# The quantile-based skew logistic distribution

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**Abstract**

We show that the quantile-based skew logistic distribution possesses kurtosis measures based on $L$-moments and on quantiles which are skewness invariant. We furthermore derive closed-form expressions for method of $L$-moments estimators for the distribution's parameters together with asymptotic standard errors for these estimators.

*Key words:*
Exponential distribution, Logistic distribution, $L$-moment, Quantile function, Skewness-invariant kurtosis measure

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## 1. Introduction

The logistic distribution holds a special place among continuous symmetric distributions due to the simplicity of its probability density, cumulative distribution and quantile (inverse cumulative distribution) functions, given in standard form (in effect, with location and scale parameters set as zero and one) by $g(z) = \frac{\exp[-z]}{(1+\exp[-z])^2}$, $G(z) = \frac{1}{1+\exp[-z]}$ and $Q(p) = \log\left[\frac{p}{1-p}\right]$ respectively for $z \in \Re$ and $p \in (0, 1)$. Various asymmetric generalizations of the logistic distribution have been proposed in the literature. For instance, the real-valued random variable $Z$ is said to have a skew logistic distribution based upon the skewing methodology introduced by Azzalini (1985), if its

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probability density function is given by $f(z) = 2g(z)G(\lambda z)$ for $z \in \Re$ where $\lambda \in \Re$ is a shape parameter. The properties of this density-based skew logistic distribution, which we will denote $SLD_{DB}$, have been studied by Wahed and Ali (2001) and Nadarajah (2009).

In this paper we consider the quantile-based skew logistic distribution introduced by Gilchrist (2000) in his book on quantile modeling and henceforth denoted $SLD_{QB}$. Gilchrist used the $SLD_{QB}$ throughout his book to illustrate important concepts and methods in quantile modeling. However, the properties of the $SLD_{QB}$ have not been presented in detail in the literature before. We do so in Section 2. In Section 3 we show that the parameters of the $SLD_{QB}$ can easily be estimated using method of $L$-moments estimation. Closed-form expressions for method of $L$-moments estimators and their standard errors are derived. Section 4 presents an example in which the $SLD_{QB}$ is fitted to a data set using method of $L$-moments estimation.

## 2. Definition and properties

### *2.1. Genesis and special cases*

The standard exponential distribution has quantile function $Q_0(p) = -\log[1-p]$, while its reflection, the standard reflected exponential distribution, has quantile function $Q_0(p) = \log[p]$. Adding the quantile functions of the standard reflected exponential and standard exponential distributions gives $Q_0(p) = \log[p] - \log[1-p] = \log\left[\frac{p}{1-p}\right]$, the quantile function of the standard symmetric logistic distribution. A skew form of the standard logistic distribution is obtained by taking a weighted sum of the quantile functions of the standard reflected exponential and standard exponential distributions,

$$Q_0(p) = (1-\delta)\log[p] - \delta\log[1-p], \quad p \in (0,1), \tag{1}$$

where $\delta \in [0,1]$. Including location and scale parameters through the linear transformation $Q(p) = \alpha + \beta Q_0(p)$ gives the quantile function of the $SLD_{QB}$, presented in Definition 1.

**Definition 1.** *A real-valued random variable $X$ is said to have a quantile-based skew logistic distribution, denoted $X \sim SLD_{QB}(\alpha, \beta, \delta)$, if its quantile function is given by*

$$Q(p) = \alpha + \beta\left((1-\delta)\log[p] - \delta\log[1-p]\right), \quad p \in (0,1), \tag{2}$$

*where $\alpha$, $\beta$ and $\delta$ are respectively location, scale and shape parameters.*

The parameter $\delta$ controls the level of skewness and thus the distributional shape of the $SLD_{QB}$ through the allocation of weight to each tail of the $SLD_{QB}$. The $SLD_{QB}$ is symmetric for $\delta = \frac{1}{2}$ and negatively (positively) skewed for $\delta < \frac{1}{2}$ ($\delta > \frac{1}{2}$). The reflected exponential, the symmetric logistic and the exponential distributions are all special cases of the $SLD_{QB}$ for $\delta = 0$, $\delta = \frac{1}{2}$ and $\delta = 1$ respectively. The $SLD_{QB}$ possesses infinite support, $(-\infty, \infty)$, for $0 < \delta < 1$ and half-infinite support, $(-\infty, \alpha]$ and $[\alpha, \infty)$, for $\delta = 0$ and $\delta = 1$ respectively.

