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The applications of fractal geometry and self-similarity to art music

by

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Abstract

Title: The applications of fractal geometry and self-similarity to art music

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The aim of this research study is to investigate different practical ways in which fractal geometry and self-similarity can be applied to art music, with reference to music composition and analysis. This specific topic was chosen because there are many misconceptions in the field of fractal and self-similar music.

Analyses of previous research as well as the music analysis of several compositions from different composers in different genres were the main methods for conducting the research. Although the dissertation restates much of the existing research on the topic, it is (to the researcher's knowledge) one of the first academic works that summarises the many different facets of fractal geometry and music.

Fractal and self-similar shapes are evident in nature and art dating back to the 16th century, despite the fact that the mathematics behind fractals was only defined in 1975 by the French mathematician, Benoit B. Mandelbrot. Mathematics has been a source of inspiration to composers and musicologists for many centuries and fractal geometry has also infiltrated the works of composers in the past 30 years. The search for fractal and self-similar structures in music composed prior to 1975 may lead to a different perspective on the way in which music is analysed.

Basic concepts and prerequisites of fractals were deliberately simplified in this research in order to collect useful information that musicians can use in composition and analysis. These include subjects such as self-similarity, fractal dimensionality and scaling. Fractal shapes with their defining properties were also illustrated because their structures have been likened to those in some music compositions.

This research may enable musicians to incorporate mathematical properties of fractal geometry and self-similarity into original compositions. It may also provide new ways to view the use of motifs and themes in the structural analysis of music.

Keywords

- 1/f-noise
- Algorithmic composition
- Fractal geometry
- Fractal music
- Fragmentation
- Iteration
- Mathematics
- Repetition
- Self-similarity
- Symmetry
- Transformation

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CHAPTER 1: INTRODUCTION

1.1 Background to the study

1.1.1 Music and mathematics through the ages

The integration of music and mathematics is certainly not a new phenomenon – it has existed for more than two millennia. One of the earliest examples was the discovery that interval ratios are directly related to the length of an instrument's string (frequency). This was determined by the Greek philosopher and mathematician, Pythagoras (c. 500 BC). Medieval universities taught what was called the *quadrivium*, which consisted of four parts: arithmetic, geometry, astronomy and music (Barbera 2001:642).

One of the first examples of algorithmic composition (i.e. music composed with the use of an algorithm or formula) dates back to the Medieval Benedictine monk, Guido d'Arezzo, with his music treatise entitled *Micrologus de disciplina artis musicae* (c.1026). His composition method consisted of assigning a note to each vowel of a text. In addition, it is thanks to D'Arezzo that we have music notation on staves as well as *solfege* (Tufró 2009).

In the middle of the 17th century, the Jesuit polymath, Athanasius Kircher, invented two "machines" (see Figure 1) that enabled non-musicians to compose four-part hymns. The first, *Arca Musaritmica* (music-making ark), was a wooden box containing wooden strips with melodic and rhythmic sequences on them. Equipped with a set of rules, the "composer" could pull strips from the box to create polyphonic compositions. A more elaborate version of the machine came in the form of the *Organum Mathematicum,* which, in addition to composing music, could also aid in calculating problems relating to arithmetic, geometry and planetary movement. Bumgardner (n.d.:2) referred to the *Organum Mathematicum* as the 17th-century equivalent of today's laptop (Bumgardner n.d.:1-2; Loughridge 2013).

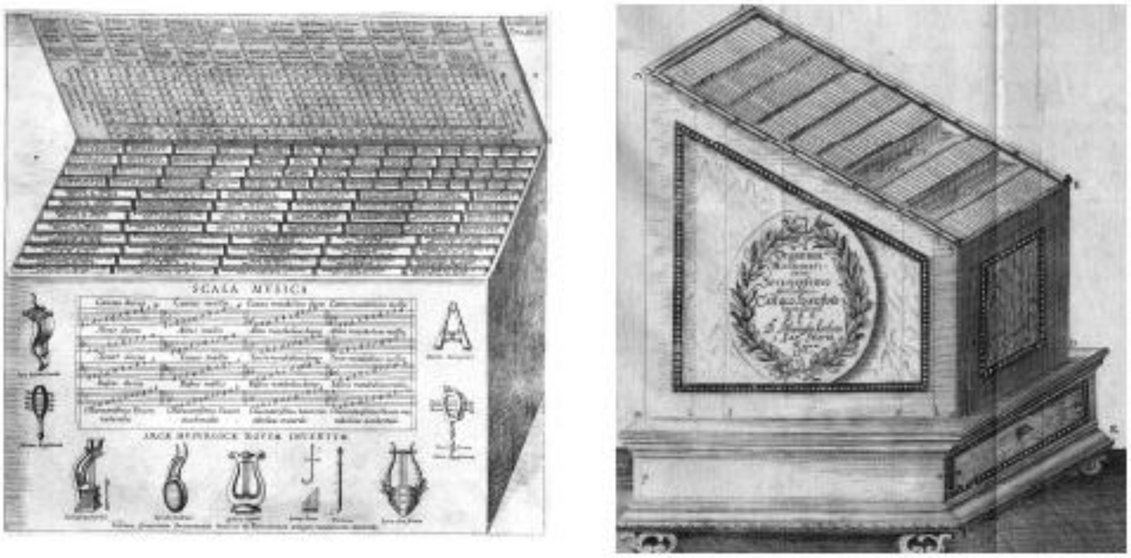

Figure 1: Kircher's two composition "machines": *Arca Musaritmica* (left) and *Organum Mathematicum* (right)

In 1792, a year after Mozart's death, his *Musikalisches Würfelspiegel* was published. This "Musical Dice Game" consisted of 272 bars of pre-composed music that could be put together in different orders to create a minuet and trio. Two six-sided dice are rolled and, adding the numbers together, the corresponding measure is looked up in a table. This is done for each bar of the minuet. For the trio, a single dice is rolled and the same procedure followed. By completing the process, the "composer" will have a minuet and trio of 16 bars each in the style of Mozart. There are a total of 11^{16} (or 45,949,729,863,572,161) different possible permutations (Peterson 2001).

Composers began using mathematics as a composition technique or a basis for a certain composition more frequently during the 20th century. Mathematical properties that were often used included number sequences, mathematical formulae, geometric figures and statistics. Twelve-tone music, led by Arnold Schönberg, Alban Berg and Anton Webern, can also be regarded as mathematical because it relies on the permutations of a series.

The rhythms in Messiaen's compositions are often derived from prime numbers or other number sequences. Xenakis used not only mathematics in his music, but also science and architecture, and many of his compositions are governed by statistics and probability theory, specifically stochastic theory (Tatlow & Griffiths 2001:231-235).

Twentieth-century musicologists started to do extensive analysis of works by Bach, Mozart and Bartók, and found significant mathematical structures in some of their compositions. It is said that Bach's music is mathematical, but it is not possible to prove whether or not Bach consciously used mathematical structures in his compositions. The Fibonacci series and the golden ratio are two of the most common mathematical properties in music, and feature extensively in the music of Debussy and Bartók.

Developments in the sciences led to new theories and have since been incorporated into music composition. Fractal geometry, like algebra, calculus and geometry, is a branch of mathematics which can be applied to music. This research study explored how the new branch of fractal geometry and its related self-similarity can be applied in the analysis and composition of music.

1.1.2 What are fractal geometry and self-similarity?

The first question that arises from the topic of the dissertation is: What are fractal geometry and self-similarity? The term "fractal" is used to describe geometric objects that reveal new detail with every magnification of that object. If the object is magnified at a specific point, a smaller replica of its overall shape is seen – a property known as self-similarity (Brothers 2004).

As an example to illustrate this, the researcher chose the Julia set (Figure 2). The Julia set is not merely a beautiful, colourful picture, but also contains many mathematical intricacies. Discovered by the French mathematician Gaston Julia (1893-1978), this set, when visually represented as below, contains various levels of self-similarity. Note how the blue spirals as well as the pink-purple circular shapes occur at different places in the figure on different scales (in other words, in different sizes). If one zooms into a part of the figure, it will present a copy of the entire object (Chen 2004).

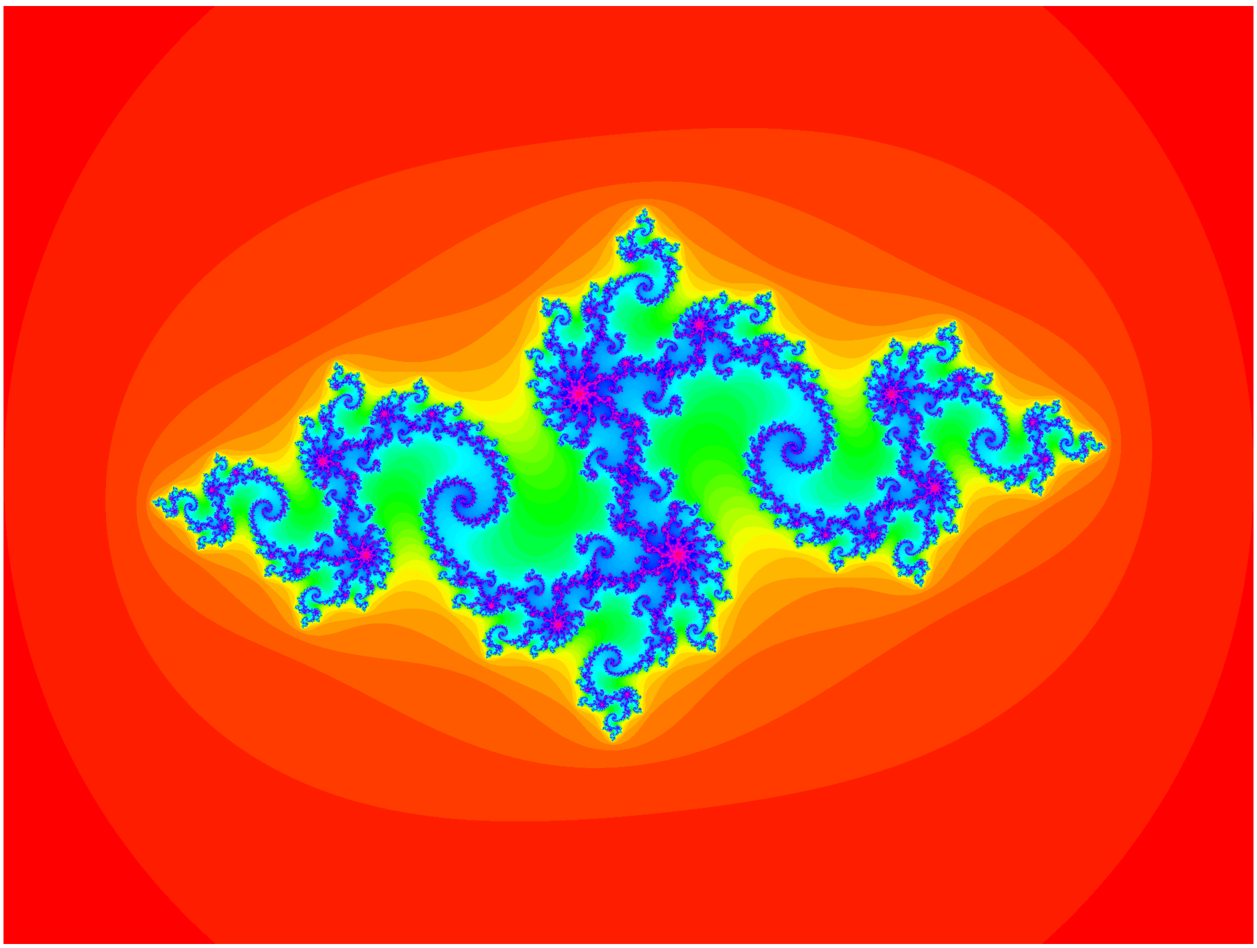

Figure 2: Julia set (Chen 2004)

The term “fractal” was coined in 1975 by the French mathematician Benoit B. Mandelbrot (1924-2010). In his book, *Fractals: form, chance, and dimension*, Mandelbrot (1977:4) explained the etymology of the term as follows:

> *Fractal* comes from the Latin adjective *fractus*, which has the same root as *fraction* and *fragment* and means “irregular or fragmented;” it is related to *frangere* which means “to break”.

The term thus accurately describes the irregular, fragmented yet self-similar nature of fractal objects.

Towards the end of the 19th century, mathematicians became increasingly aware of non-Euclidean shapes (i.e. shapes that were not perfect circles or squares), but were still not able to mathematically define all of them. In addition, there was no specific term under which these shapes could be categorised.

It is necessary to stress that fractals were not *identified* in 1975, but that a term was finally conceived to describe specific non-Euclidean shapes with a self-similar structure. Many other mathematicians had identified irregular shapes prior to Mandelbrot’s coinage of the term. Examples are Koch, Cantor, Peano and Sierpinski, to name a few (Mandelbrot 1977:4). The contributions of these mathematicians as well as the concepts relating to fractal geometry and self-similarity are discussed in greater detail in Chapter 2. Nevertheless, the researcher used the year 1975 as a date to separate the conscious use of fractals in art and music from an intuitive utilisation.

1.1.3 Fractal geometry and self-similarity in music

In the first part of the background earlier in this chapter, it was briefly illustrated how mathematics has impacted on music composition for hundreds of years. It can thus be accurately assumed that a new branch of mathematics would result in an altered view of music analysis and new techniques for music composition.

To analyse a composition by using fractal geometry, one must consider the specific composer’s intention with the music. It is highly unlikely that composers such as Bach, Mozart and Beethoven consciously used fractals in their music, since fractals were only defined and understood centuries later. It is debatable whether or not it is in fact valid to analyse music in this fashion. Music examples were analysed and discussed in this dissertation to test whether or not fractal structures exist in some classical compositions. The validity of finding such structures in music compositions prior to the 20th century was also investigated.

A number of 20th-century composers have *intentionally* incorporated algorithms or ideas from fractal geometry in their music compositions. Examples include György Ligeti, Charles Dodge, Tom Johnson and Gary Lee Nelson. Some of their compositions were built only on the *idea* of fractals and chaos theory, while others contain intricate fractal structures. Some works of these composers and their composition methods are discussed in Chapter 4.

Fractal geometry is still a new concept when compared to other branches of mathematics. Because mathematical concepts have been used to analyse and compose music for many years, it is only natural for a new mathematical concept, such as fractal geometry, to be applied to music in a similar fashion. It was not the researcher's intent to define a new method for music analysis, but rather to investigate a new manner of thinking about music analysis.

1.2 Problem statement

Despite the research that has been conducted on fractal music, the following problems can still be identified:

- Fractal geometry and its different applications to music are not well-known subjects among musicians or mathematicians. This afforded the researcher the opportunity to create a better awareness of the topic among individuals in these fields, as well as the connection between music and fractals.
- Preliminary research has shown that there is still some ambiguity to the term "fractal music".
- Most writings on fractal and self-similar music focus on a single subject, but there are few sources that give a concise summary of all the developments in fractal music.
- There are different types of fractal and self-similar music, but they are not easily distinguished from one another.

- Music examples of fractal and self-similar music and their analyses are scarce.
- The literature dealing with fractal and self-similar concepts is predominantly explained from a mathematical or scientific perspective, limiting its understanding for musicians unfamiliar with mathematical terminology and formulae.
- Not all findings from existing research are accurate, which opened up the field for further research.

It can be concluded that the novelty of fractal geometry inhibits both musicians' and mathematicians' understanding of the possible applications of fractal geometry and self-similarity to music.

1.3 Motivation for the study

The researcher has always been interested in the relationship between music and mathematics, and how music can be composed or analysed mathematically. Being raised in a family in which there has always been a fine balance between science and art, the researcher found it easy to approach an interdisciplinary field such as fractal music. Her father, a medical doctor and freelance painter, and her mother, a private mathematics tutor and amateur pianist, instilled a love for both the sciences and the arts from an early age.

The researcher started reading extensively on the applications of various mathematical concepts to music composition in high school and has been fascinated by it ever since. As an undergraduate BMus student at the University of Pretoria, she also enrolled for courses in Algebra and Calculus.

For her BMus essay in 2008, the researcher studied some of the compositions by the Greek composer, Iannis Xenakis (1922–2001), who incorporated elements of mathematics, science and architecture in his music. The best example of this is his orchestral work, *Metastaseis* (1953–1954), which has the same contour as the Philips Pavilion that Xenakis designed for the Brussels fair. Making use of graphic notation, Xenakis composed the initial draft of *Metastaseis* on the same graph paper used by architects for their designs. The sweeping contour of the music is similar to that of the pavilion (Matossian 1986:61; Xenakis 1992:10).

While researching Xenakis's composition methods, the researcher came across information on fractal music, a topic previously unknown to her. Although Xenakis did not specifically compose fractal music, many of his composition methods were linked to similar sub-sections in mathematics and physics such as white noise[1] and Brownian motion[2]. (These noise forms are discussed in detail in Chapter 2.)

This dissertation is aimed at individuals who, like the researcher, share an interest in both music and mathematics and would like to learn more about the utilisation of fractal and self-similar geometry in music. There are numerous compositions that rely on mathematical and scientific principles and an increasing number of books and journals are being published on this topic. The researcher's preliminary research has shown that there is a general lack of familiarity with fractal and self-similar music. There are many misconceptions about fractal and self-similar music: How can fractals be used to compose music? And can fractal or self-similar structures be found in examples of Western art music through analysis? The researcher's aim was to answer such questions in the dissertation.

1.4 Research question

These questions led directly to the research question of the study, namely How can fractal geometry and self-similarity be further explained and applied to the composition and analysis of art music?

The main question led to a number of sub-questions that are discussed in each chapter of the dissertation:

Chapter 2: THE HISTORY AND MATHEMATICS OF FRACTAL GEOMETRY AND SELF-SIMILARITY

- How are fractal geometry and self-similarity defined in mathematics?
- Do examples of fractals exist in nature and art?

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[1] White noise refers to a noise type which is highly uncorrelated and unpredictable, such as the static on a television.

[2] Brownian motion, also called brown noise, is the opposite of white noise as it is correlated and predictable.

- Who were the pioneers of the mathematics behind fractal geometry and self-similarity?
- What are the different types of fractals that can possibly be used in music composition and analysis?

Chapter 3: LITERATURE REVIEW

- Who were the predominant researchers in the field of fractal and self-similar music and how did their approaches to the subject matter differ?
- Are these researchers' findings still valid today?
- What are the different methods used to find fractal or self-similar features in a composition?
- What is the validity of fractals and self-similarity in music?

Chapter 4: FRACTALS AS A COMPOSITION TOOL

- What properties of fractal geometry and self-similarity can be applied to music composition?
- What methods can be used to compose fractal music?
- How do different types of fractal music differ?

Chapter 5: FRACTAL GEOMETRY AND SELF-SIMILARITY IN MUSIC ANALYSIS: MUSIC PRIOR TO 1975

- How can art music be analysed by using fractal geometry?
- What is the significance of analysing a composition with fractals?
- If fractal structures can be found in music prior to 1975, how can they be explained?

Chapter 6: CONCLUSION AND SUGGESTIONS FOR FURTHER STUDY

1.5 Objectives of the study

From the literature review and the references, it will be clear that much research has already been conducted on fractal music. Nevertheless, it is hoped that the researcher's extended research will lead to new insights in order to better understand fractal music and create a greater awareness among musicians and mathematicians of its applications.

The primary objective of the study was to define fractal and self-similar music in order to compose and analyse such works. Hence the aims of the study are to

- establish a better understanding of fractal and self-similar music by summarising the most important research conducted thus far in an extended literature review
- simplify difficult mathematical concepts that are essential in understanding fractals and self-similarity, but described in a too complex manner in many existing sources
- explain the properties of fractal and self-similar music
- give examples of how fractal music can be composed
- discuss the occurrence of fractal and self-similar properties in selected compositions from different style periods in Western art music.

1.6 Delimitation of the study

It was assumed that the readers of the study have an understanding of music terminology. Therefore, the basic musical terms used in the dissertation are not defined in detail unless absolutely necessary. Similarly, the basic concepts of arithmetic, algebra and geometry used in mathematics are not defined. However, such concepts are briefly explained in the context of the dissertation when referred to.

By contrast, the mathematical properties of fractal geometry and self-similarity are defined and explained in more detail. These definitions and explanations were deliberately simplified from their complex mathematical definitions and were restricted to the fundamentals – in order to understand how they can be applied to music.

Although there are algebraic applications of fractals, the study only deals with the geometry of fractals and its subsequent applications to music. Recommendations for further reading on more scientific applications of fractals are included in the study.

Discussions of fractal compositions from the 20th century were included, but many times without examples of the scores. This was largely due to the unavailability scores because of copyright. In other cases, such as the music of Charles Dodge, the music was composed electronically and no music score could be found.

There are some harmonic and structural analyses of compositions displaying fractal or self-similar properties in the dissertation. These discussions focus mainly on fractals in the form structure and rhythms of the compositions. Owing to the scope of the dissertation, the compositions that are discussed in the dissertation are limited. Selected works from different time periods are discussed. Works are divided into two groups: music composed prior to Mandelbrot's coinage of the term "fractal" in 1975 and subsequent works. Music examples before 1975 include mainly works from the Renaissance, Baroque and Classical eras.

1.7 Research design

Different research designs were used for the study, namely interdisciplinary research, an extended literature review and analysis of compositions (Mouton 2006:179-180; Hofstee 2006:121-122). These designs were chosen because the researcher deemed them to be the best way to draw a meaningful conclusion to the research question.

The study of fractal music can already be classified as an interdisciplinary field, because the topic relies on the application of a specific mathematical concept (in this case, fractal geometry and self-similarity) to music. This also justifies the use of many non-musical sources for the study.

The purpose of conducting an extended literature review was to summarise as much of the existing research, in order to establish the current understanding of the topic. In so doing, it was possible to draw new conclusions on the subject matter. Lastly, small-scale harmonic and structural analysis of compositions and the discussion thereof show the practical applications of fractal geometry to music.

Figure 3 outlines the research design for the dissertation.

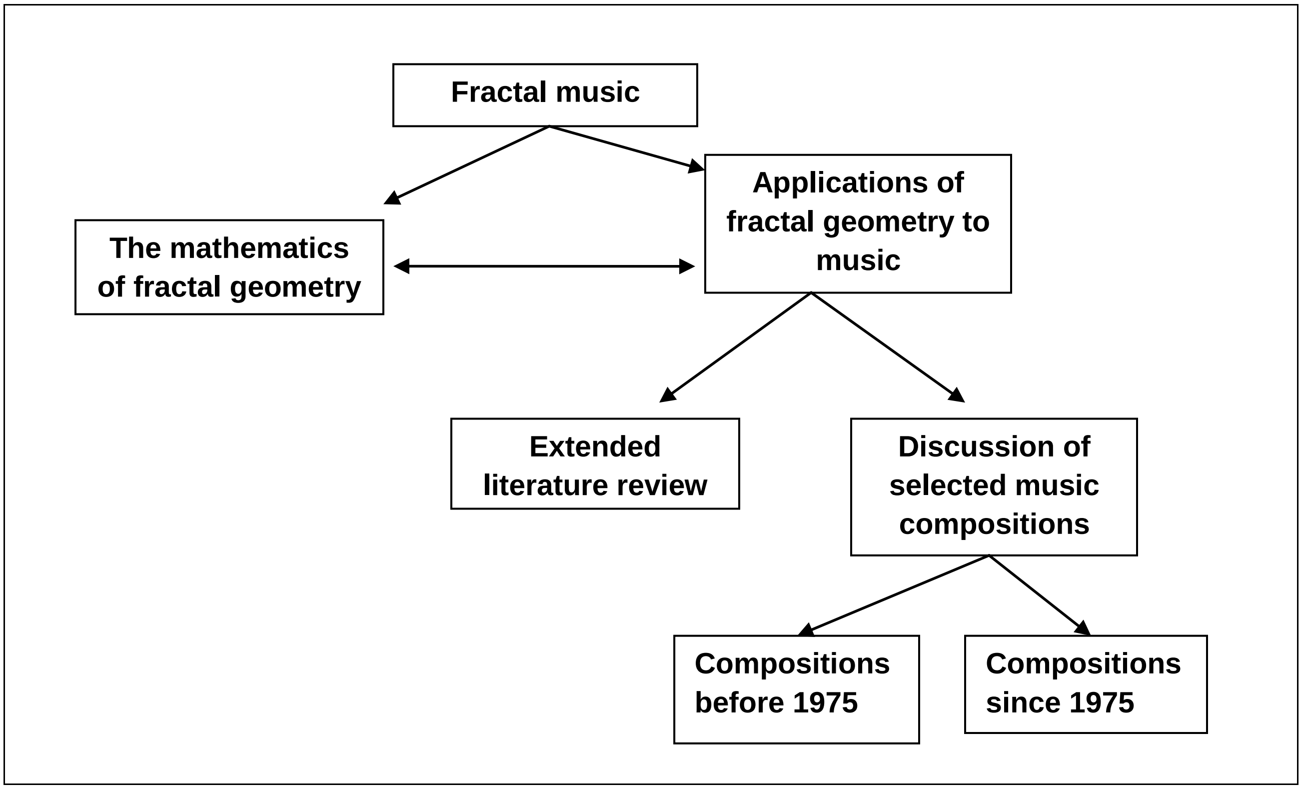


Figure 3: Outline of the research design

1.8 Research methodology

The data and information necessary for the research were collected from a vast range of books, journals and internet resources. Similar data and information from different sources were compared to ensure their validity and reliability. Where necessary, dictionaries were used to find the apt definitions or explanations of unfamiliar mathematical and scientific concepts.

Since the study is interdisciplinary, scientific books and journals dealing with fractals were scanned for the word "music". Likewise, books on music composition and analysis were skimmed for terms like "fractal", "self-similarity" and "iteration"[3]. Internet and library searches were also conducted with a variety of combinations of these words.

[3] Iteration relies on the repetition of a formula or procedure.

In this research, a great part of research methodology relied on unifying the findings in existing literature with their associated mathematical formulae and music examples.

1.9 Summary of chapters

Chapter 2 starts with a discussion of the similarities between music and geometry in general. Thereafter, the mathematical properties of fractals and self-similarity that support fractal music are discussed in detail. It also provides a historical overview of the mathematical developments that led to the classification of fractal geometry as a mathematical discipline. Specific reference is made to the pioneers in the field of fractal geometry, such as Koch, Cantor, Sierpinski, Peano and Mandelbrot. This serves as background to understanding how such fractals can be applied to the composition and analysis of music. Fractal noise forms, such as white, brown and 1/f noise[4], are also be defined because of their close relationship with fractal music.

Chapter 3 provides an extended literature review of the most significant research on fractal and self-similar music since Mandelbrot's coinage of the term in 1975. The first subsection focuses on the distribution of 1/f noise in music. The researcher also explains how such distributions may be used to distinguish between different genres of music or even the stylistic differences between the works of different composers. Furthermore, the prerequisites for fractality are studied, so that the concept can be applied in subsequent chapters. Finally, the validity for searching for fractal and self-similar structures in music prior to the 20th century is investigated.

Chapter 4 will describe some of the most common methods for composing fractal music, such as dice and spinners, 1/f noise and L-systems[5]. Music examples from research papers as well as a number of original works by contemporary composers such as Tom Johnson, Gary Lee Nelson and Charles Dodge are included. It is also explored how fractals were used metaphorically in some compositions by György Ligeti.

The main focus of Chapter 5 is to see if there are any compositions from previous centuries that contain elements of fractals or self-similarity. These include compositions from the Renaissance

[4] 1/f noise (pronounced one-over-f noise) lies between the two extremes of white and brown noise.
[5] L-systems or Lindenmayer systems are methods of rewriting a string of symbols to create increasingly longer chains.

to the 21st century. Music examples are included, analysed briefly and discussed with reference to different fractal elements.

Chapter 6 contains a summary of the entire dissertation. Conclusions and recommendations for further reading on related topics are also included.

CHAPTER 2: THE HISTORY AND MATHEMATICS OF FRACTAL GEOMETRY AND SELF-SIMILARITY

2.1 Introduction

Before any of the applications of self-similarity and fractal geometry to music can be investigated, it is necessary to explain some of the mathematical terms and their properties in greater detail. This chapter deals with the mathematical concept of fractals and their origins. It is also shown that fractal shapes occur in many natural phenomena. Concepts such as fractal dimension, scaling and transformation are defined so that they can be applied to music in subsequent sections of the dissertation. Some of the most common fractals in mathematics are given as examples, together with their pioneers.

2.2 Music and geometry

Music and geometry share many of the same terms and concepts which is important throughout the course of this study. Musical themes or motifs, like geometric figures, can undergo certain transformations in which the original is discernible but presented differently for variation.

Geometric transformations include scaling, translation, reflection, rotation and shearing. The geometric and music examples provided below show how each of these geometric transformations is applicable to music. The image of a treble clef was chosen by the researcher to illustrate the various transformations a geometric object can undergo.

2.2.1 Transposition and translation

One of the simplest tools used by composers to bring variety into music is through transposition. In the case of large-scale compositions, entire sections are often repeated in a different key. Alternatively, shorter motifs or themes can be transposed to create sequences. The first 8 bars from the first movement of Mozart's Piano Sonata No. 16 in C major, K 545 (Figure 4) is given as an example. The C major scale in the right hand in bar 5 was transposed diatonically three times, each a second lower creating sequences.

Figure 4: Mozart, First movement from Piano Sonata No. 16 in C major, K 545, bars 1-8, illustrating transposition (Mozart 1938)

The geometric equivalent of transposition is known as translation. Translation occurs when all the coordinates or points of an object are moved by a fixed distance in the same direction, up, down or sideways. Figure 5 below shows the treble clef being moved to the right.

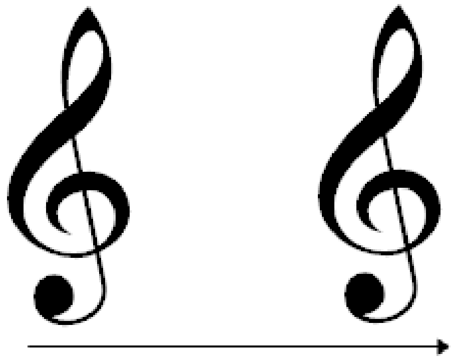


Figure 5: Geometric translation

2.2.2 Scaling

Scaling (also known as dilation in mathematics) is enlarging or reducing the size of an object while its dimensions remain the same, as is depicted in Figure 6.

Figure 6: Geometric scaling

The musical equivalent of scaling is augmenting or diminishing note values. To illustrate this, an excerpt from Berlioz's *Symphonie Fantastique* is given below. The fifth movement, entitled *Songe d'une Nuit du Sabbat,* changes frequently in tempo, time signature and character. Bar 127 marks the beginning of the *Dies Irae*. The bassoons carry the melody in dotted minims. In the second half of bar 147, the horns take over the melody (in thirds), at double the speed in dotted crotchets. A further reduction appears from the last beat of bar 156, where the woodwinds bring out a modified form the melody. Instead of presenting the melody in quavers only (which would have been four times the speed of the original melody), it is stated in alternating quaver and crotchet beats for further rhythmic variation (Caltabiano n.d.).

Figure 7: Berlioz, *Symphonie Fantastique,* Fifth movement (*Songe d'une Nuit du Sabbat*), bars 127-162, illustrating the diminution of note values (Adapted from Berlioz 1900)

This shows how the diminution of note values can be used to vary thematic material. The same can also be done by augmenting the note values.

2.2.3 Reflection

Melodic inversion and retrograde are the same as the *reflection* of an object around the x-axis and y-axis respectively. This is thus an object's mirror image.

Figure 8 shows the right hand from the first bar of Bach's Two-Part Invention No. 14 in B♭ major, BWV 785. This melody consists of a head (marked (a)) and a tail, which is the melodic inversion of the head.

Figure 8: Bach, Two-Part Invention No. 14 in B♭ major, BWV 785, right hand, bar 1, illustrating melodic inversion (Adapted from Bach n.d.)

In geometry, inversion can be likened to horizontal mirroring or reflection around the x-axis:

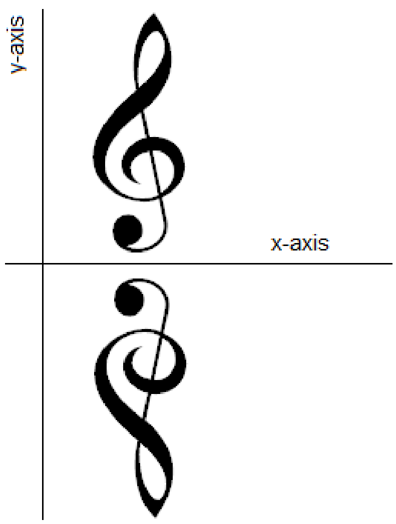


Figure 9: Reflection around the x-axis creating a horizontal mirror

A mirror image of an object can also be obtained through reflection around the y-axis or vertical mirroring:

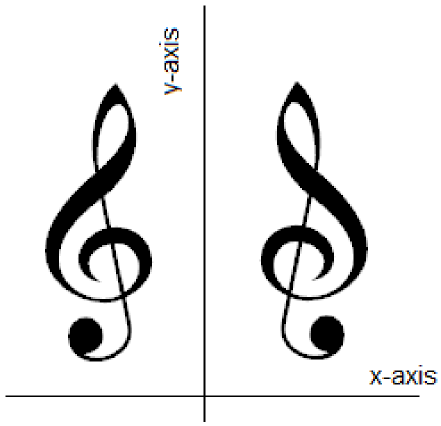


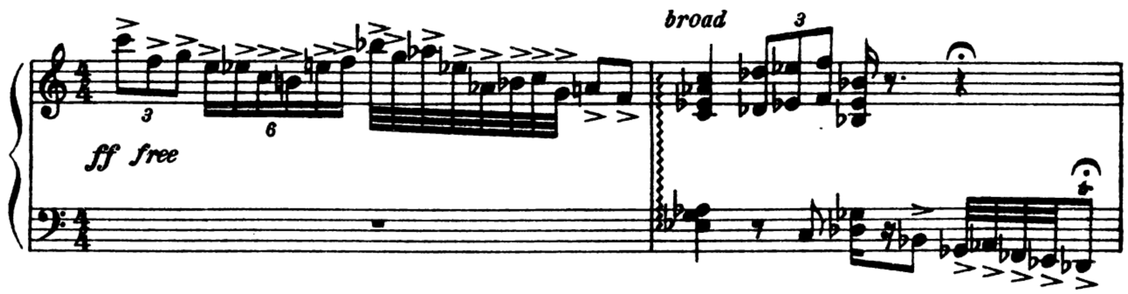
Figure 10: Reflection around the y-axis creating a vertical mirror

The musical equivalent of vertical mirroring is retrograde, where material is presented backwards. Figure 11 shows the first 20 bars from Haydn's *Menuetto al Rovescio* from his Piano Sonata in A major, Hob. XVI:26. Bars 11-20 is an exact retrograde of the first ten bars, thus creating a perfect vertical mirror and palindrome.

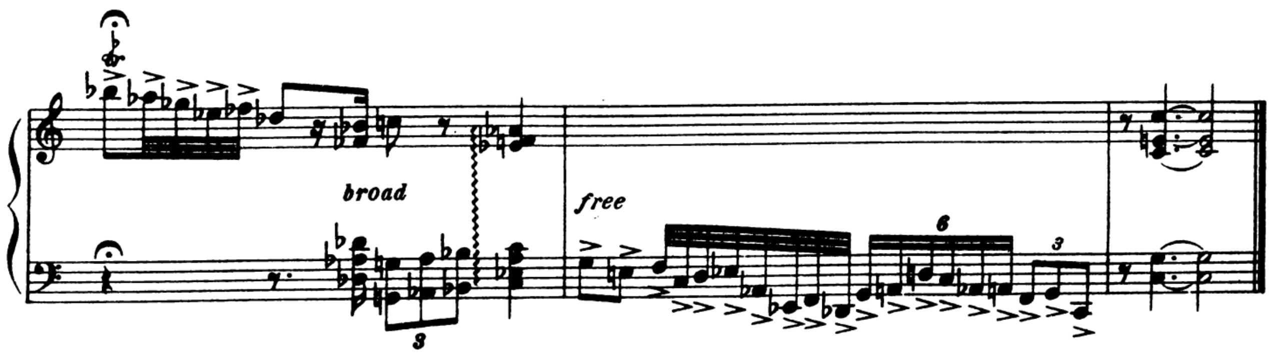
Figure 11: Haydn, *Menuetto al Rovescio* from Piano Sonata in A major, Hob. XVI:26, bars 1-20, illustrating retrograde (Haydn 1937)

2.2.4 Rotation and retrograde inversion

In music, retrograde inversion occurs when a motif or theme is repeated with both a horizontal and vertical mirror, i.e. retrograde inversion. Hindemith's *Ludus Tonalis* is an example of this. The *Postludium* that ends the piece is an exact retrograde of the *Praeludium* that was heard in the beginning, except for the added C major chord in the last bar of the composition. Figure 12 shows (a) the opening and (b) closing measures of *Ludus Tonalis*.



