THE TRACE OF NUCLEAR ELEMENTS IN BANACH ALGEBRAS

by

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0. PRELIMINARY CONVENTIONS AND NOTATION

0.1 Conventions

Throughout this paper functional and operator will assume to be linear and bounded. We also assume that the reader is familiar with the definition of the dual operator of an operator T.

0.2 Notation

Let E and F be normed spaces.

$(E)_1$	: the unit ball of E.
dim E	: the dimension of E.
$\overline{\text{span}(F)}$	: the closure of the linear span of F in E.
$\mathcal{L}(E,F)$	: the class of all operators $T:E \longrightarrow F$.
$\mathcal{L}(E)$	: the class of all operators $T:E \longrightarrow E$.
$\mathcal{F}(E,F)$	: the class of all finite rank operators in $\mathcal{L}(E,F)$.
$\mathcal{N}(E,F)$	: the class of all nuclear operators in $\mathcal{L}(E,F)$.
$\mathcal{C}(E,F)$	: the class of all compact operators in $\mathcal{L}(E,F)$.
E^*	: the dual space of E.
T'	: the dual operator of $T \in \mathcal{L}(E,F)$.
$e_{\mathcal{L}(E,F)}(T)$	: the set of eigenvalues of $T \in \mathcal{L}(E,F)$.
$\sigma_{\mathcal{L}(E,F)}(T)$	: the spectrum of $T \in \mathcal{L}(E,F)$.
ran(T)	: the rank of $T \in \mathcal{L}(E,F)$.
tr(T)	: the trace of an operator T.
det(A)	: the determinant of a nxn matrix A.
ℓ_∞	: the space of all bounded sequences.
c_0	: the space of all bounded sequences converging to zero.
ℓ_1	: the space of all absolute summable sequences.
x^*	: the adjoint of an element x in a *-algebra.
$\sigma_{\mathcal{M}}(u)$	: the spectrum of an element u in an algebra $\mathcal{M}$.
m.a.p.	: metric approximation property.
q.a.p.	: quasi-approximation-property.

1. INTRODUCTION

The classical Ascoli's theorem has interested many mathematicians and has been the object of many modifications and generalisations. In its fundamental form it gives necessary and sufficient conditions for a family of continuous functions, defined on a compact topological space, to be compact in the topology of uniform convergence. M Freundlich [5] introduced and studied the concept of completely continuous elements in a commutative normed algebra, defining them as those elements for which the corresponding regular representations are compact operators. K Vala [14] took this up again, giving a different definition of a compact element in a normed algebra. He called an element u of a normed algebra compact if the wedge operator $u \wedge u : x \longrightarrow uxu$ is compact. This definition generalises the notion of a compact operator, since in ([15], Theorem 3) it is shown that the set of all compact elements in the Banach algebra of operators on a Banach space E, coincides with the class of all compact operators. A special class of compact elements, called finite-dimensional, is formed by elements u for which the wedge operator $:x \longrightarrow uxu$ is of finite rank. J Puhl [10] again gives a different definition of finite elements in a Banach algebra. He defines an element u to be one-dimensional if there exists a non-zero functional f_u on the Banach algebra, such that $uxu = \langle f_u, x\rangle u$ for all x, and a finite element to be the sum of a finite number of one- dimensional elements. In this paper it is shown that these two different definitions, given by Vala and Puhl respectively, match.

The main purpose of the paper, however, is to introduce the trace of finite elements of a Banach algebra, which includes the notion of the trace of finite rank operators.

Since these generalisations depend heavily on the study of operators in the Banach algebra $\mathcal{L}(E)$. Chapter 2 contains a comprehensive account of standard material concerning operators on a Banach space E, which will be used in the chapters that follow.

As most of the results proved in the rest of the paper require that the Banach algebra $\mathcal{A}$ is semi-prime, and a useful condition which is equivalent for $\mathcal{A}$ to be semi-prime is proved in Theorem 3.

The third chapter is devoted to a study of one-dimensional elements in a Banach algebra $\mathcal{M}$ and in Theorem 3.22 it is shown how the two different definitions, given by Vala and Puhl respectively, match. In Lemma 3.9 we prove the existence of a unique complex number tr(u) such that $u^2 = tr(u)u$, which will be called the trace of u, provided that $\mathcal{M}$ is semi-prime. We also show that the spectrum of a one-dimensional element u consists only of two elements, namely 0 and tr(u). Another important result, concerning the correspondence between the one-dimensional elements in a semi-prime Banach algebra $\mathcal{M}$ and the minimal left (right) ideals of $\mathcal{M}$, is shown in Proposition 3.18 (Remark 3.19).

In the next chapter we define a finite element as used by Puhl, and in Theorem 4.23 it is shown how this coincides with the definition given by Vala. It is also shown in Proposition 4.21 that the spectrum of such an element is finite. We proceed in Chapter 5 to prove that a finite element has a well defined trace and some important results concerning this concept are given in Theorem 5.9.

In the last chapter the definition of a nuclear element is given in a natural way. The spectrum of a nuclear element is shown to be either finite or countable, which can accumulate only in the origin. Furthermore, it will be shown that if the Banach algebra fulfils certain conditions, then the trace admits an extension to the nuclear elements. One such condition is that the Banach algebra $\mathcal{M}$ be semi-prime, having the approximation property. Another condition is that $\mathcal{M}$ is semi-prime such that for each finite element u and $\epsilon > 0$ we can find another finite element x with $\|x\| \leq 1+\epsilon$ and $xu = u$ or $ux = u$. Yet, another condition is the following:
We say $\mathcal{M}$ has quasi-approximation-property (q.a.p.) if for each minimal idempotent $q \in \mathcal{M}$, the Banach space $\mathcal{M}q$(resp. $q\mathcal{M}$) has the approximation property. Commutative Banach algebras and C^*-algebras have q.a.p. as is shown in Theorem 6.14. We also prove in Theorem 6.11 that the trace of a nilpotent nuclear element is zero.

As far as the sources used in this work are concerned, detailed references are given throughout the chapters.

2. FINITE RANK- AND NUCLEAR OPERATORS, INCLUDING THE TRACE FUNCTIONAL

In this chapter, some basic results concerning finite rank - and nuclear operators on a normed space are given, which will be needed in the rest of the paper. The trace of a finite rank operator on a normed space E is introduced and it is shown that if E fulfils a certain condition, then the trace admits an extension to the nuclear operators.

Definition 2.1

Let E *and* F *be normed spaces.*
An operator $T \in \mathcal{L}(E,F)$ *is of finite rank if* $\dim[T(E)] < \infty$;
if $\dim[T(E)] = n$ *we say that* T *is of rank* n *and write* $\mathrm{ran}(T) = n$.

Remark 2.2

If $y \in F$ and $a \in E^*$ we denote the mapping $T:E \longrightarrow F$ defined by $Tx := \langle a,x\rangle y$ by the tensor product notation $T = a \otimes y$. Clearly $a \otimes y$ is linear and bounded and has rank at most one. Similarly for $a_i \in E^*$ and $y_i \in F$, $i=1,\ldots,n$ we denote by $\sum_{i=1}^{n} a_i \otimes y_i$ the operator $T:E \longrightarrow F$ defined by $Tx = \sum_{i=1}^{n} \langle a_i,x\rangle y_i$. In this case T has rank at most n.

Before proving that any $T \in \mathcal{L}(E,F)$ of finite rank can be represented in this way, we need a lemma.

Lemma 2.3 ([12], p 32)

Let $a_i \in E^*$, $i=1,\ldots,n$. *Then* $\{a_i : i=1,\ldots,n\}$ *is linearly independent if and only if there exists* $\{x_i : i=1,\ldots,n\}$ *in* E *with* $\langle a_i,x_j\rangle = \delta_{ij}$.

Proof

Let $\{a_i : i=1,\ldots,n\}$ be linearly independent.

If $n = 1$, nothing has to be proved. Suppose that the result is true for $n - 1$ and consider n linearly independent elements $a_1,\ldots,a_n$. By the induction hypothesis there are elements $x_1,\ldots,x_{n-1} \in E$ with $\langle a_i,x_j\rangle = \delta_{ij}$, $i,j=1,\ldots,n-1$. For each $x \in E$

$$(x - \sum_{i=1}^{n-1} \langle a_i, x\rangle x_i) \in \bigcap_{i=1}^{n-1} a_i^{-1}(0) := N$$

There exists $y_0 \in N$ with $\langle a_n, y_0\rangle \neq 0$, since if this were not true, then for all $x \in E$

$$\langle a_n, x\rangle - \sum_{i=1}^{n-1} \langle a_i, x\rangle\langle a_n, x_i\rangle = 0 \text{ , i.e.}$$

$$\langle a_n, x\rangle = \sum_{i=1}^{n-1} \langle a_i, x\rangle\langle a_n, x_i\rangle = \sum_{i=1}^{n-1} \langle\langle a_n, x_i\rangle a_i, x\rangle$$

wich implies that $a_n = \sum_{i=1}^{n-1} \langle a_n, x_i\rangle a_i$. This contradicts the fact that $\{a_1, \ldots, a_n\}$ is linearly independent.

Define $x_n := \frac{y_0}{\langle a_n, y_0\rangle}$. Then $\langle a_i, x_n\rangle = \delta_{in}$ for all $i=1,\ldots,n-1,n$.

Conversely, let $\sum_{i=1}^{n} \lambda_i a_i = 0$. Then for each $j=1,\ldots,n$ we have $\sum_{i=1}^{n} \lambda_i \langle a_i, x_j\rangle = 0$ which implies that $\lambda_j = 0$ for each $j=1,\ldots,n$. □

Proposition 2.4

If E *is a normed space, then the operator* $T \in \mathcal{L}(E,F)$ *is of rank* $n \geq 1$ *if and only if* T *has a representation* $Tx = \sum_{i=1}^{n} \langle a_i, x\rangle y_i$ *for all* $x \in E$, *where* $\{a_i : i=1,\ldots,n\}$ *and* $\{y_i : i=1,\ldots,n\}$ *are linearly independent subsets in* E^* *and* F *respectively.*

Proof

Let $\dim[T(E)] = n$. Then we can find a basis $\{y_1, \ldots, y_n\}$ such that for each $x \in E$, Tx can be written as $Tx = \sum_{i=1}^{n} \alpha_i^{(x)} y_i$, $\alpha_i^{(x)} \in \mathbb{C}$. For each $i=1,\ldots,n$ define $\langle a_i, x\rangle := \alpha_i^{(x)}$.

Clearly a_i is linear, since T is. We show that a_i is bounded.

For each $x \in E$, there exists $c > 0$ such that

$$\sum_{i=1}^{n} |\langle a_i, x\rangle| = \sum_{i=1}^{n} |\alpha_i^{(x)}| \leq c\|\sum_{i=1}^{n} \alpha_i^{(x)} y_i\| \leq c\|T\|\|x\|$$ (cf [6], Lemma 2.4-1) which implies that $|\langle a_i, x\rangle| \leq c\|T\|\|x\|$, i.e. $\|a_i\| \leq c\|T\|$ for

$i=1,\dots,n$. We still have to show that a_i is linearly independent. There exists $x_j \in E$ with $Tx_j = y_j$, $j=1,\dots,n$. Then

$$y_j = \sum_{i=1}^{n} \langle a_i,x_j\rangle y_i \quad \text{for each} \quad j=1,\dots,n.$$

Since $\{y_i:i=1,\dots,n\}$ is linearly independent, this implies that

$$\langle a_i,x_j\rangle = \delta_{ij}.$$

By Lemma 2.3 it follows that $\{a_1,\dots,a_n\}$ is linearly independent.

On the other hand, let $\{a_1,\dots,a_n\}$ and $\{y_1,\dots,y_n\}$ be linearly independent in E^* and F respectively with $Tx = \sum_{i=1}^{n} \langle a_i,x\rangle y_i$ for all $x \in E$. It is easy to see that T is linear and bounded. Clearly $\dim[T(E)] \leq n$.

By Lemma 2.3 there exists $\{x_1,\dots,x_n\}$ in E with $\langle a_i,x_j\rangle = \delta_{ij}$.

Then $Tx_j = \sum_{i=1}^{n} \langle a_i,x_j\rangle y_i = y_j$, i.e. $y_j \in T(E)$, $j=1,\dots,n$.

This implies that $\dim[T(E)] = n$. □

<u>Definition 2.5</u>

$\mathcal{F}_n(E,F) = \{T \in \mathcal{L}(E,F):T$ *is of rank* $n\}$ *and*

$\mathcal{F}(E,F) = \bigcup_{n=0}^{\infty} \mathcal{F}_n(E,F) = \{T \in \mathcal{L}(E,F):T$ *is of finite rank*$\}$.

<u>Definition 2.6</u>

If $T \in \mathcal{L}(E)$ *we define the wedge operator* $T\wedge T$ *on* $\mathcal{L}(E)$ *by* $T\wedge T(R) := TRT$ *for all* $R \in \mathcal{L}(E)$.

Next we want to prove an important theorem concerning finite rank operators, from which K Vala [14] generalises the concept of finite elements in a Banach algebra.

<u>Lemma 2.7</u>

Let $\{a_1,\dots,a_m\}$, $\{y_1,\dots,y_n\}$ *be linearly independent subsets of the normed spaces* E^* *and* F *respectively. Then* $\{a_i \otimes y_j:i=1,\dots,m,\ j=1,\dots n\}$ *is linearly independent in* $\mathcal{L}(E,F)$.

Proof

We first show that if $\sum_{i=1}^{n} a_i \otimes y_i = 0$ with $\{y_1,\ldots,y_n\}$ linearly independent, then $a_i = 0$, $i=1,\ldots,n$.

If $\sum_{i=1}^{n} a_i \otimes y_i = 0$ then $\sum_{i=1}^{n} \langle a_i,x\rangle y_i = 0$ for all $x \in E$ from which it immediately follows that $a_i = 0$, $i=1,\ldots,n$ since $\{y_1,\ldots,y_n\}$ is linearly independent. Now let $\sum_{j=1}^{n} (\sum_{i=1}^{m} \lambda_{ij}a_i)\otimes y_j = 0$.

By applying the discussion above we obtain $\sum_{i=1}^{m} \lambda_{ij}a_i = 0$, $j=1,\ldots,n$, and since $\{a_1,\ldots,a_m\}$ is linearly independent it follows that

$$\lambda_{ij} = 0, \quad i=1,\ldots,m, \ j=1,\ldots,n. \qquad \square$$

Corollary 2.8

If $0 \neq a \in E^*$ *and* $\{y_1,\ldots,y_n\}$ *is linearly independent in* F, *then* $\{a \otimes y_i : i=1,\ldots,n\}$ *is linearly independent in* $\mathcal{L}(E,F)$.

Theorem 2.9 ([1], p 17)

(a) *The operator* $T \in \mathcal{L}(E)$ *is of rank one (zero) if and only if the wedge operator is of rank one (zero).*

(b) T *is of finite rank if and only if* $T \wedge T$ *is of finite rank. In this case* $\operatorname{ran}(T) \leq \operatorname{ran}(T \wedge T)$.

Proof

(a) Clearly, if $T = 0$, $T \wedge T = 0$.

Now let $T \neq 0$ be of rank one. Then by Proposition 2.4 $T = a \otimes y$, $0 \neq a \in E^*$ and $0 \neq y \in E$. Consider any $R \in \mathcal{L}(E)$ and $x \in E$. Then

$$\begin{aligned} T \wedge T(R)x &:= (a \otimes y)R(a \otimes y)x \\ &= (a \otimes y)\langle a,x\rangle Ry = \langle a,x\rangle\langle a,Ry\rangle y \\ &= \langle a,Ry\rangle(a \otimes y)x = \langle a,Ry\rangle Tx \\ &= (r_T \otimes T)(R)x \end{aligned}$$

where we define the functional $r_T : \mathcal{L}(E) \longrightarrow \mathbb{C}$ by $\langle r_T,R\rangle := \langle a,Ry\rangle$.

We still have to show that $r_T \neq 0$.

There exists $x_o \in E$ such that $\langle a,x_o\rangle \neq 0$.

Define $R_o:\text{span}\{y\} \longrightarrow E$ by $R_o(\alpha y) := \alpha x_0$. Clearly R_o is linear and bounded and by Hahn Banach it can be extended to E such that $R_o y = x_0$. Therefore $\langle \Gamma_T, R_o \rangle \neq 0$.

On the other hand, if $T\Delta T = 0$ and $T \neq 0$ we can find $x_0 \in E$ with $Tx_0 \neq 0$. Let $y = Tx_0$. Again by Hahn Banach there exists $R_o \in \mathcal{L}(E)$ with $R_o y = x_0$. Then $T\Delta T(R_o)x_0 := TR_oTx_0 = TR_oy = Tx_0 \neq 0$ which contradicts the fact that $T\Delta T = 0$. If $T\Delta T$ is of rank one, we can find $R_o \in \mathcal{L}(E)$ such that $T\Delta T(R_o) = TR_oT \neq 0$, hence $T \neq 0$ and it follows that $\dim[T(E)] \geq 1$.
If $\dim[T(E)] > 1$, we can find at least two linearly independent vectors $\{Tx_1, Tx_2\}$ in $T(E)$. Choose $a \in E^*$ with $T'a \neq 0$ and let

$$R_1 := a \otimes x_1 \ ; \ R_2 := a \otimes x_2 .$$

Clearly R_1 and R_2 are linear and bounded and $T\Delta T(R_i) = T'a \otimes Tx_i$, $i=1,2$.
By Corollary 2.8 $\{T\Delta T(R_1), T\Delta T(R_2)\}$ is linearly independent in contradiction to the fact that $\dim[T\Delta T(\mathcal{L}(E))] = 1$.

(b) Let $T \neq 0$ be of finite rank $n \geq 2$. By Proposition 2.4 we can write $T = \sum_{i=1}^{n} a_i \otimes y_i$ with $\{a_i : i=1,\ldots,n\}$ and $\{y_i : i=1,\ldots,n\}$ linearly independent in E^* and E respectively. Then for all $R \in \mathcal{L}(E)$ and $x \in E$ it follows that

$$T\Delta T(R)x = \left(\sum_{i=1}^{n} a_i \otimes y_i\right)R\left(\sum_{j=1}^{n} a_j \otimes y_j\right)x = \left(\sum_{i=1}^{n} a_i \otimes y_i\right) \sum_{j=1}^{n} \langle a_j, x\rangle Ry_j$$

$$= \sum_{i,j=1}^{n} \langle a_i, Ry_j\rangle (a_j \otimes y_i)x \text{ , i.e.}$$

$$(T\Delta T)(R) = \sum_{i,j=1}^{n} \langle a_i, Ry_j\rangle (a_j \otimes y_i).$$

Therefore, $\{(T\Delta T)R : R \in \mathcal{L}(E)\} \subseteq \text{span}\{a_j \otimes y_i : i,j=1,\ldots,n\}$ which implies that $\dim[(T\Delta T)(\mathcal{L}(E))] \leq n^2$.
Furthermore, there exists $\{x_1,\ldots,x_n\}$ such that $\langle a_i, x_j\rangle = \delta_{ij}$ by Lemma 2.3. By choosing R_k any operator such that $R_k y_j = \delta_{kj} x_k$, it follows that

$$(T\Delta T)(R_k) = \sum_{i,j=1}^{n} \langle a_i, R_k y_j\rangle (a_j \otimes y_i) = \sum_{i=1}^{n} \langle a_i, x_k\rangle (a_k \otimes y_i)$$
$$= (a_k \otimes y_k).$$

Hence $a_k \otimes y_k \in T\Delta T[\mathcal{L}(E)]$ for $k=1,\dots,n$, which implies that $ran(T\Delta T) \geq n$.

Therefore $n = ran(T) \leq ran(T\Delta T) \leq n^2$.

Conversely, if $2 \leq ran(T\Delta T) \leq n$, then clearly $T \neq 0$.
If $\dim[T(E)] > n$, then similarly as in the case for $n = 1$ we get a contradiction to the fact that $T\Delta T[\mathcal{L}(E)]$ has at most n linearly independent vectors. Consequently $ran(T) \leq n$. □

From the proof of this theorem we immediately derive a corollary on which J Puhl [10] based his definition of finite elements in a Banach algebra.

<u>Corollary 2.10</u> ([10], p 658)

If E *is a normed space and* $0 \neq T \in \mathcal{L}(E)$, *then* $T \in \mathcal{F}_1(E)$ *if and only if there exists a non-zero functional* $r_T : \mathcal{L}(E) \longrightarrow \mathbb{C}$ *such that* $TRT = \langle r_T, R\rangle T$ *for all* $R \in \mathcal{L}(E)$.

