Uncertainty Relations and Entanglement in Fermion Systems

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Abstract

The violation of uncertainty relations is used as a signature of entanglement for both pure and mixed states of two identical fermions. In the case of fermions with a four dimensional single particle Hilbert space we obtain several different types of uncertainty-related entanglement criteria based on local uncertainty relations, on the sum of variances of projectors, and on various entropic measures. Within the latter approach we consider either entropic uncertainty relations involving a single observable or relations based upon the sum of entropies associated with more than one observable. We extend the projector based entanglement criterion to the case of two-fermion and three-fermion systems with a six dimensional single particle Hilbert space.

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I. INTRODUCTION

Entanglement is one of the most fundamental features of quantum systems [1, 2]. In addition to its revolutionary technological implications, current research in quantum entanglement is providing deep new insights in connection with fundamental issues such as the justification of the basic tenets of equilibrium quantum statistical mechanics [3] and the origins of the quantum-to-classical transition [4] and the thermodynamic arrow of time [5, 6] (some aspects of this last point are, however, the subject of controversy [7]).

The concept of entanglement in systems of indistinguishable particles is more controversial than it is in the case of systems composed of distinguishable subsystems. However, there is by now a general agreement among researchers working in this field that in systems of two or more identical fermions quantum correlations between the particles that are solely due to the antisymmetric character of the fermionic state do not contribute to the state's amount of entanglement [8–11]. For instance, an N -fermion pure state of Slater rank equal to one (that is, a state whose wave function can be represented as a single Slater determinant) has to be regarded as non-entangled [11]. Here, by entanglement in systems of identical fermions we mean entanglement between particles and not entanglement between modes. The characterization of the entanglement features of systems of identical fermions constitutes a subject that is currently attracting the attention of several researchers. For example, the entanglement between pairs of electrons in a conducting band [8], the entanglement features exhibited by the eigenstates of a soluble two-electrons model atom [12], the entanglement dynamics in scattering processes involving two electrons [13], the different families of entangled states in systems of three fermions [14], and the role of entanglement in time-optimal evolutions of fermionic systems [15, 16] have recently been investigated. A closed analytical expression for the amount of entanglement exhibited by a (mixed) state of two-identical fermions (as measured by the concurrence) is known only for systems with a single-particle Hilbert space of dimension 4, which is the fermionic case of smallest dimensionality admitting the phenomenon of entanglement [10] (these systems can be mapped into systems of spin-3/2 particles). There is a need for separability criteria and entanglement indicators that are relatively easy to implement in practice and which can be extended to higher dimensional two-fermion systems or to

situations involving more than two fermions [17]. In fact, the field of separability criteria and entanglement indicators for systems of identical fermions is a largely unexplored one.

The development of criteria for the detection of entanglement based on the violation of appropriate uncertainty relations has been the focus of some interesting recent investigations [18–23]. One can establish uncertainty relations that are satisfied by all separable (pure or mixed) states and, consequently, any violation of these relations indicates that the quantum state under consideration is entangled. Since these uncertainty-based entanglement criteria are directly formulated in terms of the expectation values of quantum mechanical observables, they provide potentially valuable tools for the experimental detection of entanglement, complementing other ways of detection. The use of uncertainty relations for entanglement characterization has so far only been established for systems consisting of distinguishable subsystems. The aim of this paper is the application of various uncertainty relations to the study of entanglement in fermionic systems. Besides its practical usefulness for the identification of families of entangled states, the characterization of entanglement via uncertainty relations is also of considerable conceptual interest because it establishes a connection between two fundamental ingredients of quantum mechanics, namely the uncertainty relations and entanglement.

The organization of the paper is as follows. In Section II a brief review of entanglement in systems of identical fermions is given. Section III deals with the connection between entanglement and local uncertainty relations for pure and mixed states of two fermions with a four dimensional single particle Hilbert space. In Section IV the violation of variance-based uncertainty relations of projectors is used as a signature of entanglement for systems of two fermions, for both four dimensional and six dimensional single particle Hilbert spaces, and also for systems of three fermions with a six dimensional Hilbert space. In Section V we investigate entanglement detection through uncertainty relations based on q -entropies (of which the Shannon entropy is a special, limit case). Finally, in Section VI some conclusions are given.

II. ENTANGLEMENT IN SYSTEMS OF IDENTICAL FERMIONS

As already mentioned, in this work we are going to consider entanglement between particles (as opposed to entanglement between modes). The minimum quantum correlations between the particles that are required to guarantee the antisymmetric character of the quantum state of N identical fermions do not contribute to the amount of entanglement of this state. Consequently, the correlations exhibited by states described by one single Slater determinant, which are due to the indistinguishability of the particles, do not constitute genuine manifestations of entanglement. There are profound physical reasons for this. On the one hand, the correlations exhibited by such states cannot be used as a resource to implement non-classical information transmission or information processing tasks. On the other hand, the non-entangled character of states represented by one Slater determinant is consistent with the possibility of associating complete sets of properties to both parts of the composite system [9]. Hence, a pure state of identical fermions must be regarded as non-entangled (separable) if it is of the form of a single Slater determinant, i.e. in the case of two fermions

$$|\psi_{sl}\rangle = \frac{1}{\sqrt{2}}\{|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle\} \quad (1)$$

with $|\phi_1\rangle$ and $|\phi_2\rangle$ orthonormal, single-particle states.

A convenient entanglement criterion for pure states $|\psi\rangle$ of systems of M identical fermions can be formulated in terms of the trace of the square of the single particle reduced density matrix $\rho_1 = \text{Tr}_{2,\dots,M}(\rho)$, where $\rho = |\psi\rangle\langle\psi|$ [17]. The criterion is

$$\begin{cases} \text{Tr}(\rho_1^2) = \frac{1}{M} & \text{non - entangled,} \\ \frac{1}{N} \leq \text{Tr}(\rho_1^2) < \frac{1}{M} & \text{entangled,} \end{cases} \quad (2)$$

where N is the dimension of the single particle state space and $M \leq N$.

Mixed states of systems of M identical fermions are non-entangled when they can be written as a mixture of Slater determinants,

$$\rho_{sl} = \sum_i \lambda^{(i)} |\psi_{sl}^{(i)}\rangle \langle \psi_{sl}^{(i)}|, \quad (3)$$

where the states $|\psi_{sl}^{(i)}\rangle$ can be represented as single Slater determinants, and $0 \leq \lambda_i \leq 1$ with $\sum_i \lambda_i = 1$.