(a.) Opening measures from Hindemith's *Ludus Tonalis*



(b.) Last three bars from Hindemith's *Ludus Tonalis*

Figure 12: Hindemith, *Ludus Tonalis*, first and last bars, illustrating retrograde inversion (Hindemith 1943)

Retrograde inversion is similar to spinning an object around a fixed point, known in mathematics as *rotation.*

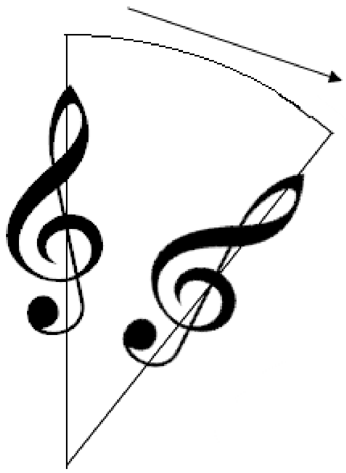


Figure 13: Rotation around a fixed point

If two or more of the aforementioned transformation techniques are combined simultaneously, this is called *shearing* in mathematics*.* Figure 14 shows an excerpt from Bach's *Contrpunctus VIII* in which the theme is repeated in the top voice in melodic inversion with the note values doubled. The alto enters in bar 3 also with the theme inverted, but in the same note values as the original theme. This example illustrates the simultaneous utilisation of melodic inversion and rhythmic augmentation.

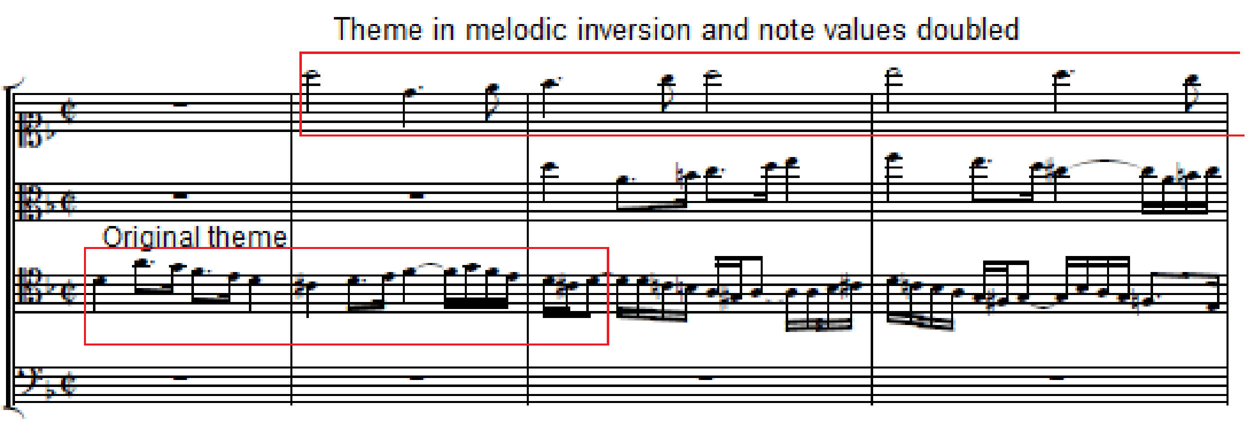


Figure 14: Bach, *Contrapunctus VIII* from *Die Kunst der Fuge*, bars 1-4, illustrating simultaneous use of two transformations, inversion and augmentation (Adapted from Bach n.d.)

From the above transformations it is clear how closely linked geometric construction and music composition are, and this establishes a connection between the two different fields. Scaling is of particular significance in this dissertation because it is an important feature of fractals.

2.3 A short history of fractal geometry

Now that a direct connection between music and geometry has been established, the history and evolution of fractal ideas can be discussed. The easiest way to describe and define fractals is by looking at examples in nature and art.

2.3.1 Examples of fractals and self-similarity in nature and art

For centuries, Euclidean geometry used to be applied in order to define the physical structures of objects. It is named after the Greek (Alexandrian) mathematician, Euclid (c.325 BC–c.265 BC), who wrote *The Elements,* one of the first known treatises on geometry. He defined shapes such as circles, triangles, rectangles and squares which formed the foundation for geometry as it is known today (Hawking 2005:1,7). These theories have served as “a model of what pure mathematics is about” (Clapham & Nicholson 2005:155). Any geometrical structures that did not fit the characteristics of Euclidean geometry were simply regarded as “non- Euclidean”.

In the book, *Art and physics: parallel visions in space, time and light,* Leonard Shlain (1991:30-31) pointed out that “Euclid made some … assumptions that he did not state in *The Elements*. For example, he organized space as if its points could be connected by an imaginary web of straight lines *that in fact do not exist in nature.*” It is exactly such images and objects that cannot be defined by means of Euclid’s geometry that form the core of this dissertation.

Nevertheless, examples of fractals and self-similarity have existed in nature around us for millennia, even though scientists were not able to define them. Figure 15 shows a number of fractals commonly found in nature: lightning, clouds, fern leaves and Romanesco broccoli. Mandelbrot is famous for his statement: “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”

Lightning (Figure 15 (a)) indeed does not move in a straight line, but instead branches out continuously, with each smaller branch resembling the larger bolt. Similarly, the shape of clouds (Figure 15(b)) cannot be defined accurately with the use of Euclidean geometry, and self-similarity is evident. A fern leaf's structure (Figure 15(c)) is also self-similar because it comprises many smaller leaves with the same shape and structure as itself. Finally, a Romanesco broccoli (Figure 15(d)) is cone-shaped, but consists of smaller cones with the same shape and level of irregularity (Frame, Mandelbrot & Neger 2014).

Figure 15: Natural fractals: (a) lightning; (b) clouds; (c) fern leaves; and (d) Romanesco broccoli (Frame et.al 2014)

Intricate self-similar patterns appear to be commonly found in nature. This has inspired many painters over the centuries. The Italian Renaissance polymath, Leonardo da Vinci, created paintings with a blotting method that created self-similar cloud-like shapes. *The Deluge* (Figure 16) is one such example (Frame et.al 2014).

Figure 16: Da Vinci, *The Deluge* (Kuhn 2009:67)

Da Vinci also wrote about water and sound waves in a way that echoes self-similarity (Shlain 1991:76):

> Just as a stone thrown into water becomes the centre and cause of various circles, sound spreads in circles in the air. Thus everybody placed in the luminous air spreads out in circles and fills the surrounding space with *infinite likeness of itself and appears all in all in every part*.

The phrase in italics can possibly be replaced with “infinite self-similarity on all scales”. This proves that Da Vinci and probably some of his contemporaries had noticed self-similar and fractal structures in nature around them.

The Japanese artist, Katsushika Hokusai (1760-1849), made woodblock prints inspired by Mount Fuji during the 1830s and 1840s. His print, *In the Hollow of a Wave off the Coast of Kanagawa*, better known as *The Great Wave* (see Figure 17) presents "fractal aspects of nature with a sophistication rarely matched, even today" (Frame et.al 2014, Kuhn 2009:67). Many of his prints, such as this one, are characterised by waves with jagged edges that become increasingly smaller towards the edges of the waves.

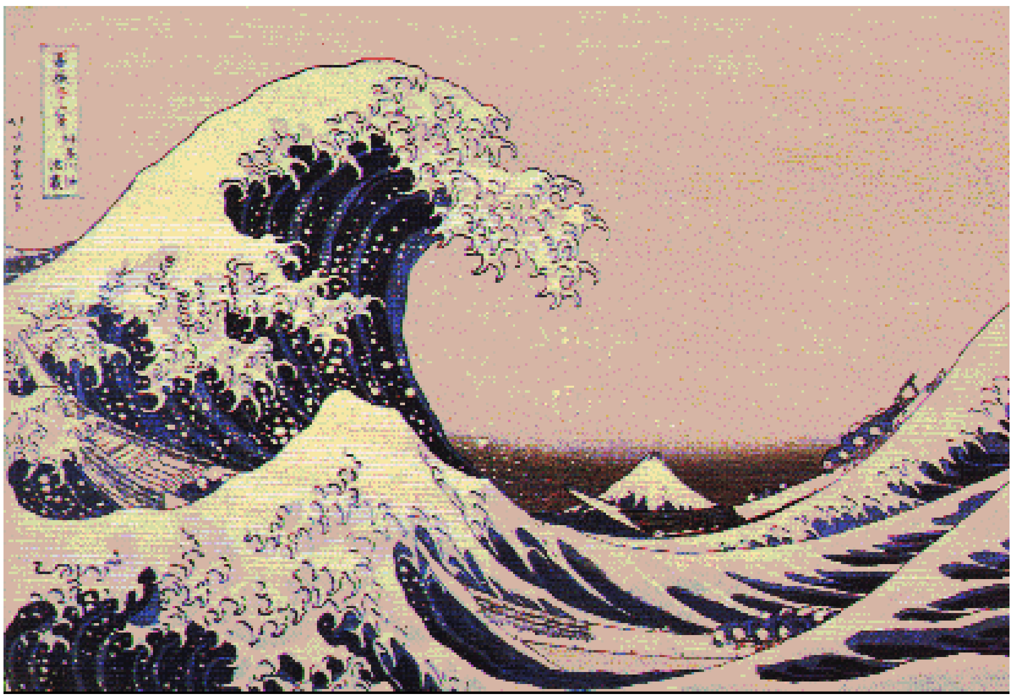


Figure 17: Hokusaki, *The Great Wave* (Frame et.al 2014)

In the late 19th and early 20th century, there was a greater awareness among artists and scientists alike of so-called "non-Euclidean shapes and objects" - artists wanted to create them, while scientists wanted to understand them.

One such example is the Dutch graphic artist, M.C. Escher (1898-1972). While he is best known for drawings that exploit optical illusions, many of his works are characterised by utmost precision, symmetry and self-similarity, in particular. His drawings have been analysed mathematically by mathematicians such as H.S.M. Coxeter (1979). The title of an article by Coxeter, "The Non-Euclidean Symmetry of Escher's Picture 'Circle Limit III'" hints at a consciousness of the presence of self-similarity in the artist's work.

In a letter to Coxeter, Escher described his desire to create drawings in which a single image (or motif) becomes "infinitely smaller". At first he was not sure how to go about obtaining such images, but with the help of Coxeter, Escher succeeded in creating a great number of such drawings and sculptures (Coxeter 1979:19). One such example is *Circle Limit I* (Figure 18) in which the same picture or "motif" is presented at different rotations and scales, becoming increasingly smaller towards the edge of the circle.

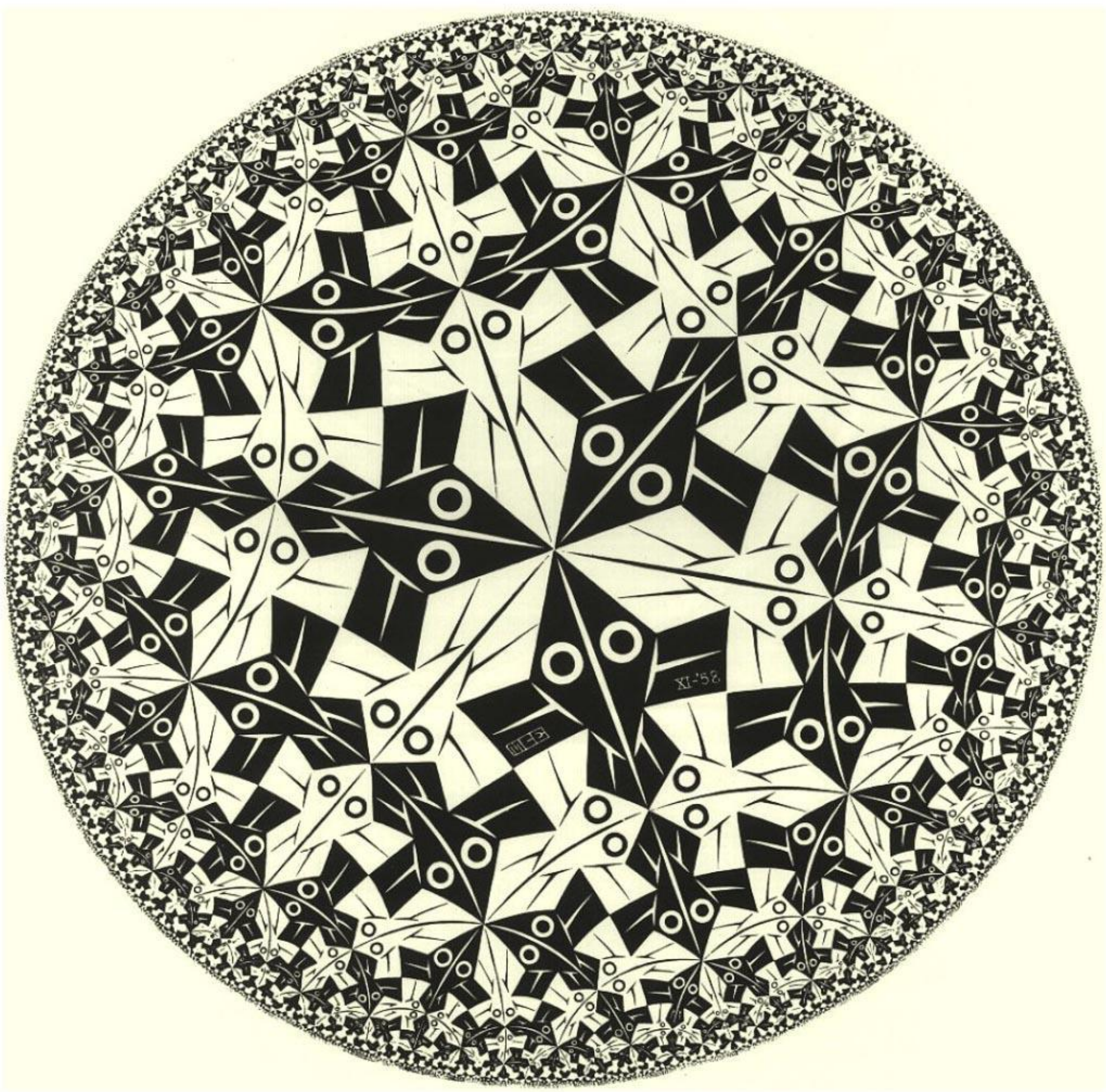


Figure 18: Escher, *Circle Limit I*

The symmetry and self-similarity in Escher's drawings are fairly obvious. According to Escher, his greatest inspiration was the music of J.S. Bach, since the composer also used a single theme or motif and expanded on it by making use of repetition, reflection and other transformation techniques. In a letter to the art historian E.H. Gombrich, in 1961, Escher compared his drawings to the works of Bach (Schattschneider 1990:254):

> It may be that the canon is near to my (anti)symmetric plane-filling mania. Bach played with repetition, superposition, inversion, mirroring, acceleration and slowing down of his themes in a way which is, in many regards, comparable with my translation and glide-mirroring of my "themes" of recognizable figures. And that's perhaps why I love his music particularly.

The 48 preludes and fugues from Bach's *Das wohltemperiete Klavier* provide many examples of such transformations in music. Bach also used the same transformations that Escher used in his art. Take, for instance, the first two bars from the Fugue No. 3 in C# major WTC II, BWV 872:

Figure 19: Bach, Fugue No. 3 in C# major, WTC II, BWV 872, bars 1–2 (Bach 1866)

The fugue starts in the bass with a six-note subject. Before the subject is completed, the soprano follows with the answer (now in the dominant key), and shortly after, the alto with the subject in inversion (back in the tonic key). It is thus a single melody that that was transformed by transposition and inversion to create a variety of different melodic figures.

In his studies, Escher also found patterns in architecture that closely resembled some known fractals. He found that the patterns on the pulpit of the 12th-century Ravello cathedral in Italy (designed by Nicola di Bartolomeo) resembled the Sierpinski triangle which was only defined mathematically in the early 1900s. Compare the two images below:

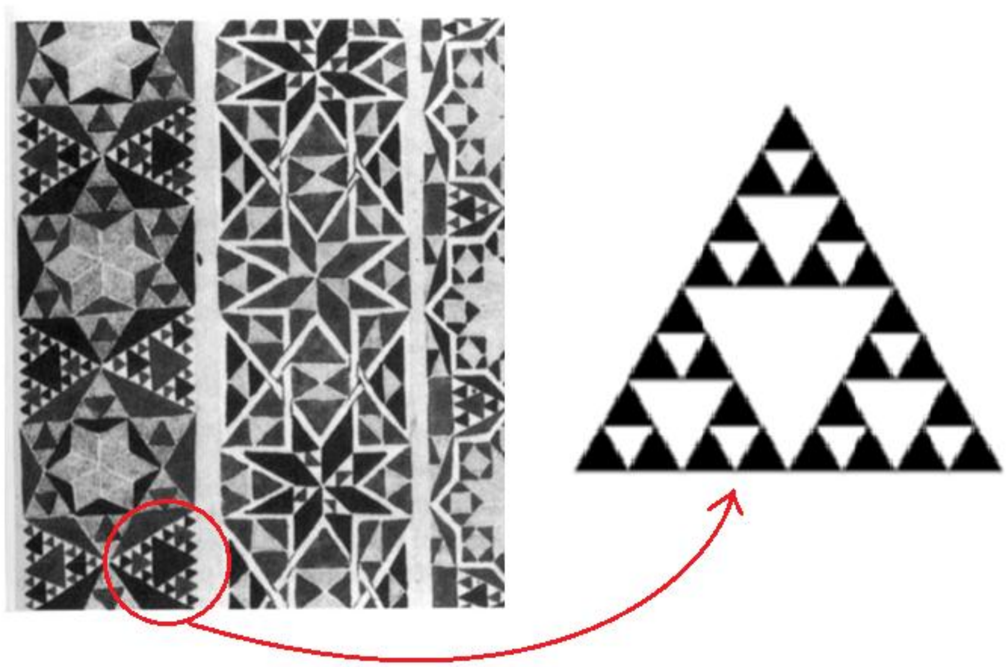


Figure 20: Patterns on the pulpit of the Ravello cathedral (left) resembling the Sierpienski triangle (right) (Adapted from Peitgen, Jürgens & Saupe 1992:79)

Another example of fractals in art is Dali's painting, *Visage of War* (1940), in which each eye socket and mouth of the faces contains another face with the same construction (Frame et.al 2014).

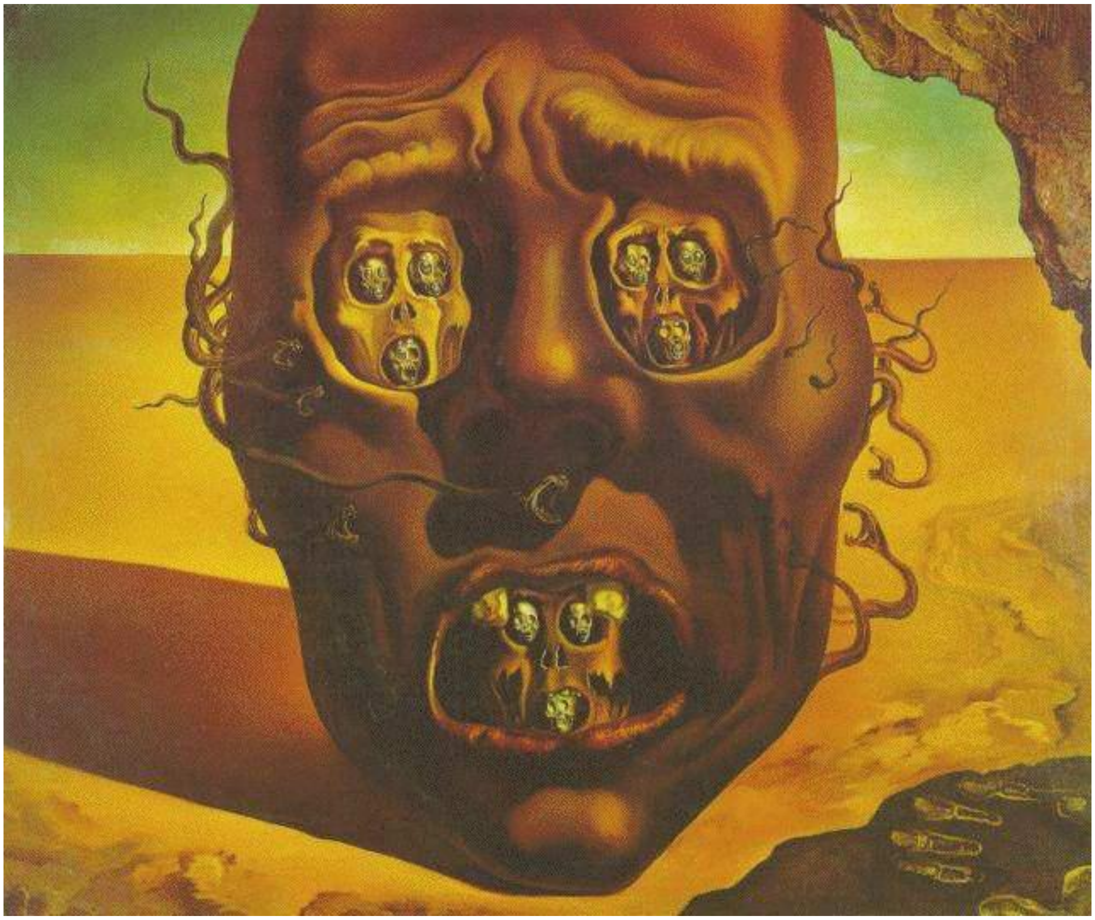


Figure 21: Dali, *Visage of War*

All of this is evidence of the existence of self-similar and fractal-like structures as well as scientists' and artists' interest in them. Since fractal-like and self-similar shapes were present in art and architecture before scientists' delineation of them, one can anticipate that such structures can also be found in music. Such music examples are discussed and analysed in subsequent chapters in this dissertation.

2.3.2 The era of "mathematical monsters"

Between 1875 and 1925, mathematicians turned their focus to shapes and objects that were too irregular to be described by traditional Euclidean geometry. This era was classified as the era of "mathematical monsters". Since fractal geometry was only properly defined in 1975 by Mandelbrot, these early fractals were merely known as self-similar or non-Euclidean shapes at the time (Mandelbrot & Blumen 1989:4).

Some of the most important fractal structures are explained in the next section. The information is given in chronological order and includes the dates of the founders, so that the reader can put each fractal in its specific time frame.

2.3.2.1 Cantor set

One of the earliest examples of a fractal is the Cantor set, named after Georg Cantor (1885-1918). The Cantor set was first published in 1883 to illustrate a perfect, no-where dense subject. In order to reconstruct the Cantor set, start with a line segment with a length of one unit. Divide this line into three equal parts and remove the middle third. There are two smaller line segments each with one-third the length of the original line segment (Figure 22). Continue to divide each of these lines into thirds and remove their middle thirds. If this process is carried out infinitely, it results in the Cantor set or Cantor comb, which is essentially only a set of points (Peitgen et al. 1992:67). The Cantor set or comb is regarded as fractal since each subsequent stage is a smaller replica of its predecessor.

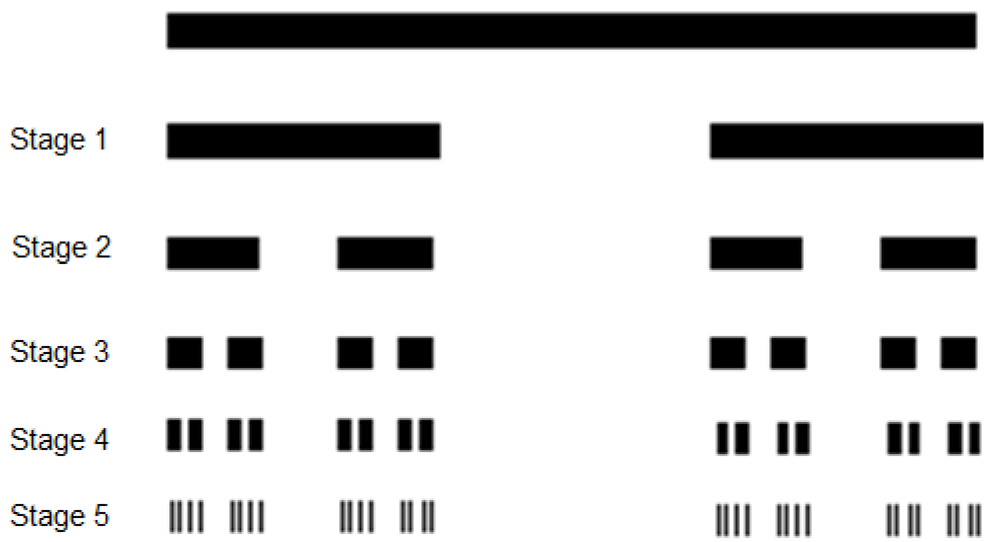


Figure 22: The first five stages of the Cantor set (Adapted from Peigen et al. 1992)

2.3.2.2 Koch snowflake

The Koch snowflake was named after the Swedish mathematician, Helge von Koch (1870-1924). It is an example of a fractal curve that is nowhere smooth and fragmented at all scales.

To construct the Koch snowflake, one starts with an equilateral triangle[6]. Each of its three sides is then divided into thirds of equal length. The middle third of each side is replaced with another equilateral triangle and the base is removed. This process is repeated with each line segment. The resulting structure is known as the Koch island or snowflake. If only part of the island is considered, it is known as the Koch curve. The Koch snowflake has a unique property – although its area can be determined, its length is infinite.

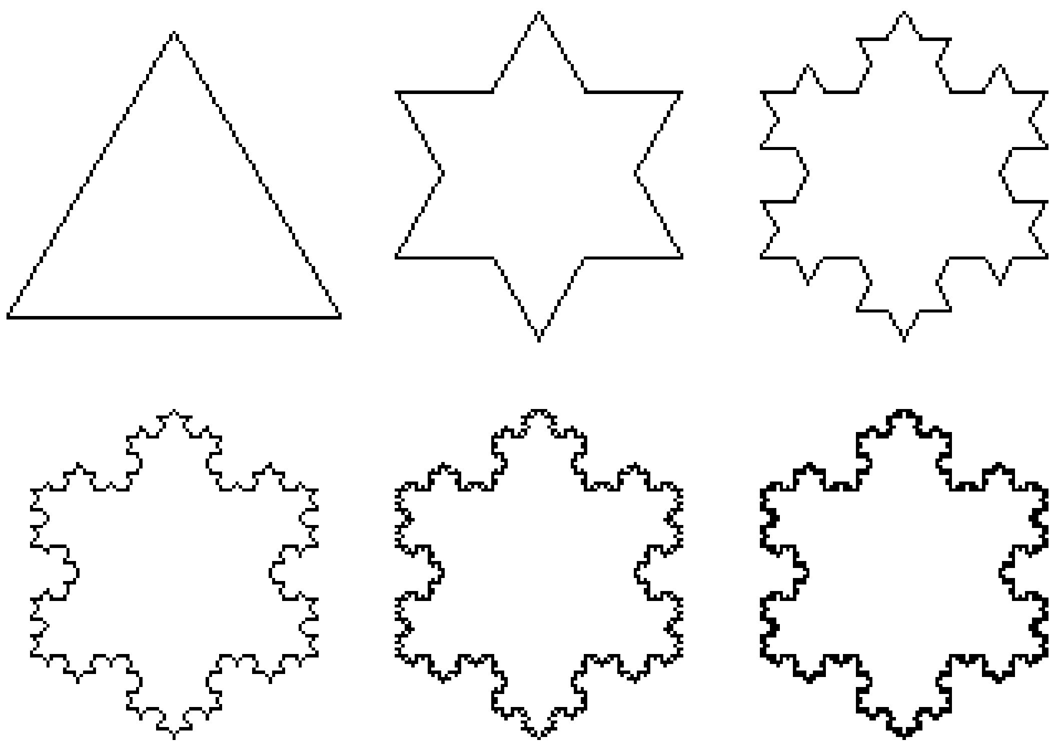


Figure 23: Steps for creating the Koch snowflake (Addison 1997:20)

[6]When all three sides of a triangle are of equal length, it is called equilateral.

2.3.2.3 Sierpinski triangle and carpet

In the previous section, it was shown how decorations on the pulpit of the 11th-century Ravello cathedral resemble the Sierpinski triangle. This self-similar shape was first introduced mathematically in c. 1916 by the Polish mathematician Waclaw Sierpinski (1882-1969), after whom it was named.

The Sierpinski triangle is constructed as follows: start with a black equilateral triangle. Find the centre of each of the sides, connect the points to form four smaller triangles and eliminate the middle triangle. The result is three smaller black triangles, similar to the initial one. When this process is carried out continuously with each of the black triangles, it results in what is referred to as the Sierpinski gasket or triangle (Peitgen et al. 1992:78).

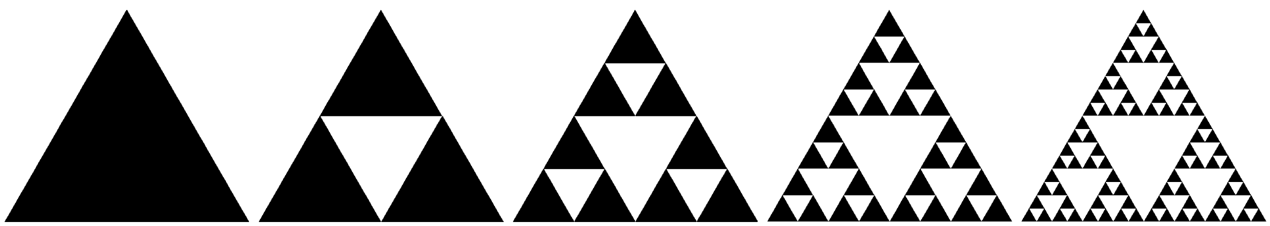


Figure 24: Steps in creating the Sierpinski gasket or triangle (Adapted from Peitgen et al. 1992:79)

The Sierpinski carpet is a variation of the triangle, where the same iterative process is carried out with squares instead of triangles (Peitgen et al. 1992:81).

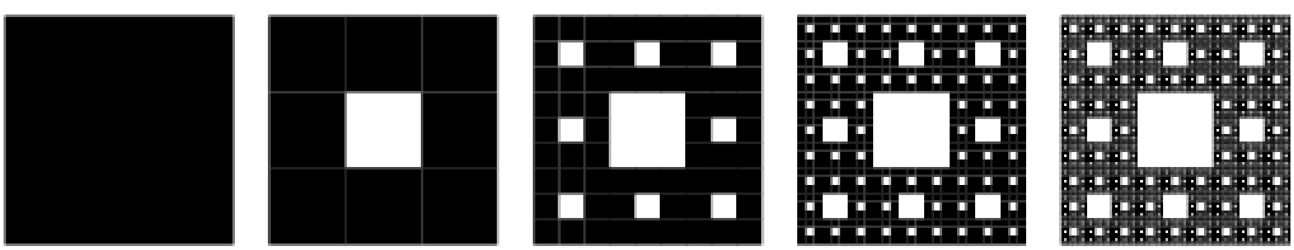


Figure 25: Steps in creating the Sierpinski carpet (Adapted from Peitgen et al. 1992:81)

2.3.2.4 Lindenmayer systems

Lindenmayer systems, also known as L-systems, are a form of string rewriting systems named after the Hungarian biologist, Aristid Lindenmayer (1925-1989). The method was first introduced in 1968 to describe “natural growth processes” in plants. It relies on substituting chains of symbols repeatedly in order to create longer, self-similar chains. (A chain or string in this context refers to a sequence of symbols.) L-systems had a huge impact in the field known as morphogenesis, which is concerned with the “development of structural features and patterns in organisms” (Peitgen et al. 2004:330-331; Prusinkiewicz 1986:445; Jennings 2011).

All L-systems consist of the following three essential elements: an alphabet lists the symbols that will be used in creating a string; an axiom gives the starting point or “seed” for the algorithm; and a set of productions provides the rules employed to produce more strings. The set of productions can be repeatedly applied to each subsequent string in order to create longer strings. According to Jennings (2011), “the languages of most L-systems […] contain an infinite number or strings.”

An example would be to examine the L-system growth pattern of algae. The alphabet consists of only two symbols, A and B. The starting point (or axiom) is B and contains two substitution rules: A → AB and B → A. This means that in each subsequent stage, A is replaced by AB, while B is replaced by A (Manousakis 2006:22-23; Jennings 2011.)

Figure 26 summarises the different elements necessary for the algae L-system growth pattern and shows the different stages of reproduction. It is easy to see how the string rapidly increases in length and that the strings display self-similarity.

Alphabet:	A, B
Axiom:	B
Rules:	A → AB
	B → A
n=0:	B
n=1:	A
n=2:	AB
n=3:	ABA
n=4:	ABAAB
n=5:	ABAABABA
n=6:	ABAABABAABAAB
n=7:	ABAABABAABAABABAABABA

Figure 26: Lindenmayer system growth pattern for algae (Adapted from Manousakis 2006:22-23)

The L-system for the growth pattern of algae possesses another interesting quality. If the number of symbols in each chain is counted, it gives the following number sequence: 1, 1, 2, 3, 5, 8, 13, 21. This series of numbers is known as the Fibonacci sequence, which, like fractals, occurs in many natural phenomena. Consequently, this particular L-system is also known as a "Fibostring" (Jennings 2011).

Lindenmayer systems seem abstract when presented as in Figure 26, but can easily be interpreted graphically through a process known as "turtle graphics".

2.3.2.5 Turtle interpretation of L-systems

According to Mason and Saffle (1994:32), the term "turtle graphics" comes from Logo®, a language developed at Massachusetts Institute of Technology (MIT). The imaginary turtle can be thought of as a pen that can be turned on and off and can carry out several instructions to create an image.

As an example, the turtle can follow five simple steps in order to create a square (Jennings 2011):

1. Touch the pen to the paper.
2. Move forward 1 step, turn left 90°.
3. Move forward 1 step, turn left 90°.
4. Move forward 1 step, turn left 90°.
5. Move forward 1 step.

This is a simple example to illustrate how the turtle works, but it can perform a vast range of more complex instructions resulting in equally intricate images. The four basic commands that the turtle can perform are as follows (Peitgen et al. 2004:351):

F Move forward by a certain fixed step length and draw a line from the old to the new position.

f Move forward by a certain fixed step length, but do *not* draw a line.

\+ Turn left (counter-clockwise) by a fixed angle.

\- Turn right (clockwise) by a fixed angle.

When instructing the Logo® turtle to act on such instructions based on L-systems, there is a vast array of self-similar images that can result from this. Figures 27 and 28 show the L-system and turtle interpretation of the Hilbert curve. The axiom and set of rules contain the symbols X and Y to make substitution possible. For each iteration, a simplified version without these signs is given, since the turtle only carries out symbols +, -, *F* and *f*.

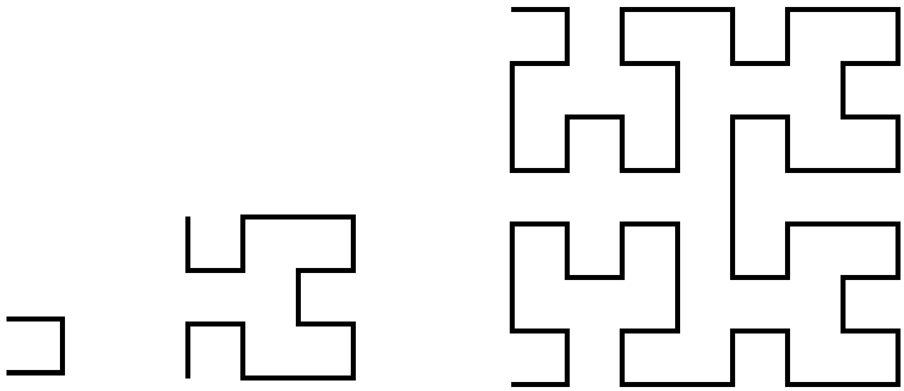


Figure 27: Graphic representation for the first three iterations of the Hilbert curve, created with turtle graphics

Axiom/seed:	-X
Production rules:	X → -YF+XFX+FY-
	Y → +XF-YFY-FX+
n=0 :	-X
n=1:	--YF+XFX+FY-
(Simplified:	--F+F+F-)
n=2:	--+XF-YFY-FX+F+-YF+XFX+FY-F-YF+XFX+FY-+F+XF-YFY-FX+-
(Simplified:	--+F-F-F+F+-F+F+F-F-F+F+F-+F+F-F-F+-)
n=3:	--+-YF+XFX+FY-F-+XF-YFY-FX+F+XF-YFY-FX+-F-YF+XFX+FY-+F+- +XF-YFY-FX+F+-YF+XFX+FY-F-YF+XFX+FY-+F+XF-YFY-FX+-F-+XF- YFY-FX+F+-YF+XFX+FY-F-YF+XFX+FY-+F+XF-YFY-FX+-+F+-YF- XFX+FY-F--+XF-YFY-FX+YF+XF-YFY-FX+-F-YF+XFX+FY-+-
(Simplified:	--F+F+F-FF-F-F+F+F-F-FF-F+F+FF+F-F-F+FF+F+F-F-F+F+FF+F-F-FFF- F-F+FF+F+F-F-F+F+FF+F-F-F+FF-F+F-F-F-F-F+F+F-F-FF-F+F+F-)

Figure 28: First three iterations of the L-system of the Hilbert curve

2.4 Definitions relating to fractals and self-similarity

Now that the reader is more familiar with what fractal objects look like, it will be easier to define many of the terms relating to fractal geometry and self-similarity.

2.4.1 Self-similarity

As mentioned in the introductory chapter, self-similarity is one of the most important properties of fractal geometry, and is mentioned throughout this dissertation. Self-similarity can be easily explained with the use of the following illustration of a fern leaf:

Figure 29: Self-similarity of a fern leaf

Consider the overall structure of a fern leaf: each of the smaller leaves of the fern is a replica of the overall shape but on a smaller scale. In theory, this process of magnification can be carried out infinitely, but with the example of objects in nature, a limit is reached after a certain number of magnifications. This is logical if one considers that any physical object will decompose into molecules and atoms after a while, which are not necessarily self-similar or fractal in structure (Peitgen et al. 2004:138).

