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COLLECTIVELY COMPACT AND COLLECTIVELY STRICTLY SINGULAR SETS OF LINEAR OPERATORS

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Collectively Compact and Collectively Strictly Singular Sets of Linear Operators

by

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Summary

In this thesis the concept of collectively compact sets of operators is studied. As a reason for the study of such operators it is shown how collectively compact sets of operators are applicable to an approximation theory for Fredholm integral equations of the second kind where the kernel is continuous. In this case the integral operator mapping $C[a,b]$ into $C[a,b]$ is compact and the set of numerical-integral operators approximating the integral operator is collectively compact. Convergence theorems and error bounds are given for this type of situation.

Once the importance of the concept of collective compactness has been established, properties of such sets of operators are studied. A characterisation of collectively compact sets of operators in terms of countable subsets is given. In addition, a comparison between totally bounded sets and collectively compact sets of compact operators is done since the approximation theory mentioned above is applicable to sets of operators that are collectively compact but not totally bounded. Perturbation theorems involving perturbations of semi-Fredholm operators with collectively compact sets of operators are also studied.

The concept of collectively strictly singular sequences of operators is defined and perturbation theorems for perturbations of semi-Fredholm operators with collectively strictly singular sequences of operators are given. It is probable that the concept of collective strict singularity might be applicable in establishing an approximation theory for Fredholm integral equations of the second kind with measurable, discontinuous kernel where the integral operator maps the Lebesgue space $\mathcal{L}_1$ into $\mathcal{L}_1$.

The concept of collectively strictly cosingular sequences of operators naturally arises and is therefore defined. It is noted that analogous perturbation theorems to the ones proved for collectively strictly singular sequences of operators could easily be proven by suitably dualising the proofs for the above-mentioned theorems.

Kollektief kompakte en kollektief streng singuliere versamelings van linêere operatore

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Opsomming

In hierdie verhandeling word die konsep van 'n kollektief kompakte versameling van operatore bestudeer. Ter motivering vir die bestudering van hierdie klas van operatore word daar aangetoon hoe hierdie operatore aangewend kan word om 'n approksimasieteorie te ontwikkel vir Fredholm-integraalvergelykings van die tweede tipe. Ons kyk in besonder na die geval waar die kern van die integraaloperator kontinu is sodat die operator, wat $C[a,b]$ in $C[a,b]$ afbeeld, kompak is. Konvergensiestellings en foutafskattings vir hierdie situasie word afgelei.

Eienskappe van kollektief kompakte versamelings van operatore word beskou sowel as 'n karakterisering van sulke versamelings in terme van aftelbare deelversamelings. Omdat die bogenoemde approksimasieteorie veral toepaslik is op versamelings van operatore wat kollektief kompak maar nie totaal begrens is nie, word 'n vergelyking getref tussen totaal begrensde versamelings en kollektief kompakte versamelings van kompakte operatore. 'n Perturbasieteorie vir perturbasies van semi-Fredholm-operatore met kollektief kompakte versamelings van operatore word ook bespreek.

Die begrip van 'n kollektief-streng-singuliere ry van operatore word gedefinieer en perturbasiestellings vir perturbasies van semi-Fredholm-operatore met versamelings van kollektief-streng-singuliere operatore word gegee. 'n Moontlike toepassing van die konsep van kollektiewe-streng-singulariteit bestaan in die ontwikkeling van 'n approksimasieteorie vir integraaloperatore met 'n meetbare, nie-kontinue kern waar die integraaloperator die Lebesgue ruimte $\mathcal{L}_1$ afbeeld op $\mathcal{L}_1$.

Die begrip van 'n kollektief-streng-kosinguliere ry van operatore volg op 'n natuurlike wyse uit die idee van kollektiewe-streng-singulariteit en 'n duale perturbasieteorie kan maklik ontwikkel word.

Contents

Chapter 1

Introduction

In this thesis we will consider collectively compact sets of operators, their application, their properties and perturbations of semi-Fredholm operators with this class of operators. We will then consider two related classes of operators, namely collectively strictly singular and collectively strictly cosingular sequences of operators. Perturbation theorems analogous to those proved for collectively compact sets of operators will be established and a possible application of collectively strictly singular sets of operators will be discussed.

In this chapter we will consider some basic definitions and results which are needed in the following chapters. By no means should this chapter be seen as exhaustive and the reader is referred to any introductory work on functional analysis such as [K] or [KR] for a more complete discussion of operator theory.

1.1 Sets and functions.

Definition 1.1.1 (Complete metric space.) *A complete metric space X is a metric space in which every Cauchy sequence is convergent in X.*

Definition 1.1.2 (ε-net, Totally bounded set.) *Let A be any subset of a metric space X and let $\varepsilon > 0$ be given. A subset M of X is called an ε-net of A if each point of A is distant by less than ε from at least one point of M. The set A is called totally bounded if for every $\varepsilon > 0$ there exists a finite ε-net for A.*

Definition 1.1.3 (Compact set.) *A set A in a metric space X is said to be compact if every sequence in A has a subsequence converging in A.*

Take note that a set is called relatively compact if its closure is compact. The following Theorem is well known. For a proof of the theorem we refer the reader to, for example, [S; 25-A].

Theorem 1.1.4 *A metric space X is compact if and only if it is complete and totally bounded.*

Definition 1.1.5 (Uniform continuity.) *Let X and Y be metric spaces with metrics d_1 and d_2 respectively. Then a function $f : X \to Y$ is uniformly continuous if for each $\varepsilon > 0$, there exists $\lambda > 0$ such that $d_2(f(x), f(x')) < \varepsilon$ for all $x, x' \in X$ with $d_1(x, x') < \lambda$.*

Definition 1.1.6 (Equicontinuity.) *Let X be a compact metric space with metric d, and let $C[X]$ be the nonempty set of continuous real or complex functions defined on X. A subset B of $C[X]$ is called equicontinuous if for each $\varepsilon > 0$ there exists a $\lambda > 0$ such that for every $f \in B$, $|f(x) - f(x')| < \varepsilon$ for all $x, x' \in X$ with $d(x, x') < \lambda$.*

Theorem 1.1.7 (Ascoli's Theorem) *If X is a compact metric space, then a closed subspace of $C[X]$ is compact if and only if it is bounded and equicontinuous.*

For a proof of this well known Theorem we refer the reader to, for example, [Mc; p.336] or [S; 25-C].

1.2 Normed spaces and Banach spaces.

Definition 1.2.1 (Normed space, Banach space.) *Let $\mathbb{X}$ be a vector space over the field of real or complex numbers. A norm on $\mathbb{X}$, denoted by $\|\ \|$, is a real-valued function on $\mathbb{X}$ such that the following properties hold for all $x, y \in \mathbb{X}$ and all scalars α:*

1. $\|x\| \geq 0$
2. $\|x\| = 0$ *if and only if* $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$.

The vector space $\mathbb{X}$ together with a norm on $\mathbb{X}$ is called a normed linear space or a normed space. If $\mathbb{X}$ is a complete normed space, we call it a Banach space.
A norm on $\mathbb{X}$ induces a metric d on $\mathbb{X}$ which is given by

$$d(x,y) = \|x - y\|.$$

Throughout this thesis we assume $\mathbb{X}$ and $\mathbb{Y}$ to be normed linear spaces unless stated otherwise.

Definition 1.2.2 *If $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{X}$, then*

$$A + B = \{x + y : x \in A, y \in B\}.$$

Proposition 1.2.3 *Let $A_i \subseteq \mathbb{X}$ for each $i \in \{1, \dots, n\}$. Then*

$$\overline{\sum_{k=1}^{n} \overline{A_k}} = \overline{\sum_{k=1}^{n} A_k}$$

.

Proof: $\sum_{k=1}^{n} \overline{A_k}$ clearly will contain $\sum_{k=1}^{n} A_k$. Since $\overline{\sum_{k=1}^{n} A_k}$ is the smallest closed subset containing $\sum_{k=1}^{n} A_k$, $\overline{\sum_{k=1}^{n} A_k} \subseteq \overline{\sum_{k=1}^{n} \overline{A_k}}$. Conversely, choose $x = \sum_{k=1}^{n} x_k \in \sum_{k=1}^{n} \overline{A_k}$. For every k, choose a sequence $(x_k^{(m)}) \subset A_k$ such that $x_k^{(m)} \to x_k$ as $m \to \infty$. Then $\sum_{k=1}^{n} x_k^{(m)} \in \sum_{k=1}^{n} A_k$ and we leave it as an exercise that $\sum_{k=1}^{n} x_k^{(m)} \to \sum_{k=1}^{n} x_k$ as $m \to \infty$. Therefore $\sum_{k=1}^{n} x_k \in \overline{\sum_{k=1}^{n} A_k}$ and hence $\overline{\sum_{k=1}^{n} \overline{A_k}} \subset \overline{\sum_{k=1}^{n} A_k}$. □

Definition 1.2.4 *For $x \in \mathbb{X}$ and M a subset of $\mathbb{X}$, we define the distance between x and M as*

$$d(x, M) = \inf_{m \in M} \|x - m\|.$$

Definition 1.2.5 *For r a positive number, we define the closed r-ball in $\mathbb{X}$ as the set*

$$\{x \in \mathbb{X} : \|x\| \leq r\}.$$

We will denote the closed unit ball in $\mathbb{X}$ by $B_{\mathbb{X}}$.

Before defining further concepts in normed linear spaces, we take note of the following result stated here without proof. The forward implication is obvious and the reverse implication depends on Riesz's Lemma. For a detailed proof we refer the reader to [K; 2.5-5].

Theorem 1.2.6 *A normed linear space is finite dimensional if and only if the closed unit ball is compact.*

Definition 1.2.7 (Quotient space.) *Let M be a closed subspace of $\mathbb{X}$. Define an equivalence relation R on $\mathbb{X}$ by xRy if $x - y \in M$. Let $\mathbb{X}/M$ denote the corresponding set of equivalence classes and let the coset $[x]$ denote the set of elements equivalent to x. Thus*

$$[x] = \{x + m : m \in M\} = x + M.$$

Vector addition and scalar multiplication on $\mathbb{X}/M$ are defined by

$$[x] + [y] = [x + y]$$

$$\alpha[x] = [\alpha x].$$

The norm on $\mathbb{X}/M$ is defined by

$$||[x]|| = d(x, M).$$

It is easy to verify that $\mathbb{X}/M$ is a normed space. Quotient spaces are therefore a useful way of forming new normed spaces out of old ones. In fact if $\mathbb{X}$ is a Banach space then so is $\mathbb{X}/M$ [S; 46-A].

Definition 1.2.8 (Deficiency.) *If V is a vector space and M is a subspace of V, then the dimension of V/M is called the deficiency or codimension of M in V.*

1.3 Linear operators.

Definition 1.3.1 (Linear operator.) *Let $\mathbb{X}$ and $\mathbb{Y}$ be vector spaces over the same space of scalars. An operator T with domain in $\mathbb{X}$ and range in $\mathbb{Y}$ is called linear if for all x and y in $\mathbb{X}$ and all scalars α and β,*

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

We will denote the identity operator on $\mathbb{X}$ by I where $Ix = x$ for all $x \in \mathbb{X}$. Note that a linear operator has the property that

$$T(0) = T(0.0) = 0T(0) = 0.$$

Therefore, apart from preserving the linearity, a linear operator is a mapping that also preserves the origin.

Definition 1.3.2 (kernel index, deficiency index and index.) *Let T be a linear operator with domain in $\mathbb{X}$ and range in $\mathbb{Y}$. The following notations will be used:*
$D(T)$ denotes the domain of T.
$R(T)$ denotes the range of T.
$N(T)$ denotes the subspace $\{x \in D(T) : Tx = 0\}$, referred to as the kernel or null space of T.
The dimension of $N(T)$ will be written $\alpha(T)$ and is referred to as the ***kernel index*** *of T.*
The deficiency of $R(T)$ in Y will be written $\beta(T)$ and is referred to as the ***deficiency index*** *of T.*
If at least one of $\alpha(T)$ or $\beta(T)$ is finite we define the ***index*** *of T by $i(T) = \alpha(T) - \beta(T)$ with the understanding that for any real number r, $\infty - r = \infty$ and $r - \infty = -\infty$.*

Definition 1.3.3 (Injective, surjective linear operators.) *A linear operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is called injective if distinct elements in $D(T)$ are mapped onto distinct elements in $R(T)$.*
The operator T is called surjective if $R(T) = \mathbb{Y}$.

Since a linear operator has the property that $T0 = 0$, T is injective if and only if $N(T) = \{0\}$.

Definition 1.3.4 (Compact operator) *A linear operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is called a compact operator if it maps bounded sets onto relatively compact sets.*

Definition 1.3.5 (Continuous operator.) *An operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$, not necessarily linear, is called continuous if T is continuous at every $x \in D(T)$.*

Definition 1.3.6 (Bounded linear operator.) *A linear operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is bounded if there exists a positive number m such that*

$$||Tx|| \leq m||x|| \text{ for all } x \in D(T).$$

We will denote the set of all such bounded linear operators by $B(\mathbb{X}, \mathbb{Y})$. The set all bounded linear operators defined on $\mathbb{X}$ we will denote by $B[\mathbb{X}, \mathbb{Y}]$. In the case where $\mathbb{X} = \mathbb{Y}$, we will write $B[\mathbb{X}]$. The norm of the operator T is defined as

$$||T|| = \sup\{||Tx|| : x \in D(T), ||x|| = 1\}.$$

Note that if T is a linear operator mapping $D(T) \subset \mathbb{X}$ into $\mathbb{Y}$, then T is continuous if and only if T is bounded.

Definition 1.3.7 (Operator Convergence.) *A sequence (T_n) of operators $T_n \in B[\mathbb{X}, \mathbb{Y}]$ is said to **converge in norm** to $T : \mathbb{X} \to \mathbb{Y}$ if*

$$\|T_n - T\| \to 0 \quad as \quad n \to \infty.$$

*A sequence (T_n) of operators $T_n \in B[\mathbb{X}, \mathbb{Y}]$ is said to **converge pointwise** to $T : \mathbb{X} \to \mathbb{Y}$ if*

$$\|T_n x - Tx\| \to 0 \quad as \quad n \to \infty$$

for every $x \in \mathbb{X}$.

Note the following well known result.

Theorem 1.3.8 *If $\mathbb{Y}$ is a Banach space then $B[\mathbb{X}, \mathbb{Y}]$ is a Banach space.*

Proof: Suppose $\mathbb{Y}$ is complete and let (T_n) be a Cauchy sequence in $B[\mathbb{X}, \mathbb{Y}]$. Given $\varepsilon > 0$ there exists a positive integer N such that

$$\|T_m - T_n\| < \varepsilon \text{ for all } m, n \geq N.$$

Hence, for all $x \in \mathbb{X}$ and $m, n \geq N$ we obtain

$$\begin{aligned} \|T_m x - T_n x\| &\leq \|T_m - T_n\|\|x\| \\ &< \varepsilon\|x\|. \end{aligned} \tag{1.1}$$

It is easy to see that this will imply $(T_n x)$ is a Cauchy sequence in $\mathbb{Y}$, and since $\mathbb{Y}$ is complete, $T_n x \to y \in \mathbb{Y}$. Thus for every $x \in \mathbb{X}$, there exists a $y \in \mathbb{Y}$ such that $T_n x \to y$. Therefore we can define a linear operator $T : \mathbb{X} \to \mathbb{Y}$ by letting $Tx = \lim_{n\to\infty} T_n x$ for each $x \in \mathbb{X}$. Since (T_n) converges pointwise to T, it follows from equation 1.1 when fixing $n \geq N$ and letting $m \to \infty$, that for all $x \in \mathbb{X}$

$$\begin{aligned} \|Tx - T_n x\| &\leq \|T - T_n\|\|x\| \\ &< \varepsilon\|x\|. \end{aligned} \tag{1.2}$$

Hence $T - T_n$ is bounded for all $n \geq N$ and thus so is

$$T = (T - T_n) + T_n.$$

Therefore $T \in B[\mathbb{X}, \mathbb{Y}]$ and it follows from equation 1.2 that

$$\|T - T_n\| = \sup_{\|x\|=1} \|Tx - T_n x\| < \varepsilon \quad \text{for} \quad n \geq N.$$

Thus $T_n \to T$ and therefore $B[\mathbb{X}, \mathbb{Y}]$ is complete. □

Definition 1.3.9 (Bounded below.) *A linear operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is called bounded below if there exists a positive number m such that $\|Tx\| \geq m\|x\|$ for all $x \in D(T)$.*

Definition 1.3.10 (Inverse.) *The inverse of an injective linear operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$, written T^{-1}, is the map from $R(T)$ into $\mathbb{X}$ defined by $T^{-1}(Tx) = x$. It is clear that T^{-1} is a linear map.*

The following characterisation of an operator that is bounded below is used extensively throughout this thesis. The proof was taken from [G2; I.3.7].