<u>Definition 2.11</u>

An operator $T \in \mathcal{L}(E)$ *is minimal idempotent if* $T \neq 0$ *is idempotent such that* $T[\mathcal{L}(E)]T$ *is a division algebra.*

<u>Lemma 2.12</u>

If $Q \in \mathcal{L}(E)$ *is minimal idempotent, then* Q *is a projection of rank one.*

<u>Proof</u> The result directly follows by Gelfand Mazur and Theorem 2.9. □

Before proving that the spectrum of a finite rank operator T is finite, we show that the spectrum of T equals the set of eigenvalues of T.

<u>Lemma 2.13</u> ([7], p 96)

If $T \in \mathcal{F}(E)$, *then* $(T-I)$ *is one to one iff it is onto.*

<u>Proof</u>

Let $B := (T-I)|_{T(E)}$. Clearly $B \in \mathcal{L}[T(E)]$ and since $T(E)$ is finite dimensional, B is one to one iff it is onto. Clearly $(T-I)$ is one to one iff B is one to one. Next we show that $(T-I)$ is onto iff B is onto.

Let $(T-I)$ be onto and consider any $v \in T(E)$. We can find $y \in E$ with $v = Ty$. Then there exists $x \in E$ such that $(T-I)x = y$. If we define $u := Tx$, then $Bu = v$.

On the other hand, let B be onto and $y \in E$ arbitrary. Then $Ty \in T(E)$ and consequently there exists $u \in T(E)$ such that $Bu = Ty$. Let $x := u-y$. Then $(T-I)x = (T-I)u - (T-I)y = y$. □

<u>Proposition 2.14</u> ([7], p 96)

If $T \in \mathcal{F}(E)$, *then* $\sigma(T) = e(T)$.

<u>Proof</u>

If $0 \neq \lambda \in \mathbb{C}$, then $\frac{T}{\lambda} \in \mathcal{F}(E)$ and by Lemma 2.13 it follows that $(\frac{T}{\lambda} - I)$ is one to one iff it is onto. Consequently $\lambda \in \sigma(T)$ iff $\lambda \in e(T)$.

Next let us consider the case $\lambda = 0$. If E is finite dimensional, then T is one to one iff it is onto, hence $0 \in \sigma(T)$ iff $0 \in e(T)$. If E is infinite dimensional, then T cannot be one to one since $T(E)$ is finite dimensional. Therefore $0 \in e(T) \subseteq \sigma(T)$. Consequently in all cases we have $\sigma(T) = e(T)$. □

<u>Lemma 2.15</u> ([2], p 80)

If $\lambda_1,\dots,\lambda_n$ *are different eigenvalues of* T *and* $0 \neq x_i \in E$ *is the corresponding eigenvector of each* λ_i, *then* $\{x_1,\dots,x_n\}$ *is linearly independent.*

<u>Proof</u>

Clearly, the lemma holds for $n = 1$.

Suppose it is true for $(n-1)$ and let $\lambda_1,\dots,\lambda_n$ be different eigenvalues of T. Then for each $i=1,\dots,n$ there exists $0 \neq x_i \in E$ with $Tx_i = \lambda_i x_i$. If $\{x_1,\dots,x_n\}$ is linearly dependent, we can find scalars $\beta_1,\dots,\beta_{n-1}$ with $x_n = \sum_{i=1}^{n-1} \beta_i x_i$. Therefore $\lambda_n x_n = Tx_n = \sum_{i=1}^{n-1} \beta_i \lambda_i x_i$.

Hence $x_n = \sum_{i=1}^{n-1} \frac{\beta_i \lambda_i x_i}{\lambda_n}$. From the linear independence of $\{x_1,\dots,x_{n-1}\}$ it follows that $\beta_i = \frac{\beta_i \lambda_i}{\lambda_n}$ for all i. Since $x_n \neq 0$, not all $\beta_i = 0$.

Assume $\beta_j \neq 0$. Then $\lambda_j = \lambda_n$ contradicting our assumption. □

Proposition 2.16

The spectrum of a finite rank operator is finite.

Proof

Let E, F be Banach spaces and consider any $T \in \mathcal{F}(E,F)$.
Assume that $\sigma(T)$ is not finite, i.e. there exists at least a countable number of different spectrum values $\{\lambda_i : i \in \mathbb{N}\}$. By Proposition 2.14, for each $i = 1,2,\ldots$ there exists $0 \neq x_i \in E$ with $Tx_i = \lambda_i x_i \in T(E)$ which implies $x_i \in T(E)$. Since $\{x_i : i=1,2,\ldots\}$ is linearly independent by Lemma 2.15, this contradicts the fact that T is of finite rank. □

Lemma 2.17

If $T \in \mathcal{F}(E,F)$, *then* $\sum_{i=1}^{n} \langle a_i, y_i\rangle$ *is independent on the specific representation of* $T = \sum_{i=1}^{n} a_i \otimes y_i$.

Proof

Let $T = \sum_{i=1}^{n} a_i \otimes y_i$ be any representation where $a_i \in E^*$, $y_i \in F$. We can find linearly independent vectors $\{x_1,\ldots,x_m\} \subseteq F$ such that $y_i \in \overline{\operatorname{span}\{x_1,\ldots,x_m\}}$, $(i=1,\ldots,n)$. Let $y_i = \sum_{j=1}^{m} \xi_{ij} x_j$.

Then there exists a representation $T = \sum_{j=1}^{m} b_j \otimes x_j$ with $b_j \in E^*$.

Therefore,

$$\sum_{j=1}^{m} b_j \otimes x_j = \sum_{i=1}^{n} a_i \otimes y_i = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \xi_{ij} a_i \right] \otimes x_j$$

which implies that the functionals $b_1,\ldots,b_n \in E^*$ are uniquely determined by $b_j = \sum_{i=1}^{n} \xi_{ij} a_i$, since $\{x_j : j=1,\ldots,m\}$ is linearly independent.

We conclude that

$$\sum_{i=1}^{n} \langle a_i, y_i\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_{ij} \langle a_i, x_j\rangle = \sum_{j=1}^{m} \langle \sum_{i=1}^{n} \xi_{ij} a_i, x_j\rangle = \sum_{j=1}^{m} \langle b_j, x_j\rangle . \quad \square$$

Definition 2.18

If $T \in \mathcal{F}(E,F)$, *the trace of* T *is defined by*

$$tr(T) := \sum_{i=1}^{n} \langle a_i, y_i \rangle$$

where $T = \sum_{i=1}^{n} a_i \otimes y_i$, $a_i \in E^*$ *and* $y_i \in F$ $(i=1,\ldots,n)$.

Lemma 2.19

If $R \in \mathcal{L}(E,F)$, $T \in \mathcal{F}(F,G)$ *and* $S \in \mathcal{L}(G,H)$, *then* $STR \in \mathcal{F}(E,H)$.

Proof

Let $T = \sum_{i=1}^{n} a_i \otimes y_i$, $a_i \in F^*$, $y_i \in G$. Then for all $x \in E$

$$STRx = S\left(\sum_{i=1}^{n} a_i \otimes y_i\right)Rx = \sum_{i=1}^{n} \langle a_i, Rx \rangle Sy_i$$

$$= \sum_{i=1}^{n} \langle R'a_i, x \rangle Sy_i$$

$$= \left(\sum_{i=1}^{n} R'a_i \otimes Sy_i\right)x \text{ with } R'a_i \in E^*,\ Sy_i \in H,\ i=1,\ldots,n.$$

Therefore $STR \in \mathcal{F}(E,H)$. □

Proposition 2.20

If $0 \neq T \in \mathcal{F}_1(E,F)$, *then* $T^2 = tr(T)T$.

Proof

Let $T = a \otimes y$, $a \in E^*$, $y \in F$ and consider any $x \in E$. Then

$$tr(T)Tx = \langle a,y \rangle (a \otimes y)x = \langle a,y \rangle \langle a,x \rangle y = \langle a,x \rangle (a \otimes y)y$$
$$= (a \otimes y)(a \otimes y)x$$
$$= T^2 x$$ □

Lemma 2.21

All eigenvalues of a nilpotent operator are zero.

Proof

Let $T \in \mathcal{L}(E,F)$ be nilpotent.

Suppose that $\lambda_0 \in \mathbb{C}$ is a non-zero eigenvalue of T with associated eigenvector $x_0 \neq 0$, i.e. $Tx_0 = \lambda_0 x_0$. Then by induction and substitution we get the contradiction

$$T^n x_0 = \lambda_0^n x_0 \neq 0 \quad \text{for all} \quad n \in \mathbb{N}. \qquad \square$$

Lemma 2.22 ([8], p 264)

If $A = (a_{ij})$ *is a* $n \times n$ *matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$, *then*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i \quad \textit{and} \quad \det(A) = \prod_{i=1}^{n} \lambda_i.$$

Proof

If λ is an eigenvalue of A, there exists $x \neq 0$ such that $(A-\lambda I_n)x = 0$ which implies that $\det(A-\lambda I_n) = 0$. Put $P(\lambda) := \det(A-\lambda I_n)$. Then by expanding the determinant of $(A-\lambda I_n)$ in terms of elements in the first row, it follows that $P(\lambda) = (a_{11} - \lambda)B_{11} + \sum_{j=2}^{n} a_{1j}B_{1j}$ where B_{1j} is the cofactor of the $(1,j)$ element of the matrix $(A-\lambda I_n)$. In each of the cofactors $B_{12}, \ldots, B_{1n}$ there are only $n-2$ elements $(a_{ii} - \lambda)$ involving λ so that the largest power of λ that can be obtained by expansion of these cofactors is λ^{n-2}. Consequently

$$P(\lambda) = (a_{11}-\lambda)B_{11} + \{\text{terms in } \lambda \text{ of degree } (n-2) \text{ or less}\}$$

By expanding B_{11} again in terms of elements in the first row, the same argument as above can be applied and by repetition it follows that

$P(\lambda)$

$= (a_{11}-\lambda)(a_{22}-\lambda)\ldots(a_{nn}-\lambda) + \{\text{terms in } \lambda \text{ of degree } (n-2) \text{ or less}\}$

$= (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}(a_{11}+a_{22}+\ldots+a_{nn}) + \{\text{terms in } \lambda \text{ of degree } (n-2) \text{ or less}\}$

Since the constant term in $P(\lambda)$ would involve no λ, it is given by

$$P(0) := \det(A).$$

Therefore

$$P(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}\operatorname{tr}(A) + \ldots + \det(A).$$

But as the characteristic polynomial $P(\lambda) = 0$ is of degree n, it has exactly n roots given by the eigenvalues of A. This implies that

$$P(\lambda) = (\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda)$$
$$= (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}(\lambda_1+\lambda_2 + \ldots + \lambda_n) + \ldots + (\lambda_1\lambda_2 \ldots \lambda_n)$$

The result follows by comparing these. $\square$

Corollary 2.23

The trace of a finite rank nilpotent operator is zero.

Next we turn to the concept of nuclear operators, and introduce a nuclear norm which dominates the norm $\|\cdot\|$ on $\mathcal{L}(E,F)$.

Definition 2.24

An operator $T \in \mathcal{L}(E,F)$ *is called nuclear if* $Tx = \sum_{i=1}^{\infty} \langle a_i, x\rangle y_i$ *with* $a_i \in E^*$ *and* $y_i \in F$ *for all* $i \in \mathbb{N}$ *such that* $\sum_{i=1}^{\infty} \|a_i\|\|y_i\| < \infty$.

We write $T = \sum_{i=1}^{\infty} a_i \otimes y_i$.

Definition 2.25

$N(E,F) = \{T \in \mathcal{L}(E,F): T$ *is nuclear*$\}$. *Put* $\nu(T) := \inf \sum_{i=1}^{\infty} \|a_i\|\|y_i\|$ *where the infimum is taken over all so-called nuclear representations described above.*

Theorem 2.26

$(N(E,F),\nu)$ *is a Banach space and if* $R \in \mathcal{L}(D,E)$, $S \in \mathcal{L}(F,G)$ and $T \in N(E,F)$, *then* $STR \in N(D,G)$ *such that* $\nu(STR) \leq \|S\|\nu(T)\|R\|$.

Proof

We first show that ν is a norm on $N(E,F)$.

Let $T = \sum_{i=1}^{\infty} a_i \otimes y_i$ be any nuclear representation. If $\nu(T) = 0$, then $\|T\| \leq \sum_{i=1}^{\infty} \|a_i\|\|y_i\|$, since $\|\cdot\|$ is continuous on $\mathcal{L}(E,F)$ and consequently $\|T\| \leq \nu(T) = 0$ which implies $T = 0$.

Clearly, the converse also holds.

It is obvious that $\lambda T \in N(E,F)$ for all $\lambda \in \mathbb{C}$ and $\nu(\lambda T) = |\lambda|\nu(T)$.

Finally, if $U = \sum_{j=1}^{\infty} b_j \otimes x_j \in N(E,F)$ we have

$$\nu(T+U) \leq \sum_{i=1}^{\infty} \|a_i\|\|y_i\| + \sum_{j=1}^{\infty} \|b_j\|\|x_j\| \quad \text{for all such representations}$$

of T and U. Fix the representation $U = \sum_{j=1}^{\infty} b_j \otimes x_j$. It follows that

$\nu(T+U) \leq \nu(T) + \sum_{j=1}^{\infty} \|b_j\|\|x_j\|$. Since this is true for each such representation of U, we have

$$\nu(T+U) \leq \nu(T) + \nu(U).$$

Now we show that $\mathcal{N}(E,F)$ is complete with respect to this norm.

Let $(T_i) \subseteq \mathcal{N}(E,F)$ be any ν-Cauchy sequence. Then (T_i) is also a $\|\cdot\|$-Cauchy sequence, since $\|\cdot\| \leq \nu(\cdot)$. Therefore, there exists $T \in \mathcal{L}(E,F)$ such that $\|T-T_n\| \xrightarrow[\infty]{n} 0$.

We prove that $T \in \mathcal{N}(E,F)$ and $\nu(T - T_n) \xrightarrow[\infty]{n} 0$.

We can find an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that

$$\nu(T_n - T_m) < \frac{1}{2^{k+2}} \text{ for all } n,m \geq n_k.$$

For each $k \in \mathbb{N}$, we can choose a nuclear representation

$$T_{n_{k+1}} - T_{n_k} = \sum_{i=1}^{\infty} a_i^{(k)} \otimes y_i^{(k)} \text{ with } \sum_{i=1}^{\infty} \|a_i^{(k)}\|\|y_i^{(k)}\| \leq \frac{1}{2^{k+2}}.$$

Then for all $p \in \mathbb{N}$

$$T_{n_{k+p}} - T_{n_k} = \left[T_{n_{k+p}} - T_{n_{k+p-1}}\right] + \left[T_{n_{k+p-1}} - T_{n_{k+p-2}}\right] + \ldots + \left[T_{n_{k+1}} - T_{n_k}\right]$$

$$= \left[\sum_{i=1}^{\infty} a_i^{(k+p-1)} \otimes y_i^{(k+p-1)}\right]$$

$$+ \left[\sum_{i=1}^{\infty} a_i^{(k+p-2)} \otimes y_i^{(k+p-2)}\right] + \ldots + \left[\sum_{i=1}^{\infty} a_i^{(k)} \otimes y_i^{(k)}\right]$$

$$= \sum_{j=k}^{k+p-1} \left[\sum_{i=1}^{\infty} a_i^{(j)} \otimes y_i^{(j)}\right].$$

Now, let $p \to \infty$ and consider the $\|\cdot\|$-limit. Then

$$\|\cdot\|\lim_{p\to\infty} (T_{n_{k+p}} - T_{n_k}) = \|\cdot\|\lim_{p\to\infty} \sum_{j=k}^{k+p-1} \left[\sum_{i=1}^{\infty} a_i^{(j)} \otimes y_i^{(j)}\right]$$

which implies $(T - T_{n_k}) = \sum_{j=k}^{\infty} \left[\sum_{i=1}^{\infty} a_i^{(j)} \otimes y_i^{(j)}\right]$.

Furthermore,

$$\sum_{j=k}^{\infty} \sum_{i=1}^{\infty} \|a_i^{(j)}\|\|y_i^{(j)}\| \leq \sum_{j=k}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^k}\left[\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right]$$

$$\leq \frac{1}{2^{k+1}}$$

Hence, $(T-T_{n_k}) \in \mathcal{N}(E,F)$ from which it follows that

$$T = (T-T_{n_k}) + T_{n_k} \in \mathcal{N}(E,F).$$

Furthermore, $\nu(T-T_{n_k}) \leq \sum_{j=k}^{\infty} \sum_{i=1}^{\infty} \|a_i^{(j)}\|\|y_i^{(j)}\| \leq \frac{1}{2^{k+1}} \xrightarrow[\infty]{k} 0.$

Consequently $\nu(T-T_n) \xrightarrow[\infty]{n} 0.$

Now, let $R \in \mathcal{L}(D,E)$, $S \in \mathcal{L}(F,G)$ and $T = \sum_{i=1}^{\infty} a_i \otimes y_i$, $a_i \in E^*$, $y_i \in F$ with $\sum_{i=1}^{\infty} \|a_i\|\|y_i\| < \infty$. Then for each $x \in D$

$$STRx = S\left[\sum_{i=1}^{\infty} a_i \otimes y_i\right]Rx = \sum_{i=1}^{\infty} \langle a_i, Rx\rangle Sy_i$$

$$= \sum_{i=1}^{\infty} \langle R'a_i, x\rangle Sy_i$$

$$= \left[\sum_{i=1}^{\infty} R'a_i \otimes Sy_i\right]x$$

with $R'a_i \in D^*$, $Sy_i \in G$ for each $i \in \mathbb{N}$.

Furthermore,

$$\sum_{i=1}^{\infty} \|R'a_i\|\|Sy_i\| \leq \|S\|\left[\sum_{i=1}^{\infty} \|a_i\|\|y_i\|\right]\|R'\| < \infty$$

which implies that $STR \in \mathcal{N}(D,G)$ and

$$\nu(STR) \leq \sum_{i=1}^{\infty} \|R'a_i\|\|Sy_i\| \leq \|S\|\nu(T)\|R\|. \qquad \square$$

<u>Proposition 2.27</u>

$\mathcal{F}(E,F)$ is dense in $\mathcal{N}(E,F)$ with respect to the nuclear norm.

<u>Proof</u>

Let $U \in \mathcal{N}(E,F)$ be any nuclear operator with arbitrary nuclear representation $U = \sum_{i=1}^{\infty} a_i \otimes y_i$. For each $n \in \mathbb{N}$, put

$$T_n := \sum_{i=1}^{n} a_i \otimes y_i$$

Since $\sum_{i=1}^{\infty} \|a_i\|\|y_i\| < \infty$, it follows that $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} \|a_i\|\|y_i\| = 0$ for each nuclear representation of U.

Therefore,

$$\nu(U-T_n) \leq \sum_{i=n+1}^{\infty} \|a_i\|\|y_i\| \xrightarrow[\infty]{n} 0, \text{ i.e.}$$

$$\nu(\cdot)\lim_{n\to\infty} T_n = U. \qquad \square$$

Before proving that the trace can be extended to the nuclear operators on a Banach space E, we need to introduce a property which E should have.

Definition 2.28

Let E and F be Banach spaces and $A \subseteq E$ a subset of E.

If (T_n) is a sequence in $\mathcal{L}(E,F)$, then $T_n \to T$ uniformly on A if for each $\epsilon > 0$, we can find $n_\epsilon \in \mathbb{N}$ such that $\|T_nx - Tx\| < \epsilon$ for all $n \geq n_\epsilon$ and for all $x \in A$. This is equivalent to saying that

$$\lim_{n\to\infty} \sup_{x\in A} \|T_nx - Tx\| = 0.$$

Example 2.29

$T_n \longrightarrow T$ uniformly on $(E)_1$ if and only if $\|T - T_n\| \xrightarrow[\infty]{n} 0$, i.e. iff $T_n \longrightarrow T$ in the norm topology of $\mathcal{L}(E,F)$.

Proposition 2.30

Let E be a Banach space. Then the following are equivalent:

(a) *For any Banach space F, every compact $K \subseteq E$ and every $T \in \mathcal{L}(E,F)$ there exists a sequence $(T_n) \subseteq \mathcal{F}(E,F)$ such that $T_n \longrightarrow T$ uniformly on K.*

(b) *For every compact $K \subseteq E$ and every $T \in \mathcal{L}(E)$ there exists a sequence $(T_n) \subseteq \mathcal{F}(E)$ such that $T_n \longrightarrow T$ uniformly on K.*

(c) *For every compact $K \subseteq E$ there exists a sequence $(L_n) \subseteq \mathcal{F}(E)$ such that $L_n \longrightarrow I$ uniformly on K.*

(d) *For every compact $K \subseteq E$ and $\epsilon > 0$ there exists $L \in \mathcal{F}(E)$ such that*

$$\|x - Lx\| \leq \epsilon \quad \text{for all } x \in K.$$

Proof

Clearly (a) => (b) => (c) and (c) <=> (d).