Scientists distinguish between two main types of self-similarity, namely statistical and exact self-similarity. The iterations of fractal objects in nature will always differ slightly from the large object, although they are still self-similar (Solomon 2002). This is known as *statistical self-similarity*. Etlinger (n.d.:1) also referred to this as a *discrete spectrum.* The fern leaf in Figure 29 is thus an example of statistical self-similarity.

Mathematically speaking, it is possible to have an object that is self-similar on an infinite number of scales without any differences. This is known as *exact* or *strict self-similarity* (Solomon 2002). The best example of exact self-similarity is found in the construction of the Koch curve[7]. This shape can be accurately drawn with the use of computers, without any approximations or slight changes resulting in exact self-similarity.

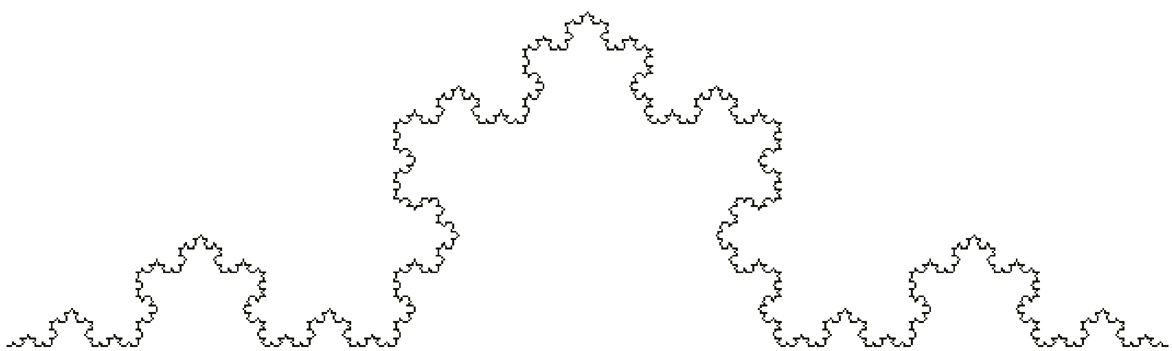


Figure 30: Self-similarity of the Koch curve

[7] Note that the object here is referred to as the Koch *curve* and not the Koch *island* or *snowflake,* as in the previous section. This is because only a part of the shape is presented and it is not closed.

A term closely linked, but often confused with self-similarity, is *self-affinity.* If an object's reductions are "distorted or skewed" in any way, the object is called *self-affine* instead of self-similar. In self-affine shapes, the "reductions are still linear but the reduction ratios in different directions are different. For example, a relief is nearly self-affine, in the sense that to go from a large piece to a small piece one must contract the horizontal and vertical coordinates in different ratios" (Mandelbrot & Blumen 1989:4, Peitgen et al. 2004:138).

It is also possible for self-similarity to occur only at one specific point (Peitgen et al. 2004:138), say, in an onion. An onion is not self-similar, but it does consist of concentric, self-similar rings. Onions are thus self-similar only around their most central part. Another example of this would be Russian dolls that fit into one another. Brothers (2004) referred to this as *limited self-similarity*. Objects or images that display limited self-similarity are *not* true fractals.

2.4.2 Fractal dimension

From elementary geometry it is general knowledge that a line has a dimension of 1, a square a dimension of 2 and a cube a dimension of 3. This next section will describe how one can calculate the dimension of a fractal. This is important because most fractals have dimensions that are not integers[8].

The following is a simple formula to calculate the dimension of any shape:

$$\boxed{k = n^d},$$

where d is the dimension; $\frac{1}{n}$ is the size of a smaller section of the whole; and k is the number of copies of n needed to recreate the original shape. Take, for example a cube (Figure 31), and divide it into 27 smaller cubes, which is $\frac{1}{3}$rd the size of the original cube. Nine of these smaller cubes are thus needed to recreate the original cube. Therefore k = 9 and n = 3.

$27 = 3^d$

$d = 3$

The dimension of a cube is therefore 3.

[8] Integers, also called whole numbers, refer to the set {...-3,-2,-1,0,1,2,3...}. These numbers do not contain decimals or fractions.

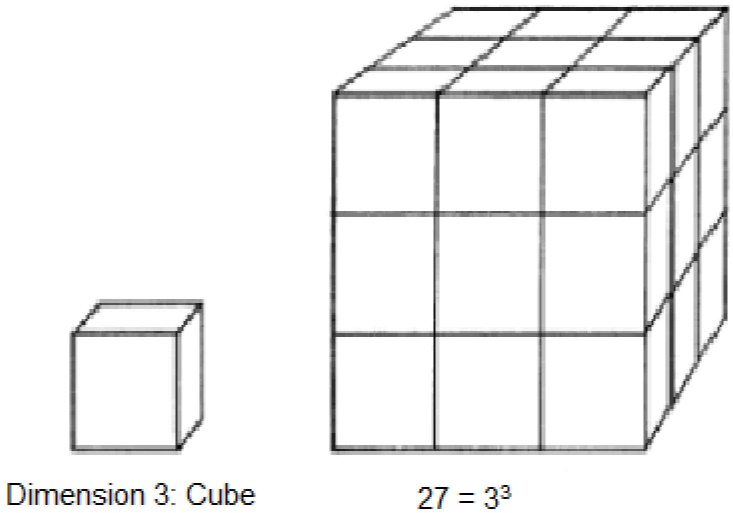


Figure 31: Dimension of a cube (Adapted from Barcellos 1984:99)

The same formula can also be applied to calculate the dimension of a fractal shape, like the Koch curve. In the first iteration of the Koch curve (Figure 32) there are four smaller line segments, each $\frac{1}{3}$rd the length of the initial line segment. Hence k = 4 and n = 3.

Therefore $4 = 3^d$

$$d = \frac{\log 4}{\log 3} = 1.26$$

The dimension is not an integer and lies between 1 and 2 – one of the characteristics of fractals. This seems somewhat abstract at first, but a shape's fractal dimension is useful to mathematicians because it shows the degree of fragmentation of the shape. The higher the fractal dimension, the more fragmented the outline of the fractal will be.

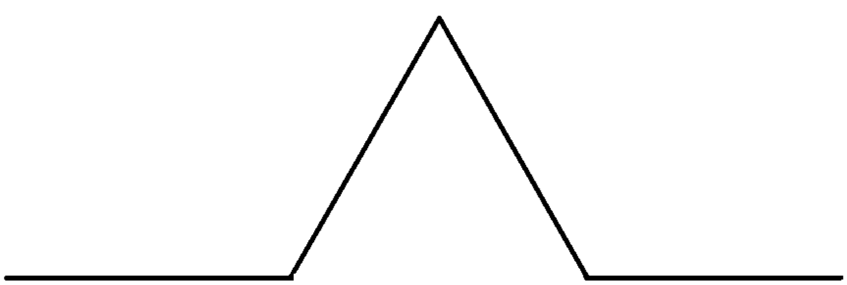


Figure 32: First iteration of the Koch curve

2.4.3 Fragmentation

This leads to another important property of fractals, that is, fragmentation on all scales. If one considers a Euclidean shape like a circle, and continuously enlarges it at a specific point P, the arc of the circle will tend to look increasingly more like a straight line. That means that it is possible to draw a tangent at point P on the circle and, in fact, at any point on the circle (Figure 33). In mathematics, it is said that the circle is everywhere differentiable.

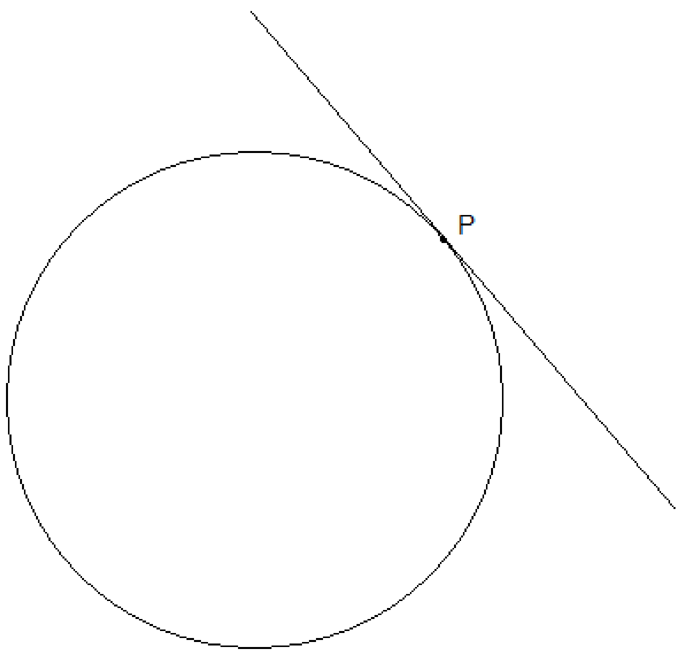


Figure 33: Tangent line drawn on a circle illustrating differentiability

The complete opposite is true of fractals. Take, for example, the Koch curve (Figure 30), which consists entirely of small jagged corners. It is not possible to draw a tangent at a corner, since it can lie at various angles. If one enlarges the snowflake at a specific point, the same amount of detail is obtained. One can thus conclude that it is impossible to draw a tangent line and the Koch curve is therefore nowhere differentiable. This substantiates the contention that fractals have some type of fragmentation or irregularity, irrespective of the scale (Mandelbrot & Blumen 1989:3-4).

To summarise, fractals adhere to the following three important properties: self-similarity, a fractal dimension that lies between 1 and 2, and fragmentation on all scales. These properties can be used and adapted to describe fractal music.

2.5 Noise

It is not only shapes or images that can have fractal properties. The next section will focus on fractal noise forms that had a huge impact on the fractal analysis of compositions.

In the early 1960s, Benoit B. Mandelbrot (1924-2010) was working at the Thomas J. Watson Research Centre as a research scientist. He was approached by The International Business Machines Corporation (IBM) to help solve a problem the company had encountered: the flow of computer data that IBM had tried to transmit through telephone lines was disrupted by a type of white noise, which caused the signal to break up.

Instead of using traditional analytical techniques, Mandelbrot examined the visual shapes that the white noise had generated and made a startling discovery: the graph had an internal self-similar structure on several different time scales. “Regardless of the scale of the graph, whether it represented data over the course of one day or one hour or one second, the pattern of disturbance was surprisingly similar” (Anonymous 1994). This marked the discovery of self-similar noise, a type of fractal that would be used abundantly in the composition and analysis of music in the following years.

In the sciences, the term “noise” refers to an unwanted random addition to a signal, which can be heard as acoustic noise when converted into sound, or can display itself as unwanted random data in signal processing. There are three main groups in which noise is generally categorised, namely white, brown and 1/f (pink) noise.

The graph of any type of noise can be converted to a logarithmic graph which presents itself as a straight line at a specific angle. In so doing, it is much easier to distinguish different noise types from one another. The graphs in Figures 34-36 display the time-frequency graph of a noise type on the right and its analogous logarithmic graph on the left (Voss 1988:40).

White noise, such as the static on a radio or snow on a television, is completely uncorrelated. This means that there is no relationship in the fluctuation of frequencies. White noise presents itself logarithmically as a straight line parallel to the x-axis (Figure 34), since it is a flat type of noise.

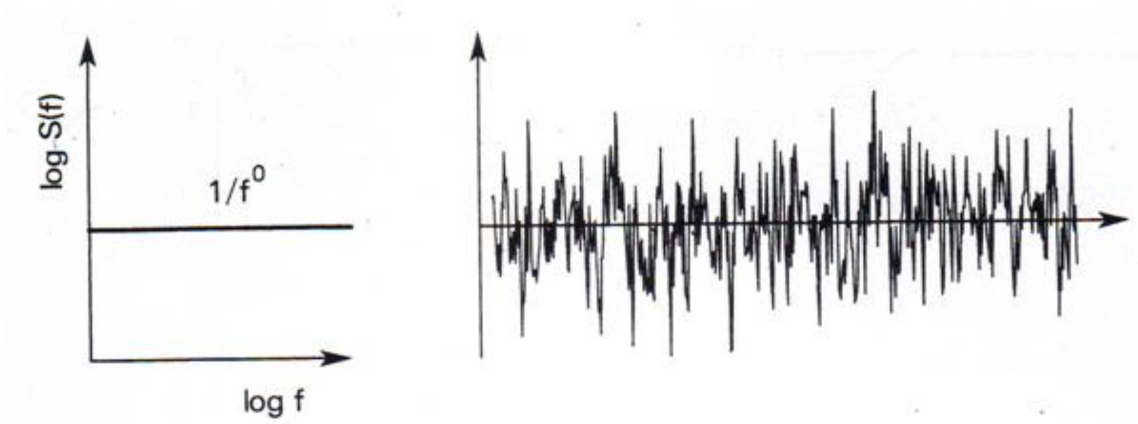


Figure 34: White or 1/f^{0}noise (Voss 1988:40)

The opposite of white noise, namely brown[9] or Brownian noise, is highly correlated. Its logarithmic graph shows a steep negative slope as in Figure 35.

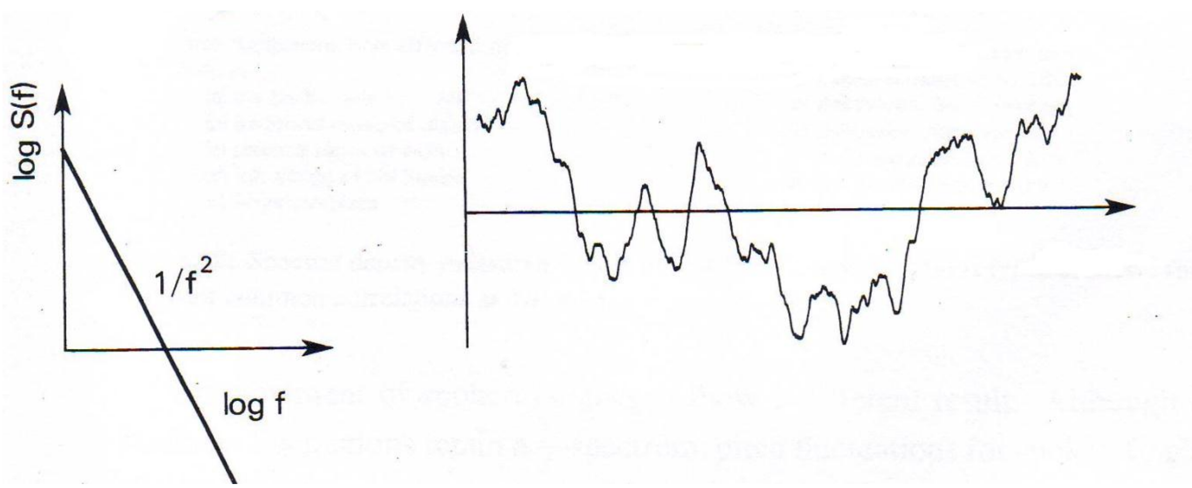


Figure 35: Brown or 1/f^{2} noise (Voss 1988:40)

[9] Brown or Brownian noise was not name after the colour, but the scientist Robert Brown, who discovered Brownian motion in c. 1827 (Frame et al. 2014)

Between these two extremes lies pink or 1/f noise[10], which is neither as correlated as brown noise nor as random as white noise. 1/f noise also displays a negative slope, but not as steep as that of brown noise (Figure 36). From these graphs it is easy to see that a noise form with high correlation has a steeper slope than that of a more random noise form (Voss & Clarke 1975:317).

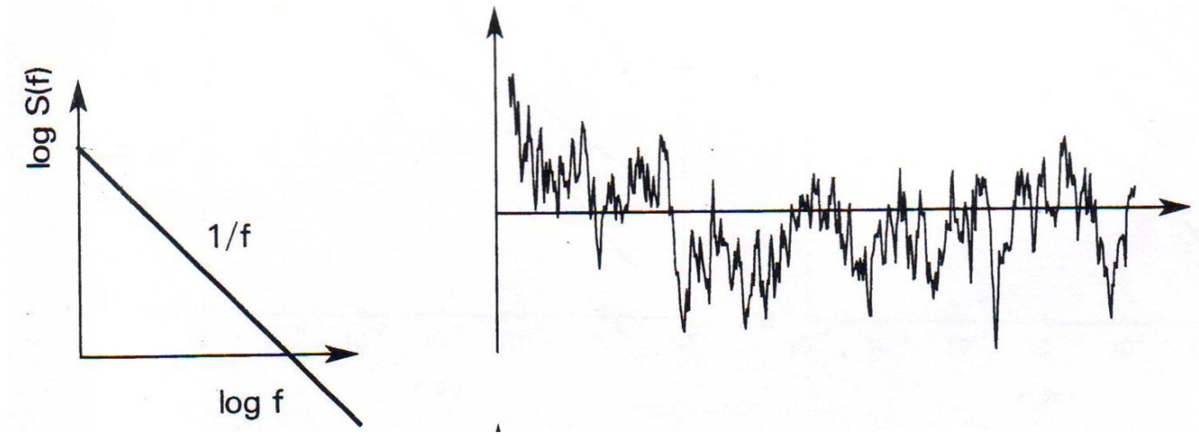


Figure 36: Pink or 1/f noise (Voss 1988:40)

For the purposes of this study, 1/f noise is deemed important because it is a noise form that displays self-similarity and can be regarded as fractal. Richard Voss and John Clarke as well as Kenneth and Andrew Hsü conducted some of the most fundamental research on the relationship between noise and music. Their research and findings are discussed in detail the literature review.

2.6 Conclusion

The ways in which geometric shapes and musical motifs can be transformed are markedly similar and suggest that musical motifs and themes may be altered in a way similar to geometric objects in order to create self-similar or fractal music compositions. These transformations include transposition (translation), melodic inversion (horizontal mirroring), retrograde (vertical mirroring), retrograde inversion (rotation) and combinations of two or more of the aforementioned transformations (shearing).

[10] 1/f noise is pronounced as “one-over-f noise”.

From the historical overview of fractals and self-similarity in nature, art and mathematics, it becomes clear that we have been surrounded by and fascinated with fractals for centuries. Thanks to mathematicians like Cantor, Koch, Sierpinski, Lindenmayer and Mandelbrot, we are also able to better understand such strange shapes and objects. The existence of fractals in art works created prior to the mathematical definition of fractal geometry, suggests that similar patterns may be found in music compositions.

In a nutshell, an object must encompass several important features in order to be called fractal. Firstly, a fractal must adhere to the property of scaling – in other words, it must have the ability to be enlarged or shrunk. Secondly, the degree of the irregularity and/or fragmentation must be irregular at all scales and thus also self-similar at all scales. Thirdly, all fractals have a fractal dimension, which, in most circumstances, is not an integer and lies somewhere between 1 and 2 (Mandelbrot 1982:1).

There are three fractal noise wave forms, namely white, brown and 1/f (pink) noise. However, only 1/f noise displays self-similarity similar to that of fractals. Its importance and possible applications to music compositions will be investigated in the chapters to follow.

CHAPTER 3: LITERATURE REVIEW

3.1 Introduction

This chapter highlights the most significant research conducted to date on the topic of fractals and self-similarity and their applications to music. Different scholars' published works are discussed to illustrate how fractal geometry and music are related. The different approaches are discussed chronologically, starting with a brief overview of sources concerned with fractal geometry in general. Thereafter, different approaches are analysed to find a connection between music and 1/f noise. This is followed by a discussion of studies by musicologists at Yale University who worked closely with Mandelbrot in the past. In conclusion, the validity of integrating fractal geometry and self-similarity with music is highlighted.

3.2 Fractal geometry in general

In order to fully understand the characteristics of fractal geometry and self-similarity, the researcher read many books and articles dealing with the mathematics of the subject matter. Because Mandelbrot is regarded as the so-called "father of fractals", his works are considered among of the most important. His book, *Fractals: form, chance and dimension,* published for the first time in 1977, summarised all of his early research. In 1982, *The fractal geometry of nature* was published, followed seven years later by an article entitled "Fractal geometry: what is it, and what does it do?*"* in collaboration with Blumen.

In 1992, Peitgen, Jürgens and Saupe collaborated in a book entitled *Chaos and fractals: new frontiers of science.* This is a source of a vast array of subjects connected to fractal geometry and self-similarity, with explanations of the discoveries of pioneering mathematicians such as Koch, Sierpinski and Cantor, as discussed in the previous chapter.

In 2012, two years after his death, Mandelbrot's memoir *The fractalist: memoir of a scientific maverick* was published. The afterword, written by Michael Frame from IBM, contains much valuable information on Mandelbrot's views regarding the use of fractals in music.

3.3 1/f noise distribution in music

The first link between music and fractal geometry was made in the late 1970s, when it was found that the distribution of the some elements in music, such as melody or rhythm, resembled the structure of 1/f noise. Over the years, many theories have been developed on this subject matter. This subsection deals with the different approaches by Voss and Clarke in the 1970s, the Hsüs in the 1990s and, most recently, Ro and Kwon in 2009.

3.3.1 Voss and Clarke

The physicist, Richard F. Voss, was one of the first researchers to study the correlation between fractals and music and is regarded as "one of the pioneers of the modern study of fractals" (Crilly, Earnshaw & Jones 1993:1). For his PhD research at the University of California in the 1970s, Voss experimented specifically with the relationship between music and noise wave forms (Gardner 1992:3). Together with his supervisor, John Clarke, Voss published their findings in several journals. Their first article, "'1/f noise' in music and speech" was published in the journal, *Nature,* in 1975, followed three years later by "'1/f noise' in music: music from 1/f noise*"*. The main aim of their experiments and research was to detect whether or not music exhibited similar fractal fluctuations as 1/f noise.

By the end of 1980, fractals had made their way into an article entitled "Science news of the year" in *Science News* journal (Anonymous 1980:406). The specific article reviewed some of the "important news stories" of that year. Among them, two entries relate to fractal geometry and also music:

- "It was found that music can be be analyzed [sic] as a structure with fractal dimensions."
- "Fractals were applied to the geometry of protein structure."

The first refers to Thomsen's article, "Making music – fractally*",* published in March 1980 in the same journal. In the article, Thomsen restated much of the research done by Voss in 1975.

3.3.1.1 The discovery of 1/f noise distribution in music

Music can be described as sound that fluctuates in frequency (pitch) and amplitude (loudness or volume) at different times. Voss and Clarke (1975, 1977) wanted to examine whether the fluctuation of these elements in different types of music were the same as the fluctuation of 1/f noise. Since 1/f noise is fractal, a piece of music mimicking 1/f noise could possibly also be regarded as fractal.

In conducting their research, Voss and Clarke (1977:258) used electronic equipment to measure the *power spectrum* or *spectral density* of different types of music. This showed what the behaviour of a varying quantity (in this case an audio signal) is over time. Another element that was considered was the *autocorrelation function*, which measured how the fluctuating quantities were related at two different times.

As stated, Voss and Clarke's first article dates back to 1975 – the same year in which Mandelbrot coined the term "fractal". This article, entitled "'1/f noise' in music and speech", does not contain any direct reference to fractal geometry, but showed how the distribution of pitches in certain types of classical music is similar to that of 1/f-noise (Voss & Clarke 1975).

In Voss and Clarke's measurements of music and speech, "the fluctuating quantity of interest was converted to a voltage whose spectral density was measured by an interfaced PDP-11 computer using a Fast-Fourier Transform algorithm that simulates a bank of filters. The most familiar fluctuating quantity associated with music is the audio signal …" In their first musical experiment, the audio signal of Bach's first Brandenburg Concerto, BWV 1046 was analysed. They analysed the concerto's sound spectrum, averaged over the entire work, but found no correlations to 1/f noise. The log-log graph in Figure 37(a) shows how the concerto displayed sharp peaks at high frequencies and was thus not similar to 1/f noise (Voss & Clarke 1975:317).

Despite the initial setback, they decided to test the fluctuations in loudness and pitch separately. This proved to be a more useful technique, since these quantities fluctuate more slowly (Thomsen 1980:190). Figure 37(b) shows the spectral density of the "instantaneous loudness" of the Brandenburg Concerto. This was obtained by measuring the spectral density of the output voltage after it had been squared, $S_v^2(f)$. It is clear that this graph follows the slope of 1/f noise much closer than in the previous example (Voss & Clarke 1975:317). The peaks between 1 and 10 Hz were attributed to the rhythmic structure of the music (Voss & Clarke 1977:260).

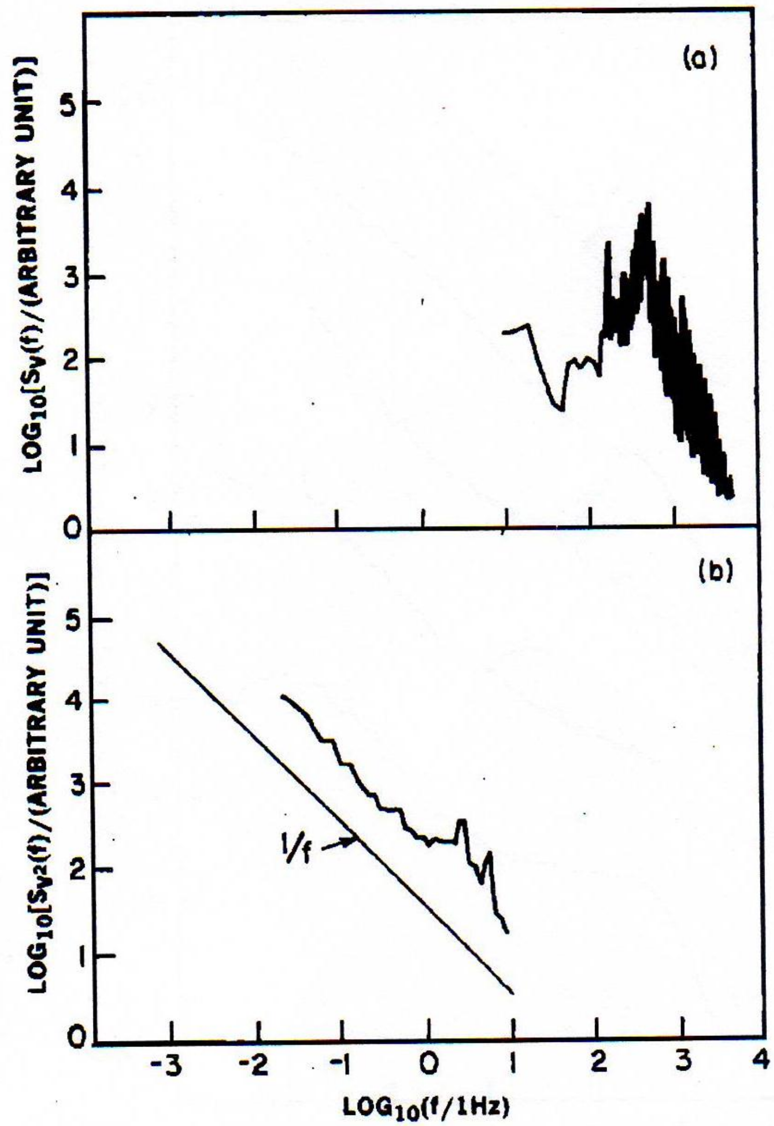


Figure 37: Spectral density of Bach's First Brandenburg Concerto (a) S_v(*f*) against *f* (b) S_v^2(*f*) against *f* (Voss & Clarke 1975:317)

Voss and Clarke continued with the same method of spectral density analysis to determine whether the instantaneous loudness in other genres of music would exhibit graphs similar to the one of the Brandenburg Concerto. The spectral density averaged over an entire recording of Scott Joplin Piano Rags displayed a downward contour similar to 1/f noise, but still contained many sharp peaks (Figure 38 (a)). They attributed this to the "pronounced rhythm" in the compositions. In order to test even more types of music, the output from different radio stations was also analysed and averaged over a period of 12 hours. The radio stations that were used, included a classical station, a rock station and a news and talk station. The log-log graphs of each of these can be compared in Figure 38 (Voss and Clarke 1975:318).

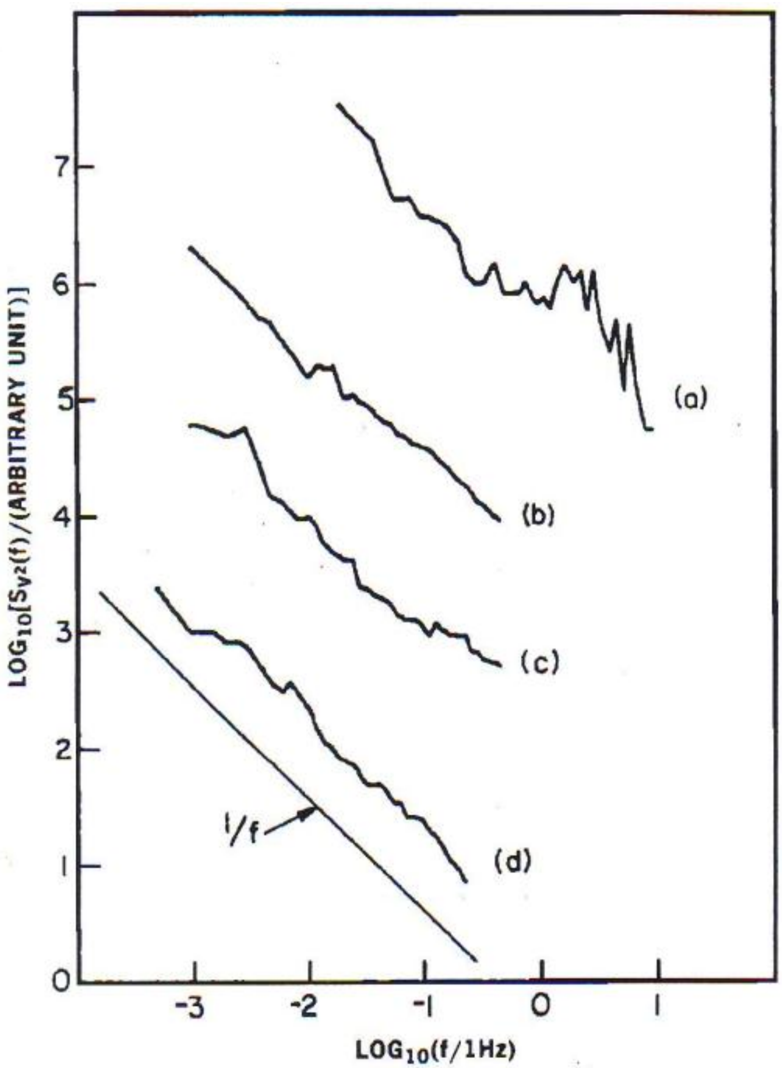


Figure 38: Spectral density of the instantaneous loudness compared to 1/f noise: (a) Scott Joplin Piano Rags; (b) classical station; (c) rock station; (d) news and talk station

In order to test the power spectrum of pitch fluctuations, the audio signal of different radio stations was also measured over a period of 12 hours. The classical station once again showed the closest resemblance to 1/f noise, while the jazz and blues, rock, and news and talk deviated from 1/f noise more (Figure 39) (Voss & Clarke 1975:318).

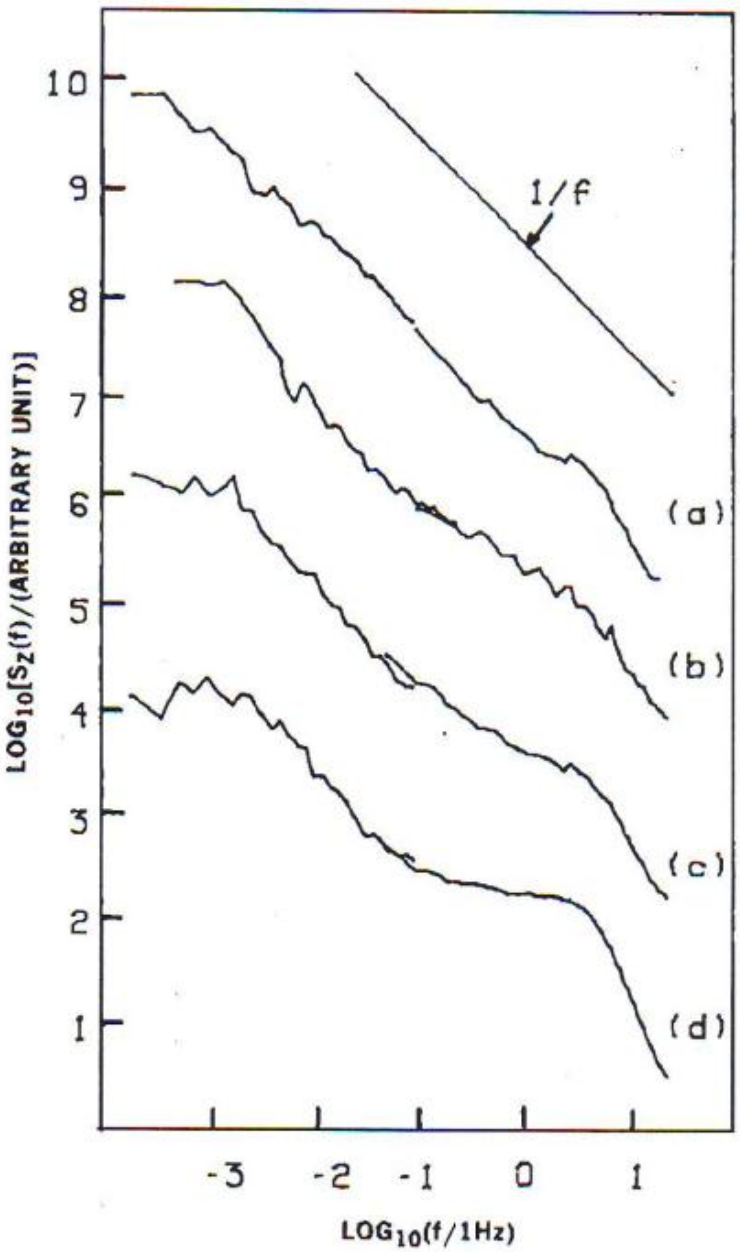


Figure 39: Power spectrum of pitch fluctuations compared to 1/f noise: (a) classical station; (b) jazz and blues station; (c) rock station; (d) news and talk station

From these experiments on the instantaneous loudness of the music as well as the frequency fluctuations, Voss and Clarke concluded that most melodies fluctuate in a similar manner as 1/f noise, which in turn shows that they are fractal in nature. There was, however, an exception in the works of 20th-century composers such as Stockhausen, Jolas and Carter. In their music, the "melody fluctuations approach white noise at low frequencies". Figure 40 clearly illustrates that the spectral densities in the compositions by 20th-century composers deviate greatly from 1/f noise (Voss 1989:367.)

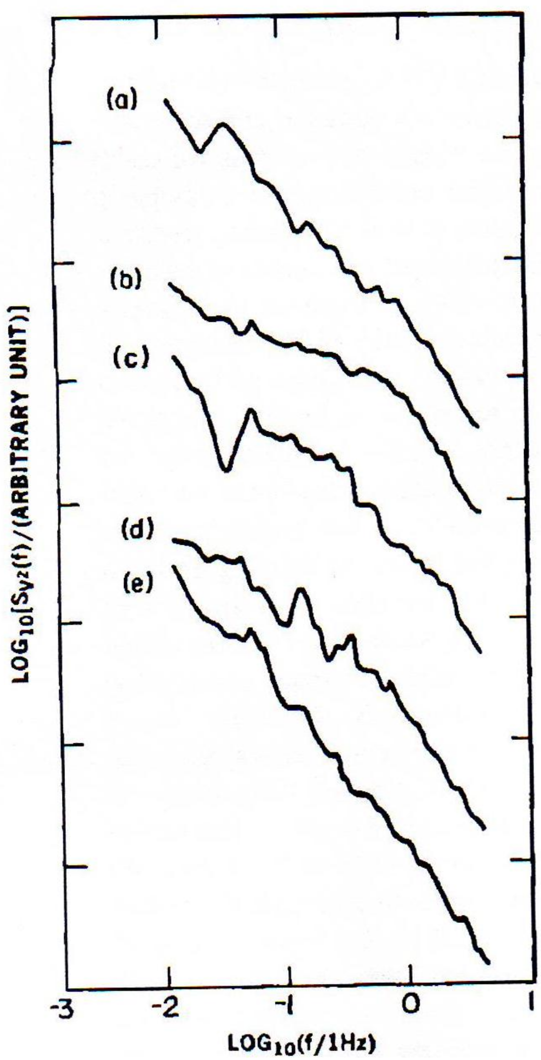


Figure 40: Audio power fluctuation spectral densities, $S_v^2(f)$ versus *f* for (a) Davidovsky's *Synchronism* I, III and III; (b) Babbit's String Quartet, No. 3; (c) Jolas's Quartet, No. 3; (d) Carter's Piano Concerto in two movements; and (e) Stockhausen's *Momente* (Voss & Clarke 1975:261)

Voss and Clarke's research has proven that "almost all musical melodies mimic 1/f noise". This was further proven by their analysis of melodies from other cultures. Figure 41 illustrates the spectral densities found in (a) the Ba-Benzele Pygmies[11]; (b) traditional music of Japan; (c) classical ragas of India; (d) folk songs of old Russia; and (e) American blues.

