



Closed-form expressions of the run-length distribution of the nonparametric double sampling precedence monitoring scheme

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Abstract

A significant challenge in statistical process monitoring (SPM) is to find exact and closed-form expressions (CFEs) (i.e. formed with constants, variables and a finite set of essential functions connected by arithmetic operations and function composition) for the run-length properties such as the average run-length (*ARL*), the standard deviation of the run-length (*SDRL*), and the percentiles of the run-length (*PRL*) of nonparametric monitoring schemes. Most of the properties of these schemes are usually evaluated using simulation techniques. Although simulation techniques are helpful when the expression for the run-length is complicated, their shortfall is that they require a high number of replications to reach reasonably accurate answers. Consequently, they take too much computational time compared to other methods, such as the Markov chain method or integration techniques, and even with many replications, the results are always affected by simulation error and may result in an inaccurate estimation. In this paper, closed-form expressions of the run-length properties for the nonparametric double sampling precedence monitoring scheme are derived and used to evaluate its ability to detect shifts in the location parameter. The computational times of the run-length properties for the CFE and the simulation approach are compared under different scenarios. It is found that the proposed approach requires less computational time compared to the simulation approach. Moreover, once derived, CFEs have the added advantage of ease of implementation, cutting off on complex convergence techniques. CFE's can also easily be built into mathematical software for ease of computation and may be recalled for further work.

Keywords Shewhart · Control chart · Order statistic · Closed-form expression · Distribution-free · Nonparametric · Precedence · Exact run-length distribution · Performance analysis

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1 Introduction

Monitoring schemes are evaluated using metrics based on the number of samples required to detect abnormalities in the process; see, e.g., Montgomery (2020). The distribution of the number of samples before detecting the first out-of-control (OOC) signal is called the run-length distribution. Many researchers have proposed different methods for the computation of the properties of the run-length distribution; see, e.g., Chakraborti et al. (2009) and Teoh et al. (2016). When it is difficult to find the run-length distribution (as is the case with some nonparametric schemes such as the Mann–Whitney, or the Wilcoxon rank sum monitoring scheme, the cumulative sum (CUSUM), exponentially weighted moving average (EWMA), generally weighted moving average (GWMA), and homogeneously weighted moving average (HWMA) types monitoring schemes), researchers have suggested the use of simulation techniques; see, e.g., Letshedi et al. (2021) and Qiu (2018). For some early use of simulations in statistical process monitoring (SPM), readers are referred to Pignatiello and Runger (1990) and Fu and Hu (1999) to cite a few. The advantage of simulation techniques is that regardless of the complexity of the run-length distribution, there is always a way to compute the properties of the run-length distribution using Monte Carlo simulation; see, e.g., Qiu (2018). However, the drawback of simulation techniques is that, in general, they require many iterations (i.e. simulation runs) and it takes a long time to compute the run-length properties of a monitoring scheme; see, e.g., Silberschatz et al. (2009). Moreover, regardless of the number of replications, the results are negatively affected by the simulation error, which is dependent on the number of simulations; see, e.g., Oberkampf et al. (2002). However, closed-form computational methods such as Markov chain approaches have the advantage of reducing the computational time and error. The Markov chain approach has often been used to study or derive closed-form expressions of the run-length distribution of many proposed monitoring schemes, including the CUSUM and EWMA schemes. However, for the Markov chain, the point of departure is approximating the problem, thereafter obtaining an exact solution to the problem; see, e.g., Champ and Rigdon (1991), Areepong and Sukparungsee (2013), and Sukparungsee and Areepong (2016). Weiß (2011) advocates for the extension of the use of the Markov chain approach for calculating other performance measures like the expected conditional delay, the steady-state *ARL*, or the worst-case *ARL*. Chakraborti et al. (2004) further studied the median monitoring scheme by Janacek and Meikle (1997) and derived closed-form expressions for the run-length distribution. Chakraborti et al. (2009) derived the exact run-length distributions of the precedence scheme with runs-rules using conditioning and some results from the theory of runs and scans; see Balakrishnan and Koutras (2011). Malela-Majika et al. (2021, 2022) used the Markov chain approach to derive closed-form expressions of the run-length characteristics of the precedence schemes to investigate their abilities to detect shifts in the location process parameter. The percentiles of the run-length distribution for both the optimal average run-length (*ARL*)-based and median run-length (*MRL*)-based for the double sampling \bar{X} schemes

with estimated process parameters were investigated by Teoh et al. (2016). The authors reported that the percentiles of their monitoring scheme changes with the Phase I and II sample sizes and optimal shift, even though the same value of the nominal in-control ARL (ARL_0) and in-control MRL (MRL_0) is attained. Apart from simulation and exact closed-form expressions, authors have also used numerical procedures using integral equations (see, e.g., Crowder (1987)), run-length generating functions (see, e.g., Shmueli and Cohen (2003)), and discrete-time Markov chains (see, e.g., Zantek (2008) and Li et al. (2014)) to study the run-length properties.

It is important to note that for simplicity, this paper focuses on the double sampling precedence scheme for the median as it has a symmetric distribution and hence symmetric control limits. However, the precedence scheme belongs to a general class of distribution-free schemes where any order statistics (including the sample median) can be used as a charting statistic. A key advantage of distribution-free charts is that one does not need to assume any particular underlying process distribution and the in-control probability calculations remain valid for all continuous distributions; see Chakraborti et al. (2004). Thus, distribution-free charts are said to be in-control robust, where the term "robustness" highlights the ability of any statistical procedure under ideal and non-ideal condition; see e.g., Balakrishnan et al. (2006, p. 7299) and Human et al. (2011). To facilitate the investigation of the performance of the double sampling precedence scheme introduced by Malela-Majika et al. (2021), this paper introduces closed-form expressions for the run-length properties and the average sample size (ASS) using calculus. The expression of the Stage 2 probability of the in-control process is derived using mathematical statistics techniques. The implementation is simplified by making use of already existing integral functions (like the incomplete Beta function) in mathematical software like Mathcad, Matlab, R, etc. The computational times of the run-length properties of the double sampling scheme using the proposed method and simulation techniques are also compared.

The remainder of this paper is organised as follows: Sect. 2 presents a brief background of the double sampling precedence scheme as well as its operation. Section 3 introduces the different closed-form expressions of the run-length properties and the ASS; these expressions are the key contribution of this paper. The in-control and out-of-control performances of the double sampling precedence scheme are investigated in Sect. 4. Moreover, a comparative study of the computational time is presented in Sect. 4. The application and implementation of the double sampling (DS) precedence monitoring scheme are given in Sect. 5 through an example. Section 6 gives the concluding remarks and suggested future research ideas.

2 Double sampling precedence scheme

2.1 Preliminaries and notations

The DS precedence scheme was first introduced by Malela-Majika et al. (2021). Let X_i , $i = 1, 2, \dots, m$, where $m > 1$ be a sequence of m Phase I observations from the in-control (IC) process with an unknown cumulative distribution function

(c.d.f.) denoted as $F_X(x)$ and let $Y_{tk}, t = 1, 2, \dots; k = 1, 2, \dots, n$, where $n > 1$ be a t^{th} Phase II test sample of size n with c.d.f. $G_Y(y)$. Assume that when the process is IC, $F_X(x) = G_Y(x)$. Otherwise, the process is said to be out-of-control then $F_X(x) = G_Y(x + \delta)$ where $\delta \in \mathbb{R}/\{0\}$ is the change or shift in the location parameter. Note that the test samples Y_{tk} are assumed to be independent and identically distributed (i.i.d.) of one another and of the reference sample X . Then, the proposed DS precedence monitoring scheme is a two-stage precedence scheme that is divided into five charting regions defined as follows and illustrated in Fig. 1:

- **Charting regions in Stage 1**

$$A = (-\infty, X_{(a_2:m)}] \cup [X_{(b_2:m)}, +\infty), \quad B = (X_{(a_2:m)}, X_{(a_1:m)}) \cup (X_{(b_1:m)}, X_{(b_2:m)})$$

and $C = (X_{(a_1:m)}, X_{(b_1:m)})$.

- **Charting regions in Stage 2**

$$D = (-\infty, X_{(c_1:m)}] \cup [X_{(c_2:m)}, +\infty) \text{ and } E = (X_{(c_1:m)}, X_{(c_2:m)}),$$

where $X_{(\ell:m)}$ represents the ℓ^{th} , where $\ell \in \{a_2, a_1, b_1, b_2, c_1, c_2\}$, order statistic of the Phase I reference sample of size m , and ℓ represents the position of the order statistic in the reference sample and it is also referred to as the charting constant(s).