Theorem 1.3.11 *Let $T : \mathbb{X} \to \mathbb{Y}$ be a linear operator. T is bounded below if and only if T^{-1} exists and is continuous.*

Proof: Suppose that T is bounded below and let $\|Tx\| \geq m\|x\|$ for all $x \in \mathbb{X}$. Then if $x \neq 0$ it implies $Tx \neq 0$. Hence T is injective and T^{-1} exists. Since

$$\|T^{-1}Tx\| = \|x\| \leq m^{-1}\|Tx\|,$$

T^{-1} is bounded and therefore continuous. On the other hand if T^{-1} is continuous, then

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| \quad \textit{for all } x \in \mathbb{X}.$$

The result then follows by taking $m = 1/\|T^{-1}\|$. □

Definition 1.3.12 (Closed linear operator.) *Let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a linear operator. Then T is called a closed linear operator if its graph*

$$G(T) = \{(x, y) : x \in D(T), y = Tx\}$$

is closed in $\mathbb{X} \times \mathbb{Y}$. $\mathbb{X} \times \mathbb{Y}$ is defined as the normed space of all ordered pairs (x, y), $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ where vector addition and scalar multiplication are defined for all $x_1, x_2 \in \mathbb{X}$ and all $y_1, y_2 \in \mathbb{Y}$ and all α and β as follows:

$$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2).$$

The norm is given by

$$\|(x, y)\| = \|x\| + \|y\|.$$

The following Lemma was taken from [G2; IV.1.1].

Lemma 1.3.13 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed linear operator. Then T is bounded below if and only if T is injective and $R(T)$ is closed.*

Proof: Suppose T is bounded below. Then T is injective since T^{-1} exists by Theorem 1.3.11. To show $R(T)$ is closed we consider a sequence $(x_n) \subset \mathbb{X}$ such that $Tx_n \to y \in \mathbb{Y}$. Since T is bounded below, there exists a positive number m such

$$\|Tx_n - Tx_k\| \geq m\|x_n - x_k\|.$$

Thus (x_n) is a Cauchy sequence which converges to some x in the Banach space $\mathbb{X}$. Since T is closed, $x \in D(T)$ and $Tx = y$ by the sequential characterisation of a closed operator [K; 4.13-3]. Hence $R(T)$ is closed. Conversely suppose that T is injective and $R(T)$ is closed. Then $R(T)$ is complete since it is a closed subspace of a complete space. It is also easy to see that T^{-1} is a closed linear operator. Hence we can apply the Closed Graph Theorem to show that T^{-1} is continuous. The result then follows by Theorem 1.3.11. □

Definition 1.3.14 (Minimum modulus.) *If $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is such that $N(T)$ is closed, the minimum modulus of T, written $\gamma(T)$, is defined by*

$$\gamma(T) = \inf_{x \in D(T)} \frac{\|Tx\|}{d(x, N(T))}$$

where 0/0 is defined to be ∞. It is obvious that $\gamma(T) \geq 0$.

Definition 1.3.15 (Induced injective operator.) *The injective operator $\hat{T}$ induced by $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is the operator $\hat{T} : D(T)/N(T) \to Y$ defined by $\hat{T}[x] = Tx$.*

It is clear from the definition that $\hat{T}$ is injective and linear and that $R(T) = R(\hat{T})$.

The following Lemma was taken from [G2; IV.1.6].

Lemma 1.3.16 *Let $\mathbb{X}$ and $\mathbb{Y}$ be complete with $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ a closed linear operator and let $\hat{T}$ be the injective operator induced by T. Then $R(T) = R(\hat{T})$ is closed if and only if $\gamma(T) > 0$.*

Proof: By Lemma 1.3.13, $R(T) = R(\hat{T})$ is closed if and only if $\hat{T}$ is bounded below which, by Theorem 1.3.11, holds if and only if $\hat{T}$ has a bounded inverse. This in turn holds if and only if

$$\begin{aligned}
\gamma(T) &= \inf\left\{\frac{\|Tx\|}{d(x, N(T))} : x \in D(T) - N(T)\right\} \\
&= \inf\left\{\frac{\|\hat{T}[x]\|}{\|[x]\|} : 0 \neq [x] \in D(T)/N(T) = D(\hat{T})\right\} \\
&= \left(\sup\left\{\frac{\|[x]\|}{\|\hat{T}[x]\|} : 0 \neq \hat{T}[x] \in R(\hat{T}) = D(\hat{T}^{-1})\right\}\right)^{-1} \\
&= \left(\sup\left\{\frac{\|\hat{T}^{-1}\hat{T}[x]\|}{\|\hat{T}[x]\|} : 0 \neq \hat{T}[x] \in D(\hat{T}^{-1})\right\}\right)^{-1} \\
&= \left(\sup\left\{\frac{\|\hat{T}^{-1}y\|}{\|y\|} : 0 \neq y \in D(\hat{T}^{-1})\right\}\right)^{-1} \\
&= (\|\hat{T}^{-1}\|)^{-1} \\
&> 0
\end{aligned}$$

□

The following well known result will be needed throughout this thesis. The main structure of the proof was taken from [A1; 1.7].

Lemma 1.3.17 *Let $\mathbb{X}$ be a Banach space and (T_n) a sequence of operators $T_n \in B[\mathbb{X}, \mathbb{Y}]$ such that $T_n \to T$ pointwise where $T \in B[\mathbb{X}, \mathbb{Y}]$. Then $(\|T_n\|)$ is bounded and $T_n \to T$ uniformly on totally bounded sets.*

Proof: Since T_n converges pointwise to a bounded operator, $(T_n x)$ is bounded for every $x \in \mathbb{X}$. So there exists a scalar c_x such that

$$\|T_n x\| \leq c_x < \infty \quad ; \quad n = 1, 2, \ldots .$$

Therefore, by the Uniform-Boundedness Principle, $(\|T_n\|)$ is bounded. Let us assume that $\|T_n\| < b$ for every n and let $\|T\| < b_1$. Also let S be any totally bounded set in $\mathbb{X}$ and let $\varepsilon > 0$ be given. Then there exists a finite ε-net for S, say S_ε. Hence for every $x \in S$ there exists an $x_\varepsilon \in S_\varepsilon$ such that $\|x - x_\varepsilon\| < \varepsilon$. Since pointwise convergence was given and S_ε is a finite set, we can choose N_ε such that $||T_n x_\varepsilon - T x_\varepsilon|| < \varepsilon$ for all $n > N_\varepsilon$ and all $x_\varepsilon \in S_\varepsilon$. Now

$$\begin{aligned} ||(T_n - T)x|| &\leq ||(T_n - T)(x - x_\varepsilon)|| + ||(T_n - T)x_\varepsilon|| \\ &\leq ||T_n - T|| ||x - x_\varepsilon|| + ||(T_n - T)x_\varepsilon|| \\ &\leq (||T_n|| + ||T||)||x - x_\varepsilon|| + ||(T_n - T)x_\varepsilon|| \\ &< (b + b_1)\varepsilon \ + \varepsilon \quad \text{for all} \quad n > N_\varepsilon \quad \text{and all} \quad x \in S \end{aligned}$$

giving the result. □

The following proposition is well known:

Proposition 1.3.18 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T, T_n \in B[\mathbb{X}, \mathbb{Y}]$ where (T_n) is a sequence of operators. Then $||T_n - T|| \to 0$ if and only if $T_n \to T$ uniformly on any bounded set.*

Note that the pointwise convergence of $T_n \to T$ need not be uniform on all the bounded sets. Therefore, if $T_n x \to Tx$ for all $x \in \mathbb{X}$ it does not imply that T_n will converge to T in norm. Consider the following example:

Example 1.3.19 *Let $T_n : \ell^2 \to \ell^2$ be defined by*

$$T_n x = (x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots)$$

where $x = (x_1, x_2, x_3, \ldots) \in \ell^2$. Then $T_n x \to x = Ix$ as $n \to \infty$ for each $x \in \ell^2$ since

$$||x - T_n x|| = \Big(\sum_{k=n+1}^{\infty} |x_k|^2\Big)^{\frac{1}{2}} \to 0$$

as $n \to \infty$ if $\sum_{k=1}^{\infty} |x_k|^2 < \infty$. However (T_n) does not converge to I in norm since each T_n is bounded with dim $T_n(\ell^2) < \infty$ making each T_n a compact operator whereas I is not compact because dim $\ell^2 = \infty$. □

However, take note that when $\mathbb{X}$ is finite dimensional, the bounded and totally bounded sets will coincide [S; section 25]. We also note the following interesting relationship between pointwise and norm operator convergence [A1; 5.2].

Lemma 1.3.20 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T, T_n \in B[\mathbb{X}, \mathbb{Y}]$ for $n = 1, 2, \ldots$. Then $T_n \to T$ in norm if and only if $T_n \to T$ pointwise and (T_n) is totally bounded.*

The following Lemma was taken from [G1] and proved in more detail. It will be needed mainly in Chapter 4.

Lemma 1.3.21 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed linear operator with closed range. Let (y_n) be a bounded sequence in $R(T)$. Then there exists a bounded sequence (x_n) such that $Tx_n = y_n$.*

Proof: We know that $y_n = Tv_n$ for some $v_n \in D(T)$. By the definition of $\gamma(T)$ the following inequality holds for some b:

$$\infty > b \geq \|y_n\| = \|Tv_n\| \geq \gamma(T) d(v_n, N(T)) \geq 0.$$

Therefore, since $\gamma(T) > 0$ by Lemma 1.3.16,

$$d(v_n, N(T)) = \inf_{k \in N(T)} \|v_n - k\| \leq \frac{b}{\gamma(T)} < \infty.$$

Hence for every v_n we can choose a $k_n \in N(T)$ such that $(x_n) = (v_n - k_n)$ is a bounded sequence. Clearly $Tx_n = y_n$. □

The following Lemma is contained in [Y; 3.1]. For the purposes of this thesis it will however be sufficient to consider the following.

Lemma 1.3.22 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed linear operator with closed range. If $\alpha(T) < \infty$ and (x_n) is a bounded sequence such that (Tx_n) converges, then (x_n) has a convergent subsequence.*

Proof: Since $R(T)$ is closed we may assume that $Tx_n \to Tx \in R(T)$. Now $\hat{T}^{-1}$ exists and is continuous by Theorem 1.3.11 and Lemma 1.3.13, so therefore

$$x_n + N(T) = \hat{T}^{-1} T x_n \to \hat{T}^{-1} T x = x + N(T) \text{ in } \mathbb{X}/N(T).$$

Thus there exists $(z_n) \in N(T)$ such that $x_n + z_n \to x$. Since both (x_n) and $(x_n + z_n)$ are bounded, (z_n) is a bounded sequence in the finite dimensional space $N(T)$. Therefore, by the compactness of the unit ball, there exists a subsequence $(z_{n'})$ of (z_n) such that $z_{n'} \to z \in N(T)$. Hence

$$x_{n'} = x_{n'} + z_{n'} - z_{n'} \to x - z.$$

So there exists a convergent subsequence of (x_n). □

The following Lemma, which is due to Krein, Krasnoselskii and Milman, depends on Borsuk's antipodal-mapping theorem [GK] and is proved in Gokhberg and Krein [GK; 1.1].

Lemma 1.3.23 *If M and N are subspaces of $\mathbb{X}$ with dim $M >$ dim N then there exists an $m \in M$ such that*

$$1 = ||m|| = d(m, N).$$

Note that the Lemma need not hold if dim M=dim $N < \infty$. Take for example $\mathbb{X}$ to be the real plane and M and N any two nonperpendicular lines through the origin. Then it is clear that there does not exist any $m \in M$ such that

$$||m|| = d(m, N).$$

1.4 Linear Functionals.

Definition 1.4.1 (Linear functional.) *A linear functional f is a linear operator with domain in a vector space $\mathbb{V}$ and range in the scalars.*

Definition 1.4.2 (Conjugate space.) *The set of all bounded linear functionals on $\mathbb{X}$, denoted by $\mathbb{X}'$, is a normed space with norm defined by*

$$||f|| = \sup\{|f(x)| : x \in \mathbb{X}, ||x|| = 1\}.$$

In other words $\mathbb{X}' = B[\mathbb{X}, K]$, where K is the space of scalars.

Note that since both $\mathbb{R}$ and $\mathbb{C}$ are complete, it follows by Theorem 1.3.8 that $\mathbb{X}'$ is a Banach space. A conjugate space is also often referred to in the literature as a dual space.

Definition 1.4.3 (Conjugate operator.) *Let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a linear operator with domain of T dense in $\mathbb{X}$. The conjugate T' of T is defined to be the operator with*

$$D(T') = \{y' \in \mathbb{Y}' : y'T \text{ continuous on } D(T)\}.$$

For $y' \in D(T')$, let T' be the operator which maps y' onto $\overline{y'T}$, the unique continuous linear extension of $y'T$ to all of $\mathbb{X}$. T' is obviously linear and $D(T')$ is a subspace of $\mathbb{Y}'$.

Note that the condition that $D(T)$ is dense in $\mathbb{X}$ is not too restrictive since if $D(T)$ is not dense in $\mathbb{X}$ we could redefine T as the map from $\mathbb{X}_1 = \overline{D(T)}$ into $\mathbb{Y}$. Also, note that $T' \in B[\mathbb{Y}', \mathbb{X}']$ if $T \in B[\mathbb{X}, \mathbb{Y}]$.

Definition 1.4.4 (Orthogonal complement.) *If $W \subset \mathbb{Y}$, we define the orthogonal complement of W by*

$$W^{\perp} = \{y' \in \mathbb{Y}' : y'w = 0 \ \ \textit{for all} \ \ w \in W\}.$$

Chapter 2

Collectively Compact Operator Approximation Theory

2.1 Introduction.

The aim of this chapter is to provide a motivation for the study of collectively compact sets of operators by illustrating their usefulness in the approximation of Fredholm integral equations of the second kind.

In the first section we define the concept of a collectively compact set of operators and consider only the elementary properties which will be needed in this chapter. A more detailed study of collectively compact sets of operators will be presented in the following chapter. In section 2.3 we take a brief look at Fredholm integral equations of the second kind and then in section 2.4 we consider the general theory of finding approximate solutions for such integral equations. In paragraphs 2.5 and 2.6 we will consider the work done by Anselone [A1] on using collectively compact sets of operators to obtain convergence theorems and error bounds for approximate solutions of Fredholm integral equations.

2.2 Collectively Compact Sets of Linear Operators.

Definition 2.2.1 *A set* $\mathcal{K} \subset B[\mathbb{X}, \mathbb{Y}]$ *is called collectively compact if the set*

$$\begin{aligned} \mathcal{K}B_{\mathbb{X}} &= \{Kx : K \in \mathcal{K}, x \in B_{\mathbb{X}}\} \\ &= \bigcup_{K \in \mathcal{K}} KB_{\mathbb{X}} \end{aligned}$$

is relatively compact. A sequence of operators in $B[\mathbb{X}, \mathbb{Y}]$ *is collectively compact whenever the corresponding set is collectively compact.*

Let us consider those properties of collectively compact sets of operators which will be needed in the approximation theory. These well known properties were verified independently by the author. Note that further properties will be given in the following chapter.

Theorem 2.2.2 *Every subset of a collectively compact set of operators is collectively compact.*

Proof: Let $\mathcal{K} = \{K_\lambda : \lambda \in I\}$, where I is an index set, be a collectively compact set of operators and let $\{K_{\lambda_\mu} : \mu \in J\}$, where J is an index set, be any subset of $\mathcal{K}$. Then

$$\bigcup_\mu K_{\lambda_\mu} B_{\mathbb{X}} \subset \bigcup_\lambda K_\lambda B_{\mathbb{X}}.$$

Therefore

$$\overline{\bigcup_\mu K_{\lambda_\mu} B_{\mathbb{X}}} \subset \overline{\bigcup_\lambda K_\lambda B_{\mathbb{X}}},$$

which is compact by definition and since a closed subset of a compact set is compact [RR; III, Lemma 6(ii)], $\overline{\bigcup_\mu K_{\lambda_\mu} B_{\mathbb{X}}}$ is compact. □

Take note that Theorem 2.2.2 implies that every operator in a collectively compact set of operators is compact.

Theorem 2.2.3 *Assume* $K, K_n \in B[\mathbb{X}, \mathbb{Y}]$. *If* (K_n) *is collectively compact and* $K_n \to K$ *pointwise, then* K *is compact.*

Proof: Since $K_n \to K$,

$$KB_{\mathbb{X}} \subset \overline{\bigcup_n K_n B_{\mathbb{X}}}.$$

Therefore

$$\overline{KB_{\mathbb{X}}} \subseteq \overline{\bigcup_n K_n B_{\mathbb{X}}},$$

which is compact by definition. Since a closed subset of a compact set is compact, $\overline{KB_{\mathbb{X}}}$ is compact. □

2.3 Fredholm Integral Equations of the Second Kind.

Let $C[a,b]$ be the space of real or complex continuous functions $x(t)$, $t \in [a,b]$ with norm $\|x\| = \max_{t\in[a,b]}|x(t)|$.