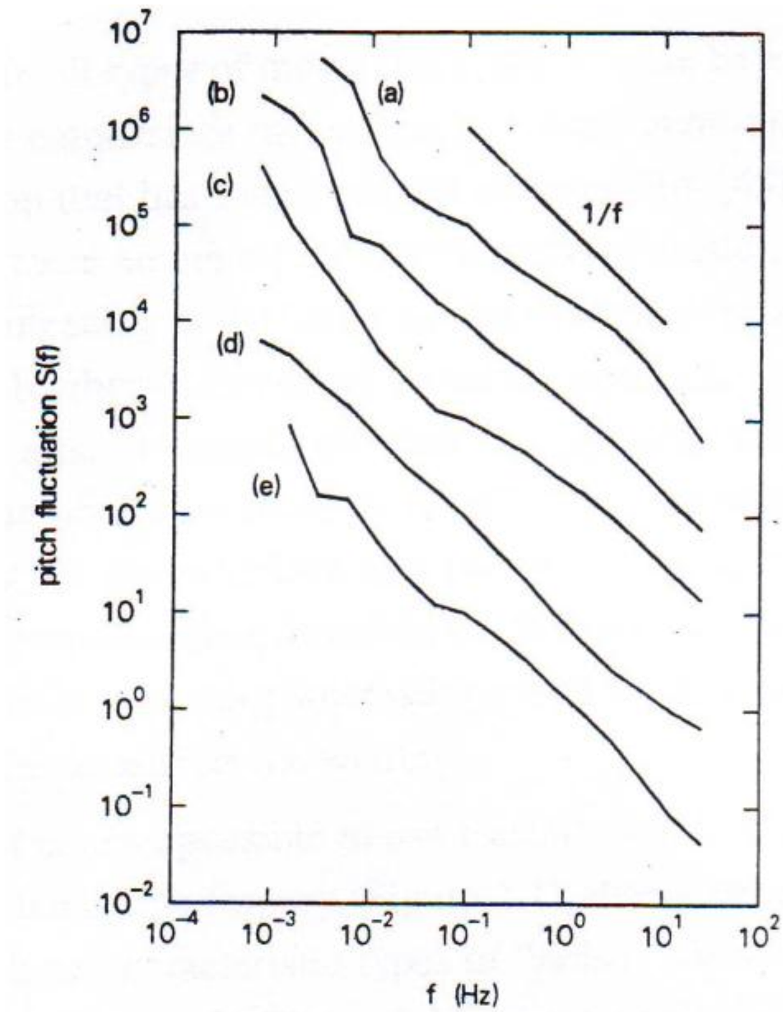


Figure 41: Pitch fluctuations from different musical cultures: (a) the Ba-Benzele Pygmies; (b) traditional music of Japan; (c) classical ragas of India; (d) folk songs of old Russia; (e) American blues

[11] The Ba-Benzele Pygmies are an indigenous nomadic people of Western Africa. Their music is characterised by complex polyphony.

Figure 42 illustrates the pitch fluctuations in different genres of music over a period of nearly seven centuries, starting with Medieval music, Beethoven's third symphony, Piano compositions by Debussy, Richard Strauss' *Ein Heldenleben* and finally *Sgt. Pepper* by the Beatles.

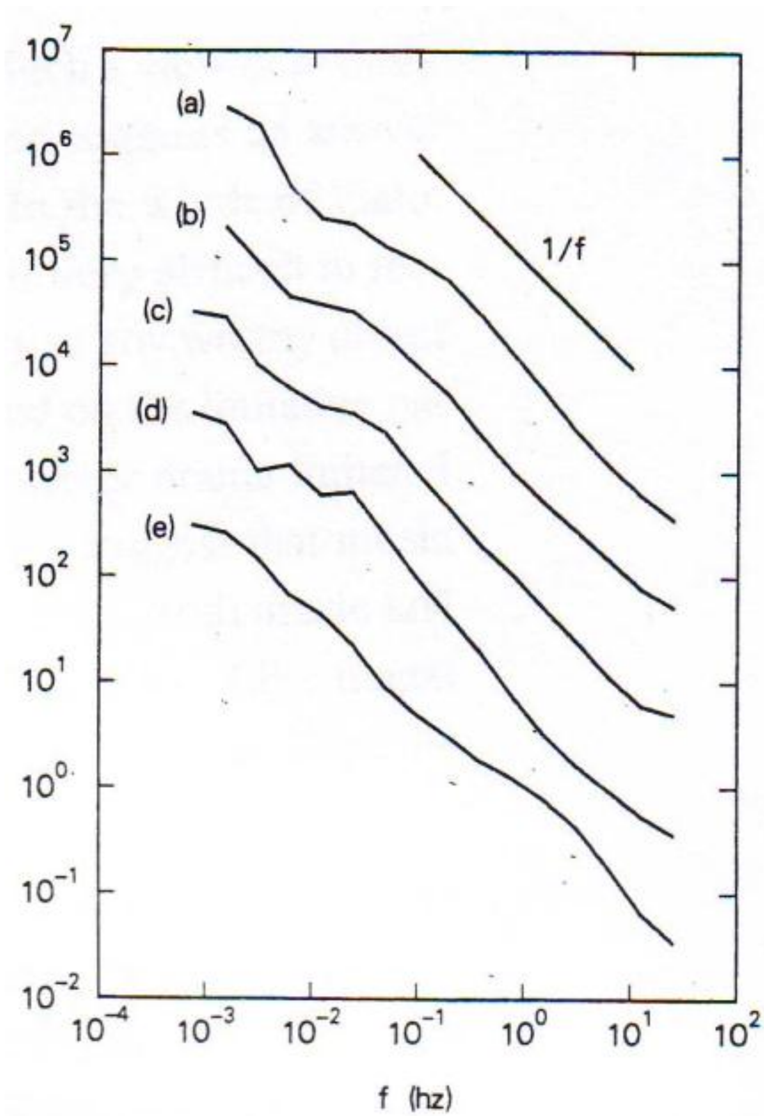


Figure 42: Pitch fluctuations in Western music: (a) Medieval music up to 1300; (b) Beethoven's third symphony; (c) Debussy piano works; (d) Strauss's *Ein Heldenleben*; (e) the Beatles' *Sgt. Pepper*

The downwards slope of the fluctuations seen in the figures above show that many different genres of music mimic the behaviour of 1/f noise. The scientific community was greatly affected by Voss and Clarke's discovery that music displayed fractal 1/f distributions. Their research made its way to an entry in *Electrical Engineering Times* in 1979, entitled "IBM researcher discovers 1/f distribution in music". The author of that particular article, whose name was not printed, summarised the findings as follows:

> [Voss] found that the spectral distribution of pitch and volume variations of music corresponded to the 1/f distribution commonly associated with electrical noise […] The correlations between notes extended over the whole musical composition; every note is correlated with every other note. A 1/f composition thus has a unity which, although subtle, is perceived by listeners […] Voss analysed music from Bach to rock and roll, and found they all conform closely to 1/f statistics.

3.3.1.2 Criticism of Voss and Clarke's research

Although Voss and Clarke's published works were diligently researched, they failed to include specific compositions (other than Bach's first Brandenburg Concerto), but only mentioned the composers. In addition, no music examples were ever included to substantiate their research findings. In the researcher's opinion, these should have been included in order to give musicians a better frame of reference.

In 1992, the Australian musicologist, Nigel Nettheim, published an article in the *Journal of New Music Research* as a criticism of Voss and Clarke's claim that "much music is well modelled by 1/f noise". Nettheim (1992) critiqued the aforementioned researchers for not producing "enough evidence […] for the reliability" of some methods they used in their research. He also believed that too many assumptions were made in their work (Nettheim 1992:135).

In conclusion, Voss and Clarke's extensive research shows that most types of music display some form of 1/f distribution, which is also a type of fractal. For their experiments, they utilised music from various genres and time periods, but Classical music (excluding music from the 20th century) always displayed the closest resemblance to 1/f noise. Numerous other scholars have used Voss and Clarke's findings as the base for their own research and articles.

3.3.2 Kenneth and Andrew Hsü

Kenneth J. Hsü and his son, Andrew, have also done research on 1/f noise and music. At the time of their study, the elder Hsü was an academic geologist at the *Zurich Technische Hochschule* and his son a concert pianist at the Zurich Conservatorium (Crilly et al. 1993:2). Three of their most important articles include "Fractal geometry of music" (1990); "Self-similarity of the '1/f noise' called music" (1990); and "Fractal geometry of music: from bird songs to Bach" (1993).

From the list of sources in their articles, it is evident that the Hsüs based much of their research on the works of Voss et al. and Mandelbrot, but their methods and approaches were slightly modified.

3.3.2.1 The fractal geometry of melody

The Hsüs had a different approach to testing for 1/f fluctuations in music compared to Voss and Clarke. Where Voss and Clarke tested the pitch fluctuations in compositions, Kenneth and Andrew Hsü believed that measuring how often a specific interval between successive notes occurred in a composition would be more accurate (Hsü & Hsü 1990:938). One of the main reasons for this, was their view that the intervals between successive notes "are the building blocks of music, not the individual notes" (Hsü 1993:23).

The Hsüs adapted Mandelbrot's formula to determine the fractal dimension of an object or set so that it could be applied to music. Mandelbrot's original formula for fractal shapes is as follows:

$$F = \frac{c}{M^D}$$

where F is the frequency; M is the "intensity of events"; c is a constant of proportionality; and D is the fractal dimension.

The adapted formula by the Hsüs is

$$F = \frac{c}{i^D}$$

where F is the incidence frequency of notes and i is the interval between two successive notes. The variable used in the place of i is calculated easily in terms of semitones, where 0 represents a repeated note, 1 a minor second, and so forth (Figure 43).

Repeated note	0
Minor second	1
Major second	2
Minor third	3
Major third	4
Perfect fourth	5
Tritone	6
Perfect fifth	7
Minor sixth	8
Major sixth	9
Minor seventh	10
Major seventh	11
Octave	12

Figure 43: Numerical values assigned to respective note intervals (Hsü & Hsü 1990:939)

One of the first compositions analysed by the Hsüs using this method was Bach's first Invention in C major, BWV 772. The intervals between successive notes were measured separately for the right and left hand. After notating their results in a table, they found that "a fractal relation is established for $2 \leq i \leq 10$"; i.e. there is a fractal relation for the intervals between (and including) a whole-tone and minor seventh. They also calculated that the fractal dimension is 2.4184, which results in the following equation:

$$F = \frac{2.15}{i^{2.4184}}$$

However, the Hsüs did not explain in their article how the constant (2.15) and the fractal dimension (2.4184) were calculated, or why only a certain range of intervals (major second to minor seventh) were included (Hsü & Hsü 1990:939). The researcher tried to duplicate the application of the above formula, but was unable to achieve the same results as in the article. This may suggest that the article did not supply sufficient information for other researchers to use or duplicate.

The same method was applied to various other compositions (Hsü & Hsü 1990:940-941) such as:

- Bach: Invention No. 13 in A minor, BWV 784
- Bach: *Adagio* movement from *Toccata* in F-sharp minor, BWV 910
- Mozart: First movement from Sonata in F major, KV 533
- Six Swiss children's songs
- Mozart: First movement from Sonata in A major, KV 331
- Stockhausen: *Capricorn*

The results of their experiments were then plotted on log-log graphs, like Voss and Clarke. Although their methodologies were different, both groups of researchers came to the same conclusion: the music of Stockhausen does not imitate 1/f noise in the same fashion as classical music and folk tunes (Hsü & Hsü 1990:939-940).

Figure 44 shows the log-log graphs for the above-mentioned compositions. From the graphs, it is clear that the Hsüs' research showed a 1/f distribution in all of the compositions analysed, except for Stockhausen's *Capricorn*. They ultimately reached the conclusion that "the intervals between successive acoustic frequencies [i.e. pitches] in classical music have a fractal distribution".

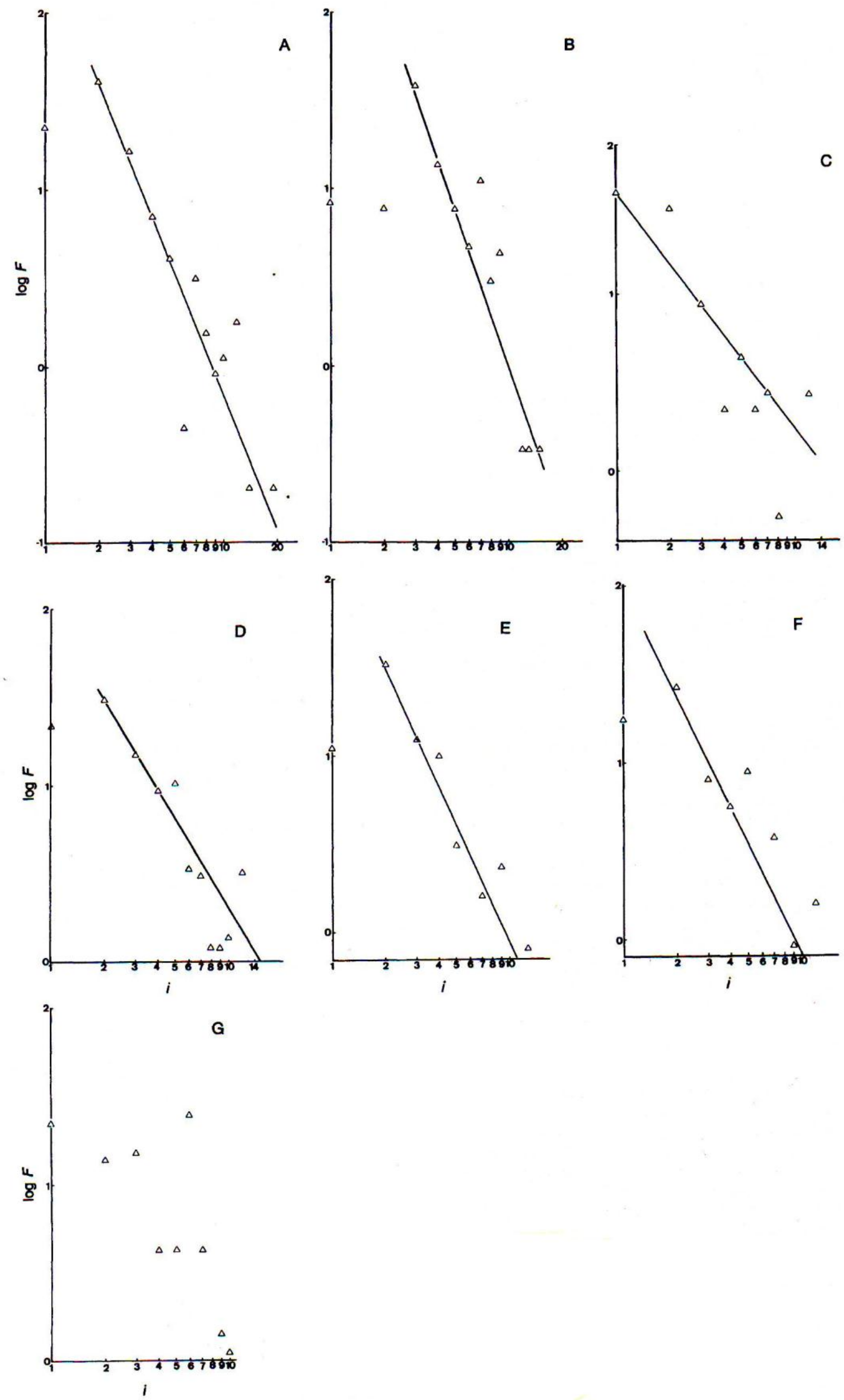


Figure 44: Fractal geometry of sound frequency where F is the percentage frequency of incidence of note interval i: (A) Bach BWV 772; (B) Bach BWV 784; (C) Bach BWV 910; (D) Mozart KV 533; (E) Six Swiss children's songs; (F) Mozart KV 331; (G) Stockhausen *Capricorn*

3.3.2.2 The fractal geometry of amplitude

The Hsüs also wanted to determine whether the "loudness of music" had a fractal distribution. They criticised Voss and Clarke's method of using a recording and instead wanted to consider the intentions of the composer, rather than the interpretation of a performer. Again, they used note-counting methods, similar to what they had done in the case of melodies. For the amplitude they measured how many notes were struck simultaneously as this would produce a louder sound (Hsü & Hsü 1990:941).

Their method, however, is ambiguous. The following is a quotation from their article (Hsü & Hsü 1990:941):

> One possibility for evaluating the amplitude is to analyse the number of notes that are played simultaneously, because more notes sounding together should make the sound louder. We again chose Bach's *Toccata*, because four melodies are played simultaneously in that fugue. But we did not find a fractal distribution, which indicates that this is not an effective way of evaluating loudness. Recognizing, then, that the intensity of the sound is greatest when a note is first struck on a keyboard, we analysed the number of notes that are struck simultaneously and found an apparent fractal distribution of amplitude.

It is clear that the Hsüs hinted towards two different methods of calculating the possible fractal distribution of loudness, but the researcher could not find any clear distinction between the two methods in any of their published works. Despite this, the Hsüs concluded that amplitude (or loudness) in classical music displayed a fractal distribution.

The Hsüs' note-counting method was also applied in literature: Ali Eftekhari (2006) from the Electrochemical Research Centre in Iran, published an article entitled "Fractal geometry of texts: an initial application to the works of Shakespeare". He found that "by counting the number of letters applied in a manuscript, it is possible to study the whole manuscript statistically" (Eftekhari 2006:1). Using a formula similar to the one employed by the Hsüs, Eftekhari was able to show that the literary works of Shakespeare also have a fractal dimension (Eftekhari 2006:11).

Although some of the methods by the Hsüs appear somewhat strange, it is clear that their research, in turn, also influenced other scholars.

3.3.2.3 Measurement and reduction of a musical composition

In an anecdote about a conversation between Mozart and Emperor Joseph II while discussing Mozart's opera, *Die Entführung aus dem Serail,* it is said that the Emperor complained that the music consisted of "too many notes". Mozart's reply was simply that "there are as many notes as there should be". While agreeing that all notes in any composition are essential, the Hsüs wondered if it would be possible to reduce the number of notes in such a way that the remaining notes would still be reminiscent of the original composition.

In their first article, they had already proven that music exhibits fractal features such as self-similarity and scale independency. Taking this into consideration, they argued that it should be possible to reduce a composition to fewer notes and still recognise the composition or, at least, the composer (Hsü & Hsü 1991:3507).

To prove this, the Hsüs drew on a particular physics phenomenon known as the coastline paradox. The British physicist, Lewis F. Richardson, discovered that the length of the common frontier between Spain and Portugal differed from source to source by as much as 20% (Hsü & Hsü 1991:3507). Further investigation showed that this was because of the use of different lengths of "measuring sticks": a coastline (or any border) will increase in length when smaller measuring sticks are used. (This is similar to what was explained earlier about the length of the Koch curve.) Hence a coastline might have an infinite length. This has become known as the coastline paradox or Richardson effect. Length is thus not an accurate way of measuring a coastline and the fractal dimension should rather be used (Mandelbrot 1967:636).

Since the outline of a coastline looks similar on different scales, owing to its self-similarity, the Hsüs considered the possibility of applying similar techniques to music. Figure 45 illustrates the similarity in the fractal distribution between a log-log graph of the West Coast of Britain and Bach's Invention No. 1 in C major, BWV 772. Both graphs have a similar descending slope characteristic of 1/f noise or fractals (Hsü & Hsü 1991:3507).

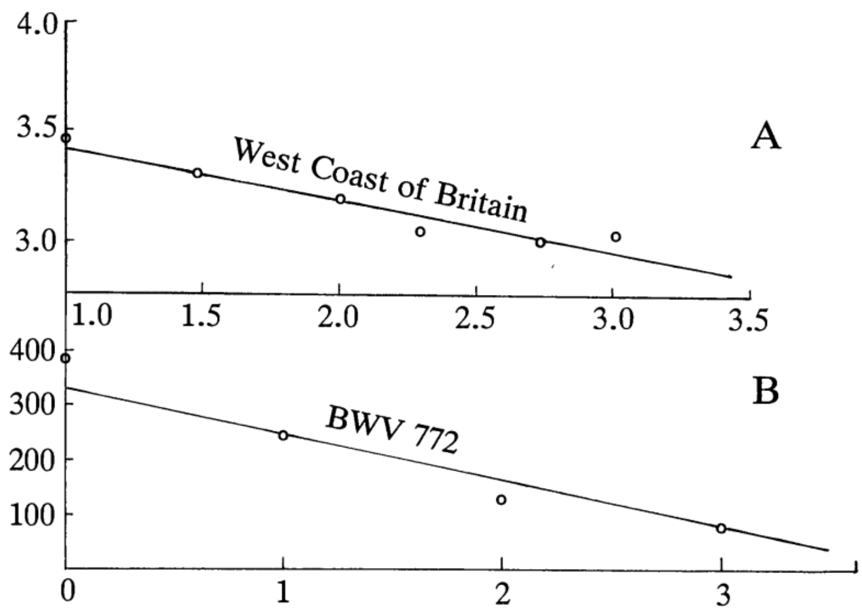


Figure 45: Resemblance between (A) the West Coast of Britain and (B) Bach's Invention No. 1 in C major, BWV 772 (Hsü & Hsü 1991:3507)

In one of their articles, entitled "Self-similarity of the 1/f noise called music", they posed some important questions: "Could we compare the fractal geometry of music to that of a coastline? If a coastline has no definite length, could we state that Mozart's music has no definite number of notes or note intervals"? (Hsü & Hsü 1991:3507).

They tested their research hypothesis by analysing Bach's Invention No. 1 in C major, BWV 772. Firstly, the parts for the right hand and left hand were digitised individually and presented visually on the same graph (see Figure 46).

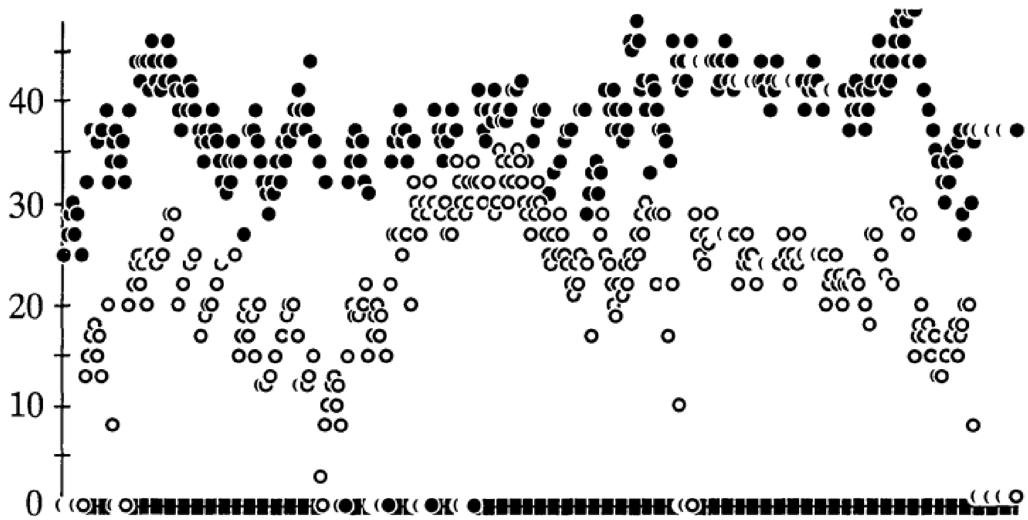


Figure 46: Digitised score of Bach's Invention No. 1 in C major, BWV 772; ○ right hand, ● left hand (Hsü & Hsü 1991:3508)

Next, the Hsüs eliminated half of the notes in the composition and plotted the results on another graph. They proceeded to halve the number of notes and plotted the results on different graphs. The final reduction, $^{1}/_{64}$th of the original notes, resulted in the three key notes on which the composition is built. As expected, the basic contour of the music remains the same, but it is less detailed (see Figure 47). They claim that "… to a novice, the half- or quarter-Bach sounds like Bach…" (Hsü & Hsü 1991:3508).

The Hsüs' findings in this experiment are not that surprising, since most compositions are based on specific harmonies and are further embellished by non-chordal tones. If one were to eliminate certain notes in a composition, it is highly likely that one would eliminate some of these non-chordal tones and be left with nothing but the basic harmony notes on which the composition is based. They wrote that "…reduced music gives the skeleton and enhances our understanding of great music" (Hsü & Hsü 1991:3509).

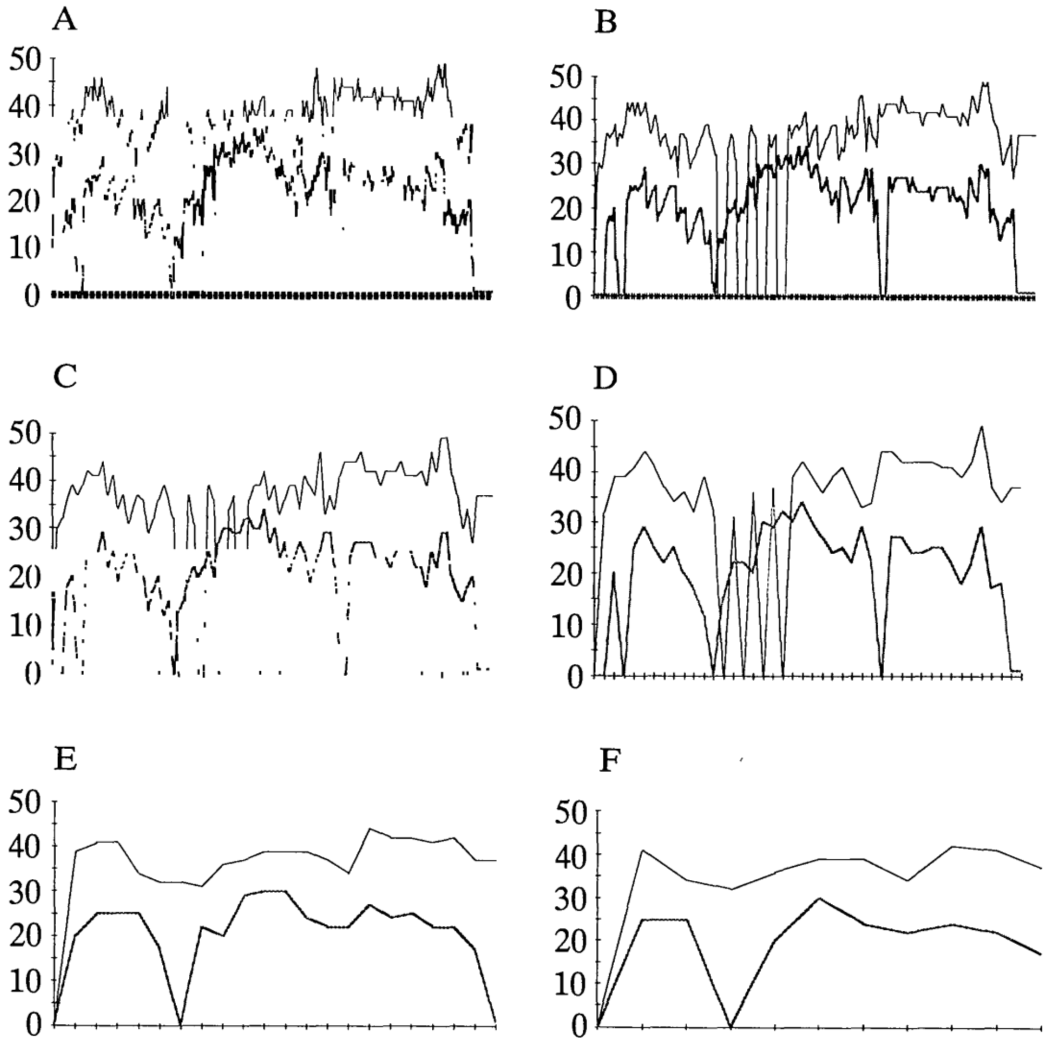


Figure 47: Reductions of Bach's Invention No. 1 in C major, BWV 772; (A) original composition, (B) $^1/_2$ reduction, (C) $^1/_4$ reduction, (D) $^1/_8$ reduction, (E) $^1/_{16}$ reduction, (F) $^1/_{32}$ reduction (Hsü & Hsü 1991:3508)

Whether or not the different scales of the composition were indeed fractal was another question that relied on "measuring" the composition. The actual "measurement" of a composition's length, however, leaves much to be desired and contains many ambiguities. The Hsüs stated that the length of a composition can be expressed as a mathematical formula, just like the length of a coastline: "… the length of a music score, $\sum i$, can be represented by the sum of all note intervals in a composition …" (Hsü 1993:29). The numbers used for the calculations should be as in the table given earlier in Figure 43.

They explained their method with the use of an example: "Take, for example, the first five notes of the BWV 772 for the right hand; the $\sum i$ value is 1 + 1 + 1 + 2 = 5." These numbers were, however, obtained from the digitised version of the score in Figure 46 and not from the original score. If one tries to apply the same method by using the actual score, and the interval numbers are as in Figure 43, the answer is different, namely 2 + 2 + 1 + 3 = 8.

Figure 48: Bach, Invention No. 1 in C major, BWV 772, first five notes with interval numbers

The digitised score used by the Hsüs clearly interpreted the distance between intervals in diatonic steps, instead of semi-tones as conducted earlier. For a half-reduction, the sum of intervals was found to be 2 + 1 = 3, but the researcher found that it should be 4 + 2 = 6, as in Figure 49.

Figure 49: Bach, Invention No. 1 in C major, BWV 772, half-reduction of the first five notes with interval numbers

This deviation from their original method has a major impact on the Hsüs' findings and the credibility thereof. Hsü calculated the sum of all intervals (without reduction) for the right hand of the Invention to be 380, but the researcher calculated the total as 585 – a difference of more than 60%.

3.3.2.4 Criticism of the Hsüs' research

In 1993, Henderson-Sellers and Cooper (1993) published an article, "Has classical music a fractal nature? – A reanalysis", in which they re-evaluated and criticised Hsüs' articles. Henderson-Sellers and Cooper (1993:277) found that "there is no inherent fractal nature in classical music; although the converse is not true. In other words, it is feasible to use fractal ideas to compose musical pieces – an area of much interest in recent years."

In 1993, Henderson-Sellers and Cooper made a number of negative remarks about the Hsüs' interval-counting method to determine the length of a composition. Henderson-Sellers and Cooper (1993:278) acknowledged the fact that a coastline's length is infinite, since its length depends on the measuring stick used, but they disputed its application to a musical composition:

> In contrast, the number of notes in a Mozart sonata is countable (in the mathematical as well as the practical sense) and therefore *does* have definite "length," despite the Hsüs' conjecture that "Mozart's music has no definite number of notes." One major difference is that notes (or rather their representations) are discrete and therefore countable whereas a coastline is continuous.

Another error that Henderson-Sellers and Cooper (1993) found in the Hsüs' research was regarding their application of Mandelbrot's formula for fractal dimension to a composition. In the original formula,

$$F = \frac{c}{M^D}$$

the variables on both sides of the equation have the same "underlying basic dimension", for example, length. In the Hsüs' application of the formula, the left- and right-hand sides of the formula did not have the same dimension (Henderson-Sellers & Cooper 1993:278).

Henderson-Sellers and Cooper (1993) also stated that although all fractals have a power law representation, the converse is not necessarily true: not everything with a power law representation is necessarily fractal, as the Hsüs believed them to be.

In their article, Henderson-Sellers and Cooper (1993:279) asked an important question: "If music is not fractal in nature, what has the analysis of Hsü and Hsü (1990) taught us?" Their answer was that "an analysis of the use of specific intervals by a composer, and the frequency of use of specific notes and specific intervals, may be a good guideline to a composer's 'fingerprint'."

In fact, the Hsüs' method of counting the distribution of specific intervals is not new: "… the relative frequency of melodic intervals is often used as a stylistic criterion by ethnomusicologists …" such as Marcia Herndon and Mervyn McLean. Two of the most common ways in which this method is applied are to count how often the various intervals appear in the music and to "compare the distribution of rising and falling intervals" (Cook 1992:191). Note-counting methods may thus be useful in characterising music from different cultures or genres and comparing them with one another.

The pianist, Rosalyn Tureck, addressed a letter to the editors of *Proceedings of the National Academy of Sciences of the United States of America,* the journal in which the Hsüs published their first article, "Fractal geometry of music*".* In it, Tureck strongly criticised the terminology used and assumptions made in the Hsüs' article. She concluded the letter as follows (Tureck 2009):

> The frequency of incidents of note intervals of music is by no means a methodology for determining the relationships that constitute an artistic composition. If one is looking for a fractal geometry of music, which I applaud wholeheartedly, then the steps taken must be based on something immeasurably more solid and accurate than what appears in this article.

The researcher agrees with Tureck in the sense that the Hsüs made too many assumptions and that their research methodology was not solid enough. They also failed to use their formulae uniformly. They also appear to have tried to force some mathematical formulae to fit the music where they simply did not belong, as is the case with their application of Mandelbrot's formula for determining the fractal dimension.

Similar to Voss and Clarke's research, the Hsüs also failed to include any music examples in their studies. Tureck (2009) wrote as follows:

> "… in presenting any kind of claims having to do with musical structure, it is imperative when dealing with specific motives and compositions to include examples of those motives and/or compositions … It is as necessary for the reader to have the musical material for verification as to have the mathematical formulations for consideration."

Tureck (2009) further criticised the Hsüs' article for the incorrect use of some of the music terminology. For example, they refer to a "small third" or "large third" instead of a minor and major third. They also referred to "the first movement" of Bach's invention, but since inventions do not consist of different movements, "an informed reader cannot equate their claims with any part of the Inventions except by guesswork".

3.3.3 Spectral density analysis for genre classification

Despite their flaws, many of the methods used by Voss, Clarke and the Hsüs are still being used as the base for the spectral analysis of music. It has already been implied that spectral analysis may be more useful in the stylistic analysis of compositions. The researcher found an increase in the amount of research that has been done in classifying genres with the use of spectral analysis since 2009.

3.3.3.1 Ro and Kwon

In 2009, Wosuk Ro and Younghun Kwon from the Department of Applied Physics of Hanyang University in South Korea, published an article entitled "1/f noise analysis of songs in various genre of music". Ro and Kwon based their research on Voss and Clarke's findings that "there is 1/f behaviour in music and speech". Ro and Kwon's research was distinctly different from that of Voss and Clarke in the sense that they tried to use spectral density analysis to compare different genres of music. Their hypothesis was that one may be able to distinguish between different genres of music through the calculation of 1/f behaviour in a composition below 20 Hz. They analysed 20 "songs" from seven different genres: classical music, hip-hop, newage, jazz, rock, Korean traditional music (pansori) and Korean modern music (trot) (Ro & Kwon 2009:2305.)

In previous research, 1/f behaviour was often measured across the entire frequency range of a composition. Instead, Ro and Kwon (2009) were more interested in low frequency range, specifically frequencies below 20 Hz as this had not been done before. Consequently it was found that "1/f noise analysis of songs would show the characteristics of each genre of music. So it implies that we may classify each genre of music by score of 1/f noise analysis in the region below 20 Hz" (Ro & Kwon 2009:2306).

They were highly specific in giving the degree of 1/f behaviour for each genre. This is illustrated in Figure 50. The data shows that classical music – more than any other genre tested by Ro and Kwon – mimics 1/f noise closest below 20 Hz.

Rank	Genre	Degree of 1/f behaviour in the region below 20 Hz (%)
1	Classic	77.3
2	Trot	65.4
3	Jazz	63.9
4	Newage	63.6
5	Rock	36.6
6	Hip-hop	23.5
7	Pansori	4.45

Figure 50: The degree of 1/f behaviour for seven different genres of music in the region below 20 Hz (Adapted from Ro & Kwon 2009:2308)

To illustrate this further, Figure 51 shows the log-log plot for the Larghetto second movement of Chopin's first Piano Concerto, Op. 11. The straight lines indicate a strong 1/f like behaviour below 20 Hz.

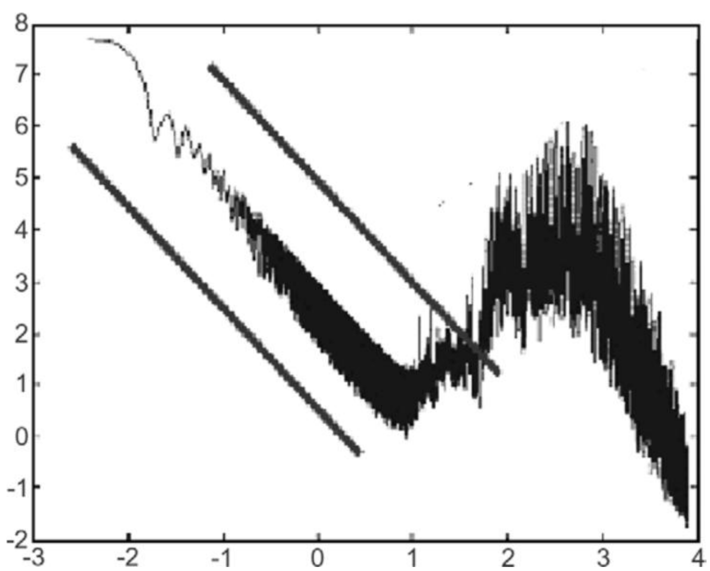


Figure 51: Log-log plot of frequency analysis in the Larghetto second movement of Chopin's first Piano Concerto, Op. 11 showing high 1/f correlation in the low frequency range (Ro & Kwon 2009:2306)

In stark comparison, Figure 52 shows the log-log plot of frequency analysis for *Song of Simchung,* a Pansori song for rice offering. This piece of traditional Korean music, unlike the slow movement of Chopin's first Piano Concerto, shows virtually no 1/f correlation in the low frequency range.

