu and Spedding,[2] the synthetic *c* chart combines a *c* sub-chart designed with two control limits, LCL_c and UCL_c, and a CRL sub-chart having a single lower control limit, $H_c\in\{1,2,\ldots\}$. The control limits, LCL_c and UCL_c, of the *c* sub-chart are

$$LCL_c=\lceil max(0,c_0-K_c\sqrt{c_0})\rceil, \tag{1}$$

$$UCL_c=\lfloor c_0+K_c\sqrt{c_0}\rfloor, \tag{2}$$

where $\lceil\ldots\rceil$ and $\lfloor\ldots\rfloor$ denote the rounded up and rounded down integers, respectively, c_0 is the *known* in-control average number of nonconformities, and $K_c>0$ is a constant. The central line of the *c* sub-chart is $CL_c=c_0$. The CRL is defined as the number of inspected samples between two consecutive nonconforming samples, inclusive of the nonconforming sample at the end. The idea of this chart is that the distribution of CRL will change when the fraction of nonconforming in the *c* sub-chart, $\theta=P(Y_i\notin[LCL_c,UCL_c])$, changes; that is, the *CRL* decreases as θ increases and increases as θ decreases. An example with three CRL samples is displayed in Figure 1, which shows how the CRL value is determined, assuming that a process starts at $t=0$, and the white and black dots represent conforming and nonconforming samples, respectively. Here, $CRL_1=5$, $CRL_2=2$, and $CRL_3=4$. If we assume that we have a sequence $Y_j, Y_{j+1},\ldots,Y_{i-1},Y_i$ where $Y_j\notin[LCL_c,UCL_c]$ and $Y_k\in[LCL_c,UCL_c]$ for $k\in\{j+1,j+2,\ldots,i-1\}$ and $Y_i\notin[LCL_c,UCL_c]$, then the synthetic *c* chart considers the process as out-of-control at time *i* if $CRL=i-j\leq H_c$. This is to say, the synthetic *c* chart signals if and only if *both* the plotting statistics plot outside the control limits *and* the *CRL* is less than or equal to the lower control limit H_c.

Similarly, a synthetic version of the popular *u* chart (for the average number of nonconformities per unit) can also be proposed by assuming $Y_{i,j}\sim P(u_1)$, where u_1 is an out-of-control average number of nonconformities *per unit*, by defining $\overline{Y}_i=\frac{1}{n}\sum_{j=1}^{n}Y_{i,j}$ and by using the following control limits, LCL_u and UCL_u for the *u* sub-chart:

$$LCL_u=max\left(0,u_0-K_u\sqrt{\frac{u_0}{n}}\right), \tag{3}$$

$$UCL_u=u_0+K_u\sqrt{\frac{u_0}{n}}, \tag{4}$$

where u_0 is the *known* in-control average number of nonconformities per unit and $K_u>0$ is a constant.

2.2. *The synthetic np* chart with known parameter p_0

Let $\{Y_{i,1},\ldots,Y_{i,n}\}$, $i=1,2,\ldots$, be a sample of $n\geq 1$ independent random variables $Y_{i,j}\sim B(p_1)$, that is, a Bernoulli random variable of parameter p_1, where p_1 is an out-of-control proportion of nonconforming units. Let $Y_i=\sum_{j=1}^{n}Y_{i,j}$ be the number of nonconforming units in the sample $\{Y_{i,1},\ldots,Y_{i,n}\}$. Similar to the synthetic *c* chart, the synthetic *np* chart combines a *np* sub-chart designed with two control limits, LCL_{np} and UCL_{np}, and a CRL sub-chart having a single lower control limit, $H_{np}\in\{1,2,\ldots\}$. The control limits, LCL_{np} and UCL_{np} of the *np* sub-chart are

$$LCL_{np}=\left\lceil max\left(0,np_0-K_{np}\sqrt{np_0(1-p_0)}\right)\right\rceil, \tag{5}$$

$$UCL_{np}=\left\lfloor np_0+K_{np}\sqrt{np_0(1-p_0)}\right\rfloor, \tag{6}$$

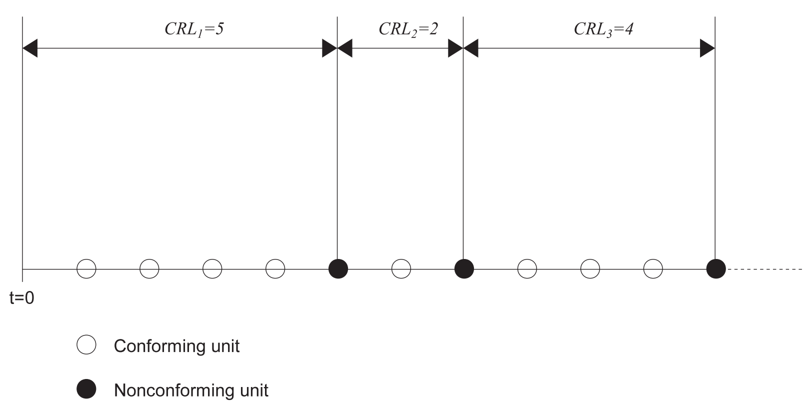

Figure 1. Conforming run length

where p_0 is the *known* in-control proportion of nonconforming units and $K_{np} > 0$ is a constant. The central line of the *np* sub-chart is $CL_{np} = np_0$. If we assume that we have a sequence $Y_j, Y_{j+1}, \ldots, Y_{i-1}, Y_i$ where $Y_j \notin [LCL_{np}, UCL_{np}]$ and $Y_k \in [LCL_{np}, UCL_{np}]$ for $k \in \{j+1, j+2, \ldots, i-1\}$ and $Y_i \notin [LCL_{np}, UCL_{np}]$, then the synthetic *np* chart considers the process as out-of-control at time *i* if $CRL = i - j \leq H_{np}$.

Similarly, a synthetic *p* chart can also be proposed by defining $\overline{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{i,j}$ and by using the following control limits, LCL_p and UCL_p for the *p* sub-chart:

$$LCL_p = max\left(0, p_0 - K_p\sqrt{\frac{p_0(1-p_0)}{n}}\right), \tag{7}$$

$$UCL_p = p_0 + K_p\sqrt{\frac{p_0(1-p_0)}{n}}, \tag{8}$$

where $K_p > 0$ is a constant.

2.3. *Choice of control chart parameters*

For an $\overline{X}$ chart with known process parameters (μ_0,σ_0), having $\mu_0 - K\frac{\sigma_0}{\sqrt{n}}$ and $\mu_0 + K\frac{\sigma_0}{\sqrt{n}}$ as control limits, it is common to choose $K = 3$. This ensures an in-control $ARL_0 = 370.4$. As in Braun,[41] the same choice, that is, $K = 3$, is also commonly used for both the *c* and *np* charts with known parameters c_0 and p_0 (but unfortunately, without ensuring a constant value for ARL_0, because of the discrete nature of the *c* and *np* charts). This choice implicitly assumes that both the Binomial distribution and the Poisson distribution are well approximated by the normal distribution.

Our approach is based on an analog between the synthetic $\overline{X}$ chart and the synthetic *c* and *np* charts. For the synthetic $\overline{X}$ chart with known process parameters (μ_0,σ_0), having $\mu_0 - K\frac{\sigma_0}{\sqrt{n}}$ and $\mu_0 + K\frac{\sigma_0}{\sqrt{n}}$ as control limits for the $\overline{X}$ sub-chart, there is an infinite number of couples (*H*,*K*) that ensures a particular value for ARL_0. It is of course impossible to investigate all these couples. For this reason, we have decided to only select three specific couples, all of them corresponding to $n = 5$ and $ARL_0 = 370.4$. These couples, dedicated to the synthetic $\overline{X}$ chart, have been chosen in order to optimally detect a specific shift δ in the mean:

- the couple ($H = 47, K = 2.639$) is optimal for detecting a shift $\delta = 0.25$, that is, a small shift;
- the couple ($H = 7, K = 2.322$) is optimal for detecting a shift $\delta = 0.75$, that is, a medium shift; and
- the couple ($H = 2, K = 2.085$) is optimal for detecting a shift $\delta = 1.5$, that is, a large shift.

Consequently, in the remainder of this paper, we also suggest to adopt these values as references for the chart parameters (H_c,K_c) and (H_{np},K_{np}) for the synthetic *c* and *np* charts, respectively (for the synthetic *u* and *p* charts, the same values can also be chosen for (H_u,K_u) and (H_p,K_p)).

2.4. *Run length properties*

In this section, we assume that both the in-control average number of nonconformities c_0 and the in-control proportion of nonconforming p_0 are *known*. Let *LCL* and *UCL* be the control limits of either the synthetic *c* sub-chart or the synthetic *np* sub-chart (i.e., *LCL* stands for either LCL_c or LCL_{np}, and *UCL* stands for either UCL_c or UCL_{np}). Let *H* be the lower limit of the CRL sub-chart (i.e., *H* stands for either H_c or H_{np}). In order to obtain the run length properties of the synthetic *c* and *np* charts, the general Markov chain approach proposed by Davis and Woodall[5] for the synthetic $\overline{X}$ chart can be used. As for run rules type charts, this method consists of defining a transition probability matrix $\mathbf{P}_{(H+2,H+2)}$ with exactly $H + 1$ transient states and one absorbing state. Each transient state corresponds to a sequence of *H* samples (i) with no nonconforming sample, (ii) with a nonconforming sample at the first sample, and (iii) with a nonconforming sample at the second sample, …, ($H + 1$) with a nonconforming sample at the *H*th sample. The matrix $\mathbf{P}_{(H+2,H+2)}$ has the following structure:

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} = \left(\begin{array}{cccccc|c} 1-\theta & \theta & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1-\theta & \ddots & & 0 & \theta \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 1-\theta & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1-\theta & \theta \\ 1-\theta & 0 & \cdots & \cdots & \cdots & 0 & \theta \\ \hline 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right), \tag{9}$$

where $\mathbf{0}_{(H+1,1)} = (0,0,\ldots,0)^T$, $\mathbf{Q}_{(H+1,H+1)}$ is the matrix of transient probabilities, vector $\mathbf{r}_{(H+1,1)}$ satisfies $\mathbf{r} = 1 - \mathbf{Q}1$ (i.e., row probabilities of $\mathbf{P}$ must sum to 1) with $1_{(H+1,1)} = (1,1,\ldots,1)^T$ and $\theta = P(Y_i \notin [LCL, UCL])$. More specifically, we have for the synthetic *c* chart

$$\theta = 1 - F_P(UCL_c|c_1) + F_P(LCL_c - 1|c_1), \tag{10}$$

where $F_P(x|\lambda)$ is the c.d.f. of the Poisson distribution with parameter λ, that is,

$$F_P(x|\lambda) = \sum_{z=0}^{x} \frac{e^{-\lambda}\lambda^z}{z!}, \tag{11}$$

and for the synthetic *np* chart,

$$\theta = 1 - F_B\left(UCL_{np}\middle|n, p_1\right) + F_B\left(LCL_{np} - 1|n, p_1\right), \tag{12}$$

where $F_B(x|n,p)$ is the c.d.f. of the binomial distribution with parameters (n,p), that is,

$$F_B(x|n,p) = \sum_{z=0}^{x} \binom{n}{z} p^z (1-p)^{n-z}. \tag{13}$$

Using this Markov chain approach allows one to compute the run length distribution as well as the run length moments of the synthetic *c* and *np* charts. The run length probability mass function (p.m.f) $f_{RL}(\ell)$ and c.d.f. $F_{RL}(\ell)$ of the synthetic *c* and *np* charts with known parameters are defined for $\ell = \{1, 2, \ldots\}$ and are equal to

$$f_{RL}(\ell) = \mathbf{q}^T \mathbf{Q}^{\ell-1} \mathbf{r}, \tag{14}$$

$$F_{RL}(\ell) = 1 - \mathbf{q}^T \mathbf{Q}^{\ell} 1, \tag{15}$$

where the vector $\mathbf{q}_{(H+1,1)}$ of initial probabilities associated with the transient states is equal to $\mathbf{q} = (0,1,0,\ldots,0)^T$; that is, the initial state is the second one. Concerning the run length moments, the ARL and the SDRL of the synthetic *c* and *np* charts are given by

$$ARL = \mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-1} 1, \tag{16}$$

$$SDRL = \sqrt{2\mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-2} \mathbf{Q} 1 - ARL^2 + ARL}. \tag{17}$$

Using direct methods, it can be proven (Calzada and Scariano[17]) that

$$ARL = \frac{1}{\theta\left(1 - (1-\theta)^H\right)}, \tag{18}$$

$$SDRL = \sqrt{\frac{2-\theta}{\left(1-(1-\theta)^H\right)\theta^2} + \frac{\frac{1}{\theta^2} - 2\sum_{k=1}^{H} k(1-\theta)^{k-1}}{\left(1-(1-\theta)^H\right)^2}}, \tag{19}$$

where θ is calculated either from (10) or from (12), for the synthetic *c* and *np* charts, respectively. The run length properties of the synthetic *u* chart can be obtained by replacing LCL_c and UCL_c by LCL_u and UCL_u, respectively, and by replacing c_1 by nu_1 in (10). Concerning the synthetic *p* chart, the run length properties can be obtained by replacing, in (12), LCL_{np} and UCL_{np} by LCL_p and UCL_p, respectively.

