A Monotone Scheme for Hamilton-Jacobi Equations via the Nonstandard Finite Difference Method

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Abstract

A usual way of approximating Hamilton-Jacobi equations is to couple space finite element discretization with time finite difference discretization. This classical approach leads to a severe restriction on the time step size for the scheme to be monotone. In this paper, we couple the finite element method with the nonstandard finite difference method, which is based on the Mickens' rule of nonlocal approximation. The scheme obtained in this way is unconditionally monotone. The convergence of the new method is discussed and numerical results that support the theory are provided.

Keywords: Nonstandard finite difference method, Hamilton-Jacobi equation, monotone scheme

Mathematics Subject Classification: 65M06, 65M60, 70H20

1 Introduction

The general setting of this work is the Hamilton-Jacobi equation in the form

$$u_t + H(\nabla u) = 0 \tag{1}$$

$$u(0, x) = u_0(x). \tag{2}$$

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For simplicity we consider the problem (1)–(2) in two space dimensions although a generalization to arbitrary space dimension is possible. The Hamilton-Jacobi equation is one of the fundamental equations in mechanics. It is used for instance in generating motion and it is equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics (see for instance [11]). Applications of this equation also occur in Optimal Control Theory, specifically as Hamilton-Jacobi-Bellman equation, [5].

Problem (1)–(2) does not have classical solutions. Various kind of generalized solutions have been considered, [10]. Here, we consider its viscosity solution, which under the conditions $H \in C^{0,1}(\mathbb{R}^2)$ and $u^0 \in C^{0,1}(\mathbb{R}^2)$ that we assume henceforth, is the uniform limit as $\varepsilon \to 0^+$ of the (classical) solution of the following regularized problem:

$$u_t(t,x) + H(\nabla u(t,x)) - \varepsilon \Delta u(t,x) = 0 \ , \quad t \in (0,\infty), \ z \in \mathbb{R}^2. \tag{3}$$

The notation $C^{0,1}(\mathbb{R}^2)$ stands for the space of Lipschitz continuous functions on $\mathbb{R}^2$. For the precise definition of viscosity solutions and their existence and uniqueness, we refer the reader to [9, 10, 5, 8].

It is well known, see e.g. [9], that the viscosity solution of (1)–(2) depends monotonically on the initial value; that is, for any two solutions u and w, we have

$$u(0,x) \leq w(0,x) \implies u(t,x) \leq w(t,x). \tag{4}$$

The property (4) is important from the physical point of view. The purpose of this work is to design monotone numerical schemes; that is, those that replicate this property. Our general approach is along the lines of the many works, and specifically [14], where the finite element space discretization is coupled with the finite difference time discretization. However, we use the Mickens' nonstandard variant of the finite difference approach, known as the nonstandard finite difference method, [15]. The schemes employing standard finite difference techniques are monotone under restrictive conditions on the time step size. On the contrary, the nonstandard finite difference scheme presented in this work preserves the monotonicity property unconditionally, improving therefore the results of [14]. More generally, as demonstrated for a wide range of problems, [16, 18], the nonstandard finite difference method has the potential to replicate physical properties of the exact solution in the sense of the following definition of qualitative stability [2].

Definition 1 *Assume that the solution of (1)–(2) satisfies some property (P). A numerical method approximating (1)–(2) is called qualitatively stable with respect to*

(P) if the numerical solutions satisfy property (P) for all values of the involved step sizes.

The rest of the work is organized as follows. In the next section we consider a space discretization of equation (3) using the finite element method, while Section 3 is devoted to a nonstandard finite difference scheme for the obtained system of differential equations. The convergence of this new scheme is proved in Section 4. Numerical results supporting the theory are presented in Section 5. Concluding remark and our future research plan are given in the last section.