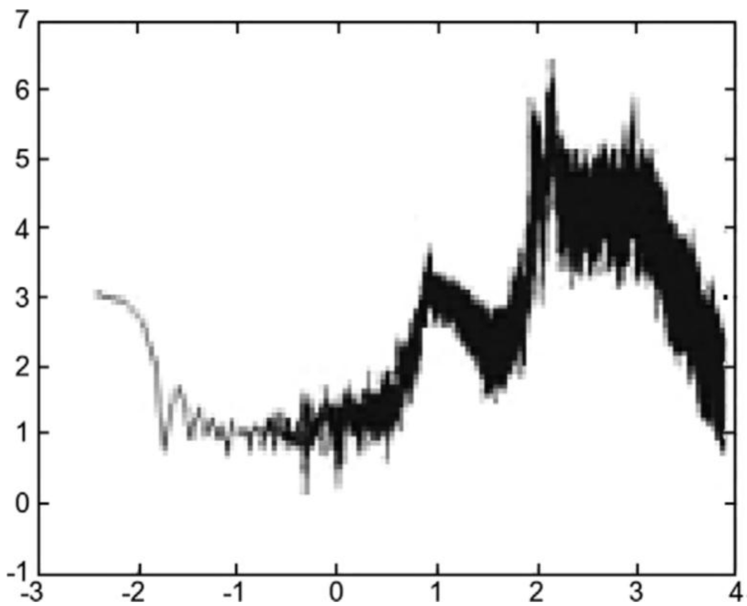


Figure 52: Log-log plot of frequency analysis in *Song of Simchung* displaying very low 1/f correlation in the low frequency range (Ro & Kwon 2009:2307)

Ro and Kwon (2009:2306-2307) also found that "1/f noise analysis of songs in the region below 20 Hz might not show the characteristic of culture but that of each genre of music". They concluded that "the score of 1/f noise analysis in that region could classify each genre of music". Unlike frequency analysis conducted between the 1970s and 1990s, Ro and Kwon (2009) were among the first key researchers to use 1/f noise analysis as a tool to identify or classify different genres of music.

3.3.3.2 Levitin, Chordia and Menon

The idea of characterising different genres of music was further investigated through the years. Most recently, in 2012, Daniel J. Levitin, Parag Chordia and Vinod Menon collaborated in the article, "Musical rhythm spectra from Bach to Joplin obey a 1/f power law". Since it had already been proven that musical pitch fluctuations follow a 1/f power law (Voss & Clarke 1975), they wished to investigate whether the same would be true for the rhythmic fluctuations in music.

Levitin et al. (2012:3716) started their article with the following statement and question:

> Musical rhythms, especially those of Western classical music, are considered highly regular and predictable, and this predictability has been hypothesized to underlie rhythm's contribution to our enjoyment of music. Are musical rhythms indeed entirely predictable and how do they vary with genre and composer?

While Ro and Kwon (2009) used 1/f distribution in music to distinguish between different genres of music, Levitin et al. (2012) used it to differentiate specifically between different subgenres in classical music. They analysed the rhythm spectra of 1 778 movements from 558 compositions of Western classical music, which covered 16 subgenres, four centuries and a total of 40 composers (Levitin et al. 2012:3716).

They found that musical rhythms also exhibited 1/f spectral structure and that composers "whose compositions are known to exhibit nearly identical pitch spectra, demonstrated distinctive 1/f rhythm spectra". They were of the opinion that these 1/f fluctuations may enhance one's aesthetic appeal to a composition (Levitin et al. 2012:3717).

Levitin et al. (2012) also tried to find a reason behind the fractal structures in the melody and rhythm of music. They wrote that "… 1/f is not merely an artefact of performance, but exists in the written scores themselves. Perhaps composers can't help but produce 1/f rhythm spectra, perhaps musical conventions require this. From a psychological standpoint, the finding suggests that composers have internalized some of the regularities of the physical world as it interacts with biological systems (including the mind) to recreate self-similarity in works of musical art" (Levitin et al. 2012:3718).

By comparing the rhythm spectra of different composers' works, Levitin et al. (2012:3716) found that "Beethoven's rhythms were among the most predictable, and Mozart's among the least". Figure 53 illustrates the hierarchy of predictability of some of the composers from their study (Levitin et al. 2012:3719).

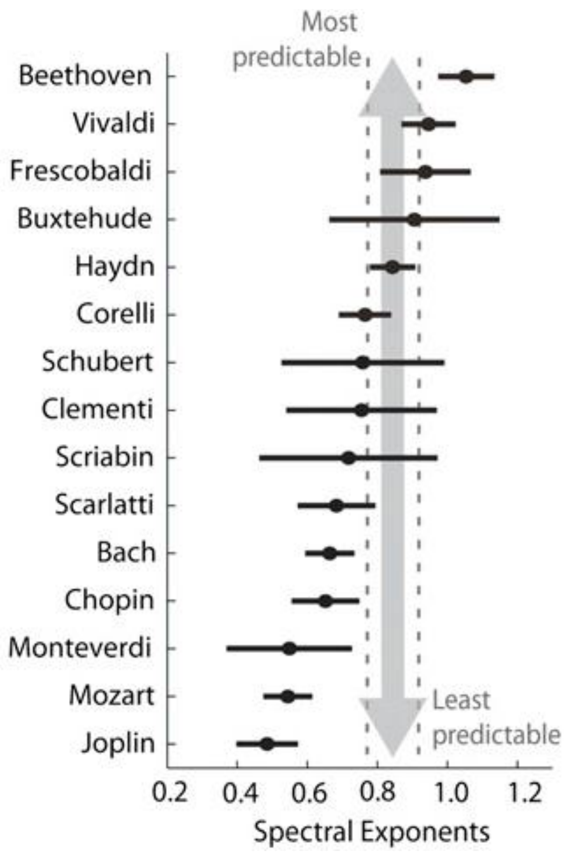


Figure 53: Distribution of spectral exponents for composers ordered from the largest mean exponent to the smallest (Levitin et al. 2012:3719)

Regarding specific classical genres, as opposed to composers, symphonies and quartets were generally indicated to have the most predictable rhythms. However, mazurkas and ragtime displayed some of the least predictable rhythms (Levitin et al. 2012:3717-3718).

Levitin et al. (2012:3718) commented that the "1/f structure [of musical rhythms] allows us to quantify the range of predictability, self-similarity, or fractal-like structure within which listeners find aesthetic pleasure. Critically, compositions across four centuries and several subgenres of Western classical music all demonstrated 1/f structure in their rhythm spectra."

According to their research, earlier experiments concerned with the measurement of pitch fluctuations could not "systematically delineate one composer's work from another." On the contrary, it was found that "1/f rhythm spectral exponents varied widely and systematically among composers." If one re-examines Figure 53, one can see the distinct rhythmic difference between Haydn, Mozart and Beethoven, despite the fact that these three composers are generally grouped together in the Classical Era.

3.4 Towards a better understanding of fractal music

3.4.1 Harlan J. Brothers

In the past ten years, some researchers have gradually moved away from the spectral analysis of music to a more practical approach – one that enables any musician with a rudimentary understanding of self-similarity and fractals to do a structural analysis of existing compositions or to compose an original work with the use of these mathematical techniques.

One of the pioneers in making the notion of fractal music more accessible to both musicians and mathematicians is Harlan J. Brothers. Whereas many of the researchers mentioned earlier were either scientists *or* musicians, Brothers is both: he is Director of Technology at The Country School in Madison, but has also studied composition at the Berklee College of Music in Boston and is a guitarist. Brothers worked closely with the mathematicians, Michael Frame and Benoit Mandelbrot, at Yale University where they started Fractal Geometry Workshops in 2004. "During an informal discussion in 2003 regarding the general lack of understanding associated with the concept of fractal music, Mandelbrot suggested to Brothers that he undertake a rigorous mathematical treatment of the subject" (Brothers 2004).

The initial purpose of the workshops at Yale University was "to train educators in the subject of fractal geometry with the goal of developing fractal-based mathematics curriculum for students in the middle school through college" (Brothers 2004). Through the years, and with the help of Mandelbrot and Frame, these workshops have been extended to include fractal music as well. An entire section of Yale University's website (http://.classes.yale.edu/fractals/) is dedicated to providing information on fractals and self-similarity.

3.4.2 Prerequisites for self-similarity and fractality

Brothers' courses on fractals and fractal music, which are available on the internet from www.brotherstechnology.com, include valuable information on what characteristics make something fractal, as well as some "misconceptions" in this field of study. These are discussed individually in the next section.

According to Brothers (2004), an object is self-similar when it meets the following three requirements:

1. It is constructed of a collection of different-size elements whose size distribution satisfies a power-law relationship spanning at least three scales;
2. It must compromise at least two similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole; and
3. Its features must possess sufficient detail that the overall structure cannot be more easily explained in Euclidean terms.

Although not stated directly, it is assumed that a music composition should also adhere to these three prerequisites in order to be regarded as self-similar. With such a wide array of prerequisites, the existence of fractal compositions seems almost impossible.

Brothers (2004) also identified some "common misconceptions" regarding self-similar and fractal music, namely transliteration, all iteration and limited self-similarity. Transliteration in music is the process in which music is directly derived from another source, for example, the outline of a graph or geometric object, through a process called mapping. He stressed that the translation of a self-similar or fractal object into music will not necessarily produce fractal music. He explained it well with the use of an example in the literature:

> This idea [of transliteration] may seem compelling at first and can produce interesting, if sometimes disconcerting, compositions. However, claiming that these compositions are fractal is something like claiming that a poem by Robert Frost, if transliterated using the Cyrillic alphabet, makes sense in Russian.

Although transliteration is a valid composition tool, it does not guarantee that the resultant music will also be fractal. Brothers referred to such examples as “fractal-inspired music” as opposed to “fractal music”. Examples of fractal-inspired music are given in the subsequent chapter.

The second misconception identified by Brothers (2004), is the idea that a composition created through iteration constitutes a fractal composition, since fractals rely on iterative processes. It should be pointed out that although fractal patterns often emerge from some iterative process, an iterative process will not guarantee a fractal pattern.

There are many shapes that are self-similar around a single point only. Examples include spirals, onions and Russian dolls whose only point of self-similarity is around the centre of the object. This is known as limited self-similarity and is not sufficient to classify an object, or a piece of music, as fractal. In order to be considered fractal, objects must “contain a minimum of two matching or similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole” (Brothers 2004).

Brothers (2004) concluded the introductory page of his website with the following words:

> Music can exhibit fractality in several ways, from rigid self-similarity of the composition to a psychological superposition of patterns on different scales. Beware of books or webpages [sic] about fractal music: many of these are the worst kind of nonsense.

3.4.3 Scaling in music

On the basis of all these rules formulated by Brothers, the possibility of fractal music existing is extremely remote. He went on to describe the different types of scaling that can be found in music, namely duration, pitch and structural scaling.

3.4.3.1 Duration scaling

One of the clearest examples of scaling in music may be found in the note values or "duration of notes" in a composition. According to Brothers (2004), if "examination reveals a power law relationship between a note's duration and the total number of such durations, then we can at least establish that a scaling phenomenon exists".

To illustrate duration scaling, Brothers provided the following two music examples which contain the same distribution of note values. That means that both of these examples contain exactly the same note values in the same distribution throughout each piece:

Figure 54: Brothers, two melodies with the same distribution of note values (Brothers 2004)

Figure 55 show the distribution of the note values in the two melodies by Brothers (2004).

Note value	**Number of occurrences**
Quaver (1/8)	16
Crotchet (1/4)	8
Minim (1/2)	4
Semibreve (1)	2

Figure 55: Table showing the distribution of note values in Brothers' two melodies (2004)

From this data the following log-log plot was generated:

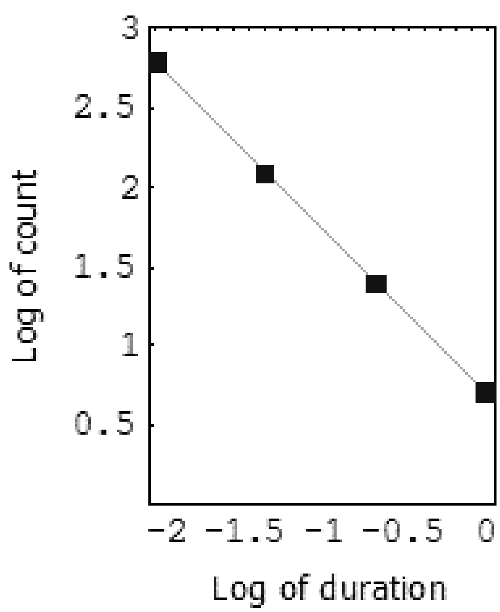


Figure 56: Log-log plot of the distribution of note values in two melodies by Brothers (2004)

The straight, negative slope of the graph shows that "both melodies obey a power law distribution". It does *not* mean, however, that both melodies are fractal in respect of note duration. The example in Figure 54(a) "simply states the sorted set of values, starting with the longest duration, proceeding to the shortest, and then wrapping back around to the longest". In addition, Brothers (2004) stressed that the note values are not evenly distributed throughout the first melody; instead each bar contains only a single duration value. He contends that the second melody (Figure 54 (b)) is a more convincing example because each bar (except for bar 4 and bar 8) contains a combination of different note values.

Brothers (2004) used the melody in Figure 54(b) as the top voice in a little counterpoint composition entitled *Go for Baroque.* The added voice contains twice as many notes as the top voice, but with the same distribution of note values (see Figures 57 & 58).

Figure 57: Brothers, *Go for Baroque,* illustrating duration scaling (2004)

Note value	Number of occurrences
Quaver	32
Crotchet	16
Minim)	8
Semibreve	4

Figure 58: Table showing the distribution of note values in Brothers' *Go for Baroque* (2004)

3.4.3.2 Pitch scaling

Another element of music to which scaling can be applied is pitch. Brothers (2004) wrote that if "examination reveals a power law relationship between a note's pitch and the total number of such pitches, then we can in a similar fashion to durations, establish that a scaling phenomenon exists".

The following six bar melody in D major (Figure 59) was written by Brothers to examine the possibility of pitch scaling. (In subsequent text and figures it will be referred to as the D major melody to avoid confusion with prior music examples.)

Figure 59: Brothers, D major melody to examine the possibility of pitch scaling (2004)

As in the case of testing for duration scaling, a table (Figure 60) was set up to summarise the relationship between each pitch and the number of occurrences. Pitches were arranged from the highest to the lowest, with each assigned a numerical value. The researcher also included the note names for clarification.

Note number (Pitch)	**Number of occurrences**
78 (F#)	2
76 (E)	3
74 (D)	9
73 (C#)	3
71 (B)	2
69 (A)	4
67 (G)	4
66 (F#)	2
62 (D)	3
57 (A)	1

Figure 60: Table showing the distribution of pitches in Brothers' D major melody, arranged from highest to lowest pitch (Adapted from Brothers 2004)

In this example, there appears to be no clear relationship in the distribution of pitches. Brothers (2004) attributed this to "the greater variety of different elements" compared to the examples of duration scaling. "Here we have 10 different values for only 33 elements; there are relatively few elements of each size on which to base our analysis" (Brothers 2004). A log-log plot of the data in the table also did not reveal any power-law relationship:

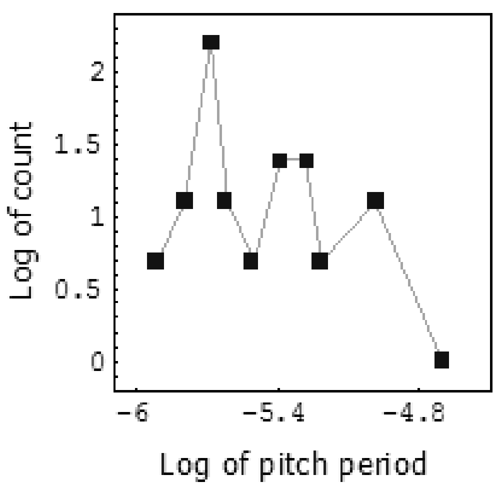


Figure 61: Log-log plot of the distribution of pitches in Brothers' D major melody (2004)

In such cases, Brothers investigated the possibility of binning the data because this might have revealed patterns that had had not been clear before. Data binning means processing the data into different intervals, called bins. The data was grouped into four separate bins:

Note number	Bin number	Number of occurrences
62 – 66	1	14
57 – 61	2	9
52 – 56	3	6
45 – 51	4	4

Figure 62: Table of binned data for Brothers' D major melody (2004)

The binned data obtained in Figure 62 was used to create the following log-log plot:

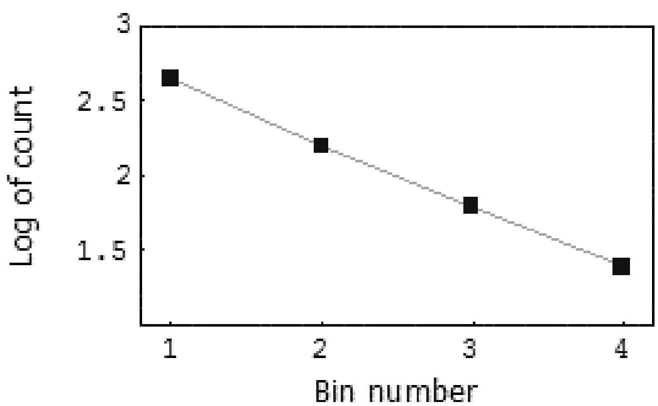


Figure 63: Log-log plot for the binned data of Brothers' D major melody (2004)

The table in Figure 62 and the log-log plot of its data in Figure 63 indicate the existence of a power law relationship in Brothers' D major melody.

3.4.3.3 Structural scaling

The third, and possibly most convincing, place to look for scaling in music is in its structure. Even if a power law relationship exists in the distribution of the durations or pitches in a composition, it will not necessarily be audible to the listener; nor will it be of much value to the performer to utilise in the interpretation of the composition. Structure, however, encompasses an entire piece of music and it is possible to hear the recurrence of a motif or section; whether presented exactly the same, slightly altered or transformed. Brothers (2004) found examples of structural scaling in compositions from the Renaissance and Baroque period, such as the second *Agnus Dei* in Josquin Des Prez's *Missa l'homme armé super voces musicales* and the first *Bourrée* from J.S. Bach's Cello Suite No. 3, BWV 1009 (Brothers 2004, 2007). These are discussed in detail in Chapter 5.

3.5 Validity of fractal musical analysis

In the article "Numbers and music" in *The new Grove dictionary of music and musicians,* Tatlow and Griffiths (2001:231) specifically addressed the credibility of the use of numbers in music analysis. The following is a quotation from this article:

> Musicology is left with a dilemma. Counting notes and pulses frequently reveals a numerical correlation between the sections of a musical work. This could imply that the composition was organized numerically at an early stage, and the temptation for the modern analyst is to assert that the numerical relationships were devised by the composer. Yet there is slender historical evidence to support this: little is known from music theory or surviving sketchbooks about the pre-compositional processes of composers before Beethoven. Without a firm historical basis it is both premature and irresponsible to draw conclusions about compositional procedure from numbers in the score. A separation must be maintained between numerical analysis, comment upon the compositional process and speculative interpretation of the numbers. There is also a need to consider whether there is any historical justification for the analytical techniques used to generate the numbers; and if so, whether the numbers in the score were created consciously by the composer and whether the numbers are wholly structural or have some further significance.

This slightly places a damper on the mathematical and fractal analysis of compositions, because there is no sure way to know if a composer thought of fractal or self-similar structures while composing. There are, however, other theories that can shed light on this use.

In 1991, Shlain published his book, *Art and physics: parallel visions in space, time, and light.* Shlain hypothesised that the arts (particularly the visual arts) anticipated many scientific discoveries. One example is that the artworks from Picasso's blue and rose period anticipated many of the theories developed by Einstein in his theory of relativity. Shlain (1991:24) stated the following in this regard:

> Art generally anticipates scientific revisions of reality. Even after these revisions have been expressed in scholarly physics journals, artists continue to create images that are consonant with these insights. Yet a biographical search of the artists' letters, comments, and conversations reveals that they were *almost never aware* of how their works could be interpreted in the light of new scientific insights into the nature of reality.

Shlain (1991:73) substantiated his hypothesis as follows: "Art reflected the thinking of the times." In addition, he believed that "the artist presented society with a new way to *see* the world before a scientist discovered a new way to *think* about the world" (Shlain 1991:73).

Shlain (1991) discussed some of the works by the impressionist painters, Cezanne, Monet and Manet to illustrate that space could be interpreted in non-Euclidean ways. He wrote:

> Their revolutionary assaults upon the conventions of perspective and the integrity of the straight line forced upon their viewers the idea that the organization of space along the lines of projective geometry was not the only way it can be envisioned.

Since this created a different way of *seeing* the world, it enabled them to *think* about it differently as well (Shlain 1991:73,118, 135).

Although Shlain's (1991) book made little reference to music compositions and fractal geometry, the researcher posits that a similar hypothesis can be used to substantiate the fractal and self-similar structures found in compositions dated prior to Mandelbrot's definition of fractals. It was mentioned that while "…visual art is an exploration of space; music is the art of the permutation of time. Like his counterpart, the composer has repeatedly expressed forms that anticipated the paradigms of his age" (Shlain 1991:271).

Shlain (1991) also touched on the philosophies of the ancient Greeks several times. Plato and Aristotle, for example, postulated that mimesis, the mimicking of nature, "was an innate impulse of the human personality" (Shlain 1991:150). He also highlighted the fact that these philosophers believed that "the essence of beauty was order, proportion and limit" (Shlain 1991:36.)

Many of these ancient philosophies still hold water and are still pursued today. If one considers the beliefs of Plato and Aristotle, the appearance of fractals and self-similarity in music that dates back to the 16th century makes sense. Without knowing it, composers, like many visual artists, mimicked fractal and self-similar structures in their compositions. The aesthetic quality of their compositions, although subjective, might be because of the level of "order, proportion and limit." Plato asked: "Is ugliness anything but lack of measure?" (Shlain 1991:151).

Shlain (1991:270) also quoted Zola: "… art *is* nature as seen through a temperament; and the nature of space, time, and light is revealed for those who want to see it through the creations of the innovative temperaments of the great artists".

3.6 Conclusion

In this chapter different research methodologies and theories of several researchers over a period of almost 40 years were examined. All studies relating to the spectral analysis of music concluded that classical music, more than any other genre of music, resembled fractal 1/f noise fluctuations. This was true of the melodic and rhythmic elements of various compositions. Although these findings are interesting, they are of little value to musicians, as was indicated by Rosalyn Tureck in her criticism of the Hsüs' research.

Nicholas Cook (1987), author of *A guide to musical analysis,* stressed that any type of analysis should either shed light on how the particular composer went about composing the work, or it should be an aid to the performer in accurately portraying the music. The researcher is of the opinion that many of the experiments conducted on the 1/f distribution are of more value to scientists than to musicians.

More recent studies on the use of 1/f noise distribution to classify genres sound promising. Scholars like Ro and Kwon (2009) and Levitin et al. (2012) found that 1/f noise distribution of music can be used to distinguish better between music of different genres and even distinct composers. This may be helpful, for example, in determining the authenticity of composers' works. To the researcher's knowledge, such research has not yet been conducted with the aid of 1/f distribution.

With the assistance of Mandelbrot and other researchers at Yale University, Brothers (2004) managed to clarify the prerequisites for fractal and self-similar music. These will be useful in the subsequent chapters where the composition and analysis of music with the help of fractal geometry and self-similarity will be discussed.

Shlain's (1991) hypothesis that the arts anticipated many scientific discoveries, coupled with the ancient Greeks' philosophy that art mimicked nature, justifies the existence of fractal and self-similar structures in music that was composed before the 20th century. Examples of such compositions are given in Chapter 5.

CHAPTER 4: FRACTALS AND SELF-SIMILARITY AS A COMPOSITION TOOL

4.1 Introduction

The research conducted by other scholars has shown that 1/f noise and fractal distributions can be found in music. This implies that fractals may be used to compose music. This chapter investigates exactly how noise forms, self-similarity and fractals can be used as an effective composition tool.

4.2 Composing music with noise

Voss and Clarke's (1975, 1977, 1988) findings that all music has some 1/f distribution has a bearing on the next section: if 1/f fluctuations could be found in existing music, then a 1/f noise source could certainly be used to compose an original piece of music. Although stochastic music has been composed before, it did not show any resemblance to 1/f noise fluctuations. Voss and Clarke (1988) decided to use sources of white, brown and 1/f noise respectively to see which of the three noise sources would produce the most interesting and musical compositions. They also proposed different methods for composing noise-based music, which is discussed in the subsections below (Voss 1988:42).

4.2.1 Using spinners to compose music based on noise forms

Voss and Clarke's (1977, 1988) research on the various methods for composing fractal music was summarised in the first chapter of Martin Gardner's book, *Fractal music, hypercards and more* (1992). The chapter entitled "White, brown, and fractal music", focused specifically on the implementation of 1/f noise in music composition. Methods included the use of dice and spinners to determine the different pitches and note values for a composition.

Gardner (1992:4-6) summarised Voss and Clarke's manual methods of composing music with the help of different noise forms. The use of a spinner, similar to one used in a board game, was one of the manual methods used to compose music based on noise forms.

In order to compose "white music" the spinner is divided into seven *unequal* segments; one for each diatonic note of a scale. The note on which the arrow lands after each spin is recorded on manuscript paper. A second spinner can be used to determine the note value of each of the pitches (Gardner 1992:4).

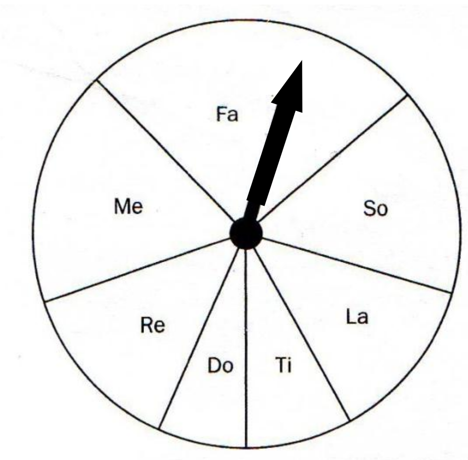


Figure 64: Spinner used to generate pitches for white music

Figure 65: White music generated with a spinner (Gardner 1992:16)

When using this method, the resultant composition is extremely random and not that musical. Voss and Clarke's research (1977) has shown that music generated by white noise is too irregular, since there is no correlation between successive notes. This can be seen in the resultant composition in Figure 65.

In order to compose a piece of music with the use of brown noise, the spinner is once again divided into seven segments, but this time they are marked with addition and subtraction rules. *+1*, *+2*, and *+3* indicate that the next note in the melody should be one, two or three notes higher than the previous one. The subtraction rules (*-1, -2, -3*) shows that the next note should be a set amount of notes lower. The *0* indicates that the subsequent note should be the same as the previous one. Since each note added to the melody is based on the previous note, the resultant melody will be highly correlated and uninteresting.

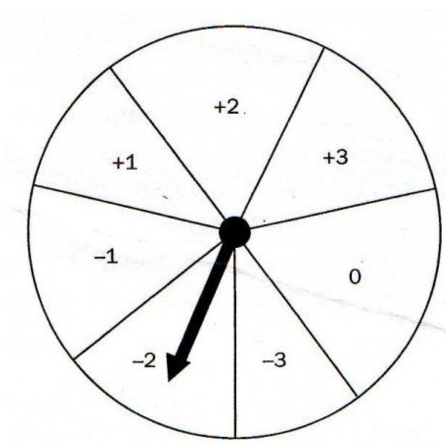


Figure 66: Spinner used to generate brown music

In contrast to music generated from white noise, Brownian motion (brown noise) is too predictable and will not be able to create interesting music with variation. This can be seen in the music example in Figure 67.

Figure 67: Brown music generated with a spinner (Gardner 1992:17)

Between these extremes of white and brown noise lies pink noise, more commonly known as 1/f noise. It is neither as random as white noise nor as predictable as Brownian motion and is thus ideal to use for generating music. By using 1/f noise, the music has an element of surprise as well as recurring ideas to bind a work to a unit (Gardner 1992:6).

Although Gardner (1992) did not explain in detail how the spinner was used to compose 1/f music, he did include a music example of the resultant music. As in stochastic composition, transition rules and rejection rules were adopted to make music correlated but interesting (Gardner 1992:5-6).

Figure 68: 1/f music generated with a spinner

4.2.2 Turning noise voltages into music

Another method explored by Voss and Clarke (1975) to compose music from noise was to use the actual noise voltages and then to convert them into music. Samples of three different noise wave forms were converted into music. So-called "Johnson noise voltage", across a resistor, was used to produce the white noise sample. This was later passed through a low-pass filter to obtain brown noise. The 1/f noise was obtained from fluctuations over a transistor. The different noise voltages were stored in a computer system and represented as notes in a 12-note chromatic scale over two octaves. Similarly, the same data was used to decide on the respective note values to be used for each pitch. The computer that was used could also perform these pieces and translate them into music notation. Figure 69 shows the music that was created from white, brown and 1/f noise respectively (Voss & Clarke 1975:262).

(a)

(b)

(c)

Figure 69: Music resultant from (a) white, (b) brown and (c) 1/f noise (Voss 1988:43)

The first melody (Figure 69(a)) was created from white noise and is not that musical in the sense that the note values cannot be grouped in bars with a single time signature and it contains many jumps of large intervals and little step-wise movement. The music thus imitates the uncorrelated, chaotic behaviour of white noise.

Music created from brown noise was notated as in Figure 69(b) and is characterised by many repeated notes. The melody wavers up and down constantly, but does not have direction. Of the three examples in Figure 69, the music resulting from brown noise contains the most stepwise movement and repeated notes, showing a high correlation as with Brownian motion.

As expected, the music resulting from 1/f noise (Figure 69(c)) shows a balance between the two extremes of the highly correlated music from brown noise and the chaotic melody from white noise. The 1/f music example contains a mixture of repeated notes, stepwise movement and some jumps.

It was easy to see that there may be higher or lower correlation between notes in the above music examples by *looking* at them, but what do they *sound* like? Voss (1988:42) remarked that "although none of the samples […] correspond to a sophisticated composition of a specific type of music, [Figure 69(c)] generated from 1/f-noise is the closest to real music. Such samples sound recognizably musical, but from a foreign or unknown culture." Voss and Clarke (1977:263) played the resultant music from the above-mentioned experiments to hundreds of people at several different universities and research laboratories over a period of two years, including professional musicians and composers as well as listeners with little theoretical knowledge of music. They came to the following conclusion:

> Our 1/f music was judged by most listeners to be far more interesting than either white music (which was "too random") or the scalelike $1/f^2$ [brown] music (which was "too correlated"). Indeed the surprising sophistication of the 1/f music (which was close to being "just right") suggests that the 1/f noise source is an excellent method for adding correlations.

After playing the music examples in Figure 69 several times on the piano, the researcher could not agree fully with Voss and Clarke's findings or the opinion of their listeners. Firstly, what constitutes "real music"? There is also no scale that can measure the level of musicality of a music example.

In the conclusion of their article, Voss and Clarke (1977:263) admitted that "there is more to music than 1/f noise". Although noise forms can be used to determine the pitch and rhythm in a music composition, there are other elements of music such as the overall structure, dynamics and timbre that were not taken into consideration.

Charles Dodge's computer-aided composition, *Profile* (1984), was directly influenced by Mandelbrot's work in fractal geometry. Dodge had been experimenting with ways to compose computer-generated music that would be aesthetically pleasing, and fractals supplied him with new ways to compose such music. *Profile* significantly differs from his earlier works in the sense that he composed the entire piece from a single computer algorithm. The work's success has led to other compositions written in a similar fashion (Dodge 1988:10). A 1/f-noise algorithm was used to generate the musical elements such as pitch, duration and volume in *Profile.* Since *Profile* is a computer-aided composition, no music example was available to insert.

4.3 Composing with Lindenmayer systems

In Chapter 2, Lindenmayer systems and their visual representation through the application of turtle graphics were briefly explained. The next subsection shows how these can also be used to compose music.

4.3.1 Prusinkiewicz

In 1986, at the International Computer Music Conference, Przemyslaw Prusinkiewicz demonstrated how the graphic representation of the Hilbert curve can be adapted as a musical representation. He used the third iteration of the Hilbert curve (as discussed in Section 2.3.2.5) to illustrate how music can be created from an L-system turtle graphic. When the Hilbert curve is laid out as in Figure 70 and moves in the direction shown by the arrow, then "the consecutive horizontal line segments are interpreted as notes" to create a melody. The horizontal line segments are numbered chronologically from the bottom left corner (Prusinkiewicz 1986:456).

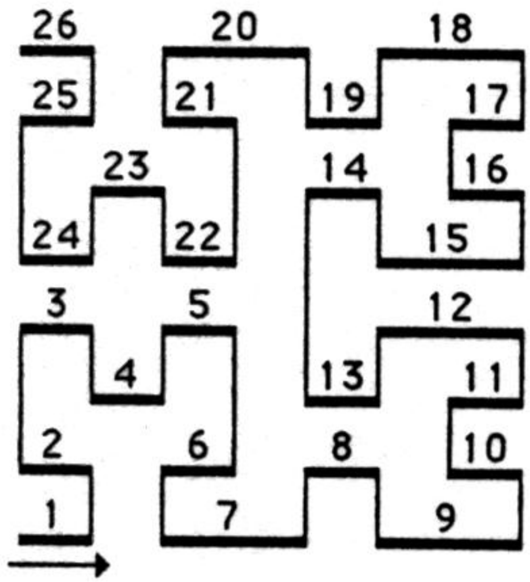


Figure 70: Third iteration of the Hilbert curve (Prusinkiewicz 1986:456)

Prusinkiewicz (1986) explained that the y-coordinates represent the pitch, while the length of each horizontal segment is proportional to the note value or duration. Since music occurs in time, the numbered line segments can be unravelled and presented chronologically.

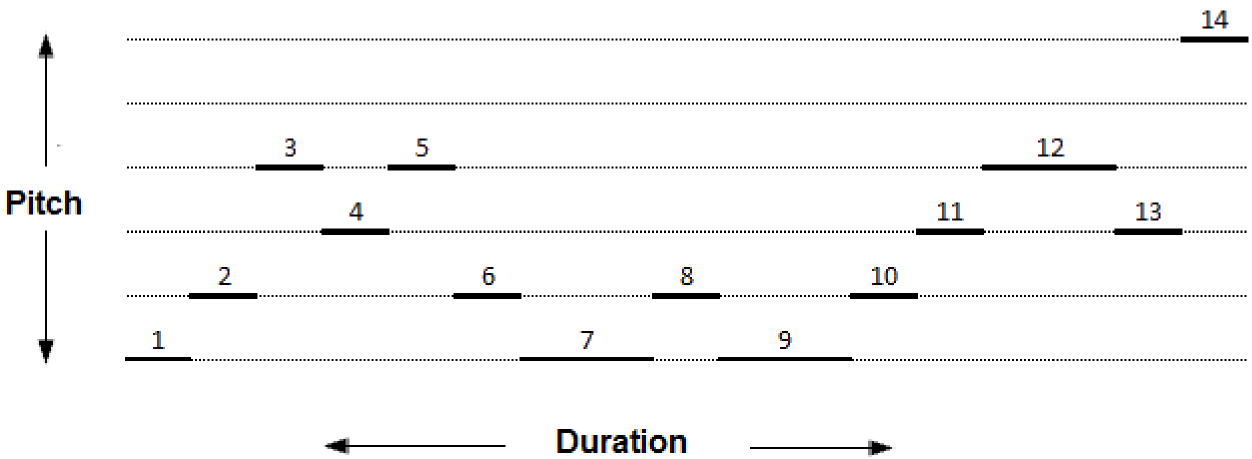


Figure 71: Unravelled form of the third iteration of the Hilbert curve (Adapted from Prusinkiewicz 1986:457)

The scale chosen for the melodic representation was C major with C as starting note (number 1). For the note values, the shorter lines were represented as quavers and the longer lines (which are twice as long) as crotchets. The resultant melody is shown in Figure 72:

Figure 72: Third iteration of the Hilbert curve represented as a melodic line (Prusinkiewicz 1986:457)

It is not clear from Prusinkiewicz's research why a 4/4 time signature was chosen or why barlines were positioned as in the example.

All of the curves used by Prusinkiewicz (1986) to generate music were read from left to right, and the first line of the curve was always horizontal. What would the musical result be if a curve were to be rotated before it was interpreted as music, or if the curve were to be read from right to left? Would it be possible to use a curve that begins with a vertical line? These questions were further investigated by other scholars.

4.3.2 Mason and Saffle

Stephanie Mason and Michael Saffle (1994) built on Prusinkiewicz's composition method by showing how transformations (rotations) of the same curve can be used to create variations of a melody. Their findings were published in 1994 in an article in the *Leonardo Music Journal.*

Mason created a program that could automatically draw a curve from any given L-system. The programme also allowed her to play the resultant music on a synthesiser. Together with Chris Cianflone, a student at the University of Minnesota at the time of their research, an experimental music program was developed based on this research (Cipra 1993:37).

Mason and Saffle (1994:32) demonstrated that a total of eight different melodies can be produced from a single geometric curve. The original curve can be read forwards or backwards (giving two “original melodies”). The entire curve can be rotated into three different positions, each of which can be read forwards and backwards, resulting in six more melodies. To illustrate this, the first iteration of the quadratic Gosper curve was used. Figure 73 displays each of the rotations with its corresponding music example. For the purposes of this example, the note values were interpreted as quavers and semi-quavers, while the C major scale was chosen to determine the pitches.