3. Synthetic *c* and *np* charts with estimated parameters

3.1. Synthetic *c* chart with estimated parameter c_0

We assume that we have a phase I data set consisting of $i = 1, \ldots, m$ samples $\{X_{i,1}, \ldots, X_{i,n}\}$ of size *n*. Let us assume that there is independence within and between samples, and $X_i = \sum_{j=1}^{n} X_{i,j} \sim P(c_0)$, that is, a Poisson distribution with parameter c_0. An estimator $\hat{c}_0$ of c_0 is

$$\hat{c}_0 = \frac{1}{m}\sum_{i=1}^{m} X_i = \frac{X}{m}, \tag{20}$$

where $X \sim P(mc_0)$, that is, a Poisson distribution with parameter mc_0 defined for $x \in \{0,1,\ldots\}$. When c_0 is estimated by $\hat{c}_0$, the control limits of the synthetic *c* chart with estimated parameter becomes

$$\widehat{LCL} = \left\lceil max\left(0, \hat{c}_0 - K_c\sqrt{\hat{c}_0}\right)\right\rceil, \tag{21}$$

$$\widehat{UCL}_c = \left\lfloor \hat{c}_0 + K_c\sqrt{\hat{c}_0}\right\rfloor \tag{22}$$

Let $\hat{\theta} = P(Y_i \notin [\widehat{LCL}_c, \widehat{UCL}_c] | X = x)$ be the probability that the number Y_i of nonconformities in sample $\{Y_{i,1}, \ldots, Y_{i,n}\}$ is outside the c sub-chart control limits $\widehat{LCL}_c$ and $\widehat{UCL}_c$, conditionally to $X = x$. If we replace $\hat{c}_0$ by $\frac{x}{m}$ in $\widehat{LCL}_c$ and $\widehat{UCL}_c$ and use the condition $X = x$, we have

$$\hat{\theta} = 1 - P\left(Y_i \leq \left\lfloor \frac{x}{m} + K_c\sqrt{\frac{x}{m}} \right\rfloor\right) + P\left(Y_i \leq max\left(0, \left\lceil \frac{x}{m} - K_c\sqrt{\frac{x}{m}} \right\rceil\right) - 1\right), \tag{23}$$

and because $Y_i \sim P(c_1)$, we have

$$\hat{\theta} = 1 - F_P\left(\left\lfloor \frac{x}{m} + K_c\sqrt{\frac{x}{m}} \right\rfloor \Big| c_1\right) + F_P\left(max\left(0, \left\lceil \frac{x}{m} - K_c\sqrt{\frac{x}{m}} \right\rceil\right) - 1 | c_1\right). \tag{24}$$

As $X \sim P(mc_0)$ is defined for $x = \{0,1,\ldots\}$, the (unconditional) run length p.m.f. $f_{RL}(\ell)$ and c.d.f. $F_{RL}(\ell)$ of the synthetic c chart with *estimated* parameter c_0 are equal to

$$f_{RL}(\ell) = \sum_{x=0}^{\infty}\left(\left(\mathbf{q}^T\hat{\mathbf{Q}}^{\ell-1}\hat{\mathbf{r}}\right) \times f_P(x|mc_0)\right), \tag{25}$$

$$F_{RL}(\ell) = 1 - \sum_{x=0}^{\infty}\left(\left(\mathbf{q}^T\hat{\mathbf{Q}}^{\ell}1\right) \times f_P(x|mc_0)\right), \tag{26}$$

where $f_P(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$ is the p.m.f. of the Poisson distribution with parameter λ and where $\hat{\mathbf{Q}}$ and $\hat{\mathbf{r}}$ are matrix $\mathbf{Q}$ and vector $\mathbf{r}$ in (9), respectively, in which θ is replaced by $\hat{\theta}$ in (24). The (unconditional) ARL of the synthetic c chart with *estimated* parameter c_0 is equal to

$$ARL = \sum_{x=0}^{\infty}\left(\widehat{ARL} \times f_P(x|mc_0)\right), \tag{27}$$

where $\widehat{ARL}$ is defined as in (18), in which θ is replaced by $\hat{\theta}$ in (24). Finally, the (unconditional) SDRL of the synthetic c chart with *estimated* parameter c_0 is equal to

$$SDRL = \sqrt{\sum_{x=0}^{\infty}\left(\left(\widehat{SDRL}^2 + \widehat{ARL}^2\right) \times f_P(x|mc_0)\right) - ARL^2}, \tag{28}$$

where $\widehat{SDRL}$ is defined as in (19), in which θ is replaced by $\hat{\theta}$ in (24). It can be easily shown that all the previous equations also hold for the synthetic u chart (as defined in (3) and (4)) with estimated parameters with the following differences: the constant c_1 in (24) must be replaced by nu_1, and the term mc_0 in $f_P(x|mc_0)$ must be replaced by mnu_0.

3.2. *Synthetic np* chart with estimated parameter p_0

We assume that we have a phase I data set composed of $i = 1, \ldots, m$ samples $\{X_{i,1}, \ldots, X_{i,n}\}$ of size n. Let us assume that there is independence within and between samples, and $X_{i,j} \sim B(p_0)$, that is, a Bernoulli random variable of parameter p_0. An unbiased estimator $\hat{p}_0$ of p_0 is

$$\hat{p}_0 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{i,j} = \frac{X}{mn}$$

where $X \sim B(mn, p_0)$, that is, a binomial random variable of parameters (mn, p_0) defined for $x \in \{0, 1, \ldots, mn-1, mn\}$. When p_0 is estimated by $\hat{p}_0$, the control limits of the synthetic np sub-chart becomes

$$\widehat{LCL}_{np} = \left\lceil max\left(0, n\hat{p}_0 - K_{np}\sqrt{n\hat{p}_0(1-\hat{p}_0)}\right)\right\rceil$$
$$\widehat{UCL}_{np} = \left\lfloor n\hat{p}_0 + K_{np}\sqrt{n\hat{p}_0(1-\hat{p}_0)}\right\rfloor$$

Similar to the synthetic c chart, let $\hat{\theta} = P(Y_i \notin [\widehat{LCL}_{np}, \widehat{UCL}_{np}] | X = x)$ be the probability that the number Y_i of nonconforming units in sample $\{Y_{i,1}, \ldots, Y_{i,n}\}$ is outside the np sub-chart control limits $\widehat{LCL}_{np}$ and $\widehat{UCL}_{np}$, conditionally to $X = x$. If we replace $\hat{p}_0$ by $\frac{x}{mn}$ in $\widehat{LCL}_{np}$ and $\widehat{UCL}_{np}$ and use the condition $X = x$, we have

$$\hat{\theta} = 1 - P\left(Y_i \le \left\lfloor \frac{x}{m} + K_{np}\sqrt{\frac{x}{m}\left(1-\frac{x}{mn}\right)} \right\rfloor \right) + P\left(Y_i \le max\left(0, \left\lceil \frac{x}{m} - K_{np}\sqrt{\frac{x}{m}\left(1-\frac{x}{mn}\right)} \right\rceil \right) - 1\right), \tag{29}$$

and because $Y_i \sim B(n,p_1)$, we have

$$\hat{\theta} = 1 - F_B\left(\left\lfloor \frac{x}{m} + K_{np}\sqrt{\frac{x}{m}\left(1-\frac{x}{mn}\right)} \right\rfloor \middle| n, p_1\right) + F_B\left(max\left(0, \left\lceil \frac{x}{m} - K_{np}\sqrt{\frac{x}{m}\left(1-\frac{x}{mn}\right)} \right\rceil \right) - 1 | n, p_1\right). \tag{30}$$

As $X \sim B(mn,p_0)$ is defined for $x = \{0, 1, \ldots, mn-1, mn\}$, the (unconditional) run length p.m.f. $f_{RL}(\ell)$ and c.d.f. $F_{RL}(\ell)$ of the synthetic *np* chart with *estimated* parameter p_0 are equal to

$$f_{RL}(\ell) = \sum_{x=0}^{mn}\left(\left(\mathbf{q}^T\hat{\mathbf{Q}}^{\ell-1}\hat{\mathbf{r}}\right) \times f_B(x|mn, p_0)\right), \tag{31}$$

$$F_{RL}(\ell) = 1 - \sum_{x=0}^{mn}\left(\left(\mathbf{q}^T\hat{\mathbf{Q}}^{\ell}\mathbf{1}\right) \times f_B(x|mn, p_0)\right), \tag{32}$$

where $f_B(x|n,p) = \binom{n}{x}p^x(1-p)^{n-x}$ is the p.m.f. of the binomial distribution with parameters (n,p). The (unconditional) ARL of the synthetic *np* chart with *estimated* parameter p_0 is equal to

$$ARL = \sum_{x=0}^{mn}\left(\widehat{ARL} \times f_B(x|mn, p_0)\right), \tag{33}$$

and the (unconditional) SDRL of the synthetic *np* chart with *estimated* parameter p_0 is equal to

$$SDRL = \sqrt{\sum_{x=0}^{mn}\left(\left(\widehat{SDRL}^2 + \widehat{ARL}^2\right) \times f_B(x|mn, p_0)\right) - ARL^2}. \tag{34}$$

It can be easily shown that all the previous equations also hold for the synthetic *p* chart (as defined in (7) and (8)) with estimated parameters.

3.3. Computational note

The computation of the p.m.f, c.d.f., ARL, and SDRL of both the synthetic *c* and *np* charts with estimated parameters involves summations over domains that can be very large (depending on *m* and *n* for the synthetic *np* chart) and can be even infinite (for the synthetic *c* chart), see (25)–(28) and (31)–(34), respectively. The time required for the computation of these summations can be greatly reduced (while keeping a high accuracy) by noticing that the terms $f_P(x|mc_0)$ (for the synthetic *c* chart) and $f_B(x|mn, p_0)$ (for the synthetic *np* chart) reach a maximum close to mc_0 (for the synthetic *c* chart) and mnp_0 (for the synthetic *np* chart) and both converge to zero when they diverge from mc_0 (for the synthetic *c* chart) and from mnp_0 (for the synthetic *np* chart). Consequently, we suggest replacing these summations over $\{0,1,\ldots,\infty\}$ (for the synthetic *c* chart) and over $\{0,1,\ldots,mn\}$ (for the synthetic *np* chart) by summations over $\{x_{min},x_{max}\}$ where,

- for the synthetic *c* chart: $\begin{cases} x_{min} = max\left(0, \left\lfloor mc_0 - a\sqrt{mc_0}\right\rfloor\right) \\ x_{max} = \left\lceil mc_0 + a\sqrt{mc_0}\right\rceil \end{cases}$; and
- for the synthetic *np* chart: $\begin{cases} x_{min} = max\left(0, \left\lfloor mnp_0 - a\sqrt{mnp_0(1-p_0)}\right\rfloor\right) \\ x_{max} = \left\lceil mnp_0 + a\sqrt{mnp_0(1-p_0)}\right\rceil \end{cases}$

Here, a is a constant ensuring that the terms $f_P(x_{min}|mc_0)$ and $f_P(x_{max}|mc_0)$ (for the synthetic *c* chart) and $f_B(x_{min}|mn, p_0)$ and $f_B(x_{max}|mn, p_0)$ (for the synthetic *np* chart) are close enough to zero (in practice, these values are less than 10^{-20}). Simulations have shown that using $a = 10$ yields excellent results. The value $a = 10$ will be used in the rest of the paper.