(c) => (a)

Consider any Banach space F. Let $T \in \mathcal{L}(E,F)$ and $K \subseteq E$ compact be given. We can find a sequence $(L_n) \subseteq \mathcal{F}(E)$ such that $\sup_{x\in K}\|L_n x - x\| \longrightarrow 0$.

Put

$$T_n := TL_n \quad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 2.19 $T_n \in \mathcal{F}(E,F)$ for all $n \in \mathbb{N}$ and

$$\sup_{x\in K} \|T_n x - Tx\| = \sup_{x\in K} \|T(L_n x - x)\| \leq \|T\| \sup_{x\in K} \|L_n x - x\| \xrightarrow[\infty]{n} 0.$$

Hence,

$$T_n \longrightarrow T \text{ uniformly on } K. \qquad \square$$

<u>Definition 2.31</u>

A Banach space E *possesses the approximation property if it satisfies any one of the conditions in Theorem* 2.30.

<u>Definition 2.32</u>

Let E *and* F *be any normed spaces.*

An operator $T \in \mathcal{L}(E,F)$ *is compact if* $\overline{T[(E)_1]}$ *is a compact subset of* F.

We write $T \in \mathcal{C}(E,F)$.

<u>Proposition 2.33</u>

Let E *and* F *be any Banach spaces. If* F *has the approximation property, then* $\overline{\mathcal{F}(E,F)} = \mathcal{C}(E,F)$.

<u>Proof</u>

Let $T \in \mathcal{C}(E,F)$ and put $K := \overline{T[(E)_1]}$. Since K is compact in F, for each $n \in \mathbb{N}$ we can find $L_n \in \mathcal{F}(F)$ such that

$$\|y - L_n y\| \leq \frac{1}{n} \quad \text{for all} \quad y \in K.$$

Put $S_n := L_n T$ for each $n \in \mathbb{N}$. Clearly $S_n \in \mathcal{F}(E,F)$ and for all $x \in (E)_1$ we have $\|Tx - S_n x\| = \|Tx - L_n(Tx)\| \leq \frac{1}{n}$.

Consequently, $\|T - S_n\| \leq \frac{1}{n}$ which implies that

$$\lim_{n\to\infty} S_n = T, \text{ i.e. } T \in \overline{\mathcal{F}(E,F)}.$$

Hence, $\mathcal{C}(E,F) \subseteq \overline{\mathcal{F}(E,F)}$.

The converse surely holds, since $\mathcal{C}(E,F)$ is closed (cf [6], Theorem 8.1-5).

$\square$

Corollary 2.34

If E *has the approximation property, then* $\overline{\mathcal{F}(E)} = \mathcal{C}(E)$.

Remark 2.35

It can be shown that conversely if $\overline{\mathcal{F}(E)} = \mathcal{C}(E)$, then E has the approximation property.

Proposition 2.36

Every Hilbert space possesses the approximation property.

Proof

Let $H \neq \{0\}$ be a Hilbert space and choose an ortonormal basis $\{e_\alpha : \alpha \in \mathcal{A}\}$. Put $F := \{\mathcal{B} \subseteq \mathcal{A} : \mathcal{B}$ is finite$\}$ and for each $\mathcal{B} \in F$, define

$$S_{\mathcal{B}}x := \sum_{\alpha \in \mathcal{B}} (x, e_\alpha)e_\alpha.$$

Clearly $S_{\mathcal{B}} \in \mathcal{F}(H)$, and by Parseval's equality we obtain

$$\|S_{\mathcal{B}}x\|^2 = \sum_{\alpha \in \mathcal{B}} |(x, e_\alpha)|^2 = \|x\|^2$$

from which it follows that $\|S_{\mathcal{B}}\| = 1$.

Now, consider any compact $K \subseteq H$ and let $\epsilon > 0$ be given. We can find a finite set $\{z_1, \ldots, z_m\}$ such that

$$K \subseteq \bigcup_{i=1}^{m} [z_i + \tfrac{\epsilon}{4}(H)_1].$$

For each $j=1,\ldots,m$ there exists $\mathcal{B}_j \in F$ with

$$\|z_j - S_{\mathcal{B}}z_j\| \leq \frac{\epsilon}{2} \quad \text{for all} \quad \mathcal{B} \supseteq \mathcal{B}_j.$$

Put $\mathcal{B}_0 = \bigcup_{j=1}^{m} \mathcal{B}_j$ and choose $L := S_{\mathcal{B}_0}$.

Then for all $x \in K$, we can find z_j such that $\|x - z_j\| < \frac{\epsilon}{4}$ and therefore

$$\begin{aligned} \|x - Lx\| &\leq \|x - z_j\| + \|z_j - Lz_j\| + \|Lz_j - Lx\| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \|L\|\frac{\epsilon}{4} \\ &= \epsilon \quad \text{since } \mathcal{B}_0 \in F. \qquad \square \end{aligned}$$

Definition 2.37

A Banach space E *is said to have the metric approximation property (m.a.p.) if it satisfies condition* (d) *of Proposition 2.30 and if in addition,* $L \in \mathcal{F}(E)$ *can always be found with* $\|L\| \leq 1$.

Lemma 2.38 ([9], p 29)

Let E *be an n-dimensional Banach space. Then there exist* $x_1,\ldots,x_n \in E$ *and* $a_1,\ldots,a_n \in E^*$ *with* $\|a_i\| = 1$, $\|x_j\| = 1$ *such that* $\langle a_i,x_j\rangle = \delta_{ij}$.

Proof

Let $\{z_1,\ldots,z_n\}$ be a fixed basis of E and $K := (E^*)_1$. Define

$$f : K^n \longrightarrow \mathbb{R} : (b_1,\ldots,b_n) \longrightarrow |\det(\langle b_i,z_j\rangle)|$$

Then $f = g\circ\varphi\circ T$ where $T : K^n \longrightarrow \mathbb{C}^{n^2} : (b_1,\ldots,b_n) \longmapsto (\langle b_i,z_j\rangle)$

$$\varphi : \mathbb{C}^{n^2} \longrightarrow \mathbb{C} : (a_{ij}) \longmapsto \det(a_{ij})$$

$$g : \mathbb{C} \longrightarrow \mathbb{R} : z \longmapsto |z|.$$

Clearly g is continuous. Since T is linear and K^n finite dimensional, T is also bounded. By

$\det(a_{ij}) = \sum_\pi (-1)^{\text{sign}(\pi)} a_{1\pi(1)}\cdots a_{n\pi(n)}$ where the sum is taken over all n! permutations of $1,\ldots,n$, it follows directly that φ is continuous, since it is the decomposition of multiplication and summation.

Consequently f is continuous and since K is compact, there exists $(a_1,\ldots,a_n) \in K^n$ for which f attains its maximum, i.e.

$$f(a_1,\ldots,a_n) \geq f(b_1,\ldots,b_n) \quad \text{for all} \quad (b_1,\ldots,b_n) \in K^n.$$

Since $\{z_1,\ldots,z_n\}$ is linearly independent it follows by Hahn Banach (cf [6], Lemma 4.6-7) that we can find $(b_1',\ldots,b_n') \in K^n$ such that

$$0 < f(b_1',\ldots,b_n') \leq f(a_1,\ldots,a_n).$$

By Cramer's rule there exists a unique $(x_1,\ldots,x_n) \in E^n$ such that

$$\sum_{i=1}^n \langle a_i,z_j\rangle x_i := z_j \ , \quad j=1,\ldots,n.$$

By the uniqueness, we notice that

$$\langle a_k,x_i\rangle = \delta_{ik}.$$

It follows from

$$\sum_{i=1}^n \langle a_i,z_j\rangle\langle b_k,x_i\rangle = \langle b_k,z_j\rangle$$

that $\quad |\det(\langle a_i,z_j\rangle)|\,|\det(\langle b_k,x_i\rangle)| = |\det(\langle b_k,z_j\rangle)|$

i.e. $\quad f(a_1,\ldots,a_n)\,|\det(\langle b_k,x_i\rangle)| = f(b_1,\ldots,b_n) \leq f(a_1,\ldots,a_n)$

This implies that

$$|\det(\langle b_k, x_i\rangle)| \leq 1 \quad \text{for all} \quad (b_1,\ldots,b_n) \in K^n.$$

For $1 \leq k \leq n$ fixed and $b_i = a_i$ if $i \neq k$ we conclude that

$$|\langle b_k, x_i\rangle| \leq 1 \quad \text{for all} \quad b_k \in K$$

since $\langle a_k, x_i\rangle = \delta_{ik}$.

Consequently $\|x_k\| \leq 1$, but from $1 = \langle a_k, x_k\rangle \leq \|a_k\|\|x_k\| \leq 1$ we obtain $\|x_k\| = \|a_k\| = 1$ for each $k = 1,\ldots,n$. □

Lemma 2.39 ([9], p 131)

If E *has the m.a.p. and* $F \subseteq E$ *is any finite dimensional subspace, then for each* $\epsilon > 0$ *there exists an operator* $A \in \mathcal{F}(E)$ *such that* $\|A\| \leq 1 + \epsilon$ *and* $Ax = x$ *for all* $x \in F$.

Proof

Let $\dim(F) = n$ and $K = (F)_1$.

Choose $\delta \in (0,1)$ such that $\frac{n\delta}{1 - \delta} \leq \epsilon$. Since K is compact we can find $U \in \mathcal{F}(E)$ with $\|U\| \leq 1$ such that $\|x - Ux\| \leq \delta$ for all $x \in K$.

Consequently, for all $x \in F$

$$|\|x\| - \|Ux\|| \leq \|x - Ux\| \leq \|I - U\|\|x\| \leq \delta\|x\|$$

which implies that $\|Ux\| \geq (1 - \delta)\|x\|$.

Therefore, the restriction of U on F is one to one, hence $U(F) = F$. This implies that $\dim[U(F)] = \dim(F) = n$. According to Lemma 2.38 there exists $x_1,\ldots,x_n \in F$ and $a_1,\ldots,a_n \in F^*$ with $\|Ux_i\| = 1$, $\|a_k\| = 1$ such that $\langle a_i, Ux_k\rangle = \delta_{ik}$. Define $V \in \mathcal{L}(E)$ by

$$V := I_E + \sum_{k=1}^{n} a_k \otimes (x_k - Ux_k).$$

Then

$$\|x_k - Ux_k\| \leq \delta\|x_k\| \quad \text{and} \quad \|x_k\| \leq \frac{\|Ux_k\|}{1 - \delta} = \frac{1}{1 - \delta}.$$

Consequently,

$$\|V\| \leq 1 + \sum_{k=1}^{n} \|a_k\|\|x_k - Ux_k\| \leq 1 + \sum_{k=1}^{n} \delta\|x_k\| \leq 1 + \frac{n\delta}{1 - \delta} \leq 1 + \epsilon$$

which implies that

$$\|VU\| \leq \|V\|\|U\| \leq (1 + \epsilon)\|U\| \leq 1 + \epsilon.$$

Furthermore, $VU \in \mathcal{F}(E)$ since $\mathcal{F}(E)$ is a bi-ideal of $\mathcal{L}(E)$.

Finally we show that $(VU)x = x$ for all $x \in F$.

$(VU)x_i = I_E(Ux_i) + \sum_{k=1}^{n} \langle a_k, Ux_i\rangle (x_k - Ux_k) = Ux_i + (x_i - Ux_i) = x_i$ for each $i=1,\ldots,n$. Since $\{x_1,\ldots,x_n\}$ is a basis of F, the result follows by the linearity of VU.

Consequently $A := VU$ satisfies the desired conditions. □

<u>Lemma 2.40</u>

If $(\lambda_i) \in \ell_1$, *there exists* $(\alpha_i) \in c_0$ *such that* $(\frac{\lambda_i}{u_i}) \in \ell_1$.

<u>Proof</u>

Since $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, for each $n \in \mathbb{N}$ we can find $i_n \in \mathbb{N}$ with $i_0 = 0$, $i_n > i_{n-1}$ such that

$$\sum_{i=1}^{\infty} |\lambda_i| - \sum_{i=1}^{i_n} |\lambda_i| = \sum_{i=i_n+1}^{\infty} |\lambda_i| \leq \frac{1}{4^n}$$

Put $\alpha_i := \frac{1}{2^{n-1}}$ for all $i_{n-1} < i \leq i_n$, $n \in \mathbb{N}$.

Clearly $(\alpha_i) \in c_0$ and

$$\begin{aligned}\sum_{i=1}^{\infty} \left|\frac{\lambda_i}{\alpha_i}\right| &= \sum_{i=1}^{i_1} |\lambda_i| + \sum_{n=1}^{\infty} 2^n \sum_{i=i_n+1}^{i_{n+1}} |\lambda_i| \\ &\leq \sum_{i=1}^{i_1} |\lambda_i| + \sum_{n=1}^{\infty} 2^n \sum_{i=i_n+1}^{\infty} |\lambda_i| \\ &\leq \sum_{i=1}^{i_1} |\lambda_i| + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &< \infty\end{aligned}$$

Hence,

$$(\frac{\lambda_i}{\alpha_i}) \in \ell_1 . \quad \square$$

<u>Proposition 2.41</u>

If E *has the approximation property, then* $tr: \mathcal{F}(E) \longrightarrow \mathbb{C}$ *is continuous with respect to the nuclear norm* ν.

Proof

Let $T = \sum_{k=1}^{m} c_k \otimes z_k$ be any finite representation of $T \in \mathcal{F}(E)$, and $\epsilon > 0$ be given. Choose a nuclear representation $T = \sum_{i=1}^{\infty} a_i \otimes x_i$ such that

$$\sum_{i=1}^{\infty} \|a_i\|\|x_i\| \leq \nu(T) + \epsilon.$$

Consider any $L \in \mathcal{F}(E)$, say $L = \sum_{j=1}^{n} b_j \otimes y_j$. Then

$$TL = \sum_{j=1}^{n} b_j \otimes Ty_j \text{ and since } b_j \text{ is continuous}$$

$$\begin{aligned} tr(TL) &= \sum_{j=1}^{n} \langle b_j, Ty_j\rangle = \sum_{j=1}^{n} \Big\langle b_j \Big[\sum_{i=1}^{\infty} \langle a_i, y_j\rangle x_i\Big]\Big\rangle \\ &= \sum_{j=1}^{n} \sum_{i=1}^{\infty} \langle a_i, y_j\rangle\langle b_j, x_i\rangle \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{n} \langle b_j, x_i\rangle\langle a_i, y_j\rangle \\ &= \sum_{i=1}^{\infty} \Big\langle a_i, \sum_{j=1}^{n} \langle b_j, x_i\rangle y_j\Big\rangle \\ &= \sum_{i=1}^{\infty} \langle a_i, Lx_i\rangle. \end{aligned}$$

Similarly $tr(TL) = \sum_{k=1}^{m} \langle c_k, Lz_k\rangle$.

Since $(\|a_i\|\|x_i\|) \in \ell_1$, there exists $(\alpha_i) \in c_0$ such that $\left(\frac{\|a_i\|\|x_i\|}{\alpha_i}\right) \in \ell_1$ by Lemma 2.40, say $\sum_{i=1}^{\infty} \frac{\|a_i\|\|x_i\|}{\alpha_i} = M$. Let

$K_o := \{z_1, \ldots, z_m, \frac{\alpha_1}{\|x_1\|} x_1, \frac{\alpha_2}{\|x_2\|} x_2, \ldots\}$ and $K_n := n\bar{K}_o$, $n \in \mathbb{N}$. Since K_o is a sequence in E converging to zero, $\bar{K}_o$ is compact, hence K_n is compact, $n \in \mathbb{N}$. Since E has the approximation property, for each $n \in \mathbb{N}$ we can find $L_n \in \mathcal{F}(E)$ such that

$$\|x - L_n x\| \leq 1 \text{ for all } x \in K_n.$$

In particular,

$$\left\|\frac{n\alpha_i}{\|x_i\|} x_i - L_n\left(\frac{n\alpha_i}{\|x_i\|} x_i\right)\right\| \leq 1 \text{ and } \|nz_k - L_n(nz_k)\| \leq 1$$

for all $i, n \in \mathbb{N}$, $k=1,\ldots,m$.

Therefore,

$$\begin{aligned}|\sum_{i=1}^{\infty} \langle a_i,x_i\rangle - tr(TL_n)| &= |\sum_{i=1}^{\infty} \langle a_i,x_i\rangle - \sum_{i=1}^{\infty} \langle a_i,L_nx_i\rangle| \\ &= |\sum_{i=1}^{\infty} \langle a_i,x_i - L_nx_i\rangle| \\ &= |\sum_{i=1}^{\infty} \frac{\|x_i\|}{n\alpha_i} \left\langle a_i,\frac{n\alpha_i}{\|x_i\|} x_i - L_n\left[\frac{n\alpha_i}{\|x_i\|} x_i\right]\right\rangle| \\ &\le \sum_{i=1}^{\infty} \frac{\|x_i\|}{n\alpha_i} \|a_i\| \left\|\frac{n\alpha_i}{\|x_i\|} x_i - L_n\left[\frac{n\alpha_i}{\|x_i\|} x_i\right]\right\| \\ &\le \frac{1}{n} \sum_{i=1}^{\infty} \frac{\|x_i\|\|a_i\|}{\alpha_i} \\ &= \frac{1}{n} M\end{aligned}$$

and similarly

$$\begin{aligned}|\sum_{k=1}^{m} \langle c_k,z_k\rangle - tr(TL_n)| &= |\sum_{k=1}^{m} \langle c_k,z_k\rangle - \sum_{k=1}^{m} \langle c_k,L_nz_k\rangle| \\ &= |\sum_{k=1}^{m} \frac{1}{n} \langle c_k,nz_k - L_n(nz_k)\rangle| \\ &\le \frac{1}{n} \sum_{k=1}^{m} \|c_k\|.\end{aligned}$$

Consequently, $\lim_{n\to\infty} tr(TL_n) = \sum_{i=1}^{\infty} \langle a_i,x_i\rangle$ and $\lim_{n\to\infty} tr(TL_n) = \sum_{k=1}^{m} \langle c_k,z_k\rangle$

from which it follows that

$$tr(T) = \sum_{i=1}^{\infty} \langle a_i,x_i\rangle.$$

Thus,

$$|tr(T)| = |\sum_{i=1}^{\infty} \langle a_i,x_i\rangle| \le \sum_{i=1}^{\infty} \|a_i\|\|x_i\| \le \nu(T) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that

$$|tr(T)| \le \nu(T). \qquad \square$$

Corollary 2.42

If E *has the approximation property and* $T = \sum_{i=1}^{\infty} a_i \otimes x_i$, $\sum_{i=1}^{\infty} \|a_i\|\|x_i\| < \infty$ *is any nuclear operator on* E, *then* $\sum_{i=1}^{\infty} \langle a_i,x_i\rangle$ *is independent of the specific representation of* T.

Proof

Since $tr:\mathcal{F}(E) \longrightarrow \mathbb{C}$ is continuous with respect to the ν-norm, it can uniquely be extended to $\mathcal{N}(E)$.

Let $T = \sum_{i=1}^{\infty} a_i \otimes x_i$ be any nuclear representation. Then

$$tr(T) = tr(\lim_{n\to\infty} \sum_{i=1}^{n} a_i \otimes x_i) = \lim_{n\to\infty} tr(\sum_{i=1}^{n} a_i \otimes x_i)$$

$$= \sum_{i=1}^{\infty} \langle a_i, x_i\rangle . \qquad \square$$

Definition 2.43

If E has the approximation property, we call the unique extension of trace to $\mathcal{N}(E)$ also the trace.

Finally we want to prove that the spectrum of a nuclear operator U is countable (perhaps finite), and can accumulate only in the origin. We first show that the set of eigenvalues of U equals the spectrum of U.

Lemma 2.44

Let $U \in \mathcal{N}(E,F)$. If F is a Banach space, then $U \in \mathcal{C}(E,F)$.

Proof

The result follows directly by the fact that $\mathcal{F}(E,F) \subseteq \mathcal{C}(E,F)$, $\mathcal{N}(E,F) \subseteq \overline{\mathcal{F}(E,F)}$ and $\mathcal{C}(E,F)$ is closed in $\mathcal{L}(E,F)$ (cf [6], Theorem 8.1-5). $\square$

Lemma 2.45 ([7], p 86)

Let X be a normed space. If $Z \subseteq X$ is a finite dimensional closed subspace of X, then there exists a closed subspace Y of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$.