Mason and Saffle (1994:33) explained that when a curve starts with a vertical line, “the first note can be read as the number of forward moves up or down from the first note of a given scale or mode. Similarly, the number of forward moves used to create the horizontal line determines the note’s duration.”

In Figure 73, numbers 1 and 2 show how the original quadratic Gosper curve can be interpreted both forwards and backwards. The music examples are melodic retrograde from one another with the second being a third higher than the first.

The curve was then rotated clockwise by 90 degrees to obtain the information for numbers 3 and 4, once again being read forwards and backwards. A 180 degree and 270 degree rotation of the original are shown in numbers 5-6 and 7-8 respectively. Numbers 5 and 6 are the melodic inversions of numbers 1 and 2. The same is true when comparing numbers 3 and 4 with 7 and 8.

The given time signatures are unclear from the context of Mason and Saffle’s (1994) article, but the resulting melodies are musically sensible. When a higher iteration of the same curve is used, longer, more interesting melodies can be created. It is also possible to use any scale or mode.

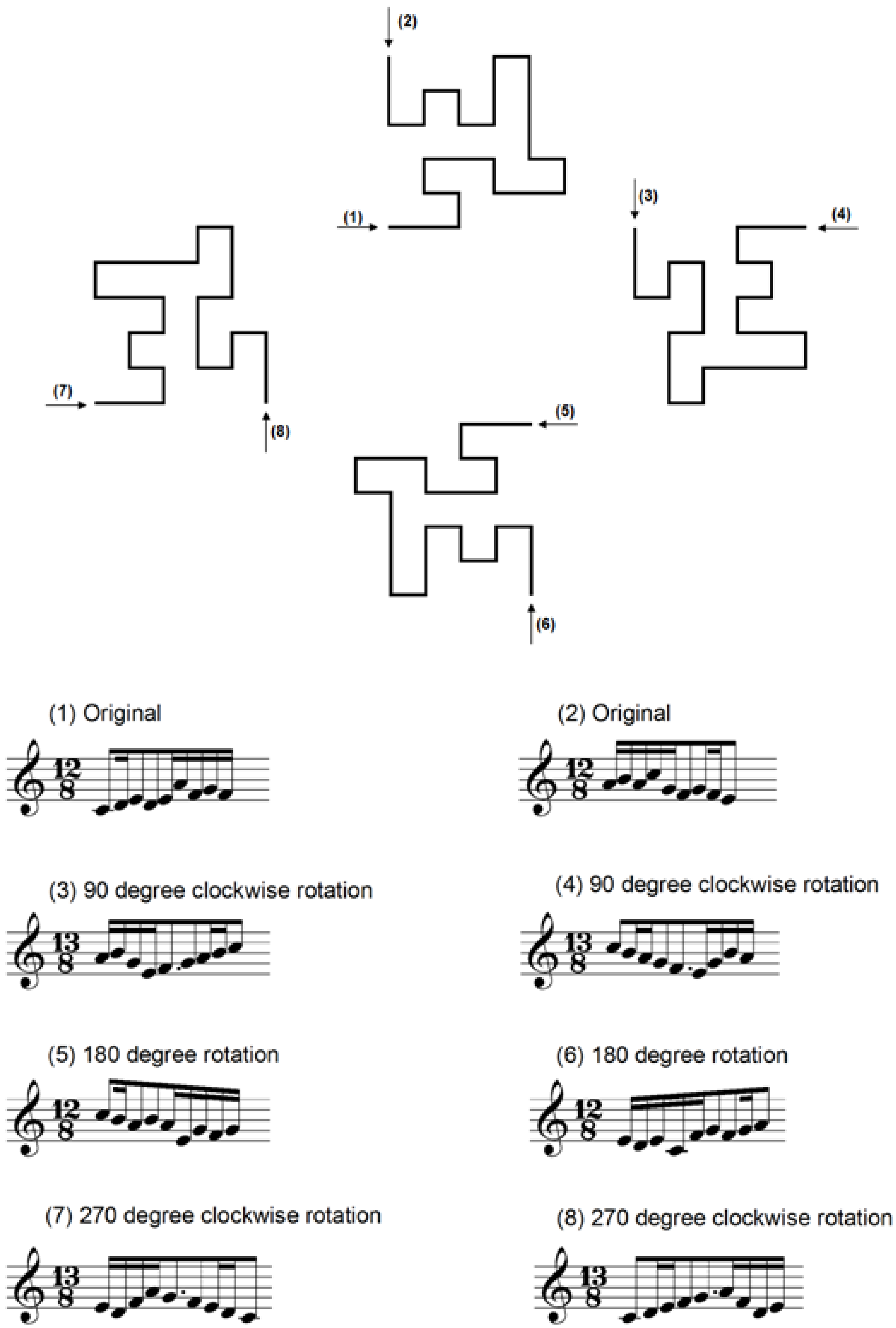


Figure 73: The first iteration of the quadratic Gosper curve read in terms of four rotations to produce eight different melodic motifs (Mason & Saffle 1994:32)

This method of composition also makes polyphonic composition possible. Two iterations of the same curve can be played simultaneously. Figure 74 shows two iterations of the quadratic Gosper curve. The curve on the left was interpreted by Mason as a melody for flute and the curve on the right as one for piano. Arrows were added to show where each melody starts: the flute starts in the lower left-hand corner, while the piano starts in the lower right-hand corner. Underneath, the resulting music is given in music notation (Cipra 1993:37; Mason & Saffle 1994:33).

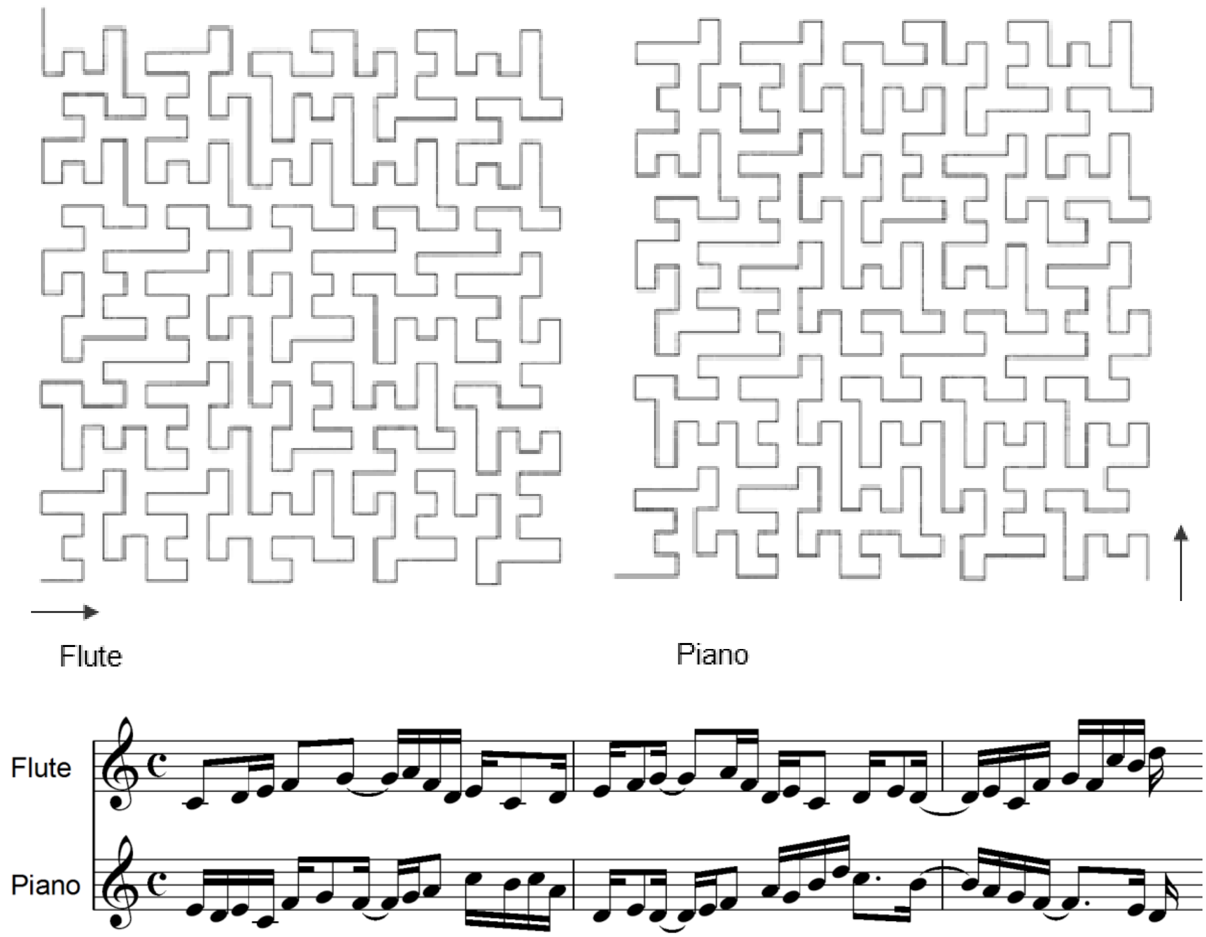


Figure 74: Two transformations of the second iteration of the quadratic Gosper curve used to compose polyphonic music for flute and piano (Adapted from Cipra 1993:37; Mason & Saffle 1994:33)

Cipra (1993:37) commented as follows on Mason's polyphonic composition from the quadratic Gosper curve:

> Composers have long played with the formal structure of music. Bach, for example, is well known for writing music that could be played backward as well as forward. Mason had gone a step further, with music that can be played sideways as well, in what she calls a "right-angle canon". To do this, she simply takes a curve and rotates it so that the pitch and duration are interchanged. When both curves are played together, using separate synthetic "voices" (Mason leans to piano and flute), the effect is surprisingly musical.

Another aspect of fractals and L-systems that can be brought into context with music composition are their self-similarity and scaling properties. Figure 75 shows the self-similarity of the quadratic Gosper curve. The first iteration of the curve (red) is *not* fractal. The second iteration (black) contains copies of the first iteration and is thus self-similar and fractal. Note how the first iteration of the curve is a larger version of the smaller second iteration of the curve. This shows that the quadratic Gosper curve in its second and higher iterations is in fact self-similar and fractal.

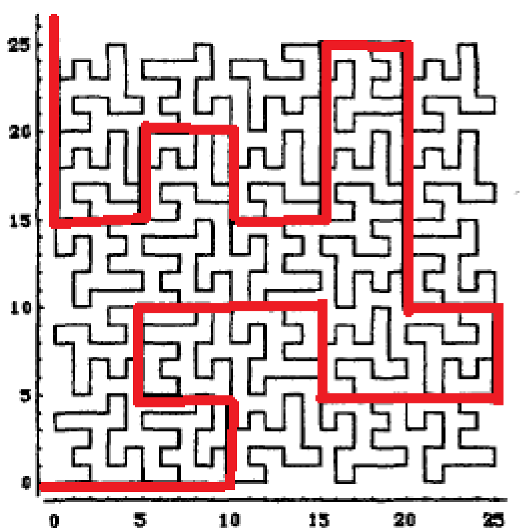


Figure 75: Self-similarity of higher iterations of the Hilbert curve

This also indicates the possibility that such scaling can also be applied in resultant music compositions. The second iteration of the quadratic curve can be interpreted melodically (as in Figure (75)). The first iteration, which is larger, will produce a melody self-similar to the first, but in longer note values. It is thus an example of duration scaling as discussed by Brothers (2004) in section 3.5.4.

Pursinkiewicz (1986:456) and other scholars such as Mason and Saffle (1994:33) commented that any FASS curves (space-filling, self-avoiding, simple and self-similar curves), fractal curves and any other curves containing 90 degree angles can be used to create melodies in a similar way to the aforementioned method.

4.3.3 Further modifications of L-systems for composition

The use of space filling curves generated from L-systems can lead to interesting music examples, but for some composers this method was still too dull. One example is Gary Lee Nelson (born 1940), a pioneer in the field of computer music. He made use of fractals, among other mathematical ideas, to compose music. His article, “Real time transformation of musical material with fractal algorithms”, explained how he went about composing such music (Nelson n.d.).

In *Summer Song* for solo flute (composed in 1991), Nelson (n.d.) extended some of the ideas used by Prusinkiewicz (1986) and Mason and Saffle (1994) as discussed above. For this composition, he used the fourth iteration of the Hilbert curve, but found it to be “too symmetrical” and that the music it produced contained “rather dull repetitive patterns with little variety in pitch or rhythm” (Nelson n.d.).

He found a simple solution for this: instead of letting the turtle turn by an angle of 90 degrees, he widened it to 101 degrees, thus creating a slightly warped version of the original curve (see Figure 76). He wrote that the result was “much better suited to make the piece [he] envisioned.”

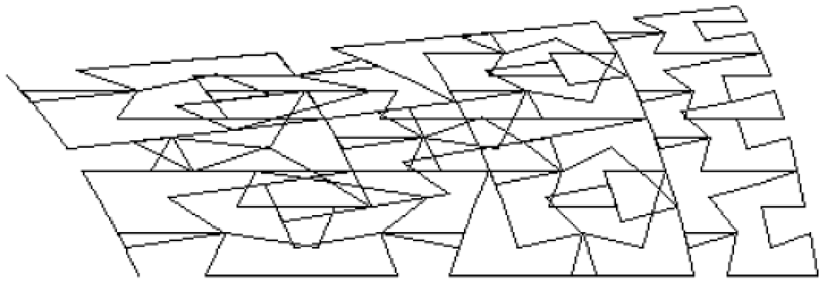


Figure 76: Warped version of the Hilbert curve used for Nelson's *Summer Song* (Nelson n.d.:6)

Also, instead of using a tonal scale, Nelson used a "projection of the interval series 2 2 3 over two and a half octaves". This means that the "scale" consists of two intervals of a major second followed by a minor third. Figure 77 shows the scale as given by Nelson (n.d.:7) in his article:

Figure 77: Scale used in Nelson's *Summer Song* (Nelson n.d.:7)

Once again the vertical position of lines was read as pitch and the length of the lines as duration. Although Nelson did not include a music example of the music produced, he inserted the following graph illustrating the contour of the melody:

Figure 78: Warped version of the Hilbert curve, unravelled to produce the pitch and time contour for Nelson's *Summer Song* (Nelson n.d.:7)

4.4 Self-similar structures in the music of Tom Johnson

The American minimalist composer, Tom Johnson, was interested in creating self-similar structures in his music. In a lecture presented at the *Mathematics, music and other sciences* (MaMuX) seminar in Paris in October 2006, Johnson explained his fascination with self-similarity and how he went about using such structures in his music. Johnson's lecture was entitled "Self-similar structures in my music: an inventory". He defined self-similar music as "music that somehow contains itself within itself and does so on at least three different levels of time" (Johnson 2006:2).

Johnson (2006:2) described the three main influences for his self-similar compositions to be Martin Gardner's regular publications in *Scientific America* in the 1970s; Mandelbrot's book on fractals; and his meeting in 1979 with the mathematician, David Feldman, who taught Johnson the fundamentals about group theory and self-replicating melodic loops.

Johnson (2006:2-23) stressed that true self-similarity is extremely rare in compositions. Although most of his music is characterised by minimalism and symmetry, looking back he found only a few pieces to be self-similar.

One of the earliest examples of self-similarity in Johnson's works was *Symmetries* which consists of 49 symmetrical "drawings" created with a music typewriter. Figure 79 shows the three self-similar drawings in the set. These drawings were not meant solely as visual representations of symmetry or self-similarity. According to Johnson (2006:5): "At the time I typed these images I was thinking of a pure visual music, hoping that others would do their own realizations, or would simply imagine sounds implied by the drawings."

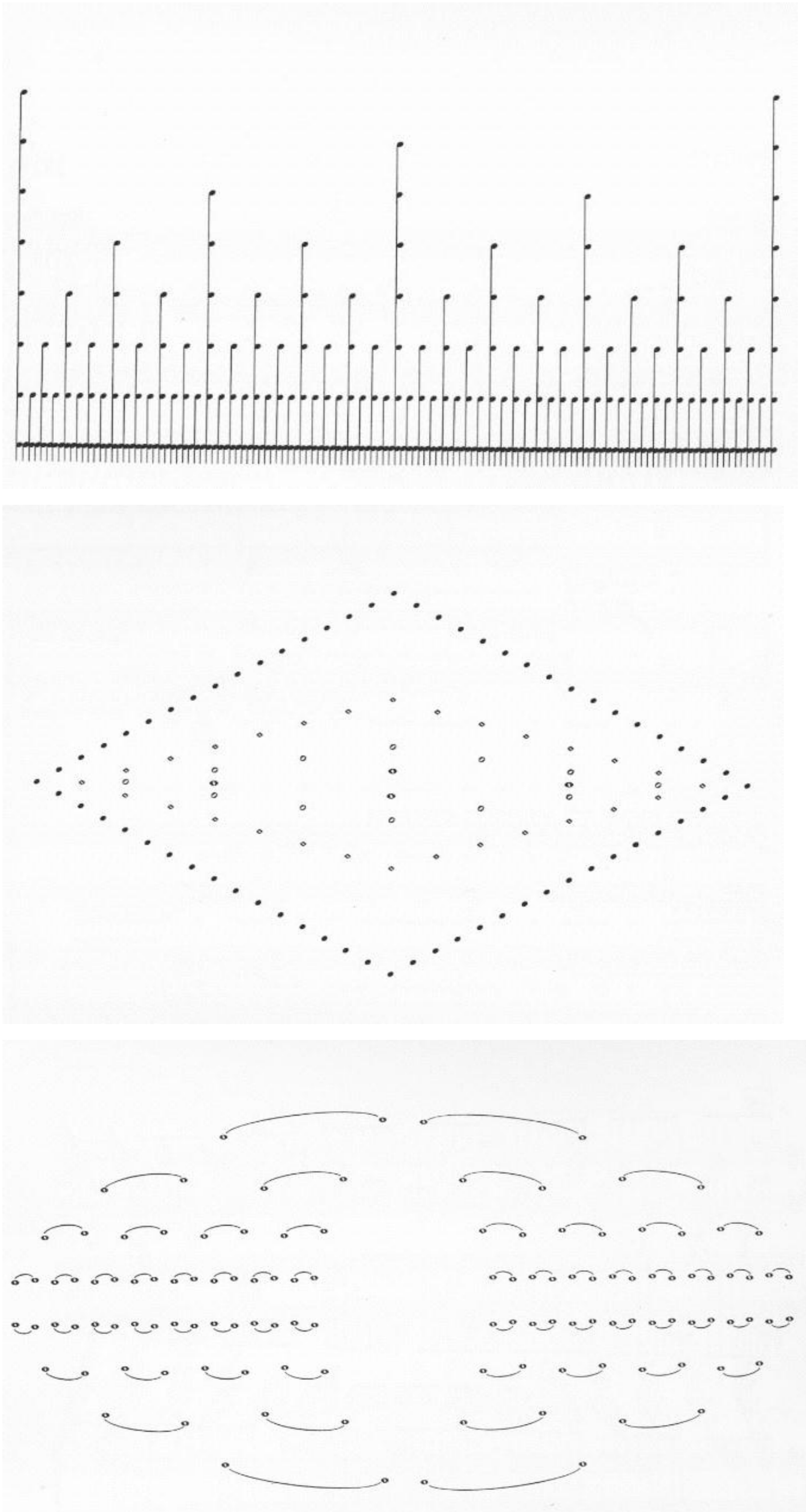


Figure 79: Three self-similar drawings from *Symmetries* created with a music typewriter (Johnson 2006:4-5)

Rational Melodies, published in 1983, were "constructed with rational systems, following rigorous rules" (Johnson 2006:6). *Rational Melody VIII* (Figure 80) has a clear self-similar structure. The melody starts with four pitches descending chromatically in minims. In the second line, a diminution of the melody (in crotchets) is inserted before each minim. The process is repeated twice in quavers and semi-quavers. Each iteration of the four-note melody thus contains itself in different note values, finally giving a total of four levels of self-similarity. The process can be carried out further, but Johnson stated that the semi-quaver iteration was as far as he wanted to go (Johnson 2006:6-7).

Figure 80: Johnson, excerpt from *Rational Melody VIII* (Johnson 2006:6)

The same composition method was applied to the second movement of *Counting Keys* for piano. Here the note sequence is C-B-A-F#-D and the notes repeat themselves on a scale of 1:5. Johnson (2006:9) commented that the score is not necessary and that he performed it for many years without notating it.

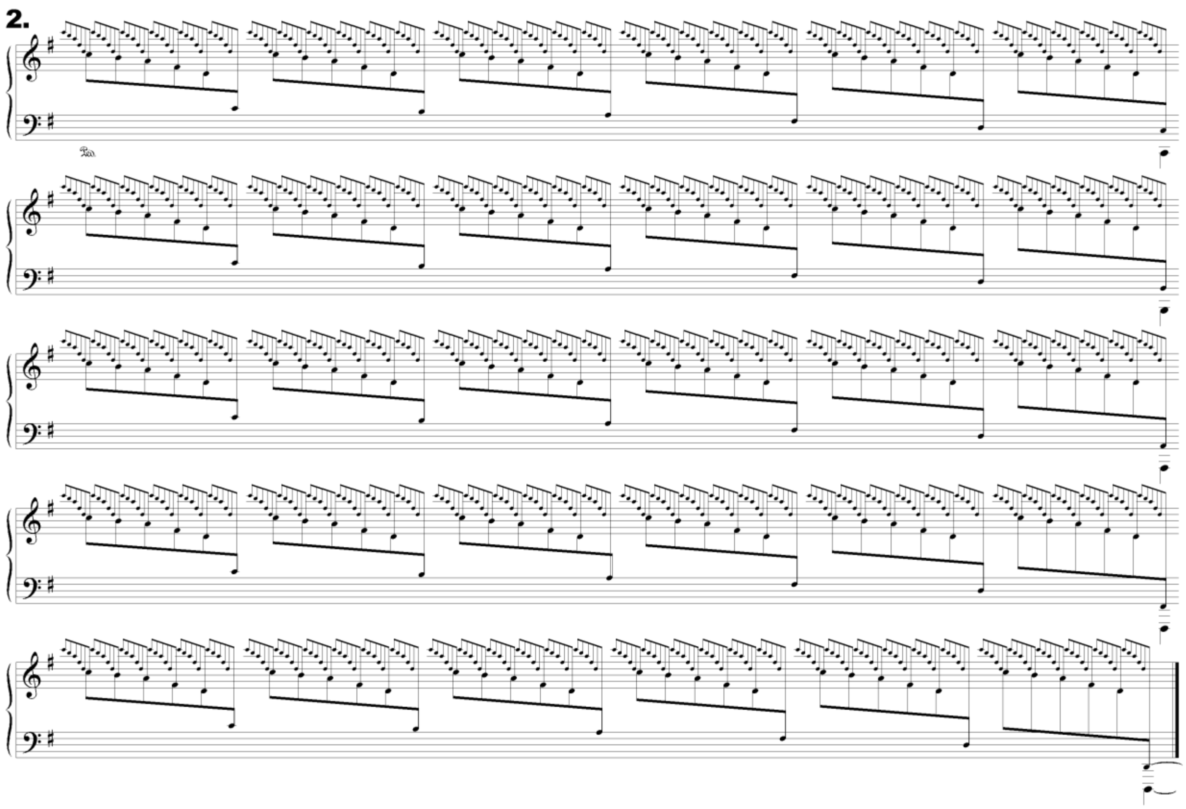


Figure 81: Johnson, second movement from *Counting Keys* (2006:9)

Counting Duets, written in 1982, is a piece for speakers[12]. The first person starts by repeating the numbers 1-2-3. The second speaker repeats the reverse of the pattern, 3-2-1, but numbers by the first and second speaker must be in unison –i.e. when they speak together, the same number must be uttered (see the first two lines in Figure 82). More speakers are added and the 1-2-3 and 3-2-1 patterns alternate with each added speaker. Each added voice is at half the tempo of the previous – hence the emergence of a different rhythmic pattern. A self-similar structure is also evident as a result (Johnson 2006:7).

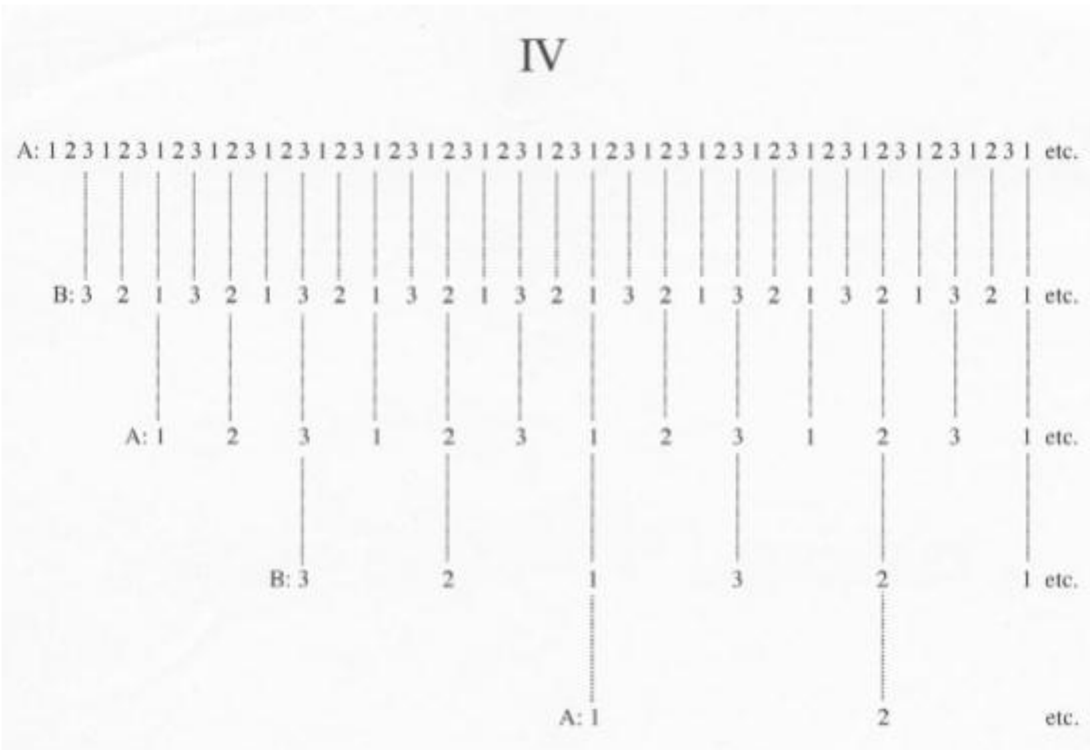


Figure 82: Johnson, *Counting Duets*, fourth movement (Johnson 2006:7)

In *Counting Duets,* the slower speakers had to say their number in unison with another speaker, creating doubling. In other works by Johnson, this simultaneous doubling was deliberately avoided to create different rhythmic effects and self-similarities. One such example is the non-pitched choral piece *1 2 3,* composed in 2002. The composer explained that the “simultaneities are systematically avoided” in order to create a texture known as “one-dimensional tiling” (Johnson 2002:13).

[12] The word “speaker” here refers to a person speaking.

In the first part of *1 2 3,* it takes three bars (or 24 beats) for each of the eight voices to have its turn. Johnson explained that the ratio between the voices is 8 : 8 : 4 : 4 : 4 : 4 : 2 : 2 (Figure 83).

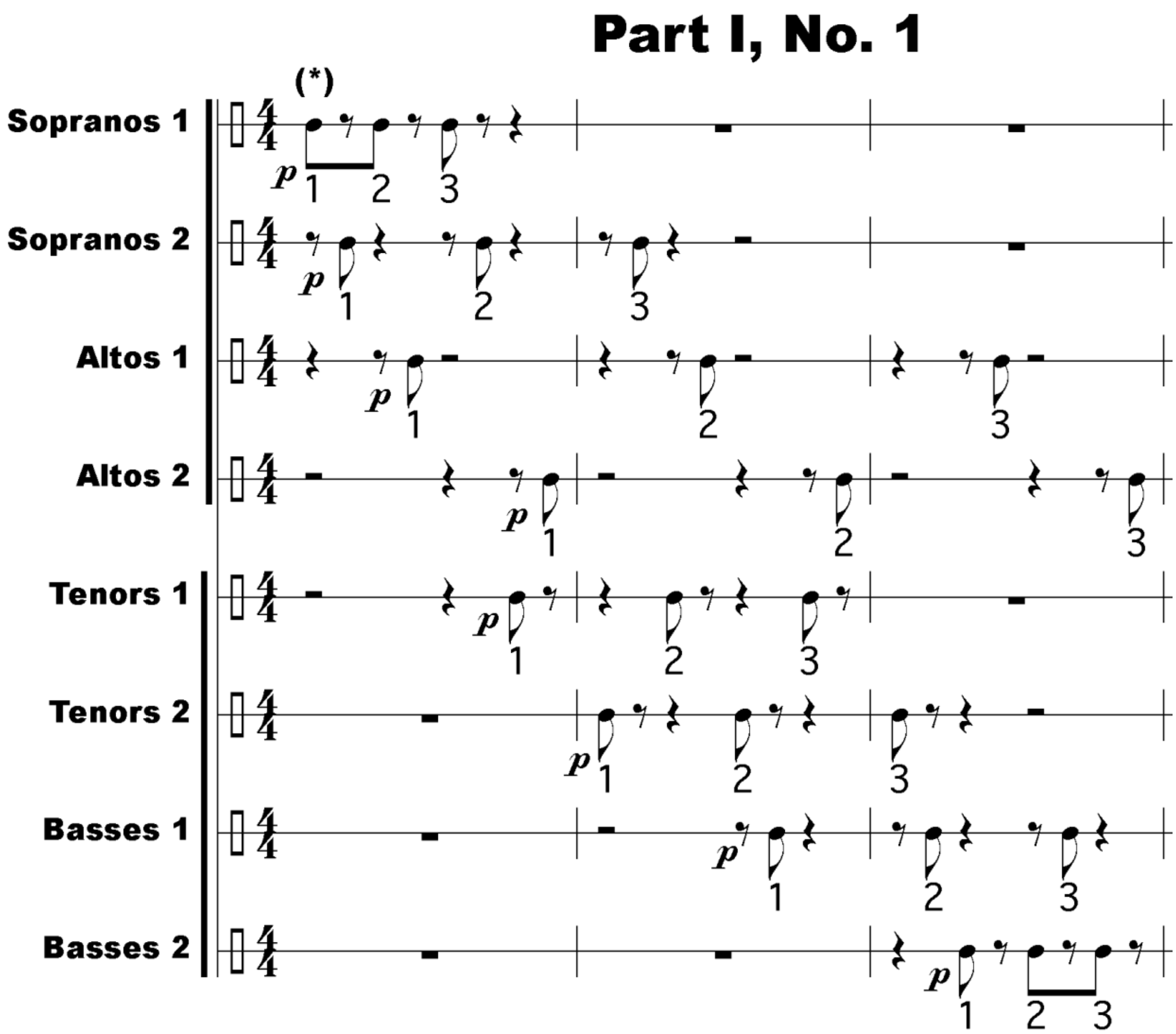


Figure 83: Johnson, *1 2 3,* Part I, No. 1 (2006:13)

Tilework for Violin (2003) is the only piece in Johnson's *Tilework* series that has a self-similar structure. The first note of bars 57-60 gives an ascending melodic pattern moving in whole tones that continue throughout the rest of the piece: A♭-B♭-C-D. The first appearance of the pattern is marked in red in Figure 84 (bars 57-60). The same pattern is repeated an octave lower, every three bars (bars 57, 60, 63 and 66) and is marked in blue. An even slower-moving iteration, another octave lower, is presented every nine bars (bars 59, 68, 77 and 87). It is marked in yellow in Figure 84. It is evident that the same material is presented in three different octaves and at three different tempos. There are thus three scale levels present, making the piece self-similar.

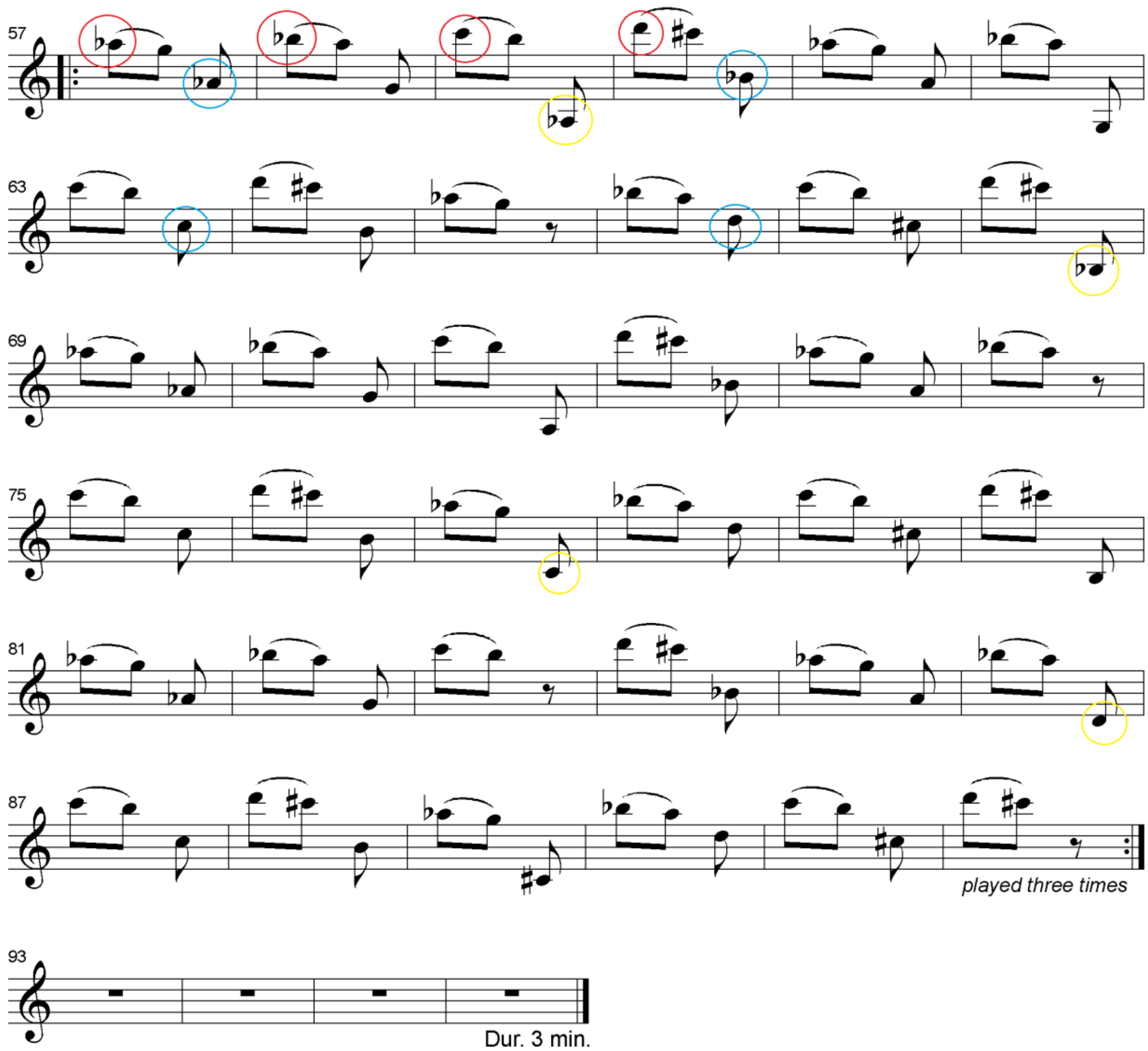


Figure 84: Johnson, *Tilework for Violin,* bars 57-96 (Johnson 2006:14)

4.5 Coastlines and mountains

The research conducted by the Hsüs (1993) proposed a method for the measurement of a composition similar to that used to measure irregular coastlines. In the next section, a more practical application is investigated: how can the irregular contour of coastlines and mountains be used as a composition tool?

4.5.1 Mandelbrot and coastlines

In 1967, almost a decade prior to the coinage of the word "fractal", Mandelbrot published an article entitled "How long is the West Coast of Britain? Statistical self-similarity and fractional dimension*"*. This research showed how the structure of coastlines and mountain ranges is self-similar and fractal in nature and had a great impact on the view of not only nature, but also art and music.

In 1961, the British meteorologist, L.F. Richardson, conducted research on the irregularity of the outline of several coastlines around the world and was further investigated by Mandelbrot in the 1960s. Their research indicated that length was not an accurate way to measure a coastline, since the length depended on the size of the measuring stick being used as well as the scale of the map. The more detailed the map is, the greater the length of the coastline is. This has led to an interesting paradox that the length of a coastline is infinite. This is also known as the Richardson effect.

Coastlines, like fern leafs, also reveal new detail with each magnification and are self-similar. This made it possible for scientists to "measure" a coastline – not according to its length, but its dimension.

A line has a dimension of 1, while a square has a dimension of 2. The dimension of a coastline lies between 1 and 2, like many other self-similar and fractal objects. Simply by looking at the west coast of Britain, Mandelbrot could see that it is one of the most irregularly shaped coastlines in the world and thus chose it for his research in 1967. Its dimension has been calculated to be 1.25. In strong contrast, the dimension of South Africa's coastline is 1.02 (close to 1) and is one of the smoothest in the world (Figure 85.) From this information it is clear that the dimension accurately describes the level of irregularity for a coastline or other self-similar shape. The same was also to be found true for the dimension of mountains. Mandelbrot called this dimension between 1 and 2 the "fractional dimension" because it is a fraction and not a whole number. This later became known as the "fractal dimension".