2 Finite element space discretization

In this section we refer essentially to [14]. Standard concepts and notation on the finite element method can also be found in [7]. Let $\mathcal{T}_h$ be a triangulation of $\mathbb{R}^2$ consisting of a countable set of triangles which satisfy the usual compatibility conditions. The generic triangle of $\mathcal{T}_h$ is denoted by T, h_T is the diameter of T, $h = \sup\limits_{T\in\mathcal{T}_h} h_T$ and ρ_T is the diameter of the largest ball in T. The triangulation is assumed to be regular, that is, there exists a constant c, which is independent of h and such that we have $\frac{h_T}{\rho_T} \leq c$ for all T in $\mathcal{T}_h$. Let $\{X_i : i = 1, 2, ...\}$ be the set of nodes on $\mathcal{T}_h$. The edge connecting X_i and X_k is denoted by E_{ik}. For any node X_i, we denote by N_i the index set of its neighbor vertexes (vertexes connected to X_i by an edge), while I_i is the index set of the triangles with common vertex X_i: a triangle corresponding to an index $k \in I_i$ is denoted by T_k. With each node X_i we associate the basis function ϕ_i defined as a continuous piecewise linear function on $\mathbb{R}^2$ such that $\phi_i(X_i) = 1$ and $\phi_i(X_k) = 0$, $k \neq i$. Note that ϕ_i has "small" support in the sense that $\text{supp}\phi_i = V_i = \bigcup\limits_{k\in I_i} T_k$. We denote by $\mathcal{V}_h$ the finite element space which is spanned by the basis functions $(\phi_i)_i$.

An approximation $v_h(t, x)$ to the solution of (3) is sought such that $v_h(t, \cdot) \in \mathcal{V}_h$, i.e. $v_h(t, x) = \sum\limits_{i=1}^{\infty} v_{h,i}(t)\phi_i(x)$, where v_h satisfies the variational equation

$$\frac{d}{dt}\iint_{\mathbb{R}^2} v_h w dx_1 dx_2 + \iint_{\mathbb{R}^2} H(\nabla v_h) w dx_1 dx_2 = -\varepsilon \iint_{\mathbb{R}^2} \nabla v_h \nabla w dx_1 dx_2 \qquad (5)$$

for all functions $w \in \mathcal{V}_h$. In the sequel, $v_{h,i}$ is abbreviated to v_i wherever this does not lead to confusion. Replacing the test functions w by the basis functions ϕ_i,

$i = 1, 2, ...$, and approximating the integral in the first term by the "mass lumping" quadrature we obtain

$$\frac{d}{dt}v_i(t)\iint_{V_i}\phi_i dx_1 dx_2 + \iint_{V_i} H(\nabla v_h)\phi_i dx_1 dx_2 = -\varepsilon \iint_{V_i} \nabla v_h \nabla \phi_i dx_1 dx_2 \ .$$

After some standard technical manipulations, see e.g. [14], the above equation can be written in the following equivalent form

$$\frac{d}{dt}v_i = -\sum_{j\in I_i} H\left(\nabla v_h|_{T_j}\right)\eta_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k - v_i), \tag{6}$$

where μ denotes the area and

$$\begin{aligned} \eta_{ij} &= \frac{\mu(T_j)}{\mu(V_i)} \ , \quad j \in I_i \\ a_{ik} &= \frac{1}{2}(\cot\theta_{ik}^{(1)} + \cot\theta_{ik}^{(2)}) \ , \quad k \in N_i, \end{aligned}$$

$\theta_{ik}^{(1)}$ and $\theta_{ik}^{(2)}$ being the angles opposite the edge E_{ik} in the two adjacent triangles which contain E_{ik}. For the monotonicity of the scheme discussed in the next section the coefficients a_{ik} need to be bounded away from zero, that is, there exists a positive constant c_0 such that we have

$$a_{ik} \geq c_0 \ , \quad i = 1, 2, ... \text{ and } k \in N_i. \tag{7}$$

It was shown in [14] that if the triangulation is such that there exists a constant c_1, $(0 < c_1 < \frac{\pi}{2})$, independent of h such that

$$\theta_{ik}^{(1)} + \theta_{ik}^{(2)} \leq \pi - c_1 \tag{8}$$

for every edge E_{ik} on $\mathcal{T}_h$, then property (7) holds.

3 A nonstandard finite difference scheme

We consider a mesh $\{t_0, t_1, ...\}$ in the time dimension with a step size Δt, that is, we have $t_n = n\Delta t$. As usual $v^n = (v_i^n)$ denotes the approximation of the solution of (6) at $t = t_n$. Our aim in this section is to design a scheme for (6) that is qualitatively stable with respect to the monotonicity on initial values. That is

$$v_i^0 \leq w_i^0 \Longrightarrow v_i^n \leq w_i^n \ , \tag{9}$$

whenever v^n and w^n are discrete solutions initiated at v^0 and w^0, respectively.