All multi-partite fermionic systems consisting of fermions with a single-particle Hilbert space of dimension $2k$ (with $k \geq 2$) can be mapped onto systems consisting of spin- s particles, with $s = (2k - 1)/2$. According to this mapping the members $\{|i\rangle, i = 1, \dots, 2k\}$ of an orthonormal basis for the single particle Hilbert space can be denoted $|s, m_s\rangle$, with $m_s = s - i + 1, i = 1, \dots, 2k$. Since each particular case considered by us corresponds to a given value of k (and, consequently s), we are going to designate these states using the compact notation $\{|m_s\rangle, m_s = -s, \dots, s\}$. Following this angular momentum representation, a basis for a system consisting of two identical fermions is given by the antisymmetric joint eigenstates $\{|j, m\rangle, -j \leq m \leq j, 0 \leq j \leq 2s\}$ of the total angular momentum operators J_z and J^2 . The antisymmetric states $|m, j\rangle$ are those having an even value of the quantum number j . These states can be classified in multiplets according to the value of j .

An exact, analytical formula for the amount of entanglement (as given by the concurrence) of general (mixed) states of two identical fermions is known only for systems with single particle Hilbert space of dimension 4 (that is, $s = 3/2$) [10]. This concurrence can be written as

$$\mathcal{C}_{\mathcal{F}}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6\} \quad (4)$$

where the λ_i 's are, in descending order of magnitude, the square roots of the eigenvalues of $\rho\tilde{\rho}$ with $\tilde{\rho} = \mathcal{D}\rho\mathcal{D}^{-1}$, where the operator $\mathcal{D}$ is given by

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{K} \quad (5)$$

where $\mathcal{K}$ is the complex conjugation operator and $\mathcal{D}$ is expressed with respect to the aforementioned total angular momentum basis, ordered as $|2, 2\rangle, |2, 1\rangle, |2, 0\rangle, |2, -1\rangle, |2, -2\rangle$ and $i|0, 0\rangle$.

In the present work we are going to investigate entanglement-related properties of systems consisting of a given, fixed number of identical fermions. Consequently, all our present developments can be formulated in terms of the first quantization formalism.

III. LOCAL UNCERTAINTY RELATIONS FOR SYSTEMS OF TWO IDENTICAL FERMIONS WITH A FOUR DIMENSIONAL SINGLE PARTICLE HILBERT SPACE

We are going to use the violation of uncertainty relations as a signature of entanglement for pure and mixed states of systems consisting of two identical fermions. In this way we are going to extend to fermionic systems the approach advanced by Hofmann and Takeuchi in [18] for systems with distinguishable subsystems. We are going to identify appropriate sets of observables such that the sum δ of their uncertainties admits a non-zero lower bound $\delta_{\min}$ for separable states. Therefore, a state violating this uncertainty relation (that is, a state exhibiting a δ -value smaller than $\delta_{\min}$) is necessarily entangled. This violation of the uncertainty relation is not only a means of entanglement detection but can also be regarded as a rough indicator of the amount of entanglement exhibited by the state under consideration, since the general tendency when entanglement is detected is that the smaller the value of δ (i.e. the greater the violation of the uncertainty relation) the more entangled the state is. In fact, in the case of composite quantum systems consisting of distinguishable subsystems, lower bounds for the concurrence have been recently established based upon the violation of appropriate uncertainty relations [22, 23].

In order to appreciate and highlight the difference between the uncertainty relations for distinguishable particles and for identical fermions, we look at the behaviour of the uncertainty δ when evaluated for arbitrary observables of the form $A^{(1)} + A^{(2)} = A \otimes I + I \otimes A$, where A is an observable corresponding to one fermion and I is the identity operator acting in the single particle Hilbert space. We are going to refer to this kind of observables as “local”. Since we are interested in finding lower bounds for separable states, let us consider a separable pure state. Let $|\psi\rangle$ be as in eq.(1), then

$$\begin{aligned} \delta(|\psi\rangle) &= \langle (A^{(1)} + A^{(2)})^2 \rangle_{|\psi\rangle} - \langle A^{(1)} + A^{(2)} \rangle_{|\psi\rangle}^2 \\ &= \langle A^2 \rangle_{|\phi_1\rangle} - \langle A \rangle_{|\phi_1\rangle}^2 + \langle A^2 \rangle_{|\phi_2\rangle} - \langle A \rangle_{|\phi_2\rangle}^2 - 2|\langle \phi_2 | A | \phi_1 \rangle|^2. \end{aligned} \quad (6)$$

When comparing the above expression with the corresponding one for factorizable states $|\phi_1\rangle|\phi_2\rangle$ of bi-partite systems with distinguishable subsystems, it is clear that the uncertainty evaluated on separable states of fermionic systems has the additional term $-2|\langle\phi_2|A|\phi_1\rangle|^2$, which makes the study of uncertainty relations more complicated in the fermion case. Since the additional term is negative, it means that the uncertainty for identical fermions is always smaller than that for distinguishable particles. Thus we can have a state which gives a non-zero value for the two individual uncertainties $\langle A^2 \rangle_{|\phi_{1,2}\rangle} - \langle A \rangle_{|\phi_{1,2}\rangle}^2$ in eq.(6), but the resultant uncertainty $\delta(|\psi\rangle)$ is zero. As an illustration let $A = S_z$ ($s = 3/2$) and $|\phi_1\rangle = \frac{1}{\sqrt{2}} [|- \frac{1}{2}\rangle - |\frac{1}{2}\rangle]$, $|\phi_2\rangle = \frac{1}{\sqrt{2}} [|- \frac{1}{2}\rangle + |\frac{1}{2}\rangle]$. Then we have $\langle S_z^2 \rangle_{|\phi_1\rangle} - \langle S_z \rangle_{|\phi_1\rangle}^2 = 1/4 = \langle S_z^2 \rangle_{|\phi_2\rangle} - \langle S_z \rangle_{|\phi_2\rangle}^2$, however, the variance of the Slater state is $\delta = 0$. This is due to the correlations between the two particles which arise from the antisymmetric character of the state. A similar situation may occur if, instead of looking at just one observable, we evaluate the sum of the uncertainties associated with a given set of observables. For example, let us consider the sum

$$\delta(|\psi\rangle) = \sum_{i=1}^3 \left\{ \langle (A_i^{(1)} + A_i^{(2)})^2 \rangle_{|\psi\rangle} - \langle A_i^{(1)} + A_i^{(2)} \rangle_{|\psi\rangle}^2 \right\}, \quad (7)$$

where the observables $A_i^{(1)} + A_i^{(2)} = A_i \otimes I + I \otimes A_i$, $i = 1, 2, 3$, corresponding to the composite system, are given in terms of the non-commuting (one fermion) observables A_1 , A_2 and A_3 represented (in the single-particle basis $\{|-\frac{3}{2}\rangle, |-\frac{1}{2}\rangle, |\frac{1}{2}\rangle, |\frac{3}{2}\rangle\}$) by the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (8)$$

We have that $\delta(|\psi\rangle) = 0$ for the state,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[\left| -\frac{3}{2} \right\rangle \left| -\frac{1}{2} \right\rangle - \left| -\frac{1}{2} \right\rangle \left| -\frac{3}{2} \right\rangle \right]. \quad (9)$$

Therefore, there are separable two-fermion states with vanishing $\delta(|\psi\rangle)$ even though the three (one-fermion) non-commuting operators A_1 , A_2 , and A_3 don't share a common eigenvector and, consequently, the uncertainty sum $\sum_{i=1}^3 (\langle A_i^2 \rangle - \langle A_i \rangle^2)$ has a non-trivial (that is, non-zero) lower bound over the single-particle states. We are going to show that, in spite of this difficulty, it is possible to find sets of non-commuting local observables for

two-fermion systems such that the corresponding sum of variances admits a non-trivial, finite lower bound over the set of separable fermion states.