4. Performance analysis

4.1. Synthetic c chart

Because the in-control case is always the *worst case* when evaluating control charts with estimated parameters (i.e., the larger the magnitude of the out-of-control condition, the more similar are the results between the known and the estimated parameters cases), we will only confine our paper to the in-control case. In Tables I–III, we present the in-control ARL and SDRL for the synthetic *c* chart, for $c_0=\{5,10,\ldots,100\}$ and $m=\{10,20,50,100,200,\infty\}$ (results for the *u* chart can be deduced from those of the *c* chart by assuming that $c_0=nu_0$). The value $m=\infty$ corresponds to the known parameter case, and the other *m* values correspond to the estimated parameters case. The selected control chart parameters (H_c,K_c) are defined as in Section 2.3, that is, $(H_c=2,K_c=2.085)$ in Table I, $(H_c=7,K_c=2.322)$ in Table II, and $(H_c=47,K_c=2.639)$ in Table III. Several interesting remarks can be drawn from the analysis of these tables:

Table I. In-control ARL and SDRL values for $(H_c=2,\ K_c=2.085)$, $c_0=\{5,\ 10,\ \ldots,\ 100\}$ and $m=\{10,\ 20,\ 50,\ 100,\ 200,\ \infty\}$

c_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$	$m=\infty$
5	(608.8, 1180.0)	(613.4, 1084.9)	(529.4, 800.1)	(464.6, 663.7)	(404.3, 535.8)	(342.8, 365.9)
10	(361.7, 513.5)	(404.1, 550.9)	(434.2, 559.5)	(425.8, 527.1)	(399.8, 467.8)	(364.7, 388.7)
15	(332.7, 438.3)	(370.4, 463.1)	(387.4, 460.6)	(398.6, 468.8)	(417.8, 493.6)	(690.3, 724.4)
20	(315.3, 401.0)	(350.6, 418.2)	(378.4, 432.6)	(396.6, 446.6)	(419.0, 465.6)	(477.4, 505.2)
25	(302.3, 381.4)	(337.9, 397.8)	(366.9, 415.7)	(382.3, 426.2)	(395.5, 435.4)	(418.7, 444.6)
30	(299.6, 374.2)	(334.0, 390.9)	(360.5, 402.3)	(375.1, 413.3)	(386.9, 422.7)	(414.5, 440.2)
35	(295.5, 363.7)	(329.5, 383.9)	(360.7, 403.9)	(373.9, 413.0)	(385.8, 423.7)	(439.7, 466.3)
40	(295.4, 362.9)	(329.2, 382.7)	(359.7, 404.0)	(373.0, 414.3)	(382.9, 423.7)	(487.0, 515.1)
45	(287.4, 350.0)	(324.3, 373.4)	(354.5, 392.7)	(365.7, 398.3)	(370.1, 398.5)	(269.1, 289.3)
50	(287.0, 348.6)	(321.9, 370.5)	(354.2, 394.1)	(368.2, 405.3)	(375.1, 410.8)	(321.8, 344.1)
55	(285.4, 345.7)	(321.0, 367.0)	(351.4, 387.4)	(364.1, 396.2)	(372.1, 402.6)	(388.3, 413.2)
60	(284.8, 345.3)	(320.8, 367.8)	(351.3, 389.1)	(365.8, 401.0)	(373.4, 407.6)	(471.7, 499.4)
65	(284.4, 343.0)	(319.2, 364.4)	(350.9, 386.8)	(364.7, 397.8)	(372.1, 404.1)	(313.9, 335.9)
70	(282.9, 341.8)	(318.0, 362.2)	(349.2, 382.9)	(361.6, 391.5)	(369.5, 397.7)	(389.6, 414.4)
75	(282.9, 340.1)	(318.9, 364.0)	(349.3, 384.1)	(362.5, 394.1)	(369.4, 399.3)	(484.1, 512.2)
80	(279.2, 335.8)	(316.7, 360.4)	(349.3, 384.5)	(362.7, 395.2)	(370.4, 401.8)	(347.3, 370.6)
85	(281.4, 339.0)	(317.5, 361.5)	(348.7, 383.4)	(362.8, 395.0)	(370.6, 401.9)	(435.4, 461.9)
90	(282.0, 337.6)	(317.3, 360.0)	(348.1, 381.8)	(362.0, 393.2)	(369.7, 400.1)	(325.9, 348.4)
95	(276.6, 330.5)	(313.9, 355.8)	(347.1, 380.1)	(361.2, 391.6)	(369.1, 398.6)	(410.4, 436.0)
100	(277.7, 332.5)	(315.0, 356.8)	(346.5, 379.1)	(360.3, 390.0)	(368.0, 396.8)	(317.0, 339.1)

Table II. In-control ARL and SDRL values for $(H_c=7,\ K_c=2.322)$, $c_0=\{5,\ 10,\ \ldots,\ 100\}$ and $m=\{10,\ 20,\ 50,\ 100,\ 200,\ \infty\}$

c_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$	$m=\infty$
5	(549.3, 906.0)	(531.5, 793.3)	(557.1, 761.3)	(597.1, 778.2)	(656.3, 809.0)	(793.5, 859.4)
10	(401.8, 627.9)	(414.0, 583.0)	(432.3, 569.3)	(460.7, 580.9)	(480.2, 569.1)	(517.3, 569.0)
15	(342.3, 498.6)	(368.4, 498.6)	(376.6, 468.6)	(370.0, 434.7)	(364.8, 418.6)	(211.0, 241.3)
20	(327.8, 459.4)	(355.1, 460.1)	(373.4, 452.3)	(390.7, 464.3)	(407.8, 474.1)	(442.5, 489.7)
25	(299.3, 406.7)	(339.2, 436.7)	(370.5, 457.1)	(381.4, 461.4)	(380.4, 451.9)	(344.3, 385.0)
30	(299.0, 400.4)	(334.2, 422.1)	(365.1, 440.8)	(376.1, 446.3)	(376.1, 442.0)	(312.7, 351.2)
35	(294.2, 390.5)	(329.6, 412.3)	(360.1, 430.6)	(373.3, 439.0)	(375.8, 438.0)	(310.8, 349.2)
40	(288.7, 381.4)	(324.5, 403.2)	(355.5, 423.7)	(369.9, 433.9)	(374.7, 435.6)	(326.9, 366.5)
45	(283.5, 371.6)	(319.3, 394.4)	(352.4, 416.7)	(365.9, 425.8)	(373.2, 430.4)	(357.1, 398.8)
50	(284.6, 372.2)	(320.9, 393.1)	(350.6, 410.2)	(363.1, 416.5)	(371.2, 421.5)	(400.6, 445.1)
55	(280.1, 363.5)	(317.8, 388.7)	(350.9, 412.3)	(365.6, 423.3)	(374.4, 430.6)	(458.0, 506.2)
60	(278.9, 361.3)	(316.1, 385.2)	(347.3, 404.5)	(360.4, 412.6)	(366.9, 414.9)	(274.1, 309.6)
65	(279.4, 361.5)	(315.7, 384.3)	(348.4, 407.0)	(362.8, 418.2)	(370.9, 424.7)	(327.1, 366.7)
70	(275.5, 354.3)	(313.2, 379.1)	(344.7, 399.5)	(359.4, 409.6)	(367.5, 415.2)	(392.9, 436.9)
75	(274.5, 353.8)	(312.0, 378.3)	(345.2, 402.0)	(360.3, 413.5)	(368.8, 420.3)	(474.2, 523.4)
80	(273.5, 351.4)	(310.9, 376.0)	(346.0, 401.6)	(360.7, 413.2)	(369.1, 420.1)	(321.8, 360.9)
85	(273.3, 350.1)	(311.0, 375.2)	(344.5, 398.2)	(358.5, 408.2)	(366.6, 414.2)	(394.6, 438.8)
90	(274.2, 352.0)	(310.2, 373.9)	(343.6, 397.4)	(358.3, 408.3)	(366.0, 413.7)	(484.8, 534.6)
95	(271.3, 347.1)	(310.6, 374.6)	(343.7, 398.1)	(358.8, 410.1)	(367.1, 416.8)	(349.3, 390.4)
100	(271.5, 348.0)	(308.8, 371.9)	(343.2, 397.0)	(358.4, 409.3)	(367.4, 417.1)	(432.5, 479.1)

Table III. In-control ARL and SDRL values for ($H_c = 47$, $K_c = 2.639$), $c_0 = \{5, 10, \dots, 100\}$ and $m = \{10, 20, 50, 100, 200, \infty\}$

c_0	$m = 10$	$m = 20$	$m = 50$	$m = 100$	$m = 200$	$m = \infty$
5	(2307.7, 10789.4)	(1044.7, 5088.3)	(492.5, 1235.4)	(409.2, 738.0)	(379.2, 661.9)	(153.1, 197.6)
10	(463.3, 867.7)	(426.8, 749.8)	(419.3, 684.2)	(410.8, 628.7)	(404.2, 557.6)	(427.8, 524.0)
15	(382.7, 651.1)	(391.3, 622.3)	(380.7, 550.2)	(384.4, 530.3)	(401.8, 538.2)	(502.5, 609.9)
20	(340.7, 551.7)	(361.4, 558.5)	(373.9, 544.7)	(367.5, 512.2)	(353.7, 479.3)	(257.3, 324.3)
25	(318.7, 500.4)	(351.0, 520.1)	(364.7, 508.3)	(369.8, 498.1)	(379.4, 501.2)	(493.3, 599.4)
30	(301.7, 461.3)	(337.4, 487.9)	(358.1, 492.2)	(366.1, 490.3)	(373.1, 489.1)	(378.7, 467.1)
35	(295.6, 443.6)	(329.7, 469.4)	(355.4, 483.5)	(363.1, 483.3)	(365.4, 478.9)	(330.3, 410.6)
40	(289.8, 433.7)	(323.4, 455.9)	(351.1, 471.9)	(361.6, 475.8)	(364.4, 473.6)	(312.7, 389.8)
45	(285.6, 423.8)	(321.5, 449.7)	(348.3, 464.3)	(360.2, 470.9)	(364.6, 471.0)	(312.8, 390.0)
50	(282.7, 418.7)	(318.0, 442.3)	(345.9, 459.7)	(358.8, 467.6)	(364.2, 469.0)	(325.5, 404.9)
55	(277.9, 407.7)	(312.8, 432.8)	(344.0, 454.7)	(356.3, 461.9)	(362.9, 464.9)	(348.6, 432.1)
60	(277.8, 406.5)	(311.5, 428.4)	(342.5, 449.2)	(354.8, 455.4)	(361.5, 458.0)	(381.7, 470.6)
65	(275.8, 402.1)	(310.1, 424.8)	(341.3, 446.8)	(355.6, 456.9)	(363.8, 462.7)	(425.2, 521.0)
70	(271.9, 394.7)	(309.3, 423.5)	(341.8, 447.5)	(354.8, 455.8)	(361.4, 458.8)	(480.2, 584.4)
75	(272.8, 395.4)	(308.2, 419.9)	(340.4, 442.2)	(353.5, 450.8)	(361.2, 456.2)	(297.1, 371.5)
80	(270.2, 391.1)	(307.2, 418.6)	(339.8, 442.6)	(353.7, 452.7)	(361.5, 458.1)	(346.1, 429.0)
85	(268.9, 387.6)	(306.3, 415.8)	(339.0, 439.6)	(353.3, 450.3)	(361.6, 456.4)	(405.7, 498.5)
90	(269.0, 388.1)	(306.0, 415.1)	(338.9, 439.5)	(353.0, 449.8)	(360.2, 453.9)	(478.4, 582.2)
95	(266.3, 382.8)	(303.9, 411.2)	(337.6, 437.3)	(353.3, 450.2)	(361.8, 457.2)	(326.8, 406.5)
100	(266.3, 382.5)	(303.5, 410.2)	(338.0, 436.3)	(352.2, 446.8)	(360.3, 452.5)	(390.6, 480.9)

Table IV. Minimum values m^* of m, for $c_0 = \{5, 10, \dots, 100\}$, ($H_c = 2$, $K_c = 2.085$), ($H_c = 7$, $K_c = 2.322$), and ($H_c = 47$, $K_c = 2.639$) for the Shewhart and synthetic c charts, satisfying $\Delta = \frac{\left|ARL_{0,m^*} - ARL_{0,\infty}\right|}{ARL_{0,\infty}} < 0.05$

c_0	$H_c = 2$, $K_c = 2.085$	$H_c = 7$, $K_c = 2.322$	$H_c = 47$, $K_c = 2.639$	Shewhart
5	410	740	6650	590
10	10	310	20	150
15	8110	$>$10,000	1100	470
20	510	340	2420	10
25	230	20	1890	$>$10,000
30	290	10	70	30
35	680	1520	20	10
40	2470	20	20	$>$10,000
45	$>$10,000	40	20	4770
50	20	420	20	1710
55	150	2290	40	1310
60	4930	10	230	1490
65	20	20	1680	2510
70	230	390	$>$10,000	9070
75	$>$10,000	$>$10,000	20	10
80	30	20	40	20
85	2750	550	1310	30
90	20	$>$10,000	$>$10,000	80
95	1340	40	30	1740
100	20	3290	770	$>$10,000

- Because of the discrete nature of the synthetic c chart, it is impossible to have a common in-control ARL_0 value in the case where $m = \infty$ (i.e., known parameter case) for all the values of c_0 considered. The in-control ARL values are sometimes smaller and sometimes larger than 370.4 (see the rightmost column of Tables I–III). This is a well known drawback of attributes charts in general.
- For a particular value of c_0, the in-control ARL values can be very different in the known and in the estimated parameter case. For instance, in Table I, if $c_0 = 5$, the in-control ARL is $ARL_0 = 342.8$ ($SDRL_0 = 365.9$) when $m = \infty$, whereas when $m = 10$, we have $ARL_0 = 608.8$

($SDRL_0=1180.0$). When m increases, the difference in terms of in-control ARLs, between the known and the estimated parameter case, tends to decrease but not always in a monotonic way (unlike $\overline{X}$-type charts where this difference always reduces monotonically with m).