The choices of the charting constants i.e., $1 \leq a_2 < a_1 < b_1 < b_2 \leq m$ and $1 \leq c_1 < c_2 \leq m$, of the DS precedence scheme are selected such that in Stage 1, $a_2 = m - b_2 + 1$, $a_1 = m - b_1 + 1$ and in Stage 2, $c_1 = m - c_2 + 1$. This relationship between the aforementioned parameters is a result of symmetrically placed control limits. Note that the Stage 1 charting constants i.e., a_2, a_1, b_1 and b_2 , must be selected such that the attained IC ASS value is equal to some specific values denoted as ASS_0 in this paper; while the Stage 2 charting constants i.e., c_1 and c_2 , are selected such that the attained IC ARL is equal to some pre-specified nominal ARL values denoted as ARL_0 . Thus, Stage 1's outer lower and upper control limits (denoted as *OLCL* and *OUCL*) and inner lower and upper control limits (denoted as *ILCL* and *IUCL*) as well as the Stage 2's *LCL* and *UCL* of the DS precedence scheme are estimated from the IC Phase I reference sample of size m as follows:

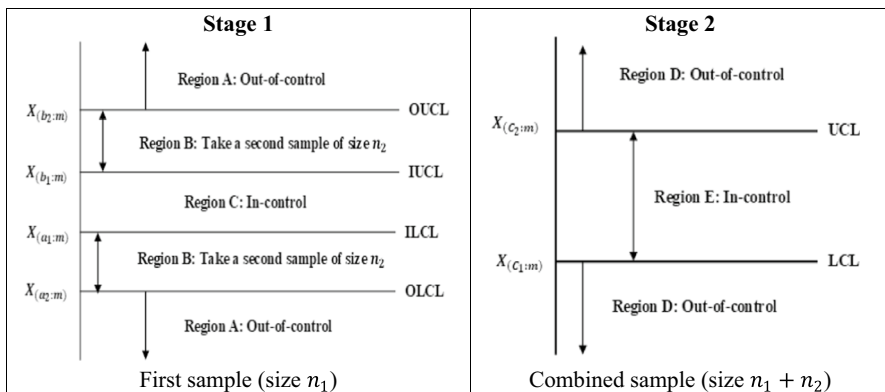


Fig. 1 Charting regions of the Phase II DS precedence scheme

$$\widehat{OLCL} = X_{(a_2:m)}, \widehat{ILCL} = X_{(a_1:m)}, \widehat{IUCL} = X_{(b_1:m)}, \widehat{OUCL} = X_{(b_2:m)}, \widehat{LCL} = X_{(c_1:m)}$$

and $\widehat{UCL} = X_{(c_2:m)}$.

In Phase II, at each sampling time, a sample of size n is collected i.e. Y_{tk} ($k = 1, 2, \dots, n$), and split into two subsamples, i.e. Y_{1tp} and Y_{2tq} ($p = 1, 2, \dots, n_1$ and $q = 1, 2, \dots, n_2$), of sizes n_1 and n_2 , respectively. Note that, $n_2 + n_1 = n$ and we assume that $n_2 > n_1$.

There are practical advantages to using a two-stage or a sequential sampling approach, and they include the following:

- **Early detection:** Sequential sampling allows for the early detection of issues or anomalies in a process. By taking smaller samples at each point and continuously evaluating the data, one can identify problems as soon as they emerge rather than waiting until the end of a large batch.
- **Cost efficiency:** It can be more cost-effective to take smaller sequential samples rather than one large sample at each point. Large samples can be expensive and time-consuming to collect and analyse, whereas smaller samples reduce resource requirements.
- **Resource allocation:** Sequential sampling allows one to allocate resources more efficiently. If one identifies a problem early, one can focus resources on investigating and addressing that specific issue rather than collecting and analysing a full large sample.
- **Reduced waste:** If a process is going out-of-control or is producing non-conforming products, taking a large sample at each point may result in a significant amount of waste. Sequential sampling helps minimise waste by detecting issues promptly.
- **Real-time control:** It enables real-time process control and adjustment. With sequential sampling, one can make immediate adjustments to the process if deviations are detected, ensuring product quality and process stability.
- **Improved productivity:** Continuous monitoring through sequential sampling can lead to improved productivity as it reduces the chances of producing defective products or processes.

2.2 Phase II operation of the DS precedence scheme

Assume the Phase I analysis is completed, and the estimated control limits for Phase II (based on the Phase I sample) are available. Thus, the Phase II operation procedure of the DS precedence scheme is as follows:

1. Take a sample of size n_1 and compute $Y_{(j:n_1)}$ at the t^{th} sampling time point of the first sample. For simplicity, in this paper, it is assumed that n_1 is odd i.e., $n_1 = 2r + 1$ (where r is a positive integer) so that $j = r + 1$ corresponds to the unique test sample median of the Y_{1tp} sample in Stage 1.
2. If $Y_{(j:n_1)} \in C$, the process is considered to be IC.
3. If $Y_{(j:n_1)} \in A$, the process is said to be OOC.

4. If $Y_{(j:n_1)} \in B$, take a second sample of size n_2 .
5. At the t^{th} sampling timepoint, compute the plotting statistic of the combined sample, $Y_{(h:n)}$. We assume that n_2 is even so that $n = n_1 + n_2$ gives an odd number i.e., $n = 2s + 1$ where s is a positive integer, so that $h = s + 1$ corresponds to the unique test sample median.
6. The process is declared OOC at Stage 2, if $Y_{(h:n)} \in D$, i.e. $Y_{(h:n)} \geq UCL$ or $Y_{(h:n)} \leq LCL$. Otherwise, the process is said to be IC.

3 Run-length properties of the double sampling precedence monitoring scheme

First, we focus on the Stage 1 and Stage 2 conditional probabilities. The conditional probabilities that at Stage 1 the statistic $Y_{(j:n_1)}$ plots in region A , B and C are defined by

$$p_A = 2I\left(1 - GF^{-1}\left(U_{(a_2:m)}\right), j, n_1 - j + 1\right), \tag{1a}$$

$$p_B = 2\left[I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n_1 - j + 1\right) - I\left(GF^{-1}\left(U_{(a_2:m)}\right), j, n_1 - j + 1\right)\right] \tag{1b}$$

and

$$p_C = I\left(GF^{-1}\left(U_{(m-a_1+1:m)}\right), j, n_1 - j + 1\right) - I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n_1 - j + 1\right), \tag{1c}$$

respectively, where $I(\cdot, \dots)$ denotes the incomplete beta function and $GF^{-1}(\cdot)$ is the conversion function (see Malela-Majika (2022)), and $U_{(a_1:m)}$ and $U_{(a_2:m)}$ represent the a_1^{th} and a_2^{th} order statistics of a sample of size m from the Uniform (0,1) distribution.

The conditional probability that the process is IC is given by

$$p_0 = p_{01} + p_{02}, \tag{2}$$

where p_{01} and p_{02} are the conditional probabilities that the process is IC at Stage 1 and Stage 2, respectively, with

$$p_{01} = P\left(Y_{(j:n_1)} \in C | X_{(a_1:m)} = x_{(a_1:m)}, X_{(m-a_1+1:m)} = x_{(m-a_1+1:m)}\right) = p_C \tag{3a}$$

and

$$p_{02} = P\left(Y_{(j:n_1)} \in B \text{ and } Y_{(h:n)} \in E | X_{(a_1:m)} = x_{(a_1:m)}, X_{(a_2:m)} = x_{(a_2:m)}, X_{(c_1:m)} = x_{(c_1:m)}, X_{(m-c_1+1:m)} = x_{(m-c_1+1:m)}\right). \tag{3b}$$

These probabilities are conditional with regard to the Phase I order statistics. The derivations of the expression of p_{02} is provided in the Appendix. Then, the expressions of the unconditional ARL , standard deviation of the run-length ($SDRL$) and ASS values at each sampling time are given by

$$ARL = \int_0^1 \int_0^t \left(\frac{1}{1-p_0} \right) f_{ab}(s, t) ds dt, \tag{4a}$$

$$SDRL = \int_0^1 \int_0^t \left(\frac{\sqrt{p_0}}{1-p_0} \right) f_{ab}(s, t) ds dt \tag{4b}$$

and

$$ASS = \int_0^1 \int_0^t (n_1 + n_2 p_B) f_{ab}(s, t) ds dt, \tag{4c}$$

where $f_{ab}(s, t) = \frac{m!}{(a-1)!(b-a-1)!(m-b)!} t^{a-1} (t-s)^{b-a-1} (1-t)^{m-b}$ is the joint probability density function (pdf) of the a^{th} and b^{th} order statistics in a random sample of size m from the Uniform (0,1) distribution.