Proof

Let $\{z_1,\ldots,z_n\}$ be a basis for Z with $\|z_i\| = 1$, $i=1,\ldots,n$.

By Lemma 2.38 there exists $a_1,\ldots,a_n \in X^*$ such that

$$\langle a_i, z_j\rangle = \delta_{ij}.$$

Let $Y := \bigcap_{i=1}^{n} N(a_i) = \bigcap_{i=1}^{n} a_i^{-1}(0)$.

Then Y is closed, since each a_i is continuous and $Y \cap Z = \{0\}$ since

$\{z_1,\ldots,z_n\}$ is a basis for Z and $\langle a_i,z_j\rangle = \delta_{ij}$. Also, if $x \in X$, then $z := \sum_{i=1}^{n} \langle a_i,x\rangle z_i \in Z$ and $(x - z) \in Y$. Thus $X = Y + Z$. □

Lemma 2.46 ([7], p 146)

Let X *be a normed space,* $T \in \mathcal{K}(X)$ *and* $Y,Z \subseteq X$ *closed subsapces of* X *such that* $Z \subsetneq Y$ *and* $(T - I)(Y) \subseteq Z$. *Then there exists* $y \in Y$ *such that* $\|y\| = 1$ and $\|Ty - Tz\| \geq \frac{1}{2}$ $\forall z \in Z$.

Proof

By the Riesz lemma (cf [6], Theorem 2.5-4) we can find $y \in Y$ such that $\|y\| = 1$ and

$$\inf_{\tilde{z}\in Z}\|y - \tilde{z}\| \geq \frac{1}{2}.$$

Let $B := T - I$.

Then for any $z \in Z$, $(z + Bz - By) = (Tz - By) \in Z$ since $B(Y) \subseteq Z$.

Hence,

$$\|Ty - Tz\| = \|y - (z + Bz - By)\| \geq \inf_{\tilde{z}\in Z} \|y - \tilde{z}\| \geq \frac{1}{2}. \qquad □$$

Lemma 2.47 ([7], p 147)

If $T \in \mathcal{K}(X)$, *then*

(a) $N(T - I) = \{x \in X:(T - I)x = 0\}$ *is finite dimensional and* $(T - I)(X)$ *is closed in* X.

(b) $(T - I)$ *is one to one iff it is onto.*

Proof

Let $B := T - I$.

(a) Let, if possible, $\{x_i:i=1,2,\ldots\}$ be an infinite linearly independent subset of $N(B)$ and let $Y_n := \mathrm{span}\{x_1,\ldots,x_n\} \subseteq N(B)$ for each $n \in \mathbb{N}$.

Then, being finite dimensional, each Y_n is a closed subspace of X.

Clearly, for each $n \in \mathbb{N}$

$$Y_n \subsetneq Y_{n+1} \text{ and } B(Y_{n+1}) = \{0\} \subseteq Y_n$$

By Lemma 2.46 there exists a sequence (y_n) in X such that

$$\|y_n\| = 1 \text{ and } \|Ty_n - Ty_m\| \geq \frac{1}{2} \text{ for } n,m=1,2,\ldots, \quad n > m.$$

This contradicts the compactness of T.

Hence, $N(B)$ is a finite dimensional closed subset of X. By Lemma 2.45, there exists a closed subspace $Y \subseteq X$ such that

$$X = Y + N(B) \text{ and } Y \cap N(B) = \{0\}.$$

Let $A := B|_Y$. Then $A(Y) = B(X)$ and moreover, A is one to one, since $N(B) \cap Y = \{0\}$.

We prove that $A(Y)$ is closed in X.

Let (y_n) be a sequence in Y such that $Ay_n \longrightarrow x \in X$.

First we show that (y_n) is bounded. For, if it is not, we can assume, by passing to a subsequence of (y_n), that $\|y_n\| \longrightarrow \infty$.

Let $z_n := \dfrac{y_n}{\|y_n\|}$. Then $\|z_n\| = 1$ for each $n \in N$ and $Az_n \longrightarrow 0$. Since T is compact and (z_n) is a bounded sequence in Y, there exists a subsequence (z_{n_j}) such that $Tz_{n_j} \longrightarrow z \in X$ (cf [6], Theorem 8.1-3) Now, $z_{n_j} = Tz_{n_j} - Az_{n_j} \longrightarrow z - 0 = z$ and since Y is closed, $z \in Y$. Hence, $Az_{n_j} \longrightarrow Az$ which implies that $Az = 0$, and A being one to one we see that $z = 0$. But this is impossible since $\|z_{n_j}\| = 1$ for $j=1,2,\ldots$ and $z_{n_j} \longrightarrow z$.

Thus, for the bounded sequence (y_n) there exists a subsequence (y_{n_j}) in Y such that $Ty_{n_j} \longrightarrow y \in X$ since T is compact. As before,

$$y_{n_j} = Ty_{n_j} - Ay_{n_j} \longrightarrow (y - x) \in Y \text{ since } Y \text{ is closed.}$$

Hence, $Ay_{n_j} \longrightarrow A(y - x)$ since A is continuous and consequently

$$x = A(y - x) \in A(Y).$$

We have thus proved that $A(Y) = B(X)$ is closed in X.

(b) Consider the following chains of subspaces of X:

$$N(B) \subseteq N(B^2) \subseteq \ldots \subseteq N(B^n) \subseteq N(B^{n+1}) \subseteq \ldots$$

and $\ldots \subseteq B^{n+1}(X) \subseteq B^n(X) \subseteq \ldots \subseteq B^2(X) \subseteq B(X)$.

Note that $N(B^n) = (B^n)^{-1}(0)$ is closed since B^n is continuous for each $n \in \mathbb{N}$. We show that $B^n(X)$ is closed as well, for each $n \in \mathbb{N}$.

For any even $n \in \mathbb{N}$, $I - B^n = I - (T - I)^n$ and for any odd $n \in \mathbb{N}$, $I + B^n = I + (T - I)^n$ are polynomials in T with no term of degree zero.

Hence, for each $n \in \mathbb{N}$, either $(I - B^n)$ or $(I + B^n)$ is compact and by using (a) above it follows that $B^n(X)$ is closed for all $n \in \mathbb{N}$. Furthermore, $B[N(B^{n+1})] \subseteq N(B^n)$ and $B[B^n(X)] = B^{n+1}(X)$ for all $n \in \mathbb{N}$.

Now, since T is compact, Lemma 2.46 shows that not all the inclusions in either of the two chains above can be proper; otherwise a sequence (y_n) in X would exist such that $\|y_n\| = 1$ and $\|Ty_n - Ty_m\| \geq \frac{1}{2}$ for all $n,m=2,3,\ldots,\ n>m$. Thus, we can find $n_0, m_0 \in \mathbb{N}$ such that

$$N(B^{n_0}) = N(B^{n_0+1}) \text{ and } B^{m_0+1}(X) = B^{m_0}(X).$$

Now, let B be one to one. We have $B^{m_0+1}(X) \supseteq B^{m_0}(X)$.

We show that $B^{m_0}(X) \supseteq B^{m_0-1}(X)$.

Let $y \in B^{m_0-1}(X)$. Then $By \in B^{m_0}(X) = B^{m_0+1}(X)$ which implies $By = B(B^{m_0}x)$ for some $x \in X$. Since B is one to one, $y = B^{m_0}x \in B^{m_0}(X)$. Proceeding in this manner we see that

$$B^{m_0+1}(X) \supseteq B^{m_0}(X) \supseteq B^{m_0-1}(X) \supseteq \ldots \supseteq B(X) \supseteq X,$$

i.e. B is onto.

Conversely, let B be onto. We have $N(B^{n_0+1}) \subseteq N(B^{n_0})$.

We show that $N(B^{n_0}) \subseteq N(B^{n_0-1})$.

Let $y \in N(B^{n_0}) \subseteq X$. There exists $x \in X$ with $y = Bx$. Then $x \in N(B^{n_0+1}) \subseteq N(B^{n_0})$. Hence $B^{n_0-1}y = B^{n_0}x = 0$, i.e. $y \in N(B^{n_0-1})$.

Again proceeding in this manner we see that

$$N(B^{n_0+1}) \subseteq N(B^{n_0}) \subseteq N(B^{n_0-1}) \subseteq \ldots \subseteq N(B) \subseteq \{0\},$$

i.e. B is one to one since $\{0\} \subseteq N(B)$. □

<u>Proposition 2.48</u>

If X *is a Banach space and* $T \in \mathcal{N}(X)$, *then* $\sigma(T)\backslash\{0\} = e(T)\backslash\{0\}$.

<u>Proof</u>

Clearly $T \in \mathscr{C}(X)$. Let $0 \neq \lambda \in \sigma(T)$. Since $(T - \lambda I) = \lambda(\frac{T}{\lambda} - I)$ is not invertible, it is either not one to one or not onto. Since $\frac{T}{\lambda} \in \mathscr{C}(X)$, it follows from Lemma 2.47(b) that $(T - \lambda I)$ is not onto iff it is not one to one. Hence, $\lambda \in e(T)$, i.e. $\sigma(T)\setminus\{0\} \subseteq e(T)\setminus\{0\}$.

Consequently, $\sigma(T)\setminus\{0\} = e(T)\setminus\{0\}$. □

Proposition 2.49

If $U \in \mathcal{N}(E)$, *then the spectrum of* U *is either finite or countable, with zero the only point of accumulation.*

Proof

Let U be a nuclear operator. If U is of finite rank, it follows by Proposition 2.16 that $\sigma(u)$ is finite. If U is not of finite rank, then each non-zero spectral value of U is an eigenvalue. First we show that for every $\epsilon > 0$, at most a finite number of eigenvalues lie outside $\epsilon(\mathbb{C})_1$.

Suppose the contrary holds for some $\delta > 0$, i.e. there exists at least a countable number of different eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ such that $|\lambda_n| \geq \delta$ for all $n \in \mathbb{N}$. Also for each $n \in \mathbb{N}$, there exists an eigenvector $x_n \neq 0$ with $Ux_n = \lambda_n x_n$ and the set $\{x_i : i \in \mathbb{N}\}$ is linearly independent (cf [6], Theorem 7.4-3). For each $n \in N$, let $M_n = \overline{\text{span}\{x_1, \ldots, x_n\}}$.

Then every $x \in M_n$ has a unique representation

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n .$$

Since $Ux_j = \lambda_j x_j$ it follows that

$$(U-\lambda_n I)x = \alpha_1(\lambda_1-\lambda_n)x_1 + \ldots + \alpha_{n-1}(\lambda_{n-1}-\lambda_n)x_{n-1}$$

which implies that

$$(U-\lambda_n I)x \in M_{n-1} \quad \text{for all} \quad x \in M_n .$$

Since each M_n is closed (cf [6], Theorem 2.4-3) it follows by Lemma 2.46 that there exists a sequence (y_n) with $y_n \in M_n$, $\|y_n\| = 1$, $\|y_n - x\| \geq \frac{1}{2}$ for all $x \in M_{n-1}$. We show that $\|Uy_n - Uy_m\| \geq \frac{1}{2}\delta$ for all $n > m$ which implies that (Uy_n) has no convergent subsequence. Since (y_n) is bounded, this would contradict the compactness of U (cf [6], Theorem 8.1-3) and therefore the fact that U is nuclear.

Let $\tilde{x}: = \lambda_n y_n - Uy_n - Uy_n + Uy_m$. Then $Uy_n - Uy_m = \lambda_n y_n - \tilde{x}$.

Let $m < n$. Then $y_m \in M_m \subseteq M_{n-1}$ which implies that $Uy_m \in M_{n-1}$ since $Ux_j = \lambda_j x_j$. Therefore, $\tilde{x} \in M_{n-1}$ since $(\lambda_n y_n - Uy_n) = -(U-\lambda_n I)y_n \in M_{n-1}$ from which it follows that $x: = \frac{\tilde{x}}{\lambda_n} \in M_{n-1}$.

Hence, $\|Uy_n - Uy_m\| = \|\lambda_n y_n - \tilde{x}\| = |\lambda_n| \|y_n - x\| \geq \frac{1}{2}|\lambda_n| \geq \frac{1}{2}\delta$.

Consequently, for each $n \in \mathbb{N}$ there exists at most a finite number λ_i for which $|\lambda_i| \geq n$. Now, for each $n \in \mathbb{N}$, let

$$A_n := \{\lambda_i : |\lambda_i| \geq n\} \text{ and } B_n := \{\lambda_i : |\lambda_i| \geq \frac{1}{n}\}.$$

Clearly both sets are countable and consequently

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

is countable which implies that U has at most a countable number of eigenvalues.

Now, let $\epsilon > 0$ be given and consider $A := \{\lambda_n \in \sigma(T) : |\lambda_n| \geq \epsilon\}$. If we rearrange the set of eigenvalues in decreasing order, considering the absolute value of each eigenvalue, and assume A to contain m_0 elements, then

$$|\lambda_n| < \epsilon \text{ for all } n \geq m_0 + 1.$$

Since $\epsilon > 0$ is arbitrary it follows that $\lambda_n \longrightarrow 0$. □

3. ONE-DIMENSIONAL ELEMENTS

This chapter is devoted to a study of one-dimensional elements in a Banach algebra. The trace of such an element is also introduced. In Proposition 3.26 it is shown that the spectrum of a one-dimensional element u consists only of zero and the trace of u. We also prove in Proposition 3.18 that there exists a unique correspondence between the one-dimensional elements and the minimal left ideals.

As was mentioned, K Vala [14] called an element $0 \neq u \in \mathcal{A}$ one-dimensional if the wedge operator $u \wedge u : \mathcal{A} \to \mathcal{A} : x \to uxu$ is of rank one. This definition is motivated by Theorem 2.9. We now turn to the definition given by J. Puhl [10] which generalises Corollary 2.10.

<u>Definition 3.1</u>

Let $\mathcal{A}$ *be a Banach algebra. An element* $0 \neq u \in \mathcal{A}$ *is one-dimensional if there exists a non-zero functional* $\tau_u : \mathcal{A} \longrightarrow \mathbb{C}$ *such that* $uxu = \langle \tau_u, x \rangle u$ *for all* $x \in \mathcal{A}$. *We write* $u \in \mathcal{F}_1$.

Most of the results proved in the rest of the paper, require that the Banach algebra $\mathcal{A}$ possesses a certain property, introduced in the next definition.

<u>Definition 3.2</u>

An algebra $\mathcal{A}$ *is semi-prime if* $\{0\}$ *is the only bi-ideal* J *of* $\mathcal{A}$ *with* $J^2 = \{0\}$.

Before proving a condition which is equivalent for $\mathcal{A}$ to be semi-prime, the following lemmas:

<u>Lemma 3.3</u>

If $u \in \mathcal{A}$ *is fixed such that* $uxu = 0$ *for all* $x \in \mathcal{A}$, *then*

$$J_u := \{ \sum_{i=1}^{n} (\lambda_i u + a_i u + ub_i + c_i ud_i) : \lambda_i \in \mathbb{C},\ a_i, b_i, c_i, d_i \in \mathcal{A};\ n \in \mathbb{N}\}$$

is the bi-ideal generated by u.

Proof

First we show that J_u is a subalgebra. Let

$$x = \sum_{i=1}^{n} (\lambda_i u + a_i u + ub_i + c_i ud_i) \quad \text{and} \quad y = \sum_{j=1}^{m} (\alpha_j u + e_j u + uf_j + m_j un_j)$$

be any two elements of J_u. Then by assumption

$$xy = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i\alpha_j u^2 + \sum_i\sum_j \lambda_i ue_j u + \sum_i\sum_j \lambda_i u^2 f_j + \sum_i\sum_j \lambda_i um_j un_j$$

$$+ \sum_i\sum_j \alpha_j a_i u^2 + \sum_i\sum_j a_i ue_j u + \sum_i\sum_j a_i u^2 f_j + \sum_i\sum_j a_i um_j un_j$$

$$+ \sum_i\sum_j \alpha_j ub_i u + \sum_i\sum_j ub_i e_j u + \sum_i\sum_j ub_i uf_j + \sum_i\sum_j ub_i m_j un_j$$

$$+ \sum_i\sum_j \alpha_j c_i ud_i u + \sum_i\sum_j c_i ud_i e_j u + \sum_i\sum_j c_i ud_i uf_j + \sum_i\sum_j c_i ud_i m_j un_j$$

$$= \sum_i\sum_j (\lambda_i\alpha_j u)u + \sum_i\sum_j (\lambda_i u)u(f_j) + \sum_i\sum_j (\alpha_j a_i u)u + \sum_i\sum_j (a_i u)u(f_j)$$

$$= \sum_i\sum_j \left[(\lambda_i\alpha_j u)u + (\lambda_i u)u(f_j) + (\alpha_j a_i u)u + (a_i u)u(f_j)\right]$$

from which it is clear that $xy \in J_u$. Furthermore, if $\gamma \in \mathbb{C}$ then clearly $(x+\gamma y) \in J_u$.

Finally, if $m \in \mathcal{M}$ and $x \in J_u$ as defined above, then

$$mx = \sum_{i=1}^{n} \left[(\lambda_i m)u + (ma_i)u + mub_i + (mc_i)ud_i\right] \in J_u \quad \text{and similarly} \quad xm \in J_u.$$

□

Lemma 3.4

If the conditions of Lemma 3.3 *are satisfied, then* $J_u^3 = \{0\}$.

Proof

If $x = \sum_{i=1}^{n} (\lambda_i u + a_i u + ub_i + c_i ud_i)$, $y = \sum_{j=1}^{m} (\alpha_j u + e_j u + uf_j + m_j un_j)$ and

$z = \sum_{k=1}^{p} (\gamma_k u + r_k u + ut_k + v_k uw_k) \in J_u$ it can easily be shown that $xyz = 0$, by using the fact that $uxu = 0$ for $x \in \mathcal{M}$. □

Lemma 3.5

$\{\sum_{i=1}^{n} j_i k_i : j_i, k_i \in J_u, \ n \in \mathbb{N}\}$ *is the ideal* $\mathcal{I}(J_u^2)$ *generated by* J_u^2.

Proof

It directly follows that $\mathcal{I}(J_u^2)$ is a subalgebra as well as a linear subspace of $\mathcal{A}$, since J_u is.

Furthermore, if $x = \sum_{i=1}^{n} j_i k_i \in \mathcal{I}(J_u^2)$ and $m \in \mathcal{A}$, then

$mx = \sum_{i=1}^{n} m j_i k_i \in \mathcal{I}(J_u^2)$ since $m j_i \in J_u$. □

Lemma 3.6

Let $\mathcal{A}$ be a semi-prime Banach algebra.

If $J_u^n = \{0\}$ *for any* $n > 2$, *then* $J_u = \{0\}$.

Proof

Without loss of generality we prove it for $n = 3$, since the result will then follow by using induction.

Consider any $x = \sum_{i=1}^{n} j_i k_i$, $y = \sum_{\ell=1}^{m} a_\ell b_\ell \in \mathcal{I}(J_u^2)$.

Then since $J_u^3 = \{0\}$ it follows that

$$0 = \sum_{i=1}^{n} \sum_{\ell=1}^{m} j_i k_i a_\ell b_\ell = xy \in \mathcal{I}(J_u^2)$$

which implies $\mathcal{I}(J_u^2) = \{0\}$. Consequently, $J_u^2 \subseteq \mathcal{I}(J_u^2) = \{0\}$.

Since $\mathcal{A}$ is semi-prime, it necessarily follows that $J_u = \{0\}$. □

Proposition 3.7 ([10], p 657)

$\mathcal{A}$ *is semi-prime if and only if the following holds:*

If $uxu = 0$ *for all* $x \in \mathcal{A}$, *then* $u = 0$.

Proof

Let $uxu = 0$ for all $x \in \mathcal{A}$ implies that $u = 0$. Suppose there exists a non-zero bi-ideal J with $J^2 = \{0\}$. Then we can find $0 \neq u \in J$ and by assumption there exists $x_0 \in \mathcal{A}$ with $ux_0u \neq 0$, which contradicts the fact that $J^2 = \{0\}$, since $\mathcal{A}J \subseteq J$. Therefore $\mathcal{A}$ is semi-prime.

