Vibration of a Reissner–Mindlin–Timoshenko plate–beam system

A. Labuschagne [a,*], N.F.J. van Rensburg [a], A.J. van der Merwe [b]

[a] Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa

[b] Department of Mechanical Engineering, Cape Peninsula University of Technology, Box 652, Cape Town 8000, South Africa

ABSTRACT

In this paper, we consider a plate–beam system in which the Reissner–Mindlin plate model is combined with the Timoshenko beam model. Natural frequencies and vibration modes for the system are calculated using the finite element method. The interface conditions at the contact between the plate and beams are discussed in some detail. The impact of regularity on the enforcement of certain interface conditions is an important feature of the paper.

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1. Introduction

In applications, structures consisting of linked systems of elastic bodies are encountered. The modelling and control of such systems are clearly of great practical importance, as pointed out in early contributions [1–4].

Our concern in this paper is systems where beams are connected to plates. Examples of recent publications are [5–9]. In all these papers, classical plate and beam theories are used. As pointed out in the conclusion of [7], it is an ongoing process to find better models. The aim is to determine a sufficiently accurate model for a given application, without it being too complex.

The limitations of the Kirchhoff and Euler–Bernoulli theories are well known – even if rotary inertia is included – and plate–beam models involving improved theories need to be considered. Combining the Reissner–Mindlin plate model and the Timoshenko beam model can be seen as a first step towards a better model, while still avoiding the complexities (not to mention computational effort) of a fully three dimensional model.

A Reissner–Mindlin–Timoshenko (**RMT**) plate–beam system is extremely complex compared to a Kirchhoff–Euler–Bernoulli (**KEB**) system. This is due to the presence of five partial differential equations instead of two (for a single beam) and the intricate geometrical constraints at the interfaces. Implementation of the Finite Element Method poses a number of difficulties not present in the case of a plate–beam system using classical theories. The first difficulty concerns decisions to enforce certain interface conditions. Secondly, the assembly of mass and stiffness matrices is more involved.

In [7] a KEB plate–beam system is investigated, and it is shown that introducing rotary inertia into the model does not cause significant change in the eigenvalues. An initial aim of this paper is to compare the eigenvalues of the RMT plate–beam system with those of the KEB plate–beam system, to determine the influence of shear.

Secondly, we investigate the effect of the regularity of the solution (and test functions) on the finite element calculations and results. We compare different options for dealing with the forced interface conditions.

Finally, we compare the natural frequencies of an RMT plate–beam system as the thickness of the supporting beams is increased, with the natural frequencies of a rigidly supported Reissner–Mindlin plate. In [7] a similar comparison is done for the KEB plate–beam system.

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* Corresponding author.
E-mail address: anneke.labuschagne@up.ac.za (A. Labuschagne).

Section 2 contains the RMT plate–beam model. A dimensionless form of the model is derived in Section 3. In Section 4 we show how the variational form of the model is obtained directly from the different equations of motion. This method for obtaining the variational form deals efficiently with the boundary conditions and the interface conditions. Also in Section 4, we discuss the regularity of the solution or, rather, the possible lack of regularity. Implementation of the forced interface conditions is postponed to the discrete version of the problem, where different options are considered. The associated eigenvalue problem and its Galerkin approximation are derived in Section 5 and options for dealing with the interface conditions are presented. Numerical results are presented in Section 6.

2. Mathematical model

2.1. The Reissner–Mindlin model for a plate

Consider small transverse vibration of a uniform plate with thickness h and density ρ. For a right hand system of unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$, the reference configuration for the plate is a domain Ω in the $\boldsymbol{e}_1\boldsymbol{e}_2$-plane. The transverse displacement of $\boldsymbol{x}$ at time t is denoted by $w(\boldsymbol{x}, t)\,\boldsymbol{e}_3$.

In the Reissner–Mindlin model, the transverse line segment at $\boldsymbol{x}$ is free to rotate. The angle between the line segment ("material line") and the perpendicular to the plane is denoted by $\psi(\boldsymbol{x}, t)$. The angle between the projection of the material line in the plane and the unit vector $\boldsymbol{e}_1$ is $\phi(\boldsymbol{x}, t)$ (see [10, Sec. 3.2, Sec. 3.5]). Consequently the orientation of the line segment is given by the vector

$$\sin\psi\cos\phi\,\boldsymbol{e}_1 + \sin\psi\sin\phi\,\boldsymbol{e}_2 + \cos\psi\,\boldsymbol{e}_3 \approx \psi\cos\phi\,\boldsymbol{e}_1 + \psi\sin\phi\,\boldsymbol{e}_2 + \boldsymbol{e}_3.$$

This linear approximation is used to derive the equations of motion (see [11] and [10, p. 152]). We use the notation

$$\boldsymbol{\psi} = \psi_1\,\boldsymbol{e}_1 + \psi_2\,\boldsymbol{e}_2 = \psi\cos\phi\,\boldsymbol{e}_1 + \psi\sin\phi\,\boldsymbol{e}_2.$$

Equations of motion

$$\rho h \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{1}$$

$$\rho I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} \boldsymbol{M} - \boldsymbol{Q}, \tag{2}$$

where $I = h^3/12$ is the length moment of inertia. The vector $\boldsymbol{Q}$ represents a shear force density (force per unit length) and q the transverse external load. The tensor (matrix) $\boldsymbol{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, represents moment densities (moment per unit length) and div $\boldsymbol{M}$ is a vector with components

$$[\operatorname{div} \boldsymbol{M}]_i = \partial_1 M_{i1} + \partial_2 M_{i2} \quad \text{for } i = 1 \text{ and } 2.$$

Constitutive equations

Hooke's law is used, as well as the assumptions that the vectors $\boldsymbol{\psi}$ and ∇w are small (see [10, p. 61], [11]).

$$\boldsymbol{Q} = \kappa^2 Gh(\nabla w + \boldsymbol{\psi}), \tag{3}$$

$$\boldsymbol{M} = \frac{1}{2} D \begin{bmatrix} 2(\partial_1\psi_1 + \nu\partial_2\psi_2) & (1-\nu)(\partial_1\psi_2 + \partial_2\psi_1) \\ (1-\nu)(\partial_1\psi_2 + \partial_2\psi_1) & 2(\partial_2\psi_2 + \nu\partial_1\psi_1) \end{bmatrix}, \tag{4}$$

where G is the shear modulus and κ^2 a correction factor. D is a measure of stiffness for the plate and is given by $D = EI/(1-\nu^2)$, where E is Young's modulus and ν Poisson's ratio. The value of κ^2 depends on ν and ranges almost linearly from 0.76 to 0.91 as ν increases from 0 to 0.5. (See [11], where a detailed explanation is given. It is also mentioned that Reissner used $\kappa^2 = 5/6$.)