The average number of observations to signal (ANOS) is given by

$$ANOS = \int_0^1 \int_0^t \left(\frac{n_1 + n_2 p_B}{1-p_0} \right) f_{ab}(s, t) ds dt. \tag{5}$$

Let R be the conditional run-length of a scheme. Then, the $(100\rho)^{th}$ percentile (with $0 < \rho < 1$) of the run-length distribution (denoted as ℓ_ρ) is given by

$$P(R \leq \ell_\rho - 1) \leq \rho \text{ and } P(R \leq \ell_\rho) > \rho. \tag{6}$$

Hence, the conditional and unconditional c.d.f. of the DS precedence monitoring scheme are defined by

$$F_R(\ell) = P(R \leq \ell) = 1 - p_0^\ell, \tag{7a}$$

$$F_R(\ell) = \int_{-\infty}^{+\infty} \int_0^{+\infty} (1 - p_0^\ell) f_{ab}(s, t) ds dt, \tag{7b}$$

where $\ell \in \{1, 2, \dots\}$ and p_0 is defined in Eq. (2). Henceforth, in this paper, the $(100\rho)^{th}$ percentile of the unconditional run-length distribution will be denoted as P_ρ . For instance, the 50th percentile (i.e. the median) of the run-length distribution is denoted as $P_{0.5}$.

Note that when the process is IC, the conversion function $GF^{-1}(q) = q$ (where q represents values u and s taken by the uniform random variable U and S). However, when the process is out-of-control, $GF^{-1}(q) \neq q$. For more details on how to

derive the out-of-control expressions of $GF^{-1}(q)$ for different distributions, readers are referred to Malela-Majika et al. (2021).

Remark It is important to note that conditional performance measures are conditional on the IC Phase I dataset and, more specifically, the order statistics used to estimate the Phase II control limits. Because the actual underlying distribution is assumed to be unknown, the Phase II control limits are then estimated as shown in Sect. 2. The ‘hat-notation’ indicates point estimators. These point estimates of the control limits introduce additional uncertainty and variability, which must be accounted for. Therefore, the average performance over all possible values of the order statistics is obtained by calculating the unconditional performance measures.

4 Performance analysis of the double sampling precedence monitoring scheme

4.1 IC performances

In this subsection, we study the behaviour of the DS precedence scheme when the process is assumed to be IC. Balakrishnan et al. (2006, p. 7299) defined a robust statistical procedure as a procedure that performs well under ideal conditions (i.e., the condition under which it is designed and proposed) and under departures from the ideal. However, the terms robust and distribution-free are often confused in the literature. Distribution-free schemes have the same IC run-length distribution for all continuous distributions and, in some cases, only for all symmetric continuous distributions (e.g., schemes based on the signed-rank statistic). Thus, by definition, a distribution-free statistic or monitoring scheme is robust, but a robust monitoring scheme, like the EWMA scheme for small values of the smoothing parameter (see Randles and Wolfe (1979) and Human et al. (2011)), is not distribution-free. One of the advantages of a distribution-free (or nonparametric) scheme is that the IC properties like the IC ARL do not depend on the underlying process distribution. The expectation in this paper is that for all continuous distributions, the proposed scheme will yield the same IC run-length characteristics. In this paper, to check the IC robustness of the DS precedence scheme we investigated its IC run-length properties under the normal, Student’s t , gamma, double exponential and Weibull distributions. These distributions were chosen to check the effects of symmetry, skewness, light and heavy tails on the performance of the DS precedence monitoring scheme. Table 1 presents the charting constants along with the corresponding attained ASS , ARL and $SDRL$ values when $\delta = 0$, $m \in \{50, 100, 500\}$, $n_1 \in \{5, 7\}$ and $n_2 \in \{6, 8, 12\}$ for pre-specified $ASS_0 = 6$ and 10, and nominal $ARL_0 = 250$ and 300. Note that the Stage 1 charting constants a_2, a_1, b_1 and b_2 are determined such that the attained IC ASS is satisfactorily close to the pre-specified ASS_0 value, and the Stage 2 charting constants c_1 and c_2 are determined such that the attained IC ARL is close or equal to the nominal ARL_0 value. Since Eq. (4c) can be expressed in terms of two charting constants through the relationship $a_2 = m - b_2 + 1$ and $a_1 = m - b_1 + 1$, one can

Table 1 Charting constants and attained *ASS*, *ARL*, *SDRL* and $P_{0.5}$ values when $\delta = 0$, $m \in \{50, 100, 500\}$ and $n_1 \in \{5, 7\}$, $n_2 \in \{6, 8, 12\}$ and $ASS_0 \in \{6, 10\}$ under the $N(0, 1)$, $t(5)$, $G(3, 1)$, $DE(0, 1)$ and $WE(2, 1)$ distributions when $ARL_0 = 250$ and 300

<i>m</i>	n_1	n_2	Stage 1				Stage 2		<i>ASS</i>	<i>ARL</i>	<i>SDRL</i>	$P_{0.5}$
			a_2	a_1	b_1	b_2	c_1	c_2				
50	5	6	11	15	36	40	8	43	6	321.9	2027.0	86
	5	8	11	15	36	40	9	42	6	327.8	1233.9	83
	5	6	7	24	27	44	8	43	10	281.1	1080.2	71
	5	8	12	23	28	39	9	42	10	280.5	1191.1	67
100	5	6	37	43	58	64	15	86	6	271.2	566.2	113
	5	8	27	32	69	74	15	84	6	309.5	676.7	127
	5	6	16	48	53	85	15	86	10	249.3	534.1	106
	5	8	12	41	60	89	17	84	10	271.5	604.8	112
500	5	6	21	115	386	480	73	428	6	309.3	355.7	194
	5	8	31	104	397	470	84	417	6	308.5	354.1	189
	5	6	46	230	271	455	70	431	10	290.7	337.8	184
	5	8	21	199	302	480	80	421	10	307.5	360.0	193
50	7	8	10	24	27	41	10	41	10	248.6	1019.1	58
	7	12	8	15	36	43	12	39	10	239.9	2473.3	49
100	7	8	14	35	66	87	19	82	10	255.8	578.8	103
	7	12	18	31	69	82	22	79	10	263.2	678.6	97
500	7	8	117	188	313	384	88	413	10	300.9	354.1	202
	7	12	107	165	336	394	104	397	10	301.8	356.9	184

use numerical integration and root finding programs in software such as Mathcad to determine the charting constants. For instance, for $m = 100$, $(n_1, n_2) = (5, 6)$ so that $j = 3$ and $h = 6$, and for a pre-specified $ASS_0 = 6$ and a nominal $ARL_0 = 250$, using Eq. (4c) and the constraint $IC\ ASS = ASS_0$, it is found that $b_1 = 58$ and $b_2 = 64$, and $a_2 = m - b_2 + 1 = 37$ and $a_1 = m - b_1 + 1 = 43$, is the only combination of Stage 1 charting constants that satisfies the constraint $IC\ ASS = 6$. Then, at Stage 2, it is found that $c_1 = 15$ (with $c_2 = m - c_1 + 1 = 86$) so that the DS precedence scheme with charting constants $(a_2, a_1, b_1, b_2, c_1, c_2) = (37, 43, 58, 64, 15, 86)$ yields an attained $IC\ ARL = 271.2$.