Conversely, assume $\mathcal{M}$ to be semi-prime and $uxu = 0$ for all $x \in \mathcal{M}$. Let

$$J_u := \{ \sum_{i=1}^{n} (\lambda_i u + a_i u + u b_i + c_i u d_i) : \lambda_i \in \mathbb{C} \text{ and } a_i, b_i, c_i, d_i \in \mathcal{M} \}.$$

The result then follows by Lemma 3.3, Lemma 3.4 and Lemma 3.6. □

Lemma 3.8

Let $\mathcal{M}$ *be semi-prime.*

(a) *If* $u,v \in \mathcal{F}_1$ *with* $uv \neq 0$, *then* $uv \in \mathcal{F}_1$.

(b) *If* $u \in \mathcal{F}_1$ *and* $\alpha \in \mathbb{C}$ *with* $\alpha u \neq 0$, *then* $\alpha u \in \mathcal{F}_1$.

(c) *If* $u \in \mathcal{F}_1$ *and* $x,y \in \mathcal{M}$ *with* $ux \neq 0$, $yu \neq 0$, *then* $ux, yu \in \mathcal{F}_1$.

Proof

(a) Let $u,v \in \mathcal{F}_1$. Then for all $x \in \mathcal{M}$ we have $uvxuv = \langle \Gamma_u, vx \rangle uv$.

Define Γ_{uv} on $\mathcal{M}$ by $\langle \Gamma_{uv}, x \rangle := \langle \Gamma_u, vx \rangle$.

Clearly Γ_{uv} is linear and for all $x \in \mathcal{M}$

$|\langle \Gamma_{uv}, x \rangle| \leq \|\Gamma_u\| \|vx\| \leq \|\Gamma_u\| \|v\| \|x\|$ which implies that $\|\Gamma_{uv}\| \leq \|\Gamma_u\| \|v\|$.

We now show that $\Gamma_{uv} \neq 0$.

Since $uv \neq 0$ it follows by Proposition 3.7 that there exists $x_0 \in \mathcal{M}$ with

$0 \neq uvx_0uv = \langle \Gamma_u, vx_0 \rangle uv =: \langle \Gamma_{uv}, x_0 \rangle uv$. Hence $uv \in \mathcal{F}_1$.

(b) Let $u \in \mathcal{F}_1$ and $\alpha \in \mathbb{C}$ such that $\alpha u \neq 0$. Then

$$\alpha u x \alpha u = \langle \Gamma_u, \alpha x \rangle \alpha u \text{ for all } x \in \mathcal{M}.$$

Let $\Gamma_{\alpha u}$ be defined by $\langle \Gamma_{\alpha u}, x \rangle := \langle \Gamma_u, \alpha x \rangle$.

Then we can find $y_0 \in \mathcal{M}$ with

$0 \neq \alpha u y_0 \alpha u = \langle \Gamma_u, \alpha y_0 \rangle \alpha u = \langle \Gamma_{\alpha u}, y_0 \rangle \alpha u$ which implies $\langle \Gamma_{\alpha u}, y_0 \rangle \neq 0$.

Clearly $\Gamma_{\alpha u}$ is also linear and bounded.

(c) Let $u \in \mathcal{F}_1$ and $x,y \in \mathcal{M}$ such that $ux \neq 0$ and $yu \neq 0$. The result then follows directly by defining the functionals Γ_{ux} and Γ_{yu} on $\mathcal{M}$ by

$$\langle \Gamma_{ux}, m \rangle := \langle \Gamma_u, xm \rangle, \ \langle \Gamma_{yu}, m \rangle := \langle \Gamma_u, my \rangle$$

respectively. □

The following lemma is needed to define the concept of the trace of a one-dimensional element in a semi-prime Banach algebra.

Lemma 3.9

Suppose $\mathcal{A}$ *to be semi-prime and let* $u \in \mathcal{F}_1$. *Then there exists* a unique $\alpha_u \in \mathbb{C}$ *such that* $u^2 = \alpha_u u$.

Proof

If $u^2 = 0$, everything is clear (choose $\alpha_u = 0$).

If $u^2 \neq 0$, then by Lemma 3.8 $u^2xu^2 = \langle \Gamma_{u^2}, x\rangle u^2$ for all $x \in \mathcal{A}$.

By Proposition 3.7 we can choose $x_0 \in \mathcal{A}$ such that $u^2x_0u^2 \neq 0$. Then $\langle \Gamma_{u^2}, x_0\rangle \neq 0$. Consequently,

$0 \neq \langle \Gamma_{u^2}, x_0\rangle u^2 = u^2x_0u^2 = \langle \Gamma_u, ux_0u\rangle u$ which implies that

$$u^2 = \frac{\langle \Gamma_u, ux_0u\rangle}{\langle \Gamma_{u^2}, x_0\rangle} u = \frac{\langle \Gamma_u, ux_0u\rangle}{\langle \Gamma_u, ux_0\rangle} u$$

Therefore we choose $\alpha_u = \dfrac{\langle \Gamma_u, ux_0u\rangle}{\langle \Gamma_u, ux_0\rangle}$. It is clear that α_u is unique. □

Remark 3.10

Let $\mathcal{A}$ *have an identity and be semi-prime. If* $u \in \mathcal{F}_1$ *then* $\alpha_u = \langle \Gamma_u, 1\rangle$.

Definition 3.11

Let $\mathcal{A}$ *be a semi-prime Banach algebra.*

If $u \in \mathcal{F}_1$, *the trace of* u *is defined by* $u^2 = \mathrm{tr}(u)u$.

Proposition 3.12 ([10], p 657)

If $\mathcal{A}$ *is* semi-prime and *commutative, then* $0 \neq u \in \mathcal{F}_1$ *if and only if there exists a non-zero functional* σ_u on $\mathcal{A}$ such that

$$ux = \langle \sigma_u, x\rangle u \quad \text{for all } x \in \mathcal{A}.$$

Proof

By definition there exists a non-zero functional Γ_u on $\mathcal{A}$ with

$\langle \Gamma_u, x\rangle u = uxu = u^2x = \mathrm{tr}(u)ux$ for all $x \in \mathcal{A}$.

Since $\mathcal{A}$ is semi-prime there exists $x_0 \in \mathcal{A}$ such that

$0 \neq ux_0u = \mathrm{tr}(u)ux_0$ which implies that $\mathrm{tr}(u) \neq 0$. The result now follows directly by defining $\langle \sigma_u, x\rangle := \frac{1}{\mathrm{tr}(u)}\langle \Gamma_u, x\rangle$, $x \in \mathcal{A}$. □

<u>Definition 3.13</u>

An element $u \in \mathcal{M}$ *is minimal idempotent if* $u \neq 0$ *is idempotent such that* $u\mathcal{M}u$ *is a division algebra.*

<u>Theorem 3.14</u> ([3], p 157)

If $u \in \mathcal{M}$ *is minimal idempotent then* $u \in \mathcal{F}_1$.

<u>Proof</u>

Clearly $u \neq 0$ and since $u\mathcal{M}u$ is a division Banach Algebra, by Gelfand Mazur there exists a unique surjective isometric algebra isomorphism $\varphi : u\mathcal{M}u \to \mathbb{C} : uxu \to \lambda_{uxu}$ with λ_{uxu} the only element in the spectrum of uxu. Clearly u is the identity of $u\mathcal{M}u$ and consequently $(uxu - \lambda_{uxu}u)$ is not invertible in $u\mathcal{M}u$ from which it follows that $uxu = \lambda_{uxu}u$. If we define f_u on $\mathcal{M}$ by $\langle f_u, x\rangle := \lambda_{uxu}$ it is clear that $f_u \neq 0$ is linear and bounded, since $\sigma(uxu)$ is non-empty and bounded. □

<u>Definition 3.15</u>

A minimal left ideal of $\mathcal{M}$ *is a left ideal* $J \neq \{0\}$ *such that* $\{0\}$ *and* J *are the only left ideals contained in* J.

We are now going to prove that the one-dimensional elements and the minimal left ideals are in one to one correspondence.

<u>Lemma 3.16</u>

Let A *be an algebra with identity* e. *If every element* $x \neq 0$ *of* A *has a left inverse, i.e. there exists* $y \in A$ *with* $yx = e$, *then* A *is a division algebra.*

<u>Proof</u>

Consider any $0 \neq x \in A$. Then we can find left inverses $0 \neq y, z \in A$ for x and y respectively with $yx = e$, $zy = e$. Consequently

$$zyx = ex = x \text{ which implies}$$
$$ze = x, \text{ i.e. } z = x.$$

From this it follows that $xy = zy = e$ which shows that y is also a right inverse of x. □

Lemma 3.17 ([11], p 45)

If J is a minimal left ideal of $\mathcal{A}$ with $J^2 \neq \{0\}$, then J contains a minimal idempotent u such that $J = \mathcal{A}u$.

Proof

We first consider a unital algebra.

By assumption, there exists $0 \neq x_0 \in J$ with $Jx_0 \neq \{0\}$ a left ideal contained in J. Since J is minimal it follows that $Jx_0 = J$.

Therefore we can find an element $0 \neq u \in J$ with $ux_0 = x_0$.

Furthermore, $J(u-1)$ is also a left ideal contained in J and consequently either $J(u-1) = \{0\}$ or $J(u-1) = J$. If $J(u-1) = J$, there exists $a \in J$ with $au-a = u$ which gives the contradiction $x_0 = ux_0 = aux_0-ax_0 = 0$. Consequently, $J(u-1) = \{0\}$ which shows that u is idempotent.

We still have to prove that u is minimal.

Clearly $u\mathcal{A}u$ is a subalgebra of $\mathcal{A}$. Let $0 \neq umu \in u\mathcal{A}u$ be any non-zero element, with $m \in \mathcal{A}$. Then

$$0 \neq umu = u(umu) \in Ju\mathcal{A}u \subseteq J\mathcal{A}u \subseteq J$$

from which it follows that $Ju\mathcal{A}u = J$. Choose $b \in J$ such that $bumu = u$. Then

$u = u^2 = ubumu = (ubu)(umu)$ which implies that

ubu is a left inverse of umu since u is the identity of $u\mathcal{A}u$. It follows by Lemma 3.16 that $u\mathcal{A}u$ is a division algebra.

Clearly $\mathcal{A}u = J$, since if $\mathcal{A}u = \{0\}$ we get the contradiction $u = u^2 = 0$.

If $\mathcal{A}$ does not have an identity we consider $\mathcal{A}_1 = \{(x,\lambda) : x \in \mathcal{A}, \lambda \in \mathbb{C}\}$ with identity $(0,1)$. Let $J' = \{(x,0) : x \in J\}$ denote the ideal imbedded in $\mathcal{A}_1$. If $J \subseteq \mathcal{A}$ is a minimal left ideal in $\mathcal{A}$ with $J^2 \neq \{0\}$, then J' is a minimal left ideal in $\mathcal{A}_1$ with $(J')^2 \neq \{0\}$. Thus by the first part of the proof there exists a minimal idempotent $(u,0) \in J'$ such that $J' = \mathcal{A}_1(u,0)$. It remains to show that $J = \mathcal{A}u$ with u minimal idempotent. Since $(u,0)^2 = (u,0)$ it follows that $u^2 = u$.

Furthermore we show that $u\mathcal{A}u$ is a division algebra, i.e. $u\mathcal{A}u = \mathbb{C}u$.

Since $(u,0)\mathcal{A}_1(u,0) = \mathbb{C}(u,0)$ it follows that if $uxu \in u\mathcal{A}u$ then

$$(uxu,0) = (u,0)(x,0)(u,0) \in (u,0)\mathcal{A}_1(u,0) = \mathbb{C}(u,0).$$

Hence $uxu \in \mathbb{C}u$ and therefore $u\mathcal{M}u \subseteq \mathbb{C}u$.

Clearly the converse also holds, since if $\lambda u \in \mathbb{C}u$, then $\lambda u = u(\lambda u)u \in u\mathcal{M}u$.

Finally, since $u \in J$, $\mathcal{M}u \subseteq J$ (J is a left ideal in $\mathcal{M}$).

Conversely, if $x \in J$, then $(x,0) \in J'$ which implies that $(x,0) = (y,\lambda)(u,0)$ since $J' = \mathcal{M}_1(u,0)$.

Thus $(x,0) = (yu + \lambda u,0)$, i.e. $x = yu + \lambda u = (y+\lambda u)u \in \mathcal{M}u$.

Hence $J = \mathcal{M}u$. □

<u>Propostion 3.18</u> ([10], p 658)

Let $\mathcal{M}$ *be semi-prime.*

(a) *If* J *is a minimal left ideal, then* J *contains a minimal idempotent* u *such that* $J = \mathcal{M}u$.

(b) *If* $u \in \mathcal{F}_1$, *then* $J := \mathcal{M}u$ *is a minimal left ideal.*

<u>Proof</u>

(a) It follows directly from Lemma 3.17, since $J \neq \{0\}$ and $\mathcal{M}$ is semi-prime.

(b) Clearly J is a left ideal since $\mathcal{M}J \subseteq J$.

Now let $\{0\} \neq I \subseteq J$ be a non-zero ideal contained in J. Then we can find $0 \neq z \in I \subseteq \mathcal{M}u$ which implies that there exists $z_0 \in \mathcal{M}$ such that

$$0 \neq z = z_0u.$$

Since $\mathcal{M}$ is semi-prime we can choose $y_0 \in \mathcal{M}$ such that

$0 \neq z_0uy_0z_0u = \langle\Gamma_u,y_0z_0\rangle z_0u$ from which it follows that $\langle\Gamma_u,y_0z_0\rangle \neq 0$.

Now consider any $x \in \mathcal{M}$ with $xu \in J$. Then

$$\langle\Gamma_u,y_0z_0\rangle xu = x\langle\Gamma_u,y_0z_0\rangle u = xuy_0z_0u$$

from which it follows that $xu = \dfrac{xuy_0}{\langle\Gamma_u,y_0z_0\rangle} z \in I$

since I is a left ideal. This yields $I = J$. □

<u>Remark 3.19</u>

The above proposition can also be proved for minimal right ideals if $\mathcal{M}$ *is semi-prime, i.e.*

(a) *If* J *is a minimal right ideal, then* J *contains a minimal idempotent* u *such that* $J = u\mathcal{M}$.

(b) *If* $u \in \mathcal{F}_1$, *then* $u\mathcal{M}$ *is a minimal right ideal.*

Example 3.20 ([9], p 658)

Let K *be a disconnected completely regular Hausdorff space and consider the Banach algebra of all complex valued bounded continuous function on* K, *denoted by* $C_b(K)$ *with the surpremum norm. Then the one-dimensional elements are of the form*

$$f_{t_0}(t) = \begin{cases} \alpha_f & \text{for } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_f \in \mathbb{C}$ *is fixed and* $t_0 \in K$ *is an isolated point of* K.

Proof

Let $f \in C_b(K)$ be one-dimensional. Then by Lemma 3.9 and Remark 3.10

$$f^2 = \alpha_f f \,, \qquad \alpha_f = \langle r_f, 1\rangle \,, \text{ i.e.}$$

$$[f(t)]^2 = \alpha_f f(t) \quad \text{for each } t \in K$$

which implies that either $f(t) = \alpha_f \neq 0$ or $f(t) = 0$ for each $t \in K$. We show that there exists exactly one $t_0 \in K$ for which $f(t_0) = \alpha_f$ and $f(t) = 0$, $t \neq t_0$. Suppose there exist $t_0 \neq t_1$ with $f(t_0) = f(t_1) = \alpha_f$. Since K is completely regular we can find a continuous function $g:K \longrightarrow [0,1]$ such that $g(t_0) = 1$, $g(t_1) = 0$. Then $(fgf)(t_0) = \alpha_f^2$ and $(fgf)(t_1) = 0$. But since $(fgf)(t_0) = \langle r_f,g\rangle f(t_0) = \langle r_f,g\rangle\alpha_f$ we have $\alpha_f = \langle r_f,g\rangle$ from which we get a contradiction

$$(fgf)(t_1) = \langle r_f,g\rangle f(t_1) = \alpha_f^2 \neq 0.$$

Hence, the one-dimensional elements of $C_b(K)$ are of the form defined above. □

Next we show that the two different definitions of a one-dimensional element, given by K Vala and J Puhl respectively, match, after giving a lemma.

Lemma 3.21 ([10], p 659)

Let $\mathcal{M}$ *be semi-prime*

If $u \in \mathcal{M}$ *is a given non-zero element of* $\mathcal{M}$ *such that* $\dim(u\mathcal{M}u) < \infty$, *there exists a minimal idempotent* $p \in \mathcal{M}u$ (resp $u\mathcal{M}$).

<u>Proof</u>

We can find $0 \neq v \in u\mathcal{M}u$ such that $\dim(v\mathcal{M}v)$ is as small as possible. First we show that for any $y \in \mathcal{M}$ with $vyv \neq 0$, there exists $z \in \mathcal{M}$ such that

$$v = vzvyv.$$

Let $y \in \mathcal{M}$ with $vyv \neq 0$. Then $vyv\mathcal{M}vyv \subseteq v\mathcal{M}v$, but since the dimension of $v\mathcal{M}v$ is as small as possible and $vyv\mathcal{M}vyv \neq \{0\}$ by Proposition 3.7 it follows that

$$vyv\mathcal{M}vyv = v\mathcal{M}v. \tag{3.1}$$

Consequently, there exists $z \in \mathcal{M}$ with

$$vyvzvyv = vyv \tag{3.2}$$

which implies that

$$(vzvyv-v)\mathcal{M}(vzvyv-v) \subseteq v\mathcal{M}v.$$

Equality is impossible, since by (3.1) and (3.2) it would imply the false assertion

$$\{0\} = vy[(vzvyv - v)\mathcal{M}(vzvyv - v)]yv = vyv\mathcal{M}vyv = v\mathcal{M}v$$

Consequently,

$$(vzvyv - v)\mathcal{M}(vzvyv - v) = \{0\}$$

and since $\mathcal{M}$ is semi-prime, it follows that $v = vzvyv$.

By using this we show that $\mathcal{M}v$ is a minimal left ideal and the result then follows by Proposition 3.18.

Consider any left ideal $\{0\} \neq J \subseteq \mathcal{M}v$ contained in $\mathcal{M}v$. We can find $0 \neq y_0 \in J$ such that $y_0 = mv$ with $m \in \mathcal{M}$. .Since $\mathcal{M}$ is semi-prime, there exists $x_0 \in \mathcal{M}$ such that $y_0x_0y_0 \neq 0$. This implies that $0 \neq v(x_0m)v \in v\mathcal{M}v$, hence we can choose $z \in \mathcal{M}$ such that

$$v = vzvx_0mv.$$

Consequently,

$$\mathcal{M}v = \mathcal{M}vzvx_0mv = \mathcal{M}vzvx_0y_0 \subseteq \mathcal{M}y_0 \subseteq J.$$

Thus, there is a minimal idempotent $p \in \mathcal{M}v \subseteq \mathcal{M}u$.. □

The following theorem proves that the two different definitions of a one-dimensional element, given by Vala and Puhl respectively, match.

Theorem 3.22 ([10], p 660)
If $\mathcal{M}$ *is semi-prime and* $0 \neq u \in \mathcal{M}$ *is given, then* $u \in \mathcal{F}_1$ *if and only if* $u\mathcal{M}u : x \longrightarrow uxu$ *is of rank* one.

Proof
If $u \in \mathcal{F}_1$, it follows directly that $u\mathcal{M}u$ is of rank one, since $0 \neq u$ can be written as $u = u \frac{x_0}{\langle \Gamma_u, x_0 \rangle} u \in u\mathcal{M}u$ by Proposition 3.7.

Conversely, if $u\mathcal{M}u$ is of rank one, there exists a minimal idempotent $p \in \mathcal{M}u$, say $p = mu$ with $m \in \mathcal{M}$. Then

$$(u-up)\mathcal{M}(u-up) \subseteq u\mathcal{M}u$$

Equality is impossible since it would imply the false assertion

$$\{0\} = (u-up)\mathcal{M}(u-up)p = u\mathcal{M}up = u\mathcal{M}umu = u\mathcal{M}u.$$

The last equality follows since $u\mathcal{M}umu \subseteq u\mathcal{M}u$ and $u\mathcal{M}umu \neq \{0\}$ by the fact that $\mathcal{M}$ is semi-prime.
Consequently, by the one-dimensionality of $u\mathcal{M}u$ it follows that

$$(u - up)\mathcal{M}(u-up) = \{0\}.$$

Since $\mathcal{M}$ is semi-prime we have $0 \neq u = up$. It follows by Lemma 3.8(c) that $up \in \mathcal{F}_1$, since $p \in \mathcal{F}_1$. Hence $u \in \mathcal{F}_1$. □

Finally we show that the spectrum of a one-dimensional element contains only two elements.