The equations of motion and the constitutive equations, above, are known as the *Reissner–Mindlin plate model*. The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [10, p. 152] and [11]). In our approach, these partial differential equations are not used.

2.2. The Timoshenko model for a beam

Consider small transverse vibration of a uniform beam with density ρ_b and cross sectional area A. The reference configuration for the beam is the interval $[0, \ell]$ on the real line. The transverse displacement of x at time t is denoted by $w_b(x, t)$ and the angle due to rotation of a cross section by $\phi_b(x, t)$.

Equations of motion

$$\rho_b A \partial_t^2 w_b = \partial_x V + P, \tag{5}$$

$$\rho_b I_b \partial_t^2 \phi_b = \partial_x M_b + V + L, \tag{6}$$

where I_b denotes the area moment of inertia, V the shear force and M_b the bending moment. (See [12, p. 337–9], [13, p. 322–3] and [14, p. 336–8] for more detail.) External loads are a force density P and a moment density L. In the plate–beam model, L and P are due to interaction with the plate.
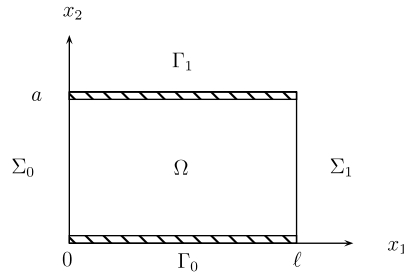
Fig. 1. Reference configuration of the plate–beam system.

Constitutive equations

Hooke's law is used, as well as the assumptions that $\partial_x w_b$ and ϕ_b are small.

$$V = \kappa_b^2 G_b A(\partial_x w_b - \phi_b), \tag{7}$$

$$M_b = E_b I_b \partial_x \phi_b, \tag{8}$$

where E_b denotes Young's modulus, G_b the shear modulus and κ_b^2 is a correction factor which depends on the shape of the cross section. The values of κ_b^2 range between 0.5 and 1 (see [15] or [16, p. 173]).

The equations of motion and the constitutive equations above are known as the *Timoshenko beam model*. The constitutive equations may be substituted into the equations of motion, leading to a system of two partial differential equations.

2.3. The RMT plate–beam system

Consider small transverse vibration of a thin rectangular plate supported by identical beams at two opposing sides and rigidly supported at the remaining sides. The beams are supported at their endpoints. Assume, furthermore, the case of free vibration, i.e. $q = 0$. The displacement for the system is measured with respect to the equilibrium state. (Due to gravity, the equilibrium state is not the same as the undeformed state.) It is assumed that the plate remains in contact with the beams and supporting structure at all times.

The reference configuration for the plate is the rectangle Ω, where $0 \leq x_1 \leq \ell$ and $0 \leq x_2 \leq a$. The plate is rigidly supported at $x_1 = 0$ and $x_1 = \ell$. These sections of the boundary of Ω are denoted by Σ_0 and Σ_1 respectively. The plate is supported by beams at $x_2 = 0$ and $x_2 = a$ and these sections are denoted by Γ_0 and Γ_1 respectively. Fig. 1 depicts this reference configuration. The shaded areas denote the beams.

For the mathematical model, we use the Reissner–Mindlin plate theory and the Timoshenko beam theory.

On Ω, the equations of motion (1) and (2) are satisfied, and on Γ_0 and Γ_1, the two sets of equations of motion are given by (5) and (6). In (5) P is a force density and in (6) L represents a moment density, both transmitted from the plate to a beam.

Boundary conditions on Σ_0 and Σ_1

On these sections of the boundary, the conventional homogeneous boundary conditions for a rigidly supported plate are used, i.e.

$$w = 0, \quad \psi_2 = 0 \quad \text{and} \quad \boldsymbol{Mn} \cdot \boldsymbol{n} = 0, \tag{9}$$

where $\boldsymbol{n}$ is the unit exterior normal (see [10, p. 66]). The third condition reduces to $M_{11} = 0$.

Interface conditions on Γ_0 and Γ_1

On Γ_0 and Γ_1 the interaction between the plate and the beams is considered. The interface conditions are given in [17] for a general case. For this special case, they reduce to

$$w_b(\cdot, t) = w(\cdot, 0, t) \quad \text{on } \Gamma_0, \qquad w_b(\cdot, t) = w(\cdot, a, t) \quad \text{on } \Gamma_1, \tag{10}$$

$$\phi_b(\cdot, t) = -\psi_1(\cdot, 0, t) \quad \text{on } \Gamma_0, \qquad \phi_b(\cdot, t) = -\psi_1(\cdot, a, t) \quad \text{on } \Gamma_1. \tag{11}$$

The interface conditions for the force densities and moment densities on Γ_0 and Γ_1 are given by

$$\boldsymbol{Q} \cdot \boldsymbol{n} = -P, \tag{12}$$

$$\boldsymbol{Mn} \cdot \boldsymbol{\tau} = L, \tag{13}$$

$$\boldsymbol{Mn} \cdot \boldsymbol{n} = 0, \tag{14}$$

where $\boldsymbol{\tau}$ is the unit tangent oriented in such a way that Ω is on the left hand side of $\boldsymbol{\tau}$. For a detailed explanation of the moments $\boldsymbol{Mn} \cdot \boldsymbol{n}$ and $\boldsymbol{Mn} \cdot \boldsymbol{\tau}$, see [10, p. 66].

Conditions at the endpoints of Γ_0 and Γ_1

At the endpoints of Γ_0 and Γ_1 we have the obvious boundary conditions for supported beams, namely

$$w_b = 0 \quad \text{and} \quad M_b = 0. \tag{15}$$

Remarks

1. Note the difference in sign convention for measuring the angles ψ_1 and ϕ_b in the plate and beam models.

2. Care should be taken to also incorporate the difference between sign conventions for moments in the plate and beam models. The beam equations for Γ_1 are derived for a beam oriented from left to right. When applying the interface condition (13) on Γ_1, the moment L has to be replaced by $-L$.
3. The angles ψ and ϕ_b do not present physical realities but convenient averages, and it is not obvious what the geometrical constraints should be. The interface conditions (10) and (11) are based on the idea of rigid rotations and conform to standard practices.

3. Dimensionless form

We introduce the dimensionless variables

$$\tau = \frac{t}{t_0}, \qquad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell},$$

where t_0 must still be specified.

The dimensionless reference configuration for the plate is a rectangle Ω where $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq a/\ell$.