Regarding, as already mentioned, our two-fermion system as a system of two spin- s particles, we are now going to focus on the uncertainties associated with the three components J_x, J_y, J_z of the total angular momentum of the system. The sum of the corresponding uncertainties evaluated on a pure state $|\psi\rangle$ of the two fermions is then

$$\delta(|\psi\rangle) = \langle\psi|J^2|\psi\rangle - \langle\psi|J_x|\psi\rangle^2 - \langle\psi|J_y|\psi\rangle^2 - \langle\psi|J_z|\psi\rangle^2 \quad (10)$$

where

$$\begin{aligned} J^2 &= J_x^2 + J_y^2 + J_z^2 \\ J_x &= S_x \otimes I + I \otimes S_x \\ J_y &= S_y \otimes I + I \otimes S_y \\ J_z &= S_z \otimes I + I \otimes S_z, \end{aligned} \quad (11)$$

S_x, S_y, S_z are the angular momentum operators corresponding to one fermion and I denotes the identity operator for the single-particle Hilbert space.

The sum of uncertainties δ corresponding to a general mixed state ρ of the two-fermion system reads

$$\delta(\rho) = \text{Tr}(\rho J_x^2) - [\text{Tr}(\rho J_x)]^2 + \text{Tr}(\rho J_y^2) - [\text{Tr}(\rho J_y)]^2 + \text{Tr}(\rho J_z^2) - [\text{Tr}(\rho J_z)]^2. \quad (12)$$

Our aim is to determine an uncertainty relation for separable mixed states ρ_{sl} of two-fermion systems based on the quantity (12). In order to achieve this end it is enough to establish a lower bound for (12) valid for pure states $|\psi_{sl}\rangle$ having Slater rank 1. To see that the uncertainty limit for separable pure states also applies to mixed separable states, we note that the uncertainty of a mixture is always equal to or greater than the averaged uncertainties of the components. Thus for a general mixture of pure states of Slater rank 1 we have,

$$\delta(\rho_{sl}) \geq \sum_i \lambda^{(i)} \delta(|\psi_{sl}^{(i)}\rangle). \quad (13)$$

In the $s = 3/2$ case (corresponding to a single-particle Hilbert space of dimension 4) we are going to establish that for any pure state $|\psi_{sl}\rangle$ of the Slater form,

$$\delta(|\psi_{sl}\rangle) \geq 2s - 1. \quad (14)$$

Consequently, we also have

$$\delta(\rho_{sl}) \geq 2s - 1, \quad (15)$$

for general non-entangled mixed states ρ_{sl} of the two-fermion system.

The following is a list of the antisymmetric total angular momentum eigenstates for $s = 3/2$ (which, according to the angular momentum representation, constitute a basis of the Hilbert space describing the two-fermion system) with the value for the uncertainty δ in brackets: $|2, 2\rangle$ ($\delta = 2$), $|2, 1\rangle$ ($\delta = 5$), $|2, 0\rangle$ ($\delta = 6$), $|2, -1\rangle$ ($\delta = 5$), $|2, -2\rangle$ ($\delta = 2$) and $|0, 0\rangle$ ($\delta = 0$). The singlet state $|0, 0\rangle$ has $\delta = 0$ and is maximally entangled, $\text{Tr}(\rho_1^2) = 1/4$. What we want to find is the minimum δ_{min} for $\delta(|\psi_{sl}\rangle)$ among all separable pure states $|\psi_{sl}\rangle$, such that for $0 \leq \delta < \delta_{min}$ the state is certainly entangled (not separable) and for $\delta_{min} \leq \delta \leq 6$ the state is either entangled or separable. This bound is

$$\min_{\{|\psi_{sl}\rangle\}} \delta(|\psi_{sl}\rangle) = 2. \quad (16)$$

In order to prove this we first establish the maximum of $|\langle\psi_{sl}|0, 0\rangle|$ among separable states, which turns out to be $1/\sqrt{2}$. When expressing a general state $|\psi\rangle$ in the total angular momentum basis (which comprises the $j = 0$ singlet and the $j = 2$ multiplet) we see that the maximum of $|c_0|^2$ is $1/2$, where c_0 is the coefficient of the singlet state. After an uncertainty inequality (28) holding for general two-fermion states is established, we then use the previous result to obtain the uncertainty relation (30) for separable states. To see the first part, let

$$|\psi_{sl}\rangle = \frac{1}{\sqrt{2}} [|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle] \quad (17)$$

with normalized, orthogonal single particle states

$$|\phi_1\rangle = \alpha_1 \left| \frac{3}{2} \right\rangle + \alpha_2 \left| \frac{1}{2} \right\rangle + \alpha_3 \left| -\frac{1}{2} \right\rangle + \alpha_4 \left| -\frac{3}{2} \right\rangle \quad (18)$$

$$|\phi_2\rangle = \beta_1 \left| \frac{3}{2} \right\rangle + \beta_2 \left| \frac{1}{2} \right\rangle + \beta_3 \left| -\frac{1}{2} \right\rangle + \beta_4 \left| -\frac{3}{2} \right\rangle. \quad (19)$$

Then

$$\langle 0, 0 | \psi_{sl} \rangle = \frac{1}{\sqrt{2}} [\alpha_3 \beta_2 - \alpha_2 \beta_3 + \alpha_1 \beta_4 - \alpha_4 \beta_1]. \quad (20)$$

Using the Schwartz inequality and the normalization condition of the states $|\phi_1\rangle$ and $|\phi_2\rangle$ we obtain,

$$\begin{aligned} |\langle 0, 0 | \psi_{sl} \rangle| &\leq \frac{1}{\sqrt{2}} [|\alpha_3| |\beta_2| + |\alpha_2| |\beta_3| + |\alpha_1| |\beta_4| + |\alpha_4| |\beta_1|] \\ &\leq \frac{1}{\sqrt{2}}. \end{aligned} \quad (21)$$

Equality can be achieved by $|\psi_{sl}\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{3}{2} \right\rangle \left| -\frac{3}{2} \right\rangle - \left| -\frac{3}{2} \right\rangle \left| \frac{3}{2} \right\rangle \right]$ and thus

$$\max |\langle 0, 0 | \psi_{sl} \rangle| = \frac{1}{\sqrt{2}}. \quad (22)$$

Now any two-fermion state $|\psi\rangle$ (whether separable or not and with a four-dimensional single particle state space) can be expressed as

$$|\psi\rangle = c_0 |0, 0\rangle + c_1 |\tilde{\psi}\rangle \quad (23)$$

with $\langle \tilde{\psi} | \tilde{\psi} \rangle = 1$ (normalized) and $|\tilde{\psi}\rangle$ being a linear combination of the five members of the $j = 2$ multiplet. Therefore we have

$$\begin{aligned} \langle \tilde{\psi} | 0, 0 \rangle &= 0 \\ |c_0|^2 + |c_1|^2 &= 1. \end{aligned} \quad (24)$$