- For $m<\infty$, depending on the value of c_0, the in-control ARL values are either smaller or larger than the in-control ARL values corresponding to $m=\infty$. In the previous example with $c_0=5$, the in-control ARL values corresponding to $m<\infty$ are all larger than the in-control ARL value ($ARL_0=342.8$) corresponding to $m=\infty$. But, for instance, in Table I, case $c_0=20$, the corresponding in-control *ARL* values (315.3, 350.6, 378.4, 396.6, and 419.0) are *smaller* than the in-control ARL value ($ARL_0=477.4$) corresponding to $m=\infty$. This result (already emphasized in Braun[41]) is quite different from what can be observed in the case of $\overline{X}$-type charts where the in-control *ARL* values corresponding to $m<\infty$ are always larger than the in-control ARL value corresponding to $m=\infty$.

Because the in-control ARL values are different in the *known* and in the *estimated* parameter case, an interesting question is how large the number m of phase I samples should be in order to have approximatively the same in-control ARL values in both the known and estimated parameter cases if we keep, as in Tables I–III, the same control chart parameters ($H_c=2, K_c=2.085$), ($H_c=7, K_c=2.322$) and ($H_c=47, K_c=2.639$)? In Table IV, we have computed, for $c_0=\{5,10,\ldots,100\}$, the minimum values m^* of m satisfying $\Delta=\frac{|ARL_{0,m^*}-ARL_{0,\infty}|}{ARL_{0,\infty}}<0.05$, that is, such that the relative difference between the in-control ARL_{0,m^*} (estimated parameter case) and the in-control $ARL_{0,\infty}$ (known parameter case) is not larger than 5%. As it can be noticed in Table IV:

- Depending on the value of c_0, the value of m^* satisfying $\Delta<0.05$ can be very large and, in some cases, larger than 10,000.
- When c_0 increases, there is no particular trend for m^*. For example, in the case ($H_c=2, K_c=2.085$), if $c_0=5$, then we need at least $m^*=410$ samples in order to have $\Delta<0.05$. This value decreases down to $m^*=10$ if $c_0=10$ and then increases to $m^*=8110$ if $c_0=15$. This phenomenon has already been emphasized in the case of the c chart in Castagliola and Wu[42] and is due to the discrete nature of the control limits.
- For a particular value c_0, the selected chart parameters (H_c,K_c) have no influence on m^*. For example, if $c_0=5$, the value $m^*=410$ for ($H_c=2, K_c=2.085$) is smaller than the value $m^*=740$ for ($H_c=7, K_c=2.322$), but if $c_0=20$, this is the opposite with $m^*=510$ for ($H_c=2, K_c=2.085$), which is larger than $m^*=340$ for ($H_c=7, K_c=2.322$).

As presented in Castagliola and Wu,[42] for the same values of $c_0=\{5,10,\ldots,100\}$, the rightmost column of Table IV lists the minimum values of m^* for the Shewhart c chart. As it can be noticed, there is no particular trend concerning which control chart requires a smaller number of samples. For instance, if $c_0=25$, then depending on the choice of (H_c,K_c), the values of m^* for the synthetic c chart are 230, 20, and 1890, whereas for the Shewhart c chart, this value is larger than 10,000. On the other hand, if $c_0=85$, then depending on the choice of (H_c,K_c), the values of m^* for the synthetic c chart are 2750, 550, and 1310, whereas for the Shewhart c chart, this value reduces to 30.

In conclusion, if we want to use the control chart parameters ($H_c=2, K_c=2.085$), ($H_c=7, K_c=2.322$), and ($H_c=47, K_c=2.639$) for the synthetic c chart with estimated parameter c_0, having run length performances close to that of the synthetic c chart with known parameter, then we must pay the price to obtain a large number m of phase I samples, sometimes larger than 10,000. But from a practical point of view, obtaining so many samples could be hard to handle in practice (i.e., completing the phase I chart implementation can take too long a time) and can be very costly. In order to relax this constraint on the number of phase I samples, we suggest to compute alternative chart parameters (H'_c,K'_c), which take the value of m into account and allow the in-control ARL value corresponding to the estimated parameter case to be *as close as possible* to the in-control ARL value corresponding to the known parameter case.

These values are in Tables V–VII for $c_0=\{5,10,\ldots,100\}$ and for $m=\{10,20,50,100,200\}$. For example, in Table V, if $c_0=5$ and $m=10$, the chart parameters are ($H'_c=84, K'_c=2.49$), and in this case, the in-control ARL is $ARL_0=342.8$ ($SDRL_0=1102.2$). Referring to Table I, the in-control ARL corresponding to $m=\infty$ (i.e., the known parameter case) is also $ARL_0=342.8$. Consequently, the use of the new chart parameters ($H'_c=84, K'_c=2.49$) instead of ($H_c=2, K_c=2.085$) allows to obtain the same in-control ARL as for the known parameter case.

4.2. Synthetic *np* chart

Tables VIII–X are the counterparts of Tables I–III but for the synthetic *np* charts. In Table VIII, we present the in-control ARL and SDRL for the synthetic *np* chart for $n=\{25,50,75,100\}$, $p_0=\{0.01,0.02,0.05,0.1,0.15,0.2\}$, $m=\{10,20,50,100,200,\infty\}$, and chart parameters ($H_{np}=2, K_{np}=2.085$), whereas in Tables IX and X, the chart parameters are ($H_{np}=7, K_{np}=2.322$) and ($H_{np}=47, K_{np}=2.639$), respectively. Conclusions for Tables VIII–X are similar to those for Tables I–III; that is,

- There is no common in-control ARL value in the known parameter case for all the values of n and p_0 considered;
- The in-control ARL values can be very different in the known versus estimated parameter case; and
- Depending on the values of n and p_0, the in-control ARL values for $m<\infty$ are either smaller or larger than the in-control ARL values corresponding to $m=\infty$.

A simple example illustrating these remarks can be given, for instance, for the synthetic *np* chart with ($H_{np}=2, K_{np}=2.085$), see Table VIII, when $n=75$ and $p_0=0.05$. For these values, the in-control ARL is $ARL_0=449.7$ ($SDRL_0=476.6$) when $m=\infty$ and $ARL_0=714.5$ ($SDRL_0=1521.3$) when $m=10$. In this example, the in-control ARL corresponding to the estimated parameter case is *larger* than the one corresponding to the known parameter case. But if we choose a second example with $n=75$, $p_0=0.15$, the in-control ARL is $ARL_0=411.5$ ($SDRL_0=437.2$) when $m=\infty$ and $ARL_0=345.8$ ($SDRL_0=470.2$) when $m=10$. In this second example, the in-control ARL corresponding to the estimated parameter case is *smaller* than the one corresponding to the known parameter case.

Table V. Chart parameters (K'_c, H'_c) and in-control $(ARL_0, SDRL_0)$, for $c_0 = \{5, 10, \ldots, 100\}$ and $m = \{10, 20, 50, 100, 200\}$, corresponding to known parameter case ($H_c = 2$, $K_c = 2.085$)

	$(H'_c, K'_c, ARL_0, SDRL_0)$				
c_0	$m = 10$	$m = 20$	$m = 50$	$m = 100$	$m = 200$
5	(84, 2.49, 342.8, 1102.2)	(69, 2.51, 342.8, 1053.8)	(87, 2.67, 342.9, 833.9)	(4, 2.18, 342.4, 477.4)	(95, 2.74, 342.9, 514.3)
10	(39, 2.57, 364.7, 696.7)	(49, 2.62, 364.6, 600.5)	(63, 2.65, 364.7, 624.0)	(92, 2.73, 364.7, 607.6)	(45, 2.60, 364.6, 492.4)
15	(24, 2.64, 690.5, 1140.2)	(25, 2.65, 690.4, 1059.1)	(82, 2.84, 690.2, 1136.7)	(95, 2.87, 690.3, 1025.4)	(70, 2.84, 690.7, 940.0)
20	(36, 2.66, 477.2, 770.9)	(70, 2.75, 477.4, 749.3)	(44, 2.68, 477.6, 682.9)	(66, 2.74, 477.2, 655.5)	(60, 2.72, 477.4, 644.2)
25	(55, 2.71, 418.7, 666.5)	(68, 2.73, 418.8, 631.4)	(55, 2.69, 418.7, 593.7)	(22, 2.54, 418.7, 540.0)	(75, 2.74, 418.5, 575.2)
30	(40, 2.67, 414.2, 624.9)	(30, 2.61, 414.3, 580.7)	(15, 2.48, 414.6, 528.6)	(67, 2.72, 414.4, 564.4)	(62, 2.71, 414.5, 548.7)
35	(94, 2.81, 439.6, 691.7)	(97, 2.80, 439.8, 655.7)	(92, 2.78, 439.7, 620.9)	(54, 2.70, 439.7, 579.5)	(17, 2.51, 439.7, 530.2)
40	(98, 2.84, 487.0, 768.8)	(60, 2.75, 486.9, 695.7)	(32, 2.64, 486.9, 632.9)	(94, 2.80, 486.9, 662.3)	(64, 2.74, 486.9, 627.8)
45	(62, 2.67, 269.2, 405.8)	(84, 2.69, 269.3, 390.4)	(6, 2.24, 269.1, 316.9)	(2, 2.02, 269.0, 297.2)	(3, 2.04, 249.1, 276.6)
50	(43, 2.65, 321.8, 470.5)	(86, 2.73, 321.5, 463.1)	(9, 2.35, 321.9, 385.8)	(14, 2.42, 321.8, 390.2)	(89, 2.71, 321.8, 430.3)
55	(44, 2.69, 388.2, 566.6)	(22, 2.56, 388.3, 508.6)	(91, 2.76, 388.2, 531.6)	(18, 2.50, 388.2, 471.8)	(28, 2.57, 388.4, 478.4)
60	(51, 2.75, 471.7, 687.2)	(7, 2.40, 471.7, 572.2)	(27, 2.61, 471.7, 592.3)	(41, 2.67, 471.9, 598.3)	(20, 2.55, 471.3, 565.2)
65	(19, 2.52, 313.9, 430.2)	(93, 2.74, 313.9, 448.0)	(74, 2.69, 314.0, 423.9)	(17, 2.45, 313.8, 381.2)	(28, 2.53, 313.9, 386.7)
70	(20, 2.57, 389.6, 532.5)	(74, 2.75, 389.9, 545.4)	(9, 2.39, 389.5, 457.0)	(2, 2.10, 389.6, 423.2)	(64, 2.70, 389.6, 502.7)
75	(5, 2.37, 483.9, 610.9)	(16, 2.55, 484.2, 611.5)	(89, 2.80, 484.1, 647.7)	(37, 2.66, 484.2, 602.6)	(18, 2.54, 484.1, 567.3)
80	(37, 2.65, 347.3, 493.0)	(69, 2.72, 347.6, 481.5)	(60, 2.68, 347.2, 457.7)	(33, 2.58, 347.2, 434.9)	(41, 2.61, 347.2, 435.7)
85	(25, 2.63, 435.3, 599.6)	(19, 2.56, 435.5, 553.9)	(29, 2.61, 435.5, 543.6)	(62, 2.72, 435.1, 560.6)	(73, 2.74, 435.5, 560.4)
90	(48, 2.68, 325.6, 468.0)	(53, 2.67, 325.9, 443.3)	(41, 2.61, 325.5, 419.0)	(56, 2.65, 325.8, 421.7)	(2, 2.06, 325.9, 352.0)
95	(47, 2.72, 410.4, 584.2)	(33, 2.64, 410.8, 539.2)	(31, 2.61, 410.4, 512.6)	(37, 2.63, 410.3, 512.2)	(49, 2.67, 410.4, 514.2)
100	(57, 2.70, 317.0, 460.3)	(20, 2.51, 317.1, 406.2)	(24, 2.52, 317.0, 394.6)	(39, 2.59, 317.0, 399.5)	(26, 2.52, 317.0, 388.9)