Using $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, the dimensionless variables are

$$w^*(\boldsymbol{\xi}, \tau) = \frac{w(\boldsymbol{x}, t)}{\ell}, \qquad \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) = \boldsymbol{\psi}(\boldsymbol{x}, t),$$

$$\boldsymbol{Q}^*(\boldsymbol{\xi}, \tau) = \frac{\boldsymbol{Q}(\boldsymbol{x}, t)}{\ell G \kappa^2}, \qquad \boldsymbol{M}^*(\boldsymbol{\xi}, \tau) = \frac{\boldsymbol{M}(\boldsymbol{x}, t)}{\ell^2 G \kappa^2} \quad \text{and} \quad q^*(\boldsymbol{\xi}, \tau) = \frac{q(\boldsymbol{x}, t)}{G \kappa^2}.$$

The dimensionless constants are given by

$$h_p = \frac{h}{\ell}, \qquad I_p = \frac{h^3}{12\ell^3} \quad \text{and} \quad \beta_p = \frac{\ell^3 G \kappa^2}{EI}.$$

The constant h_p denotes the dimensionless thickness of the plate and I_p the dimensionless length moment of inertia.

We choose $t_0 = \ell \sqrt{\frac{\rho}{G\kappa^2}}$ (for convenience) and use the original notation for the corresponding dimensionless quantities.

The equations of motion for the plate model and constitutive equations in dimensionless form are presented below.

Reissner–Mindlin plate model

$$h_p \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{16}$$

$$I_p \partial_t^2 \boldsymbol{\psi} = \operatorname{div} \boldsymbol{M} - \boldsymbol{Q}, \tag{17}$$

$$\boldsymbol{Q} = h_p(\nabla w + \boldsymbol{\psi}), \tag{18}$$

$$\boldsymbol{M} = \frac{1}{2\beta_p(1 - \nu^2)} \begin{bmatrix} 2(\partial_1 \psi_1 + \nu \partial_2 \psi_2) & (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) \\ (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) & 2(\partial_2 \psi_2 + \nu \partial_1 \psi_1) \end{bmatrix} \tag{19}$$

Classical plate model

For classical plate theory, ψ_i is replaced by $-\partial_i w$ and the constitutive equation for $\boldsymbol{Q}$ is redundant. This is sometimes referred to as the *Kirchhoff plate model*. Generally the rotary inertia term in (17) is ignored.

In addition, to deal with the beam model, set $\xi = x/\ell$ and

$$w_b^*(\xi, \tau) = \frac{w_b(x, t)}{\ell}, \qquad \phi_b^*(\xi, \tau) = \phi_b(x, t), \qquad P^*(\xi, \tau) = \frac{P(x, t)}{\ell G \kappa^2},$$

$$V^*(\xi, \tau) = \frac{V(x, t)}{\ell^2 G \kappa^2}, \qquad M_b^*(\xi, \tau) = \frac{M_b(x, t)}{\ell^3 G \kappa^2} \quad \text{and} \quad L^*(\xi, \tau) = \frac{L(x, t)}{\ell^2 G \kappa^2}.$$

Note that the parameters of the plate are used for the scaling.

Timoshenko beam model

$$\eta_1 \partial_t^2 w_b = \partial_x V + P, \tag{20}$$

$$\eta_1 \partial_t^2 \phi_b = \alpha_b(\partial_x M_b + V + L), \tag{21}$$

$$V = \eta_2(\partial_x w_b - \phi_b), \tag{22}$$

$$\beta_b M_b = \eta_2 \partial_x \phi_b. \tag{23}$$

The dimensionless constants α_b and β_b are

$$\alpha_b = \frac{A\ell^2}{I_b}, \quad \beta_b = \frac{AG_b \kappa_b^2 \ell^2}{E_b I_b}.$$

The constant α_b is subject to significant variation. If r denotes the radius of gyration, we have $\alpha_b = A\ell^2/I_b = \ell^2/r^2$. However, the ratio β_b/α_b does not vary much. It depends on the elastic constants and the shear correction factor κ_b^2 that is determined by the shape of the cross section. Realistic values for β_b/α_b range between $1/6$ and $1/2$.

The two additional dimensionless constants η_1 and η_2 express ratios for the material properties and the geometrical properties of the plate and the beams:

$$\eta_1 = \left(\frac{\rho_b}{\rho}\right)\left(\frac{A}{\ell^2}\right) \quad \text{and} \quad \eta_2 = \left(\frac{G_b}{G}\right)\left(\frac{\kappa_b^2}{\kappa^2}\right)\left(\frac{A}{\ell^2}\right).$$

Euler–Bernoulli beam model

For the classical beam model, ϕ_b is replaced by $\partial_x w_b$, the rotary inertia term in (21) is ignored and the constitutive equation for V is redundant.

The vibration problem for the plate–beam system is given by the following equations.

Problem RMT

Equations of motion for the plate: (16) and (17) on Ω.
Constitutive equations for the plate: (18) and (19) on Ω.
Equations of motion for the beams: (20) and (21) on Γ_0 and Γ_1.
Constitutive equations for the beams: (22) and (23) on Γ_0 and Γ_1.
Boundary conditions: (9) on Σ_0 and Σ_1.
Interface conditions: (10) to (14) on Γ_0 and Γ_1.
Endpoint conditions: (15) at the endpoints of Γ_0 and Γ_1.

Simplified model

A simplified model is obtained when the Kirchhoff plate model and the Euler–Bernoulli beam model are used. Formally, this model problem can be derived from Problem RMT by replacing ψ_i by $-\partial_i w$ and ϕ_b by $\partial_1 w_b$ and ignoring the rotary inertia terms. We refer to this as **Problem KEB** for the purpose of comparison.

4. Variational form of Problem RMT

For any function v,

$$\iint_\Omega (\operatorname{div} \mathbf{Q}) v \, dA = -\iint_\Omega \mathbf{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v \, ds. \tag{24}$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \, \phi_2]^T$ we have

$$\iint_\Omega \operatorname{div} \mathbf{M} \cdot \boldsymbol{\phi} \, dA = -\iint_\Omega \operatorname{tr}(\mathbf{M}\boldsymbol{\Phi}) dA + \int_{\partial\Omega} \mathbf{M}\mathbf{n} \cdot \boldsymbol{\phi} \, ds, \tag{25}$$

where $\boldsymbol{\Phi} = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$ and "tr" denotes the trace of the matrix.