Definition 3.23
Let $x,y \in \mathcal{M}$. *The quasi-product of* x *and* y *is defined by*

$$x \circ y = x + y - xy.$$

We say that x *is quasi-invertible if there exists* $z \in \mathcal{M}$ *such that* $x \circ z = z \circ x = 0$ *and* z *is then called the quasi-inverse of* x.

Definition 3.24
If $\mathcal{M}$ *has an identity, the spectrum of* $x \in \mathcal{M}$ *is defined by*

$$\sigma_{\mathcal{M}}(x) := \{\lambda \in \mathbb{C} : (x-\lambda 1) \text{ is not invertible in } \mathcal{M}\}$$

and if $\mathcal{M}$ *does not have an identity, it is defined by*

$$\sigma_{\mathcal{M}}(x) := \{0\} \cup \{\lambda \neq 0 : \tfrac{x}{\lambda} \text{ is not quasi-invertible in } \mathcal{M}\}.$$

We define the spectral radius of x *by* $r_{\mathcal{M}}(x) := \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{M}}(x)\}$.

Proposition 3.25 ([10], p 658)

Let $\mathcal{M}$ *be semi-prime with* $\dim(\mathcal{M}) \geq 2$. *If* $u \in \mathcal{F}_1$ *then*

$$\sigma_{\mathcal{M}}(u) = \{0, tr(u)\}.$$

Proof

If $\mathcal{M}$ does not have an identity, then clearly $0 \in \sigma_{\mathcal{M}}(u)$.

If $\mathcal{M}$ has an identity and we suppose that $0 \notin \sigma(u)$, then u^{-1} exists in $\mathcal{M}$. We show that $u\mathcal{M}u = \mathcal{M}$ which contradicts the fact that $\dim(u\mathcal{M}u) = 1$. Choose any $x \in \mathcal{M}$ and consider $y = u^{-1}xu^{-1}$. Then $x = uyu \in u\mathcal{M}u$. Furthermore,

if $0 \neq \lambda \in \sigma_{\mathcal{M}}(u)$, then we can find a quasi-inverse $y \in \mathcal{M}$ such that $0 = u(\frac{u}{\lambda} \circ y) = \frac{tr(u)}{\lambda} u - (\frac{tr(u)}{\lambda} - 1)uy$

which implies that $\lambda \neq tr(u)$.

On the other hand, if $\lambda \neq 0$ and $\lambda \neq tr(u)$, then

$$\frac{u}{\lambda} \circ \frac{u}{tr(u) - \lambda} = \frac{u}{tr(u) - \lambda} \circ \frac{u}{\lambda} = 0$$

which implies that $\lambda \notin \sigma_{\mathcal{M}}(u)$. □

4. FINITE ELEMENTS

In this chapter we define finite elements in a Banach algebra, in the sense of J. Puhl [10], and in Theorem 4.22 it is shown how this coincides with the definition given by Vala [14]. We also prove in Proposition 4.20 that the spectrum of a finite element is finite.

<u>Definition 4.1</u>

An element $u \in \mathcal{M}$ *is finite if* $u = 0$ *or* $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathcal{F}_1$.

We write $u \in \mathcal{F}$.

<u>Proposition 4.2</u>

Let $\mathcal{M}$ *be a semi-prime Banach algebra.* $\mathcal{F}$ *is a bi-ideal of* $\mathcal{M}$.

<u>Proof</u>

Consider any $u = \sum_{i=1}^{n} u_i$, $v = \sum_{j=1}^{m} v_j \in \mathcal{F}$, $\alpha \in \mathbb{C}$ and $m \in \mathcal{M}$. Then

$$uv = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j, \quad \alpha u = \sum_{i=1}^{n} \alpha u_i,$$

$$um = \sum_{i=1}^{n} u_i m \quad \text{and} \quad mu = \sum_{i=1}^{n} mu_i .$$

If $u_i v_j = \alpha u_i = u_i m = mu_i = 0$ for all $i=1,\ldots,n$ and $j=1,\ldots,m$ there is nothing to prove, since it is clear that $u+v \in \mathcal{F}$. If some i and j exist such that $u_i v_j \neq 0$, $\alpha u_i \neq 0$, $mu_i \neq 0$ and $u_i m \neq 0$, then it follows by Lemma 3.8 that uv, αu, um and $mu \in \mathcal{F}$. □

<u>Definition 4.3</u>

If $\mathcal{M}$ *has minimal left ideals, the smallest left ideal containing all of them is called the left socle of* $\mathcal{M}$, *denoted by* $\mathrm{Soc}_\ell(\mathcal{M})$. *The right socle is similarly defined in terms of right ideals, denoted by* $\mathrm{Soc}_r(\mathcal{M})$. *If* $\mathcal{M}$ *has both minimal left and minimal right ideals, and if the left socle coincides with the right socle, it is called the socle of* $\mathcal{M}$, *denoted by* $\mathrm{Soc}(\mathcal{M})$.

Before proving that the socle of $\mathcal{M}$ coincides with the class of finite elements of $\mathcal{M}$, we prove that the socle of a semi-prime algebra always exists.

Lemma 4.4 ([3], p 155)
Let $\mathcal{M}$ *be any algebra. If* J *is a minimal left ideal of* $\mathcal{M}$ *and* $x \in \mathcal{M}$, *then either* $Jx = \{0\}$ *or* Jx *is a minimal left ideal.*

Proof
Suppose that $Jx \neq \{0\}$ and I is a left ideal with $\{0\} \neq I \subseteq Jx$.
Define $H := \{a \in J : ax \in I\}$.
Then $H \neq \{0\}$, since if this were not true we would have had the contradiction $Jx = \{0\}$. Clearly H is a left ideal contained in J, and by the mini-mality of J we have $H = J$. This implies that $Jx \subseteq I$ which proves that Jx is a minimal left ideal. □

Lemma 4.5
Let $\{J_\alpha\}$ *be the set of all minimal left ideals of* $\mathcal{M}$. *Then*

$$\mathrm{Soc}_\ell(\mathcal{M}) = \mathrm{span}\{\bigcup_\alpha J_\alpha\} .$$

Proof
Clearly $\mathrm{span}\{\bigcup_\alpha J_\alpha\}$ is a left ideal containing all J_α which directly implies that $\mathrm{Soc}_\ell(\mathcal{M}) \subseteq \mathrm{span}\, \{\bigcup_\alpha J_\alpha\}$. The converse is also true since $\mathrm{Soc}_\ell(\mathcal{M})$ is a linear subspace. □

Lemma 4.6 ([3], p 156)
If $\mathcal{M}$ *is any algebra possessing minimal left ideals, then* $\mathrm{Soc}_\ell(\mathcal{M})$ *is a bi-ideal.*

Proof
Let $J_1,\dots,J_n$ be minimal left ideals and consider any $x \in \mathcal{M}$. By Lemma 4.4 it follows for $i=1,\dots,n$ that either

$$J_i x = \{0\} \quad \text{or} \quad J_i x \text{ is a minimal left ideal.}$$

Then clearly

$$\sum_{i=1}^{n} J_i x \subseteq \mathrm{Soc}_\ell(\mathcal{M})$$

irrespective whether $J_i x = \{0\}$ for all i or $J_j x \neq \{0\}$ for some j. Since by Lemma 4.5 each element of $Soc_\ell(\mathcal{M})x$ is of the form $\sum_{i=1}^{n} J_i x$, it follows that

$$Soc_\ell(\mathcal{M})x \subseteq Soc_\ell(\mathcal{M}). \qquad \square$$

<u>Proposition 4.7</u> ([3], p 156)

The socle of a semi-prime algebra always exists.

<u>Proof</u>

Let $\mathcal{M}$ be semi-prime. Then by Proposition 3.18 and Remark 3.19 $\mathcal{M}$ possesses both minimal right and minimal left ideals. Again by Proposition 3.18 $Soc_\ell(\mathcal{M})$ contains all minimal idempotents u_α. Since $u_\alpha \in \mathcal{F}_1$, $u_\alpha \mathcal{M} \subseteq Soc_r(\mathcal{M})$ by Remark 3.19. Then $u_\alpha \mathcal{M} \subseteq Soc_\ell(\mathcal{M})$ since $Soc_\ell(\mathcal{M})$ is a bi-ideal by Lemma 4.6. Consequently $Soc_r(\mathcal{M}) \subseteq Soc_\ell(\mathcal{M})$. Similarly $Soc_\ell(\mathcal{M}) \subseteq Soc_r(\mathcal{M})$, which implies that $Soc(\mathcal{M})$ exists. $\square$

<u>Theorem 4.8</u> ([10], p 659)

If $\mathcal{M}$ *is semi-prime, then* $\mathcal{F} = Soc(\mathcal{M})$.

<u>Proof</u>

Let $u \in \mathcal{F}$, say $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathcal{F}_1$. Since each $\mathcal{M}u_i$ is a minimal left ideal and $Soc(\mathcal{M})$ is a linear subspace it follows that $\sum_{i=1}^{n} \mathcal{M}u_i \subseteq Soc(\mathcal{M})$. But since $\mathcal{M}$ is semi-prime, we can write $u_i = \frac{u_i x_i}{\langle r_{u_i}, x_i \rangle} u_i \in \mathcal{M}u_i$ for some $x_i \in \mathcal{M}$ with $u_i x_i u_i \neq 0$, $i=1,\ldots,n$ and hence $u \in Soc(\mathcal{M})$.

Conversely, consider any $u \in Soc\ \mathcal{M}$. By Lemma 4.5 and Proposition 3.18 there exist $u_1,\ldots,u_n \in \mathcal{F}_1$ such that $u \in \sum_{i=1}^{n} \mathcal{M}u_i$.

Therefore, $u = \sum_{i=1}^{n} m_i u_i$ with $m_i \in \mathcal{M}$.

If $m_j u_j \neq 0$ for some j it follows by Lemma 3.8 that $u \in \mathcal{F}$. If $m_i u_i = 0$ for all $i=1,\ldots,n$, then clearly $u \in \mathcal{F}$ by definition. $\square$

Definition 4.9

The centraliser of an element $u \in \mathcal{M}$ *is defined by*

$Y_u := \{x \in \mathcal{M} : ux = xu\}$.

Remark 4.10

Clearly Y_u *is a closed subalgebra of* $\mathcal{M}$.

Before proving that the spectrum of a finite element is finite, we need a few lemmas.

Let $u \in \mathcal{M}$ be a fixed element and define an operator D_u on the centraliser of u by

$$D_u x := uxu, \quad x \in Y_u.$$

Lemma 4.11

Let $\mathcal{M}$ *be semi-prime.*

Suppose that $\mathcal{M}$ *does not have an identity and consider* $\mathcal{M}_1 = \mathcal{M} \oplus \mathbb{C}$. *Let* $(\mathcal{F}_{\mathcal{M}})_1$ *denote the one-dimensional elements in* $\mathcal{M}$ *and* $(\mathcal{F}_{\mathcal{M}_1})_1$ *the one-dimensional elements in* $\mathcal{M}_1$. *If* $u \in (\mathcal{F}_{\mathcal{M}})_1$, *then* $(u,0) \in (\mathcal{F}_{\mathcal{M}_1})_1$.

Proof

Consider any $(x,\lambda) \in \mathcal{M}_1$. Then

$$(u,0)(x,\lambda)(u,0) = (uxu + \lambda u^2, 0) = (\langle \Gamma_u, x\rangle u + \lambda \mathrm{tr}(u)u, 0)$$
$$= [\langle \Gamma_u, x\rangle + \lambda \mathrm{tr}(u)](u,0).$$

Define $\langle \Gamma_{(u,o)}, (x,\lambda)\rangle := \langle \Gamma_u, x\rangle + \lambda \mathrm{tr}(u)$.

Clearly $\Gamma_{(u,o)}$ is linear and bounded since

$$|\langle \Gamma_{(u,o)}, (x,\lambda)\rangle| \leq \|\Gamma_u\|\|x\| + |\mathrm{tr}(u)||\lambda|$$
$$\leq (\|\Gamma_u\| + |\mathrm{tr}(u)|)(\|x\| + |\lambda|)$$
$$= (\|\Gamma_u\| + |\mathrm{tr}(u)|)\|(x,\lambda)\|_{\mathcal{M}_1}$$

which implies $\|\Gamma_{(u,o)}\| \leq \|\Gamma_u\| + |\mathrm{tr}(u)|$.

Since $\mathcal{M}$ is semi-prime, there exists x_0 with $ux_0u \neq 0$.

Consider $(x_0, \lambda_0) \in \mathcal{M}_1$. Then

$$\langle \Gamma_{(u,o)}, (x_0,\lambda_0)\rangle(u,0) = (u,0)(x_0,\lambda_0)(u,0) = (ux_0u + \lambda_0u^2,0) \neq (0,0)$$

which implies that $\Gamma_{(u,o)} \neq 0$ □

Corollary 4.12

If $u \in \mathcal{M}$ *is finite in* $\mathcal{M}$, *then* $(u,0)$ *is finite in* $\mathcal{M}_1$.

We are going to show that if $u \in \mathcal{F}$, then D_u is a finite rank operator on Y_u if $\mathcal{M}$ has an identity and $D_{(u,o)}$ is a finite rank operator on $Y_{(u,o)}$ if $\mathcal{M}$ does not have an identity.

Definition 4.13

Two elements $u,v \in \mathcal{F}_1$ *are called equivalent* $(u \sim v)$ *if there exists* $x_0 \in \mathcal{M}$ *such that* $ux_0v \neq 0$.

Remark 4.14 ([10], p 661)

The relation $\sim$ *is an equivalence relation on* $\mathcal{F}_1$ *if* $\mathcal{M}$ *is semi-prime.*

Proof

Clearly $\sim$ is reflexive since $\mathcal{M}$ is semi-prime. By using the definition of $\sim$ and applying Proposition 3.7 it follows directly that $\sim$ is symmetric.

We now show transitivity.

Let $u,v,w \in \mathcal{F}_1$ with $u \sim v$ and $v \sim w$. We can find $x_0,x_1 \in \mathcal{M}$ such that $ux_0v \neq 0$ and $vx_1w \neq 0$.

Consequently, there exists $y_0 \in \mathcal{M}$ with

$$0 \neq (ux_0v)y_0(ux_0v) = \langle \Gamma_v, y_0ux_0\rangle ux_0v$$

which implies $\langle \Gamma_v, y_0ux_0\rangle \neq 0$. Therefore,

$$0 \neq \langle \Gamma_v, y_0ux_0\rangle vx_1w = vy_0ux_0vx_1w \text{ from which}$$

it follows that $u(x_0vx_1)w \neq 0$, i.e. $u \sim w$ □

Lemma 4.15 ([10], p 661)

Let $u,v \in \mathcal{F}_1$ *with* $u \sim v$ *and* $\mathcal{M}$ *be semi-prime. Then the operator* $D_{u,v}:\mathcal{M} \longrightarrow \mathcal{M}:x \longrightarrow uxv$ *is of rank one,* $\|D_{u,v}\| \leq \|u\|\|v\|$ *and* $tr(D_{u,v}) = tr(u)tr(v)$.

Proof

Choose $x_0 \in \mathcal{M}$ with $ux_0v \neq 0$. Then there exists $y_0 \in \mathcal{M}$ such that

$$0 \neq (ux_0v)y_0(ux_0v) = \langle \Gamma_v, y_0ux_0\rangle ux_0v$$

which implies $\langle \Gamma_v, y_0ux_0\rangle \neq 0$. Then for all $x \in \mathcal{M}$

$$uxv\langle \Gamma_v, y_0ux_0\rangle = uxvy_0ux_0v = \langle \Gamma_u, xvy_0\rangle ux_0v$$

and consequently

$$D_{u,v}x = \frac{\langle \Gamma_u, xvy_0\rangle}{\langle \Gamma_v, y_0ux_0\rangle} ux_0v.$$

Hence, $D_{u,v}$ is of rank one.

Clearly $\|D_{u,v}\| \leq \|u\|\|v\|$ and by Proposition 2.20 it follows that

$$tr(D_{u,v})ux_0v = tr(D_{u,v})D_{u,v}x_0 = (D_{u,v})^2x_0 = u^2x_0v^2 = tr(u)tr(v)ux_0v$$

which implies $tr(D_{u,v}) = tr(u)tr(v)$ since $ux_0v \neq 0$. □

Corollary 4.16

Let $\mathcal{M}$ *be semi-prime and* $u \in \mathcal{F}$.

If $\mathcal{M}$ *has an identity then* D_u *is a finite rank operator on* Y_u, *and if* $\mathcal{M}$ *does not have an identity, then* $D_{(u,o)}$ *is of finite rank on* $Y_{(u,0)} \subseteq \mathcal{M}_1$.

Proof

Let $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathcal{F}_1$ be any representation. If $\mathcal{M}$ has an identity, then clearly Y_u contains the identity. It follows directly by Lemma 4.15 that D_u is of finite rank on Y_u. The case if $\mathcal{M}$ does not have identity is similar. □

Lemma 4.17 ([11], p 32)

Let $a \in \mathcal{M}$ *be fixed and define* $T_a \in \mathcal{L}(\mathcal{M})$ *by*

$$T_ax := ax \ , \quad x \in \mathcal{M}.$$

If $\mathcal{M}$ *has an identity, then* a *is invertible in* $\mathcal{M}$ *if and only if* T_a *is invertible in* $\mathcal{L}(\mathcal{M})$ *and then* $\sigma_{\mathcal{M}}(a) = \sigma_{\mathcal{L}(\mathcal{M})}(T_a)$.

Proof

If a is invertible in $\mathcal{M}$, then $T_{a^{-1}}: x \longrightarrow a^{-1}x$ exists in $\mathcal{L}(\mathcal{M})$.

Clearly $T_{a^{-1}}T_a x = T_a T_{a^{-1}} x = x$ for all $x \in \mathcal{M}$ from which it follows that T_a is invertible with inverse $T_{a^{-1}}$. On the other hand, if T_a is invertible in $\mathcal{L}(\mathcal{M})$ we can find $T \in \mathcal{L}(\mathcal{M})$ with

$$T_a T = T T_a = I.$$

Let $x = T1$. Then

$$ax = T_a T1 = 1 \text{ which implies } T_a T_x = T_{ax} = T_1 = I.$$

Consequently, $T_x = T$ and $T_x T_a = I$.

Hence, $xa = T_x T_a 1 = 1$ which shows that a is invertible.

Furthermore, $\lambda \notin \sigma(a)$ iff $(a - \lambda 1)$ is invertible in $\mathcal{M}$, iff $T_{(a-\lambda 1)} = (T_a - \lambda I)$ is invertible in $\mathcal{L}(\mathcal{M})$, iff $\lambda \notin \sigma_{\mathcal{L}(\mathcal{M})}(T_a)$. □

Corollary 4.18 ([11], p 32)

Let $a \in \mathcal{M}$ be fixed and $\mathcal{M}$ has identity. Then

$$\sigma_{\mathcal{L}(Y_a)}(T_a) = \sigma_{\mathcal{M}}(a).$$

Proof

Obviously Y_a contains the identity.

We first show that $\sigma_{Y_a}(a) = \sigma_{\mathcal{M}}(a)$. Clearly $\sigma_{\mathcal{M}}(a) \subseteq \sigma_{Y_a}(a)$.

On the other hand, if $\lambda \notin \sigma_{\mathcal{M}}(a)$ then $(\lambda 1 - a)$ is invertible in $\mathcal{M}$ and since $(\lambda 1 - a) \in Y_a$ we have

$$(\lambda 1 - a)^{-1} a = (\lambda 1 - a)^{-1} a(\lambda 1 - a)(\lambda 1 - a)^{-1} = a(\lambda 1 - a)^{-1}$$

which implies that $\lambda \notin \sigma_{Y_a}(a)$. Hence $\sigma_{Y_a}(a) \subseteq \sigma_{\mathcal{M}}(a)$.

Hence, by Lemma 4.17 $\sigma_{\mathcal{L}(Y_a)}(T_a) = \sigma_{Y_a}(a) = \sigma_{\mathcal{M}}(a)$. □

Proposition 4.19

If $\mathcal{M}$ has an identity and $u \in \mathcal{M}$ is fixed, then

$$\sigma_{\mathcal{L}(Y_u)}(D_u) = [\sigma_{\mathcal{M}}(u)]^2.$$

Proof

First we note that $D_u = T_{u^2}$ on Y_u.

Then by Corollary 4.18 and the spectral mapping theorem it follows directly that $\sigma_{\mathcal{L}(Y_u)}(D_u) = [\sigma_{\mathcal{M}}(u)]^2$. □

Corollary 4.20

If $\mathcal{M}$ *doesn't have an identity, and* $(u,0) \in \mathcal{M}_1$ *is fixed, then*

$$\sigma_{\mathcal{L}(Y_{(u,o)})}(D_{(u,o)}) = [\sigma_{\mathcal{M}_1}(u,0)]^2.$$

Proposition 4.21

Let $\mathcal{M}$ *be semi-prime and* $u \in \mathcal{F}$.