Fredholm integral equations of the second kind arise from boundary value problems such as the following:

$$Lu = u'' + pu' + qu \quad \text{where} \quad u(a) = u(b) = 0.$$

It is well known that this type of problem can be rewritten as an integral equation of the form

$$x(s) - \mu \int_a^b k(s,t)x(t)dt = y(s)$$

Here, $[a,b]$ is a given interval, x is an unknown function on $[a,b]$ and μ is a parameter. The kernel k of the equation is a given function on $[a,b] \times [a,b]$ and y is a given function on $[a,b]$. The type of equation mentioned above is called a Fredholm integral equation of the second kind. For further reading on this subject we refer the reader to [Ha]. An equation without the term $x(s)$ is of the form

$$\int_a^b k(s,t)x(t)dt = y(s)$$

and is said to be a Fredholm integral equation of the first kind.

Consider the mapping

$$x \longmapsto (Kx)(s) = \int_a^b k(s,t)x(t)dt \tag{2.1}$$

Theorem 2.3.1 *Let $[a,b]$ be a compact interval and assume that k is continuous on $[a,b] \times [a,b]$. Then the operator $K : C[a,b] \longmapsto C[a,b]$ in equation 2.1 is compact.*

Proof: The operator K is clearly linear. That K is a bounded operator can be seen by

$$\begin{aligned}
\|Kx\| &= \max_{s\in[a,b]}|Kx(s)| \\
&= \max_{s\in[a,b]}\left|\int_a^b k(s,t)x(t)dt\right| \\
&\leq \max_{s\in[a,b]}\int_a^b |k(s,t)||x(t)|dt \\
&\leq \max_{s,t}|k(s,t)|\max_t|x(t)|\int_a^b dt \\
&\leq k_0(b-a)\|x\| \text{ if } |k(s,t)|\leq k_0 \text{ for all } (s,t)\in[a,b]\times[a,b].
\end{aligned}$$

Consider the closed unit ball $B_{C[a,b]}$ in $C[a,b]$. Then $\|x_i\| \leq 1$ for each $x_i \in B_{C[a,b]}$ where i is in some indexing set. Let $y_i = Kx_i$ for each $i \in I$. Then $\{y_i : i \in I\}$ is bounded since

$$\begin{aligned}
\|Kx_i\| &\leq \|K\|\|x_i\| \\
&\leq k_0(b-a) \quad \text{for each} \quad i.
\end{aligned}$$

We now show that $\{y_i : i \in I\}$ is equicontinuous. Since k is continuous on the compact set $[a,b]\times[a,b]$, k is uniformly continuous on $[a,b]\times[a,b]$ [S; 24-F]. Hence, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \in [a,b]$ and all s_1, $s_2 \in [a,b]$ with $|s_1 - s_2| < \delta$, we have

$$|k(s_1,t) - k(s_2,t)| \leq \frac{\varepsilon}{b-a}.$$

Hence for s_1 and s_2 as before and for every i we get

$$\begin{aligned}
|y_i(s_1) - y_i(s_2)| &= \left|\int_a^b [k(s_1,t) - k(s_2,t)]x_i(t)dt\right| \\
&\leq \int_a^b |k(s_1,t) - k(s_2,t)||x_i(t)|dt \\
&\leq \int_a^b \frac{\varepsilon}{b-a}|x_i(t)|dt \\
&\leq \int_a^b \frac{\varepsilon}{b-a}dt \\
&= \frac{\varepsilon}{(b-a)}(b-a) \\
&= \varepsilon
\end{aligned}$$

So $\{y_i : i \in I\}$ is equicontinuous and by Ascoli's Theorem (1.1.7) it is relatively compact. Therefore K is a compact operator. □

Definition 2.3.2 *A linear operator $T : \mathbb{X} \longmapsto \mathbb{Y}$ is called* ***solvable*** *if for a given $y \in \mathbb{Y}$ there exists an $x \in \mathbb{X}$ such that $Tx = y$. T is called* ***uniquely solvable*** *if for every $y \in \mathbb{Y}$ there exists a unique $x \in \mathbb{X}$ such that $Tx = y$.*

It is clear from the above definitions that T will be solvable if and only if $\beta(T) = 0$, i.e. if T is surjective. Similarly T will be uniquely solvable if and only if $\alpha(T) = 0$ and $\beta(T) = 0$, i.e. if T is injective and surjective.

Definition 2.3.3 (Fredholm alternative) *A linear operator $T \subset B[\mathbb{X}]$ is said to satisfy the Fredholm alternative if T is such that either (a) or (b) holds:*

(a) $N(T)$ is trivial and $Tx = y$ is solvable for every $y \in \mathbb{X}$.

(b) $N(T)$ is non-trivial and $Tx = y$ is not solvable for every $y \in \mathbb{X}$.

Theorem 2.3.4 *If K in equation 2.1 is a compact linear operator, then the Fredholm alternative holds for $I - \lambda K$ where λ is any scalar.*

Proof: If K is compact then clearly λK is also compact. Then, since $i(I) = 0$, we can apply [G2;III.1.3 and V.2.1] to obtain $i(I - \lambda K) = 0$. But, by the definition of the index, $i(I - \lambda K) = 0$ if and only if $\beta(I - \lambda K) = \alpha(I - \lambda K)$. Therefore $I - \lambda K$ is surjective, or solvable, if and only if $I - \lambda K$ is injective, or uniquely solvable. □

2.4 Approximate Solutions of Fredholm Integral Equations.

Consider a Fredholm integral equation of the second kind:

$$y(s) = x(s) - \int_0^1 k(s,t)x(t)dt \quad s \in [0,1]. \tag{2.2}$$

Numerical integration with a general quadrature formula results in the following equations for approximate solutions of the integral equation:

$$y(s) = x_n(s) - \sum_{j=1}^{n} w_{nj}k(s,t_{nj})x_n(t_{nj}) \ s \in [0,1], n = 1,2,\ldots \tag{2.3}$$

Now the equations for $x_n(s)$ and $x(s)$ have the same general form

$$(I - K)x = y \quad \text{and} \quad (I - K_n)x_n = y,$$

where

$$(Kx)(s) = \int_0^1 k(s,t)x(t)dt$$

and

$$(K_n x_n)(s) = \sum_{j=1}^{n} w_{nj} k(s, t_{nj}) x_n(t_{nj})$$

The integral operator K is approximated by a numerical-integral operator K_n.

The approximate solution of integral equations by numerical integration has been studied by researchers such as Fredholm [F1,F2] since at least the beginning of the century. Valuable work was done on convergence theorems and error bounds for approximate solutions under various assumptions by, amongst others, Brakhage [B1,B2], Kantorovich and Krylof [KK] and Mysovskih [M1,M2,M3].

It was in fact a study of the error bounds obtained by the above-mentioned researchers that first led Anselone and Moore [AM] to the concept of a collectively compact set of operators.

2.5 Approximation by a Collectively Compact Set of Operators.

The following two sections have been included to illustrate the importance of collectively compact sets of operators in the approximation theory of integral equations as it was developed by Anselone [A1] in his book "Collectively Compact Operator Approximation Theory and Applications to Integral Equations."

It has already been shown in Theorem 2.3.1 that when the kernel $k(s,t)$ of an integral equation is continuous for $s, t \in [0,1]$, the integral operator K is a compact linear operator. We will now show that the numerical-integral operators K_n approximating K are bounded linear operators. In fact, if $k(s,t)$ is continuous for $s, t \in [0,1]$, the set $\{K_n\}$ is a collectively compact set of operators. It will also be shown that if we assume that

$$\sum_{j=1}^{n} w_{nj} x(t_{nj}) \rightarrow \int_0^1 x(t)dt$$

for each $x \in C[0,1]$, then $K_n \to K$ pointwise.

In the next section we will develop an abstract approximation theory applicable to a set of collectively compact operators $\{K_n\}$ with $K_n \to K$ pointwise where K is a compact operator. This theory will give us convergence theorems and error bounds for the approximate solutions for integral equations of the second kind under the condition that the kernel is continuous and that

$$\sum_{j=1}^{n} w_{nj} x(t_{nj}) \to \int_0^1 x(t)dt$$

for each $x \in C[0,1]$.

Consider the Fredholm integral equation 2.2 and equation 2.3. For the sake of easier notation, we define φ and φ_n, $n = 1, 2, \ldots$, on $C[0,1]$ by

$$\varphi x = \int_0^1 x(t)dt \quad \text{and} \quad \varphi_n x = \sum_{j=1}^{n} w_{nj} x(t_{nj}),$$

where $t_{nj} \in [0,1]$ and the weights w_{nj} are real or complex. Now take note that φ and φ_n are bounded linear functionals on $C[0,1]$:
Using the same proof as in [K; 2.8-6] we can show that

$$||\varphi|| = 1.$$

We also have that

$$\begin{aligned} |\varphi_n(x)| &= |\sum_{j=1}^{n} w_{nj} x(t_{nj})| \\ &\leq \sum_{j=1}^{n} |w_{nj}||x(t_{nj})| \\ &\leq \max_{t_{nj}} |x(t_{nj})| \sum_{j=1}^{n} |w_{nj}| \\ &\leq \max_{t} |x(t)| \sum_{j=1}^{n} |w_{nj}| \\ &= ||x|| \sum_{j=1}^{n} |w_{nj}|. \end{aligned}$$

Therefore
$$\|\varphi_n\| \leq \sum_{j=1}^{n} |w_{nj}|.$$
Now we choose $x \in C[0,1]$ with $x(t_{nj}) = \overline{sgn(w_{nj})}$ and $\|x\| = 1$ where sgn is defined as follows: $sgn(0) = 1$ and $sgn(w) = \frac{w}{|w|}$ if $w \neq 0$. Then
$$\begin{aligned} \|\varphi_n\| &\geq |\varphi_n(x)| \\ &= |\sum_{j=1}^{n} w_{nj}\overline{sgn(w_{nj})}| \\ &= |\sum_{j=1}^{n} |w_{nj}|| \\ &= \sum_{j=1}^{n} |w_{nj}|. \end{aligned}$$
Therefore, for every n,
$$\|\varphi_n\| = \sum_{j=1}^{n} |w_{nj}|.$$

If we assume that $\varphi_n \to \varphi$ pointwise, i.e.
$$\varphi_n x \to \varphi x \quad \text{for each} \quad x \in C[0,1] \quad \text{as} \quad n \to \infty,$$
we have by Lemma 1.3.17 that the convergence is uniform on totally bounded sets in $C[0,1]$ and that $(\|\varphi_n\|)$ is bounded. Thus
$$\|\varphi_n\| = \sum_{j=1}^{n} |w_{nj}| \leq b < \infty$$
for some b and all n. Note that the totally bounded sets in $C[0,1]$ are the bounded equicontinuous sets according to Ascoli's Theorem.

Now we can prove the following very important proposition which was taken from [A1; 2.1]:

Proposition 2.5.1 *If the kernel $k(s,t)$ of the integral equation 2.2 is continuous for $s,t \in [0,1]$ and we assume that $\varphi_n \to \varphi$ pointwise, then $K_n \to K$ pointwise.*

Proof: Since k is continuous on the compact metric space $[0,1]\times[0,1]$, k is uniformly continuous for $s,t\in[0,1]$ by [S; 24-F]. Therefore the sets $\{k_s : s\in[0,1]\}$ and $\{k_t : t\in[0,1]\}$, where $k_s(t)=k_t(s)=k(s,t)$, are bounded and equicontinuous. It is also clear that

$$(Kx)(s)=\varphi(k_s x) \quad\text{and}\quad (K_n x)(s)=\varphi_n(k_s x).$$

Now, since $\varphi_n\to\varphi$ uniformly on the bounded equicontinuous sets in $C[0,1]$, $\varphi_n\to\varphi$ uniformly on the set $\{k_s x : s\in[0,1]\}$ for every $x\in C[0,1]$. Therefore $K_n x\to Kx$ for every $x\in C[0,1]$. □

We also have the following proposition which was taken from [A1; 2.2].

Proposition 2.5.2 *If the kernel $k(s,t)$ of the integral equation 2.2 is continuous for $s,t\in[0,1]$ then the set $\{K_n\}$ is collectively compact.*

Proof: Let $\|k\|=\max_{s,t\in[0,1]}|k(s,t)|$. Then

$$\begin{aligned}
\|K_n x\| &= \max_{s\in[0,1]}|K_n x(s)|\\
&= \max_{s\in[0,1]}\left|\sum_{j=1}^{n} w_{nj}k(s,t_{nj})x(t_{nj})\right|\\
&\le \max_{s\in[0,1]}\left[\sum_{j=1}^{n}|w_{nj}||k(s,t_{nj})||x(t_{nj})|\right]\\
&\le \max_{s\in[0,1]}\left[\sum_{j=1}^{n}|w_{nj}|\max_{t\in[0,1]}|k(s,t)|\ max_{t\in[0,1]}|x(t)|\right]\\
&\le \sum_{j=1}^{n}|w_{nj}|\max_{s,t\in[0,1]}|k(s,t)|\ \|x\|\\
&\le b\|k\|\|x\|
\end{aligned}$$

It is clear that each K_n is bounded.

Therefore, to show that $\{K_n\}$ is collectively compact we need to show that

$$\bigcup_n K_n B_{C[0,1]}$$

is relatively compact. So let (y_n) be any sequence in $\bigcup_n K_n B_{C[0,1]}$. Then (y_n) is of the form $(K_m x_p)_{m,p}$ where $(x_p)\subset B_{C[0,1]}$. (y_n) is a bounded sequence since

$$\begin{aligned}
\|K_m x_p\| &\le \|K_m\|\|x_p\|\\
&\le b\|k\| \text{ for all } m,p.
\end{aligned}$$

$k(s,t)$ is continuous on the compact set $[0,1]\times[0,1]$, and therefore uniformly continuous on $[0,1]\times[0,1]$. Hence given any $\varepsilon > 0$, there exists $\lambda > 0$ such that for all s, s' and $t \in [0,1]$, with $|s-s'| < \lambda$, we have $|k(s,t)-k(s',t)| < \frac{\varepsilon}{b}$. Hence for $s, s' \in [0,1]$ with $|s-s'| < \lambda$ and for every m, p we get

$$\begin{aligned} |K_m x_p(s) - K_m x_p(s')| &= |\sum_{j=1}^{m} w_{mj}[k(s,t_{mj}) - k(s',t_{mj})]x(t_{mj})| \\ &\leq \sum_{j=1}^{m} |w_{mj}||k(s,t_{mj}) - k(s',t_{mj})||x(t_{mj})| \\ &< \sum_{j=1}^{m}[\frac{\varepsilon}{b}|w_{mj}||x(t_{mj})|] \\ &\leq \frac{\varepsilon}{b}\sum_{j=1}^{m}|w_{mj}| \\ &\leq b\frac{\varepsilon}{b} \\ &= \varepsilon \end{aligned}$$

Therefore $(K_m x_p)_{m,p}$ is equicontinuous and by Ascoli's Theorem (1.1.7), $\bigcup_n K_n B_{C[0,1]}$ is relatively compact. So $\{K_n\}$ is collectively compact. □

2.6 Convergence Theorems and Error Bounds.

We are now going to develop an abstract approximation theory to obtain approximations and error bounds for the solutions of integral equations of the second kind when the kernel $k(s,t)$ is continuous for $s, t \in [0,1]$ and

$$\sum_{j=1}^{n} w_{nj}x(t_{nj}) \to \int_0^1 x(t)dt$$

for each $x \in C[0,1]$.

The following proposition was taken from [A1; 1.8] and will allow us to move from pointwise convergence to norm convergence in the case where $\mathbb{X}$ is infinite dimensional.

Proposition 2.6.1 *Let $\mathbb{X}$ be a Banach space and let $T, T_n \in B[\mathbb{X}]$, $n = 1, 2, \ldots$, and $T_n x \to Tx$ for all $x \in \mathbb{X}$. Then*

$$\|(T_n - T)K\| \to 0$$

for each compact operator $K \in B[\mathbb{X}]$. The convergence is uniform with respect to K, for K in any collectively compact set $\mathcal{K} \subset B[\mathbb{X}]$.

Proof: Let $\mathcal{K} \subset B[\mathbb{X}]$ be any collectively compact set of operators. Then $\mathcal{K}B_{\mathbb{X}}$ will be relatively compact and hence totally bounded. Then by Lemma 1.3.17, $\|(T_n - T)Kx\| \to 0$ uniformly for $K \in \mathcal{K}$ and $x \in B_{\mathbb{X}}$. Therefore, by Proposition 1.3.18, $\|(T_n - T)K\| \to 0$ for each compact operator K. □

The following Corollary follows easily and was proved in [A1; 1.9].

Corollary 2.6.2 *Let $\mathbb{X}$ be a Banach space and let $K, K_n \in B[\mathbb{X}]$, $n = 1, 2, \ldots$. Assume that $K_n \to K$ pointwise where $\{K_n\}$ is collectively compact. Then*

$$\|(K_n - K)K\| \to 0 \quad \textit{and} \quad \|(K_n - K)K_n\| \to 0.$$

Proof: By Theorem 2.2.2 each K_n will be compact and by Theorem 2.2.3, K will also be compact. So the result follows by Proposition 2.6.1. □

The following two Lemmas are needed to prove Theorem 2.6.5.
Lemma 2.6.3 is well known and for a proof we refer the reader to [K; 7.3-1].