Obviously we have $c_0 = \langle 0, 0 | \psi \rangle$ and therefore we know that for separable states the maximum possible value for $|c_0|^2$, is

$$\max |c_0|^2 = \frac{1}{2}. \quad (25)$$

The allowed values for $|c_0|^2$ for separable states $|\psi\rangle$ are then

$$0 \leq |c_0|^2 \leq \frac{1}{2}. \quad (26)$$

Now we have to find an expression for the variance (10) from which we can then obtain an uncertainty relation for general bipartite states with $s = 3/2$ and then using the previous information we establish an uncertainty relation for separable states. This means we have to simplify eq.(10). By observing that J_x, J_y, J_z and J^2 have vanishing matrix elements (in the basis $|jm\rangle$) connecting states with different values of j and by making use of the normalization condition,

$$\begin{aligned} \langle \psi | J^2 | \psi \rangle &= 6|c_1|^2 = 6(1 - |c_0|^2) \\ \langle \psi | J_x | \psi \rangle &= (1 - |c_0|^2) \langle \tilde{\psi} | J_x | \tilde{\psi} \rangle \end{aligned} \quad (27)$$

and similarly for J_y and J_z . So eq.(10) becomes

$$\begin{aligned} \delta(|\psi\rangle) &= 6(1 - |c_0|^2) - (1 - |c_0|^2)^2 \left[\langle \tilde{\psi} | J_x | \tilde{\psi} \rangle^2 + \langle \tilde{\psi} | J_y | \tilde{\psi} \rangle^2 + \langle \tilde{\psi} | J_z | \tilde{\psi} \rangle^2 \right] \\ &\geq 6(1 - |c_0|^2) - 4(1 - |c_0|^2)^2. \end{aligned} \quad (28)$$

This last inequality holds because the maximum possible value of $\langle \tilde{\psi} | J_x | \tilde{\psi} \rangle^2 + \langle \tilde{\psi} | J_y | \tilde{\psi} \rangle^2 + \langle \tilde{\psi} | J_z | \tilde{\psi} \rangle^2$ is 4. For example the state $|2, 2\rangle$ achieves this. Expanding eq.(28) results in

$$\delta(|\psi\rangle) \geq 2(1 + |c_0|^2 - 2|c_0|^4). \quad (29)$$

This inequality holds for any two-fermion state with $s = 3/2$, regardless of it being separable or not. Now for separable states we have that $|c_0|^2$ lies in the interval $[0, 1/2]$. Within this interval we have $(1 + |c_0|^2 - 2|c_0|^4) \geq 2$. This expression achieves the value 2 only in the extremes of the interval, $|c_0|^2 = 0$ or $|c_0|^2 = 1/2$. Therefore, for separable states we have that

$$\delta(|\psi\rangle) \geq 2 \quad (|\psi\rangle \text{ separable}). \quad (30)$$

The equality case can be achieved, for instance, by the state $|2, 2\rangle$.

FIG. 1: Concurrence (thick line), δ' and δ'_M (dashed line) for the state in eq.(32).

FIG. 2: Concurrence and δ' (dashed line) for the state in eq.(35).

In order to illustrate this method of entanglement detection we consider three examples of two-fermion states (with $s = 3/2$). In each case we compute the concurrence of the states and the relative violation of the local uncertainty relation, which is defined as [18]

$$\delta' = 1 - \frac{\delta}{\delta_{min}}, \quad (31)$$

where δ_{min} is the uncertainty limit for separable states, in this case $\delta_{min} = 2$.

For the first example, let ρ be a mixture of a maximally entangled state and a maximally mixed state of the fermion system,

$$\rho = p|0, 0\rangle\langle 0, 0| + \frac{1-p}{6} I, \quad (32)$$

where

$$I = |0, 0\rangle\langle 0, 0| + \sum_{m=-2}^2 |2, m\rangle\langle 2, m| \quad (33)$$

is the identity operator acting on the six-dimensional Hilbert space corresponding to the two-fermion system. The plot of the concurrence as a function of p depicted in Figure 1 indicates that all states with p larger than $p^* = 0.4$ are entangled. In comparison, we have $\delta(p) = 5 - 5p$ and so the uncertainty based criterion detects entanglement for $3/5 < p \leq 1$. Thus, the interval of p -values where entanglement is detected is $2/3$ of the total range of p -values corresponding to entangled states.

As a second illustration of the criterion we now consider the family of states

$$\rho = p|0, 0\rangle\langle 0, 0| + \frac{1-p}{2} (|2, -2\rangle\langle 2, -2| + |2, 2\rangle\langle 2, 2|). \quad (34)$$

The expression for $\delta(p) = 6 - 6p$ and thus the minimal p for which entanglement is detected is $p_{min} = 2/3$. Evaluating the concurrence for these states shows that they are entangled from $p^* = 0.5$ and hence $2/3$ of entangled states are detected.

As third example we take

$$|\psi\rangle = \frac{\sin \theta}{\sqrt{2}} \left[\left| -\frac{3}{2} \right\rangle \left| \frac{3}{2} \right\rangle - \left| \frac{3}{2} \right\rangle \left| -\frac{3}{2} \right\rangle \right] + \frac{\cos \theta}{\sqrt{2}} \left[\left| -\frac{1}{2} \right\rangle \left| \frac{1}{2} \right\rangle - \left| \frac{1}{2} \right\rangle \left| -\frac{1}{2} \right\rangle \right]. \quad (35)$$

From Figure 2 we can see that for $\theta \in (0, \pi)$, $\theta \neq \pi/2$ all states are entangled. In comparison, the uncertainty measure detects entanglement for $1.74071 < \theta \leq 2.97167$, thus about 0.392 of the entangled states are detected.

IV. CHARACTERIZATION OF ENTANGLEMENT VIA THE VARIANCES OF PROJECTOR OPERATORS

Here we are going to formulate separability criteria for systems of identical fermions based upon the uncertainties of observables that are not of the form $A^{(1)} + A^{(2)} = A \otimes I + I \otimes A$. Instead, we are going to consider families of observables of the form $M_i = |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle$ are appropriate pure states of the N -fermion system under consideration. This kind of approach was recently applied by Ghne [20] to composite quantum systems with distinguishable subsystems.