*2.2. Functions*

The quantile and probability density functions of the standard $SLD_{QB}$ with $\alpha = 0$ and $\beta = 1$ are illustrated graphically in Fig. 1 for selected values of $\delta$. The $SLD_{QB}$ is a quantile-based distribution defined through its quantile function. Akin to other quantile-based distributions such as Tukey's lambda distribution and its generalizations (Tukey, 1960; Ramberg and Schmeiser, 1974; Freimer et al., 1988) and the Davies distribution (Hankin and Lee, 2006), closed-form expressions do not exist for either the cumulative distribution function or the probability density function of the $SLD_{QB}$, except of course for its special cases mentioned above.

**Theorem 1.** *The quantile density function of the $SLD_{QB}$ is*

$$q(p) = \beta\left(\tfrac{1-\delta}{p} + \tfrac{\delta}{1-p}\right), \quad p \in (0,1), \tag{3}$$

*while its density quantile function is*

$$f_p(p) = \tfrac{p(1-p)}{\beta(\delta p + (1-\delta)(1-p))}, \quad p \in (0,1). \tag{4}$$

**Proof.** The expressions in (3) and (4) follow immediately from the definitions of the quantile density function and density quantile function - see Parzen (1979) or Gilchrist (2000) - in that $q(p) = \frac{dQ(p)}{dp}$ and $f_p(p) = \frac{1}{q(p)}$. ■

*2.3. Moments*

**Theorem 2.** *The mean, variance, skewness moment ratio and kurtosis moment ratio of $X \sim SLD_{QB}(\alpha, \beta, \delta)$ are*

$$\mu = \alpha + \beta(2\delta - 1), \tag{5}$$

$$\sigma^2 = \beta^2((2\delta - 1)^2 + \tfrac{\pi^2}{3}\omega), \tag{6}$$

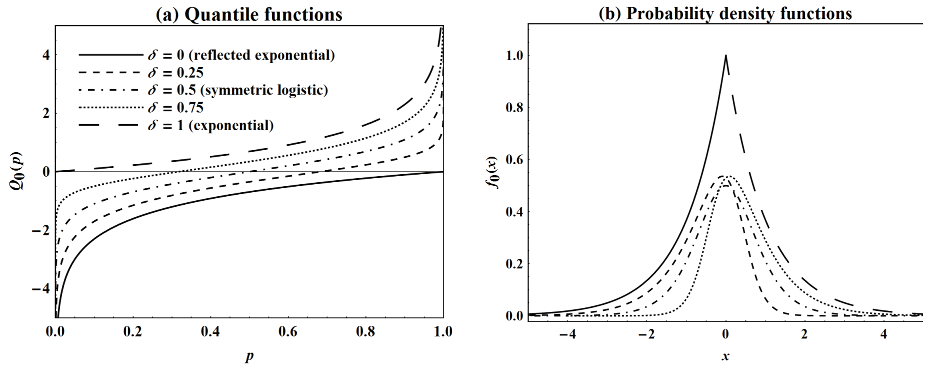


**Fig. 1.** The quantile and probability density functions of the standard $SLD_{QB}$ with $\alpha = 0$, $\beta = 1$ and $\delta = 0, 0.25, 0.5, 0.75, 1$. The line types indicated in graph (a) also apply to graph (b).

$$\alpha_3 = \tfrac{\beta^3}{\sigma^3}(2(2\delta - 1)(1 - \omega(4 - 3\zeta(3)))) \tag{7}$$

*and*

$$\alpha_4 = \tfrac{\beta^4}{\sigma^4}(9 + \omega(2((2\delta - 1)^2\pi^2 - 4) + (9\omega - 4)(16 - \tfrac{\pi^4}{15}))) \tag{8}$$

*respectively, where* $\omega = \delta(1 - \delta)$.