Lemma 2.6.3 *Let $\mathbb{X}$ be a Banach space and let $T \in B[\mathbb{X}]$ with $\|T\| < 1$. Then $(I - T)^{-1}$ exists as a bounded linear operator on the whole space $\mathbb{X}$ with*

$$(I - T)^{-1} = \sum_{m=0}^{\infty} T^m$$

where the series converges in norm.

Note that in fact

$$\begin{aligned} \|(I - T)^{-1}\| &= \|\sum_{m=0}^{\infty} T^m\| \\ &\leq \sum_{m=0}^{\infty} \|T\|^m \\ &= \frac{1}{1 - \|T\|} \end{aligned}$$

Lemma 2.6.4 (A1; 1.2.) *Let $\mathbb{X}$ be a Banach space and let $B, T \in B[\mathbb{X}]$. If*

$$BT = I - A \quad \textit{and} \quad \|A\| < 1,$$

then T^{-1} exists and is bounded. In fact

$$T^{-1} = (I-A)^{-1}B, \quad T^{-1} - B = (I-A)^{-1}AB \quad \textit{on} \quad T\mathbb{X},$$

$$\|T^{-1}\| \leq \frac{\|B\|}{1-\|A\|} \quad \textit{and} \quad \|T^{-1} - B\| \leq \frac{\|A\|\|B\|}{1-\|A\|}.$$

Proof: If $BT = I - A$ then $(I-A)^{-1}BT = I$, whence

$$T^{-1} = (I-A)^{-1}B \quad \text{on} \quad T\mathbb{X}$$

and using Lemma 2.6.3

$$\begin{aligned} \|T^{-1}\| &\leq \|(I-A)^{-1}B\| \\ &\leq \|(I-A)^{-1}\|\|B\| \\ &\leq \frac{\|B\|}{1-\|A\|}. \end{aligned}$$

Also

$$\begin{aligned} T^{-1} - B &= (I-A)^{-1}B - B \\ &= (I-A)^{-1}(B - (I-A)B) \\ &= (I-A)^{-1}AB \quad \text{on} \quad T\mathbb{X} \end{aligned}$$

and then clearly

$$\begin{aligned} \|T^{-1} - B\| &\leq \|(I-A)^{-1}AB\| \\ &\leq \|(I-A)^{-1}\|\|AB\| \\ &\leq \frac{\|AB\|}{1-\|A\|} \end{aligned}$$

□

The following Theorem on the existence and approximation of operator inverses taken from [A1; 1.10] will allow us to prove Theorem 2.6.6 and Theorem 2.6.7.

Theorem 2.6.5 *Let* $\mathbb{X}$ *be a Banach space and let* $K, L \in B[\mathbb{X}]$. *Assume there exists*
$(I-K)^{-1} \in B[\mathbb{X}]$ *and that*

$$\Delta = \|(I-K)^{-1}(L-K)L\| < 1.$$

Then $(I-L)^{-1}$ *exists and is bounded,*

$$\|(I-L)^{-1}\| \leq \frac{1 + \|(I-K)^{-1}\|\|L\|}{1-\Delta},$$

and

$$\|(I-L)^{-1}y - (I-K)^{-1}y\| \leq \frac{\|(I-K)^{-1}\|\|Ly - Ky\| + \Delta\|(I-K)^{-1}y\|}{1-\Delta}$$

for $y \in (I-L)\mathbb{X}$. *(If* L *is compact then* $(I-L)\mathbb{X} = \mathbb{X}$.*)*

Proof: $(I-K)^{-1}$ exists and we can express it in terms of the so-called resolvent operator $(I-K)^{-1}K$:

$$(I-K)^{-1} = I + (I-K)^{-1}K.$$

This would suggest

$$B = I + (I-K)^{-1}L \tag{2.4}$$

as an approximate inverse for $I - L$. Now

$$\begin{aligned} B(I-L) &= (I + (I-K)^{-1}L)(I-L) \\ &= I - L + (I-K)^{-1}(I-L)L \\ &= I + (I-K)^{-1}(-LI + KL + LI - LL) \\ &= I - (I-K)^{-1}(L-K)L \qquad (2.5) \end{aligned}$$

and letting $A = (I-K)^{-1}(L-K)L$ we have $\|A\| \leq \Delta < 1$. Then by Lemma 2.6.4 $(I-L)^{-1}$ exists and is bounded with

$$\begin{aligned} \|(I-L)^{-1}\| &\leq \frac{\|I + (I-K)^{-1}L\|}{1 - \|A\|} \\ &\leq \frac{1 + \|(I-K)^{-1}\|\|L\|}{1-\Delta}. \end{aligned}$$

More calculation using equations 2.5 and 2.4 yields

$$\begin{aligned}(I-L)^{-1}-(I-K)^{-1} &= (I-A)^{-1}B-(I-K)^{-1}\\ &= (I-A)^{-1}[B-(I-A)(I-K)^{-1}]\\ &= (I-A)^{-1}[B-(I-K)^{-1}+A(I-K)^{-1}]\\ &= (I-A)^{-1}[(I-K)^{-1}(L-K)+A(I-K)^{-1}]\end{aligned}$$

on $(I-L)\mathbb{X}$ and the theorem follows. □

The following two theorems taken from [A1; 1.11 and 1.12] now give us convergence results and error bounds. To express Theorems 2.6.6 and 2.6.7 in concise form, let $y \in \mathbb{X}$ be a fixed element and let $x = (I-K)^{-1}y$ and $x_n = (I-K_n)^{-1}y$ whenever the inverse operators exist.

Theorem 2.6.6 *Let $\mathbb{X}$ be a Banach space and let $K, K_n \in B[\mathbb{X}]$, $n = 1, 2, \ldots$. Assume that $K_n \to K$ pointwise where $\{K_n\}$ is collectively compact and K is compact. Suppose that $(I-K)^{-1}$ exists and define*

$$\Delta_n = \|(I-K)^{-1}\|\|(K_n-K)K_n\|.$$

Then $\Delta_n \to 0$ and if $\Delta_n < 1$, $(I-K_n)^{-1}$ exists,

$$\|(I-K_n)^{-1}\| \leq \frac{1+\|(I-K)^{-1}\|\|K_n\|}{1-\Delta_n},$$

and

$$\|x_n - x\| \leq \frac{\|(I-K)^{-1}\|\|K_n y - Ky\| + \Delta_n\|x\|}{1-\Delta_n} \to 0.$$

The estimates for $\|(I-K_n)^{-1}\|$ are bounded uniformly in n.

Proof: Since K is compact and $i(I) = 0$, $i(I-K) = 0$ by [G2; V.2.1]. Therefore $\beta(I-K) = \alpha(I-K)$ and hence $I-K$ is injective if and only if $I-K$ is surjective. So $(I-K)^{-1}$ is bounded by the Open Mapping Theorem. Also, by Corollary 2.6.2, $\|(K_n-K)K_n\| \to 0$. Hence

$$\Delta_n = \|(I-K)^{-1}\|\|(K_n-K)K_n\| \to 0.$$

Therefore by letting $L = K_n$ in Theorem 2.6.5 we get the existence of $(I-K_n)^{-1}$ with

$$\|(I-K_n)^{-1}\| \leq \frac{1+\|(I-K)^{-1}\|\|K_n\|}{1-\Delta_n}$$

whenever $\Delta_n < 1$. Also by Theorem 2.6.5

$$\|x_n - x\| \le \frac{\|(I-K)^{-1}\|\|K_n y - Ky\| + \Delta_n\|x\|}{1-\Delta_n} \to 0$$

since $\|K_n y - Ky\| \to 0$ and $\Delta_n \to 0$. □

Note that one can get other approximations for $\|x_n - x\|$ using Theorem 2.6.6 which are sometimes useful. For instance

$$\begin{aligned} \|x_n\| &= \|(I-K_n)^{-1}y\| \\ &\le \|(I-K_n)^{-1}\|\|y\| \\ &\le \left(\frac{1+\|(I-K)^{-1}\|\|K_n\|}{1-\Delta_n}\right)\|y\|, \end{aligned}$$

and

$$\begin{aligned} \|x_n - x\| &\le \frac{\|(I-K)^{-1}\|\|K_n y - Ky\| + \Delta_n\|x\|}{1-\Delta_n} \\ &\le \frac{\|(I-K)^{-1}\|\|K_n y - Ky\| + \Delta_n\|(I-K)^{-1}\|\|y\|}{1-\Delta_n} \\ &\le \frac{\|(I-K)^{-1}\|(\|K_n y - Ky\| + \Delta_n\|y\|)}{1-\Delta_n} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This bound for $\|x_n - x\|$ is very useful when considering a set of elements y, since the estimate involves y but not x or x_n. Using the identity

$$\begin{aligned} x_n - x &= (I-K_n)^{-1}y - x \\ &= (I-K_n)^{-1}[y - (I-K_n)x] \end{aligned}$$

and Theorem 2.6.6 we obtain the following estimate:

$$\begin{aligned} \|x_n - x\| &= \|(I-K_n)^{-1}[y-(I-K_n)x]\| \\ &\le \|(I-K_n)^{-1}\|\|y-(I-K_n)x\| \\ &\le \frac{(1+\|(I-K)^{-1}\|\|K_n\|)\|y-(I-K_n)x\|}{1-\Delta_n} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

The next Theorem is analogous to Theorem 2.6.6, but with the roles of K and K_n interchanged:

Theorem 2.6.7 *Let $\mathbb{X}$ be a Banach space and let $K, K_n \in B[\mathbb{X}]$, $n = 1, 2, \ldots$. Assume that $K_n \to K$ pointwise where $\{K_n\}$ is collectively compact and K is compact. Whenever $(I - K_n)^{-1}$ exists, define*

$$\Delta^n = \|(I - K_n)^{-1}\|\|(K_n - K)K\|.$$

For a particular n assume that $(I-K_n)^{-1}$ exists and $\Delta^n < 1$. Then $(I-K)^{-1}$ exists,

$$\|(I - K)^{-1}\| \leq \frac{1 + \|(I - K_n)^{-1}\|\|K\|}{1 - \Delta^n},$$

and

$$\|x_n - x\| \leq \frac{\|(I - K_n)^{-1}\|\|K_n y - Ky\| + \Delta^n \|x_n\|}{1 - \Delta^n}.$$

Moreover, $(I-K_n)^{-1}$ exists for all n sufficiently large, $\Delta^n \to 0$, the estimates for $\|(I - K)^{-1}\|$ are bounded uniformly with respect to n and the estimates for $\|x_n - x\| \to 0$ as $n \to \infty$.

Proof: Use Theorem 2.6.5, replacing K by K_n and L by K, Corollary 2.6.2 and Theorem 2.6.6. □

Note that Theorem 2.6.7 gives the existence of $(I - K)^{-1}$ and related inequalities under conditions on K_n for a single n. This is of considerable importance in practical applications. The Theorem also yields the following estimates for $\|x\|$ and $\|x_n - x\|$:

$$\begin{aligned}
\|x_{\not{n}}\| &\leq \left(\frac{1 + \|(I - K_n)^{-1}\|\|K\|}{1 - \Delta^n}\right)\|y\|, \\
\|x_n - x\| &\leq \frac{\|(I - K_n)^{-1}\|(\|K_n y - Ky\| + \Delta^n\|y\|)}{1 - \Delta^n} \to 0, \\
\|x_n - x\| &\leq \frac{(1 + \|(I - K_n)^{-1}\|\|K\|)\|y - (I - K)x_n\|}{1 - \Delta^n} \to 0.
\end{aligned}$$

The derivation of these estimates is analogous to the derivation of the estimates obtained using Theorem 2.6.6.

2.7 Summary.

In this chapter we defined a collectively compact set of operators and illustrated their usefulness by considering their application to the approximation theory of Fredholm integral equations as set out in [A1].

Chapter 3

Collectively Compact Sets of Operators

3.1 Introduction.

In this chapter we will study collectively compact sets of operators in more detail. Firstly we will look at additional properties of collectively compact sets of operators. In the next section we will then consider certain characterisations of collectively compact sets of operators. The work in that section was largely motivated by the work done by Higgins [H] in his unpublished paper "A Characterization of Collectively Compact Sets of Linear Operators". Lastly we will consider a characterisation of totally bounded sets of compact operators in terms of collective compactness. This section is included because the approximation theory discussed in Chapter 2 is most appropriately applied to sets which are collectively compact but not totally bounded.

3.2 Elementary Properties of Collectively Compact Sets of Operators.

Some properties were already given in Section 2.2. Let us now consider further properties which will be needed in the following sections. The results in this section are well known and were verified independently by the author.

Theorem 3.2.1 *Any finite set of compact operators is collectively compact.*

Proof: Let $\{K_n\}$, $n = 1, 2, \ldots, k$ be any finite collection of compact operators, Therefore $\overline{K_n B_{\mathbb{X}}}$ is compact for each n and by [RR; III, Lemma 6(i)], $\bigcup_n \overline{K_n B_{\mathbb{X}}}$ is compact. □

Theorem 3.2.2 *1. Finite unions of collectively compact sets of operators are collectively compact,*

2. Finite sums of collectively compact sets of operators are collectively compact,

3. Scalar multiples of collectively compact sets of operators are collectively compact.

Proof:

1. Let $\mathcal{K}_n = \{K_\lambda^n : \lambda \in I\}$, where $1 \le n \le k$ and I is an index set. Assume that each $\mathcal{K}_n$ is collectively compact. Then $\mathcal{K}_n B_{\mathbb{X}}$ is relatively compact and therefore totally bounded for each n. Hence, by [RR; III, Lemma 3(iii)], $\bigcup_n \mathcal{K}_n B_{\mathbb{X}}$ is totally bounded and therefore relatively compact.

2. Let $\mathcal{K}_n = \{K_\lambda^n : \lambda \in I\}$, where $1 \le n \le k$ and I is an index set. Assume that each $\mathcal{K}_n$ is collectively compact. Then $\overline{\mathcal{K}_n B_{\mathbb{X}}}$ is compact for each n and by [RR; III, Lemma 7(ii)], $\sum_{n=1}^k \overline{\mathcal{K}_n B_{\mathbb{X}}}$ is compact since $\mathbb{Y}$ is a convex space. If $\sum_{n=1}^k \overline{\mathcal{K}_n B_{\mathbb{X}}}$ is compact it is closed, so
$$\sum_{n=1}^{k} \overline{\mathcal{K}_n B_{\mathbb{X}}} = \overline{\sum_{n=1}^{k} \overline{\mathcal{K}_n B_{\mathbb{X}}}} = \overline{\sum_{n=1}^{k} \mathcal{K}_n B_{\mathbb{X}}}$$
by Proposition 1.2.3. Hence $\overline{\sum_{n=1}^k \mathcal{K}_n B_{\mathbb{X}}}$ is compact and, since $\overline{(\sum_{n=1}^k \mathcal{K}_n) B_{\mathbb{X}}} \subset \overline{\sum_{n=1}^k \mathcal{K}_n B_{\mathbb{X}}}$, finite sums of collectively compact sets of operators are collectively compact.

3. Let $\mathcal{K}$ be any collectively compact set of operators and λ any scalar. Then $\overline{\mathcal{K} B_{\mathbb{X}}}$ is compact and by [RR; III, Lemma 7(i)], $\lambda \overline{\mathcal{K} B_{\mathbb{X}}} = \overline{\lambda \mathcal{K} B_{\mathbb{X}}}$ will be compact. Hence $\lambda \mathcal{K}$ is collectively compact.

□

Theorem 3.2.3 *A collectively compact set of operators is bounded.*

Proof: Let $\mathcal{K} = \{K_\lambda : \lambda \in I\}$, where I is an index set, be a collectively compact set of operators. Then $\overline{\mathcal{K}B_{\mathbb{X}}}$ is compact and therefore bounded. Hence, for some scalar α, $\mathcal{K}B_{\mathbb{X}} \subset \alpha B_{\mathbb{Y}}$. Since

$$\begin{aligned} \|K_\lambda\| &= \sup\{\|K_\lambda x\| : \|x\| \leq 1\} \\ &\leq \sup\{\|K_\lambda x\| : \|x\| \leq 1, \lambda \in I\} \\ &\leq \alpha \quad \text{for each} \quad \lambda \in I, \end{aligned}$$

$\mathcal{K}$ is bounded. □

3.3 Characterisations of Collectively Compact Sets of Operators.

The following Lemma was taken from [H; 2.1, 2.2]. The proof of the second part of the Lemma was simplified by L.E. Labuschagne and privately communicated to the author.