A. Bipartite states of two fermions with a four dimensional single particle Hilbert space

We are interested in finding an entanglement detection criterion involving the sum $\delta_M = \sum_i \delta^2(M_i)$ of the variances of an appropriate set of projector operators $\{M_i\}$. First, let's introduce the following notation:

$$|m_1 m_2| = \frac{1}{\sqrt{2}} [|m_1\rangle|m_2\rangle - |m_2\rangle|m_1\rangle]. \quad (36)$$

We now define six orthogonal basis states each exhibiting the same amount of entanglement and all consisting of a linear combination of two Slater determinants which are orthogonal as well,

$$\begin{aligned}
|\psi_1\rangle &= a \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + b \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\
|\psi_2\rangle &= b \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - a \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\
|\psi_3\rangle &= a \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + b \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} \\
|\psi_4\rangle &= b \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - a \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} \\
|\psi_5\rangle &= a \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + b \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\
|\psi_6\rangle &= b \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - a \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix},
\end{aligned} \tag{37}$$

where a and b are assumed to be real and $|a| \geq |b|$. Then we let $M_i = |\psi_i\rangle\langle\psi_i|$, $i = 1, \dots, 6$. The proof of the uncertainty relation (42) is as follows. We let $|\phi_{sl}\rangle$, $|\phi_1\rangle$ and $|\phi_2\rangle$ be the same as in eqs. (17), (18) and (19). The idea is to find the minimum of

$$\sum_{i=1}^6 \delta^2(M_i)_{|\phi_{sl}\rangle\langle\phi_{sl}|} = 1 - \sum_{i=1}^6 (|\langle\phi_{sl}|\psi_i\rangle|^2)^2 \tag{38}$$

for all separable states $|\phi_{sl}\rangle$, so that when $\sum_{i=1}^6 \delta^2(M_i)_{|\psi\rangle\langle\psi|}$ is less than that minimum, we know with certainty that the state $|\psi\rangle$ is entangled. This amounts to finding the maximum for $\sum_{i=1}^6 (|\langle\phi_{sl}|\psi_i\rangle|^2)^2$. This can be determined by finding first the maximum of $|\langle\phi_{sl}|\psi_i\rangle|^2$ for each i (here we follow a procedure similar to the one used in [20] for systems of two distinguishable qubits). Due to symmetry, it is only necessary to find the maximum for one i . Now,

$$\begin{aligned}
|\langle\phi_{sl}|\psi_1\rangle| &= |a\alpha_1\beta_2 - a\alpha_2\beta_1 + b\alpha_3\beta_4 - b\alpha_4\beta_3| \\
&\leq |a| [|\alpha_1||\beta_2| + |\alpha_2||\beta_1|] + |b| [|\alpha_3||\beta_4| + |\alpha_4||\beta_3|] \\
&\leq |a|,
\end{aligned} \tag{39}$$

where the last inequality follows from the Schwartz inequality together with the normalization of $|\phi_1\rangle$ and $|\phi_2\rangle$. As mentioned, due to symmetry, this last inequality will hold for all $|\psi_i\rangle$,

$$|\langle \phi_{sl} | \psi_i \rangle|^2 \leq a^2 \quad \forall i. \quad (40)$$

The inequality (40) leads to the inequality

$$\sum_{i=1}^6 (|\langle \phi_{sl} | \psi_i \rangle|^2)^2 \leq a^4 + (1 - a^2)^2. \quad (41)$$

This upper bound is achieved, for instance, by the state $|\phi'_{sl}\rangle = \left| \frac{3}{2} \frac{1}{2} \right\rangle$.

From (41) we finally obtain,

$$\sum_{i=1}^6 \delta^2(M_i)_{|\phi_{sl}\rangle\langle\phi_{sl}|} \geq 2a^2b^2. \quad (42)$$

At least one entangled state violates this inequality, namely $|\psi_1\rangle$ which gives zero for the right hand side of equation (42). To compare the efficiency of this entanglement criterion with the previous one we use the same families of mixed states ρ as in equations (32) and (34). When applying the criterion to those states we consider the case $a = b = 1/\sqrt{2}$, which is the one maximizing the right hand side of equation (42). The results corresponding to the first illustration are shown in Figure 1 where we again use the relative violation (31) with $\delta_{min} = 1/2$,

$$\delta'_M = 1 - 2\delta_M, \quad (43)$$

and since $\delta_M(p) = -\frac{5}{6}(-1 + p^2)$, entanglement is detected for $\sqrt{\frac{2}{5}} < p \leq 1$. Thus the range of entanglement detection in this case is very close to the one of the first uncertainty relation established. For the second density matrix (34), $\delta_M(p) = \frac{1}{2} + p - \frac{3p^2}{2}$ and so entanglement is detected from $p_{min} = 2/3$, which is exactly the same as for the local uncertainty relation.

B. Two-fermion systems with a six dimensional single particle Hilbert space

As mentioned before, the particular case of systems of two identical fermions with a 4-dimensional single particle Hilbert space (the simplest fermion system admitting entanglement) is the only one for which we have a closed, analytical expression for the concurrence. No such expression is known for fermion systems of higher dimensionality, nor any necessary and sufficient separability criteria for general mixed states of those systems. It is thus of considerable interest to explore the application of separability criteria based upon uncertainty

relations of a two-fermion system with single-particle Hilbert space of dimension larger than 4. Here we are going to consider fermions with single-particle Hilbert space of dimension 6. The dimension of the concomitant two-fermion system is then 15 and, again, we have to choose a basis of orthogonal states $\{|\psi_i\rangle, i = 1, \dots, 15\}$, all exhibiting the same amount of entanglement. We are going to use an orthonormal basis comprising five “triplets” of states of the form

$$\begin{aligned} & a|m_1m_2| + a|m_3m_4| - \frac{a}{2}|m_5m_6|, \\ & a|m_1m_2| - \frac{a}{2}|m_3m_4| + a|m_5m_6|, \\ & -\frac{a}{2}|m_1m_2| + a|m_3m_4| + a|m_5m_6|, \end{aligned} \quad (44)$$

where the six m_i ’s appearing in (44) are all different and $a = 2/3$. The three Slater determinants $|m_im_j|$ appearing in each of the “triplets” are different from the Slater determinants appearing in the other four triplets. Thus the basis reads

$$\begin{aligned} |\psi_1\rangle &= \frac{2}{3} \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} + \frac{2}{3} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \frac{1}{3} \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} \\ |\psi_2\rangle &= \frac{2}{3} \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} - \frac{1}{3} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \frac{2}{3} \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} \\ |\psi_3\rangle &= -\frac{1}{3} \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} + \frac{2}{3} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \frac{2}{3} \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} \\ &\vdots \end{aligned} \quad (45)$$

The derivation of the uncertainty relation (47) follows along similar lines as the one in the previous subsection. The general separable state is as in eq.(17), however, the two orthogonal single particle states are of dimension six. The overlap between a member $|\psi_i\rangle$ of the above basis and any separable state $|\phi_{sl}\rangle$ is always less or equal to a . This leads (note that $a^2 < \frac{1}{2} < 2a^2$) to the inequality

$$\sum_{i=1}^{15} (|\langle \phi_{sl} | \psi_i \rangle|^2)^2 \leq 2a^4 + (1 - 2a^2)^2 = \frac{33}{81} \quad (46)$$

which, in turn, implies the following lower bound for the sum of the uncertainties evaluated on a separable state,

$$\sum_{i=1}^{15} \delta^2(M_i)_{|\phi_{sl}\rangle\langle\phi_{sl}|} \geq \frac{48}{81}. \quad (47)$$

The separable state $|\phi'_{sl}\rangle = \left|\frac{5}{2}\frac{3}{2}\right\rangle$ attains this lower bound.