**Proof.** Let $Z = \frac{X-\alpha}{\beta} \sim SLD_{QB}(0, 1, \delta)$ with quantile function given in (1). Then, for example, the fourth order uncorrected moment of $Z$ is given by

$$\begin{aligned}
E\left[Z^4\right] &= \int_0^1 (Q_0(p))^4\, dp \\
&= \int_0^1 ((1-\delta)\log[p] - \delta\log[1-p])^4\, dp \\
&= (1-\delta)^4\Psi(4,0) - 4(1-\delta)^3\delta\Psi(3,1) + 6(1-\delta)^2\delta^2\Psi(2,2) \\
&- 4(1-\delta)\delta^3\Psi(1,3) + \delta^4\Psi(0,4) \\
&= 2\Big(12\phi_4 - 2\omega\phi_2(24 - \pi^2 - \tfrac{\pi^4}{15} - 6\zeta(3)) \\
&\quad +3\omega^2(24 - \tfrac{4\pi^2}{3} - \tfrac{\pi^4}{90} - 8\zeta(3))\Big),
\end{aligned}$$

where $\zeta(a)$ is Riemann's zeta function, $\omega = \delta(1-\delta)$, $\phi_i = ((1-\delta)^i + (-1)^i\delta^i)$ for $i = 1, 2, ...$ and

$$\begin{aligned}\Psi(j,k) &= \int_0^1 (\log[p])^j \, (\log[1-p])^k \, dp \\ &= \tfrac{\partial^{j+k}}{\partial u^j \partial v^k} \left[\int_0^1 p^u (1-p)^v dp\right]\bigg|_{u=v=0} \\ &= \tfrac{\partial^{j+k}}{\partial u^j \partial v^k} \left[B(u+1, v+1)\right]\bigg|_{u=v=0}\end{aligned}$$

for $j, k = 0, 1, 2, ...$ with $B(a, b)$ the beta function. Likewise it can be shown that $E[Z] = -\phi_1 = (2\delta - 1)$, $E\left[Z^2\right] = 2(\phi_2 - \omega(2 - \frac{\pi^2}{6}))$ and $E\left[Z^3\right] = -3(2\phi_3 - \omega\phi_1(\frac{\pi^2}{3} + 2\zeta(3) - 6))$. Then, since $X = \alpha + \beta Z$, the expressions in (5) to (8) are found (after extensive simplification) using $\mu_r' = E\left[X^r\right]$ and $\mu_r = E\left[(X - E[X])^r\right]$ for $r = 1, 2, 3, 4$. Specifically the mean and variance of $X$ are given by $\mu = \mu_1'$ and $\sigma^2 = \mu_2$, and the skewness and kurtosis moment ratios of $X$ are respectively obtained with $\alpha_3 = \frac{\mu_3}{\mu_2^{1.5}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$. ■

*2.4. L-moments*

**Lemma 1.** *Hosking (1990) showed that the rth order L-moment of $X$ can be written as $\lambda_r = \int_0^1 Q(p) P_{r-1}^*(p) dp$, where*

$$P_r^*(p) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} p^k \tag{9}$$

*is the rth order shifted Legendre polynomial.*

**Theorem 3.** *If $X \sim SLD_{QB}(\alpha, \beta, \delta)$, then*

*(i) the first order L-moment, called the L-location, is*

$$\lambda_1 = \alpha + \beta(2\delta - 1), \tag{10}$$

*(ii) while the rth order L-moment is*

$$\lambda_r = \tfrac{\beta(2\delta-1)^{r \bmod 2}}{r(r-1)}, \quad r = 2, 3, 4, ....$$