Test functions

Choose spaces of test functions $T_1(\Omega)$, $T_2(\Omega)$ and $T(I)$, with

$$T_1(\Omega) = \left\{ v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\},$$

$$T_2(\Omega) = \left\{ \boldsymbol{\phi} = [\phi_1 \, \phi_2]^T \mid \phi_1, \phi_2 \in C^1(\bar{\Omega}), \phi_2 = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1 \right\},$$

$$T(I) = \{ v \in C^1[0, 1] \mid v(0) = v(1) = 0 \}.$$

Combining the first equation of motion (16) for the plate, with (24) yields that

$$h_p \iint_\Omega \partial_t^2 w v \, dA + \iint_\Omega \mathbf{Q} \cdot \nabla v \, dA - \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v \, ds = 0 \tag{26}$$

for each $v \in T_1(\Omega)$.

It follows from the first equation of motion (20) for the beam, using integration by parts, that

$$\eta_1 \int_0^1 \partial_t^2 w_b v \, dx + \int_0^1 V v' \, dx = \int_0^1 P v \, dx \tag{27}$$

for each $v \in T(I)$. The subscripts "0" and "1" will be used to distinguish between quantities associated with the two different beams.

To accommodate the interface condition (10), choose $v_0 = v(\cdot, 0)$ and $v_1 = v(\cdot, a)$, where a denotes the dimensionless width of the plate. Denote this test space by T_w:

$$T_w = \{[v \, v_0 \, v_1]^T \mid v \in T_1(\Omega), v_0 \in T(I), v_1 \in T(I), v_0 = v(\cdot, 0), v_1 = v(\cdot, a)\}.$$

The fact that $\mathbf{Q} \cdot \mathbf{n} = -P$ on both Γ_0 and Γ_1 (interface condition (12)), results in some cancellations when adding (26) and (27) (for both beams). Therefore,

$$h_p \iint_\Omega \partial_t^2 w v \, dA + \eta_1 \int_0^1 \partial_t^2 w_{b0} v_0 dx + \eta_1 \int_0^1 \partial_t^2 w_{b1} v_1 dx + \iint_\Omega \mathbf{Q} \cdot \nabla v \, dA + \int_0^1 V_0 v_0' dx + \int_0^1 V_1 v_1' dx = 0$$

for each $[v \, v_0 \, v_1]^{\mathrm{T}} \in T_w$.

The final form of this variational equation is obtained from the constitutive equations (18) for $\mathbf{Q}$ and (22) for V_0 and V_1:

$$h_p \iint_\Omega \partial_t^2 w v \, dA + \eta_1 \int_0^1 \partial_t^2 w_{b0} v_0 dx + \eta_1 \int_0^1 \partial_t^2 w_{b1} v_1 dx + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \nabla v \, dA$$

$$+ \eta_2 \int_0^1 (\partial_x w_{b0} - \phi_{b0}) v_0' dx + \eta_2 \int_0^1 (\partial_x w_{b1} - \phi_{b1}) v_1' dx = 0 \tag{28}$$

for each $[v \, v_0 \, v_1]^{\mathrm{T}} \in T_w$.

A similar calculation is performed for the remaining equations of motion. Combining the second equation of motion (17) for the plate with the Green formula (25) yields

$$I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_\Omega \mathrm{tr}(\mathbf{M}\boldsymbol{\Phi}) dA - \int_{\partial\Omega} \mathbf{M}\mathbf{n} \cdot \boldsymbol{\phi} \, ds + \iint_\Omega \mathbf{Q} \cdot \boldsymbol{\phi} \, dA = 0 \tag{29}$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

It follows from the second equation of motion (21) for the beam, using integration by parts, that

$$\frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_b \chi \, dx + \int_0^1 M_b \chi' dx - \int_0^1 (V + L) \chi \, dx = 0 \tag{30}$$

for each $\chi \in C^1[0, 1]$. (M_b is zero at the endpoints of the beam.)

The test functions χ_0 and χ_1 must satisfy the conditions $\chi_0 = -\phi_1(\cdot, 0)$ and $\chi_1 = -\phi_1(\cdot, a)$ in order to accommodate the interface condition (11). Denote this test space by T_ψ with

$$T_\psi = \{[\boldsymbol{\phi} \, \chi_0 \, \chi_1]^{\mathrm{T}} | \boldsymbol{\phi} \in T_2(\Omega), \, \chi_0 \in C^1[0, 1], \, \chi_1 \in C^1[0, 1], \, \chi_0 = -\phi_1(\cdot, 0), \, \chi_1 = -\phi_1(\cdot, a)\}.$$

As before, when adding (29) and (30), (for both beams) some cancellation of terms occurs. Note that $\boldsymbol{\phi} = (\boldsymbol{\phi} \cdot \mathbf{n})\mathbf{n} + (\boldsymbol{\phi} \cdot \boldsymbol{\tau})\boldsymbol{\tau}$ and consequently,

$$\int_{\partial\Omega} \mathbf{M}\mathbf{n} \cdot \boldsymbol{\phi} \, ds = \int_{\partial\Omega} \Big((\boldsymbol{\phi} \cdot \mathbf{n})\mathbf{M}\mathbf{n} \cdot \mathbf{n} + (\boldsymbol{\phi} \cdot \boldsymbol{\tau})\mathbf{M}\mathbf{n} \cdot \boldsymbol{\tau} \Big) ds.$$

The natural boundary condition on Σ_0 and Σ_1 is $\mathbf{M}\mathbf{n} \cdot \mathbf{n} = 0$. Also, for $\boldsymbol{\phi} \in T_2(\Omega)$, $\phi_2 = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and therefore $\boldsymbol{\phi} \cdot \boldsymbol{\tau} = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$. On Γ_0 and Γ_1 the interface conditions (13) and (14) are used.

Consequently,

$$I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_\Omega \mathrm{tr}(\mathbf{M}\boldsymbol{\Phi}) dA + \iint_\Omega \mathbf{Q} \cdot \boldsymbol{\phi} \, dA + \frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b0} \chi_0 dx + \frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b1} \chi_1 dx$$

$$+ \int_0^1 M_{b0} \chi_0' dx + \int_0^1 M_{b1} \chi_1' dx - \int_0^1 V_0 \chi_0 dx - \int_0^1 V_1 \chi_1 dx = 0$$

for each $[\boldsymbol{\phi} \, \chi_0 \, \chi_1]^{\mathrm{T}} \in T_\psi$.

The constitutive equations (18) and (19) for $\mathbf{Q}$ and $\mathbf{M}$, and (22) and (23) for V_1, V_2, M_{b0} and M_{b1} are used to obtain the final form of this variational equation.