From the results displayed in Table 1, we can observe that an increase in m (i.e. Phase I sample size), results in smaller *SDRL* values. However, the $IC\ ARL$ values converge toward the desired nominal ARL_0 value. This implies a stable performance of the proposed DS precedence scheme for large Phase I sample sizes. In terms of the 50th percentile (i.e. $P_{0.5}$), when $m = 100$, $(n_1, n_2) = (5, 6)$ and $ASS_0 = 6$, there is 50% chance that the proposed DS precedence scheme gives a signal on the 113th sample in the prospective phase when $\delta = 0$ for $ARL_0 = 321.9$. The results tabulated below are thus indicating that the IC run-length properties of the DS precedence scheme are the same for all continuous distributions considered in this paper, and

Table 2 ARL_1 profile of the proposed scheme when $m \in \{50, 100, 500\}$, $ASS_0 \in \{6, 10\}$, $n_1 = 5$ and $n_2 = 6$ for a nominal $ARL_0 = 320$

Distribution	m	$(n_1, n_2, ASS_0) = (5, 6, 6)$						$(n_1, n_2, ASS_0) = (5, 6, 10)$					
		0.25	0.50	0.75	1.00	1.50	2.00	0.25	0.50	0.75	1.00	1.50	2.00
$N(0,1)$	50	178.50	45.05	9.10	3.25	1.20	1.02	150.55	36.06	8.36	2.93	1.19	1.01
	100	120.18	25.28	6.62	2.67	1.17	1.01	116.48	24.26	6.49	2.66	1.17	1.01
	500	100.35	20.87	6.26	2.70	1.20	1.00	94.83	19.52	5.90	2.53	1.16	1.01
$t(5)$	50	185.70	48.90	8.61	2.42	1.11	1.00	159.08	41.02	8.45	2.39	1.08	1.00
	100	119.15	22.31	4.95	1.93	1.05	1.00	114.85	21.55	4.87	1.93	1.05	1.00
	500	88.81	15.86	4.19	1.84	1.06	1.00	86.45	15.32	4.12	1.79	1.04	1.00
$G(3,1)$	50	487.66	183.23	41.04	9.87	1.71	1.12	428.55	149.58	38.82	8.98	1.62	1.04
	100	248.46	62.64	16.50	5.47	1.42	1.02	232.71	60.66	16.34	5.43	1.41	1.02
	500	161.96	40.75	11.94	4.38	1.30	1.00	155.82	39.18	11.74	4.37	1.32	1.01
$DE(0,1)$	50	262.87	140.40	63.33	20.07	2.94	1.14	208.04	122.37	49.94	20.03	2.45	1.15
	100	192.50	80.34	25.76	8.54	1.64	1.06	183.94	77.57	24.79	8.38	1.64	1.06
	500	169.47	51.80	15.65	5.32	1.39	1.05	165.55	51.68	16.16	5.63	1.40	1.04
$WE(2,1)$	50	311.19	100.71	22.13	6.08	1.52	1.04	293.88	83.93	19.68	6.06	1.46	1.04
	100	186.60	46.08	12.06	4.31	1.38	1.00	180.17	43.17	11.89	4.34	1.38	1.02
	500	139.79	33.34	10.21	3.99	1.35	1.02	133.43	31.90	9.68	3.86	1.33	1.02

Table 3 ARL_1 profile of the proposed scheme when $m \in \{50, 100, 500\}$, $ASS_0 \in \{6, 10\}$, $n_1 = 5$ and $n_2 = 8$ for a nominal $ARL_0 = 320$

Distribution	m	$(n_1, n_2, ASS_0) = (5, 8, 6)$						$(n_1, n_2, ASS_0) = (5, 8, 10)$					
		0.25	0.50	0.75	1.00	1.50	2.00	0.25	0.50	0.75	1.00	1.50	2.00
$N(0,1)$	50	185.57	39.49	7.89	2.66	1.15	1.01	143.33	30.82	6.81	2.42	1.11	1.00
	100	128.72	22.58	5.72	2.32	1.12	1.01	115.17	20.77	5.33	2.20	1.10	1.00
	500	89.65	17.53	5.25	2.35	1.16	1.01	87.24	16.48	4.73	2.12	1.50	1.00
$t(5)$	50	184.82	38.88	6.24	1.93	1.05	1.00	148.88	33.57	5.56	1.83	1.04	1.00
	100	119.39	17.93	3.87	1.64	1.03	1.00	106.30	16.27	3.69	1.59	1.02	1.00
	500	76.22	12.29	3.42	1.63	1.05	1.00	74.73	11.66	3.14	1.50	1.02	1.00
$G(3,1)$	50	593.23	171.97	35.39	6.54	1.37	1.02	412.11	212.90	25.87	7.31	1.36	1.02
	100	259.60	58.53	13.08	3.94	1.20	1.00	243.87	49.84	12.06	3.82	1.19	1.01
	500	143.92	32.41	9.07	3.33	1.17	1.00	142.64	31.56	8.64	3.16	1.14	1.00
$DE(0,1)$	50	258.95	144.68	49.44	13.21	1.72	1.07	227.84	119.31	40.43	14.18	1.85	1.08
	100	208.05	71.32	19.19	5.79	1.32	1.03	185.91	65.42	18.03	5.38	1.32	1.02
	500	153.02	40.24	11.07	3.69	1.25	1.04	150.83	39.24	10.99	3.67	1.18	1.02
$WE(2,1)$	50	348.98	91.83	16.92	4.61	1.32	1.02	258.30	71.94	15.46	4.13	1.03	1.02
	100	199.44	41.10	10.26	3.51	1.22	1.01	171.67	36.52	9.33	3.35	1.21	1.01
	500	125.44	27.66	8.19	3.26	1.25	1.01	122.99	26.36	7.55	3.02	1.17	1.00

the DS precedence scheme is then considered to be IC robust. In the next subsection, the OOC performance is investigated across the distributions considered in this paper.

4.2 OOC performance

Since the DS precedence scheme is IC robust, we can now proceed to investigate its sensitivity under different probability distributions. Tables 2 and 3 present the OOC performance of the proposed DS precedence scheme when $m \in \{50, 100, 500\}$, $n_1 = 5$, $n_2 \in \{6, 8\}$ and $ASS_0 \in \{6, 10\}$ under the $N(0, 1)$, $t(5)$, $G(3, 1)$, $DE(0, 1)$ and $WE(2, 1)$ distributions. The findings in Tables 2 and 3 can be summarised as follows:

- For large values of m , the DS precedence scheme performs better under symmetric distributions regardless of the magnitude of the shift and the Phase I sample sizes, in the location parameter. This is highlighted across all combinations of (n_1, n_2, ASS_0) .
- An increase in the value of ASS_0 keeping n_1 and n_2 fixed will improve the performance of the DS precedence scheme.
- An increase in the Stage 1 sample size, n_1 , keeping ASS_0 and n_2 fixed does not improve the sensitivity of the DS precedence scheme. However, an increase in the Stage 2 sample size, n_2 , results in a better OOC performance as it is expected based on the design. Thus, the larger n_1 and/or n_2 , the better the performance of the DS precedence scheme.
- Under the $t(5)$ distribution, the DS precedence scheme performs better across all combinations of (n_1, n_2, ASS_0) compared to the other distributions considered except for small shifts. Following the $t(5)$ distribution, the DS precedence scheme performs better under the symmetric distributions and, under heavy-tailed distributions, it performs relatively worse. That is, the DS precedence scheme performs better under the $N(0, 1)$, $DE(0, 1)$ and $WE(2, 1)$ distributions, and in this order.
- The DS precedence scheme performs worse, especially for small shifts under positively skewed distributions, see for instance the $G(3, 1)$ results in both Table 2 and 3. In addition, the proposed scheme is ARL -biased under the $G(3, 1)$ distribution for small Phase I sample sizes. As the Phase I sample size increases, the DS precedence scheme becomes ARL -unbiased.

4.3 Comparative analysis

4.3.1 Attained ARL values using simulation and closed-form expression

In this subsection, the results found using simulation and closed-form expression (CFE) approaches are compared in terms of the ARL profile when $ASS_0 =$

5, $(n_1, n_2) = (5, 6)$, $m \in \{100, 500\}$ for $ARL_0 = 360$. In this comparison, we used 10^5 replications to implement the simulation approach. To quantify the difference between the two approaches the absolute percentage difference, denoted as $\%Diff_{ARL}$, is computed as follows:

$$\%Diff_{ARL} = \left| \frac{ARL_{sim} - ARL_{CFE}}{ARL_{sim}} \right| \times 100, \tag{8}$$

where ARL_{sim} and ARL_{CFE} represent the attained ARL value computed using simulation and CFE, respectively.

From Table 4, the difference between the results found using the two approaches varies between 0 and 4.15% which is considerably small. The smaller the shifts, the larger the difference between the two approaches. In addition, the percentage difference between the two approaches under the $N(0,1)$ distribution is smaller than the ones under non-normal distributions. This is due to the fact that the DS precedence scheme performs better under symmetric distributions than it does under heavy-tailed distributions. The larger the Phase I sample size, the smaller the difference between the results found using the two approaches.