**Proof.** (i) The $L$-location equals the mean, $\lambda_1 = \mu$, derived in Theorem 2.

For (ii), note that $\int_0^1 P_{r-1}^*(p)dp = 0$ for $r > 1$ and $P_{r-1}^*(p) = (-1)^{r-1}P_{r-1}^*(1-p)$ so that

$$
\begin{aligned}
\lambda_r &= \int_0^1 (\alpha + \beta\left((1-\delta)\log[p] - \delta\log[1-p]\right)) P_{r-1}^*(p)dp \\
&= \beta(1-\delta)\int_0^1 \log[p]P_{r-1}^*(p)dp - (-1)^{r-1}\beta\delta\int_0^1 \log[1-p]P_{r-1}^*(1-p)dp \\
&= -\beta(2\delta-1)^{r \bmod 2}\int_0^1 \log[p]P_{r-1}^*(p)dp \\
&= -\beta(2\delta-1)^{r \bmod 2}\sum_{k=0}^{r}\left((-1)^{r-1-k}\tbinom{r-1}{k}\tbinom{r-1+k}{k}\int_0^1 \log[p]p^k dp\right) \\
&= \tfrac{\beta(2\delta-1)^{r \bmod 2}}{r(r-1)},
\end{aligned}
$$

where the final result is obtained (after simplification) using $\int_0^1 \log[p]p^k dp = -(k+1)^{-2}$ from expression (2.6.3.2) in Prudnikov et al. (1986). ■

The $L$-scale of the $SLD_{QB}$ is simply

$$\lambda_2 = \tfrac{1}{2}\beta, \tag{11}$$

resulting in basic expressions for the $L$-moment ratios,

$$\tau_r = \tfrac{\lambda_r}{\lambda_2} = \tfrac{2(2\delta-1)^{r \bmod 2}}{r(r-1)}, \quad r = 3, 4, 5, ....$$

In particular, the $L$-skewness and $L$-kurtosis ratios for the $SLD_{QB}$ are

$$\tau_3 = \tfrac{1}{3}(2\delta - 1) \tag{12}$$

and $\tau_4 = \frac{1}{6}$. Comparing the $L$-moments of the $SLD_{QB}$ to its conventional moments, it is evident that it is more expedient to characterize the $SLD_{QB}$ with its $L$-moments.

**Remark 1.** *Because $\delta \in [0, 1]$, the $SLD_{QB}$ has $\tau_3 \in [-\frac{1}{3}, \frac{1}{3}]$. In effect, the minimum and maximum values attainable by the L-skewness ratio of the $SLD_{QB}$ are the values for the reflected exponential distribution ($\tau_3 = -\frac{1}{3}$) and the exponential distribution ($\tau_3 = \frac{1}{3}$).*

**Remark 2.** *The L-kurtosis ratio of the $SLD_{QB}$ is skewness-invariant in that $\tau_4$ is constant and independent of the value of $\delta$.*

*2.5. Quantile-based measures of location, spread and shape*

**Theorem 4.** *The median, spread function (MacGillivray and Balanda, 1988), $\gamma$-functional (MacGillivray, 1986) and ratio-of-spread functions (MacGillivray and Balanda, 1988) of $X \sim SLD_{QB}(\alpha, \beta, \delta)$ are respectively*

$$me = Q(\tfrac{1}{2}) = \alpha + \beta(2\delta - 1)\log[2], \tag{13}$$

$$S(u) = Q(u) - Q(1-u) = \beta \log\left[\tfrac{u}{1-u}\right], \quad \tfrac{1}{2} < u < 1, \tag{14}$$

$$\gamma(u) = \tfrac{Q(u)+Q(1-u)-2me}{S(u)} = -\tfrac{(2\delta-1)\log[4u(1-u)]}{\log\left[\frac{u}{1-u}\right]}, \quad \tfrac{1}{2} < u < 1, \tag{15}$$