If $\mathcal{M}$ *has identity, then* $\sigma_{\mathcal{M}}(u)$ *is finite, otherwise* $\sigma_{\mathcal{M}_1}(u,o)$ *is finite in* $\mathcal{M}_1 = \mathcal{M} \oplus \mathbb{C}$.

Proof

Let $\mathcal{M}$ have identity. It follows directly by Corollary 4.16 and Proposition 4.19 that $\sigma_{\mathcal{M}}(u)$ is finite, since $\sigma_{\mathcal{L}(Y_u)}(D_u)$ is finite by Proposition 2.16.

If $\mathcal{M}$ does not have an identity, it follows by Corollary 4.16, Corollary 4.20 and Proposition 2.16 that $\sigma_{\mathcal{M}_1}(u,0)$ is finite. □

Lemma 4.22 ([10]), p 660)

If $\mathcal{M}$ *is semi-prime and* $u \in \mathcal{M}$ *is a non-zero element such that* $\dim(u\mathcal{M}u) < \infty$, *there exists an idempotent* $p \in \mathcal{F} \cap u\mathcal{M}$ *such that* $u = pu$ (resp. $u = up$).

Proof

First we show that every subset of orthogonal idempotents of $u\mathcal{M}$ is finite and then make use of Lemma 3.21.

Let, if possible, $\{p_i : i=1,2,\ldots\}$ be an infinite set of non-zero orthogonal idempotents in $u\mathcal{M}$. Then for each $p_i \in u\mathcal{M}$ we can find $x_i \in \mathcal{M}$ with $p_i = ux_i$.

Choose a sequence of distinct scalars $\lambda_i \in \mathbb{C}$ such that

$$|\lambda_i| \leq \frac{1}{2^i \|x_i\|} \quad \text{for all} \quad i \in \mathbb{N}$$

and let $x := \sum_{i=1}^{\infty} \lambda_i x_i$. Then $x \in \mathcal{A}$ is well defined, since the series is absolute convergent, and

$$ux = \sum_{i=1}^{\infty} \lambda_i ux_i = \sum_{i=1}^{\infty} \lambda_i p_i$$

We show that $\lambda_j \in \sigma_{\mathcal{A}}(ux)$, $j \in \mathbb{N}$. Suppose there exists an inverse $z \in \mathcal{A}$ with $1 = (ux-\lambda_j 1)z = (\sum_{i=1}^{\infty} \lambda_i p_i - \lambda_j 1)z$. Then we have the contradiction

$$0 \neq p_j = (\sum_{i=1}^{\infty} \lambda_i p_j p_i - \lambda_j p_j)z = (\lambda_j p_j^2 - \lambda_j p_j)z = 0.$$

Therefore, $\{\lambda_j : j=1,2,\ldots\}$ is an infinite set of $\sigma_{\mathcal{A}}(ux)$ in contradiction to the fact that $D_{ux} : Y_{ux} \longrightarrow \mathcal{A} : y \longrightarrow (ux)y(ux)$ is of finite rank and hence $[\sigma_{\mathcal{A}}(ux)]^2 = \sigma_{\mathcal{L}(Y_{ux})} D_{ux}$ is finite .

Then by Lemma 3.21 and the Gram Schmidt orthogonalisation process there exists a non-empty set of orthogonal minimal idempotents of $u\mathcal{A}$.
Choose a maximal orthogonal set $\{p_i : i=1,2,\ldots,n\}$ of minimal idempotents.

Then $p_i \in \mathcal{F}_1 \cap u\mathcal{A}$, $i=1,\ldots,n$. If we put $p := \sum_{i=1}^{n} p_i$, it is clear that $p \in \mathcal{F} \cap u\mathcal{A}$ is idempotent. If $pu \neq u$, then

$$(pu-u)\mathcal{A}(pu-u) \subseteq u\mathcal{A}u \quad \text{which implies that}$$

$$\dim[(pu-u)\mathcal{A}(pu-u)] \leq \dim[u\mathcal{A}u] < \infty.$$

Again by Lemma 3.21 there is a minimal idempotent $q \in (pu-u)\mathcal{A} \subseteq u\mathcal{A}$.

Clearly $p_j q = 0$ for each $j=1,\ldots,n$.

If we define $w := q - \sum_{i=1}^{n} qp_i$, it is clear that $w \in u\mathcal{A}$.

Furthermore,

$$wq = q^2 - \sum_{i=1}^{n} q(p_i q) = q \neq 0 \quad \text{which implies} \quad w \neq 0,$$

and $w^2 = w$ is minimal, since q and each p_i are minimal idempotent.

Consequently $w \in \mathcal{F}_1 \cap u\mathcal{A}$.

It can also easily be checked that $p_j w = wp_j = 0$ for each $j=1,\ldots,n$.

Hence $\{p_1,\ldots,p_n,w\} \subseteq \mathcal{F}_1 \cap u\mathcal{A}$ is an orthogonal set of minimal idempotents in contradiction with the maximality of $\{p_1,\ldots,p_n\} \subseteq \mathcal{F}_1 \cap u\mathcal{A}$.

Therefore, $u = pu$. □

The same result as for one-dimensional elements, mentioned previous to Theorem 3.22, is now proved for finite elements.

Theorem 4.23 ([10], p 661)

Let $\mathcal{A}$ be semi-prime and $u \in \mathcal{A}$ a non-zero element.

Then $u \in \mathcal{F}$ if and only if the wedge operator $u \wedge u : x \to uxu$ is of a finite rank.

Proof

Let $u = \sum_{i=1}^{n} u_i,\ u_i \in \mathcal{F}_1$.

Then for all $x \in \mathcal{A}$

$$u \wedge u(x) = uxu = \sum_{i,j=1}^{n} u_i x u_j$$

By using Lemma 4.15 it immediately follows that $u \wedge u$ is of finite rank. Conversely, if $u \wedge u$ has finite rank, then there exists an idempotent $p \in \mathcal{F}$ such that $u = pu$. Since $\mathcal{F}$ is a bi-ideal it follows that $u \in \mathcal{F}$. □

5. THE TRACE OF FINITE ELEMENTS

In this section we introduce the notion of a trace of finite elements and the results are analogous to that of the classical operator theory.

The following lemma gives rise to a well defined trace for finite elements.

<u>Lemma 5.1</u> ([10], p 662)

Let $\mathcal{M}$ *be semi-prime and* $u_1,\ldots,u_n \in \mathcal{F}_1$ *such that* $\sum_{i=1}^{n} u_i = 0$.

Then $\sum_{i=1}^{n} \mathrm{tr}(u_i) = 0$.

<u>Proof</u>

Let $A_1,\ldots,A_s$ be a disjoint decomposition of $\{1,\ldots,n\}$ induced by $\sim$ on $\mathcal{F}_1$. For any fixed $k \in \{1,\ldots,s\}$ we get

$$D_k x := \sum_{i,j\in A_k} u_i x u_j = \Big(\sum_{i=1}^{n} u_i\Big) x \Big(\sum_{j\in A_k} u_j\Big) = 0 \quad \text{for all} \quad x \in \mathcal{M}.$$

Then by Lemma 4.15 and Definition 2.18 it follows that

$$\Big[\sum_{i\in A_k} \mathrm{tr}(u_i)\Big]^2 = \sum_{i,j\in A_k} \mathrm{tr}(u_i)\mathrm{tr}(u_j) = 0.$$

Consequently,

$$\sum_{i=1}^{n} \mathrm{tr}(u_i) = \sum_{k=1}^{s}\Big[\sum_{i\in A_k} \mathrm{tr}(u_i)\Big] = 0. \qquad \square$$

<u>Definition 5.2</u>

If $u = \sum_{i=1}^{n} u_i,\ u_i \in \mathcal{F}_1$ *is any representation of* $u \in \mathcal{F}$, *the trace of* u *is defined by*

$$\mathrm{tr}(u) := \sum_{i=1}^{n} \mathrm{tr}(u_i).$$

Before proving a few properties concerning the trace on $\mathcal{F}$, we need a few lemmas.

Definition 5.3

Let $\mathcal{M}$ *be a* C^**-algebra.*

An element $a \in \mathcal{M}$ *is positive if* a *is selfadjoint and* $\sigma_{\mathcal{M}}(a) \subset [0,\infty)$.

We write $a \geq 0$.

The functional $f: \mathcal{M} \longrightarrow \mathbb{C}$ *is said to be positive if* $\langle f,a \rangle \geq 0$ *for all* $a \geq 0$.

Lemma 5.4 ([13], p 17)

Let $\mathcal{M}$ *be a* C^**-algebra.*

(a) *If* $x \in \mathcal{M}$ *is normal, then* $\sigma_{\mathcal{M}}(x) \subseteq \{\lambda \in \mathbb{C}: |\lambda| \leq \|x\|\}$, and $r_{\mathcal{M}}(x) = \|x\|$.

(b) *If* $x \in \mathcal{M}$ *is unitary, then* $\sigma_{\mathcal{M}}(x) \subseteq \{\lambda \in \mathbb{C}: |\lambda| = 1\}$.

(c) *If* $s \in \mathcal{M}$ *is selfadjoint, then* $\sigma_{\mathcal{M}}(s) \subseteq \mathbb{R}$.

Proof

(a) Clearly $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$, since x is normal and xx^* selfadjoint. Therefore,

$$r_{\mathcal{M}}(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\| \quad \text{(cf [4], Theorem 2.38).}$$

(b) Observe first that if x is an element of a C^*-algebra, then the inequality $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ implies that $\|x\| \leq \|x^*\|$ and hence $\|x\| = \|x^*\|$, since $x^{**} = x$. Let $x \in \mathcal{M}$ be unitary. Then $\|x\|^2 = \|x^*x\| = 1$ which implies that $\|x^*\| = \|x\| = 1$. By (a) we have $\sigma(x^*) = \sigma(x) \subseteq \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$. But since $x^* = x^{-1}$, we also have $\sigma(x^{-1}) = \sigma(x)$ and it is clear that $\sigma(x^{-1}) = \{0 \neq \lambda \in \mathbb{C}: \frac{1}{\lambda} \in \sigma(x)\}$. Consequently, $\sigma(x) \subseteq \{\lambda \in \mathbb{C}: |\lambda| = 1\}$.

(c) (cf [4], Theorem 4.27). □

Lemma 5.5 ([13], p 7)

If $\mathcal{M}$ *is an algebra with identity, then for any* $x,y \in \mathcal{M}$

$$\sigma_{\mathcal{M}}(xy) \cup \{0\} = \sigma_{\mathcal{M}}(yx) \cup \{0\}.$$

Proof

Suppose $\lambda \notin \sigma_{\mathcal{M}}(xy) \cup \{0\}$. Then $(xy - \lambda 1)^{-1} = u$ exists.

Since $xyu = 1 + \lambda u = uxy$, we have

$$(yux - 1)(yx - \lambda 1) = y(1 + \lambda u)x - \lambda yux - yx + \lambda 1 = (yx - \lambda 1)(yux - 1) = \lambda 1$$

which implies that

$$\frac{1}{\lambda}(yux - 1)(yx - \lambda 1) = 1, \quad \frac{1}{\lambda}(yx - \lambda 1)(yux - 1) = 1.$$

Hence, $(yx - \lambda 1)$ is invertible, i.e. $\lambda \notin \sigma_{\mathcal{M}}(yx) \cup \{0\}$. By symmetry

$$\sigma_{\mathcal{M}}(xy) \cup \{0\} = \sigma_{\mathcal{M}}(yx) \cup \{0\}. \qquad \square$$

Lemma 5.6 ([13], p 23)

Let $\mathcal{M}$ *be a* C^**-algebra. Then*

$a \in \mathcal{M}$ *is positive iff there exists a selfadjoint* $s \in \mathcal{M}$ *with* $a = s^2$.

Proof

Let $a \in \mathcal{M}$ be any positive element, and consider the closed commutative C^*-algebra C_a generated by a. Let the set of multiplicative functionals on C_a be denoted by M_{C_a}. Since M_{C_a} is homeomorphic to $\sigma_{\mathcal{M}}(a)$, it follows that the Gelfand transform $\Gamma : C_a \longrightarrow C[M_{C_a}]$, defined by

$$(\Gamma x)(f) = f(x), \quad x \in C_a \text{ and } f \in M_{C_a},$$

is a $*$-isometric algebra isomorphism of C_a onto $C[\sigma_{\mathcal{M}}(a)]$ (cf [4], Theorem 4.30).

Let $s := a^{\frac{1}{2}}$. Since $\sigma_{\mathcal{M}}(a) \subseteq [0,\infty)$, the square root function $\sqrt{\ }$ is continuous on $\sigma_{\mathcal{M}}(a) \approx M_{C_a}$ and consequently $\sqrt{a}$ is well defined.

Furthermore, since the involution is continuous on $\mathcal{M}$, we have

$$s^* = s \quad \text{since} \quad \overline{\lambda} = \lambda \quad \text{if} \quad \lambda \in [0,\infty).$$

Therefore $a = a^{\frac{1}{2}}a^{\frac{1}{2}} = s^2$ with $s \in \mathcal{M}$ selfadjoint. On the other hand, if $a = s^2$, $s \in \mathcal{M}$ selfadjoint, it is clear that a is selfadjoint and $\sigma_{\mathcal{M}}(a) = \{\lambda^2 : \lambda \in \sigma_{\mathcal{M}}(s)\} \subseteq [0,\infty)$, since $\sigma_{\mathcal{M}}(s) \subseteq \mathbb{R}$. $\square$

Corollary 5.7

If $\mathcal{M}$ *is a* C^**-algebra and* $a \geq 0$, *then there exists* $x \in \mathcal{M}$ *such that* $a = x^*x$.

Lemma 5.8 ([13], p 23)
Let $\mathcal{M}$ be a C^*-algebra. If $v \in \mathcal{M}$, then vv^* is positive.

Proof
First we show that $vv^* = s^2$ for some selfadjoint $s \in \mathcal{M}$.
Let $P := \{x \in \mathcal{M}: x$ is selfadjoint and $\sigma_{\mathcal{M}}(x) \subseteq [0,\infty)\}$.

We prove that $P \cap (-P) = \{0\}$, with $-P = \{y \in \mathcal{M}: y = -x,\ x \in P\}$.
We first note that if $y \in -P$, then by the spectral mapping theorem $\sigma_{-P}(y) = \sigma_{-P}(-x) = -[\sigma_P(x)] \subseteq (-\infty,0]$. Therefore, if $x \in P \cap (-P)$ it follows that $\sigma_{\mathcal{M}}(x) = \{0\}$. This implies that $r_P(x) = 0$, i.e. $x = 0$.

Now, let $v \in \mathcal{M}$ and consider vv^*. Then $vv^* = (vv^*)^+ - (vv^*)^-$ and since both the positive and negative parts of vv^* are positive elements of $\mathcal{M}$, it follows by Lemma 5.6 that we can find selfadjoint elements $s,t \in \mathcal{M}$ such that $vv^* = s^2 - t^2$. It can easily be checked that $st = 0$, since

$$(vv^*)^+ = \tfrac{1}{2}(|vv^*| + vv^*) \text{ and } (vv^*)^- = \tfrac{1}{2}(|vv^*| - vv^*) \text{ where}$$
$$|vv^*| := [(vv^*)(vv^*)^*]^{\frac{1}{2}} = [(vv^*)^2]^{\frac{1}{2}}.$$

Therefore,

$$(tv)(tv)^* = tvv^*t = (ts)(st) - t^4 = -t^4 \in -P \tag{5.1}$$

and if we put $tv := k_1 + ik_2$ with $k_1,k_2 \in \mathcal{M}$ selfadjoint and making use of (5.1) it follows that

$$(tv)^*(tv) = (k_1 - ik_2)(k_1 + ik_2) + (k_1 + ik_2)(k_1 - ik_2) - (tv)(tv)^*$$
$$= 2k_1^2 + 2k_2^2 - (tv)(tv)^* \in P$$

But since $\sigma[(tv)(tv)^*] = \sigma[(tv)^*(tv)] \subseteq [0,\infty)$ by Lemma 5.5, it follows that $(tv)(tv)^* \in P \cap (-P) = \{0\}$ which implies that $-t^4 = 0$. Hence $t = 0$. Therefore $vv^* = s^2$, with $s \in \mathcal{M}$ selfadjoint. It follows by Lemma 5.6 that $vv^* \geq 0$. □

Theorem 5.9 ([10], p 663)
Let $\mathcal{M}$ *be a semi-prime Banach algebra. Then the trace has the following properties:*

(a) *The trace is a linear functional on* $\mathcal{S}$.

(b) *If* $u \in \mathcal{S}$ *and* $x \in \mathcal{M}$, *then* $tr(ux) = tr(xu)$.

(c) *If* $u \in \mathcal{S}$ *is nilpotent, then* $tr(u) = 0$.

(d) *If* $\mathcal{M}$ *is a* $*$*-algebra and* $u \in \mathcal{S}$, *then* $tr(u^*) = \overline{tr(u)}$.

(e) *If* $\mathcal{M}$ *is a* C^**-algebra, then* Γ_{vv^*} *is a positive functional on* $\mathcal{M}$ *for all* $v \in \mathcal{S}_1$.

Proof

(a) This is obvious, since if $v \in \mathcal{F}_1$ and $\alpha \in \mathbb{C}$, then $\alpha v \in \mathcal{F}_1$ by Lemma 3.8. Hence, $tr(\alpha v)\alpha v = \alpha^2 v^2$ and $\alpha tr(v)v = \alpha v^2$ which implies that $tr(\alpha v) = \alpha tr(v)$ if $\alpha \neq 0$. If $\alpha = 0$, everything is trivial.

Therefore,

if $u = \sum_{i=1}^{n} u_i$, $v = \sum_{j=1}^{m} v_j$ with $u_i, v_j \in \mathcal{F}_1$ and $\alpha \in \mathbb{C}$, then

$$tr(u + \alpha v) = \sum_{i=1}^{n} tr(u_i) + \sum_{j=1}^{m} tr(\alpha v_j) = \sum_{i=1}^{n} tr(u_i) + \sum_{j=1}^{m} \alpha tr(v_j)$$

$$= tr(u) + \alpha tr(v)$$

(b) First we show for $v \in \mathcal{F}_1$, $x \in \mathcal{A}$ that $tr(xv) = tr(vx)$. Clearly $xv, vx \in \mathcal{F}_1$ by Lemma 3.8. Therefore,

$$\langle \Gamma_v, x\rangle xv = xvxv = tr(xv)xv \quad \text{and} \quad \langle \Gamma_v, x\rangle vx = vxvx = tr(vx)vx$$

which implies that $tr(xv) = \langle \Gamma_v, x\rangle = tr(vx)$.

Now it follows for $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathcal{F}_1$ that

$$tr(xu) = \sum_{i=1}^{n} tr(xu_i) = \sum_{i=1}^{n} tr(u_i x) = tr(ux).$$

(c) Let $A_1, \ldots, A_s$ have the same meaning as in the proof of Lemma 5.1. For each fixed k, let

$$D_k x := ux\Big(\sum_{j \in A_k} u_j\Big), \quad u = \sum_{i=1}^{n} u_i, \quad x \in \mathcal{A}$$

Then $D_k x = \Big(\sum_{i=1}^{n} u_i\Big)x\Big(\sum_{j \in A_k} u_j\Big) = \sum_{i,j \in A_k} u_i x u_j$ is a finite rank operator by Lemma 4.15 with

$$tr(D_k) = \sum_{i,j \in A_k} tr(u_i)tr(u_j) = \Big[\sum_{i \in A_k} tr(u_i)\Big]^2.$$

On the other hand, since u is nilpotent it is obvious that $D_k \in \mathcal{F}(\mathcal{A})$ is nilpotent. Hence we also have $tr(D_k) = 0$ by Corollary 2.23.

Consequently,

$$\sum_{i \in A_k} tr(u_i) = 0 \quad \text{for each fixed } k, \text{ from which}$$

it follows that

$$\operatorname{tr}(u) = \sum_{k=1}^{s} \Big[\sum_{i\in A_k} u_i\Big] = 0.$$

(d) We first prove the result for one-dimensional elements.
If $v \in \mathscr{F}_1$, then $v^* \in \mathscr{F}_1$, since $\langle \Gamma_v, x^*\rangle v = vx^*v$ implies that

$$\overline{\langle \Gamma_v, x^*\rangle} v^* = v^* x v^*$$

Let $\langle \Gamma_{v^*}, x\rangle := \overline{\langle \Gamma_v, x^*\rangle}$, $x \in \mathscr{A}$.