4.3.2 Simulation versus closed-form expression computational time comparison

In this subsection, we present a head-to-head comparison between the time (in seconds) taken to compute the ARL value for a specific shift using simulation and the proposed CFE approaches when $ASS_0 = 5$, $(n_1, n_2) = (5, 6)$, $m \in \{50, 100, 500\}$ under the $N(0,1)$, $t(5)$, $G(3,1)$, $DE(0,1)$ and $WE(2,1)$ distributions for $ARL_0 = 320$ using SAS®9.4 and Mathcad 8 prime, respectively. Note that when the process is IC, the computational time, denoted as τ_c , depends on the time required to determine the six charting constants and the computation of the IC ARL value. However, when the process is OOC ($\delta \neq 0$), the charting constants found when the process was deemed to be IC, are used to compute the OOC ARL .

In addition, to quantify the difference between the time taken to compute the ARL value using simulation and the proposed CFE, we define the absolute percentage difference in computational time, $\%Diff_{\tau_c}$, of the two approaches as follows:

$$\%Diff_{\tau_c} = \left| \frac{\tau_c^{(sim)} - \tau_c^{(CFE)}}{\tau_c^{(sim)}} \right| \times 100, \tag{9}$$

where $\tau_c^{(sim)}$ and $\tau_c^{(CFE)}$ represent the amount of time taken to compute the ARL value for a specific shift using simulation and CFE, respectively. Note that the absolute percentage difference for the amount of time taken to compute other run-length characteristics can be defined and computed in a similar way.

Using Mathcad 8 prime, it can be noticed that the computational time of the IC ARL values using CFE varies between 47 and 56 sec depending on the sample sizes. The results from Table 5 reveals that the percentage difference between the computational time of the two approaches varies between 0 and 92.56% with the CFE

Table 4 ARL profile and absolute percentage difference of the DS precedence scheme using simulation and proposed CFE when $m \in \{100, 500\}$ for $ARL_0 = 370$

m	Shift	$N(0,1)$			$t(5)$			$G(3,1)$			$DE(0,1)$			$WE(2,1)$		
		sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff
100	0.00	359.71	350.24	2.63	363.28	350.24	3.59	361.93	350.24	3.23	365.42	350.24	4.15	365.30	350.24	4.12
	0.25	231.12	237.06	2.57	259.71	269.23	3.67	575.50	558.01	3.04	320.56	330.51	3.10	411.87	400.04	2.87
	0.50	79.47	81.46	2.50	107.38	110.21	2.64	255.49	262.57	2.77	240.41	235.01	2.25	151.94	155.82	2.55
	0.75	26.35	26.83	1.82	37.86	38.89	2.72	103.74	105.99	2.17	150.35	153.99	2.42	54.74	56.16	2.59
	1.00	10.13	9.97	1.58	12.70	13.01	2.44	44.72	45.58	1.92	81.60	80.09	1.85	22.01	21.52	2.23
500	1.50	2.66	2.61	1.88	2.78	2.72	2.16	9.52	9.33	2.00	21.76	22.08	1.47	5.00	5.08	1.60
	2.00	1.35	1.36	0.74	1.32	1.31	0.76	2.81	2.84	1.07	6.09	6.20	1.81	1.92	1.90	1.04
	0.00	361.46	356.67	1.33	362.25	356.67	1.54	360.93	356.67	1.18	362.82	356.67	1.70	361.86	356.67	1.43
	0.25	149.80	151.31	1.01	160.95	159.09	1.16	250.52	252.74	0.89	240.37	243.15	1.16	206.48	209.50	1.46
	0.50	40.01	40.26	0.62	44.12	43.69	0.97	85.61	86.54	1.09	109.16	111.06	1.74	63.83	64.80	1.52
500	0.75	13.71	13.81	0.73	14.27	14.08	1.33	34.39	34.11	0.81	50.69	51.27	1.14	23.84	24.10	1.09
	1.00	6.02	6.07	0.83	5.66	5.61	0.88	15.57	15.78	1.35	25.01	25.43	1.68	10.62	10.71	0.85
	1.50	2.03	2.04	0.49	1.70	1.69	0.59	4.19	4.16	0.72	6.97	7.04	1.00	3.16	3.13	0.95
	2.00	1.12	1.12	0.00	1.10	1.11	0.91	1.68	1.69	0.60	2.28	2.30	0.88	1.50	1.49	0.67

Table 5 Absolute percentage difference and amount of time taken to compute the ARL values for specific shifts using simulation and proposed CFE when $m \in \{100, 500\}$ for $ARL_0 = 360$

m	Shift	$N(0,1)$			$t(5)$			$G(3,1)$			$DE(0,1)$			$WE(2,1)$		
		sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff	sim	CFE	%Diff
100	0.00	151	47	68.87	263	48	81.75	207	50	75.85	308	49	84.09	331	49	85.20
	0.25	65	21	67.69	196	29	85.20	368	31	91.58	303	30	90.10	368	31	91.58
	0.50	48	17	64.58	68	18	73.53	148	19	87.16	190	19	90.00	129	19	85.27
	0.75	19	7	63.16	28	8	71.43	66	9	86.36	121	9	92.56	51	8	84.31
	1.00	10	5	50.00	12	6	50.00	41	6	85.37	65	8	87.69	20	6	70.00
	1.50	6	5	16.67	7	6	14.29	11	6	45.45	20	7	65.00	8	6	25.00
500	2.00	4	4	0.00	5	5	0.00	5	5	0.00	8	6	25.00	5	5	0.00
	0.00	257	51	80.16	321	52	83.80	212	50	76.42	433	55	87.30	502	56	88.84
	0.25	73	33	54.79	89	33	62.92	152	35	76.97	191	35	81.68	156	33	78.85
	0.50	27	19	29.63	38	22	42.11	75	25	66.67	93	27	70.97	59	24	59.32
	0.75	14	10	28.57	22	11	50.00	47	17	63.83	58	23	60.34	27	16	40.74
	1.00	10	7	30.00	14	7	50.00	28	10	64.29	28	13	53.57	16	9	43.75
	1.50	8	5	37.50	12	5	58.33	13	7	46.15	16	9	43.75	10	7	30.00
	2.00	8	4	50.00	12	5	58.33	11	5	54.55	13	5	61.54	9	5	44.44

approach having an upper hand i.e. it requires less time to compute the *ARL* value as compared to the simulation approach. This means one can save up to 92.56% of the time spent computing the *ARL* when using the proposed CFE instead of simulation. As the magnitude of the shift increases, τ_c decreases. The $\%Diff_{\tau_c}$ values are smaller under the normal distribution compared to non-normal distributions. This is because the simulation under the normal distribution takes less time as compared to the simulations under non-normal distributions. Astivia (2020) highlights how simulating from non-normal distributions may come with issues of estimation and mathematical tractability, and many more. However, the computational time, if it were to be studied for *rnorm* (i.e., the R function that generates random variates having a specified normal distribution), would outperform other non-normal distributions, especially for large simulations.

Note that for a large Phase I sample size, for both simulation and CFE approaches, the amount of time taken to compute the *ARL* value for a specific shift is larger than the one for small Phase I sample sizes (see Fig. 2a-d). Figure 3 compares the

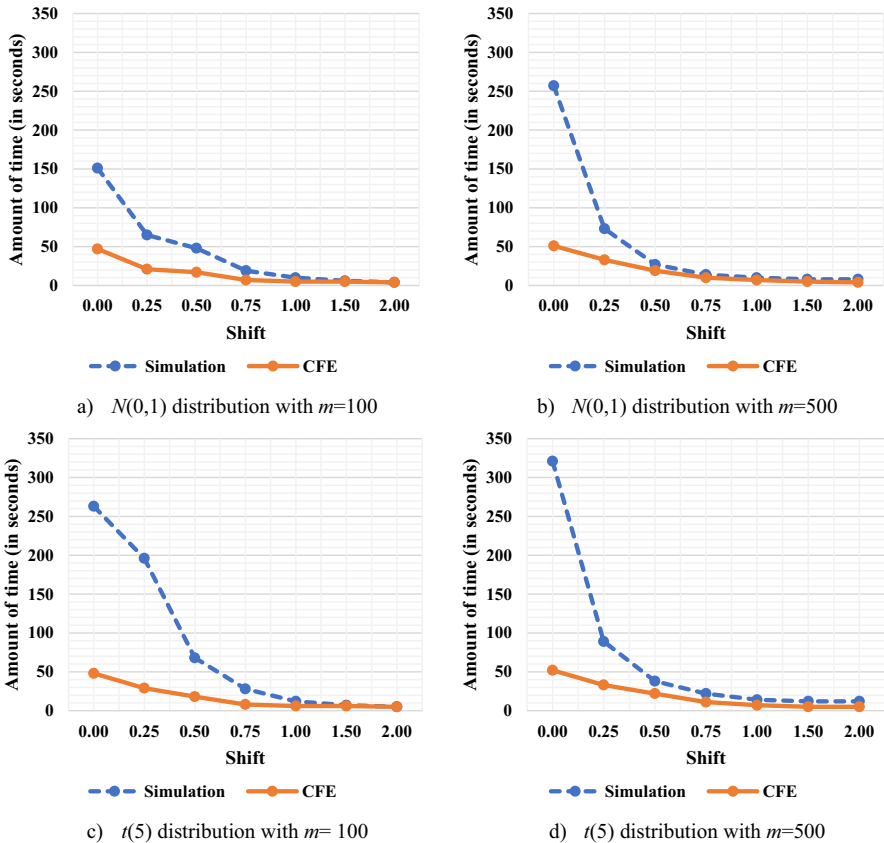


Fig. 2 Amount of time taken to compute the *ARL* value for different shift when $m \in \{100,500\}$ under the $N(0,1)$ and $t(5)$ distributions