For simplicity, we ignore for the moment the space index i and assume that we are dealing with a scalar problem the discrete solution of which is given by an explicit scheme of the form

$$v^{n+1} = g(\Delta t; v^n). \tag{10}$$

The following result is proved in [3]:

Theorem 2 *The difference scheme (10) is qualitatively stable with respect to the monotonicity on initial values if and only if*

$$\frac{\partial v^{n+1}}{\partial v^n} \equiv \frac{\partial g}{\partial v}(\Delta t; v) \geq 0 \text{ for } \Delta t > 0 \text{ and } v \in \mathbb{R}. \tag{11}$$

Since we are reduced to checking the positivity condition (11), we will in what follows adapt and exploit the favorable situation described in the following theorem:

Theorem 3 *Let w be the solution of the problem*

$$Lw = f(w)$$

where L is either the differential operator $Lz = z'$ or the identity operator $Lz = z$. Assume that the solution w is nonnegative and that the function f admits the decomposition

$$f(z) = p(z) - q(z)z \tag{12}$$

where $p(z) \geq 0$ and $q(z) \geq 0$. Then the difference scheme

$$\frac{w^{n+1} - w^n}{\Delta t} = p(w^n) - q(w^n)w^{n+1} \tag{13}$$

for $Lz = z'$ or

$$w^{n+1} = p(w^n) - q(w^n)w^{n+1} \tag{14}$$

for $Lz = z$ is qualitatively stable with respect to the positivity property of the solution w.

Proof. Obvious by re-writing (13) and (14) as

$$w^{n+1} = \frac{w^n + \Delta t p(w^n)}{1 + \Delta t q(w^n)},$$

and

$$w^{n+1} = \frac{p(w^n)}{1 + q(w^n)},$$

respectively. ∎

Remark 4 *The situation described in Theorem 3 was introduced in a more specific form by the authors in [1] in order to design schemes that preserve the positivity property of the solutions of reaction diffusion equations. The idea is also exploited for the approximation of differential models in population biology and mathematical epidemiology where the positivity of the the involved species is essential (see, for instance, [12, 17, 18]). The underlying point of these schemes is, as it can be seen from (13) and (14), that one of Mickens' rules of constructing nonstandard finite difference schemes is reinforced: the nonlinear term* $q(w)w$ *is approximated in a nonlocal way i.e. by* $q(w^n)w^{n+1}$ *and not by* $q(w^n)w^n$ *or* $q(w^{n+1})w^{n+1}$.

Coming back to the problem (6), we will use the nonstandard finite difference method, which can be defined as follows [2]:

Definition 5 *A finite difference scheme for (6) is nonstandard if it involves at least one of the following conditions:*

- *In the discrete derivative, the traditional denominator* Δt *is replaced by a nonnegative function* ψ *such that*

$$\psi(\Delta t) = \Delta t + O(\Delta t^2); \tag{15}$$

- *Nonlinear terms are approximated in a nonlocal way.*

In view of the form of the right-hand side of (6) and of Theorem 2, which requires to show the positivity condition $\dfrac{\partial v_i^{n+1}}{\partial v_i^n} \geq 0$, we propose, in the spirit of the nonlocal approximation in Theorem 3, the following nonstandard finite difference scheme for the system of equations (6):

$$v_i^{n+1} = v_i^n - \Delta t \sum_{j \in I_i} H\left(\nabla v_h^n|_{T_j}\right) \eta_{ij} + \frac{3\varepsilon \Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(v_k^n - v_i^{n+1}). \tag{16}$$

Observe that the second sum in (6) is approximated in a nonlocal way.