Table VI. Chart parameters (K'_c,H'_c) and in-control (ARL_0, $SDRL_0$), for c_0 = {5, 10, …, 100} and m = {10, 20, 50, 100, 200}, corresponding to known parameter case (H_c = 7, K_c = 2.322)

c_0	(H'_c, K'_c, ARL_0, $SDRL_0$)				
	m = 10	m = 20	m = 50	m = 100	m = 200
5	(3, 2.22, 793.4, 1522.9)	(30, 2.54, 793.7, 2538.1)	(11, 2.44, 793.6, 1602.1)	(12, 2.51, 793.6, 1502.5)	(36, 2.75, 793.6, 1096.6)
10	(61, 2.70, 517.3, 948.4)	(11, 2.43, 516.5, 843.7)	(56, 2.70, 517.0, 879.2)	(69, 2.76, 517.2, 851.8)	(86, 2.82, 517.7, 789.4)
15	(62, 2.57, 210.8, 358.4)	(47, 2.52, 211.0, 340.3)	(72, 2.59, 211.1, 317.5)	(6, 2.16, 211.0, 253.5)	(40, 2.51, 210.9, 295.7)
20	(25, 2.59, 442.3, 683.8)	(77, 2.75, 442.1, 697.3)	(60, 2.71, 442.4, 647.8)	(42, 2.66, 442.4, 603.2)	(45, 2.68, 442.9, 571.3)
25	(30, 2.58, 344.5, 526.0)	(57, 2.67, 344.4, 508.6)	(57, 2.66, 344.3, 480.3)	(72, 2.69, 344.3, 482.3)	(87, 2.72, 344.3, 480.3)
30	(39, 2.62, 313.0, 473.6)	(84, 2.71, 312.7, 471.3)	(64, 2.66, 312.7, 441.6)	(75, 2.68, 312.9, 433.3)	(40, 2.58, 312.9, 409.6)
35	(73, 2.71, 310.9, 483.2)	(13, 2.42, 311.0, 405.9)	(73, 2.68, 310.9, 435.1)	(39, 2.58, 310.7, 406.4)	(30, 2.54, 310.8, 390.0)
40	(57, 2.69, 327.0, 494.0)	(70, 2.70, 326.8, 472.8)	(37, 2.59, 327.0, 428.6)	(11, 2.38, 327.0, 392.1)	(65, 2.67, 327.1, 429.8)
45	(70, 2.74, 357.1, 541.7)	(34, 2.61, 357.1, 488.4)	(73, 2.71, 357.1, 488.6)	(92, 2.74, 357.1, 479.3)	(4, 2.21, 357.2, 400.4)
50	(68, 2.76, 400.7, 602.3)	(85, 2.77, 400.6, 576.4)	(88, 2.76, 400.6, 551.6)	(98, 2.77, 400.5, 542.6)	(20, 2.52, 400.9, 484.1)
55	(30, 2.66, 457.8, 649.4)	(71, 2.77, 457.7, 647.7)	(91, 2.79, 457.9, 625.0)	(55, 2.71, 458.1, 594.5)	(56, 2.71, 458.2, 586.6)
60	(45, 2.63, 274.0, 399.6)	(12, 2.39, 274.0, 346.8)	(55, 2.62, 274.1, 365.5)	(39, 2.56, 274.2, 351.5)	(52, 2.60, 274.0, 356.4)
65	(81, 2.75, 327.1, 489.0)	(3, 2.17, 327.2, 380.8)	(61, 2.67, 327.1, 436.5)	(84, 2.71, 326.9, 434.1)	(44, 2.61, 327.2, 415.1)
70	(51, 2.72, 393.0, 569.6)	(43, 2.67, 392.8, 530.3)	(19, 2.52, 393.0, 483.2)	(39, 2.63, 392.9, 494.3)	(52, 2.67, 393.0, 498.6)
75	(56, 2.77, 474.3, 689.0)	(4, 2.30, 474.4, 554.4)	(34, 2.65, 474.3, 598.2)	(9, 2.42, 474.2, 543.1)	(13, 2.48, 474.2, 549.0)
80	(85, 2.76, 321.8, 479.1)	(58, 2.68, 322.1, 443.3)	(21, 2.50, 321.7, 398.3)	(30, 2.55, 321.7, 401.2)	(29, 2.54, 321.8, 398.8)
85	(36, 2.67, 394.6, 556.9)	(48, 2.69, 394.6, 532.9)	(27, 2.58, 394.6, 491.7)	(47, 2.66, 394.9, 501.4)	(59, 2.69, 394.7, 503.2)
90	(87, 2.84, 484.7, 718.2)	(40, 2.70, 485.0, 643.5)	(72, 2.77, 484.8, 635.8)	(11, 2.46, 485.0, 559.6)	(73, 2.76, 484.9, 621.2)
95	(47, 2.69, 349.3, 500.0)	(15, 2.48, 349.3, 438.8)	(15, 2.46, 349.2, 420.0)	(48, 2.64, 349.3, 445.8)	(69, 2.69, 349.1, 451.6)
100	(86, 2.82, 432.8, 636.8)	(33, 2.65, 432.6, 566.3)	(59, 2.72, 432.6, 562.4)	(24, 2.57, 432.4, 523.9)	(60, 2.71, 432.6, 548.8)

Table VII. Chart parameters (K'_c,H'_c) and in-control (ARL_0, $SDRL_0$), for $c_0 = \{5, 10, \ldots, 100\}$ and $m = \{10, 20, 50, 100, 200\}$, corresponding to known parameter case ($H_c = 47$, $K_c = 2.639$)

	(H'_c, K'_c, ARL_0, $SDRL_0$)				
c_0	$m = 10$	$m = 20$	$m = 50$	$m = 100$	$m = 200$
5	(67, 2.40, 153.1, 333.4)	(15, 2.23, 153.0, 264.8)	(68, 2.42, 153.0, 311.3)	(59, 2.44, 153.1, 281.2)	(88, 2.56, 153.1, 283.2)
10	(14, 2.44, 427.8, 772.0)	(35, 2.59, 427.6, 723.5)	(56, 2.66, 427.7, 736.5)	(82, 2.75, 427.8, 712.2)	(10, 2.40, 427.7, 564.8)
15	(77, 2.76, 502.5, 897.6)	(92, 2.78, 502.6, 893.5)	(99, 2.80, 502.5, 839.8)	(75, 2.77, 502.4, 765.8)	(40, 2.66, 502.8, 646.0)
20	(59, 2.62, 257.2, 421.4)	(74, 2.64, 257.4, 403.9)	(62, 2.61, 257.3, 374.8)	(53, 2.58, 257.4, 367.6)	(11, 2.31, 257.3, 311.6)
25	(3, 2.26, 493.5, 642.7)	(4, 2.29, 493.2, 614.5)	(42, 2.68, 493.3, 676.6)	(24, 2.59, 493.4, 625.4)	(37, 2.65, 493.5, 631.3)
30	(77, 2.75, 378.7, 598.3)	(1, 1.97, 378.5, 429.5)	(98, 2.76, 378.6, 543.8)	(80, 2.73, 378.7, 519.4)	(8, 2.35, 378.8, 441.4)
35	(89, 2.75, 330.3, 515.8)	(36, 2.60, 330.3, 459.1)	(16, 2.45, 330.2, 418.4)	(53, 2.64, 330.2, 443.9)	(77, 2.70, 330.1, 435.5)
40	(15, 2.47, 312.7, 434.0)	(12, 2.41, 312.7, 401.4)	(8, 2.32, 312.7, 376.7)	(7, 2.29, 312.6, 366.3)	(61, 2.65, 312.6, 412.9)
45	(42, 2.64, 312.9, 460.9)	(9, 2.36, 313.0, 391.3)	(6, 2.27, 313.0, 368.0)	(64, 2.66, 312.7, 416.2)	(79, 2.69, 312.8, 414.5)
50	(35, 2.62, 325.5, 472.6)	(18, 2.49, 325.5, 424.1)	(62, 2.67, 325.5, 439.5)	(28, 2.54, 325.7, 408.4)	(58, 2.65, 325.5, 425.1)
55	(46, 2.68, 348.5, 509.0)	(87, 2.75, 348.6, 497.6)	(28, 2.56, 348.5, 443.5)	(18, 2.48, 348.4, 421.5)	(11, 2.39, 348.6, 410.9)
60	(45, 2.69, 382.0, 555.8)	(67, 2.73, 381.8, 535.3)	(1, 1.96, 381.7, 413.0)	(36, 2.61, 381.7, 483.3)	(93, 2.75, 381.8, 503.6)
65	(32, 2.66, 425.2, 602.8)	(11, 2.46, 425.2, 527.0)	(57, 2.71, 425.5, 557.6)	(4, 2.25, 425.5, 476.1)	(75, 2.74, 425.1, 552.3)
70	(53, 2.76, 479.5, 700.2)	(86, 2.81, 480.4, 677.0)	(97, 2.81, 480.5, 650.5)	(72, 2.76, 479.9, 625.8)	(28, 2.61, 480.4, 581.3)
75	(37, 2.62, 297.1, 423.6)	(74, 2.70, 297.2, 415.4)	(56, 2.64, 297.1, 391.9)	(89, 2.70, 297.2, 395.1)	(41, 2.58, 297.0, 375.9)
80	(19, 2.54, 346.2, 472.1)	(36, 2.62, 346.1, 461.3)	(5, 2.26, 346.0, 395.7)	(31, 2.57, 346.2, 430.2)	(25, 2.53, 346.1, 423.4)
85	(37, 2.68, 405.8, 573.4)	(85, 2.78, 405.7, 566.9)	(60, 2.71, 405.7, 530.6)	(63, 2.71, 405.7, 522.7)	(57, 2.69, 405.8, 515.6)
90	(5, 2.37, 478.3, 599.0)	(97, 2.83, 478.4, 667.9)	(9, 2.43, 478.1, 555.9)	(19, 2.55, 478.4, 571.0)	(41, 2.67, 478.2, 590.0)
95	(51, 2.69, 326.9, 470.7)	(69, 2.71, 326.9, 450.9)	(98, 2.74, 326.8, 441.2)	(78, 2.70, 326.7, 428.8)	(3, 2.14, 326.7, 358.2)
100	(30, 2.64, 390.1, 540.6)	(95, 2.79, 390.5, 545.1)	(67, 2.72, 390.7, 513.7)	(71, 2.72, 390.5, 507.3)	(26, 2.56, 390.7, 473.7)