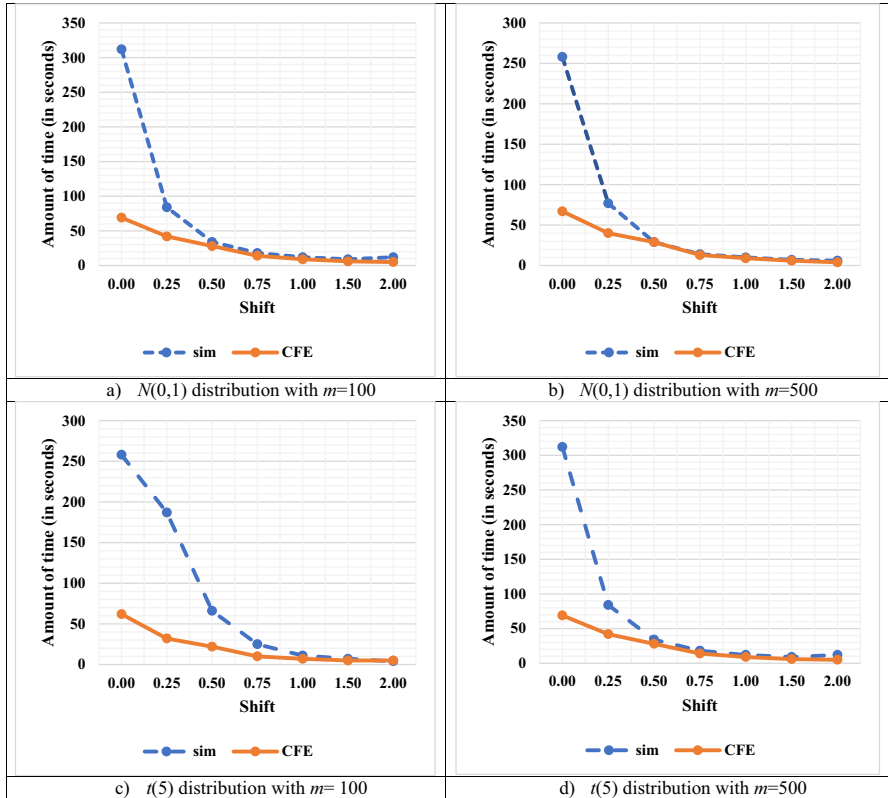


Fig. 3 Amount of time taken to compute the 95th percentile ($P_{0.95}$) for different shift when $m \in \{100,500\}$ under the $N(0,1)$ and $t(5)$ distributions

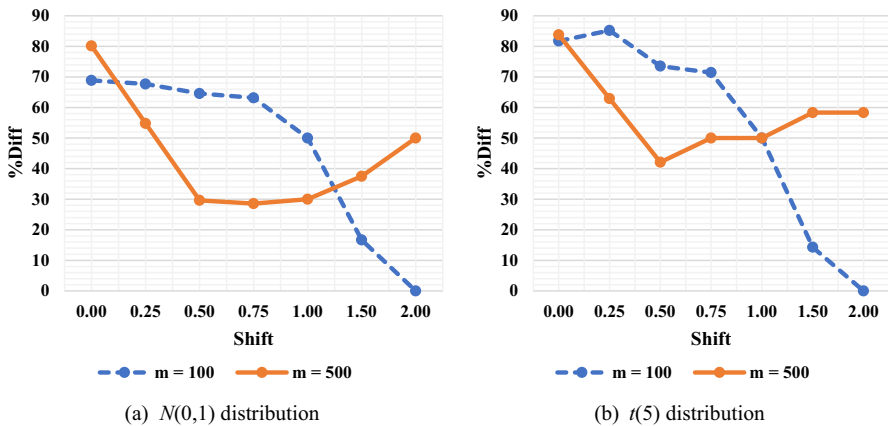


Fig. 4 Absolute percentage difference required to compute the ARL value using simulation and the proposed CFE approaches when $m \in \{100,500\}$ under the $N(0,1)$ and $t(5)$ distributions

computational times between the Monte Carlo simulation approach and the use of the CFE in the computation of the 95th percentile points (e.g. $P_{0.95}$) of the run-length distribution. Before we discuss the results displayed in this figure, one should know that all the characteristics of the run-length distribution can be computed using one specific procedure (e.g., PROC UNIVARIATE in SAS®IML9.4). In other words, the amount of time taken to compute the *ARL* will then be the same as that of computing the percentiles. Note that $P_{0.95}$ is computed using the CFE defined in Eq. (7b). Figure 3a–d show that for small shifts, the high percentile points are computed faster using CFE as compared to simulations. However, for moderate and large shifts, the computational time is almost the same for both methods. Figure 4a–b show that for small shifts the required amount of time is significantly reduced for small Phase I sample sizes. However, for moderate and large shifts the amount of time is significantly reduced for large Phase I sample sizes.

5 Illustrative example

In this section, real-world data from iron ore mining are used to illustrate the application and implementation of the DS precedence scheme in a froth flotation process (see Mukherjee et al. (2019)). Froth flotation is often used to improve the iron concentration of low-grade iron ores. Low-grade iron ores contain high concentration of impurities such as silicon dioxide (known as silica), phosphorus and alumina containing minerals, which are undesired. Froth flotation helps to remove the main impurity.

This example is implemented in two phases, namely Phase I and Phase II. The Phase I regime is used to estimate the control limits of the IC process. In Phase II, the control limits found in Phase I are used to monitor the percentage of silica that is present on each iron ore sample at the end of the flotation process.

5.1 Phase I analysis

The data from the flotation plant were collected between 01 June 2017 and 31 July 2017. In this example, 67 master samples, each of size $n = 9$ (i.e. 603 observations) were considered and the null hypothesis of the Shapiro–Wilk test for normality was rejected with p -value = 2×10^{-16} , which implies that the data are clearly not normally distributed. Each master sample was divided into two samples of sizes $n_1 = 3$ and $n_2 = 6$ to be used in Stages 1 and 2, respectively. The first step in the implementation of the DS precedence scheme is to find the charting constants that yield an attained ARL_0 closer to 370. Thus, it was found that when $m = 603$, $(n_1, n_2) = (3, 6)$ and $ASS_0 = 5$, the combination of the charting constants $(a_2, a_1, b_1, b_2, c_1, c_2) = (162, 238, 366, 442, 24, 580)$ corresponding to the control limits $OLCL = 1.38$, $ILCL = 1.58$, $IUCL = 1.90$, $OUCL = 2.20$, $LCL = 1.05$ and $UCL = 4.55$, respectively, so that the DS precedence monitoring scheme yields an attained $ARL_0 = 354.88$. The second step is to construct the DS