*and*

$$R(u,v) = \tfrac{S(u)}{S(v)} = \tfrac{\log\left[\frac{u}{1-u}\right]}{\log\left[\frac{v}{1-v}\right]}, \quad \tfrac{1}{2} < v < u < 1. \tag{16}$$

**Proof.** The expressions in (13) to (16) follow directly using the quantile function of the $SLD_{QB}$ in (2). ■

**Remark 3.** *The ratio-of-spread functions of the $SLD_{QB}$ does not depend on the value of $\delta$. Thus, in terms of van Zwet's ordering $\leqslant_S$(van Zwet, 1964), extended by Balanda and MacGillivray (1990) to asymmetric distributions, the kurtosis of the $SLD_{QB}$ is the same for all $\delta \in [0,1]$. Hence, akin to its L-kurtosis ratio, the ratio-of-spread functions of the $SLD_{QB}$ is skewness-invariant. In fact, any quantile-based kurtosis measure of the general form*

$$\tfrac{\sum_{j=1}^{n_1} a_j S(u_j)}{\sum_{k=1}^{n_2} b_k S(u_k)} = \tfrac{\sum_{j=1}^{n_1} a_j (Q(u_j)-Q(1-u_j))}{\sum_{k=1}^{n_2} b_k (Q(u_k)-Q(1-u_k))},$$

*where $a_j : j = 1, 2, ..., n_1$ and $b_k : k = 1, 2, ..., n_2$ are constants with $n_1$ and $n_2$ positive integers, is skewness invariant for the $SLD_{QB}$. See Jones et al. (2011) and van Staden (2013) for more detailed discussions on skewness-invariance.*

## 3. Method of *L*-moments estimation

In the absence of a closed-form expression for the probability density function of the $SLD_{QB}$ and due to the complexity of the expressions of the moments of the $SLD_{QB}$, maximum likelihood estimation and method of moments estimation for the parameters of the $SLD_{QB}$ are unappealing. However, closed-form expressions are available for method of $L$-moments estimators as well as for their asymptotic standard errors.

**Lemma 2.** *Let $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$ denote an ordered data set of sample size $n$. Then, following Hosking (1990), the rth order sample L-moment, $l_r = \binom{n}{r}^{-1} \sum\limits_{1\leq i_1<i_2<...<i_r\leq n}\sum ... \sum r^{-1} \sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} x_{i_{r-k}:n}$ for $r = 1, 2, ..., n$, is an unbiased estimator of the rth order theoretical L-moment, $\lambda_r$. The rth order sample L-moment ratio is $t_r = \frac{l_r}{l_2}$ for $r = 3, 4, ..., n$.*

**Theorem 5.** *If $X \sim SLD_{QB}(\alpha, \beta, \delta)$, then*

*(i) the method of L-moments estimators for $\alpha$, $\beta$ and $\delta$ are $\widehat{\alpha} = l_1 - 6l_3$, $\widehat{\beta} = 2l_2$ and $\widehat{\delta} = \frac{1}{2}(1 + 3t_3) = \frac{1}{2}(1 + 3\frac{l_3}{l_2})$,*

*(ii) with asymptotic covariance matrix given by*

$$n \; var \begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\delta} \end{bmatrix} \approx \begin{bmatrix} \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} \\ \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} \\ \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} \end{bmatrix},$$

*where*

$$\Theta_{1,1} = \frac{1}{15}\beta^2 \left(57 + (125\pi^2 - 1308)\omega\right), \tag{17}$$

$$\Theta_{2,2} = \frac{4}{3}\beta^2 \left(1 - (\pi^2 - 8)\omega\right), \tag{18}$$

$$\Theta_{3,3} = \frac{1}{15} \left(8 - (397 + 160\omega - 20\pi^2(\omega + 2))\omega\right), \tag{19}$$

$$\Theta_{1,2} = \Theta_{2,1} = -\beta^2(2\delta - 1), \tag{20}$$

$$\Theta_{1,3} = \Theta_{3,1} = -\frac{1}{5}\beta \left(7 + (25\pi^2 - 253)\omega\right) \tag{21}$$