As an illustration of this separability criterion let us look at the following family of states having Slater rank equal to 2,

$$|\phi\rangle = \frac{1}{\sqrt{2}} [|\phi_1\phi_2\rangle + |\phi_3\phi_4\rangle], \quad (48)$$

with the $|\phi_i\rangle$'s being four normalized, orthogonal single-particle states. The overlap between this family of states and $|\psi_1\rangle$ is less or equal to $\frac{4}{3\sqrt{2}}$. The minimum value adopted by the quantity $\sum_{i=1}^{15} \delta^2(M_i)$ when evaluated on states belonging to the family (48) is $\frac{33}{162}$ and it is achieved by the state $|\phi'\rangle = \frac{1}{\sqrt{2}} \left[\left|\frac{5}{2}\frac{3}{2}\right\rangle + \left|\frac{1}{2}-\frac{1}{2}\right\rangle\right]$. Thus we obtain

$$\sum_{i=1}^{15} \delta^2(M_i)_{|\phi\rangle\langle\phi|} \geq \frac{33}{162}. \quad (49)$$

So all the states within the family (48) whose sum of uncertainties lies between $\frac{33}{162}$ and $\frac{48}{81}$ are identified as entangled.

As a second example we apply the criterion to mixed states of the form

$$\rho = p|\varphi_i\rangle\langle\varphi_i| + \frac{1-p}{15} I, \quad (50)$$

where $0 \leq p \leq 1$, $|\varphi_i\rangle$ is an entangled pure state, and $I = \sum_{i=1}^{15} |\psi_i\rangle\langle\psi_i|$ is the identity operator acting on the Hilbert space associated with the two-fermion system, the $|\psi_i\rangle$'s being the members of an orthonormal basis of this space (for instance, the basis (45)). We are going to consider the following three different cases for the state $|\varphi_i\rangle$.

1) $|\varphi_1\rangle = \frac{1}{\sqrt{3}} \left[\left|\frac{5}{2}\frac{3}{2}\right\rangle + \left|\frac{1}{2}-\frac{1}{2}\right\rangle - \left|-\frac{3}{2}-\frac{5}{2}\right\rangle\right]$, in which case entanglement is detected in the interval $1 \geq p > \sqrt{\frac{207}{482}} \approx 0.655$, since $\delta_M = \frac{14}{15} - \frac{964}{1215}p^2$.

2) $|\varphi_2\rangle = |\psi_1\rangle$, then $\delta_M = -\frac{14}{15}(-1 + p^2)$ and entanglement is detected for $1 \geq p > \sqrt{\frac{23}{63}} \approx 0.604$.

3) $|\varphi_3\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{5}{2} \frac{3}{2} \right\rangle + \left| \frac{1}{2} -\frac{1}{2} \right\rangle \right]$, with $\delta_M = \frac{1}{270}(252 - 197p^2)$ and entangled states are detected for $1 \geq p > \sqrt{\frac{92}{197}} \approx 0.683$.

C. Systems of three fermions with a six dimensional single particle Hilbert space

In the case of three fermions with $s = 5/2$, a general separable state (Slater determinant) is of the form

$$|\phi_{sl}\rangle = |\phi_1\phi_2\phi_3| = \frac{1}{\sqrt{6}} [|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle - |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle + |\phi_3\rangle|\phi_1\rangle|\phi_2\rangle - |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle], \quad (51)$$

with the three orthogonal single particle states being

$$\begin{aligned} |\phi_1\rangle &= \alpha_1 \left| \frac{5}{2} \right\rangle + \alpha_2 \left| \frac{3}{2} \right\rangle + \alpha_3 \left| \frac{1}{2} \right\rangle + \alpha_4 \left| -\frac{1}{2} \right\rangle + \alpha_5 \left| -\frac{3}{2} \right\rangle + \alpha_6 \left| -\frac{5}{2} \right\rangle \\ |\phi_2\rangle &= \beta_1 \left| \frac{5}{2} \right\rangle + \beta_2 \left| \frac{3}{2} \right\rangle + \beta_3 \left| \frac{1}{2} \right\rangle + \beta_4 \left| -\frac{1}{2} \right\rangle + \beta_5 \left| -\frac{3}{2} \right\rangle + \beta_6 \left| -\frac{5}{2} \right\rangle \\ |\phi_3\rangle &= \gamma_1 \left| \frac{5}{2} \right\rangle + \gamma_2 \left| \frac{3}{2} \right\rangle + \gamma_3 \left| \frac{1}{2} \right\rangle + \gamma_4 \left| -\frac{1}{2} \right\rangle + \gamma_5 \left| -\frac{3}{2} \right\rangle + \gamma_6 \left| -\frac{5}{2} \right\rangle. \end{aligned} \quad (52)$$

In this section we are extending the previous one to three fermions. Since the dimension of the system is 20, we need to construct 20 orthogonal states whose entanglement is the same and where there is no overlap between the two Slater determinants that each state consists of. So the basis consists of ten pairs of states which are unique combinations of non-overlapping Slater determinants and where the members of each pair are permutations of each other:

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}} \left(\left| \frac{5}{2} \frac{3}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{3}{2} \frac{5}{2} \right\rangle \right) \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}} \left(\left| \frac{5}{2} \frac{3}{2} \frac{1}{2} \right\rangle - \left| \frac{1}{2} \frac{3}{2} \frac{5}{2} \right\rangle \right) \\ &\vdots \end{aligned} \quad (53)$$

Once again, let $M_i = |\psi_i\rangle\langle\psi_i|$, $i = 1, \dots, 20$ and so we obtain exactly the same equation as in (38), except that the summation goes to 20. A bound on the maximum overlap of $|\psi_i\rangle$ and $|\phi_{sl}\rangle$ is

$$\begin{aligned}
|\langle \phi_{sl} | \psi_i \rangle| &= \frac{1}{\sqrt{2}} |\alpha_1 \beta_2 \gamma_3 - \alpha_1 \beta_3 \gamma_2 - \alpha_2 \beta_1 \gamma_3 + \alpha_2 \beta_3 \gamma_1 + \alpha_3 \beta_1 \gamma_2 - \alpha_3 \beta_2 \gamma_1 \\
&\quad + \alpha_4 \beta_5 \gamma_6 - \alpha_4 \beta_6 \gamma_5 - \alpha_5 \beta_4 \gamma_6 + \alpha_5 \beta_6 \gamma_4 + \alpha_6 \beta_4 \gamma_5 - \alpha_6 \beta_5 \gamma_4| \\
&\leq \frac{1}{\sqrt{2}},
\end{aligned} \tag{54}$$

where the last inequality follows from the Schwartz inequality and normalization. A state that realizes the maximum is $|\phi'_{sl}\rangle = \left|\frac{5}{2}\frac{3}{2}\frac{1}{2}\right|$ and so together with normalization

$$\sum_{i=1}^{20} (|\langle \phi_{sl} | \psi_i \rangle|^2)^2 \leq \frac{1}{2} \tag{55}$$

and hence

$$\sum_{i=1}^{20} \delta^2(M_i)_{|\phi_{sl}\rangle\langle\phi_{sl}|} \geq \frac{1}{2} \quad \forall |\phi_{sl}\rangle. \tag{56}$$