Lemma 3.3.1 *Let $(T_n) \subset B[\mathbb{X}, \mathbb{Y}]$ be a sequence of compact operators and let $T \in B[\mathbb{X}, \mathbb{Y}]$ be such that $\{(T_n - T)x_n\}$ is relatively compact for every sequence $(x_n) \subset B_{\mathbb{X}}$. Then*

1. *for every increasing sequence of integers (n_i), the set $K = \{T_{n_1} - T_{n_2}, T_{n_2} - T_{n_3}, ...\}$ is collectively compact,*
2. *T is compact.*

Proof:

1. Let (y_n) be any sequence in $KB_{\mathbb{X}}$. In order to show that (y_n) has a convergent subsequence, there are two cases to consider:
Case 1. The difference between two compact operators is again compact and therefore $T_{k_i} - T_{k_{i+1}}$ is a compact operator for each k_i. Choosing any fixed k_i, $(T_{k_i} - T_{k_{i+1}})B_{\mathbb{X}}$ is relatively compact and therefore, if infinitely many elements of (y_n) are contained in $(T_{k_i} - T_{k_{i+1}})B_{\mathbb{X}}$, (y_n) has a convergent subsequence.
Case 2. Suppose that for each k_i, $(T_{k_i} - T_{k_{i+1}})B_{\mathbb{X}}$ contains only finitely many elements of (y_n). Then there exists a subsequence (n_{i_j}) of (n_i) and elements $x_j \in B_{\mathbb{X}}$ such that $((T_{n_{i_j}} - T_{n_{i_{j+1}}})x_j)$ forms a subsequence of (y_n). Now $T_{n_{i_j}} - T_{n_{i_{j+1}}} = (T_{n_{i_j}} - T) + (T - T_{n_{i_{j+1}}})$ and

$\{(T_{n_{i_j}} - T)x_j\}$ and $\{(T - T_{n_{i_{j+1}}})x_j\}$ are both relatively compact. Therefore $\{(T_{n_{i_j}} - T_{n_{i_{j+1}}})x_j\}$ is relatively compact which implies that (y_n) has a convergent subsequence.

2. Let (z_n) be any bounded sequence in $T(B_{\mathbb{X}})$. For each n choose $x_n \in B_{\mathbb{X}}$ with $Tx_n = z_n$. We want to show that (Tx_n) has a convergent subsequence. Since T_1 is compact, there exists a subsequence $(x_n^{(1)})$ of x_n such that $(T_1 x_n^{(1)})$ converges to say y_1. Since T_2 is compact, there exists a subsequence $(x_n^{(2)})$ of $(x_n^{(1)})$ such that $(T_2 x_n^{(2)})$ converges to say y_2. Proceeding in this way we obtain a subsequence $(x_n^{(k)})$ of $(x_n^{(k-1)})$ such that $(T_k x_n^{(k)})$ converges to say y_k. Now, for each k, there exists an N_k such that $\|T_k x_n^{(k)} - y_k\| \leq 2^{-k}$ for all $n \geq N_k$. Select the N_k's such that $N_1 \leq N_2 \leq N_3 \leq \ldots$ and set $v_k = x_{N_k}^{(k)}$. Now (v_k) is a subsequence of (x_n) and $\|T_k v_k - y_k\| \leq 2^{-k}$ for all k. In fact

$$\|T_k v_m - y_k\| \leq 2^{-k} \quad \text{for all} \quad m \geq k \tag{3.1}$$

and since $\|T_k v_k - y_k\| \leq 2^{-k} \to 0$ as $k \to \infty$, it is clear that (y_k) has a convergent subsequence if and only if $(T_k v_k)$ has a convergent subsequence. It was also given that $\{T_k v_k - T v_k\}$ is relatively compact. Hence $(T_k v_k)$ has a convergent subsequence if and only if $(T v_k)$, and therefore $(T x_k)$, has a convergent subsequence. It therefore suffices to show that (y_k) has a convergent subsequence:
Let us suppose that (y_k) does not have a convergent subsequence. Then there exists an $\epsilon > 0$ such that for some subsequence of (y_k), $\|y_{n_p} - y_{n_k}\| \geq \epsilon$ for all $p \neq k$. By the first part of the Theorem, the family $\{T_{n_i} - T_{n_{i+1}}\}$ is collectively compact which implies that$((T_{n_i} - T_{n_{i+1}})v_{n_{i+1}})$ has a convergent subsequence. However, by equation 3.1, we have that

$$\begin{aligned}
& \|((y_{n_i} - y_{n_{i+1}}) - (T_{n_i} - T_{n_{i+1}})v_{n_{i+1}}\| \\
= \; & \|y_{n_i} - T_{n_i} v_{n_{i+1}} - (y_{n_{i+1}} - T_{n_{i+1}} v_{n_{i+1}})\| \\
\leq \; & \|y_{n_i} - T_{n_i} v_{n_{i+1}}\| + \|y_{n_{i+1}} - T_{n_{i+1}} v_{n_{i+1}}\| \\
\leq \; & 2^{-i} + 2^{-(i+1)} \\
= \; & \frac{3}{2^{(i+1)}} \to 0 \; as \; i \to \infty.
\end{aligned}$$

Therefore $(y_{n_i} - y_{n_{i+1}})$ has a convergent subsequence which gives us a contradiction.

□

The reverse implication of the following theorem was proved by Higgins in [H; 2.3]. The forward implication follows easily.

Theorem 3.3.2 *Let $(T_n) \subset B[\mathbb{X}, \mathbb{Y}]$ be a sequence of compact operators and let $T \in B[\mathbb{X}, \mathbb{Y}]$. Then $(T_n - T)$ is collectively compact if and only if $\{(T_n - T)x_n\}$ is relatively compact for every sequence $(x_n) \subset B_{\mathbb{X}}$.*

Proof: If $(T_n - T)$ is collectively compact then $\bigcup_n (T_n - T)B_{\mathbb{X}}$ is relatively compact by definition. It is clear that for every sequence $(x_n) \subset B_{\mathbb{X}}$,

$$\{(T_n - T)x_n\} \subset \bigcup_n (T_n - T)B_{\mathbb{X}}$$

and since subsets of relatively compact sets are relatively compact [RR; III, Lemma 6] we have the forward implication. To get the reverse implication note that T is compact by Theorem 3.3.1 which implies that $T_n - T$ is compact for each n using Theorems 3.2.2 and 3.2.1. Let (y_n) be any sequence in $\bigcup_n (T_n - T)B_{\mathbb{X}}$. In order to show that (y_n) has a convergent subsequence, there are two cases to consider:
Case 1. If infinitely many elements of (y_n) lie in $(T_k - T)B_{\mathbb{X}}$ for some fixed k, then (y_k) clearly has a convergent subsequence since $T_k - T$ is compact.
Case 2. If for each k only finitely many elements of (y_n) lie in $(T_k - T)B_{\mathbb{X}}$, then there exists a subsequence (n_i) of (n) and elements $x_{n_i} \in B_{\mathbb{X}}$ such that $((T_{n_i} - T)x_{n_i})$ forms a subsequence of (y_n). This subsequence is given relatively compact and therefore (y_n) has a convergent subsequence. □

The following Corollary was noted by the author.

Corollary 3.3.3 *Let $(T_n) \subset B[\mathbb{X}, \mathbb{Y}]$ be a sequence of compact operators and let $T \in B[\mathbb{X}, \mathbb{Y}]$. Then $(T_n - T)$ is collectively compact if and only if (T_n) is collectively compact.*

Proof: If $(T_n - T)$ is collectively compact then T is compact by Theorem 3.3.1. Hence T is collectively compact and we can apply Theorem 3.2.2 to obtain

$$(T_n) = (T_n - T) + T$$

collectively compact. Conversely, since (T_n) and T are collectively compact, $(T_n - T)$ is collectively compact by Theorem 3.2.2. □

The following characterisation of collectively compact sets of operators was proved by Higgins in [H; 2.6].

Lemma 3.3.4 *Let $\mathcal{T} \subset B[\mathbb{X}, \mathbb{Y}]$. Then $\mathcal{T}$ is collectively compact if and only if every countable subset of $\mathcal{T}$ is collectively compact.*

Proof: We proved in Theorem 2.2.2 that subsets of collectively compact sets are collectively compact. Conversely, let (y_n) be any sequence in $\mathcal{T}(B_{\mathbb{X}})$ and for each n, let $x_n \in B_{\mathbb{X}}$ and $T_n \in \mathcal{T}$ such that $T_n x_n = y_n$. By hypothesis, $\{T_n\}$ is collectively compact so (y_n) has a convergent subsequence. Hence $\mathcal{T}$ is collectively compact. □

Note that the Lemma implies that a set $\mathcal{T}$ is collectively compact if and only if every sequence contained in $\mathcal{T}$ is collectively compact. This now allows us to consider the following important characterisation of collective compactness which was proved by Higgins in [H; 2.7].

Theorem 3.3.5 *Let $\mathcal{K} \subseteq B[\mathbb{X}, \mathbb{Y}]$ be a family of compact operators. Then $\mathcal{K}$ is collectively compact if and only if for every sequence $(K_n) \subset \mathcal{K}$ and every sequence $(x_n) \subset B_{\mathbb{X}}$, $\{K_n x_n\}$ is relatively compact.*

Proof: $\{K_n x_n\} \subset \mathcal{K}(B_{\mathbb{X}})$ which is relatively compact. Conversely, let $(K_n) \subset \mathcal{K}$ and $(x_n) \subset B_{\mathbb{X}}$. Then by Theorem 3.3.2 with $T = 0$, (K_n) is collectively compact. Therefore by Lemma 3.3.4, $\mathcal{K}$ is collectively compact. □

3.4 Collectively Compact and Totally Bounded Sets of Operators.

The fact that a collectively compact set of operators is a bounded set of compact operators follows by Theorem 2.2.2 and Theorem 3.2.3. However, the converse fails as can be seen by the following example.

Example 3.4.1 *Let $\mathbb{A}$ be any infinite dimensional Banach space. Then there exists a set of finite projections which is a bounded set of compact operators that is not collectively compact.*

By a Corollary to the Hahn-Banach Theorem we have for every $y \in \mathbb{A}$ with $\|y\| = 1$ the existence of a linear functional y' such that $y'(y) = \|y\| = 1$ and $\|y'\| = 1$ [G2; I.5.5]. Define the projection P by

$$P_y x = y'(x) y \text{ for all } x \in \mathbb{A}.$$

Then $\mathcal{P} = \{P_y : y \in \mathbb{A}, ||y|| = 1\}$ is a bounded set of compact operators which is not collectively compact:

$$||P_y|| \leq ||y'|| ||y|| = 1$$

and

$$\begin{aligned} ||P_y|| &= \sup_{x \in \mathbb{A}} \frac{||y'(x)y||}{||x||} \\ &\geq \frac{||y'(y)y||}{||y||} \\ &= \frac{||y||}{||y||} \\ &= 1. \end{aligned}$$

Hence $||P_y|| = 1$ for each $P_y \in \mathcal{P}$. Also for each $y \in \mathbb{A}$ dim $P_y(\mathbb{A}) < \infty$ since $P_y x \in \text{span}\{y\}$ for all $x \in \mathbb{A}$. So, by [K; 8.1-4], $\mathcal{P}$ is a bounded set of compact operators.

However,

$$\mathcal{P}B_{\mathbb{A}} = B_{\mathbb{A}}.$$

The one inclusion is obvious. To see the other one, choose $x \in B_{\mathbb{A}}$ arbitrarily. Let $x_o = \lambda x$ for some scalar λ such that $||x_o|| = 1$. Then

$$P_{x_o} x = x \in B_{\mathbb{A}}.$$

Since $\mathbb{A}$ is an infinite dimensional space, $B_{\mathbb{A}}$ is not relatively compact and therefore $\mathcal{P}$ is not collectively compact. □

Now we will show that a compact set of compact operators is collectively compact. The theorem was first proved for normed linear spaces in 1968 by Anselone and Palmer [AP1; 2.4] using a generalisation of Ascoli's Theorem. The result was generalised to Hausdorff linear topological spaces by DePree and Higgins [DH; 3.6] in 1970. Shortly afterwards Higgins [H] gave the shorter proof for normed linear spaces shown here. The proof uses Higgins' characterisation of collective compactness given in Theorem 3.3.5.

Theorem 3.4.2 *Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces. Then every compact set of compact operators in $B[\mathbb{X}, \mathbb{Y}]$ is collectively compact.*

Proof: Let $\mathcal{K} \subset B[\mathbb{X}, \mathbb{Y}]$ be a compact set of compact operators and let $\{K_n\}$ be any countable subset of $\mathcal{K}$. Then there exists a subsequence $(K_{n_i}) \subseteq (K_n)$ and a compact operator $K \in \mathcal{K}$ such that $\|K_{n_i} - K\| \to 0$. Therefore

$$\begin{aligned} \|(K_{n_i} - K)x_{n_i}\| &\leq \|K_{n_i} - K\|\|x_{n_i}\| \\ &\to 0 \end{aligned}$$

for any bounded sequence (x_n). Therefore $(K_{n_i} - K)x_{n_i} \to 0$ for any $(x_n) \subset B_{\mathbb{X}}$. Since K is compact, (Kx_{n_i}) has a convergent subsequence and by the above $(K_{n_i}x_{n_i})$, and also (K_nx_n) has a convergent subsequence. Hence (K_nx_n) is relatively compact and by Theorem 3.3.5, $\mathcal{K}$ is collectively compact. □

The following Corollary was first proved by Anselone and Palmer [AP1; 2.5] for normed linear spaces using Theorem 3.4.2. The result was generalised to Hausdorff linear topological spaces by DePree and Higgins [DH; 3.7].

Corollary 3.4.3 *Let $\mathbb{X}$ be a normed linear space and $\mathbb{Y}$ a Banach space. Then every totally bounded set of compact operators in $B[\mathbb{X}, \mathbb{Y}]$ is collectively compact.*

Proof: Let $\mathcal{K}$ be a totally bounded set of compact operators. Then, by [K; 2.10-2 and 8.2-2], $\overline{\mathcal{K}}$ is a compact set of compact operators and hence by Theorem 3.4.2, $\overline{\mathcal{K}}$ is collectively compact. Therefore, by Theorem 2.2.2, $\mathcal{K}$ is collectively compact. □

The converses of Theorem 3.4.2 and Corollary 3.4.3 are false as can be seen by the following example which was taken from [A1; 5.4].

Example 3.4.4 *Let $\mathbb{X} = \ell^2$ and $\mathcal{K} = \{K_n \in B[\ell^2] : n = 1, 2, \ldots\}$ where K_n is defined by*

$$K_nx = x_n\varphi_1$$

for $x = (x_1, x_2, x_3, \ldots, x_n, \ldots)$ and $\varphi_1 = (1, 0, 0, 0, \ldots)$. Then $\mathcal{K}$ is collectively compact but not totally bounded.

$\mathcal{K}B_{\ell^2}$ is bounded and $\dim \mathcal{K}\ell^2 = 1$ implying that $\overline{\mathcal{K}B_{\ell^2}}$ is compact. So $\mathcal{K}$ is collectively compact. However,

$$\begin{aligned} \|K_m - K_n\| &= \sup_{\|x\|=1} \|(K_m - K_n)x\| \\ &= \sup_{\|x\|=1} \|K_m x - K_n x\| \\ &= \sup_{\|x\|=1} \|x_m\varphi_1 - x_n\varphi_1\| \\ &= \sup_{\|x\|=1} \|(x_m - x_n)\|\|\varphi_1\| \\ &= \sqrt{2} \text{ for all } m \neq n \end{aligned}$$

Therefore there exists no finite ε-net with $\varepsilon < \sqrt{2}$ which covers $\mathcal{K}$ implying that $\mathcal{K}$ is not totally bounded. □

The approximation theory that we discussed in Chapter 2 is based on pointwise operator convergence. There is an alternative approximation theory for norm operator convergence as developed in [Ka], thereby making it important for us to distinguish between the two types of convergence. From Lemma 1.3.20 it is clear that we need to be able to distinguish between sets of operators which are collectively compact but not totally bounded in order for the approximation theory of Chapter 2 to be appropriate.
Corollary 3.4.3 showed that in general totally bounded sets of compact operators are collectively compact. The following Theorem now characterises totally bounded sets of compact operators in terms of collective compactness. It will be clear from this Theorem that the approximation theory in Chapter 2 is particularly applicable to collectively compact sequences of operators for which the sequences of conjugate operators are not collectively compact.

Definition 3.4.5 *For each $\mathcal{K} \subset B[\mathbb{X}, \mathbb{Y}]$ we define*

$$\mathcal{K}' = \{K' : K \in \mathcal{K}\}.$$

Theorem 3.4.6 *Let $\mathbb{Y}$ be a Banach space and let $\mathcal{K}$ be a set of compact operators in $B[\mathbb{X}, \mathbb{Y}]$. Then $\mathcal{K}$ is totally bounded if and only if $\mathcal{K}$ and $\mathcal{K}'$ are collectively compact.*

The proof of the forward implication of this Theorem follows quite easily by Corollary 3.4.3 and the fact that $\mathcal{K}$ is totally bounded if and only if $\mathcal{K}'$ is totally bounded since $\|K\| = \|K'\|$ [T; pg.485].

The proof of the reverse implication has an interesting history. A proof was given in 1968 by Anselone and Palmer in [AP1; 3.5] for the case where $\mathbb{X} = \mathbb{Y}$, a Hilbert space. The proof makes use of the Hilbert space spectral theorem.