Table VIII. In-control ARL and SDRL values for ($H_{np}=2$, $K_{np}=2.085$), $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$ and $m=\{10, 20, 50, 100, 200, \infty\}$

p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$	$m=\infty$
			$n=25$			
0.01	(17028.6, 3188226.0)	(2309.0, 38950.9)	(738.1, 4301.8)	(735.7, 826.3)	(760.2, 799.2)	(763.4, 799.4)
0.02	(10228.7, 2197057.6)	(1975.8, 21891.2)	(1322.5, 2663.6)	(1206.7, 2328.2)	(1066.1, 2212.2)	(66.6, 75.3)
0.05	(5881.1, 441950.1)	(1764.5, 12937.7)	(882.5, 3020.2)	(518.8, 1361.9)	(442.0, 548.9)	(437.7, 464.2)
0.10	(3717.6, 19936.6)	(2025.3, 9211.2)	(1063.0, 2702.0)	(725.1, 1802.7)	(516.8, 962.3)	(455.8, 483.0)
0.15	(459.3, 684.0)	(471.3, 654.0)	(477.0, 632.0)	(472.3, 615.1)	(456.3, 592.9)	(280.7, 301.3)
0.20	(464.8, 752.8)	(512.7, 803.8)	(668.0, 956.8)	(787.7, 1040.8)	(924.0, 1111.1)	(1134.0, 1178.5)
			$n=50$			
0.01	(12150.6, 3182832.7)	(1779.1, 22060.4)	(1217.6, 2425.4)	(1112.1, 2140.4)	(1081.0, 2115.8)	(65.4, 74.1)
0.02	(8406.4, 731146.3)	(1880.6, 13849.2)	(981.5, 2121.4)	(1043.5, 1507.6)	(1108.2, 1533.7)	(1599.7, 1653.2)
0.05	(3444.5, 14337.2)	(1785.6, 7503.6)	(1061.7, 2350.3)	(776.3, 1756.2)	(529.6, 1181.0)	(357.1, 380.8)
0.10	(535.0, 1015.3)	(630.8, 1121.4)	(637.2, 997.8)	(588.4, 764.3)	(571.6, 625.4)	(575.7, 606.6)
0.15	(383.5, 563.1)	(416.1, 560.7)	(404.6, 513.9)	(373.0, 471.3)	(331.2, 415.2)	(261.1, 281.0)
0.20	(337.8, 443.1)	(366.9, 449.0)	(374.6, 432.5)	(362.6, 416.7)	(342.1, 399.8)	(210.9, 228.5)
			$n=75$			
0.01	(9173.4, 1886763.6)	(1748.3, 16414.7)	(722.4, 2927.7)	(405.1, 1358.7)	(327.2, 430.5)	(324.2, 346.6)
0.02	(6748.1, 432961.3)	(1598.2, 10555.4)	(954.5, 2080.0)	(964.7, 1490.3)	(1015.7, 1515.2)	(1666.2, 1720.7)
0.05	(714.5, 1521.3)	(707.6, 1471.1)	(754.2, 1506.4)	(717.9, 1402.6)	(597.2, 1092.1)	(449.7, 476.6)
0.10	(377.3, 536.2)	(392.8, 510.9)	(380.8, 451.0)	(367.9, 429.6)	(348.7, 414.5)	(201.6, 218.8)
0.15	(345.8, 470.2)	(368.8, 457.8)	(387.5, 444.3)	(401.8, 442.2)	(412.2, 448.2)	(411.5, 437.2)
0.20	(308.4, 402.6)	(352.3, 439.2)	(396.3, 480.0)	(431.2, 515.5)	(478.9, 556.5)	(597.7, 629.2)
			$n=100$			
0.01	(7363.7, 1090099.2)	(1741.4, 14829.8)	(917.8, 1948.2)	(975.3, 1409.7)	(1071.2, 1447.5)	(1494.8, 1546.3)
0.02	(5976.9, 127400.8)	(1646.1, 10120.0)	(972.3, 2058.2)	(903.0, 1616.5)	(795.2, 1509.6)	(198.6, 215.5)
0.05	(580.4, 1034.2)	(627.2, 1042.3)	(572.7, 886.7)	(508.5, 707.3)	(458.3, 542.8)	(437.2, 463.7)
0.10	(346.6, 467.2)	(389.0, 496.4)	(434.3, 535.5)	(484.5, 578.1)	(539.1, 615.6)	(627.3, 659.6)
0.15	(311.4, 408.4)	(353.0, 435.7)	(384.0, 451.9)	(407.1, 467.2)	(422.6, 466.5)	(432.2, 458.6)
0.20	(301.3, 376.6)	(340.0, 402.2)	(374.0, 428.4)	(398.1, 450.5)	(427.0, 473.9)	(478.4, 506.3)
0.10	(627.3, 659.6)	(627.3, 659.6)	(627.3, 659.6)	(627.3, 659.6)	(627.3, 659.6)	(627.3, 659.6)
0.15	(432.2, 458.6)	(432.2, 458.6)	(432.2, 458.6)	(432.2, 458.6)	(432.2, 458.6)	(432.2, 458.6)
0.20	(478.4, 506.3)	(478.4, 506.3)	(478.4, 506.3)	(478.4, 506.3)	(478.4, 506.3)	(478.4, 506.3)

Table IX. In-control ARL and SDRL values for ($H_{np}=7$, $K_{np}=2.322$), $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$ and $m=\{10, 20, 50, 100, 200, \infty\}$

p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$	$m=\infty$
			$n=25$			
0.01	(19682.6, 5739672.4)	(1429.1, 39697.7)	(401.0, 3678.4)	(234.7, 591.1)	(232.5, 265.3)	(232.5, 264.7)
0.02	(11169.4, 3839976.4)	(1394.5, 21014.3)	(609.2, 1928.9)	(629.5, 868.5)	(708.3, 890.2)	(847.5, 915.8)
0.05	(6045.1, 1204194.0)	(1374.6, 13474.5)	(722.6, 1938.6)	(531.1, 1492.2)	(335.3, 1083.7)	(136.1, 159.1)
0.10	(5461.1, 58554.2)	(1805.8, 12242.4)	(892.7, 2079.6)	(824.7, 1403.0)	(815.2, 1387.3)	(141.5, 165.1)
0.15	(867.7, 1680.6)	(999.3, 1798.0)	(996.7, 1787.8)	(929.4, 1720.5)	(819.4, 1601.5)	(237.7, 270.3)
0.20	(630.2, 1168.6)	(630.0, 1116.5)	(556.2, 958.2)	(464.6, 763.3)	(381.5, 537.9)	(341.5, 382.0)
			$n=50$			
0.01	(9910.7, 4834585.4)	(1334.6, 24824.9)	(559.5, 1670.8)	(613.6, 814.2)	(670.1, 825.6)	(779.9, 845.1)
0.02	(7817.2, 1684253.9)	(1483.6, 14069.5)	(510.2, 1865.0)	(446.0, 663.2)	(467.5, 525.9)	(477.7, 527.1)
0.05	(5455.1, 129198.0)	(1374.9, 9720.5)	(742.4, 1775.7)	(697.4, 1046.4)	(747.4, 1038.5)	(1065.3, 1142.8)
0.10	(588.1, 1131.9)	(491.7, 850.4)	(438.5, 633.4)	(408.0, 577.0)	(384.2, 556.7)	(177.1, 204.3)
0.15	(431.4, 763.1)	(447.4, 689.4)	(494.9, 679.3)	(530.6, 660.8)	(555.3, 635.3)	(580.3, 635.5)
0.20	(373.9, 598.1)	(424.7, 641.1)	(447.2, 630.9)	(433.5, 574.1)	(400.3, 486.4)	(375.6, 418.5)
0.01	(7768.3, 2323204.2)	(1315.1, 16163.6)	(701.5, 1962.5)	(434.9, 1423.7)	(223.2, 878.6)	(102.2, 121.3)
0.02	(5939.7, 1427754.2)	(1311.8, 12221.6)	(580.2, 1803.7)	(474.0, 763.5)	(484.8, 547.8)	(497.1, 547.6)
0.05	(2306.2, 12557.0)	(1319.2, 5081.3)	(791.5, 1881.5)	(703.4, 1050.5)	(749.5, 965.0)	(960.3, 1033.6)
0.10	(475.8, 838.9)	(528.2, 881.1)	(516.1, 800.7)	(477.5, 682.6)	(430.1, 538.3)	(405.0, 449.8)
0.15	(332.9 ARL, 490.5)	(381.8, 524.0)	(398.0, 516.0)	(396.5, 498.6)	(398.0, 486.9)	(321.7, 360.9)
0.20	(319.5, 455.1)	(358.0, 475.3)	(375.1, 469.9)	(389.2, 487.5)	(407.7, 517.4)	(817.7, 884.7)
			$n=100$			
0.01	(6575.4, 1791230.0)	(1328.9, 12824.0)	(530.1, 1953.0)	(425.5, 655.9)	(438.9, 493.8)	(447.0, 494.5)
0.02	(5548.7, 636777.1)	(1333.2, 10874.2)	(618.7, 1724.3)	(543.2, 816.8)	(578.8, 672.9)	(624.1, 681.6)
0.05	(583.7, 1046.1)	(514.9, 859.4)	(508.0, 772.8)	(548.6, 818.6)	(619.5, 902.8)	(1123.3, 1203.2)
0.10	(374.0, 571.9)	(395.3, 552.6)	(380.1, 477.4)	(371.6, 443.6)	(362.7, 432.8)	(192.3, 221.0)
0.15	(323.0, 446.3)	(358.8, 468.2)	(396.3, 494.0)	(430.0, 523.4)	(471.3, 555.4)	(545.5, 598.8)
0.20	(301.6, 410.7)	(335.6, 429.8)	(382.0, 468.2)	(407.1, 491.7)	(442.9, 522.7)	(523.4, 575.4)

Table X. In-control ARL and SDRL values for ($H_{np}=47$, $K_{np}=2.639$), $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$ and $m=\{10, 20, 50, 100, 200, \infty\}$

p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$	$m=\infty$
			$n=25$			
0.01	(21078.6, 43052425.3)	(1198.1, 51986.2)	(227.4, 1482.6)	(86.5, 637.0)	(56.2, 145.7)	(54.9, 71.5)
0.02	(13562.5, 6999996.8)	(1113.7, 25969.2)	(231.3, 1425.1)	(160.5, 400.3)	(161.1, 209.4)	(162.2, 208.9)
0.05	(6655.9, 3575578.3)	(1060.5, 15197.8)	(371.9, 1303.5)	(347.9, 562.8)	(383.6, 553.4)	(486.7, 591.7)
0.10	(6352.5, 445371.7)	(1330.2, 13596.3)	(521.0, 1650.3)	(349.9, 879.2)	(296.6, 433.9)	(292.5, 366.0)
0.15	(3566.3, 20544.7)	(1473.3, 8798.5)	(712.0, 2063.0)	(494.6, 1181.5)	(408.7, 602.4)	(400.0, 491.9)
0.20	(496.5, 821.9)	(526.8, 811.8)	(600.3, 848.6)	(665.7, 881.1)	(726.4, 906.5)	(781.5, 924.9)
			$n=50$			
0.01	(10143.5, 17237428.1)	(930.7, 18388.9)	(213.8, 1201.8)	(150.7, 350.4)	(150.1, 195.5)	(150.8, 194.8)
0.02	(7482.7, 6985147.7)	(770.6, 14014.0)	(339.2, 1164.3)	(186.2, 671.2)	(115.8, 318.9)	(98.9, 129.3)
0.05	(5633.5, 1380226.6)	(1094.4, 10012.7)	(449.0, 1269.5)	(313.5, 809.8)	(223.8, 440.6)	(198.6, 253.5)
0.10	(1485.8, 5869.1)	(986.5, 3645.6)	(642.6, 1604.2)	(495.3, 1063.4)	(371.1, 708.1)	(299.4, 374.2)
0.15	(485.9, 864.1)	(476.1, 780.7)	(471.7, 725.3)	(516.9, 757.3)	(587.2, 814.7)	(774.3, 916.8)
0.20	(401.2, 746.8)	(448.8, 795.8)	(479.0, 790.9)	(473.4, 717.3)	(451.0, 601.3)	(442.4, 540.9)
			$n=75$			
0.01	(7292.9, 6297830.9)	(783.8, 16722.7)	(284.2, 999.2)	(282.0, 518.2)	(295.7, 524.3)	(518.9, 628.6)
0.02	(5055.6, 5296151.5)	(865.5, 10507.5)	(364.7, 1074.3)	(262.7, 737.8)	(163.1, 472.5)	(102.3, 133.7)
0.05	(5313.4, 99607.5)	(1145.8, 9487.5)	(522.9, 1327.9)	(408.0, 866.0)	(307.0, 668.1)	(181.1, 232.2)
0.10	(487.2, 951.0)	(485.9, 918.4)	(489.0, 893.8)	(473.8, 762.3)	(464.1, 623.5)	(484.7, 589.4)
0.15	(396.2, 708.7)	(415.6, 676.6)	(408.5, 613.7)	(389.1, 549.9)	(379.3, 502.4)	(342.8, 425.2)
0.20	(338.0, 537.9)	(376.3, 566.3)	(403.5, 582.0)	(428.6, 608.1)	(472.2, 657.2)	(698.5, 831.9)
			$n=100$			
0.01	(6050.2, 7913163.8)	(819.5, 11914.4)	(306.7, 1025.9)	(188.8, 657.2)	(111.9, 311.0)	(93.6, 122.4)
0.02	(6045.4, 3519398.5)	(838.0, 10561.6)	(378.3, 1093.5)	(278.7, 717.8)	(191.0, 492.2)	(124.3, 161.6)
0.05	(1973.0, 11694.0)	(1021.5, 4485.2)	(547.1, 1346.2)	(447.6, 876.0)	(365.0, 723.3)	(208.2, 265.2)
0.10	(431.3, 817.1)	(440.4, 784.7)	(398.4, 627.9)	(373.5, 542.6)	(356.8, 515.4)	(193.8, 247.7)
0.15	(341.7, 549.6)	(369.1, 564.8)	(404.1, 595.6)	(417.4, 584.9)	(424.8, 559.0)	(433.2, 530.3)
0.20	(310.9, 490.5)	(351.0, 522.5)	(388.7, 555.2)	(398.9, 553.0)	(392.6, 523.4)	(364.3, 450.3)