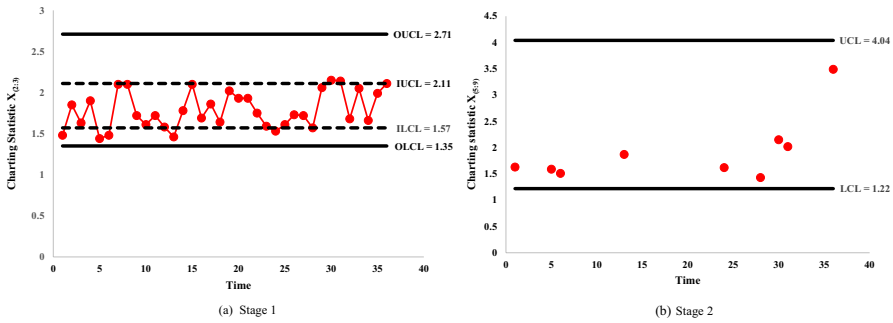


Fig. 5 Phase I analysis

precedence scheme using the control limits found in Step 1 and plot the charting statistics. This Phase I analysis detected 31 samples that plotted OOC which were subsequently removed from the process (this plot is not shown in this paper to conserve space). Since the Phase I sample is not IC, then the control limits must be revised using the reduced Phase I sample. Thus, after removing the 31 samples (i.e. 279 observations), the remaining data were analysed again. It was found that when $m = 324$, $(n_1, n_2) = (3, 6)$ and $ASS_0 = 5$, the combination of the charting constants $(a_2, a_1, b_1, b_2, c_1, c_2) = (43, 97, 228, 282, 15, 310)$ corresponding to the control limits $OLCL = 1.35$, $ILCL = 1.57$, $IUCL = 2.11$, $OUCL = 2.71$, $LCL = 1.22$ and $UCL = 4.04$, respectively, so that the DS precedence scheme yields an attained $ARL_0 = 341.98$. Figure 5 shows that the revised Phase I sample is now IC as in Stage 1 the charting statistics plot either in region B or C, and in Stage 2 they all plot IC (see Fig. 5). Since the Phase I sample is now considered to be IC, the percentage of the silica can be monitored continuously in Phase II.

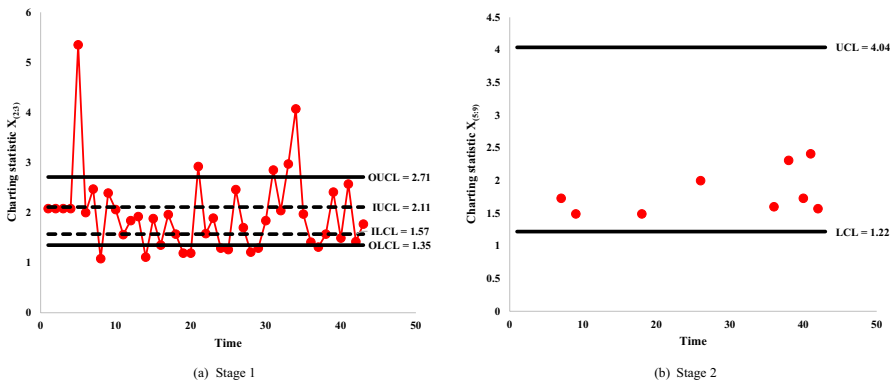


Fig. 6 Phase II analysis

5.2 Phase II analysis

In Phase II, the revised control limits found in Phase I are used to monitor the percentage of silica. The Phase II data contain 43 master samples of size 9 divided such that $n_1 = 3$ and $n_2 = 6$. Figure 6 shows that the charting statistic of the fifth sample plotted beyond the *OUCL*. Therefore, the DS precedence scheme will give a signal on the fifth sample (see Fig. 6).

In practice, the advantage of the proposed approach over the simulation approach is due to the computation of the control limits in Phase I. It takes a smaller amount of time to compute the control limits of the DS precedence scheme using CFE than using simulations. However, when it comes to monitor the process in Phase II both approaches take the same amount of time.

6 Conclusion

This paper introduced a new closed-form expression (CFE) approach to compute the characteristics of the run-length distribution of the DS precedence scheme. It is found that the CFE approach considerably reduces the amount of time taken to compute the run-length properties of the DS precedence monitoring scheme. Moreover, the run-length characteristics found using the proposed CFE approach are noticeably closer to the ones found using simulation. However, quality operators who are willing to use order statistics monitoring schemes are recommended to use the proposed approach instead of simulations. The reason behind this is that to reach more accurate results, one must use a very large number of replications which will consequently increase the amount of time needed to compute the properties of the monitoring scheme. In future, researchers are recommended to derive more CFE for other types of schemes such as nonparametric EWMA and CUSUM schemes using other order statistics such as the exceedance, minimum, and maximum statistics.

Appendix

This Appendix provides the derivations of the in-control probability at Stage 2 denoted as p_{02} . It is known that

$$p_{02} = \text{Prob}\left(Y_{(j:n_1)} \in B \text{ and } Y_{(h:n)} \in E\right) = \text{Prob}\left(Y_{(j:n_1)} \in B_1 \text{ and } Y_{(h:n)} \in E\right) \\ + \text{Prob}\left(Y_{(j:n_1)} \in B_2 \text{ and } Y_{(h:n)} \in E\right),$$

where

$$B_1 = \left\{X_{(a_2:m)} < Y_{(j:n_1)} \leq X_{(a_1:m)}\right\}, B_2 = \left\{X_{(b_1:m)} \leq Y_{(j:n_1)} < X_{(b_2:m)}\right\} \text{ and} \\ E = \left\{X_{(c_1:m)} \leq Y_{(h:n)} < X_{(c_2:m)}\right\}$$

Then, because of symmetry:

$$Prob\left(Y_{(j:n_1)} \in B_1 \text{ and } Y_{(h:n)} \in E\right) = Prob\left(Y_{(j:n_1)} \in B_2 \text{ and } Y_{(h:n)} \in E\right).$$

Therefore,

$$p_{02} = 2Prob\left(Y_{(j:n_1)} \in B_1 \text{ and } Y_{(h:n)} \in E\right) \tag{A1}$$

Now

$$\begin{aligned} & Prob\left(Y_{(j:n_1)} \in B_1 \text{ and } Y_{(h:n)} \in E\right) \\ &= Prob\left(X_{(a_2:m)} < Y_{(j:n_1)} \leq X_{(a_1:m)} \text{ and } X_{(c_1:m)} \leq Y_{(h:n)} < X_{(c_2:m)}\right) \\ &= Prob\left(Y_{(j:n_1)} \leq X_{(a_1:m)}, Y_{(h:n)} < X_{(c_2:m)}\right) \\ &\quad - Prob\left(Y_{(j:n_1)} \leq X_{(a_1:m)}, Y_{(h:n)} < X_{(c_1:m)}\right) \\ &\quad - Prob\left(Y_{(j:n_1)} \leq X_{(a_2:m)}, Y_{(h:n)} < X_{(c_2:m)}\right) \\ &\quad + Prob\left(Y_{(j:n_1)} \leq X_{(a_2:m)}, Y_{(h:n)} < X_{(c_1:m)}\right) \end{aligned}$$

Define

$$L(l_1, l_2, l_3, l_4) = Prob\left(Y_{(l_1:n_1)} \leq X_{(l_3:m)}, Y_{(l_2:n)} < X_{(l_4:m)}\right)$$

Then

$$\begin{aligned} Prob\left(Y_{(j:n_1)} \in B_1 \text{ and } Y_{(h:n)} \in E\right) &= L(j, h, a_1, c_2) - L(j, h, a_1, c_1) \\ &\quad - L(j, h, a_2, c_2) + L(j, h, a_2, c_1) \end{aligned}$$

Let $\{Y_{11}, Y_{12}, \dots, Y_{1n_1}\}$ be the random variables obtained at the first sample. Let $\{Y_{21}, Y_{22}, \dots, Y_{2n_2}\}$ be the random variables obtained at the second sample. Finally, let $\{Y_1, Y_2, \dots, Y_n\}$ be the $n = n_1 + n_2$ random variables obtained through combining the first Stage of n_1 observations and the second Stage of n_2 observations.