P.M. Anselone attempted to extend the result to Banach spaces. A proof was published by him in 1968 [A2; 5.1] for normed linear spaces and sets of compact operators with the property that $\dim K\mathbb{X} \leq n$ for each $K \in \mathcal{K}$.

In the same year Anselone and Palmer published a paper entitled "Spectral Properties of Collectively Compact sets of Linear Operators." in which they used this result together with spectral theory to prove Theorem 3.4.6 for the case where $\mathbb{X} = \mathbb{Y}$, a complex uniformly smooth Banach space and $\mathcal{K}$ a set of normal operators [AP2; 4.7].

In 1969 Palmer [P] proved the following Theorem for the case where $\mathbb{X}$ and $\mathbb{Y}$ are normed linear spaces. This Theorem implies the reverse implication of Theorem 3.4.6.

Theorem 3.4.7 *Let $\mathcal{K} \subset B[\mathbb{X}, \mathbb{Y}]$ and assume that*

1. *$\mathcal{K}B_{\mathbb{X}}$ is totally bounded in $\mathbb{Y}$, and*
2. *$\mathcal{K}'f$ is totally bounded in $\mathbb{X}'$ for each $f \in \mathbb{Y}'$.*

Then $\mathcal{K}$ is totally bounded in $B[\mathbb{X}, \mathbb{Y}]$.

Note that Palmer's proof does not depend on spectral theory.

In 1970, Anselone [A3] obtained another proof for Theorem 3.4.7 for the Banach space case. This proof will be shown below.

It should however be noted that in 1980, Tacon [Ta; 4] gave a far more elegant direct proof for Theorem 3.4.6 with his proof depending on the nonstandard hull of a Banach space. The proof requires preliminary work outside the scope of this thesis so we refer the interested reader to [Ta] for a detailed discussion of this proof.

Before we can discuss Anselone's proof of Theorem 3.4.7 we need to consider a few additional concepts.

Definition 3.4.8 (Tangent Functional.) *A linear functional $x' \in \mathbb{X}'$ is called a tangent functional to $x \in \mathbb{X}$ if*

$$\|x'\| = \|x\| \quad \textit{and} \quad x'(x) = \|x\|^2.$$

Note that if $\mathbb{X}$ is a normed space the Hahn-Banach Theorem ensures the existence of at least one tangent functional for each $x \in \mathbb{X}$. For each $x \in \mathbb{X}$ let x^t denote the set of all the tangent functionals to x.

Lemma 3.4.9 *A set S in a metric space is totally bounded if and only if for each $\varepsilon > 0$ and each infinite set $\hat{S} \subset S$ there exists an infinite set $S_\varepsilon \subset \hat{S}$ with diameter less than ε.*

Proof: Let S be totally bounded, let $\varepsilon > 0$ be given and assume that $\hat{S}$ is an infinite subset of S. Then $\hat{S}$ is totally bounded. So it is a finite union of sets each with diameter less that ε. One of these sets, say S_ε, must be infinite. Conversely, assume that S is not totally bounded. Then there exists $\varepsilon > 0$ and an infinite set $\hat{S} \subset S$ such that

$$d(x, y) \geq \varepsilon \text{ for } x, y \in \hat{S}, \; x \neq y.$$

Clearly then $\hat{S}$ has no infinite subset with diameter less than ε. □

Lemma 3.4.10 *Let $\mathbb{X}$ be a normed space. Let $S \subset \mathbb{X}$ be totally bounded and $\varepsilon > 0$. Then for each $x \in S$ there exists $f_x \in \mathbb{X}'$ such that*

$$|\|f_x\| - \|x\|| < \varepsilon \;, \quad |f_x(x) - \|x\|^2| < \varepsilon$$

and

$$F = \{f_x : x \in S\} \text{ is finite.}$$

Proof: Let b be a bound for S and let $\delta = \min(\varepsilon, \frac{\varepsilon}{3b})$. Let $S_\delta \subset S$ be a finite δ-net for S. For each $x \in S$ choose $x_\delta \in S_\delta$ such that $\|x - x_\delta\| < \delta$ and choose $x'_\delta \in x^t_\delta$. Define $f_x = x'_\delta$. Then

$$\begin{aligned} |\|f_x\| - \|x\|| &= |\|x'_\delta\| - \|x\|| \\ &= |\|x_\delta\| - \|x\|| \\ &\leq \|x_\delta - x\| \\ &< \delta. \end{aligned}$$

The rest follows from the definition of a tangent functional and the way f_x was defined. □

Lemma 3.4.11 *Let $T \in B[\mathbb{X}, \mathbb{Y}]$ and $\varepsilon > 0$. For each $y \in TB_{\mathbb{X}}$, choose $f_y \in \mathbb{Y}'$ such that*

$$|f_y(y) - \|y\|^2| \leq \varepsilon.$$

Let $F = \{f_y : y \in TB_{\mathbb{X}}\} \subset \mathbb{Y}'$. Then

$$\|T\|^2 \leq \sup_{f \in F} \|T'f\| + \varepsilon.$$

Proof: Let $x \in B_{\mathbb{X}}$ and let $y = Tx$. Then

$$f_y(y) = f_y(Tx) = (T'f_y)x$$

and

$$|f_y(y)| \leq \|T'f_y\| \leq \sup_{f \in F} \|T'f\|.$$

Hence

$$\|Tx\|^2 = \|y\|^2 \leq |f_y(y)| + \varepsilon \leq \sup_{f \in F} \|T'f\| + \varepsilon.$$

□

Proof of Theorem 3.4.7 : Fix $\varepsilon > 0$ and assume that $\hat{\mathcal{K}}$ is an infinite subset of $\mathcal{K}$. By the first hypothesis of the Theorem, the set

$$S = \{Kx - Lx : K, L \in \mathcal{K}\ ,\ \ x \in B_{\mathbb{X}}\}$$

is totally bounded. Define F as in Lemma 3.4.10. By the second hypothesis of the Theorem and Lemma 3.4.9 there exists an infinite set $\mathcal{K}_\varepsilon \subset \hat{\mathcal{K}}$ such that

$$\|(K - L)'f\| < \varepsilon \ \text{ for } \ K, L \in \mathcal{K}_\varepsilon,\ \ f \in F.$$

Hence, Lemma 3.4.11 implies that

$$\|K - L\|^2 < 2\varepsilon \ \text{ for } \ K, L \in \mathcal{K}_\varepsilon.$$

Therefore, by Lemma 3.4.9, $\mathcal{K}$ is totally bounded. □

3.5 Summary.

In this Chapter we took a closer look at collectively compact sets of operators and some of their elementary properties. We also considered some important characterisations of this class of operators done by Higgins in [H]. We then used one of these characterisations to give an easier proof for a result needed to give conditions under which a totally bounded set of compact operators is collectively compact. Lastly we considered a Theorem which characterises totally bounded sets of compact operators in terms of collectively compact sets of operators.

Chapter 4

Perturbation Theorems

4.1 Introduction.

The perturbation theorems given in this chapter are analogous to those given in [G2] and [Ka] and were taken from [G1]. Apart from the obvious importance of these perturbation theorems on their own, the results in this chapter were the motivation for the perturbation results for strictly singular operators discussed in the next chapter. Take note that in [G1], Goldberg defines a sequence (K_n) of bounded linear operators mapping Banach space $\mathbb{X}$ into Banach space $\mathbb{Y}$ as converging to zero compactly if (K_n) is collectively compact and $K_n x \to 0$ for all $x \in \mathbb{X}$. We will however not use the concept of compact convergence, so the statement of theorems has been changed accordingly.

4.2 Perturbations of Semi-Fredholm Operators by Collectively Compact Sets of Operators.

Definition 4.2.1 (Semi-Fredholm operators.) *A closed linear operator* $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ *with closed range is called a*

1. ϕ_+*-operator if* $\alpha(T) < \infty$.
2. ϕ_-*-operator if* $\beta(T) < \infty$.

Definition 4.2.2 (Fredholm operator.) *A closed linear operator which has a finite index is called a Fredholm operator.*

Lemma 4.2.3 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed linear operator with closed range and $\{K_n\}$ a collectively compact set of operators such that $K_n x \to 0$ for all $x \in \mathbb{X}$. If $\alpha(T) < \infty$ and $N(T)$ is complemented in $\mathbb{X}$ by a closed subspace M, then there exists a p and $c > 0$ such that for $n \geq p$, $T_M + K_n$ is injective and $\gamma(T_M + K_n) \geq c$, where T_M is the restriction of T to $M \cap D(T)$.*

Proof: Let us assume that $(\gamma(T_M + K_n))$ has a subsequence converging to zero. Without loss of generality we may assume that

$$\gamma(T_M + K_n) = \inf_{m \in M} \frac{\|(T + K_n)m\|}{d(m, N(T_M + K_n))} \to 0.$$

By the definition of the minimum modulus we may now, for every n, select $z_n \in M$ and $k_n \in N(T_M + K_n) = N(T + K_n) \cap M$ such that

$$\left| \frac{\|(T + K_n)z_n\|}{\|z_n - k_n\|} - \gamma(T_M + K_n) \right| < \frac{1}{n} \to 0.$$

Note that now $\frac{\|(T+K_n)z_n\|}{\|z_n - k_n\|} \to 0$. Let $m_n = \frac{z_n - k_n}{\|z_n - k_n\|}$, then $\|m_n\| = 1$ and

$$\begin{aligned} (T + K_n)m_n &= (T + K_n)\frac{(z_n - k_n)}{\|z_n - k_n\|} \\ &= \frac{(T + K_n)z_n}{\|z_n - k_n\|} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

So we have the existence of a sequence $(m_n) \subset M$ such that $\|m_n\| = 1$ and $(T + K_n)m_n \to 0$. Since $\{K_n\}$ is collectively compact, $\{K_n m_n\}$ is relatively compact and therefore any subsequence of $(K_n m_n)$ has a convergent subsequence. Hence the same is true of (Tm_n), otherwise $(T + K_n)m_n$ could not converge to zero. Thus, by Lemma 1.3.22 any subsequence of (m_n) has a converging subsequence which implies that $\{m_n\}$ is totally bounded. Therefore, by Lemma 1.3.17, and by the assumption that $K_n x \to 0$ for all $x \in \mathbb{X}$, (K_n) converges to zero uniformly on $\{m_n\}$. So $(K_n m_n)$ converges to zero. Therefore (Tm_n) converges to zero. This is impossible since T_M is injective and therefore by Lemma 1.3.13, T_M is bounded below. So $\gamma(T_M + K_n) \geq c$ for some $c > 0$.

Also there exists a p such that $T_M + K_n$ is injective for $n \geq p$, since if not, by passing to a subsequence if necessary, we could choose elements m_n

of norm one from $N(T_M + K_n) = N(T + K_n) \cap M$ for every $n \geq p$ with the property that $(T + K_n)m_n \to 0$. Now we can apply the above argument to give us a contradiction. $\square$

Note that in the above Lemma T is a ϕ_+-operator. By [G2; V.2.1] $T + K_n$ is a ϕ_+-operator for every $n \in \mathbb{N}$, since each K_n is compact by Theorem 2.2.2.

Lemma 4.2.4 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed linear operator with $R(T)$ closed and $\overline{D(T)} = \mathbb{X}$. Let $\{K_n\}$ be a collectively compact set of operators such that $K_n x \to 0$ for all $x \in \mathbb{X}$. If $R(T)$ is complemented in $\mathbb{Y}$ by a closed subspace W, then there exists a p and $c > 0$ such that for $n \geq p$, $T_1' + K_n'$ is injective and $\gamma(T_1' + K_n') \geq c$, where T_1' is the restriction of T' to $W^\perp \cap D(T')$.*

Proof: $\mathbb{Y} = R(T) \oplus W$ and by [G2; IV.1.11], $\mathbb{Y}' = R(T)^\perp \oplus W^\perp$. Assume $\gamma(T_1' + K_n')$ has a subsequence converging to zero. For simplicity assume that

$$\gamma(T_1' + K_n') = \inf_{w' \in W^\perp} \frac{\|(T_1' + K_n')w'\|}{d(w', N(T_1' + K_n'))} \to 0.$$

By the definition of the minimum modulus we may now, for every n, select $w_n' \in W^\perp$ and $k_n' \in N(T_1' + K_n') = N(T' + K_n') \cap W^\perp$ such that

$$\left| \frac{\|(T' + K_n')w_n'\|}{\|w_n' - k_n'\|} - \gamma(T_1' + K_n') \right| < \frac{1}{n} \to 0.$$

Note that now $\frac{\|(T'+K_n')w_n'\|}{\|w_n'-k_n'\|} \to 0$. Let $y_n' = \frac{w_n'-k_n'}{\|w_n'-k_n'\|}$, then $\|y_n'\| = 1$ and

$$\begin{aligned} (T' + K_n')y_n' &= (T' + K_n')\frac{(w_n' - k_n')}{\|w_n' - k_n'\|} \\ &= \frac{(T' + K_n')w_n'}{\|w_n' - k_n'\|} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now we have the existence of a sequence $(y_n') \subset W^\perp$ such that $\|y_n'\| = 1$ and $(T' + K_n')y_n' \to 0$.

Since $\|y_n'\| = \sup_{\|y\|=1} |y_n'(y)|$, we can choose $y \in \mathbb{Y}$ so that $\|y\| = 1$ and $|y_n'y| \geq \frac{1}{2}$. Suppose $y_n'y = re^{i\theta}$, where $r = |y_n'y|$ and let $y_n = e^{-i\theta}y$. Then

$$y_n'y_n = e^{-i\theta}y_n'y = r = |y_n'y| \geq \frac{1}{2}.$$

Now $y_n = Tv_n + w_n$ for some $w_n \in W$ and $v_n \in \mathbb{X}$. Also, $R(T)$ is closed and complemented by W, therefore by [G2; II.1.14] there exists a projection P along W onto $R(T)$ such that $Py_n = Tv_n$ and since (y_n) and P are bounded, (Tv_n) is bounded. Hence, by Lemma 1.3.21, there exists a bounded sequence (x_n) such that $Tx_n = Tv_n$. Furthermore, $y'_n v \to 0$ for all $v \in \mathbb{Y}$. To see this, note firstly that $y'_n v = 0$ for all $v \in W$ since $y'_n \in W^\perp$. Secondly, $y'_n v \to 0$ for all $v \in R(T)$ since

$$y'_n Tx = (T' + K'_n)y'_n x - y'_n K_n x \to 0.$$

Therefore, since $\mathbb{Y} = R(T) \oplus W$, $y'_n v \to 0$ for all $v \in \mathbb{Y}$. Now

$$\begin{aligned} \frac{1}{2} &\leq y'_n y_n \\ &= y'_n(Tx_n + w_n) \\ &= y'_n Tx_n + y'_n w_n \\ &= ((T' + K'_n)y'_n(x_n) - y'_n K_n x_n \end{aligned} \tag{4.1}$$

Since (x_n) is bounded, $\{K_n x_n\}$ is relatively compact and therefore totally bounded in $\mathbb{Y}$ [S; 25-A]. This, together with the observation that $y'_n v \to 0$ for all $v \in \mathbb{Y}$ implies that $y'_n K_n x_n \to 0$ by Lemma 1.3.17. Therefore equation 4.1 cannot hold since $(T' + K'_n)y'_n \to 0$. So we have a contradiction.

Also, by the above argument, there exists a p such that $T' + K'_n$ is injective on $W^\perp$ for $n \geq p$, since otherwise, by the same argument as in the proof of Lemma 4.2.3, a sequence (y'_n) with the above properties would exist and this leads to a contradiction. □

Note that the type of operator T in the above Lemma is a generalisation of a ϕ_--operator. $T'_1 + K'_n$ is a ϕ_+-operator since $T'_1 + K'_n$ is injective implies that $\alpha(T'_1 + K'_n) = 0$ and Lemma 1.3.16 implies that $R(T'_1 + K'_n)$ is closed.

Theorem 4.2.5 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed operator with $R(T)$ closed and $\{K_n\}$ a collectively compact set of operators such that $K_n x \to 0$ for all $x \in \mathbb{X}$. If $\alpha(T) < \infty$, i.e. T is a ϕ_+-operator, then there exists a p such that*

1. *$T + K_n$ has a closed range and $\alpha(T + K_n) \leq \alpha(T)$ for all $n \geq p$.*
2. *$\alpha(T + K_n) = \alpha(T)$ for all $n \geq p$, if and only if $\inf_{n \geq p}[\gamma(T + K_n)] > 0$.*

Proof:

1. $\mathbb{X} = M \oplus N(T)$ for some closed subspace M by [G2; II.1.16] since $N(T)$ is finite dimensional and hence closed. Let p and $c > 0$ be as in Lemma 4.2.3 and let $n \geq p$. Then $(T+K_n)M$ is closed by Lemma 4.2.3 and Lemma 1.3.16. The finite dimensionality of $N(T)$ implies that

$$\begin{aligned} R(T+K_n) &= (T+K_n)\mathbb{X} \\ &= (T+K_n)M + (T+K_n)N(T) \\ &= (T+K_n)M + K_nN(T) \end{aligned}$$

is closed.