As an example of entanglement detection, let us consider the following family of states

$$|\Phi\rangle = a|\phi_1\phi_2\phi_3\rangle + b|\phi_4\phi_5\phi_6\rangle \tag{57}$$

with the $|\phi_i\rangle$'s being six orthogonal single particle states. The above procedure is repeated, this time for the family given in eq.(57). Then

$$\sum_{i=1}^{20} \delta^2(M_i)_{|\Phi\rangle\langle\Phi|} \geq 1 - \frac{1}{4} [|a+b|^4 + |a-b|^4] \tag{58}$$

and the specific case $a = \sqrt{\frac{2}{3}}$, $b = \frac{1}{\sqrt{3}}$ [14, 24] gives the value $\frac{1}{18}$ for the right hand side. Thus all the states whose sum of uncertainties lies between $\frac{1}{18}$ and $\frac{1}{2}$ are identified as entangled.

V. SEPARABILITY CRITERIA FOR TWO-FERMION SYSTEMS WITH A FOUR DIMENSIONAL SINGLE PARTICLE HILBERT SPACE BASED ON ENTROPIC UNCERTAINTY RELATIONS

Entropic uncertainty relations provide an alternative way to develop entanglement criteria based upon uncertainty relations. The application of this procedure to the detection of entanglement in quantum systems consisting of distinguishable subsystems was investigated by Gühne and Lewenstein in [19] and by Giovannetti in [25]. Here we are going to explore the use of entropic uncertainty relations to identify entangled states of systems constituted by

identical fermions. Within this approach the statistical variances are replaced by entropic measures as a means of estimating the uncertainties associated with the measurement of observables. If $\{p_i\}$ denotes the probabilities of obtaining the different eigenvalues of an observable A when measuring it on a system prepared in a given state, one can use the concomitant Shannon entropy,

$$H[A] = - \sum_i p_i \ln p_i, \quad (59)$$

to characterize the uncertainty associated with the measurement. Alternatively, one can consider the Rényi or the Tsallis families of q -entropies [25–27], respectively given by

$$H_q^{(R)}[A] = \frac{1}{1-q} \ln \left(\sum_i p_i^q \right), \quad (60)$$

and

$$H_q^{(T)}[A] = \frac{1 - \sum_i p_i^q}{q-1}. \quad (61)$$

The Shannon measure is a particular member of the above two entropic families, corresponding to the limit case $q \rightarrow 1$.

Given a suitable set of observables $\{A_k\}$ we want to establish the minimum value

$$H_{\min} = \min \sum_k H[A_k]_{|\psi_{sl}\rangle}, \quad (62)$$

over all separable pure states $|\psi_{sl}\rangle$, adopted by the sum of the entropies $H[A_k]$ associated with the measurement of the observables A_k . If the entropic measure H used in (62) is convex, then

$$\sum_k H[A_k]_{\rho} \geq H_{\min}, \quad (63)$$

for any separable mixed state ρ .

We are going to consider the set of observables

$$\hat{X} = S_x \otimes S_x \quad \hat{Y} = S_y \otimes S_y \quad \hat{Z} = S_z \otimes S_z \quad (64)$$

associated with a system of two identical fermions. In order to determine the minimum value of the corresponding entropic sum (62), it is convenient to represent the general separable state $|\psi_{sl}\rangle$ as

$$|\psi_{sl}\rangle = \frac{|\chi\rangle}{\langle\chi|\chi\rangle}, \quad (65)$$

where

$$|\chi\rangle = |\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle, \quad (66)$$

and the single-particle states $|\phi_1\rangle$ and $|\phi_2\rangle$ are parameterized as

$$\begin{aligned} |\phi_1\rangle &= \cos\alpha_3 \left|-\frac{3}{2}\right\rangle + \sin\alpha_3 \cos\alpha_2 e^{i\gamma_3} \left|-\frac{1}{2}\right\rangle + \sin\alpha_3 \sin\alpha_2 \cos\alpha_1 e^{i\gamma_2} \left|\frac{1}{2}\right\rangle \\ &\quad + \sin\alpha_3 \sin\alpha_2 \sin\alpha_1 e^{i\gamma_1} \left|\frac{3}{2}\right\rangle \\ |\phi_2\rangle &= \cos\beta_3 \left|-\frac{3}{2}\right\rangle + \sin\beta_3 \cos\beta_2 e^{i\delta_3} \left|-\frac{1}{2}\right\rangle + \sin\beta_3 \sin\beta_2 \cos\beta_1 e^{i\delta_2} \left|\frac{1}{2}\right\rangle \\ &\quad + \sin\beta_3 \sin\beta_2 \sin\beta_1 e^{i\delta_1} \left|\frac{3}{2}\right\rangle. \end{aligned} \quad (67)$$

The six parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in [0, \pi/2]$, whereas the phase-parameters lie in the interval $[0, 2\pi]$. Note that here we don't require that $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal. The state (65) has Slater rank 1 even if the states $|\phi_1\rangle$ and $|\phi_2\rangle$ are non-orthogonal. The entropic uncertainties (Shannon entropy) corresponding to the measurement of the observables $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ on the state (65) are calculated and added together. This procedure leads to two functions, $H[\hat{X}] + H[\hat{Y}]$ and $H[\hat{X}] + H[\hat{Y}] + H[\hat{Z}]$, of the 12 parameters characterizing the state (65), that have to be minimized in order to obtain the separability criteria. Since the resultant expressions are much too complicated to establish the corresponding minima H_{xy} and H_{xyz} analytically, we implemented instead a numerical minimization. The result of this numerical optimization is $H_{xy} = 0.693147$ and $H_{xyz} = 1.38629$. This means that states such that $H[\hat{X}] + H[\hat{Y}]$ or $H[\hat{X}] + H[\hat{Y}] + H[\hat{Z}]$ are smaller than these values are necessarily entangled.

In what follows we are going to consider separability criteria for systems of two identical fermions based, instead of on the Shannon entropy, on Tsallis' q -entropy. The separability

FIG. 3: Numerical lower bounds in eq.(69). The upper curve is $H_{xyz}(q)$ and the lower one $H_{xy}(q)$.