Theorem 6 *Let the triangulation* $\mathcal{T}_h$ *be regular and satisfy the condition (7). Let* $H \in C^{0,1}(\mathbb{R}^2)$. *Then there exists a constant* c_3 *independent of* Δt *and* h *such that if* $\varepsilon \geq c_3 h$ *the scheme (16) is monotone.*

Proof. Let $m \in N_i$ and let $X_{m'}$ and $X_{m''}$ be the nodes opposite to E_{im} in the two adjacent triangles T' and T'' containing E_{im}. We have

$$\begin{aligned} \frac{\partial v_i^{n+1}}{\partial v_m^n} &= -\Delta t\left[\nabla H\cdot\nabla\phi_m|_{T'}\frac{\mu(T')}{\mu(V_i)}+\nabla H\cdot\nabla\phi_m|_{T''}\frac{\mu(T'')}{\mu(V_i)}\right] \\ &\quad +\frac{3\varepsilon\Delta t}{\mu(V_i)}a_{im}-\frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i}a_{ik}\frac{\partial v_i^{n+1}}{\partial v_m^n} \end{aligned}$$

Hence

$$\begin{aligned} \left(1+\frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i}a_{ik}\right)\frac{\partial v_i^{n+1}}{\partial v_m^n} &\geq \frac{\Delta t}{\mu(V_i)}\left[3\varepsilon a_{im}-\frac{1}{2}|H|_{1,\infty}(|E_{im'}|+|E_{im''|})\right] \\ &\geq \frac{\Delta t}{\mu(V_i)}\left[3\varepsilon c_0-h|H|_{1,\infty}\right], \end{aligned}$$

where $|H|_{1,\infty}=\sup||\nabla H||_2$. Setting $c_3=\frac{|H|_{1,\infty}}{3c_0}$ we obtain $\frac{\partial v_i^{n+1}}{\partial v_m^n}\geq 0$ whenever $\varepsilon\geq c_3 h$.

In a similar way

$$\frac{\partial v_i^{n+1}}{\partial v_i^n}=1-\Delta t\sum_{j\in I_i}\nabla H\cdot\nabla\phi_i|_{T_j}\frac{\mu(T_j)}{\mu(V_i)}-\frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i}a_{ik}\frac{\partial v_i^{n+1}}{\partial v_i^n}.$$

Hence

$$\left(1+\frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i}a_{ik}\right)\frac{\partial v_i^{n+1}}{\partial v_i^n}=1-\frac{\Delta t}{\mu(V_i)}\nabla H\cdot\sum_{j\in I_i}\nabla\phi_i|_{T_j}\mu(T_j). \tag{17}$$

It is easy to see that in any triangulation $\mathcal{T}_h$ we have

$$\sum_{j\in I_i}\nabla\phi_i|_{T_j}\mu(T_j)=0. \tag{18}$$

Indeed, for any constant vector in $z=(z_1,z_2)\in\mathbb{R}^2$ we have

$$z\sum_{j\in I_i}\nabla\phi_i|_{T_j}\mu(T_j)=\iint_{V_i}z\nabla\phi_i=-\iint_{V_i}\nabla z\phi_i dx_1dx_2=0,$$

which implies (18). Substituting (18) in (17) we obtain $\frac{\partial v_i^{n+1}}{\partial v_i^n}>0$. This completes the proof. ■

Remark 7 *With the notation of the proof of Theorem 6 in mind, we assume that the second term in the right hand side of (16) is increasing with respect to* v_m^n*. In this case, Theorem 6 is a straightforward consequence of Theorem 3 and the monotonicity of the scheme (16) occurs then without the relation* $\varepsilon \geq c_3 h$*. This relation is essential in the general setting of Theorem 6 where the monotonicity of the mentioned term cannot be monitored.*

Mickens' rule of nonlocal approximation is normally applied to nonlinear terms, see [15], [3]. Here we apply it to a linear term. Usually, the two conditions in Definition 5 are considered independently. It is interesting that in our case the scheme formulated in (16) through nonlocal approximations admits an equivalent formulation using a renormalization of the denominator of the discrete derivative. More precisely, (16) is equivalent to

$$\frac{v_i^{n+1} - v_i^n}{\psi_i(\Delta t)} = -\sum_{j \in I_i} H\left(\nabla v_h^n|_{T_j}\right)\eta_{ij} + \frac{3\varepsilon}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(v_k^n - v_i^n), \tag{19}$$

where

$$\psi_i(\Delta t) = \frac{\Delta t}{1 + \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum\limits_{k \in N_i} a_{ik}}$$

has the asymptotic behavior stated in (15).