Also by Lemma 4.2.3, $M \cap N(T+K_n) = \{0\}$. Hence

$$\mathbb{X} = M \oplus N(T) \supset M \oplus N(T+K_n),$$

which implies that $\alpha(T) \geq \alpha(T+K_n)$.

2. As above, $\mathbb{X} = M \oplus N(T)$ for some closed subspace M. Suppose $\alpha(T+K_n) = \alpha(T)$ for all $n \geq p$. Then $\mathbb{X} = M \oplus N(T+K_n)$ and for $x = m_n + z_n$ where $m_n \in M$ and $z_n \in N(T+K_n)$, we have by Lemma 4.2.3 that

$$\begin{aligned} \|(T+K_n)x\| &= \|(T+K_n)m_n\| \\ &= \frac{\|(T+K_n)m_n\|}{d(m_n, N(T+K_n))} d(m_n, N(T+K_n)) \\ &\geq \inf_{v \in M} \frac{\|(T+K_n)v\|}{d(v, N(T+K_n))} d(m_n, N(T+K_n)) \\ &= \gamma(T_M + K_n) d(m_n, N(T+K_n)) \\ &\geq c\, d(m_n, N(T+K_n)) \\ &= c\, d(x, N(T+K_n)). \end{aligned}$$

Thus $\gamma(T+K_n) \geq c > 0$ for all $n \geq p$ and hence $\inf_{n \geq p}[\gamma(T+K_n)] > 0$. Conversely, assume that $\gamma(T+K_n) \geq c > 0$ for all $n \geq p$ but that $\alpha(T+K_n) \neq \alpha(T)$. Then, by passing to a subsequence if necessary, we have from the first part of the Theorem, $\alpha(T+K_n) < \alpha(T)$ for all $n \geq p$ and by Lemma 1.3.23 there exists a sequence $(z_n) \subset N(T)$ such that $1 = \|z_n\| = d(z_n, N(T+K_n))$. Hence for $n \geq p$,

$$\begin{aligned} 0 &< c \\ &= c\, d(z_n, N(T+K_n)) \\ &\leq \|(T+K_n)z_n\| \qquad (4.2) \\ &= \|K_n z_n\|. \end{aligned}$$

Since $N(T)$ is finite dimensional, by the compactness of the unit ball, (z_n) is totally bounded [S; 25-A]. Therefore, by Lemma 1.3.17, (K_n) converges to zero uniformly on (z_n), contradicting equation 4.2.

□

Theorem 4.2.6 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a closed operator with $R(T)$ closed and $\{K_n\}$ a collectively compact set of operators such that $K_n x \to 0$ for all $x \in \mathbb{X}$. If $R(T)$ is complemented in $\mathbb{Y}$ by a closed subspace and $D(T)$ is dense in $\mathbb{X}$, then there exists a p such that for $n \geq p$, $\alpha(T' + K_n') \leq \alpha(T')$. If $\beta(T) < \infty$, i.e. T is a ϕ_--operator, there exists a p such that:*

1. *$T + K_n$ has a closed range and $\beta(T + K_n) \leq \beta(T)$ for all $n \geq p$.*
2. *$\beta(T + K_n) = \beta(T)$ for all $n \geq p$ implies that $\inf_{n \geq p}[\gamma(T + K_n)] > 0$.*

Proof: We may assume, without loss of generality, that $\overline{D(T)} = \mathbb{X}$ by replacing $\mathbb{X}$ with $\overline{D(T)}$ when necessary. Let W and p be as in Lemma 4.2.4 with $n \geq p$. Then, since $\mathbb{Y} = R(T) \oplus W$,

$$\mathbb{Y}' = R(T)^{\perp} \oplus W^{\perp} = N(T') \oplus W^{\perp},$$

by [G2; IV.1.11 and II.3.7]. Since $T' + K_n'$ is injective on $W^{\perp}$ by Lemma 4.2.4, $W^{\perp} \cap N(T' + K_n') = \{0\}$ and hence

$$\mathbb{Y}' \supset N(T' + K_n') \oplus W^{\perp}.$$

Therefore $\alpha(T' + K_n') \leq \alpha(T')$.

1. For $n \geq p$, $(T' + K_n')W^{\perp}$ is closed by Lemma 4.2.4 and Lemma 1.3.16. Since $\alpha(T') = \beta(T) < \infty$ by [G2; II.3.7],

$$\begin{aligned} (T' + K_n')\mathbb{Y}' &= (T' + K_n')(W^{\perp} + N(T')) \\ &= (T' + K_n')W^{\perp} + (T' + K_n')N(T') \\ &= (T' + K_n')W^{\perp} + K_n' N(T') \end{aligned}$$

is closed. Thus $R(T' + K_n')$ is closed and then [G2; IV.1.2] implies that $R(T + K_n)$ is closed. Also

$$\beta(T + K_n) = \alpha(T' + K_n') \leq \alpha(T') = \beta(T)$$

for all $n \geq p$.

2. Assume that $\beta(T+K_n)=\beta(T)<\infty$ for all $n\geq p$, i.e. $\alpha(T'+K_n')=\alpha(T')$ for all $n\geq p$, where p and c are as in Lemma 4.2.4. Then

$$\begin{aligned}\mathbb{Y}' &= N(T')\oplus W^{\perp}\\ &= N(T'+K_n')\oplus W^{\perp}.\end{aligned}$$

Hence for $y'=z_n'+w_n'$, where $z_n'\in N(T'+K_n')$ and $w_n'\in W^{\perp}$, we have

$$\begin{aligned}\|(T'+K_n')y'\| &= \|(T'+K_n')w_n'\|\\ &= \frac{\|(T'+K_n')w_n'\|}{d(w_n',N(T'+K_n'))}d(w_n',N(T'+K_n'))\\ &\geq \inf_{y'\in W^{\perp}}\frac{\|(T'+K_n')y'\|}{d(y',N(T'+K_n'))}d(w_n',N(T'+K_n'))\\ &\geq \gamma(T_1'+K_n')d(w_n',N(T'+K_n'))\\ &\geq c\ d(w_n',N(T'+K_n'))\\ &= c\ d(y',N(T'+K_n')).\end{aligned}$$

Therefore, by [G2; IV.1.9], $\gamma(T+K_n)=\gamma(T'+K_n')\geq c$ for all $n\geq p$ and hence $\inf_{n\geq p}[\gamma(T+K_n)]>0$.

□

Theorem 4.2.7 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T:D(T)\subset\mathbb{X}\to\mathbb{Y}$ be a Fredholm operator. Let $\{K_n\}$ be a a collectively compact set of operators such that $K_nx\to 0$ for all $x\in\mathbb{X}$. Then $\gamma(T+K_n)\geq c>0$ for all sufficiently large n if and only if $\alpha(T+K_n)=\alpha(T)$ and $\beta(T+K_n)=\beta(T)$ for all sufficiently large n.*

Proof: We may assume, without loss of generality, that $\overline{D(T)}=\mathbb{X}$ by replacing $\mathbb{X}$ with $\overline{D(T)}$ when necessary. We may write $\mathbb{Y}=R(T)\oplus W$ where $\dim(W)<\infty$ since $\beta(T)<\infty$. Suppose that $\gamma(T+K_n)\geq c>0$ for all but a finite number of n, but that $\beta(T+K_n)\neq\beta(T)$ for infinitely many n. Then, by Theorem 4.2.6, $\beta(T+K_n)<\beta(T)$ for infinitely many n. For simplicity assume that $\beta(T+K_n)<\beta(T)$ and that $\gamma(T+K_n)\geq c>0$ for all $n\geq p$, where p is chosen so that Lemma 4.2.4 holds. Thus since $\beta(T+K_n)<\dim(W)$ for all $n\geq p$, there exists $y_n\in R(T+K_n)\cap W$ with $\|y_n\|=1$ for $n\geq p$. Since $(\|y_n\|)$ is bounded and $\gamma(T+K_n)\geq c>0$ it follows that there exists a bounded sequence (x_n) such that $y_n=(T+K_n)x_n$.

A proof for this may be constructed along the lines of Lemma 1.3.21. Now, since $\dim(W) < \infty$, the compactness of the unit ball implies that (y_n) has a convergent subsequence, say $y_{n'} \to y \in W$. Since $\{K_n\}$ is collectively compact, $(K_{n'}x_{n'})$ has a convergent subsequence, and therefore so does $(Tx_{n'})$. Thus, by Lemma 1.3.22 and by passing to a subsequence if necessary, $(x_{n'})$ may be assumed to be totally bounded. Therefore, by Lemma 1.3.17, and by the assumption that $K_n x \to 0$ for all $x \in \mathbb{X}$, (K_n) converges to zero uniformly on $\{x_{n'}\}$. So $(K_{n'}x_{n'})$ has a subsequence $(K_{n''}x_{n''})$ converging to zero. Hence

$$\begin{aligned} y &= \lim_{n'' \to \infty} y_{n''} \\ &= \lim_{n'' \to \infty} (T + K_{n''})x_{n''} \\ &= \lim_{n'' \to \infty} Tx_{n''} + \lim_{n'' \to \infty} K_{n''}x_{n''} \\ &= \lim_{n'' \to \infty} Tx_{n''} \in R(T) \end{aligned}$$

which shows that $y \in R(T) \cap W = \{0\}$. But this is impossible since $\|y\| = 1$. Therefore $\beta(T + K_n) = \beta(T)$ for all sufficiently large n. The fact that $\alpha(T + K_n) = \alpha(T)$ for all sufficently large n and the converse follows directly from the second part of Theorem 4.2.5. $\square$

We are now ready to state the most important theorem in this chapter.

Theorem 4.2.8 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ be a semi-Fredholm operator. Let $\{K_n\}$ be a collectively compact set of operators such that $K_n x \to 0$ for all $x \in \mathbb{X}$. Then there exists a p such that for all $n \geq p$,*

1. *$T + K_n$ is semi-Fredholm,*
2. *$\alpha(T + K_n) \leq \alpha(T)$,*
3. *$\beta(T + K_n) \leq \beta(T)$,*
4. *$i(T + K_n) = i(T)$.*

Proof:

1. This result is contained in Theorem 4.2.5 and Theorem 4.2.6.
2. This result is contained in Theorem 4.2.5.

3. This result is contained in Theorem 4.2.6.

4. There exists a p such that for all $\lambda \in [0,1]$ and $n \geq p$, $T + \lambda K_n$ is semi-Fredholm. If this were not the case, there would exist a subsequence $(K_{n'})$ of (K_n) and a sequence $\lambda_{n'} \in [0,1]$ such that $T + \lambda_{n'} K_{n'}$ is not semi-Fredholm. This would contradict Theorem 4.2.5 and Theorem 4.2.6 since by Theorem 2.2.2 and Theorem 3.3.5, $\lambda_{n'} K_{n'}$ is collectively compact and converges pointwise to 0.

   Let $\mathbb{Z}$ be the set of integers together with the "ideal" elements ∞ and $-\infty$. Let $[0,1]$ have the usual topology and $\mathbb{Z}$ the discrete topology. Now, given $n \geq p$, define $\varphi : [0,1] \to \mathbb{Z}$ by $\varphi(\lambda) = i(T + \lambda K_n)$. By the proof of [G2; V.1.6(iii)], φ is constant for very small variations of λ, hence it is continuous. Since $[0,1]$ is connected, so is its continuous image $\varphi([0,1])$. [If $\varphi([0,1])$ is not connected there exists a nonempty $M \subset \varphi([0,1])$ which is both open and closed in $\mathbb{Z}$. However, then $\varphi^{-1}(M) \subset \varphi^{-1}(\varphi([0,1])) = [0,1]$ is closed and open by continuity, which implies that $[0,1]$ is not connected which gives us a contradiction.] Hence $\varphi([0,1])$ consists of only one point. Therefore φ is constant on $[0,1]$. Hence $i(T) = \varphi(0) = \varphi(1) = i(T + K_n)$.

□

Chapter 5

Collectively Strictly Singular Sequences of Operators

5.1 Introduction.

The results in this chapter are a generalisation of the perturbation results obtained by Goldberg in [G1] for collectively compact operators discussed in the previous chapter. We will now consider perturbations of semi-Fredholm operators with a class of operators called collectively strictly singular operators. Collectively strictly singular operators were defined by L.E. Labuschagne. All of the results pertaining to this class were originally proved by him and then privately communicated to the author without proof. The author then subsequently independently verified 5.2.3 and 5.2.7 and also contributed to simplifying the proofs of 5.2.9, 5.2.10 and 5.2.11. As yet no work has been done on applications for this work but possibilities will be discussed in the summary. The work done in the section on collectively strictly cosingular operators is a natural analogue of section 5.2 and this work was done by the author.

5.2 Collectively Strictly Singular Sequences of Operators.

The concept of strict singularity of an operator, defined below, has proved to be very useful in Perturbation Theory and was first introduced by Kato [Ka] in 1958.

Definition 5.2.1 *Let $T \in B(\mathbb{X}, \mathbb{Y})$. T is called strictly singular if there does not exist any infinite dimensional subspace M of $D(T)$ such that T restricted to M, written $T\backslash_M$, is bounded below.*

The contrapositive of Definition 1.3.9 is that an operator $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is not bounded below if and only if for every positive number m there exists $x \in D(T)$ such that $\|Tx\| < m\|x\|$. The following Lemma follows immediately.

Lemma 5.2.2 *$T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ is not bounded below if and only if there exists a sequence $(x_n) \subset D(T)$ with $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$.*

We are now ready to prove the following Proposition.

Proposition 5.2.3 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$. Then T is strictly singular if and only if for every sequence of infinite dimensional subspaces (M_n) of $D(T)$ there exists a sequence (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$.*

Proof: T is strictly singular if and only if for every infinite dimensional $M \subset D(T), T\backslash_M$ is not bounded below. Hence, by Lemma 5.2.2, T is strictly singular if and only if for every infinite dimensional $M \subset D(T)$ there exists a sequence $(x_n) \subset M$ with $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$. From this we can show that T is strictly singular if and only if for every sequence of infinite dimensional subspaces $(M_n) \subset D(T)$ there exists a sequence (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$. To see this let $(M_k) \subset D(T)$ be any sequence of infinite dimensional subspaces. Then, for each k, there exists $(x_n^{(k)}) \subset M_k$ with $\|x_n^{(k)}\| = 1$ such that $\|Tx_n^{(k)}\| \to 0$. Hence for each k there exists N_k such that $\|Tx_n^{(k)}\| < 2^{-k}$ for all $n \geq N_k$. Select the N_k's such that $N_1 < N_2 < N_3 \ldots$. Clearly $\|Tx_{N_k}^{(k)}\| \to 0$ as $k \to \infty$. Then by letting $x_{N_k}^{(k)} = x_k$ we have the existence of (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$. To get the reverse implication, choose $M = M_n$ for every n. □

Motivated by the above result the following definition was made by L.E. Labuschagne:

Definition 5.2.4 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, Then $(T_n) \subset B[\mathbb{X}, \mathbb{Y}]$ is called a collectively strictly singular sequence of operators if for every sequence of infinite dimensional subspaces (M_n) of $\mathbb{X}$, there exists (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|T_n x_n\| \to 0$.*

Let us look at an example of a collectively strictly singular sequence of operators:

Example 5.2.5 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. If $T_n \to T \in B[\mathbb{X}, \mathbb{Y}]$ in norm where T is a strictly singular operator, then (T_n) is a collectively strictly singular sequence of operators.*

We may see this by letting (M_n) be any sequence of infinite dimensional subspaces of $\mathbb{X}$. By Proposition 5.2.3 there exists a sequence (x_n) with $x_n \in M_n$ and $||x_n|| = 1$ such that $||Tx_n|| \to 0$. Hence

$$\begin{aligned} ||T_n x_n|| &\leq ||Tx_n|| + ||(T_n - T)x_n|| \\ &\leq ||Tx_n|| + ||T_n - T|| ||x_n|| \\ &\to 0. \end{aligned}$$

□

We will need the following property of collectively strictly singular sequences of operators to prove our perturbation result.

Lemma 5.2.6 *Subsequences of collectively strictly singular sequences of operators are collectively strictly singular.*

Proof: Let (K_n) be a collectively strictly singular sequence of operators and let (K_{n_p}) be an arbitrary subsequence of (K_n). Moreover let (M_p) be any sequence of infinite dimensional subspaces of $\mathbb{X}$. (K_n) collectively strictly singular implies that for every sequence of infinite dimensional subspaces (W_n) of $\mathbb{X}$, there exists (x_n) with $x_n \in W_n$ and $||x_n|| = 1$ such that $||K_n x_n|| \to 0$. Denote M_p by W_{n_p} and let $W_k = \mathbb{X}$ for all $k \notin \{n_1, n_2, n_3, \ldots\}$. Then there exists a sequence (x_k) with $x_k \in W_k$ and $||x_k|| = 1$ such that $||K_k x_k|| \to 0$. Setting $x_{n_p} = z_p$ we have the existence of a sequence (z_p) with $z_p \in W_{n_p} = M_p$ and $||z_p|| = 1$ such that $||K_{n_p} z_p|| \to 0$. □

The following result is well known and will be needed to prove Lemma 5.2.8.