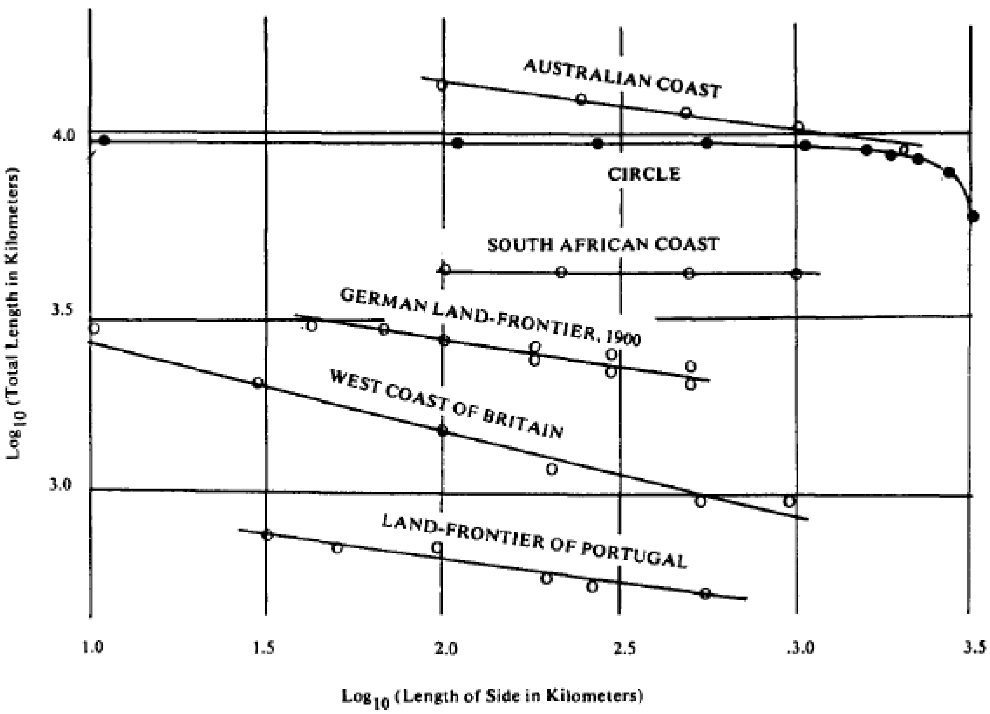


Figure 85: Richardson's data concerning the rate of increase of a coastline's length at decreasing scales (Kappraff 1986:660)

Mandelbrot discovered that any segment of a coastline is statistically self-similar to the whole. Objects can be statistically self-similar if they have "the same statistical distribution of their features, such as ins and outs, under magnification or contraction in their geometric scale … Statistical self-similarity can never be empirically verified for naturally occurring curves such as coastlines, since there are an infinite number of features associated with such curves" (Kappraff 1986:657).

Computer graphic artists have used this newly found information useful in reconstructing real-looking mountain ranges on computers. By using a simple, iterative formula it is possible to "design" a mountain range like in Figure 85, which shows a fractal mountain range that was created by Ken Musgrave. It is known as a "multifractal forgery" (Frame et al. 2014).

Figure 86: Musgrave, a fractal landscape (Frame et al. 2014)

This research by Richardson, Mandelbrot and other scientists had a huge impact on the development of ideas in geo-statistics, physics and even economics (Mandelbrot 1967:356). It is hard to imagine that it has also impacted on music composition. The remainder of this section illustrates how this research on the dimension of coastlines and mountains influenced the works of 20th- and 21st-century composers.

4.5.2 Relating music to coastlines and mountains

Mountains have been a metaphorical source of inspiration for composers for many years, for example, Grieg's *In the Hall of the Mountain King* from the Peer Gynt Suite and Mussorgsky's *Night on Bald Mountain.* From the latter half of the 20th century, composers began using the structure of mountains more directly in their work.

Composers used what is called "mapping": a process in which the contour of any shape (in this case a building, mountain or coastline) is directly translated as the contour for the melodic line(s) in a composition. One of the best examples in literature of this type of mapping is *New York Skyline* by the Brazilian composer, Heitor Villa-Lobos. Although not fractal, this piece illustrates the basic concept of mapping extremely well.

New York Skyline was composed in 1939 by superimposing the New York skyline on to graph paper and from there plotted as music notes. It is said that Villa-Lobos used the same method five years later to compose his sixth symphony. For the symphony, however, he used the contour of mountains from his homeland, Brazil. Another composition, *Melodia da Montanha* (Mountain Melody) was published in 1942 and was based on the horizon of the peaks of Serra da Piedade in Belo Horizante in the state of Minas Gerais (Melo 2007:3).

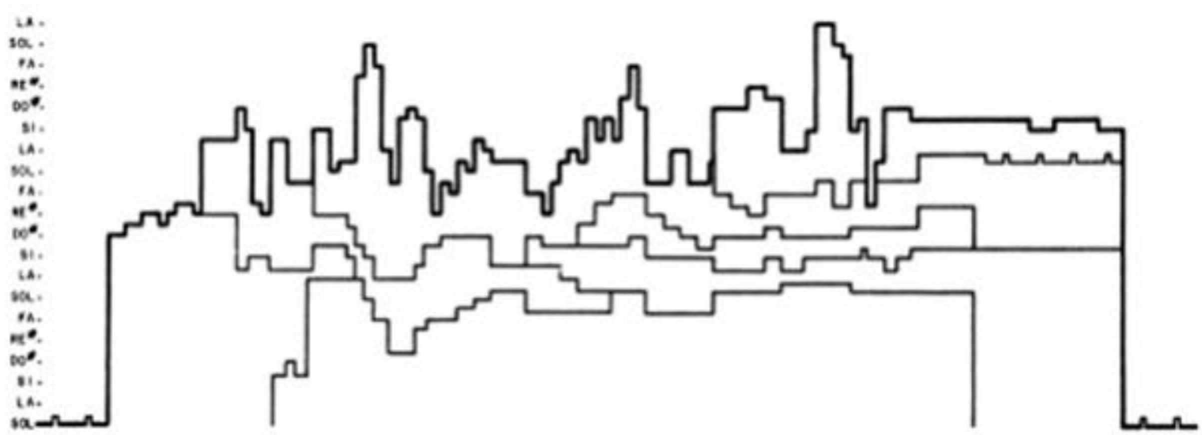


Figure 87: Graphic representation used in Villa-Lobos' *New York Skyline* (Kater 1984:105)

If it is possible to map an object as music, the resultant music from a fractal or self-similar object is expected to produce the same type of music.

4.5.2.1 Composing with coastlines

Music based on coastlines is rare, but the American composer Larry Austin composed an excellent example of this type of music. Austin's composition for instruments and tape, *Canadian Coastlines: Canonic fractals for musicians and computer band* (1981), was commissioned by the Canadian Broadcasting Company and is entirely based on the outline of the coast of Canada. The composition is for eight instruments and four channels of computer-generated sound (Clark & Austin 1989:22).

The technique that Austin was using during this time for his compositions relied on the "processing data by plots over time." Certain data was extracted from the map and used as a "seed" for the computer algorithm (Figure 88.) He admitted in an interview with Thomas Clark that "the fractal result [of the music] could not be understood in terms of the map". Unlike composers before him, Austin did not simply plot the contours of the coastline as music notes, but also made use of a 1/f-noise computer algorithm for the composition (Clark & Austin 1989:22).

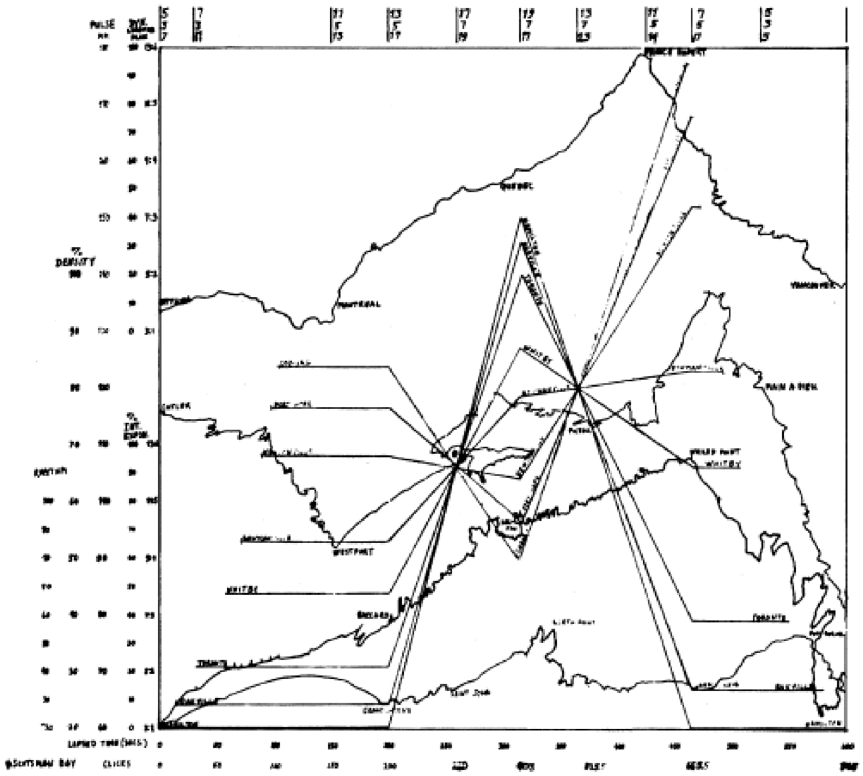


Figure 88: Time-plot obtained from a map of Canada for Austin's *Canadian Coastlines* (Clark & Austin 1989:23)

The subtitle of the composition hints at a type of canon. The eight musicians, guided by click tracks in their headphones, play at four different tempos. In addition, the four computer tracks also move at their own tempos. There are eight parts that are "canonic imitations" of one another, played at different rates. This is thus similar to the prolation or mensuration canons of the Renaissance period (Clark & Austin 1989:28).

Other compositions by Austin that also incorporate fractals include his *Sonata Concertante* (1983-1984) and *Beachcombers* (1983) in which self-avoiding random walk (similar to Brownian motion) was used. In the latter work, some actual text from Mandelbrot's book was used (Clark & Austin 1989:23).

4.5.2.2 Gary Lee Nelson's *Fractal Mountains*

Another mountain-inspired composition is *Fractal Mountains* (1988-1989), composed by Gary Lee Nelson. *Fractal Mountains* is an electronic composition in which a fractal algorithm was used to determine the time, pitch and amplitude of the piece. The piece won first prize in the international competition for micro tonal music in 1988. This composition marks Nelson's first attempt at composing with fractal algorithms (Nelson n.d.:1-2).

Instead of drawing inspiration from an actual mountain range, the data for Nelson's composition was obtained from the recursive subdivision to simulate a two-dimensional mountain contour. The points on the simulated "mountain" were plotted as pitch and duration. To construct his "mountain", Nelson started by drawing the main peaks. Through an iterative process, similar to constructing the Koch curve, each line is subdivided to create finer details of the mountain. Figure 89 shows the subdivision of the mountain's contour (Nelson n.d.:3).

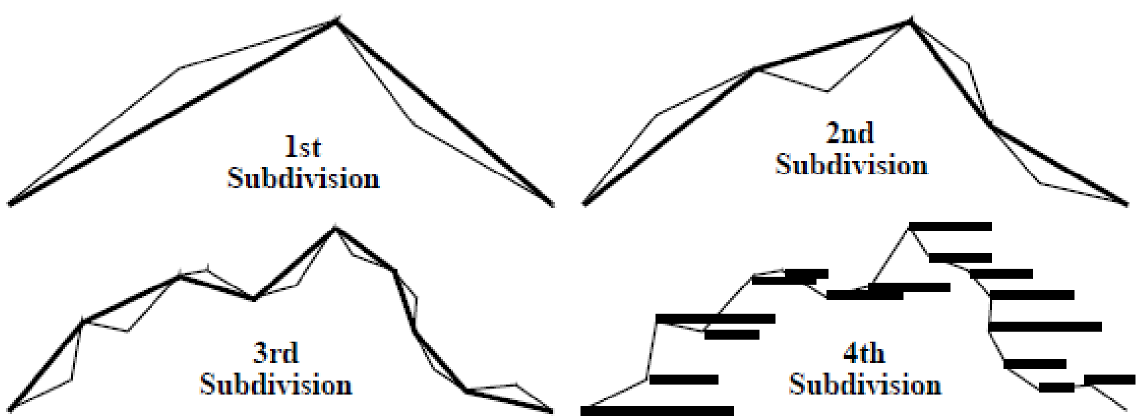


Figure 89: Nelson, *Fractal Mountains,* different subdivisions used to simulate the outline of a mountain (Nelson n.d.:3)

Each peak of the mountain outline will then represent a different pitch and the distance between the consecutive will determine the duration of each note. Unfortunately there is no sheet music to accompany this example because the entire composition was composed digitally with the use of what is called a MIDI horn. It is performed on a Yamaha TX816 and synthesisers controlled by a Macintosh computer (Nelson n.d.:1, 3). The "scale" used for this composition is completely microtonal: the octave was divided into 96 equally tempered steps, making intervals 12.5 cents each.

Fractal Mountains, also illustrates another interesting feature of fractals. Nelson (n.d.:3) wrote that the beginning of each phrase in the music contains most notes in a single channel[13]. As the music progresses, “the notes fan out among the eight channels and the richness of the micro tonal palette is revealed. Toward the end of the each phrase the notes move toward a different channel and cadence in relative consonance.” It is amazing to think that a computer-generated piece of music can simulate the same evolution of dissonance to consonance (tension and release) that is so common in Western music. This is attributed to another characteristic of fractals, namely *strange attractors*.

4.6 Fractal-inspired music of Ligeti

According to Beran (2004:92) some contemporary composers employed fractals in their music “mainly as a conceptual inspiration rather than an exact algorithm”. These composers include, for example, Harri Vuori and György Ligeti. This section explores how Ligeti incorporated fractals into his music, although more in a metaphorical sense than his contemporaries.

4.6.1 Ligeti’s interest in fractals

Ligeti started to investigate Mandelbrot’s work with fractals since the early 1980’s as well as Conlon Nancarrow’s polyrhythmic studies for player piano. Ligeti was also a great admirer of the optic illusions in Carl Escher’s art. According to Svard (2000:802) “Ligeti claimed that there are no direct links to his work from these writers, authors, or musicians, but only speaks about his interest in them to reveal the intellectual climate in which he composes.”

In 1984, Ligeti’s friend and former Nobel Prize winner Manfred Eigen showed him some of Peitgen’s computer generated pictures of fractals. “Ligeti was one amongst many creative artists to be inspired by the intricacy of these strange images, and intrigued by the mathematical

[13] Eight channels in total were used.

principles behind them" (Steinitz 2003:273). A year later, Ligeti read Mandelbrot's book *The Fractal Geometry of Nature* and they finally met in 1986. Later that year, Ligeti also met Peter Richter and invited him to give a lecture to his composition students at the Musikhochschule. Thereafter Ligeti met Heinz-Otto Peitgen with whom he had done some collateral work. The collaborations between Ligeti and these mathematicians continued for a long time. In 1996 two festivals were held in Lyons and Geneva respectively, where they held conferences on the topics "Fractals and Music" and "Music and Mathematics" (Steinitz 2003:274).

The 1980s marked an important turning point in Ligeti's life and consequently in his compositions. He came in contact with Simha Aron, who led him to explore polyrhythms of Central African music; discovered Conlon Nancarrow's compositions for player piano; and met some of the founders of fractal geometry. (Svard 2000:803.)

4.6.2 Fractals in Ligeti's compositions

4.6.2.1 *Désordre*

Ligeti's first piano etude *Désordre* is one of two pieces in which he made intentional use of ideas from self-similarity and fractal geometry, the other being the fourth movement from his Piano Concerto. In a conversation with Heinz-Otto Peitgen and Richard Steinitz in 1993, Ligeti dubbed the first etude to be self-similar and consciously based on the structure of the Koch snowflake (Steinitz 1996:8).

Désordre, meaning "disorder" is based on two prevalent notions from fractal mathematics; firstly self-similar reductions or contractions characteristic of fractals and, secondly, chaos theory. Steinitz (2003:280) explained that "*Désordre* was first conceived as a 'pulsation' study and an exercise in segregating the black keys from the white, but as Ligeti worked on it, it also became an ingenious representation of chaos theory."

In *Désordre* the right hand plays only on the white keys (heptatonic) and the left hand on the black keys (pentatonic). The combination of these two scales is not a chaotic form of polytonality, but rather a form of "combinatorial tonality" as the two scales together create an "illusion of a third or resultant tonality" (Steinitz 2003:281-282). That illustrates the principal of chaos theory:

something that might appear to be chaotic is in fact orderly. Steinitz (1996:8) explained that "although [the] right and left hands each have independent metrical cycles, as logical processes they look orderly and deterministic."

The formal structure for the right hand and left hand are different. For the right hand there are a total of fourteen iterations of the same melodic statement. Each statement is divided into three phrases of four, four and six bars each. Each successive statement of the melody is transposed one step higher on the heptatonic scale. The left hand also has a recurring melodic statement, but it is longer than the right hand: four phrases of four, four, six and four bars each (Steinitz 1996:8).

Figure 90 shows the first page of Ligeti's *Désordre.* For the first four "bars" the right and left hands are rhythmically synchronized. At the end of the first line, the right hand starts one quaver beat earlier with the statement of the melody. Consequently accents between the right and left hands no longer fall together. Ligeti continued with this process of contracting the statement of the right hand and starting one quaver beat too early with each statement. This results in an increased "chaotic" effect. Steinitz (1996:10) was of the opinion that this "resizing" of the phrase structure in *Désordre* through iteration is a fractal characteristic and also likened it to the construction of the Koch snowflake.

Figure 90: Ligeti, *Étude pour piano* No. 1 (*Désordre),* first page (Ligeti 1986)

The author could not find any evidence that Ligeti based *Désordre* on a strict mathematical execution of fractals – no algorithms were used. Therefore, it is suggested that this work rather be called fractal-inspired music as opposed to fractal music.

4.6.2.2 *L'escalier du diable*

The thirteenth etude is titled *L'escalier du diable* (the devil's staircase) and serves as another example of fractal-inspired music. This is the only one of Ligeti's etudes that has a direct reference to chaos theory in the title. The devil's staircase is closely related to self-similarity and the Cantor set. The construction of the devil's staircase relies on repeatedly removing the middle third of a square and pasting that rectangular slither next to the square. With each subsequent step, the columns get increasingly higher and also increasingly closer to one another (Figure 91).

Figure 91: First three steps for creating the devil's staircase (Peitgen et.al. 2004:210)

When this iterative process is carried out an infinite amount of times, it results in the devil's staircase:

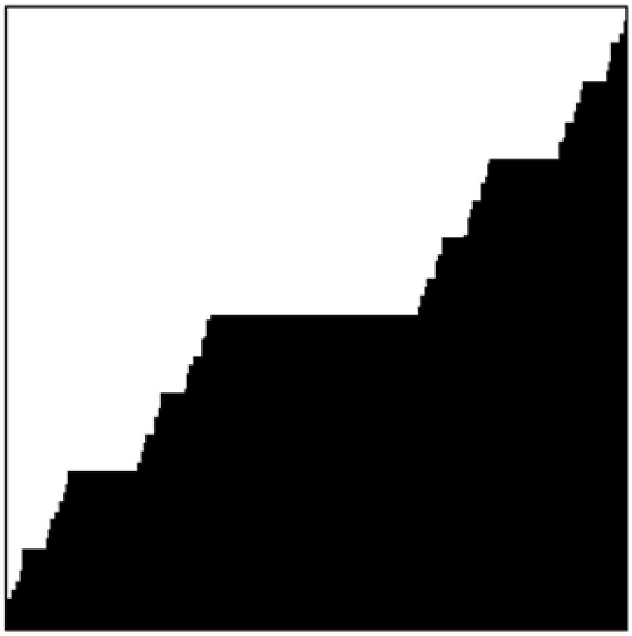


Figure 92: The complete devil's staircase (Peitgen et.al. 2004:210)

Steinitz (1996:18) described the devil's staircase as "a secondary phenomenon derived from the recursive 1/3 to 2/3 proportions within the Cantor set; that is a series of unequal steps produced by plotting […] the mathematical relationships between the eliminated and surviving segments." This is mirrored in this etude in several ways.

Firstly, the notes are grouped together asymmetrically in groups of twos and threes, playing on the "binary-ternary geometry of the devil's staircase" (Figure 93). The music is grouped in "cells": 2 + 2 + 3 / 2 + 2 + 2 + 3 / 2 + 2 + 2 +2 + 3 / 2 + 2 + 2 + 3 / 2 + 2 +3 / etc. According to Steinitz, "…the elongated 'step' of three quavers, emphasized by legato phrasing, makes a small plateau, whilst the unequal but orderly progression of the subgroups recalls the irregular staircase of the graphic image" (Steinitz 1996:19).

According to Ligeti, this etude was not directly inspired by fractal mathematics, but instead his experiences during a bicycle ride in a storm. In 1993, when Ligeti was busy with the composition of his etudes, he was living in Santa Monica, California. During that time, the coast was struck by an El Niño weather system. One day he had to make his way back to his hotel by bicycle. He had to cycle against the wind and it was during this bicycle ride that the first ideas for *L'escalier du diable* came to him. He thought of "an endless climbing, a wild apocalyptic vortex, a staircase it was almost impossible to ascend." Despite this, it is almost irrefutable that his interest and knowledge of fractals influenced this composition as well; even if it was only sub-consciously (Steinitz 2003:307-308).

Figure 93: Ligeti, *Étude pour piano* No. 13 (*L'escalier du diable*), line 1-4 (Ligeti 1998)

In an interview for the *New York Times* in 1986, Ligeti admitted that he did not make use of "direct mathematical translation" in his music. He said that "the influence is more poetic: fractals are the most complex ornaments ever, in all the arts… They provide exactly what I want to discover in my own music, a kind of organic development" (Rockwell 1986).

4.7 Conclusion

In this chapter it was explored how music can be composed with the use of fractal and self-similar algorithms and ideas. Literal translations of fractal geometry into music can be done with fractal noise forms, as proposed by Voss and Clarke. Of all noise forms, 1/f noise (or pink noise) was found to produce the most aesthetically pleasing music. This emphasises one of the keys of music composition: that there should be a balance between recurring ideas and variation.

Another method that was first presented by Prusinkiewicz (1986) is the musical interpretation of L-systems and turtle graphics. This was further developed by Mason and Saffle (1994). Tom Johnson's experiments to create self-similar compositions are perhaps more for academic use. The self-similar nature of his compositions displays characteristics of minimalism, which suggests that minimalist music might also possess self-similar qualities.

The mapping of mountains and coastlines has also been a source of inspiration to composers such as Larry Austin and Gary Lee Nelson. György Ligeti made use of fractal mathematics and chaos theory in his music in a more metaphorical sense and the researcher suggests that it should be referred to as fractal-inspired music.

Fulkerson (1992:756-757) made some comments regarding fractal composition that is important to this study:

> The model of fractal geometry excites many composers today and it is being incorporated into their compositional work, but they are still struggling with mapping problems. We can see the visual applications of this geometry, but mapped onto music it has not yet shown anything like the same interesting results. One solution is to transfer the data into timbral generation with greatly reduced pitch or harmonic movement so that the listener's ear has an adequate orientation with respect to pitch content. When applied to pitches in a chromatic context, fractal geometry has failed to create the spectacular results that it has generated in the visual domain, because it is impossible for the listener's ear to comprehend the pitch content.

Most of the compositions discussed in this dissertation cannot be identified as fractal by simply *listening* to the music. A diligent analysis of the score is necessary to point out fractal and self-similar features in each composition. It can be proposed that one might only be able to characterise music as fractal if one can *hear* the fractal structures – just as easily as one can *see* the fractal structure of the Julia set (Figure 2). Coupled with Brothers' (2004) strict prerequisites, the existence of true "fractal music" seems even more unlikely.

Nevertheless, the researcher is of the opinion that any composition that was influenced by fractals should be named appropriately as well. It is again proposed that one should rather refer to music that was based on fractal ideas as "fractal-based" or "fractal-inspired" music. If an analysis of a piece of music indeed reveals fractal structures, a term like "quasi-fractal" music may be used. These terms have been used frequently by several scholars to describe music that was composed with the use of fractals or that was inspired by fractals.

CHAPTER 5: FRACTAL GEOMETRY AND SELF-SIMILARITY IN MUSIC ANALYSIS: MUSIC PRIOR TO 1975

5.1 Introduction

This chapter explores the occurrence of fractal and self-similar structures in art music prior to 1975. Music examples include mensuration canons of the Renaissance, polyphonic compositions of Bach and, finally, the works of Classical composers like Beethoven. Each composition discussed will show a unique and different way in which various fractal-like structures are exhibited.

5.2 Rhythmic self-similarity

5.2.1 The mensuration canon

The mensuration canon is one of the best examples of self-similarity in music. It is named after the use of mensuration, which refers to the relationship between note values. The system of mensuration was used in the time period between about 1250 and 1600 (Bent 2001:435). Mensuration canons (also known as prolation canons) are prevalent in the works of the Franco-Flemish composers of the Renaissance period, such as Johannes Ockeghem and Josquin des Prez (Hindley 1990:97-98).

In a mensuration canon, the melody appears simultaneously in all voices, but in different mensurations or note values (Randel 1986:128). This is accomplished by augmentation or diminution of the note values, which is known as *scaling* in mathematics (Etlinger n.d.:1). Because scaling is involved, one can anticipate that some mensuration canons might be self-similar in structure.

5.2.1.1 Ockeghem's *Missa Prolationum*

The first example to illustrate the simultaneous use of the same thematic or melodic material in different voices in different tempos is taken from Johannes Ockeghem's *Missa Prolationum.* As the title suggests, this mass consists of mensuration canons. Figure 90 shows the *Kyrie* from the mass. The two top voices (soprano and contra tenor) share the same melodic material, but the contra tenor sings the melody in longer note values. A different melodic line is shared between the tenor and bass, where the bass is in longer note values than the tenor. Note, however, that the bass returns to the rhythmic values of the tenor from the first time the word *eleison* is sung. This gives two distinct melodic lines moving at four different tempos.

Figure 94: Ockeghem, *Kyrie* from *Missa Prolationum,* bars 1-18 (Ockeghem 1966)

The *Gloria in excelsis Deo* from the same composition has a canonic struture similar to the *Kyrie.* Again, the soprano and contra tenor sing the same melody, but the contra tenor's part is augmented in longer note values. However, there is a canon in the same note values between the tenor and bass, where the bass enters after all of the other voices.

Figure 95: Ockeghem, *Gloria in Excelsis Deo* from *Missa Prolationum*, bars 1-14 (Ockeghem 1966)

Further examples of mensuration or prolation can be found in Ockeghem's *Missa Prolationum*. This composition is a unique example of rhythmic self-similarity obtained through augmenting and diminishing the note values in the piece.

5.2.1.2 Josquin Des Prez's first *Missa l'homme armé*

A similar example can be found in Josquin des Prez's second mass entitled *Missa l'homme armé super voces musicales.* The second *Agnus Dei* consists of a three-part mensuration canon. This movement will also be used to illustrate rhythmic self-similarity obtained in a mensuration canon.

In the *Agnus Dei*, the tenor (middle voice) moves in minims and is the slowest of the three voices. It serves as the *cantus firmus* or fixed voice of the canon. (It was common in Medieval and Renaissance music for the tenor to be the slowest-moving voice.) The soprano moves three times as fast as the tenor, thus giving a ratio of 3:1 between the two voices. The bass moves only twice as fast as the tenor, giving a ratio of 2:1. Also note that the soprano's and bass's melodic line was transposed a fourth higher than the tenor (or fifth lower) starting on a D instead of an A (Brothers 2004).

Figure 96: Des Prez, *Agnus Dei II* from *Missa l'homme armé super voces musicales,* bars 1-12 (Brothers 2004)

Hence, the soprano and bass voices can be seen as smaller copies of the tenor voice, since their note values will be the same as that of the tenor when augmented. This *Agnus Dei* can thus be regarded as a rhythmically self-similar composition. In his research, Brothers (2004) referred to this type of self-similarity as rhythmic scaling because the same rhythm is presented in different levels of augmentation or diminution.

5.2.2 Other examples of rhythmic self-similarity

Rhythmic self-similarity is not found exclusively in mensuration canons, but also appears in the works of many other composers such as J.S. Bach. The fugues in *Die Kunst der Fuge,* BWV 1080, contain numerous instances of augmentation and diminution. Although these rhythmic alterations do not always occur simultaneously, Bach frequently used such rhythmic alterations of motifs or themes.

Die Kunst der Fuge is a monothematic composition consisting of various fugues in which Bach exploited various transformations. Figure 97 shows the subject that is used throughout the entire composition:

Figure 97: Subject in Bach's *Kunst der Fuge,* BWV 1080 (Alvira 2013)

Figure 98 shows the first ten bars of *Contrapunctus VI* from *Die Kunst der Fuge.* (Note that Bach had already altered the subject rhythmically in this fugue by using dotted rhythms.) The fugue starts in the bass[14] (bar 1). The soprano enters in bar 2, but in melodic inversion and diminution. In the middle of bar 3, the alto enters with the subject also in diminution. This music example shows the use of the same subject in staggered entries.

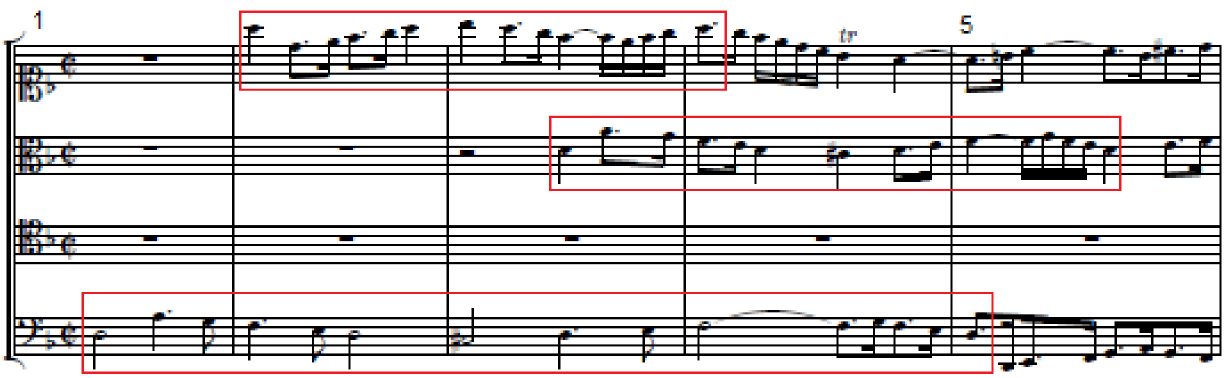


Figure 98: Bach, Contrapunctus VI, from *Die Kunst der Fuge,* BWV 1080, bars 1-5 (Adapted from Bach n.d.)

[14] Bach did not specify instrumentation for the fugues in *Die Kunst der Fuge*. The researcher therefore refers to different lines of music as soprano, alto, tenor and bass when discussing fugues from this work.

In *Contrapunctus VII a 4 per Augment et Diminu,* the tenor starts the fugue (bar 1), but in diminution. It is followed by the soprano (bar 2) in the original note values, but melodic inversion. The alto (bar 3) displays both melodic inversion and diminution. The bass enters in bar 5 with the note values doubled. In bar 6 and 7 the subject appears in the tenor and alto respectively with the note values halved.

Figure 99: Bach, Contrapunctus VII a 4 per *Augment et Diminu*, from *Die Kunst der Fuge,* BWV 1080, bars 1-12 (Adapted from Bach n.d.)

The fourth stretto in *Contrapunctus VII* is an example of three simultaneous statements of the fugue. It starts in bar 23 in the alto, where the subject's note values are halved. One beat later, the tenor enters with an augmentation of the subject. Finally, in the middle of bar 24, the soprano enters with the subject in diminished note values and melodic inversion (Alvira 2013).

Figure 100: Bach, Contrapunctus VII a 4 per *Augment et Diminu*, from *Die Kunst der Fuge,* BWV 1080, bars 21-28 (Adapted from Icking (ed.) n.d.)

The other stretti in *Contapunctus VII* are only between two voices (Alvira 2013). The excerpt in Figure 100 is therefore the best example of momentary rhythmic self-similarity that the researcher could find for this study.

Shorter examples can be found in countless compositions. In an *Allemande* by Maurice Greene (1696-1755), there is a short example of rhythmic augmentation in the anacrusis to the first bar. The right hand opens with an ascending line from A to D in semi-quavers. The left hand copies this line starting on a D and ending on an A, but in quavers. The left-hand part is thus a transposition and augmentation of the right hand for that one moment.

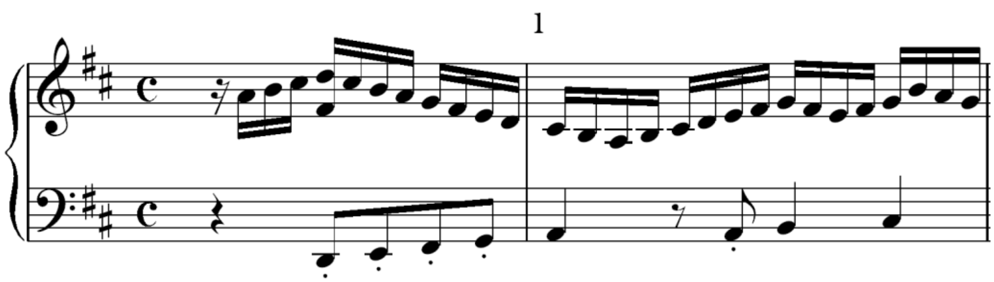


Figure 101: Greene, Allemande, bars 0^2-1 (Greene 1995)

In the *Allegro di molto* movement of C.P.E. Bach's sixth keyboard sonata there is another interesting composition technique obtained through the use of augmentation. In the second bar of the movement, the right hand plays a broken triad on the tonic of f minor. The pattern C-A♭-C repeats itself. This jump of a third is inverted in the left hand in the same bar (F-A♭-F) and the note values are augmented.

Figure 102: C.P.E. Bach, Allegro di Molto from Keyboard Sonata No. 6, bars 1-2 (Bach 1995)

This play with the same intervals in the right and left hand, but in augmented rhythms, characterises the entire movement. The same technique can be found in the works of other composers as well.

5.3 Structural self-similarity

Some researchers, such as Solomon (2002), Brothers (2004, 2007) and Lee (2004) found the structures of compositions to be self-similar or fractal-like. Some even went as far as comparing the structure of such compositions to fractals like the Cantor set and Sierpinski triangle.

5.3.1 Structural scaling in Bach

In the next few music examples, it is shown how a visual representation of a composition can look similar to the structure of some fractals. In an article in the journal, *Fractals*, Harlan Brothers showed how the phrasing of the first *Bourrée* from Bach's Cello Suite No. 3, BWV 1009 displays structural scaling. That means that the composition's structure is fractal on different scales with regard to the phrasing, and hence its structure (Brothers 2007:91).

The *Bourrée* is in binary form, with section A from bar 1 to 8 and section B from bar 9 to 28 (see Figure 106). There are repetition signs at the end of both sections, but according to Brothers (2007:93), it is customary for cellists to perform only the first repetition, thus giving an AAB structure to the entire *Bourrée*. Although the second section is not exactly twice the length of that of the first, the overall structure is short-short-long.

The first section of the *Bourrée*, marked A in the score, can be divided into a smaller AAB structure: bars 1-2 and its sequence in bars 3-4 is followed by longer phrase from bar 5 to 8. (See the red brackets in Figure 103). Next, bars 1-2 and bars 3-4 can be analysed in a similar fashion: two similar three note motifs are followed by a longer melodic line, which is yet another AAB structure. (See the blue blocks in Figure 103.) The smallest scaling unit in the *Bourrée* is the three-note rhythmic motif where two quavers are followed by a crotchet (short-short-long), giving the smallest AAB structure in the movement.

Brothers stated that one can thus see the entire *Bourrée* as an AAB structure on four different scales, where A is short and B is twice as long. This is true for the two main sections of the movement, the phrases and the main rhythmic motifs. Every A section has its own AAB structure, but on a smaller scale. This is known as structural scaling, as was briefly discussed in section 3.4.4.3, and is also a form of self-similarity.

Figure 103: Bach, *Bourrée* I from Cello Suite No. 3 in C major, BWV 1009 (Adapted from Bach 1911)

5.3.2 Bach and the Cantor set

In addition to the *Bourrée* having self-similar structural scaling, it can also be likened to the structure of a type of fractal, namely the Cantor set. In the second chapter it was explained how the Cantor set is constructed by repeatedly extracting the middle third of a line segment. However, the Cantor set can also be created by using L-systems with the following substitution map: A → ABA, B →BBB. (This means that A is replaced by ABA, and B is replaced by BBB.) In the case of the Cantor set, A represents a straight line while B represents a gap (see Figure 104) (Brothers 2007:93).

The structure of the *Bourrée* can be reproduced by using a similar iteration as the one used for the Cantor set. The substitution map is changed slightly: A → AAB instead of A → ABA. Now, compare the Cantor set (Figure 104) to a Cantor map of the first 16 bars (the first eight bars which are repeated) of the *Bourrée* (Figure 105).