We define a bilinear form b_B by

$$b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) = \iint_\Omega \mathrm{tr}(\mathbf{M}\boldsymbol{\Phi}) dA$$

$$= \frac{1}{\beta_p(1 - \nu^2)} \iint_\Omega \Big((\partial_1 \psi_1 + \nu \partial_2 \psi_2)\partial_1 \phi_1 + (\partial_2 \psi_2 + \nu \partial_1 \psi_1)\partial_2 \phi_2 \Big) dA$$

$$+ \frac{1}{2\beta_p(1 + \nu)} \iint_\Omega (\partial_1 \psi_2 + \partial_2 \psi_1)(\partial_1 \phi_2 + \partial_2 \phi_1) dA.$$

for each $\boldsymbol{\psi}, \boldsymbol{\phi}$ in $C^1(\bar{\Omega})^2$.

Finally, the second variational equation is given by

$$I_p \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi} \, dA + \frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b0} \chi_0 \, dx + \frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b1} \chi_1 \, dx$$

$$+ \frac{\eta_2}{\beta_b} \int_0^1 \partial_x \phi_{b0} \chi_0' \, dx + \frac{\eta_2}{\beta_b} \int_0^1 \partial_x \phi_{b1}' \chi_1' \, dx - \eta_2 \int_0^1 (\partial_x w_{b0} - \phi_{b0}) \chi_0 \, dx - \eta_2 \int_0^1 (\partial_x w_{b1} - \phi_{b1}) \chi_1 \, dx = 0 \tag{31}$$

for each $[\boldsymbol{\phi} \; \chi_0 \; \chi_1]^T \in T_\psi$.

Variational form of Problem RMT

Find $[w \; w_{b0} \; w_{b1}]^T$ and $[\boldsymbol{\psi} \; \phi_{b0} \; \phi_{b1}]^T$ such that, for $t > 0$, it holds that $[w(\cdot, t) \; w_{b0}(\cdot, t) \; w_{b1}(\cdot, t)]^T \in T_w$ and $[\boldsymbol{\psi}(\cdot, t) \; \phi_{b0}(\cdot, t) \; \phi_{b1}(\cdot, t)]^T \in T_\psi$ and that the system of variational equations (28) and (31) is satisfied for each $[v \; v_0 \; v_1]^T \in T_w$ and for each $[\boldsymbol{\phi} \; \chi_0 \; \chi_1]^T \in T_\psi$.

Product space formulation

At this stage, it may seem reasonable to reduce the degrees of freedom of the system (28) and (31) by applying the interface conditions (10) and (11) to obtain the variational form of Problem RMT in terms of w and $\boldsymbol{\psi}$ (the plate displacement and angles) only. However, certain regularity considerations for the weak solution should be kept in mind. To appreciate the impact of regularity, it is necessary to consider the properties of the (exact) solution of the problem.

It is well known that to obtain the well-posedness of the associated weak variational problem, only forced interface conditions should be imposed. For a weak solution, we have that the displacement and the angles belong to the Sobolev spaces $H^1(\Omega)$ and $H^1(\Omega)^2$ and one may not assume that they belong to $H^2(\Omega)$ and $H^2(\Omega)^2$. Consequently, the traces of the tangential derivatives of w and ψ_1 on Γ_0 and Γ_1 are not well defined. The product space formulation in $T_w \times T_\psi$ is a natural setting for an investigation of the problem.

In finite element applications, the tangential derivative of w on Γ_0 and Γ_1 should not necessarily be set equal to the derivatives of w_{b0} and w_{b1}. This also applies to the tangential derivative of ψ_1 and the derivatives of ϕ_{b0} and ϕ_{b1}. Although the equality of these derivatives are obvious for the smooth functions in $T_w \times T_\psi$, imposing these additional constraints may result in incorrect approximations by the finite element method, as they influence the convergence to a limit. Of course, the same considerations apply to the test functions. How this impacts on finite element calculations and results is discussed further in Sections 5.2 and 6.2.

Variational form of Problem KEB

For this simplified model the variational form can be obtained by setting $\psi_i = -\partial_i w$, $\phi_{bj} = -\partial_x w_{bj}$ for $j = 0, 1$ and choosing $\phi_i = -\partial_i v$, $\chi_j = -v_j'$ for $j = 0, 1$ in (28) and (31). The rotary inertia terms containing I_p and η_1/α_b are ignored. In this case the test functions are defined by

$$T(\Omega) = \left\{ v \in C^2(\bar{\Omega}) \mid v = 0 \text{ on } \Sigma_0 \text{ and } \Sigma_1 \right\}.$$

5. The eigenvalue problems

5.1. Problem RMTE

If the pair $[\tilde{w}(\boldsymbol{x}, t) \; \tilde{w}_{b0}(x, t) \; \tilde{w}_{b1}(x, t)]^T = T(t)[w(\boldsymbol{x}) \; w_{b0}(x) \; w_{b1}(x)]^T$ and $[\tilde{\boldsymbol{\psi}}(\boldsymbol{x}, t) \; \phi_{b0}(x, t) \; \phi_{b1}(x, t)]^T = T(t)[\boldsymbol{\psi}(\boldsymbol{x}) \; \phi_{b0}(x) \; \phi_{b1}(x)]^T$ is considered as a possible solution for the system (28) and (31), the following eigenvalue problem is obtained.

Find $[w \; w_{b0} \; w_{b1}]^T \in T_w$ and $[\boldsymbol{\psi} \; \phi_{b0} \; \phi_{b1}]^T \in T_\psi$ such that

$$\lambda \left\{ h_p \iint_\Omega wv \, dA + \eta_1 \int_0^1 w_{b0} v_0 \, dx + \eta_1 \int_0^1 w_{b1} v_1 \, dx \right\}$$

$$= h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \nabla v \, dA + \eta_2 \int_0^1 (w_{b0}' - \phi_{b0}) v_0' \, dx + \eta_2 \int_0^1 (w_{b1}' - \phi_{b1}) v_1' \, dx \tag{32}$$

for each $[v \; v_0 \; v_1]^T \in T_w$, and

$$\lambda \left\{ I_p \iint_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \frac{\eta_1}{\alpha_b} \int_0^1 \phi_{b0} \chi_0 \, dx + \frac{\eta_1}{\alpha_b} \int_0^1 \phi_{b1} \chi_1 \, dx \right\} = b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi} \, dA$$

$$+ \frac{\eta_2}{\beta_b} \int_0^1 \phi_{b0}' \chi_0' \, dx + \frac{\eta_2}{\beta_b} \int_0^1 \phi_{b1}' \chi_1' \, dx - \eta_2 \int_0^1 (w_{b0}' - \phi_{b0}) \chi_0 \, dx - \eta_2 \int_0^1 (w_{b1}' - \phi_{b1}) \chi_1 \, dx \tag{33}$$

for each $[\boldsymbol{\phi} \; \chi_0 \; \chi_1]^T \in T_\psi$.