Table XI. Minimum values m^* of m, for $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$, ($H_{np}=2$, $K_{np}=2.085$), ($H_{np}=7$, $K_{np}=2.322$) and ($H_{np}=47$, $K_{np}=2.639$) for the synthetic np chart, satisfying $\Delta = \frac{\left|ARL_{0,m^*}-ARL_{0,\infty}\right|}{ARL_{0,\infty}} < 0.05$

p_0	$n=25$	$n=50$	$n=75$	$n=100$
		($H_{np}=2$, $K_{np}=2.085$)		
0.01	50	>10,000	160	1920
0.02	>10,000	2530	20	5770
0.05	150	490	480	200
0.10	270	80	5680	480
0.15	5320	670	60	110
0.20	590	3510	740	390
		($H_{np}=7$, $K_{np}=2.322$)		
0.01	80	500	560	60
0.02	570	60	60	50
0.05	740	30	1200	6790
0.10	>10,000	7640	220	>10,000
0.15	3960	180	10	480
0.20	290	10	>10,000	570
		($H_{np}=47$, $K_{np}=2.639$)		
0.01	190	80	>10,000	300
0.02	90	270	500	520
0.05	850	270	770	1020
0.10	160	370	10	9150
0.15	170	1050	1070	80
0.20	260	20	2010	20

Table XII. Chart parameters (K'_{np},H'_{np}) and in-control (ARL_0, $SDRL_0$), for $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$, $m=\{10, 20, 50, 100, 200\}$ corresponding to known parameter case ($H_{np}=2$, $K_{np}=2.085$)

	(H'_{np}, K'_{np}, ARL_0, $SDRL_0$)				
p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$
			$n=25$		
0.01	(23, 2.04, 760.7, 266840.9)	(76, 2.72, 763.2, 32209.3)	(50, 2.98, 761.8, 3681.0)	(7, 2.75, 762.3, 6352.4)	(2, 2.39, 763.4, 813.3)
0.02	(84, 1.73, 66.6, 3154.5)	(96, 2.14, 66.6, 614.8)	(4, 1.57, 66.6, 305.6)	(54, 2.10, 66.6, 139.3)	(2, 1.37, 66.6, 75.3)
0.05	(14, 1.93, 437.2, 11761.1)	(45, 2.44, 437.6, 4159.1)	(73, 2.81, 437.2, 1853.8)	(16, 2.43, 437.5, 959.3)	(53, 2.88, 437.8, 588.5)
0.10	(78, 2.26, 456.0, 5233.8)	(20, 2.25, 455.9, 2796.4)	(12, 2.27, 455.8, 995.6)	(97, 2.81, 455.8, 1214.9)	(2, 1.96, 455.9, 499.7)
0.15	(78, 2.40, 280.5, 1030.3)	(95, 2.44, 280.8, 919.7)	(56, 2.46, 280.7, 699.1)	(51, 2.46, 280.6, 443.2)	(75, 2.63, 280.7, 404.3)
0.20	(5, 2.37, 1135.1, 2146.9)	(81, 2.73, 1134.2, 3002.7)	(89, 2.80, 1133.9, 2809.3)	(67, 2.81, 1133.9, 2896.4)	(40, 2.80, 1133.6, 2535.8)
			$n=50$		
0.01	(71, 1.67, 65.4, 2093.2)	(86, 2.07, 65.4, 472.4)	(94, 2.28, 65.4, 142.5)	(2, 1.32, 65.4, 75.8)	(2, 1.36, 65.4, 74.1)
0.02	(35, 2.32, 1601.3, 336911.3)	(27, 2.68, 1599.0, 26314.3)	(18, 2.83, 1597.7, 5407.4)	(4, 2.61, 1599.3, 6253.8)	(2, 2.45, 1599.5, 1772.6)
0.05	(62, 2.20, 357.2, 2886.6)	(7, 1.99, 357.2, 1492.5)	(4, 1.99, 357.2, 854.9)	(31, 2.59, 357.1, 871.8)	(24, 2.59, 357.2, 559.4)
0.10	(14, 2.46, 575.6, 948.7)	(31, 2.52, 576.0, 1424.8)	(5, 2.31, 575.9, 835.1)	(27, 2.57, 576.3, 1155.9)	(74, 2.77, 575.8, 1108.8)
0.15	(19, 2.38, 261.2, 493.9)	(85, 2.62, 261.0, 450.4)	(28, 2.44, 261.1, 424.8)	(14, 2.30, 261.1, 333.4)	(20, 2.43, 261.3, 376.9)
0.20	(88, 2.61, 210.8, 385.5)	(64, 2.56, 211.0, 327.2)	(94, 2.60, 210.9, 301.8)	(55, 2.51, 211.0, 325.3)	(29, 2.46, 210.9, 281.4)
			$n=75$		
0.01	(65, 2.15, 324.2, 73701.8)	(77, 2.56, 324.1, 4949.7)	(33, 2.53, 323.8, 844.8)	(59, 2.78, 324.2, 529.9)	(83, 3.00, 324.2, 433.2)
0.02	(14, 2.18, 1666.6, 105139.7)	(53, 2.80, 1666.1, 27956.6)	(22, 2.86, 1666.7, 5734.4)	(10, 2.72, 1668.8, 4015.0)	(2, 2.41, 1666.5, 1881.3)
0.05	(26, 2.31, 449.6, 1234.0)	(77, 2.53, 449.4, 2627.0)	(24, 2.47, 449.4, 1139.0)	(38, 2.62, 449.7, 968.7)	(33, 2.65, 449.9, 977.7)
0.10	(18, 2.32, 201.7, 366.0)	(81, 2.56, 201.8, 341.3)	(10, 2.21, 201.7, 276.5)	(96, 2.57, 201.6, 303.3)	(73, 2.53, 201.7, 312.6)
0.15	(10, 2.41, 411.4, 634.6)	(15, 2.46, 411.7, 622.2)	(44, 2.63, 411.5, 620.2)	(94, 2.74, 411.3, 650.2)	(56, 2.68, 411.6, 542.5)
0.20	(53, 2.75, 598.5, 1007.2)	(24, 2.62, 597.8, 868.1)	(20, 2.59, 597.7, 768.5)	(40, 2.67, 597.7, 801.1)	(42, 2.66, 597.6, 791.9)
			$n=100$		
0.01	(3, 1.84, 1495.1, 50275.0)	(38, 2.76, 1495.1, 29623.0)	(32, 2.97, 1495.0, 6107.5)	(2, 2.37, 1497.0, 2840.7)	(25, 2.97, 1494.7, 2920.7)
0.02	(17, 1.87, 198.5, 941.4)	(35, 2.21, 198.6, 1256.6)	(95, 2.61, 198.4, 550.0)	(38, 2.52, 198.6, 504.3)	(9, 2.09, 198.5, 381.7)
0.05	(11, 2.35, 437.2, 776.8)	(85, 2.58, 437.2, 1438.7)	(61, 2.63, 437.3, 1070.1)	(89, 2.75, 437.2, 781.9)	(65, 2.71, 437.2, 789.0)
0.10	(55, 2.72, 627.6, 1173.6)	(52, 2.72, 627.6, 1072.7)	(42, 2.70, 627.3, 964.2)	(92, 2.81, 627.2, 1060.7)	(97, 2.85, 627.4, 987.7)
0.15	(97, 2.77, 432.1, 776.0)	(94, 2.76, 432.3, 718.3)	(33, 2.60, 432.1, 585.8)	(5, 2.27, 432.3, 544.5)	(96, 2.78, 432.3, 607.3)
0.20	(54, 2.74, 478.4, 759.1)	(20, 2.56, 478.4, 652.7)	(8, 2.39, 478.5, 606.8)	(88, 2.77, 478.7, 676.3)	(51, 2.70, 478.2, 642.3)

Table XIII. Chart parameters (K'_{np},H'_{np}) and in-control (ARL_0, $SDRL_0$), for $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$, $m=\{10, 20, 50, 100, 200\}$ corresponding to known parameter case ($H_{np}=7$, $K_{np}=2.322$)

	(H'_{np}, K'_{np}, ARL_0, $SDRL_0$)				
p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$
			$n=25$		
0.01	(43, 1.53, 54.8, 1754.5)	(72, 1.85, 55.0, 433.6)	(38, 1.94, 54.9, 102.8)	(67, 2.55, 54.9, 279.1)	(47, 2.41, 54.9, 72.1)
0.02	(40, 1.85, 162.4, 14058.9)	(28, 2.14, 162.2, 1995.0)	(75, 2.64, 161.9, 927.7)	(63, 2.80, 162.1, 750.9)	(46, 2.57, 162.2, 211.1)
0.05	(86, 2.26, 486.6, 89653.5)	(11, 2.17, 486.6, 3999.8)	(33, 2.62, 485.8, 1556.9)	(69, 2.96, 486.6, 1804.7)	(47, 2.90, 486.6, 675.3)
0.10	(1, 1.76, 292.5, 573.8)	(26, 2.21, 292.4, 1630.0)	(50, 2.51, 292.3, 882.2)	(48, 2.58, 292.5, 589.7)	(34, 2.41, 292.6, 424.2)
0.15	(51, 2.40, 399.5, 1497.2)	(76, 2.49, 399.9, 1267.7)	(44, 2.50, 399.6, 1086.5)	(79, 2.69, 400.1, 1025.4)	(67, 2.74, 400.1, 956.1)
0.20	(58, 2.67, 781.5, 2386.6)	(13, 2.50, 781.5, 1345.9)	(41, 2.66, 781.5, 1503.5)	(50, 2.69, 781.7, 1621.7)	(14, 2.51, 781.4, 1308.4)
			$n=50$		
0.01	(38, 1.83, 150.8, 18309.3)	(71, 2.32, 150.6, 2502.9)	(48, 2.53, 150.7, 730.1)	(47, 2.64, 150.7, 350.4)	(58, 2.96, 150.8, 581.6)
0.02	(33, 1.75, 99.0, 1959.4)	(34, 2.01, 98.9, 634.6)	(66, 2.38, 99.0, 328.3)	(44, 2.35, 98.9, 165.7)	(46, 2.34, 98.9, 130.6)
0.05	(66, 2.11, 198.4, 1474.4)	(75, 2.35, 198.5, 1220.2)	(20, 2.22, 198.6, 446.9)	(68, 2.60, 198.7, 474.7)	(46, 2.51, 198.5, 264.0)
0.10	(15, 2.35, 299.2, 588.0)	(13, 2.33, 299.4, 522.6)	(88, 2.59, 299.5, 668.5)	(81, 2.63, 299.4, 634.9)	(44, 2.51, 299.4, 399.1)
0.15	(87, 2.79, 774.3, 1604.2)	(48, 2.72, 774.1, 1549.1)	(96, 2.81, 774.0, 1501.4)	(28, 2.63, 774.9, 1114.5)	(42, 2.69, 774.2, 968.2)
0.20	(96, 2.74, 442.9, 858.5)	(69, 2.69, 442.4, 782.5)	(6, 2.29, 442.3, 609.1)	(91, 2.75, 442.5, 706.0)	(46, 2.62, 442.5, 574.3)
			$n=75$		
0.01	(11, 1.91, 517.8, 45742.1)	(15, 2.27, 518.9, 5161.2)	(59, 2.92, 519.0, 2684.7)	(1, 1.62, 519.1, 638.0)	(46, 3.00, 519.2, 661.2)
0.02	(33, 1.80, 102.4, 1305.2)	(96, 2.25, 102.2, 572.6)	(60, 2.35, 102.3, 292.9)	(56, 2.44, 102.3, 228.4)	(48, 2.46, 102.3, 150.7)
0.05	(74, 2.29, 180.9, 508.2)	(67, 2.33, 181.1, 708.5)	(49, 2.39, 181.1, 433.8)	(34, 2.34, 181.1, 289.4)	(56, 2.55, 181.1, 323.0)
0.10	(55, 2.66, 484.4, 993.1)	(8, 2.34, 484.5, 796.2)	(90, 2.73, 484.7, 910.0)	(83, 2.75, 484.5, 884.4)	(46, 2.65, 484.8, 663.8)
0.15	(41, 2.59, 342.6, 606.0)	(79, 2.68, 342.7, 565.2)	(55, 2.63, 342.7, 522.8)	(24, 2.47, 342.7, 501.5)	(66, 2.67, 342.6, 461.4)
0.20	(19, 2.63, 698.5, 1080.9)	(8, 2.46, 698.2, 995.6)	(94, 2.84, 698.4, 1056.4)	(2, 2.20, 698.3, 825.4)	(58, 2.78, 698.5, 984.7)
			$n=100$		
0.01	(55, 1.81, 93.6, 2715.0)	(27, 1.96, 93.4, 520.2)	(65, 2.37, 93.6, 299.3)	(63, 2.51, 93.6, 254.9)	(49, 2.54, 93.5, 157.0)
0.02	(73, 2.01, 124.2, 1193.3)	(85, 2.29, 124.2, 754.3)	(55, 2.36, 124.3, 329.2)	(3, 1.63, 124.3, 154.5)	(40, 2.29, 124.2, 172.0)
0.05	(60, 2.44, 208.2, 479.2)	(32, 2.41, 208.1, 322.7)	(30, 2.38, 208.2, 311.8)	(67, 2.53, 208.1, 407.7)	(51, 2.52, 208.2, 323.8)
0.10	(97, 2.58, 193.9, 369.2)	(7, 2.17, 193.8, 288.9)	(92, 2.58, 193.9, 316.4)	(78, 2.57, 193.8, 303.9)	(39, 2.42, 193.8, 261.1)
0.15	(35, 2.63, 433.4, 694.4)	(28, 2.59, 433.1, 625.7)	(84, 2.74, 433.0, 658.2)	(90, 2.76, 433.3, 627.6)	(30, 2.56, 433.1, 587.1)
0.20	(74, 2.73, 364.3, 593.9)	(21, 2.52, 364.2, 505.7)	(32, 2.57, 364.2, 487.4)	(53, 2.64, 364.2, 510.2)	(60, 2.67, 364.2, 500.6)