Remark 8 *The more complex denominator function* $\psi_i(\Delta t)$ *captures the intrinsic property of the solution of the problem (6) of being monotone dependent on initial values under the condition* $\varepsilon \geq c_3 h$ *stated in Theorem 6. It would be interesting to investigate, along the lines of the methodology of the nonstandard approach (see [18]), whether there are other physical properties of (6) that are captured by* $\psi_i(\Delta t)$*.*

Remark 9 *At every* $t = t_n$ *the solution of the problem (1)-(2) is approximated by the function* $v_h^n = \sum\limits_{i=1}^{\infty} v_i^n \phi_i(x) \in \mathcal{V}_h$*. Hence the scheme (16) can equivalently be considered as a mapping* $G(\Delta t, \cdot)$ *from* $\mathcal{V}_h$ *into* $\mathcal{V}_h$ *such that* $v_h^{n+1} = G(v_h^n)$*. Due to the explicit formulation (19) of (16), the mapping* G *can also be given in an explicit form. More precisely, for any* $w_h = \sum\limits_{i=1}^{\infty} w_i \phi_i \in \mathcal{V}_h$*, we have*

$$G(\Delta t, w_h) = \sum_{i=1}^{\infty} \alpha_i \phi_i(x), \tag{20}$$

$$\alpha_i = w_i - \psi_i(\Delta t) \sum_{j \in I_i} H\left(\nabla w_h|_{T_j}\right)\eta_{ij} + \frac{3\varepsilon}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(w_k - w_i), \;\; i = 1, 2, ... \tag{21}$$

It is clear that under the conditions in Theorem 6 the mapping $G(\Delta t, \cdot)$ is monotone with respect to the usual pointwise partial order on $\mathcal{V}_h$.

Remark 10 *Numerical schemes using the standard finite difference method are typically monotone only under a restriction on the time step size. This might be a disadvantage in applications. For example in [14] the restriction is*

$$\Delta t \leq C \frac{\min_j \mu(T_j)}{\varepsilon}.$$

Since the bound of Δt involves the size of the smallest triangle in the triangulation the above inequality implies that even when the triangulation is refined only locally Δt need to be adjusted as well. Through the nonstandard approach the scheme (16) is monotone for any time step size.

4 Convergence

The convergence of the scheme is obtained through an abstract convergence result of Barles and Souganidis, [6], which is detailed in its consequences in [13] for the equation (1) as stated below. The function spaces and notations are defined in these references.

For $\rho > 0$ let a mapping $S(\rho) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ be given. The following conditions are considered in connection with the mapping $S(\rho)$:

- monotonicity, i.e., $u \leq w$ implies $S(\rho)u \leq S(\rho)w$, (22)
- invariance under translation, i.e., $S(\rho)(u + z) = S(\rho)u + z, \ z \in \mathbb{R}^2$, (23)
- consistency, i.e., $\frac{\varphi - S(\rho)\varphi}{\rho} \to H(\nabla\varphi)$ as $\rho \to 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ (24)
- rate of approximation:
  $$\left| \frac{\varphi - S(\rho)\varphi}{\rho} - H(\nabla\varphi) \right| = O(\rho(|\varphi|_{1,\infty} + |\varphi|_{2,\infty})) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2) \quad (25)$$

An approximation $u_{\Delta t}$ to the solution of (1)-(2) is constructed by using a grid $t_n = n\Delta t$ in time as follows:

$$u_{\Delta t}(t, x) = \begin{cases} u_0(x) & \text{if} \quad t = 0 \\ S(t - t_n)u_{\Delta t}(\mathbf{t}_n, \cdot)(x) & \text{if} \quad t \in (t_n, t_{n+1}], \ n = 0, 1, ... \end{cases} \quad (26)$$

Theorem 11 *If a mapping $S(\rho)$ satisfies (22), (23) and (24) and $H \in C^{0,1}(\mathbb{R}^2)$, $u_0 \in C^{0,1}(\mathbb{R}^2)$ then for any $\bar{t} > 0$ we have $u_{\Delta t} \to u$ uniformly on $[0, \bar{t}] \times \mathbb{R}^2$ as $\Delta t \to 0$. Furthermore, if $S(\rho)$ satisfies also (25) then there exists a positive constant C independent of Δt such that*