As before, an eigenvalue problem for **Problem KEB** can be obtained directly from this one by applying the required simplifying assumptions.

5.2. Galerkin approximation

We consider an approximate solution for the system (32) and (33) as

$$w^h(\boldsymbol{x}) = \sum_{i=1}^{N} w_i \gamma_i(\boldsymbol{x}), \qquad \psi_1^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{1i} \gamma_i(\boldsymbol{x}) \quad \text{and} \quad \psi_2^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{2i} \gamma_i(\boldsymbol{x})$$

in terms of a set of basis functions γ_i, $i = 1, 2, \ldots, N$ and

$$w_{bj}^h(x) = \sum_{i=1}^{r} w_i^j \delta_i(x), \qquad \phi_{bj}^h(x) = \sum_{i=1}^{r} \phi_i^j \delta_i(x) \quad \text{with } j = 0, 1$$

in terms of a set of basis functions δ_i, $i = 1, 2, \ldots, r$.

Let $\boldsymbol{w} = [w_1 \ w_2 \ \ldots \ w_N]^{\mathrm{T}}$, $\boldsymbol{w}_{bj} = [w_1^j \ w_2^j \ \ldots \ w_r^j]^{\mathrm{T}}$ for $j = 0, 1$,
$\boldsymbol{\psi}_1 = [\psi_{11} \ \psi_{12} \ \ldots \ \psi_{1N}]^{\mathrm{T}}$, $\boldsymbol{\psi}_2 = [\psi_{21} \ \psi_{22} \ \ldots \ \psi_{2N}]^{\mathrm{T}}$ and
$\boldsymbol{\phi}_{bj} = [\phi_1^j \ \phi_2^j \ \ldots \ \phi_r^j]^{\mathrm{T}}$ for $j = 0, 1$.

The *interface conditions* $w_{b0}^h = w^h(\cdot, 0)$, $w_{b1}^h = w^h(\cdot, a)$, $\phi_{b0}^h = -\psi_1^h(\cdot, 0)$ and $\phi_{b1}^h = -\psi_1^h(\cdot, a)$ yield certain relationships between the coefficients of the different components of the approximate solution. These conditions will be applied after the matrix formulation of the discrete eigenvalue problem is obtained.

The **discrete eigenvalue problem** can be represented in matrix notation as

$$\mathcal{K}\boldsymbol{z} = \lambda \mathcal{M}\boldsymbol{z}$$

where $\boldsymbol{z} = [\boldsymbol{w} \ \boldsymbol{w}_{b0} \ \boldsymbol{w}_{b1} \ \boldsymbol{\psi}_1 \ \boldsymbol{\psi}_2 \ \boldsymbol{\phi}_{b0} \ \boldsymbol{\phi}_{b1}]^{\mathrm{T}}$. $\mathcal{K}$ and $\mathcal{M}$ are the stiffness matrix and mass matrix respectively.

This discrete eigenvalue problem is obtained by substituting the approximate solutions into (32) and (33). Then the test function $[v \ v_0 \ v_1]^{\mathrm{T}}$ is chosen as $[\gamma_j \ 0 \ 0]^{\mathrm{T}}$, $[0 \ \delta_j \ 0]^{\mathrm{T}}$ and $[0 \ 0 \ \delta_j]^{\mathrm{T}}$ in (32). Finally the test function $[\phi_1 \ \phi_2 \ \chi_0 \ \chi_1]^{\mathrm{T}}$ is chosen as $[\gamma_j \ 0 \ 0 \ 0]^{\mathrm{T}}$, $[0 \ \gamma_j \ 0 \ 0]^{\mathrm{T}}$, $[0 \ 0 \ \delta_j \ 0]^{\mathrm{T}}$ and $[0 \ 0 \ 0 \ \delta_j]^{\mathrm{T}}$ in (33). (A basis for a finite dimensional subspace S^h of the product space $T_w \times T_\psi$ can be constructed from these functions.) Recall that $[v \ v_0 \ v_1]^{\mathrm{T}} \in T_w$ and $[\boldsymbol{\phi} \ \chi_0 \ \chi_1]^{\mathrm{T}} \in T_\psi$ and that only admissible basis functions may be used.

The following matrices are required for obtaining $\mathcal{K}$ and $\mathcal{M}$.

$$K_{ij}^{\Omega 11} = \iint_{\Omega} \partial_1 \gamma_j \partial_1 \gamma_i \mathrm{d}A, \qquad K_{ij}^{\Omega 22} = \iint_{\Omega} \partial_2 \gamma_j \partial_2 \gamma_i \mathrm{d}A, \qquad K_{ij}^{\Omega 12} = \iint_{\Omega} \partial_1 \gamma_j \partial_2 \gamma_i \mathrm{d}A,$$

$$L_{ij}^{\Omega 1} = \iint_{\Omega} \gamma_j \partial_1 \gamma_i \mathrm{d}A, \qquad L_{ij}^{\Omega 2} = \iint_{\Omega} \gamma_j \partial_2 \gamma_i \mathrm{d}A, \qquad M_{ij}^{\Omega} = \iint_{\Omega} \gamma_j \gamma_i \mathrm{d}A$$

$$K_{ij}^{\Gamma} = \int_0^1 \delta_j' \delta_i' \mathrm{d}x, \qquad L_{ij}^{\Gamma} = \int_0^1 \delta_j \delta_i' \mathrm{d}x, \qquad M_{ij}^{\Gamma} = \int_0^1 \delta_j \delta_i \mathrm{d}x.$$

To deal with the bilinear form b_B, we define the following matrices:

$$K_w = h_p(K^{\Omega 11} + K^{\Omega 22}),$$

$$K^{B1} = \frac{1}{\beta_p(1 - \nu^2)}\left(K^{\Omega 11} + \frac{1 - \nu}{2} K^{\Omega 22}\right),$$

$$K_\nu = \frac{1}{\beta_p(1 - \nu^2)}\left(\nu\left(K^{\Omega 12}\right)^{\mathrm{T}} + \frac{1 - \nu}{2} K^{\Omega 12}\right),$$

$$K^{B2} = \frac{1}{\beta_p(1 - \nu^2)}\left(\frac{1 - \nu}{2} K^{\Omega 11} + K^{\Omega 22}\right).$$