*and*

$$\Theta_{2,3} = \Theta_{3,2} = \frac{1}{3}\beta(2\delta - 1) \left(1 + 2(\pi^2 - 8)\omega\right), \tag{22}$$

*with $\omega = \delta(1 - \delta)$. Specifically the asymptotic standard errors of the method of L-moments estimators for $\alpha$, $\beta$ and $\delta$ are*

$$s.e.\left[\widehat{\alpha}\right] = \beta\sqrt{\tfrac{1}{15n} \left(57 + (125\pi^2 - 1308)\omega\right)},$$

$$s.e.\left[\widehat{\beta}\right] = \beta\sqrt{\tfrac{4}{3n} \left(1 - (\pi^2 - 8)\omega\right)}$$

*and*

$$s.e.\left[\widehat{\delta}\right] = \sqrt{\tfrac{1}{15n} \left(8 - (397 + 160\omega - 20\pi^2(\omega + 2))\omega\right)}.$$

**Proof.** (i) The expressions for $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\delta}$ follow from (10), (11) and (12).

(ii) We must derive the covariance matrix $\Theta = G\Lambda G^T$. The elements of $G$ are given by the partial derivatives of the parameters with respect to the $L$-moments, that is

$$G = \begin{bmatrix} \frac{\partial\alpha}{\partial\lambda_1} & \frac{\partial\alpha}{\partial\lambda_2} & \frac{\partial\alpha}{\partial\lambda_3} \\ \frac{\partial\beta}{\partial\lambda_1} & \frac{\partial\beta}{\partial\lambda_2} & \frac{\partial\beta}{\partial\lambda_3} \\ \frac{\partial\delta}{\partial\lambda_1} & \frac{\partial\delta}{\partial\lambda_2} & \frac{\partial\delta}{\partial\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 2 & 0 \\ 0 & -\frac{3\lambda_3}{2\lambda_2^2} & \frac{3}{2\lambda_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 2 & 0 \\ 0 & -\frac{2\delta-1}{\beta} & \frac{3}{\beta} \end{bmatrix},$$

where the final result is obtained using (11) and (12) and simplifying. The elements of the symmetric matrix,

$$\Lambda = \begin{bmatrix} \Lambda_{1,1} & \Lambda_{1,2} & \Lambda_{1,3} \\ \Lambda_{2,1} & \Lambda_{2,2} & \Lambda_{2,3} \\ \Lambda_{3,1} & \Lambda_{3,2} & \Lambda_{3,3} \end{bmatrix},$$

can, as shown by Hosking (1990), be determined using

$$\begin{aligned} \Lambda_{r,s} &= \lim_{n\to\infty} n \operatorname{cov}(l_r, l_s) \\ &= \int_0^1 \int_0^v \left(P_{r-1}^*(u)P_{s-1}^*(v) + P_{s-1}^*(u)P_{r-1}^*(v)\right) u(1-v)q(u)q(v)dudv \end{aligned}$$

for $r, s = 1, 2, 3$, where $P_r^*(p)$ is the $r$th order shifted Legendre polynomial in (9) and where, for the $SLD_{QB}$, $q(p)$ is the quantile density function given in (3). For example,

$$\begin{aligned} \Lambda_{3,3} &= \int_0^1 \int_0^v \left(P_2^*(u)P_2^*(v) + P_2^*(u)P_2^*(v)\right) u(1-v)q(u)q(v)dudv \\ &= 2\int_0^1 \int_0^v \left(6u^2 - 6u + 1\right)\left(6v^2 - 6v + 1\right) u(1-v) \\ &\qquad \times \beta\left(\tfrac{1-\delta}{u} + \tfrac{\delta}{1-u}\right)\beta\left(\tfrac{1-\delta}{v} + \tfrac{\delta}{1-v}\right) dudv \end{aligned}$$