Lemma 5.2.7 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. Then T is a ϕ_+-operator if and only if there exists a closed finite codimensional subspace $E \subset D(T)$ such that $T\backslash_E$ is bounded below.*

Proof: Assume T is a ϕ_+-operator. Write $\mathbb{X} = E \oplus N(T)$. (This may be done by [G2; II.1.16] since $N(T)$ is finite dimensional). Then $R(T\backslash_E) = TE =$

$T\mathbb{X} = R(T)$ which is closed. Also $T\backslash_E$ is clearly injective and hence by Lemma 1.3.13, $T\backslash_E$ is bounded below. For the other direction note that TE is closed by Lemma 1.3.13. (We may apply Lemma 1.3.13 since $\mathbb{X}$ complete and T bounded implies that T is closed by [K; 4.13-5(a)].) Hence, by [G2; IV.2.8.(i)], there exists a finite dimensional $K \subset D(T)$ such that $\mathbb{X} = E \oplus K$. Now $R(T) = T\mathbb{X} = TE + TK$ and dim $TK \leq$ dim $K < \infty$. Thus, by [G2; I.4.12], $R(T)$ is closed. Also dim $N(T) < \infty$ since codim $E < \infty$. □

Corollary 5.2.8 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. Then T is not a ϕ_+-operator if and only if for every closed finite codimensional $E \subset D(T)$ there exists a sequence $(x_n) \subset E$ with $\|x_n\| = 1$ such that $\|Tx_n\| \to 0$.*

Proof: By Lemma 5.2.7, T is not a ϕ_+-operator if and only if for every closed finite codimensional $E \subset D(T)$, $T\backslash_E$ is not bounded below. By applying Lemma 5.2.2 we obtain the result. □

Lemma 5.2.9 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. If T is a ϕ_+-operator and (K_n) is a collectively strictly singular sequence of operators then there exists $p \geq 0$ such that $\alpha(T + K_n) < \infty$ and $\gamma(T + K_n) > 0$ for all $n \geq p$. (i.e. $T + K_n$ is a ϕ_+-operator for all $n \geq p$).*

Proof: Let T be a ϕ_+-operator and (K_n) a collectively strictly singular sequence of operators. Assume that, for all n, $(T + K_n)$ is not a ϕ_+-operator. Let E be an arbitrary closed finite codimensional subspace of $\mathbb{X}$. Then $(T + K_n)\backslash_E$ is not a ϕ_+-operator for all n. (Since if $(T + K_n)\backslash_E$ was a ϕ_+-operator then by Lemma 5.2.7 there would exist a closed finite codimensional subspace $M \subset E = D((T + K_n)\backslash_E)$ such that $(T + K_n)\backslash_E$ restricted to M, which is equal to $(T + K_n)\backslash_M$, would be bounded below. This would imply that $(T + K_n)$ is a ϕ_+-operator.) Hence, by Lemma 5.2.7 and [G2; III.1.9], there exists an infinite dimensional sequence $(M_n) \subset E$ such that $\|(T + K_n)\backslash_{M_n}\| < 2^{-n}$ for every positive integer n. Since (K_n) is collectively strictly singular, there exists (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|K_n x_n\| \to 0$. Hence $K_n x_n \to 0$ and since $\|(T + K_n)x_n\| < 2^{-n}$, $\|Tx_n\| \to 0$. So, by Corollary 5.2.8, T is not a ϕ_+-operator which gives us a contradiction. Therefore $T + K_n$ is a ϕ_+-operator for some n. Also, by Lemma 5.2.6, there does not exist a subsequence (K_{n_p}) of (K_n) for which $(T + K_{n_p})$ is not ϕ_+ for all n_p since it would contradict what we have just proved. Hence there exists a $p \geq 0$ such that $T + K_n$ is a ϕ_+-operator for all $n \geq p$ and the result follows. □

Lemma 5.2.10 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. If T is a ϕ_+-operator and (K_n) is collectively strictly singular then for every $(\lambda_n) \in \ell^\infty$ there exists $p \geq 0$ such that for $n \geq p$ we have $\alpha(T + \lambda_n K_n) < \infty$ and $\gamma(T + \lambda_n K_n) > 0$.*

Proof: It suffices to show that (K_n) collectively strictly singular implies that $(\lambda_n K_n)$ is collectively strictly singular since then the result follows directly from Lemma 5.2.9. Since (K_n) is collectively strictly singular we know that for any infinite dimensional sequence (M_n) there exists a sequence (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|K_n x_n\| \to 0$. Since $(\lambda_n) \in \ell^\infty$, we have $\sup_n|\lambda_n| = M(\text{say}) < \infty$ and therefore $\|(\lambda_n K_n)x_n\| \leq M\|K_n x_n\|$. Hence $\|(\lambda_n K_n)x_n\| \to 0$. □

We are now ready to prove the following perturbation theorem.

Theorem 5.2.11 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. If T is a ϕ_+-operator and (K_n) is a collectively strictly singular sequence of operators, then for every $r > 0$ there exists $p \geq 0$ such that for any $(\lambda_n) \subset rB_{\mathbb{C}}$ we have:*

1. *$T + \lambda_n K_n$ is a ϕ_+-operator whenever $n \geq p$.*
2. *$i(T + \lambda_n K_n) = i(T)$ whenever $n \geq p$.*
3. *For any $n \geq p$, $\alpha(T + \lambda_n K_n)$ and $\beta(T + \lambda_n K_n)$ have constant values n_1 and n_2 except perhaps at isolated points where $\infty > \alpha(T + \lambda_n K_n) > n_1$ and $\beta(T + \lambda_n K_n) > n_2$.*

Proof:

1. Let T be a ϕ_+-operator and (K_n) a collectively strictly singular sequence of operators. Assume there exists $r > 0$ such that for every $p \geq 0$ there exists $(\lambda_n^{(p)}) \subset rB_{\mathbb{C}}$ with $(T + \lambda_{n_p}^{(p)} K_{n_p}) \notin \phi_+$ for some $n_p \geq p$. Now let $p_1 = 1, p_2 = n_1, p_3 = n_{p_2} (= n_2)$ etc. (This is to ensure that $n_1 < n_2 < n_3 < \ldots$). For such n_p we have, by Lemma 5.2.6, that (K_{n_p}) is collectively strictly singular since (K_n) is collectively strictly singular. Also, since $|\lambda_{n_p}^{(p)}| \leq r$, $(\lambda_{n_p}^{(p)}) \in \ell^\infty$. Hence, by Lemma 5.2.10, there exists a $q \geq 0$ such that $(T + \lambda_{n_p}^{(p)} K_{n_p})$ is a ϕ_+-operator for all $n_p \geq q$ which gives us a contradiction. Hence for every $r > 0$ there exists $p \geq 0$ such that for all $(\lambda_n) \subset rB_{\mathbb{C}}$ and any $n \geq p$ we have that $T + \lambda_n K_n$ is a ϕ_+-operator.

2. Let $\mathbb{Z}$ be the set of integers together with the "ideal" elements ∞ and $-\infty$. Let $rB_{\mathbb{C}}$ have the usual topology and $\mathbb{Z}$ the discrete topology. In the first part of this theorem we showed that for all $n \geq p$ and $\lambda \in rB_{\mathbb{C}}$, $T + \lambda K_n$ is a ϕ_+-operator. Now, given $n \geq p$, define $\varphi : rB_{\mathbb{C}} \to \mathbb{Z}$ by $\varphi(\lambda) = i(T + \lambda K_n)$. By the proof of [G2; V.1.6(iii)], φ is constant for very small variations of λ, hence it is continuous. $rB_{\mathbb{C}}$ is connected by an analogue argument to that in Theorem 4.2.8 and therefore the continuous image $\varphi(rB_{\mathbb{C}})$ is also connected. Hence $\varphi(rB_{\mathbb{C}})$ consists of only one point and this implies that φ is constant on $rB_{\mathbb{C}}$. Hence $i(T) = \varphi(0) = \varphi(\lambda) = i(T + \lambda K_n)$ where $\lambda \in rB_{\mathbb{C}}$.

3. Since $T + \lambda_n K_n$ is a ϕ_+-operator for all $n \geq p$ and for all $\lambda_n \in rB_{\mathbb{C}}$ the result follows directly by [G2; V.1.8]. □

The following Corollary shows that Theorem 5.2.11 in fact contains the classical result on small perturbations of ϕ_+-operators:

Corollary 5.2.12 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T \in B[\mathbb{X}, \mathbb{Y}]$. If T is a ϕ_+-operator and B is any bounded operator then there exists a $\rho > 0$ such that if $\|B\| < \rho$, $T + B$ is a ϕ_+-operator.*

Proof: Suppose there does not exist a fixed $\rho > 0$ such that $T + B$ is a ϕ_+-operator for any operator B with $\|B\| < \rho$. This implies the existence of a bounded sequence of operators (B_n) such that $T + \frac{1}{n}B_n$ is not ϕ_+ for any n. Assume that $\|B_n\| \leq M < \infty$ for each n. Then $(\frac{1}{n}B_n)$ is collectively strictly singular since $\|\frac{1}{n}B_n\| \leq \frac{1}{n}M \to 0$, which gives us a contradiction with Theorem 5.2.11. □

Consider the following example:

Example 5.2.13 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let (A_n) with $A_n \in B[\mathbb{X}, \mathbb{Y}]$ be a sequence of norm one operators. Let (K_n) be any sequence of strictly singular operators. Then $\frac{1}{n}A_n + K_n$ is collectively strictly singular.*

Let (M_k) be any sequence of infinite dimensional subspaces of $\mathbb{X}$. Then, by Proposition 5.2.3, for each fixed k, there exists a sequence $(x_n^{(k)})$ with $x_n^{(k)} \in M_k$ and $\|x_n^{(k)}\| = 1$ such that $\|K_k x_n^{(k)}\| \to 0$ as $n \to \infty$. Hence, for each k, there exists a N_k such that $\|K_k x_n^{(k)}\| < 2^{-k}$ for all $n \geq N_k$. Select the N_k's such that $N_1 < N_2 < N_3 \ldots$. Clearly $\|K_k x_{N_k}^{(k)}\| \to 0$ as $k \to \infty$.

Then, by letting $x_{N_k}^{(k)} = x_k$, we have the existence of (x_n) with $x_n \in M_n$ and $\|x_n\| = 1$ such that $\|K_n x_n\| \to 0$. But

$$\begin{aligned} \|(\frac{1}{n}A_n + K_n)x_n\| &\leq \|\frac{1}{n}A_n x_n\| + \|K_n x_n\| \\ &\leq \frac{1}{n} + \|K_n x_n\| \to 0. \end{aligned}$$

Hence $\frac{1}{n}A_n + K_n$ is collectively strictly singular. □

Using the above example we can illustrate that Theorem 5.2.11 also contains the dual result of the strictly singular perturbation Theorem [G2; V.2.1.]. Given a ϕ_+-operator T, if we let $A_n = 0$ and $K_n = K$ for each n, we may apply Theorem 5.2.11 to obtain the existence of a $p \geq 0$ such that $T + K_n = T + K$ is ϕ_+ for any $n \geq p$.

5.3 Collectively Strictly Cosingular Sequences of Operators.

Definition 5.3.1 (Quotient map.) *Let $M \subset \mathbb{X}$. We define the operator $Q_M : D(T) \subset \mathbb{X} \to \mathbb{X}/M$ by $Q_M(x) = [x]$ for each $x \in \mathbb{X}$.*

Q_M is clearly linear. It is easy to see by the definition of the norm of a coset that $\|[x]\| \leq \|x\|$ since any $y \in [x]$ is of the form $x - m$ where $m \in M$ so that $\|[x]\| = \inf_{y \in [x]} \|y\|$. Then

$$\begin{aligned} \|Q_M\| &= \sup_{x \in D(T)} \frac{\|Q_M x\|}{\|x\|} \\ &= \sup_{x \in D(T)} \frac{\|[x]\|}{\|x\|} \\ &\leq \sup_{x \in D(T)} \frac{\|x\|}{\|x\|} \\ &= 1 \end{aligned}$$

and therefore Q_M is continuous.

The class of strictly cosingular operators, which in a certain sense is dual to the class of strictly singular operators, was introduced by Pelczinsky in 1965 in [Pe]. In the Banach space setting Pelczinsky's definition is equivalent to the following more general definition.

Definition 5.3.2 *If $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$ then T is called strictly cosingular if there does not exist a closed infinite codimensional subspace M of Y for which $(Q_M T)' = T'\backslash_{M^\perp}$ is bounded below.*

This definition now allows us to prove the following proposition.

Proposition 5.3.3 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with $T : D(T) \subset \mathbb{X} \to \mathbb{Y}$. Then T is strictly cosingular if and only if for every sequence of closed infinite codimensional subspaces (M_n) of $\mathbb{Y}$ there exists a sequence (y'_n) with $y'_n \in {M_n}^\perp$ and $\|y'_n\| = 1$ such that $\|T'y'_n\| \to 0$.*

Proof: By applying Lemma 5.2.2 to the above definition of strict cosingularity we have that T is strictly cosingular if and only if for every closed infinite codimensional subspace M of $\mathbb{Y}$ there exists a sequence (y'_n) with $y'_n \in M^\perp$ and $\|y'_n\| = 1$ such that $\|T'y'_n\| \to 0$. From this we can show in an analogue way to the proof of proposition 5.2.3 that T is strictly cosingular if and only if for every closed infinite codimensional sequence $(M_n) \subset \mathbb{Y}$ there exists a sequence (y'_n) with $y'_n \in {M_n}^\perp$ and $\|y'_n\| = 1$ such that $\|T'y'_n\| \to 0$. To see this let $(M_k) \subset \mathbb{Y}$ be any sequence of closed infinite codimensional subspaces. Then, for each k, there exists a sequence $(y_n^{(k)'}) \subset {M_k}^\perp$ with $\|y_n^{(k)'}\| = 1$ such that $\|T'y_n^{(k)'}\| \to 0$. Hence for each k there exists N_k such that $\|T'y_n^{(k)'}\| < 2^{-k}$ for all $n \geq N_k$. Select the N_k's such that $N_1 < N_2 < N_3 \ \ldots.$ Clearly $\|T'y_{N_k}^{(k)'}\| \to 0$ as $k \to \infty$. Then by letting $y_{N_k}^{(k)'} = y'_k$ we have the existence of (y'_n) with $y'_n \in {M_n}^\perp$ and $\|y'_n\| = 1$ such that $\|T'y'_n\| \to 0$. To get the reverse implication, choose $M = M_n$ for every n. □

Motivated by the above result we now make the following definition:

Definition 5.3.4 *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. Then $(T_n) \in B[\mathbb{X}, \mathbb{Y}]$ is called a collectively strictly cosingular sequence of operators if for every sequence of closed infinite codimensional subspaces (M_n) of $\mathbb{Y}$ there exists a sequence (y'_n) with $y'_n \in {M_n}^\perp$ and $\|y'_n\| = 1$ such that $\|T'_n y'_n\| \to 0$.*

Using this definition it is now possible to dualise the perturbation results obtained in section 5.2 for ϕ_--operators and collectively strictly cosingular operators.

5.4 Summary.

In Chapter 2 it was shown how collectively compact sets of operators may be used in an approximation theory for Fredholm integral equations of the second kind. This approximation theory was applicable to the case where the kernel of the integral equation is continuous such that the integral operator K, mapping $C[a,b]$ into $C[a,b]$, is compact. In this case it was shown that the set of numerical-integral operators $\{K_n\}$ approximating K is collectively compact.

In the situation where the kernel is not continuous but only measurable and the integral operator K maps from the Lebesque space $\mathcal{L}_1$ into $\mathcal{L}_1$, the operator K will not be compact. By [G2 III.3.9] it will however be weakly compact, meaning that K will map bounded sequences onto sequences which have a weakly convergent subsequence, and since K is defined on $\mathcal{L}_1$, [G2 III.3.5] implies that K is strictly singular. It should now be possible to prove that if $\{K_n\}$ is the set of numerical-integral operators approximating K, then the set $\{K - K_n\}$ is collectively strictly singular but not collectively compact. Therefore a perturbation theory for collectively strictly singular operators might be applicable in developing an approximation theory for this type of integral equation. This would however require additional work and falls outside the scope of this thesis.

It should however be noticed that since the space $\mathcal{L}_1$ has the Dunford-Pettis property, meaning that any weakly compact mapping from $\mathcal{L}_1$ into some other Banach space is completely continuous, all the operators in question are both weakly compact and completely continuous. By complete continuity of a bounded operator we mean that it maps weakly convergent sequences onto norm convergent sequences. Therefore compositions of the type $\{(K - K_n)K\}$ are compact since the first operator maps bounded sequences onto sequences which have a weakly convergent subsequence and the second operator maps those weakly convergent subsequences onto norm convergent sequences. These compositions will possibly also be collectively compact. It is therefore possible that one could develop an approximation theory for integral equations with a measurable kernel where the integral operator maps from $\mathcal{L}_1$ into $\mathcal{L}_1$ without needing the concept of collective strict singularity.

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