Table XIV. Chart parameters (K'_{np},H'_{np}) and in-control (ARL_0, $SDRL_0$), for $n=\{25, 50, 75, 100\}$, $p_0=\{0.01, 0.02, 0.05, 0.1, 0.15, 0.2\}$, $m=\{10, 20, 50, 100, 200\}$ corresponding to known parameter case ($H_{np}=47$, $K_{np}=2.639$)

	(H'_{np}, K'_{np}, ARL_0, $SDRL_0$)				
p_0	$m=10$	$m=20$	$m=50$	$m=100$	$m=200$
			$n=25$		
0.01	(8, 1.43, 232.8, 9332.6)	(45, 2.20, 232.6, 3360.3)	(28, 2.49, 233.2, 1801.3)	(73, 2.92, 232.1, 1260.0)	(7, 2.39, 232.5, 268.2)
0.02	(38, 2.21, 847.3, 294492.6)	(63, 2.69, 847.2, 19408.6)	(67, 3.00, 846.5, 3699.2)	(6, 2.42, 847.9, 1064.1)	(7, 2.75, 847.6, 1037.0)
0.05	(27, 1.78, 136.1, 1606.5)	(67, 2.26, 136.1, 1063.0)	(2, 1.47, 136.0, 295.0)	(18, 2.24, 136.0, 434.4)	(1, 1.46, 136.1, 362.1)
0.10	(96, 2.11, 141.4, 648.8)	(44, 2.13, 141.5, 578.4)	(26, 2.16, 141.5, 335.0)	(70, 2.37, 141.5, 237.6)	(7, 1.96, 141.5, 169.4)
0.15	(97, 2.40, 237.1, 856.9)	(26, 2.27, 237.6, 492.3)	(85, 2.50, 237.5, 619.3)	(11, 2.20, 237.7, 534.4)	(85, 2.53, 237.6, 323.0)
0.20	(69, 2.62, 341.3, 585.2)	(80, 2.63, 341.4, 543.3)	(68, 2.58, 341.4, 554.5)	(94, 2.60, 341.5, 505.6)	(86, 2.56, 341.4, 518.9)
			$n=50$		
0.01	(96, 2.39, 779.0, 758660.5)	(98, 2.79, 778.8, 20972.0)	(61, 2.98, 780.3, 3301.3)	(6, 2.44, 780.0, 1029.7)	(7, 2.73, 779.9, 943.4)
0.02	(73, 2.23, 477.7, 83669.3)	(79, 2.61, 477.9, 6232.2)	(6, 2.23, 477.7, 1555.2)	(1, 1.78, 478.2, 1403.0)	(7, 2.45, 477.6, 558.7)
0.05	(62, 2.39, 1065.5, 48341.6)	(38, 2.60, 1064.7, 9668.9)	(62, 2.90, 1065.4, 3480.2)	(20, 2.73, 1065.0, 2641.4)	(7, 2.54, 1065.6, 1299.3)
0.10	(56, 2.46, 177.1, 309.2)	(40, 2.40, 176.9, 306.2)	(12, 2.20, 177.0, 293.1)	(8, 2.13, 177.0, 259.5)	(47, 2.39, 177.0, 283.0)
0.15	(79, 2.73, 579.8, 1258.1)	(43, 2.67, 580.2, 909.1)	(15, 2.50, 580.8, 885.7)	(55, 2.70, 580.1, 774.1)	(11, 2.49, 580.3, 869.6)
0.20	(68, 2.68, 375.6, 700.2)	(75, 2.67, 375.5, 677.1)	(8, 2.31, 375.7, 533.5)	(21, 2.49, 375.4, 542.8)	(2, 2.10, 375.6, 425.3)
			$n=75$		
0.01	(70, 1.91, 102.1, 7208.5)	(81, 2.22, 102.3, 997.6)	(8, 1.89, 102.0, 322.1)	(86, 2.46, 102.2, 247.6)	(70, 2.48, 102.2, 264.3)
0.02	(9, 1.90, 497.3, 7429.5)	(55, 2.55, 497.1, 5228.6)	(25, 2.59, 496.9, 1532.7)	(10, 2.50, 497.4, 1443.0)	(7, 2.41, 497.1, 590.5)
0.05	(82, 2.50, 958.9, 8512.0)	(70, 2.69, 961.3, 8794.1)	(83, 2.91, 960.3, 2925.2)	(86, 2.99, 959.8, 2226.7)	(47, 2.83, 960.4, 1407.9)
0.10	(68, 2.65, 405.0, 832.2)	(8, 2.30, 405.0, 694.7)	(96, 2.70, 405.4, 776.9)	(24, 2.52, 404.8, 601.5)	(8, 2.35, 404.9, 538.8)
0.15	(29, 2.53, 321.9, 545.6)	(37, 2.55, 321.6, 533.7)	(6, 2.25, 321.8, 420.2)	(5, 2.20, 321.7, 403.5)	(97, 2.72, 321.8, 428.5)
0.20	(70, 2.85, 818.1, 1441.8)	(37, 2.74, 817.2, 1283.7)	(45, 2.75, 817.5, 1240.3)	(56, 2.79, 817.5, 1194.9)	(73, 2.85, 817.8, 1146.2)
			$n=100$		
0.01	(67, 2.22, 447.0, 96358.4)	(35, 2.47, 447.3, 6529.2)	(3, 1.97, 447.1, 945.3)	(7, 2.37, 447.2, 849.6)	(7, 2.44, 447.3, 530.5)
0.02	(20, 2.09, 624.4, 12820.0)	(63, 2.62, 624.1, 6276.6)	(2, 1.97, 623.1, 1380.3)	(79, 2.94, 624.8, 1138.3)	(94, 2.97, 624.2, 895.1)
0.05	(86, 2.64, 1124.5, 6535.7)	(80, 2.73, 1122.9, 6048.6)	(48, 2.80, 1123.7, 3037.5)	(74, 2.96, 1123.3, 2434.8)	(52, 2.91, 1123.2, 1564.9)
0.10	(81, 2.56, 192.4, 361.0)	(9, 2.22, 192.3, 278.4)	(73, 2.54, 192.3, 311.9)	(35, 2.42, 192.3, 266.6)	(88, 2.61, 192.2, 294.5)
0.15	(54, 2.73, 546.1, 935.6)	(61, 2.74, 545.3, 870.6)	(75, 2.77, 545.3, 826.5)	(32, 2.63, 545.5, 713.7)	(7, 2.37, 545.5, 640.3)
0.20	(60, 2.76, 523.4, 841.7)	(41, 2.69, 523.2, 747.5)	(90, 2.79, 523.5, 755.4)	(57, 2.73, 523.3, 692.1)	(52, 2.72, 523.2, 688.5)

In Table XI, we have computed the minimum values m^* of m for $n = \{25,50,75,100\}$, $p_0 = \{0.01,0.02,0.05,0.1,0.15,0.2\}$, and chart parameters $(H_{np} = 2, K_{np} = 2.085)$, $(H_{np} = 7, K_{np} = 2.322)$, and $(H_{np} = 47, K_{np} = 2.639)$, satisfying $\Delta = \frac{|ARL_{0,m^*} - ARL_{0,\infty}|}{ARL_{0,\infty}} < 0.05$. As it can be noticed in Table XI, depending on the values of n and p_0, the value of m^* satisfying $\Delta < 0.05$ can be very large and, in some cases, larger than 10,000.

In order to relax this constraint on the number of phase I samples, we suggest to compute alternative chart parameters (H'_{np}, K_{np}) for the synthetic *np* chart with estimated parameters, which take the value of m into account and allow the in-control ARL value corresponding to the estimated parameters case to be *as close as possible* to the in-control ARL value corresponding to the known parameter case. These values are in Tables XII–XIV for $n = \{25,50,75,100\}$, $p_0 = \{0.01,0.02,0.05,0.1,0.15,0.2\}$, and $m = \{10,20,50,100,200\}$. For example, in Table XII, if $n = 75$, $p_0 = 0.05$, and $m = 10$, the chart parameters are $(H'_{np} = 26, K'_{np} = 2.31)$, and, in this case, the in-control ARL is $ARL_0 = 449.6$ ($SDRL_0 = 1234.0$). Referring to Table VIII, the in-control ARL corresponding to $m = \infty$ (i.e., the known parameter case) is $ARL_0 = 449.7$. Consequently, the use of the new chart parameters $(H'_{np} = 26, K'_{np} = 2.31)$ instead of $(H_{np} = 2, K_{np} = 2.085)$ allows one to obtain the same (or approximately the same) in-control ARL as for the known parameter case.

5. Conclusions

In this paper, we have compared the run length properties of the synthetic *c* and *np* charts in both the known and estimated parameter cases. From this comparison, many interesting conclusions can be drawn. The first one is that for a particular value of c_0 or (n,p_0), the in-control ARL values can be very different in the known and in the estimated parameter cases. Moreover, when m increases, the difference in terms of in-control ARLs, between the known and the estimated parameter cases, tends to decrease but not always in a monotonic way, unlike the $\overline{X}$ chart, where this difference reduces monotonically with m. The second one is that depending on the values of c_0 or (n,p_0), the in-control ARLs when $m < \infty$ are either smaller or larger than the in-control ARLs corresponding to $m = \infty$. This result is also quite different from what can be observed in the case of the $\overline{X}$ chart where the in-control ARL values corresponding to $m < \infty$ are always larger than the in-control ARL values corresponding to $m = \infty$. The third one is that if we want to use the chart parameters $(H = 2, K = 2.085)$, $(H = 7, K = 2.322)$, and $(H = 47, K = 2.639)$ for the synthetic *c* and *np* charts with estimated parameters, respectively, having run length performances close to that of the synthetic *c* and *np* charts with known parameters, then we must pay the price to obtain a large number m of phase I samples, sometimes larger than 10,000. The fourth one is that the use of alternative chart parameters (H',K') especially dedicated to the number m of phase I samples allows the in-control ARL value corresponding to the estimated parameter case to be *as close as possible* to the in-control ARL value corresponding to the known parameter case.

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