$$
\begin{aligned}
&= 2\beta^2 \Bigg( (1-\delta)^2 \int_0^1 \left(-12v^5 + 42v^4 - 56v^3 + 35v^2 - 10v + 1\right) dv \\
&\quad + \omega \int_0^1 \left(12v^5 - 30v^4 + 26v^3 - 9v^2 + v\right) dv \\
&\quad + \omega \left(-36\Xi(4,2) + 72\Xi(4,1) - 42\Xi(4,0) + 6\Xi(4,-1)\right. \\
&\qquad + 36\Xi(3,2) - 72\Xi(3,1) + 42\Xi(3,0) - 6\Xi(3,-1) \\
&\qquad \left. -6\Xi(2,2) + 12\Xi(2,1) - 7\Xi(2,0) + \Xi(2,-1)\right) \\
&\quad + \delta^2 \left(36\Xi(4,2) - 36\Xi(4,1) + 6\Xi(4,0) - 36\Xi(3,2) + 36\Xi(3,1)\right. \\
&\qquad \left. -6\Xi(3,0) + 6\Xi(2,2) - 6\Xi(2,1) + \Xi(2,0)\right)\Bigg) \\
&= \tfrac{1}{15}\beta^2 \left(2 + \omega(5\pi^2 - 53)\right),
\end{aligned}
$$

with $\omega = \delta(1-\delta)$ and where the final expression is obtained after substantial simplification using

$$
\begin{aligned}
\Xi(j,k) &= \int_0^1 v^k \int_0^v u^{j-1}(1-u)^{-1} du dv \\
&= \int_0^1 v^k B_v(j,0) dv \\
&= -\int_0^1 v^k \log[1-v] dv - \sum_{m=1}^{j-1} \tfrac{1}{m(m+k+1)} \\
&= \begin{cases} \frac{\pi^2}{6} - \sum_{m=1}^{j-1} \frac{1}{m^2} & , k=-1, \\ \frac{1}{k+1}(\psi(k+2)+C) - \sum_{m=1}^{j-1} \frac{1}{m(m+k+1)} & , k>-1, \end{cases}
\end{aligned}
\tag{23}
$$

for $j = 2, 3, 4$, where $B_z(a,b)$ is the incomplete beta function, $\psi(a)$ is the Euler psi function and $C = 0.5772156649...$ is Euler's constant, and where the final result in (23) is obtained using expressions (4.291.4) and (4.293.8) in Gradshteyn and Ryzhik (2007). Likewise it can be shown that $\Lambda_{1,1} = \frac{1}{3}\beta^2(3 + \omega(\pi^2 - 12))$, $\Lambda_{2,2} = \frac{1}{3}\beta^2(1 - \omega(\pi^2 - 8))$, $\Lambda_{1,2} = \Lambda_{2,1} = \frac{1}{2}\beta^2(2\delta - 1)$, $\Lambda_{1,3} = \Lambda_{3,1} = \frac{1}{6}\beta^2(1 + 2\omega(\pi^2 - 11))$ and $\Lambda_{2,3} = \Lambda_{3,2} = \frac{1}{6}\beta^2(2\delta - 1)$. Finally the expressions for the elements of $\Theta$ in (17) to (22) are obtained with $\Theta_{1,1} = \Lambda_{1,1} - 12(\Lambda_{1,3} - 3\Lambda_{3,3})$, $\Theta_{2,2} = 4\Lambda_{2,2}$, $\Theta_{3,3} = \beta^{-2}((2\delta-1)^2\Lambda_{2,2} - 6(2\delta-1)\Lambda_{2,3} + 9\Lambda_{3,3})$, $\Theta_{1,2} = \Theta_{2,1} = 2(\Lambda_{1,2} - 6\Lambda_{2,3})$, $\Theta_{1,3} = \Theta_{3,1} = \beta^{-1}(3(\Lambda_{1,3} - 6\Lambda_{3,3}) - (2\delta - 1)(\Lambda_{1,2} - 6\Lambda_{2,3}))$ and $\Theta_{2,3} = \Theta_{3,2} = 2\beta^{-1}(3\Lambda_{2,3} - (2\delta - 1)\Lambda_{2,2})$. ■