$$||u_{\Delta t} - u||_\infty \le C \Delta t^{\frac{1}{2}} \ .$$

Let $\mathcal{I}_h$ denote the piece-wise interpolation operator at the nodes of the triangulation $\mathcal{T}_h$, that is, for any real function φ on $\mathbb{R}^2$ the function $\mathcal{I}_h\varphi$ is linear on any triangle $T \in \mathcal{T}_h$ and $(\mathcal{I}_h\varphi)(X_i) = \varphi(X_i)$ at every node X_i of $\mathcal{T}_h$. Note that we have $\mathcal{I}_h\varphi \in \mathcal{V}_h$. We consider the mapping $S(\rho) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ defined as a composition of the operator $\mathcal{I}_h$ and the scheme (16). More precisely using the mapping G given in (20)-(21) we have

$$S(\rho)\varphi = G(\rho, \mathcal{I}_h\varphi). \tag{27}$$

Then the numerical scheme (16) is equivalent to the scheme (26) where the numerical solution is evaluated only at the points of the mesh. Therefore the convergence of the scheme (16) can be obtained through Theorem 11, where the mapping $S(\rho)$ is given by (27). To this end we only need to verify the conditions (22)-(25) for $S(\rho)$ given by (27). The essential property of the monotonicity of $S(\rho)$ follows from the monotonicity of G and $\mathcal{I}_h$. The condition (23) follows trivially from the form (20)-(21) of the mapping G. Now we consider condition (24) and its stronger form (25). Using standard techniques as in [13] and [14] one can show that if $\varepsilon = O(h)$ then

$$\left| \frac{\varphi - S(\rho)\varphi}{\rho} - H(\nabla\varphi) \right| = O(h|\varphi|_{2,\infty} + \rho). \tag{28}$$

For convergence we assume that both Δt and h approach zero. Hence the consistency condition (24) follows from (28). Moreover, if we assume that $\Delta t = O(h)$ and $h = O(\Delta t)$ the estimate (28) implies (25). Hence we have the following convergence result.

Theorem 12 *Let the family of triangulations $(\mathcal{T}_h)$ be regular and satisfy the condition (7). Let $H \in C^{0,1}(\mathbb{R}^2)$ and $u_0 \in C^{0,1}(\mathbb{R}^2)$. Then the numerical solution v_h^n obtained by (16) with $\varepsilon \ge c_3 h$ and $\varepsilon = O(h)$ converges to the exact solution u of the problem (1)-(2), i.e., for any $\bar{t} > 0$ we have*

$$\sup_{i, n \le \bar{t}/\Delta t} |u(t_n, X_i) - v_i^n| \to 0 \ \textit{as}\ \Delta t \to 0,\ h \to 0.$$

Moreover, if $0 < \inf \frac{\Delta t}{h} \leq \sup \frac{\Delta t}{h} < \infty$ *there exists a constant* C *such that*

$$\sup_{i,n \leq \bar{t}/\Delta t} |u(t_n, X_i) - v_i^n| \leq Ch^{\frac{1}{2}}.$$

Remark 13 *As a follow up to Remark 10 and in view of the other parameters* ε *and* h *that are involved in the previous results, it might be useful to comment further on how our nonstandard finite difference scheme (16) replicates the monotonicity property without any restriction on the time step size* Δt*. Notice that* $\min_j \mu(T_j) = O(h)$ *for small* h*. Thus, if in the classical approach* ε*is assumed to be equivalent to* h *as in Theorem 12, then the restrictive nature of the condition on* Δt *in Remark 10 appears further even for small values of* ε*. Equally, while the nonstandard scheme (16) is* Δt*-unconditionally convergent, the additional equivalence between* h *and* Δt *is only required to guarantee its rate of convergence* $O(h^{\frac{1}{2}})$*.*

5 Numerical results

As an example we consider the following problem which is often used in testing numerical methods, [19], [14]:

$$u_t - \sqrt{u_x^2 + u_y^2 + 1} = 0 \quad , \quad (x, y) \in (0,1) \times (0,1) \ , \ t > 0 \tag{29}$$

$$u(0, x, y) = \cos(2\pi y) - \cos(2\pi x) \tag{30}$$

with periodic boundary conditions. We use a triangulation with 6240 elements which satisfies condition (8). The numerical solution obtained with $\varepsilon = 0.01$ and $\Delta t = 0.01$ is presented on Figure 1. For comparison we consider the standard Euler scheme for the equation (6) which is monotone only for sufficiently small values of Δt, see [14] for details. The numerical solution obtained for the same value of the parameters, that is, $\varepsilon = 0.01$ and $\Delta t = 0.01$, is presented on Figure 2. The advantage of the considered nonstandard method with regard to preserving the qualitative behavior of the solution is apparent.