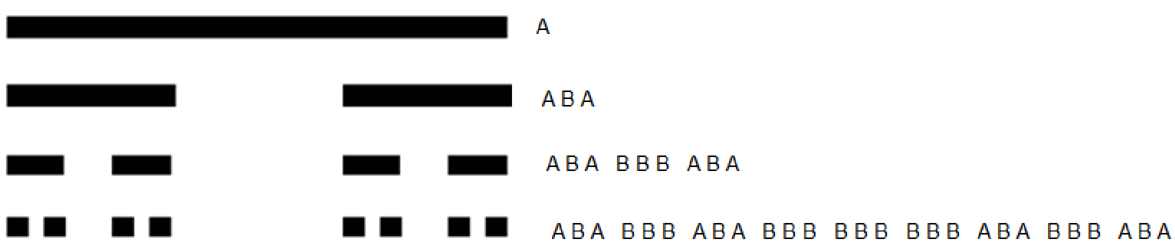


Figure 104: Initiator and first three iterations of the Cantor set created with L-systems

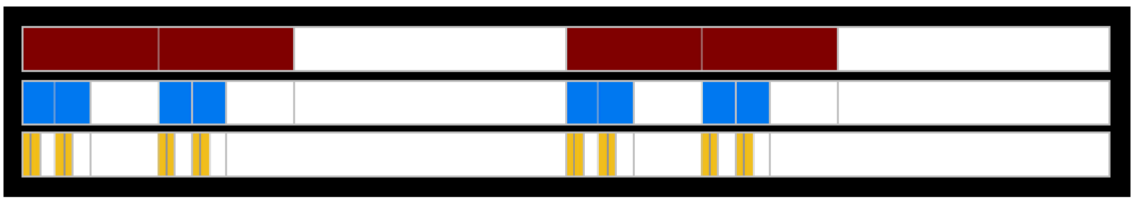


Figure 105: Cantor map of the first 16 bars of Bach's *Bourrée I* from the Cello Suite No. 3 in C major, BWV 1009 (Brothers 2007:93)

In Figure 105, the red regions each represent four beats (two measures); the blue regions, one beat; and the yellow regions, quaver beats. The white regions do not represent “nothingness”, but rather “everything that is not part of the short phrase at each scale measurement”. (See the score in Figure 103.) From these illustrations it becomes clear how the Cantor set can be used to give a visual representation of the structure of a piece of music. This visual representation also accurately displays the recursive nature of this *Bourrée* (Brothers 2007:89, 93).

5.3.3 Beethoven’s Piano Sonata No. 15 in D major, Op. 28

The next section explores how composers use a single motif or phrase as a building block to create an entire movement or, in the case of Beethoven’s Piano Sonata No. 15 in D major, Op. 28, a multi-movement work. The first theme of the first movement, *Allegro*, contains short melodic and rhythmic motives which are repeated, expanded and varied throughout all of the sonata’s movements.

5.3.3.1 Structural analysis of the first movement

The *Allegro* first movement is in sonata form. The exposition (bars 1-163) starts with the first theme in the tonic key, D major. The first theme is ten bars long (bars 1-10) and is characterised by a pedal point on the note D in the left hand; a step-wise descending melody in the right hand; and tied notes that create suspensions, anticipations and syncopated rhythms. Bars 11-20 is an exact repetition of the first theme, transposed an octave higher. Bars 21-28 is a slight variation of the first theme and is also repeated an octave higher in bars 29-39. The key remains in D major for the entirety of the first theme.

Bar 40 marks the beginning of an episode which is in A major and ends in the same key in bar 62. This leads to the second theme (bars 63-159) in the dominant key of A major. The first part of the second theme is characterised by a melody (in octaves) in the outer voices and arpeggiated harmonies in the inner voices. The melody contains many intervals of a semitone. Bars 160-163 is a single descending line that functions as a link to the development section.

The development section (bars 164-268) starts with the first theme in G major and develops mainly ideas from the first theme. After the first theme is stated in G major (bars 167-176), it is repeated an octave higher, but in G minor (bars 177-186). Thereafter the tail of the first theme is presented multiple times while the music alternates between G minor and D minor. The development section ends in the key of B minor. Bars 266-268 is marked *Adagio* – the only tempo change in the movement – and serves as a link to the recapitulation.

The recapitulation (bars 269-460) starts with the first theme in the tonic key, D major. Thereafter, the first episode is again presented in A major. The recapitulation ends with a Coda (bars 438-460) that is based on the first theme. The table in Figure 106 summarises the main sections and keys of the first movement from this sonata.

Exposition			**Development**	**Recapitulation**		
1-39	First theme	D major	**164-268**	**269-311**	First theme	D major
40-62	Episode	A major	Modulations from	**312-336**	Episode	A major
63-159	Second theme	A major	G major through	**337-434**	Second theme	
160-163	Link		G minor, D minor,	**435-437**	Link	
			B minor	**438-460**	Coda	D major

Figure 106: Analysis of the first movement from Beethoven's Piano Sonata No. 15 in D major, Op. 28 (Adapted from Harding n.d.:30-31)

5.3.3.2 Scaling of the first theme in the first movement

According to Lee (2004), the first theme of the *Allegro* is the basic building block for the entire movement, not only the melodic and rhythmic patterns, but also the underlying harmonies. Examine the descending melodic line of the first statement of the first theme. Figure 107 shows the first ten bars of the movement with the harmonic notes circled in red. When only the harmonic notes are considered and non-harmonic notes are ignored, it results in the following notes: D – A – G – D – B – A – D. (Note that the first D in the left hand is also regarded as a melodic note, since subsequent statements of the theme all start on a D.)

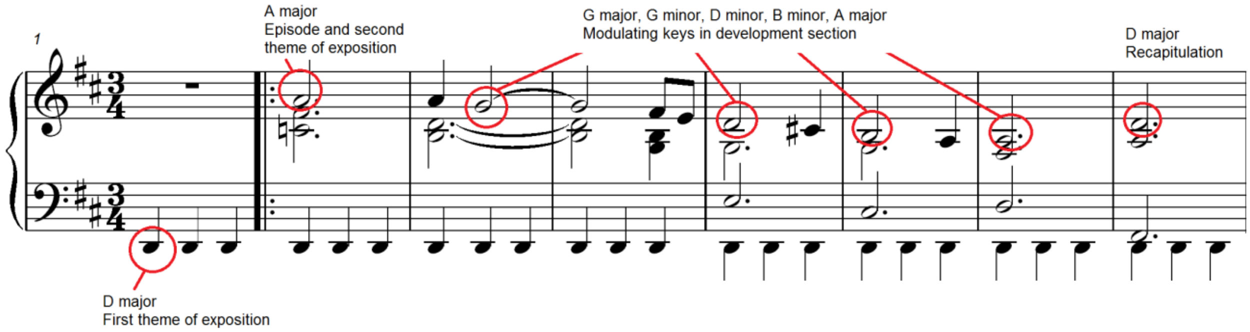


Figure 107: Beethoven, Allegro from Piano Sonata No. 15 in D major, Op. 28, bars 1-10 (Adapted from Beethoven 1975)

Lee (2004) found that the harmonic notes in the first eight bars foreshadow the key changes throughout the movement. The exposition starts in the tonic key (D major) after which it modulates to the dominant key (A major) for the episode and the second theme. The development section starts in G major and then modulates to and fro between D minor and G minor. In bar 214, the development modulates strongly to B minor, where it stays until the end of the development. The development closes with a short Adagio section of three bars on an A major seventh chord (the dominant seventh of D major) before the recapitulation starts in the tonic key again.

From this short discussion one can see that the main keys through which the first movement modulates are directly linked to the harmonic notes of the melody in the opening of the movement.

Lee (2004) showed that there is scaling between the harmonic notes in the first theme of the movement and the different keys in the entire movement. It was also pointed out that the rhythmic and melodic motifs of the first theme were altered to create the material for the rest of the composition. The following examples of the first movement justify this further.

In bar 21, it appears as if new material is presented, but in fact, old material is presented only in a different manner. Bars 21-23 (see Figure 108) contains the same material as in bars 3-5. Here it is in melodic inversion and note augmentation coupled with slightly different harmonies and a pedal point in the inner voice instead of the bass.

Figure 108: Beethoven, *Allegro* from Piano Sonata No. 15 in D major, Op. 28, comparison between bars 3-5 and 21-23 (Adapted from Beethoven 1975)

From bar 21 onwards Beethoven expands on the idea of tied notes that create suspensions (see Figure 109). Coupled with the sforzandos, it creates a hemiola-like effect in which the barlines become vague. Also note the placement of the sforzando markings: on the G, D and A. These are not only the notes on which the primary chords of D major are built, but also some of the chordal notes in the first theme discussed above.

Figure 109: Beethoven, *Allegro* from Piano Sonata No. 15 in D major, Op. 28, bars 21-28 (Adapted from Beethoven 1975)

In the episode (bars 40-62), Beethoven does away with the pedal point, but still builds on the idea of a falling second in the melody as well as tied notes (see Figure 110). The opening melody of the episode (bars 40-43) is presented and then transposed directly to E major (bars 44-47).

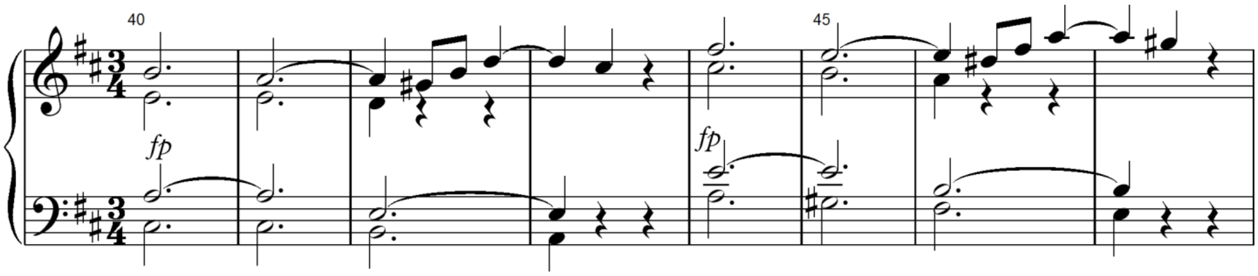


Figure 110: Beethoven, Allegro from Piano Sonata No. 15 in D major, Op. 28, bars 40-47 (Adapted from Beethoven 1975)

The tail of the first theme (bars 7-10) also plays a prominent role in the entire sonata. In the development section, the theme is stated in G major and then modulates to d minor. From bar 187 the tail of the theme is heard three times in full, then four times, but only its end. From bar 207, the last note of the tail becomes the first note of its next appearance, thus allowing the motifs to overlap. From bar 219, this motif is inverted in the tenor. Figure 111 highlights the tail of the theme as it occurs in bars 201-212.

Figure 111: Beethoven, *Allegro* from Piano Sonata No. 15 in D major, Op. 28, bars 201-212 (Adapted from Beethoven 1975)

From the music examples in Figures 107-111, it is clear that the same thematic and motivic material was used by Beethoven to create the entire first movement for his Piano Sonata No. 15 in D major, Op. 28.

5.3.3.3 Interrelationship of material between movements

Even more impressive than the similarities among sections in the first movement of Beethoven's Piano Sonata No. 15 in D major, Op. 28, is the use of the same melodic motif for the opening of the second and fourth movements and the trio of the third movement. Each of these are discussed in the next section.

The second movement (*Andante*) starts with the same notes in the melody as the first movement, only in the tonic minor (D minor). The melodic contour of the first two bars of the *Andante* is also the same as the first five bars of the *Allegro* (Figure 112). In addition, the rhythm can be seen as an altered diminution of what appeared in the first movement. The D and A are long notes, the G slightly shorter and the F and E even shorter and dotted.

Figure 112: Beethoven, *Andante* from Piano Sonata No. 15 in D major, Op. 28, bars 1-2 (Adapted from Beethoven 1975)

The *Trio* in the *Scherzo and Trio* third movement is built on the same descending melodic contour of the first theme of the *Allegro*, but in the relative minor, B minor.

Figure 113: Beethoven, Trio from third movement from Piano Sonata No. 15 in D major, Op. 28, bars 1-2 (Adapted from Beethoven 1975)

The melody in the *Rondo* (fourth and final movement) again follows the same descending contour of the theme in the first movement. The countermelody in the alto voice is canonic to this and presents the same melodic contour as well. There is also a pedal point on the tonic in the left hand, as in the first movement.

Figure 114: Beethoven, *Rondo* from Piano Sonata No. 15 in D major, Op. 28, bars 1-4 (Adapted from Schenker (ed.) 1975)

According to Ludwig Misch, “this sonata is probably the only one among Beethoven’s instrumental works that carries through the principle of thematic unity with unequivocal clarity” (Lee 2004).

Figure 115 summarises the openings from each movement in Beethoven's Piano Sonata No. 15 in D major, Op. 28 to illustrate the similarities between them.

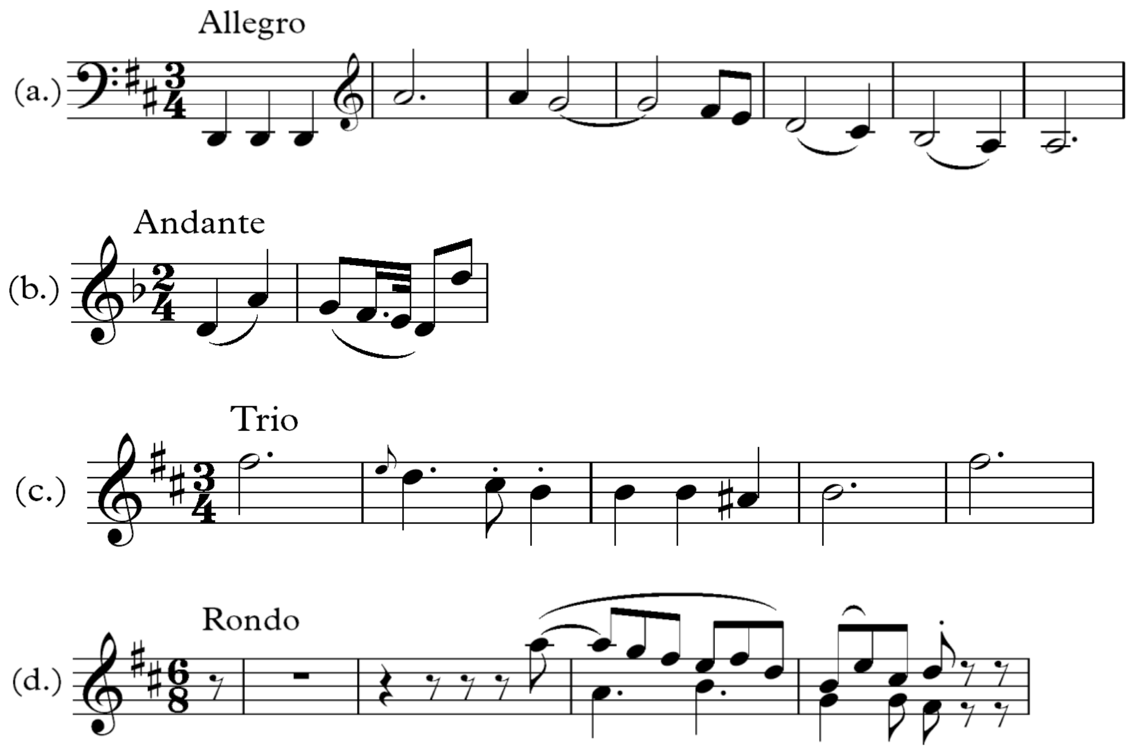


Figure 115: Comparison of the melodic contour of (a) first movement, (b) second movement, (c) trio from the third movement and (d) fourth movement of Beethoven's Piano Sonata No. 15 in D major, Op. 28 (Adapted from Beethoven 1975)

5.3.3.4 Comparison between the third movement and the Sierpinski triangle

Independent of Lee (2004), Larry Solomon's analysis of the third movement (*Scherzo and Trio*) displays a remarkable similarity to the construction of the Sierpinski triangle. The entire movement consists of three parts: the *Scherzo* (A) in the tonic key of D major, the *Trio* (B) in the relative key of B minor, followed by a return of the *Scherzo* (A). According to Solomon (2002), this can be visually represented by the first step of the Sierpinski triangle, where the three sides of a solid triangle are drawn.

The *Scherzo and Trio* each have their own inherent structures (see Figure 116). The *Scherzo* is clearly divided into two sections, separated by a repeat sign at the beginning of bar 33. Bars 1-32 is A^1, while A^2 starts in bar 33. There is, however, a return of A^1 in bar 49. This makes the structure of the *Scherzo* rounded binary:

A^1 ||: A^2 A^1 :|| ||:B^1:|| B^2 A^1 ||: A^2 A^1 :||

Solomon (2002) argued that since A^2 contains A^1 within it, the A section can be seen as a "miniature ABA within the A". If the Scherzo is then seen as ternary rather than rounded binary in form structure, it can be represented as yet another step of the Sierpinski triangle.

Solomon (2002) suggested that the structure of the Scherzo and Trio is similar to that of the Sierpinski triangle when he writes that the movement is "a combination of binary and ternary schemes similar to the Sierpinski structure". Although he describes the structural analysis of the movement in great detail, much of how it relates to the Sierpinski triangle was left to the discretion and insight of the reader.

SCHERZO.

Allegro vivace.

Figure 116: Beethoven, Scherzo and Trio from Sonata No. 15 in D major, Op. 28 (Beethoven 1975)

5.4 Arvo Pärt's *Fratres für violine und klavier*

Pärt's *Fratres für violine und klavier*[15] exhibits many examples of symmetry and self-similarity. This is not only evident in the large-scale structure of the composition, but also in its smaller divisions and subdivisions. The composition is characteristic of Pärt's tintinnabuli style that he started to employ from c.1976 (Hillier 1997:98). Although defined as a style, the tintinnabuli serves as a formula to create the entire composition.

[15]Pärt (1980) arranged many different instrumentations and versions of *Fratres.* The arrangement for violin and piano is used as frame of reference for this study, but the structure for all arrangements is the same.

5.4.1 The structure

This composition has a unique vertical and horizontal structure. Horizontally, the composition is clearly divided into nine numbered sections of six bars each. Each section is divided by a two-bar "reprise". The first section is for the solo violin, whereas the rest of the composition is written for violin and piano (www.linusakesson.net).

The harmonies in the composition always change according to a specific rhythm throughout the piece. Figure 117 shows the rhythmic structure of each segment:

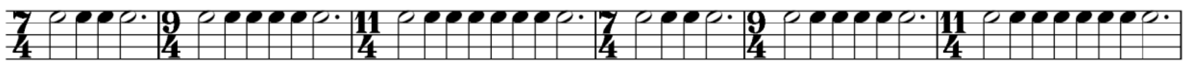


Figure 117: Rhythmic structure of *Fratres* (Adapted from Pärt 1980)

At first glance, it is easy to discern the structure of the piece, but it is more difficult to make sense of the harmonies or chords being used. Vertically, the piece consists of two melodic voices with a so-called "tintinnanbuli voice" between them, as well as two drones (on A and E) (www.linusakesson.net). To better understand the tintinnabulli style used in this composition, the first section is examined in detail.

Figure 118: Pärt, *Fratres für violine und klavier,* first page (Pärt 1980)

The first section stands in contrast to the rest of the work because the violin is not accompanied by the piano and all chords are arpeggiated. The following is a condensed version of the first section:

Figure 119: Pärt, *Fratres für violine und klavier,* reduced copy of bars 1-6 (Adapted from Pärt 1980)

By looking at this condensed representation, one can see a two characteristics of the piece that remain constant throughout:

1. The 7/4 bar and 9/4 bar are reductions of the 11/4 bar, in which an eight-chord sequence appears. This shows some level of fragmentation in the composition.
2. In the first half of each "bar", the outer voices move downwards, while the second half moves upwards. In the middle of each bar there is an octave displacement, possibly to create the illusion of continuous upward or downward movement.

One can also see the parallel movement in tenths between the two outer voices, in this case the melodic voices (M-voices). Also note that the M-voices move stepwise along the notes of the D harmonic minor scale (D E F G A B♭ C# D). This is characteristic of Pärt's tintinnabuli style.

The middle voice, called the tintinnabuli voice (T-voice) incorporates only notes from the A minor triad (A-C-E). It is used in such a way that there is never an interval of a unison or octave between the T-voice and the M-voice(s). Later in the composition, the T-voice is emphasised in the use of a drone on the notes A and E in the piano part. Although the C natural of the T-voice is never heard against a C# in the melodic voices, the change between these two notes creates interesting harmonies.

Although the key signature of the piece implies D minor, there is no traditional sense of tonality. Instead, the root of the A minor triad features as the central pitch in the composition. Throughout the composition there is movement towards and away from the central pitch.

Akesson (2007) showed that the formulation for the entire composition relies on two rotating wheels, one for the melodic voices and another for the tintinnabuli voice.

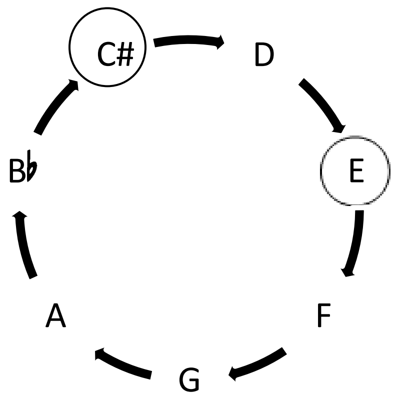


Figure 120: Rotating wheel used for the melodic voices in Pärt's *Fratres für violine und klavier* (Akesson 2007)

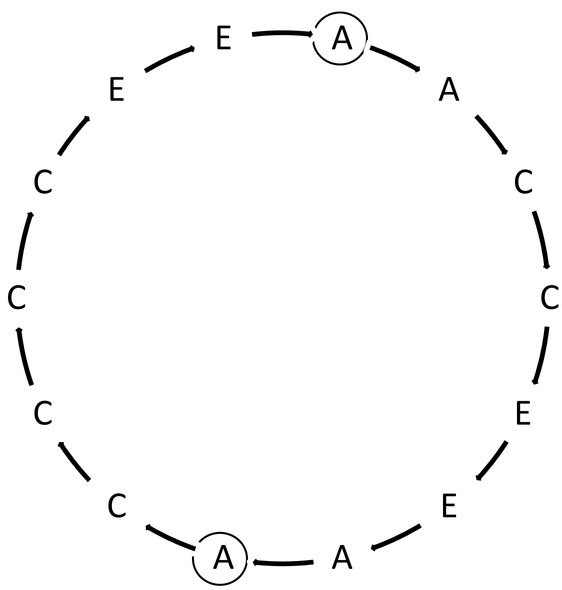


Figure 121: Rotating wheel used for the tintinnabuli voice in Pärt's *Fratres für violine und klavier* (Akesson 2007)

In the first three bars of the first section, the M-voices start on C# and E, and then move counter-clockwise for each consecutive chord. Meanwhile the T-voice starts on A (at the top of the wheel) and also moves counter-clockwise. This produces the following eight-chord sequence:

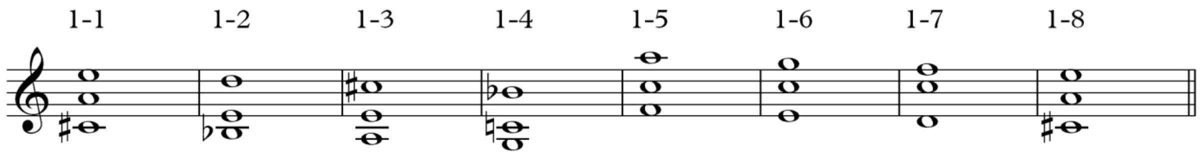


Figure 122: Pärt, *Fratres für violine und klavier,* resulting chord sequence in bar 3 (Adapted from Pärt 1980 and Akesson 2007)

Note how the melodic voices move downwards, except for the octave displacement in the middle of the chord sequence.

Bars 4-6 of the first section starts on the same chord as the first half, but slightly different harmonies are produced. This is obtained by letting the M-voices start on C# and E again, but rotating the wheel clockwise. The T-voice begins on A at the top of the wheel and continues to move clockwise.

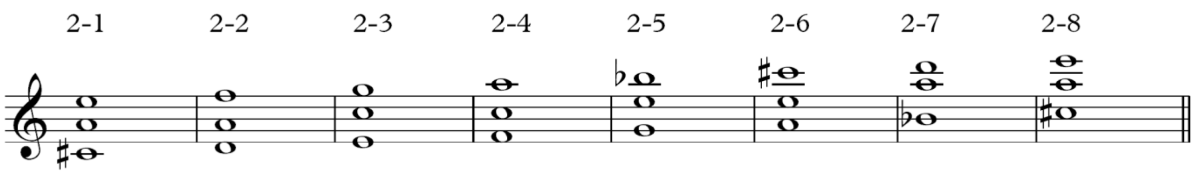


Figure 123: Pärt, *Fratres für violine und klavier,* resulting chord sequence in bar 6 (Adapted from Pärt 1980 and Akesson 2007)

5.4.2 Expansion and reduction

A central “theme” that can describe the composition of *Fratres* is the idea of expansion. Since the work is a type of variation-form, there is no development. Instead, musical ideas are expanded on different scales. This gives rise to many self-similar structures in the work. The most obvious example of this would be the expansion of a 7/4 bar with four chords, to a 9/4 bar with six chords and finally eight chords in an 11/4 bar. As already mentioned, this is found throughout the composition.

Another striking example of this is in the seventh section (Figure 124). The violin starts with single notes in the first bar, then plays double stops in the next bar and finally triple stops in the third bar. The process is reversed in the second half of the section (Zivanovic 2012:33).

Figure 124: Pärt, seventh section from *Fratres für violin und klavier* (Pärt 1980)

A similar example is found in the eighth section, but here the expansion is rhythmical and not in the texture. The rising arpeggiated melody begins in quavers, expands to triplets and further into semi-quavers in each consecutive bar.

Figure 125: Pärt, eighth section from *Fratres für violine und klavier* (Pärt 1980)

Fratres is an apt example of how a few ideas can be used to create an entire musical work. Moreover, all of these melodic and harmonic ideas are also fragmented within each section of the composition. Although no writings on the piece characterised *Fratres* as fractal or self-similar, the researcher believes that this composition can be used as another example of symmetry and self-similarity in music.

5.5 Conclusion

Polyphonic compositions of the Renaissance and Baroque have supplied some of the best examples of rhythmic self-similarity, since they enabled the composers to use two or more transformations of the same motif or theme simultaneously. Just like one is able to *see* several transformations of a shape in a fractal object at one glance, one can *hear* the transformations at the same moment in time.

Mensuration canons of the Renaissance are an excellent depiction of the same theme or motif being sung at different speeds, which displays rhythmic scaling. The canons and fugues of J.S. Bach also supply ample illustrations thereof, although it is only momentarily, mostly in the stretti of his fugues. Although only a few of Bach's compositions were highlighted in this chapter, more examples can be found in *Die Kunst der Fugue* and *Das Wohltemperierte Klavier.*

Structural self-similarity was pointed out in the *Bourrée* from Bach's first Cello Suite. Whether or not its structure truly represents that of the Cantor set is up for debate, but it is clear that the structuring of the phrases, sub-phrases and motifs mirror one another.

Beethoven's Piano Sonata No. 15 in D major, Op. 28 is a striking example of the way in which a composer built an entire work from one single theme. Different transformations such as transposition, retrograde and scaling of note values are also apparent.

The author could not find any evidence that Arvo Pärt was ever directly influenced by fractal mathematics. Therefore, *Fratres* was categorised among compositions prior to 1975, despite the date of his composition (1980); it simply signifies that the work was conceived without a direct influence of fractal geometry. *Fratres* displays many examples of expansion, fragmentation and transformation of the same chord sequences with strong self-similar structures.

CHAPTER 6: CONCLUSION

6.1 Summary of findings

Throughout this dissertation, several links between music and mathematics and specifically geometry were highlighted. Furthermore, the researcher illustrated how musical motifs or themes can be transformed in a similar way to geometric objects, which implies the possibility of applying fractal geometry to music as well. One transformation that is common to both music and fractals is that of scaling.

All fractals must adhere to specific prerequisites, namely self-similarity, fragmentation on all scales and fractal dimensionality. Brothers (2004) translated some of these properties to apply to music.

Three main noise types have been applied to both music composition and analysis: white noise, brown noise, and fractal 1/f (pink) noise. Spectral density analysis of compositions has proven that many classical compositions imitate 1/f noise, making them fractal (Voss & Clarke 1975, 1977; Hsü & Hsü 1990, 1991).

Classical music also showed the greatest resemblance to 1/f noise of a wide spectrum of genres, including jazz, blues, folk tunes from several cultures, rock and pop. Further developments led scholars to believe that the level of 1/f noise distribution in music can be used to define a specific genre in art music and even the stylistic characteristics of different composers (Ro & Kwon 2009; Levitin et al. 2012).

The application of fractal geometry and self-similarity is not limited to scientific research, but also has some noteworthy practical implications. The first is the application to music composition. Many different methods for fractal music composition were investigated in Chapter 4.

The first method is the utilisation of noise forms. Most scholars found that the use of 1/f noise was the most successful in this regard, since it is midway between chaos and predictability. Since 1/f noise is fractal, it is reasonable to assume that music created from 1/f noise is also fractal (Gardner 1992; Voss & Clarke 1977).

The second fractal that can be used to create music compositions is Lindenmayer systems and their visual interpretations with the use of turtle graphics. This method was first proposed by Prusinkiewicz in 1986 and further developed by Mason and Saffle (1994).

The coastline paradox was a source of inspiration, not only for the research of Andrew and Kenneth Hsü (1991), but also for the music of composers like Larry Austin and Gary Lee Nelson. Since the scaling of coastlines and mountains is fractal, these composers used these fractal qualities to map these natural phenomena as music.

Tom Johnson (2006) created many self-similar compositions with the utilisation of iteration and scaling. His compositions are musical experiments by means of self-similarity and have a minimalistic feel.

Some of the piano etudes by Ligeti contain metaphoric references to fractal geometry and chaos theory rather than direct translations of fractals into music; for example *Étude pour piano* No. 1 (*Désordre*) and *Étude pour piano* No. 13 (*L'escalier du diable*). These types of composition can be defined as "fractal-inspired".

All of the compositions discussed in Chapter 5 were referred to as fractal or self-similar by the composers themselves or musicologists who conducted research on their works. In Chapter 2 it was shown that Brothers (2004) firmly believed that the translation of a fractal into music does not necessarily guarantee that a piece of music be fractal. He proposed that such music should be referred to as "fractal-inspired" music. The researcher also proposed the use of terms such as "fractal-based" and "quasi-fractal" music.

The appearance of fractal or self-similar structures in music before Mandelbrot defined fractal mathematics can possibly be attributed to the fact that many fractals appear in nature. Another possible reason is that the arts, or in this case, music, reflected the scientific thinking of the times and even anticipated it (Shlain 1991).

While there is no specific method for the analysis of music with regard to fractals and self-similarity, such patterns were found in music as early as the Renaissance. A promising type of self-similar music exists in the form of mensuration canons where the same material is sung simultaneously, but at different speeds.

Solomon (2002) and Brothers (2004) attempted to show that the structure of some musical compositions can be likened to the structure of well-known fractals like the Sierpinski triangle and Cantor set. Although this amalgamation may seem somewhat far-fetched by some readers, the self-similar structures in the works they analysed cannot be refuted.

6.2 Conclusions

From the study it can be concluded that fractal geometry and self-similarity can indeed be applied in art music in the following ways:

- Fractals and self-similarity are a useful composition tools.
- Fractals can be used to categorise music in specific genres or to belong to a specific composer.
- Since the development of fractal geometry in the sciences, an increasing number of musicians have drawn inspiration for their music from fractals and self-similarity.

The different methods in which fractals can be used in music composition include:

- direct translation of noise wave forms, like 1/f noise, into music notation
- direct translation of an L-system or self-similar curve as music
- mapping of the contour of a fractal structure, such as a mountain range or coastline, as the melodic contour of a composition
- iteration of a musical idea, motif or theme in order to create self-similar musical structures
- metaphoric use of fractals in a composition

Examples of *entire* works or movements being built on fractals or self-similarity in music prior to the 20^{th} century are scarce. Nevertheless, “moments” of fractality or self-similarity can be seen in several compositions, as was shown in Chapter 5. In essence, most classical music can be defined as fractal in layperson’s terms: most classical compositions rely on the use of similar themes and motifs to create uniformity, but also inserting new material to avoid monotony. The research by the Hsüs and Voss and Clarke emphasised that these same qualities can be obtained when using fractal 1/f noise to compose.

Other than composing or analysing music with fractals and music, the researcher was surprised by the possibility that fractal distributions in a composition may be used to categorise it into a specific time period or genre. The prospect that it might be useful to verify the style of a specific composer mathematically is astounding and could be useful when trying to determine the authenticity of a specific composer's work.

Throughout the 20th century, composers were often influenced by scientific thinking of the time. If nothing else, the rise of fractal geometry in the sciences served as a new source of inspiration to composers to create music that had not been composed or heard before. Similarly, fractal geometry did not bring anything new to the way music is analysed, but it can certainly change one's thinking about music analysis. In the discussions in Chapter 5, certain motifs, themes or ideas relating to the melody, rhythm or structure of several compositions were highlighted. These can in turn be used by performers to give different interpretations of the music.

6.3 Summary of contributions

Throughout this dissertation the researcher deliberately simplified the mathematical and scientific aspects of fractal geometry and self-similarity. This was done to enable musicians who do not necessarily have a strong mathematical background to understand these concepts in order to incorporate them into music composition and analysis.

To the researcher's knowledge this is one of the first academic works in which several facets of fractals are discussed together with their musical connotations. Previous scholars addressed only a single aspect of fractal music, for example, Lindenmayer systems or 1/f noise. In this dissertation, a number of fractals that can be applied musically were highlighted.

The researcher, like Brothers (2004), posits that one should be careful not to use the term "fractal music" too loosely. As seen from the works analysed, many compositions can only be characterised as "fractal-inspired", "fractal-based" or "quasi-fractal".

Initially, the researcher endeavoured to define the term “fractal music”, but was unsuccessful for a number of reasons. It is believed that the characteristics of fractal objects should also be valid for so-called “fractal music”. This implies self-similarity, iteration, scaling and a fractal dimension.

As indicated in the research by the Hsüs, there is still ambiguity surrounding the calculation of the fractal dimension of a piece of music. Since this calculation largely relies on spectral density analysis, which could not be duplicated in this study, the fractal dimension of compositions was not considered.

In addition, spectral density analysis of compositions does not have many practical applications, except possibly to validate the genre or authenticity of the composer (as pointed out by Ro and Kwon 2009 and Levitin et al. 2012) Instead, it is an interesting mathematical analysis of music.

The researcher, like Cook (1992), believes that any analysis should be of practical use – in other words it should be valuable to the performer. When the smaller motifs and themes in a composition are analysed with regard to the larger structure and similarities between them can be found, it enables the performer to highlight these in his or her playing. Likewise, such analysis provides the analyst with a new way of thinking about the music.

The researcher was also successful in pinpointing certain discrepancies in some of the existing literature in the literature review. Although it was not possible to redo the experiments in order to correct methods, this should be use to future scholars who wish to investigate the topic further.

6.4 Suggestions for further research

As mentioned earlier in this chapter, the analysis of compositions with 1/f noise distribution may be helpful in determining the authenticity of the works by specific composers. The earlier research conducted by Ro and Kwon (2009) and Levitin et al. (2012) indicated that a certain level of 1/f distribution in a piece of music can be linked to a specific genre or even a specific composer.

Although the spectral density analyses conducted by scholars like the Hsüs (1990, 1991) and Voss and Clarke (1975, 1977) do not seem to have many practical applications for music analysts or performers, it is suggested that future musicologists should recreate their experiments in order to shed some light on aspects that were previously unclear.

6.5 Suggestions for further reading

More can be read about the aesthetic implications of fractal melodies in Michael Beauvois' article, "Quantifying aesthetic preference and perceived complexity for fractal melodies" (2007). It is a further investigation of which type of fractal noise creates the most pleasing melodies (Beauvois 2007:247).

Different methods for calculating the fractal dimension of a musical composition were discussed in the article, "Fractal dimension and classification of music" by Bigerelle and Iost[16] (2000). The complexity of the mathematical formulae used in the article prevented its use in this dissertation. These authors' research proved that "different kinds of music could be discriminated by their fractal dimension" (2007:2191).

Jonathan Foote (n.d.) developed a method for converting the sound waves from musical compositions into graphs, creating visual representations of the music. If the visualisation is self-similar it would mean that the corresponding piece of music is also self-similar.

In the introduction to *Applications of fractals and chaos: the shape of things,* Crilly et al. (1993:4) wrote the following on fractal research in the 1990s:

> The current wave of interest [in fractals and chaos] may be faddish, but there is enough solid theoretical material to ensure that the topic's importance will extend beyond any fashionable peak. They continue to have value for both technical and aesthetic applications. Fractals and Chaos are here to stay.

Today, 34 years later, fractals and chaos are an integral part of scientific research. Possibly, in another 30 years, one will be able to look back and see that fractals have also played a vital part in the development of music in the first half of the 21st century. To paraphrase Crilly et al. (1993:4) "Fractal and self-similar music are here to stay…"

[16] The surname Iost is not a spelling error; it starts with a capital i.

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