## 4. Example

Consider the concentration of polychlorinated biphenyl (PCB) in the yolk lipids of pelican eggs, recently used by Thas (2010) as an example data set with respect to goodness-of-fit testing. Fig. 2(a) shows a histogram for the data set. The values of the sample $L$-location, $L$-scale, $L$-skewness ratio and $L$-kurtosis ratio for the data set, consisting of $n = 65$ observations, are $l_1 = 210$, $l_2 = 39.7928$, $t_3 = 0.1044$ and $t_4 = 0.2125$. Table 1 presents the parameter estimates of the fitted $SLD_{QB}$ obtained with method of $L$-moments estimation. The standard error of each estimate is given in parentheses below the estimate. The $p$-values for the Kolmogorov-Smirnov ($D_n$), Anderson-Darling ($A_n$) and Cramér-von Mises ($W_n$) goodness-of-fit tests, obtained with 10 000 parametric bootstrap samples and tabulated in Table 1 along with the goodness-of-fit statistics, indicate that the $SLD_{QB}$ provides an adequate fit to the data set. This is confirmed by the probability density function of the fitted $SLD_{QB}$, plotted in Fig. 2(a), as well as the $Q$-$Q$ plot for the fitted $SLD_{QB}$, depicted in Fig. 2(b).

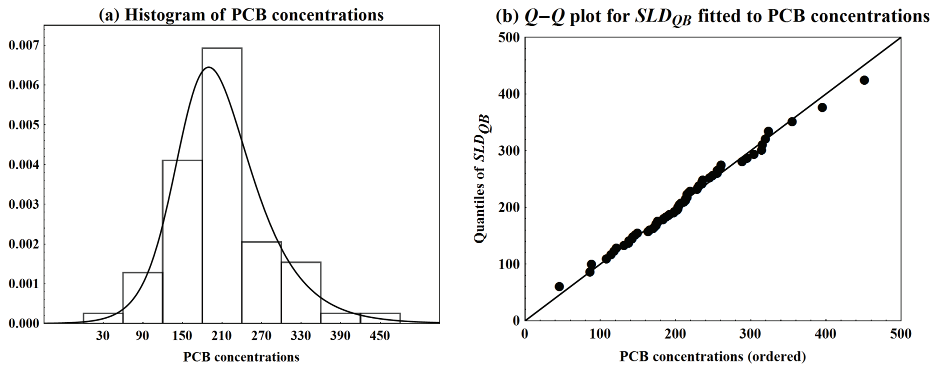


**Fig. 2.** A histogram of the concentration of polychlorinated biphenyl (PCB) in the yolk lipids of pelican eggs together with the probability density function of the fitted $SLD_{QB}$ and the corresponding $Q$-$Q$ plot for the fitted $SLD_{QB}$.

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**Table 1.** Parameter estimates with asymptotic standard errors as well as goodness-of-fit statistics with $p$-values for the $SLD_{QB}$ fitted to the polychlorinated biphenyl (PCB) concentrations.

| Parameter estimates | | | Goodness-of-fit statistics | | |
|---|---|---|---|---|---|
| $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\delta}$ | $D_n$ | $A_n$ | $W_n$ |
| 185.0633 | 79.5856 | 0.6567 | 0.4091 | 0.2595 | 0.0487 |
| (16.1699) | (8.6695) | (0.0982) | (0.7310) | (0.5326) | (0.3489) |

NOTE: Standard errors of parameter estimates and $p$-values of goodness-of-fit tests given in parentheses

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