6 Conclusion

This work is motivated by the paper [14] where a severe restriction on the time step size is imposed for the numerical scheme for Hamilton-Jacobi equations obtained through the coupling of the finite element method (in space) and the finite difference

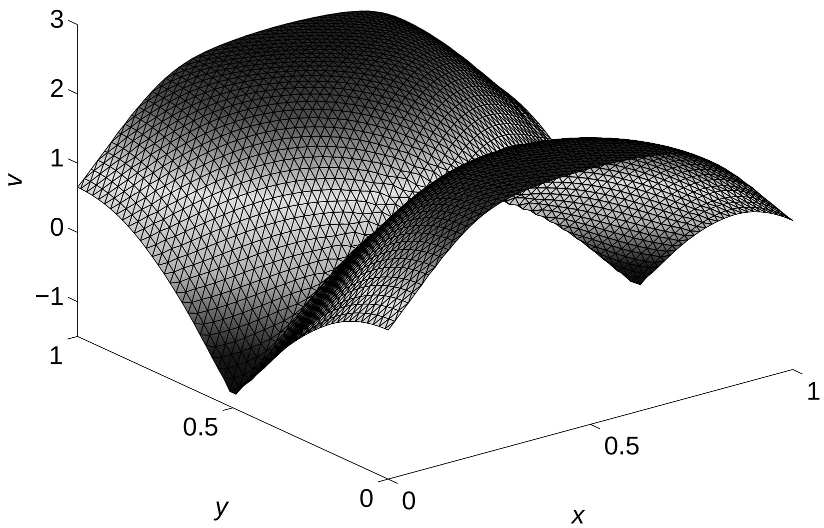

Figure 1: Numerical solution of (29)–(30) using the nonstandard method (16).

method (in time) to be monotone. We have relaxed this restriction by using Mickens' nonstandard finite difference method, [15]. More precisely, Mickens' rule of nonlocal approximation is exploited and this leads to a nonstandard scheme that replicates the monotonicity property of the Hamilton-Jacobi equations for all positive step sizes. Furthermore, the superiority of the nonstandard method to the standard one is confirmed by numerical results.

In previous works, [3, 4], it was proved that the monotonicity of nonstandard schemes is essential for their elementary stability. Our plan for future research is to investigate whether a similar result holds for the Hamilton-Jacobi equations.

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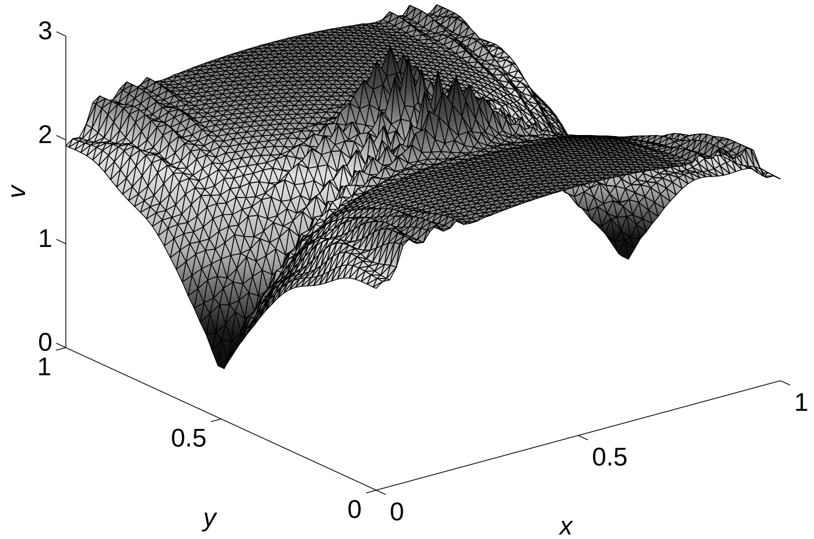

Figure 2: Numerical solution of (29)–(30) using the standard Euler time discretization.

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