FIG. 4: Minimal values p_{min} depending on q such that for $p > p_{min}$ entanglement in (70) is detected. These states are entangled for $p > 0.4$. The upper curve is $p_{xy}(q)$ and the lower one $p_{xyz}(q)$.

bounds associated with the q -entropies are obtained by recourse to a procedure similar to the previously discussed one corresponding to Shannon's measure. In fact, the Shannon-based criteria can be regarded as special instances of the Tsallis-based ones, corresponding to $q = 1$. So for a range of values of q we numerically determine the minima

$$\begin{aligned} H_{xy}(q) &= \min_{|\psi_{sl}\rangle} [H_q^{(T)}(S_x \otimes S_x)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_y \otimes S_y)_{|\psi_{sl}\rangle}] \\ H_{xyz}(q) &= \min_{|\psi_{sl}\rangle} [H_q^{(T)}(S_x \otimes S_x)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_y \otimes S_y)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_z \otimes S_z)_{|\psi_{sl}\rangle}], \end{aligned} \quad (68)$$

where $|\psi_{sl}\rangle$ is a pure state of Slater rank 1 (expressed as in (65)) with $|\phi_1\rangle, |\phi_2\rangle$ being the same as in eq.(67). Then we have

$$\begin{aligned} H_q^{(T)}(S_x \otimes S_x)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_y \otimes S_y)_{|\psi_{sl}\rangle} &\geq H_{xy}(q) \\ H_q^{(T)}(S_x \otimes S_x)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_y \otimes S_y)_{|\psi_{sl}\rangle} + H_q^{(T)}(S_z \otimes S_z)_{|\psi_{sl}\rangle} &\geq H_{xyz}(q). \end{aligned} \quad (69)$$

for all separable pure states $|\psi_{sl}\rangle$. Since the entropic measure H_q^T is convex, the inequality (69) actually holds for any separable state ρ , pure or mixed. The results corresponding to the numerical minimization (68) are shown in Figure 3.

As illustrations of the efficiency of these criteria we can consider the same families of states (32) and (34) as in Section III. However, neither the Shannon nor the Tsallis entropic criteria detect entanglement for these states. Another family of states was thus constructed

$$\rho = p \cdot \frac{1}{2}(|2, 2\rangle - |2, -2\rangle)(\langle 2, 2| - \langle 2, -2|) + \frac{1-p}{6} I, \quad (70)$$

where $\frac{1}{\sqrt{2}}(|2, 2\rangle - |2, -2\rangle)$ is maximally entangled and is a common eigenvector of $\hat{X}$, $\hat{Y}$ and $\hat{Z}$. Evaluating the concurrence shows that the state is entangled for $0.4 < p \leq 1$. The values of p from which onward the criteria detect entanglement are given in Figure 4. Thus for q between 2.5 and 2.9 the greatest range of entanglement is detected.

Next we are going to formulate another family of separability criteria for systems of two fermions. These criteria are based upon either Tsallis or Rényi entropies, and are similar to the ones proposed by Gühne and Lewenstein [19] for bipartite systems comprising distinguishable subsystems.

Let's consider the non-degenerate observable M

$$M = \sum_i \mu_i |\psi_i\rangle \langle \psi_i|, \quad (71)$$

with the $|\psi_i\rangle$'s being the states in eq.(37) with $a = 1/\sqrt{2}$. When measuring M on a separable state $|\psi_{sl}\rangle$ (that is, a state of Slater rank 1) the maximum possible value of the probability $|\langle \psi_{sl} | \psi_i \rangle|^2$ of obtaining a given eigenvalue of M is $\frac{1}{2}$. The minimum possible value of $H_q^{(T)}(M)_{|\psi_{sl}\rangle}$ is then attained when two of these probabilities are equal to $\frac{1}{2}$. Therefore (for $q > 1$) we have that any separable state ρ (pure or mixed) complies with

$$H_q^{(T)}(M)_\rho \geq \frac{1 - 2^{1-q}}{q - 1}. \quad (72)$$

This inequality provides an entanglement criterion since any state violating this inequality is necessarily entangled. For $q = 2$ this criterion turns out to be equivalent to the one considered in Section IV, which is formulated in terms of the variances of projector operators (an equivalent situation occurs in the case of distinguishable subsystems [19]).

Since the Rényi entropy is a monotonic function of the Tsallis entropy, this bound reads

$$H_q^{(R)}(M)_\rho \geq \ln 2 \quad (73)$$

and the criterion gets stronger as q increases. For $q \rightarrow \infty$ the Rényi entropy becomes very simple, namely $H_\infty^{(R)}(M)_\rho = -\ln(\max\{p \in \mathcal{P}(M)_\rho\})$, where $\mathcal{P}(M)_\rho$ is the probability

distribution associated with the measurement of the observable M .

To have an idea of the power of this criterion, we apply it to the states given by (32). The expression for the Tsallis entropy is then

$$H_q^{(T)}(M)_\rho = \frac{6^{-q}(6^q - 5(1-p)^q - (1+5p)^q)}{q-1}. \quad (74)$$

However, it is more useful to use the Rényi entropy as the violations of (72) become smaller and smaller as q increases until it becomes impossible to say clearly whether the inequality (72) is violated or not due to finite accuracy. Table I shows the minimal p from which onwards entanglement is detected for various values of q . The limit $q \rightarrow \infty$ gives the best result, namely entanglement detection for $p > 0.4$ (thus all the entangled states in the family (32) are detected) and the violation of the inequality (73) as a function of p is very clear.

TABLE I: Table of minimal values of p for which entanglement is detected in eq.(32).

q	2	4	6	8	10	20	50	∞
min p	0.632	0.513	0.473	0.454	0.443	0.421	0.408	0.4

For the family of states (34) the limit $q \rightarrow \infty$ gives entanglement detection for $p > 0.5$, so entanglement is detected within the total range of p -values corresponding to entangled states.

VI. CONCLUSIONS

We derived a separability criterion for states (pure or mixed) of two fermions with a single particle Hilbert space of dimension 4 ($s = \frac{3}{2}$) based on the violation of local uncertainty relations. The violation of these relations provides a means of entanglement detection and also a quantitative indicator of entanglement. Then we developed separability criteria for fermion systems using sums of variances of appropriate projector operators. We implemented this type of criteria for two-fermions systems with four dimensional ($s = \frac{3}{2}$) and six dimensional ($s = \frac{5}{2}$) single-particle Hilbert spaces, and for systems of three identical fermions with a six dimensional single-particle Hilbert space. In the latter two instances no analytical, closed expression for the concurrence, nor a necessary and sufficient separability criterion exists.

Therefore, the present criteria for entanglement detection provide in these cases a valuable tool for the identification of families of entangled mixed states. The violation of entropic uncertainty relations was then investigated as a means of entanglement detection. Criteria were established using the Shannon, Tsallis and Rényi entropies. We provided illustrations of families of entangled states that are detected by our separability criteria.

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