Finally, let $K_1 = K^{B1} + h_p M^{\Omega}$ and $K_2 = K^{B2} + h_p M^{\Omega}$. Then the matrices $\mathcal{K}$ and $\mathcal{M}$ for the discrete eigenvalue problem are given by

$$\mathcal{K} = \begin{bmatrix} K_w & 0 & 0 & h_p L^{\Omega 1} & h_p L^{\Omega 2} & 0 & 0 \\ 0 & \eta_2 K^{\Gamma} & 0 & 0 & 0 & -\eta_2 L^{\Gamma} & 0 \\ 0 & 0 & \eta_2 K^{\Gamma} & 0 & 0 & 0 & -\eta_2 L^{\Gamma} \\ h_p(L^{\Omega 1})^{\mathrm{T}} & 0 & 0 & K_1 & K_\nu & 0 & 0 \\ h_p(L^{\Omega 2})^{\mathrm{T}} & 0 & 0 & K_\nu^{\mathrm{T}} & K_2 & 0 & 0 \\ 0 & -\eta_2(L^{\Gamma})^{\mathrm{T}} & 0 & 0 & 0 & \frac{\eta_2}{\beta_b}K^{\Gamma} + \eta_2 M^{\Gamma} & 0 \\ 0 & 0 & -\eta_2(L^{\Gamma})^{\mathrm{T}} & 0 & 0 & 0 & \frac{\eta_2}{\beta_b}K^{\Gamma} + \eta_2 M^{\Gamma} \end{bmatrix}$$

Table 1
Eigenvalues for the plate–beam systems.

i	KEB	RMT	Shear correction (%)
1	0.2413	0.2340	3.03
2	0.8765	0.7744	11.65
3	1.3715	1.1785	14.07
4	1.7197	1.6406	4.60
5	2.6642	2.4266	8.92
6	4.2835	3.9311	8.23

and

$$
\mathcal{M} =
\begin{bmatrix}
h_p M^{\Omega} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \eta_1 M^{\Gamma} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_1 M^{\Gamma} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_p M^{\Omega} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_p M^{\Omega} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{\eta_1}{\alpha_b} M^{\Gamma} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\eta_1}{\alpha_b} M^{\Gamma}
\end{bmatrix}.
$$

We use the Hermite piecewise bicubic functions γ_i, $i = 1, 2, \ldots, N$ and Hermite piecewise cubic functions δ_i, $i = 1, 2, \ldots, r$. See e.g. [18] or [19] for detailed descriptions of these functions.

Interface conditions

It is important to note that the restrictions of each of the bicubic basis functions γ_j to Γ_0 and Γ_1 will either be zero or will be equal to one of the Hermite piecewise cubics δ_i. The relationship between γ_j and δ_i is determined by the *interface conditions* in the definitions of T_w and T_ψ, i.e. $v_0 = v(\cdot, 0)$, $v_1 = v(\cdot, a)$, $\chi_0 = -\phi_1(\cdot, 0)$ and $\chi_1 = -\phi_1(\cdot, a)$. As far as the derivatives are concerned, we will consider two options:

(a) $v_0' \neq \partial_1 v(\cdot, 0)$, $v_1' \neq \partial_1 v(\cdot, a)$, $\chi_0' \neq -\partial_1 \phi_1(\cdot, 0)$ and $\chi_1' \neq -\partial_1 \phi_1(\cdot, a)$,
and
(b) $v_0' = \partial_1 v(\cdot, 0)$, $v_1' = \partial_1 v(\cdot, a)$, $\chi_0' = -\partial_1 \phi_1(\cdot, 0)$ and $\chi_1' = -\partial_1 \phi_1(\cdot, a)$.

6. Numerical results

Parameters

For the numerical results, we consider a square plate (i.e. $a = 1$) and beams with a rectangular profile of thickness d and height $5d$. The dimensionless thickness d_b of the beams is denoted by $d_b = d/\ell$. We also assume that the plate and the beams are made of the same isotropic material, i.e. $G = \frac{E}{2(1+v)}$. For this special case the dimensionless constants reduce to

$$
\eta_1 = 5d_b^2, \qquad \eta_2 = \frac{5\kappa_b^2 d_b^2}{\kappa_p^2}, \qquad \frac{1}{\alpha_b} = \frac{25d_b^2}{12}, \qquad \frac{1}{\beta_p} = \frac{(1+v)h_p^3}{6\kappa_p^2}, \qquad \frac{1}{\beta_b} = \frac{25(1+v_b)d_b^2}{6\kappa_b^2}.
$$

We choose Poisson's ratios $v = v_b = 0.3$ and the shear correction factors $\kappa^2 = \kappa_b^2 = 5/6$. The value of h_p is fixed at $h_p = 0.05$ and $d_b/h_p = 1$, unless specified differently.

Convergence

MATLAB programs are written for calculating approximate eigenvalues and eigenfunctions of the RMT and KEB plate–beam systems. When the grid is refined, the eigenvalues form a decreasing sequence, which is in line with the theory. The results in the following tables are accurate to at least three significant digits.

6.1. Comparing the RMT and KEB systems

For beam and plate models, it is well known that the shear corrections to the eigenvalues and eigenfunctions which are introduced by the Timoshenko model and the Reissner–Mindlin model, are significantly larger than the corrections due to rotary inertia (see e.g. [14,20,10,11]).

In [7] a KEB plate–beam system is investigated and it is shown that introducing rotary inertia into the model does not cause a significant change in the eigenvalues. For a particular example, the correction to the tenth eigenvalue is only 0.1%. In Table 1 the eigenvalues for the RMT system are compared to those of the KEB system for $d_b/h_p = 1$. It should be noted that the scaling for the dimensionless form for the KEB system differs from the scaling used in [7].

Table 2
Derivatives of the first eigenfunction on Γ_0.

First eigenfunction of RMT system				
x	$\partial_1 w^h$	$(w_{b0}^h)'$	$\partial_1 \psi_1^h$	$(\phi_{b0}^h)'$
0	−5.0713	−4.7561	262.5596	−0.2038
0.05	−4.6689	−4.6945	59.2691	−2.0776
0.10	−4.4778	−4.5114	1.6054	−4.0744
0.15	−4.2476	−4.2216	−15.3787	−5.9983
0.20	−3.8883	−3.8313	−22.5621	−7.7766
0.25	−3.4109	−3.3481	−27.3887	−9.3617
0.30	−2.8394	−2.7830	−31.3024	−10.7147
0.35	−2.1942	−2.1495	−34.4493	−11.8029
0.40	−1.4939	−1.4631	−36.7642	−12.5997
0.45	−0.7563	−0.7407	−38.1800	−13.0857
0.50	−0.0000	−0.0000	−38.6563	−13.2490