Then, we can consider three scenarios:

1. When $l_3 < l_4$, we have:

$$\begin{aligned} L(l_1, l_2, l_3, l_4) &= Prob\left(Y_{(l_1:n_1)} \leq X_{(l_3:m)}, Y_{(l_2:n)} \leq X_{(l_4:m)}\right) \\ &= \int_{-\infty}^{\infty} \int_{x_1}^{\infty} Prob\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2\right) \frac{m!}{(l_3 - 1)!(l_4 - l_3 - 1)!(m - l_4)!} F(x_1)^{l_3 - 1} \\ &\quad \times (F(x_2) - F(x_1))^{l_4 - l_3 - 1} (1 - F(x_2))^{m - l_4} f(x_1) f(x_2) dx_1 dx_2. \end{aligned}$$

Now

$$\begin{aligned}
 & Prob\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2\right) \\
 &= Prob\left(\text{atleast } l_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1, Y_{(l_2:n)} < x_2\right) \\
 &= \sum_{i_1=l_1}^{n_1} Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1, Y_{(l_2:n)} < x_2\right) \\
 &= \sum_{i_1=l_1}^{n_1} Prob\left(Y_{(l_2:n)} < x_2 \mid \text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &\times Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} Prob\left(\text{atleast } l_2 \text{ of } Y_1, Y_2, \dots, Y_n \leq x_2 \mid \text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &\times Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right).
 \end{aligned} \tag{A2}$$

For a given i_1 ($l_1 \leq i_1 \leq n_1$), let $Y_{11}^*, Y_{12}^*, \dots, Y_{1(n_1-i_1)}^*$ represent the random variables among $Y_{11}, Y_{12}, \dots, Y_{1n_1}$ which are obtained after deleting the i_1 observations. Then:

$$\begin{aligned}
 & Prob\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2\right) \\
 &= \sum_{i_1=l_1}^{n_1} Prob\left(\text{atleast } l_2^* \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1(n_1-i_1)}^* \text{ and } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2\right) \\
 &\times Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right), \left(l_2^* = \max\{0, (l_2 - i_1)\}\right) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=l_2^*}^{n-i_1} Prob\left(\text{exactly } i_1 \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1(n_1-i_1)}^* \text{ and } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2\right) \\
 &\times Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=l_2^*}^{n-i_1} \sum_{i_1^*=\max\{0, i-i_1\}}^{\min\{n_1-i_1, i-i_1\}} Prob\left(\text{exactly } i_1^* \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1(n_1-i_1)}^*\right) \\
 &\leq x_2 \text{ and exactly } (i - i_1^*) \text{ among } Y_{21}, Y_{22}, \dots, Y_{2n_2} \\
 &\leq x_2 \times Prob\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i_1=l_1}^{n_1} \sum_{i=l_2}^{n-i_1} \sum_{i_1^*=\max\{0, i-n_2\}}^{\min\{n_1-i_1, i-i_1\}} \text{Prob}\left(\text{exactly } i_1^* \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1(n_1-i_1)}^* \leq x_2\right) \\
 &\times \text{Prob}(\text{exactly } (i - i_1^*) \text{ among } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2) \\
 &\times \text{Prob}(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i=l_2}^{n-i_1} \sum_{i_1^*=\max\{0, i-n_2\}}^{\min\{n_1-i_1, i-i_1\}} \left\{ \binom{n_1-i_1}{i_1^*} F(x_2)^{i_1^*} (1-F(x_2))^{n_1-i_1-i_1^*} \right\} \\
 &\times \left\{ \binom{n_2}{i-i_1^*} F(x_2)^{i-i_1^*} (1-F(x_2))^{n_2-i_1-i_1^*} \right\} \\
 &\times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1-F(x_1))^{n_1-i_1} \right\} \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i=l_2}^{n-i_1} \sum_{i_1^*=\max\{0, i-n_2\}}^{\min\{n_1-i_1, i-i_1\}} \left\{ \binom{n_1-i_1}{i_1^*} \binom{n_2}{i-i_1^*} F(x_2)^i (1-F(x_2))^{n-i-i_1} \right\} \\
 &\times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1-F(x_1))^{n_1-i_1} \right\}.
 \end{aligned}$$

2. When $l_3 = l_4$, it follows that

$$\begin{aligned}
 L(l_1, l_2, l_3, l_4) &= L(l_1, l_2, l_3, l_3) = \text{Prob}\left(Y_{(l_1:n_1)} \leq X_{(l_3:m)}, Y_{(l_2:n)} \leq X_{(l_3:m)}\right) \\
 &= \int_{-\infty}^{\infty} \text{Prob}\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_1\right) \frac{m!}{(l_3-1)!(m-l_3)!} \\
 &\quad F(x_1)^{l_3-1} (1-F(x_1))^{m-l_3} f(x_1) dx_1.
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \text{Prob}\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_1\right) \\
 &= \text{Prob}\left(\text{at least } l_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1, Y_{(l_2:n)} < x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \text{Prob}\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1, Y_{(l_2:n)} < x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \text{Prob}\left(Y_{(l_2:n)} < x_1 \mid \text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &\times \text{Prob}\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \text{Prob}\left(\text{at least } l_2 \text{ of } Y_1, Y_2, \dots, Y_n \leq x_1 \mid \text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &\times \text{Prob}\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_2=l_2}^{n_2} \text{Prob}\left(\text{exactly } i_2 \text{ of } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2\right) \\
 &\times \text{Prob}\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_2=l_2}^{n_2} \left\{ \binom{n_2}{i_2} F(x_2)^{i_2} (1 - F(x_2))^{n_2-i_2} \right\} \\
 &\times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1 - F(x_1))^{n_1-i_1} \right\}.
 \end{aligned}$$

3. When $l_3 > l_4$, From (A.2) we have

$$\begin{aligned}
 & L(l_1, l_2, l_3, l_4) \text{Prob}\left(Y_{(l_1:n_1)} \leq X_{(l_3:m)}, Y_{(l_2:n)} \leq X_{(l_4:m)}\right) \\
 &= \int_{-\infty}^{\infty} \int_{x_1}^{\infty} \text{Prob}\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2\right) \frac{m!}{(l_3-1)!(l_4-l_3-1)!(m-l_4)!} F(x_1)^{l_3-1} \\
 &\times (F(x_2) - F(x_1))^{l_4-l_3-1} (1 - F(x_2))^{m-l_4} f(x_1) f(x_2) dx_1 dx_2.
 \end{aligned}$$

From (A2) it follows that,

$$\begin{aligned}
 & \text{Prob}\left(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2\right) \\
 &= \sum_{i_1=l_1}^{n_1} \text{Prob}\left(\text{at least } l_2 \text{ of } Y_1, Y_2, \dots, Y_n \leq x_2 \mid \text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right) \\
 &\times \text{Prob}\left(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1\right)
 \end{aligned}$$

For a given i_1 ($l_1 \leq i_1 \leq n_1$), let $Y_{11}^*, Y_{12}^*, \dots, Y_{1i_1}^*$ represent the random variables among $Y_{11}, Y_{12}, \dots, Y_{1n_1}$ which are $\leq x_1$. Then:

$$\begin{aligned}
 & Prob(Y_{(l_1:n_1)} \leq x_1, Y_{(l_2:n)} \leq x_2) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=0}^{i_1} Prob(\text{at least } i_1^* \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1i_1^*}^* \text{ and at least } i_2^* \text{ among } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2) \\
 &\times Prob(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1), (l_2^* = \max\{0, (l_2 - i_1^*)\}) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=0}^{i_1} \sum_{i_2^*=l_2}^{n_2} Prob(\text{at least } i_1^* \text{ among } Y_{11}^*, Y_{12}^*, \dots, Y_{1i_1^*}^* \text{ and at least } i_2^* \text{ among } Y_{21}, Y_{22}, \dots, Y_{2n_2} \leq x_2) \\
 &\times Prob(\text{exactly } i_1 \text{ of } Y_{11}, Y_{12}, \dots, Y_{1n_1} \leq x_1) \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=0}^{i_1} \sum_{i_2^*=l_2}^{n_2} \left\{ \binom{i_1}{i_1^*} F(x_2)^{i_1^*} (1 - F(x_2))^{i_1 - i_1^*} \right\} \\
 &\times \left\{ \binom{n_2}{i_2^*} F(x_2)^{i_2^*} (1 - F(x_2))^{n_2 - i_2^*} \right\} \times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1 - F(x_1))^{n_1 - i_1} \right\} \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=0}^{i_1} \sum_{i_2^*=l_2}^{n_2} \left\{ \binom{i_1}{i_1^*} F(x_2)^{i_1^*} (1 - F(x_2))^{i_1 - i_1^*} \right\} \\
 &\times \left\{ \binom{n_2}{i_2^*} F(x_2)^{i_2^*} (1 - F(x_2))^{n_2 - i_2^*} \right\} \times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1 - F(x_1))^{n_1 - i_1} \right\} \\
 &= \sum_{i_1=l_1}^{n_1} \sum_{i_1^*=0}^{i_1} \sum_{i_2^*=l_2}^{n_2} \left\{ \binom{i_1}{i_1^*} \binom{n_2}{i_2^*} F(x_2)^{i_1^* + i_2^*} (1 - F(x_2))^{n_2 + i_1 - i_1^* - i_2^*} \right\} \\
 &\times \left\{ \binom{n_1}{i_1} F(x_1)^{i_1} (1 - F(x_1))^{n_1 - i_1} \right\}
 \end{aligned}$$

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