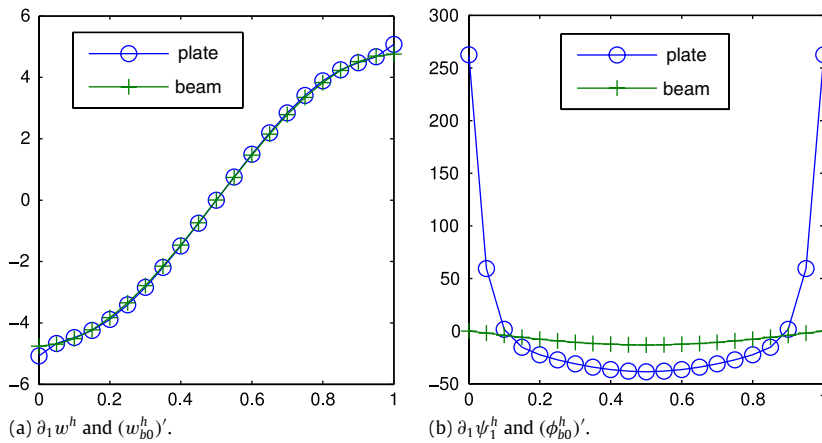(a) $\partial_1 w^h$ and $(w_{b0}^h)'$. (b) $\partial_1 \psi_1^h$ and $(\phi_{b0}^h)'$.

Fig. 2. Derivatives of the first eigenfunction on Γ_0.

Clearly the shear corrections are significant (contrary to the corrections due to rotary inertia, see [7]). It is also interesting to note that the shear corrections introduced by the RMT system do not increase monotonically for the sequence of eigenvalues. This is an unexpected result as, for a Reissner–Mindlin plate or a Timoshenko beam on their own, the shear corrections increase monotonically.

6.2. Forced interface conditions

In this section, we investigate how the regularity assumption on the tangential derivatives of w and ψ_1 influences the finite element calculations and results.

Firstly we consider the case where the tangential derivative of w on Γ_0 and Γ_1 is not set equal to the derivatives of w_{b0} and w_{b1}. The same assumption is made for the tangential derivative of ψ_1 (and the derivatives of ϕ_{b0} and ϕ_{b1}). Similar assumptions hold for the test functions. In Table 2 we present details of the relevant derivatives on Γ_0 for the first eigenfunction. Due to the spatial symmetry of the model, values are listed for $0 \leq x \leq 0.5$ and for only one of the beams. In Fig. 2 this information is displayed graphically.

Clearly, for the transverse displacements w and w_{b0}, $\partial_1 w^h \approx (w_{b0}^h)'$ on Γ_0 (excluding the endpoints). However, for the derivatives of the angles ψ_1 and ϕ_{b0}, there are significant differences. It seems that singularities for $\partial_1 \psi_1^h$ occur at the vertices of the plate.

Next, *we assume that the tangential derivatives of w and ψ_1 on Γ_0 and Γ_1 equal the derivatives of w_{bj} and ϕ_{bj}* (with similar conditions holding for the test functions). In Table 3 we present details of the tangential derivatives on Γ_0 for the first eigenfunction.

Impact of regularity assumption

Note that the values of $\partial_1 w^h$ in Table 3 compare well with those obtained for the beam in Table 2, whereas the values of $\partial_1 \psi_1^h$ in Table 3 are approximately equal to the values obtained for the beam in Table 2.

Although not shown here, under the regularity assumption, the finite element approximations for the eigenvalues are marginally larger, but not significantly so.

It is clearly desirable to have information concerning the regularity of solutions for plate–beam systems.

Table 3
Derivatives of the first eigenfunction on Γ_0.

First eigenfunction (regularity assumed)		
x	$\partial_1 w^h$	$\partial_1 \psi_1^h$
0	−4.7480	0.0006
0.05	−4.6895	−2.0817
0.10	−4.5156	−4.1123
0.15	−4.2305	−6.0416
0.20	−3.8412	−7.8221
0.25	−3.3573	−9.4100
0.30	−2.7908	−10.7662
0.35	−2.1555	−11.8573
0.40	−1.4672	−12.6564
0.45	−0.7428	−13.1439
0.50	0.0000	−13.3077

Table 4
Eigenvalues for RMT plate–beam system and the RM plate.

	Eigenvalues of RMT plate–beam system				Eigenvalues of supported RM plate
i	$d_b/h_p = 1$	$d_b/h_p = 2$	$d_b/h_p = 4$	$d_b/h_p = 8$	
1	0.2340	0.2702	0.2730	0.2733	0.2733
2	0.7744	1.5695	1.6552	1.6627	1.6643
3	1.1785	1.6619	1.6639	1.6642	1.6643
				3.0030	
				3.0030	
4	1.6406	3.2510	4.1503	4.1532	4.1540
5	2.4266	3.5914	5.8931	6.3471	6.3849
6	3.9311	4.1320	6.3844	6.3849	6.3849

6.3. Comparison of RMT system with a RM plate

In [7] it is also shown that the eigenvalues of a KEB plate–beam system tend to those of a rigidly supported Kirchhoff plate if the thickness of the supporting beams is increased. In Table 4 the eigenvalues of the RMT plate–beam system are compared to the eigenvalues of a Reissner–Mindlin plate that is rigidly supported on all four sides. The exact eigenvalues for the rigidly supported plate is presented in the last column.

It is clear that, as expected, the eigenvalues of the RMT plate–beam system tend to the eigenvalues of the rigidly supported Reissner–Mindlin plate as the ratio d_b/h_p is increased.

An interesting phenomenon in this table warrants some comment. For large values of the ratio d_b/h_p, an extra pair of eigenvalues appears for the RMT system. For $d_b/h_p = 8$ in Table 4, the double eigenvalue $\lambda \approx 3$ does not correspond to an eigenvalue of the supported plate. This was not the case for the KEB system, see [7].

To explain this phenomenon, we consider the vibration spectrum of a simply supported Timoshenko beam. It is easy to see that $\lambda = \alpha_b$ is an eigenvalue with the associated pair of eigenfunctions $w_b(x) = 0$ and $\phi_b(x) = 1$ (see [20]).

For $d_b/h_p = 8$ and $h_p = 0.05$, $d_b = 0.4$ and hence $\alpha_b = 12/(25d_b^2) = 3$. We conclude that the pair of extra eigenvalues in Table 4 is a consequence of the pure rotation mode of the Timoshenko beam model.

It should be noted that, in this case, the length to height ratio for the beam is 1:2 and a two-dimensional beam model is called for, rather than the Timoshenko model. However, it is remarkable that the Timoshenko model does capture this two-dimensional effect in the elastic support.

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