Clearly Γ_{v^*} is linear and bounded since

$$\langle \Gamma_{v^*}, x + \alpha y\rangle := \overline{\langle \Gamma_v, x^* + \bar{\alpha} y^*\rangle} = \overline{\langle \Gamma_v, x^*\rangle} + \alpha\overline{\langle \Gamma_v, y^*\rangle}$$

$$= \langle \Gamma_{v^*}, x\rangle + \alpha\langle \Gamma_{v^*}, y\rangle$$

and $|\langle \Gamma_{v^*}, x\rangle| \le \|\Gamma_v\|\|x^*\| = \|\Gamma_v\|\|x\|$ for all $x \in \mathscr{A}$ which implies $\|\Gamma_{v^*}\| \le \|\Gamma_v\|$.

Furthermore, since $v^* \neq 0$ and $\mathscr{A}$ is semi-prime, there exists $x_0 \in \mathscr{A}$ with

$$0 \neq v^* x_0 v^* = (v x_0^* v)^* = \overline{\langle \Gamma_v, x_0^*\rangle} v^* =: \langle \Gamma_{v^*}, x_0\rangle v^*$$

which implies $\Gamma_{v^*} \neq 0$.

Consequently,

$$\overline{\operatorname{tr}(v)} v^* = (\operatorname{tr}(v)v)^* = v^* v^* = \operatorname{tr}(v^*) v^*, \quad \text{hence}$$

$$\overline{\operatorname{tr}(v)} = \operatorname{tr}(v^*).$$

Now, if $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathscr{F}_1$, then $u^* = \sum_{i=1}^{n} u_i^* \in \mathscr{F}$ and

$$\operatorname{tr}(u^*) = \sum_{i=1}^{n} \operatorname{tr}(u_i^*) = \sum_{i=1}^{n} \overline{\operatorname{tr}(u_i)} = \overline{\operatorname{tr}(u)}.$$

(e) Consider any $v \in \mathscr{F}_1$. Then $0 \neq vv^* \in \mathscr{F}_1$ and for any $x \in \mathscr{A}$ with $vv^*x^*x \neq 0$ we have $vv^*x^*x \in \mathscr{F}_1$ by Lemma 3.8.
Therefore,

$$\operatorname{tr}(vv^*x^*x)vv^*x^*x = (vv^*x^*xvv^*)x^*x = \langle \Gamma_{vv^*}, x^*x\rangle vv^*x^*x$$

which implies $\operatorname{tr}(vv^*x^*x) = \langle r_{vv^*}, x^*x\rangle$. If $vv^*x^*x = 0$, there is nothing to prove.

Now, let $a \in \mathcal{M}$ be any positive element. Then by Corollary 5.7

$$a = x^*x \quad \text{with} \quad x \in \mathcal{M}.$$

Since $vv^*x^* \in \mathcal{F}_1$ and $xvv^*x^* = xv(xv)^* \in \mathcal{F}_1$ if $vv^*x^* \neq 0$ and $xvv^*x^* \neq 0$ by Lemma 3.8, it follows by (b) and Proposition 3.25 that

$$\langle r_{vv^*}, a\rangle = \langle r_{vv^*}, x^*x\rangle = \operatorname{tr}(vv^*x^*x) = \operatorname{tr}(xvv^*x^*) = \operatorname{tr}(xv(xv)^*) \in \sigma_{\mathcal{M}}[xv(xv)^*].$$

Since $(xv)(xv)^* \geq 0$ according to Lemma 5.8, it follows that

$$\langle r_{vv^*}, a\rangle \in \sigma_{\mathcal{M}}[xv(xv)^*] \subseteq [0,\infty).$$

If $vv^*x^* = 0$ or $xvv^*x^* = 0$, it is clear that $\langle r_{vv^*}, a\rangle = 0$. Therefore,

$$\langle r_{vv^*}, a\rangle \geq 0 \quad \text{for all} \quad a \geq 0. \qquad \square$$

6. NUCLEAR ELEMENTS

The definition of nuclear operators on a Banach space E gave rise to that of nuclear elements in a Banach algebra. In Theorem 6.4 we show that the definition given by J Puhl [10] implies that of K Vala [14], but even for C^*-algebras the converse is not true. It will be shown that if the Banach algebra fulfils certain conditions, then the trace admits an extension to the nuclear elements. Three such conditions are needed in the proofs of Theorems 6.6, 6.8 and 6.11. It is also shown in Proposition 6.5 that the spectrum of a nuclear element is countable at most, which can accumulate only in the origin.

<u>Definition 6.1</u>

Let $\mathcal{A}$ be a Banach algebra. An element $u \in \mathcal{A}$ is called nuclear if $u = \sum_{i=1}^{\infty} u_i$, $u_i \in \mathcal{F}_1$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. The class of all nuclear elements is denoted by $\mathcal{N}$. We define

$$\nu(u) := \inf \sum_{i=1}^{\infty} \|u_i\|$$

where the infimum is taken over all nuclear representations.

<u>Theorem 6.2</u> ([10], p 664)

$\mathcal{N}$ is a bi-ideal of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{N}$ and ν is a norm on $\mathcal{N}$ such that if $x,y \in \mathcal{A}$, $u \in \mathcal{N}$ then

$$\nu(xuy) \leq \|x\|\nu(u)\|y\|.$$

Moreover, $\mathcal{N}$ is complete with respect to this norm.

<u>Proof</u>

Clearly $\mathcal{N}$ is a subalgebra of $\mathcal{A}$, since for any $u = \sum_{i=1}^{\infty} u_i$, $v = \sum_{j=1}^{\infty} v_j \in \mathcal{N}$ we have $uv = \sum_{i,j=1}^{\infty} u_i v_j$ with $u_i v_j \in \mathcal{F}_1$ if $u_i v_j \neq 0$ by Lemma 3.8 and $\sum_{i,j=1}^{\infty} \|u_i v_j\| \leq (\sum_{i=1}^{\infty} \|u_i\|)(\sum_{j=1}^{\infty} \|v_j\|) < \infty$. This implies that $uv \in \mathcal{N}$ if $u_i v_j \neq 0$ for some i and j. If $u_i v_j = 0$ for all $i,j \in \mathbb{N}$, then clearly $uv \in \mathcal{F} \subseteq \mathcal{N}$. It is also easy to check that $\mathcal{N}$ is a linear subspace.

We show that υ defines a norm on $\mathcal{N}$.

Let $u = \sum_{i=1}^{\infty} u_i$ be any nuclear representation such that $\upsilon(u) = 0$. Then

$$\|u\| = \|\lim_{n\to\infty} \sum_{i=1}^{n} u_i\| \leq \lim_{n\to\infty} \sum_{i=1}^{n} \|u_i\| = \sum_{i=1}^{\infty} \|u_i\|$$

from which it follows that $\|u\| \leq \upsilon(u) = 0$.

Hence, $u = 0$.

Conversely, if $u = \sum_{i=1}^{\infty} u_i = 0$, then obviously $\upsilon(u) \leq 0$ which implies $\upsilon(u) = 0$.

Clearly, if $\lambda \in \mathbb{C}$, $u \in \mathcal{N}$ then $\lambda u \in \mathcal{N}$ and $\upsilon(\lambda u) = |\lambda|\upsilon(u)$. Let $u = \sum_{i=1}^{\infty} u_i$, $w = \sum_{j=1}^{\infty} w_j \in \mathcal{N}$. Then

$\upsilon(u + w) \leq \sum_{i=1}^{\infty} \|u_i\| + \sum_{j=1}^{\infty} \|w_j\|$ for all such representations of u and w.

Fix the representation $w = \sum_{i=1}^{\infty} w_i$. Then $\upsilon(u+w) \leq \upsilon(u) + \sum_{j=1}^{\infty} \|w_j\|$.

Since this is true for each such representation of w, we have $\upsilon(u+w) \leq \upsilon(u) + \upsilon(w)$.

Therefore υ is a norm on $\mathcal{N}$.

Now, if $x,y \in \mathcal{M}$, $u \in \mathcal{N}$ then clearly $xuy \in \mathcal{N}$ since $\mathcal{N}$ is a bi-ideal and

$$\upsilon(xuy) \leq \sum_{i=1}^{\infty} \|xu_i y\| \leq \|x\|\upsilon(u)\|y\|.$$

Finally we prove that $\mathcal{N}$ is complete with respect to υ. Let $(u_i) \subseteq \mathcal{N}$ be any υ-Cauchy sequence in $\mathcal{N}$. Since $\|\cdot\| \leq \upsilon(\cdot)$, it follows that

$$(u_i) \text{ is a } \|\cdot\|\text{-Cauchy sequence in } \mathcal{M}.$$

Therefore, there exists $u \in \mathcal{M}$ such that $\|u - u_n\| \xrightarrow[\infty]{n} 0$. We show that $u \in \mathcal{N}$ and $\upsilon(u - u_n) \xrightarrow[\infty]{n} 0$. Choose an increasing sequence (n_k) of natural numbers such that

$$\upsilon(u_n - u_m) < \frac{1}{2^{k+2}} \text{ for all } n,m \geq n_k.$$

Then for the subsequence $(u_{n_k})_{k=1}^{\infty} \subseteq \mathcal{N}$ we have

$$\upsilon(u_{n_{k+1}} - u_{n_k}) \leq \frac{1}{2^{k+2}} \text{ for each } k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$ we choose a representation

$$u_{n_{k+1}} - u_{n_k} = \sum_{i=1}^{\infty} v_i^{(k)} \quad \text{with} \quad \sum_{i=1}^{\infty} \|v_i^{(k)}\| < \frac{1}{2^{k+2}}, \quad v_i^{(k)} \in \mathcal{F}_1.$$

Then for all $p \in \mathbb{N}$

$$\begin{aligned} u_{n_{k+p}} - u_{n_k} &= (u_{n_{k+p}} - u_{n_{k+p-1}}) + (u_{n_{k+p-1}} - u_{n_{k+p-2}}) + \ldots + (u_{n_{k+1}} - u_{n_k}) \\ &= \sum_{i=1}^{\infty} v_i^{(k+p-1)} + \sum_{i=1}^{\infty} v_i^{(k+p-2)} + \ldots + \sum_{i=1}^{\infty} v_i^{(k)} \\ &= \sum_{j=k}^{k+p-1} \Big(\sum_{i=1}^{\infty} v_i^{(j)}\Big). \end{aligned}$$

Now let $p \longrightarrow \infty$ and consider the norm limit u. Then for each $k \in \mathbb{N}$,

$$u - u_{n_k} = \|\cdot\|\lim_{p\to\infty} (u_{n_{k+p}} - u_{n_k}) = \sum_{j=k}^{\infty} \sum_{i=1}^{\infty} v_i^{(j)}$$

and

$$\sum_{j=k}^{\infty} \sum_{i=1}^{\infty} \|v_i^{(j)}\| < \sum_{j=k}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{2^k}\left[\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right] \leq \frac{1}{2^{k+1}}$$

which implies that $(u - u_{n_k}) \in \mathcal{N}$. Since $\mathcal{N}$ is a linear subspace and $u_{n_k} \in \mathcal{N}$ for each $k \in N$, it follows that $u \in \mathcal{N}$ and furthermore

$$\nu(u - u_{n_k}) \leq \sum_{j=k}^{\infty} \sum_{i=1}^{\infty} \|v_i^{(j)}\| \leq \frac{1}{2^{k+1}} \xrightarrow[\infty]{k} 0.$$

Hence,

$$\nu(\cdot)\lim_{n\to\infty} u_n = u \in \mathcal{N}. \qquad \square$$

<u>Example 6.3</u> ([10], p 664)

(a) *If* $\mathcal{M} = \mathcal{L}(E)$, *then* $u \in \mathcal{N}$ *if and only if* $u \in \mathcal{N}(E)$.

(b) If $\mathcal{M} = \ell_\infty$, *then* $\mathcal{N} = \ell_1$.

<u>Proof</u>

(a) The result follows directly by Corollary 2.10 which states that U_i is a one-dimensional element of $\mathcal{L}(E)$ if and only if U_i is an operator of rank one on the Banach space E.

(b) Let $x = \sum_{i=1}^{\infty} x_i$, $x_i \in \ell_\infty$ one-dimensional with $\sum_{i=1}^{\infty} \|x_i\|_\infty < \infty$. By Example 3.20 it follows that each one-dimensional element x_i of ℓ_∞ is of the form

$$x_i(n) = \begin{cases} \alpha_n & \text{if} \quad n = i \\ 0 & \text{if} \quad n \neq i \end{cases}$$

by putting $K = \mathbb{N}$, and $\|x_i\|_\infty = |\alpha_i|$, $i \in \mathbb{N}$. Therefore,

$$x = (x(n)) = \left(\sum_{i=1}^{\infty} x_i(n)\right)_{n=1}^{\infty}$$

and $\sum_{n=1}^{\infty} |x(n)| \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |x_i(n)| = \sum_{n=1}^{\infty} |\alpha_n| < \infty$

which implies that $x \in \ell_1$.

Conversely, let $x = (x(n)) \in \ell_1 \subseteq \ell_\infty$.

Then $x = \sum_{n=1}^{\infty} x(n)e_n$ with $e_n = (0,\ldots,0,1,0,\ldots)$.

Clearly each $x(n)e_n \in \ell_\infty$ is one-dimensional and

$$\sum_{n=1}^{\infty} \|x(n)e_n\|_\infty = \sum_{n=1}^{\infty} |x(n)| < \infty.$$

Hence, $x \in \ell_\infty$ is nuclear. □

Next we show that Theorem 4.23 for finite elements does not necessarily hold for nuclear elements.

<u>Theorem 6.4</u> ([10], p 664)

If $u \in \mathcal{N}$, *then* $u \wedge u : x \longrightarrow uxu$ *is a nuclear operator, but even for* C^**-algebras, the converse doesn't hold.*

<u>Proof</u>

Choose a nuclear representation $u = \sum_{i=1}^{\infty} u_i$, $u_i \in \mathcal{F}_1$, $\sum_{i=1}^{\infty} \|u_i\| < \infty$. Then

for all $x \in \mathcal{A}$

$$u \wedge u(x) = \sum_{i,j=1}^{\infty} u_i x u_j = \sum_{i,j=1}^{\infty} \langle a_{ij}, x\rangle y_{ij} = \left(\sum_{i,j=1}^{\infty} a_{ij} \otimes y_{ij}\right)x \quad \text{with}$$

$a_{ij} \in \mathcal{A}^*$, $y_{ij} \in \mathcal{A}$. By Lemma 4.15 each operator $(a_{ij} \otimes y_{ij})$ is of rank one with $\|a_{ij}\|\|y_{ij}\| \leq \|u_i\|\|y_j\|$ from which it follows that

$$\nu(u \wedge u) \leq \sum_{i,j=1}^{\infty} \|u_i\|\|u_j\| = \left[\sum_{i=1}^{\infty} \|u_i\|\right]^2 < \infty.$$

Hence, $u \wedge u$ is a nuclear operator on $\mathcal{A}$.

If we consider the algebra $\mathcal{A} = \ell_\infty$ and the sequence $u := (\frac{1}{n})$, it immediately follows that $u \in \ell_\infty$, but $u \notin \ell_1$, i.e. $u \notin \mathcal{N}$. However, it is clear that the wedge operator on ℓ_∞ is nuclear, since for any $x = (x(n)) \in \ell_\infty$ we have $u \wedge u(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x(n) e_n = \sum_{n=1}^{\infty} (\frac{1}{n^2} e_n \otimes e_n)(x)$ with $\sum_{n=1}^{\infty} \|\frac{1}{n^2} e_n\| \|e_n\| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Hence $u \wedge u$ is a nuclear operator whereas u is not a nuclear element. □

Proposition 6.5 ([10], p 665)

Let $\mathcal{A}$ *be a Banach algebra.*

If $\mathcal{A}$ *has an identity, then for every* $u \in \mathcal{N}$, $\sigma_{\mathcal{A}}(u)$ *is either finite or countable and has no non-zero point of accumulation. If* $\mathcal{A}$ *doesn't have an identity and we consider* $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$, *then the same result is true for every nuclear element in* $\mathcal{A}_1$.

Proof

Let $\mathcal{A}$ have an identity and consider any nuclear representation $u = \sum_{i=1}^{\infty} u_i$, $u_i \in \mathcal{F}_1$, $\sum_{i=1}^{\infty} \|u_i\| < \infty$.

Since Y_u is a closed subalgebra of $\mathcal{A}$, D_u is a nuclear operator on Y_u, by applying the same argument as in the proof of Theorem 6.4. By Proposition 2.49 $\sigma_{\mathcal{L}(Y_u)}(D_u)$ is at most countable which can accumulate only in the origin. The result then follows by Proposition 4.19. □

We are now going to investigate conditions under which the trace on $\mathcal{F}$ admits an extension to the nuclear elements.

Theorem 6.6 ([10], p 665)

If $\mathcal{A}$ *is a semi-prime Banach algebra, having the approximation property, then every* $u \in \mathcal{N}$ *has a well defined trace.*

Proof

Let $\sum_{i=1}^{\infty} u_i = 0$, $u_i \in \mathcal{F}$, and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. We prove that $\sum_{i=1}^{\infty} tr(u_i) = 0$. Let $A_1, \ldots, A_s$ have the same meaning as in the proof of Lemma 5.1. For each fixed k, put

$$D_k x := \sum_{i,j\in A_k} u_i x u_j \quad \text{for all } x \in \mathcal{A}.$$

By Lemma 4.15 D_k is a nuclear operator on $\mathcal{A}$ and since $\mathcal{A}$ has the approximation property, it follows by Definition 2.43 that D_k has a well defined trace given by

$$tr(D_k) = \sum_{i,j\in A_k} tr(u_i)tr(u_j) = \left[\sum_{i\in A_k} tr(u_i)\right]^2$$

But since

$$D_k x = \left(\sum_{i=1}^{\infty} u_i\right) x \left(\sum_{j\in A_k} u_j\right) = 0$$

it follows that $tr(D_k) = 0$.

Therefore, $\sum_{i\in A_k} tr(u_i) = 0$ for each fixed $k=1,\ldots,s$ which implies that

$$\sum_{i=1}^{\infty} tr(u_i) = \sum_{k=1}^{s} \left[\sum_{i\in A_k} tr(u_i)\right] = 0. \qquad \square$$

Another property we are going to investigate is the following.

Property 6.7

Let $\mathcal{A}$ be a semi-prime Banach algebra such that for each $u \in \mathcal{F}$ and $\epsilon > 0$, there exists $x \in \mathcal{F}$ with $\|x\| \leq 1 + \epsilon$ and $xu = u$ or $ux = u$.

Theorem 6.8 ([10], p 665)

If $\mathcal{A}$ posseses Property 6.7, then every $u \in \mathcal{N}$ has a well defined trace.

Proof

We first show that the trace is a continuous linear mapping on $\mathcal{F}$. If $v \in \mathcal{F}$ and $\epsilon > 0$, it follows by Property 6.7 that there exists $x \in \mathcal{F}$ with $\|x\| \leq 1 + \epsilon$ such that $xv = v$. Choose a nuclear representation

$v = \sum_{i=1}^{\infty} v_i$, $v_i \in \mathcal{F}_1$ such that

$$\sum_{i=1}^{\infty} \|v_i\| \leq \upsilon(v) + \epsilon.$$

Then by Lemma 3.8 and Theorem 4.11(a) it follows that

$$|tr(v)| = |tr(xv)| = |\sum_{i=1}^{n} tr(x_i v)| = |\sum_{i=1}^{n} \langle \Gamma_{x_i}, v\rangle| = |\sum_{i=1}^{n} \langle \Gamma_{x_i}, \lim_{k\to\infty} \sum_{j=1}^{k} v_j\rangle|$$

$$= |\sum_{i=1}^{n} \lim_{k\to\infty} \sum_{j=1}^{k} \langle \Gamma_{x_i}, v_j\rangle| = |\lim_{k\to\infty} \sum_{j=1}^{k} \sum_{i=1}^{n} tr(x_i v_j)|$$

$$= \lim_{k\to\infty} |\sum_{j=1}^{k} tr(\sum_{i=1}^{n} x_i v_j)|$$

$$\leq \sum_{j=1}^{\infty} |tr(xv_j)| \leq \sum_{j=1}^{\infty} \|xv_j\|$$

$$\leq (\sum_{j=1}^{\infty} \|v_j\|)\|x\| \leq (\upsilon(v) + \epsilon)(1 + \epsilon) \longrightarrow \upsilon(v) < \infty$$

since $\epsilon > 0$ is arbitrary.

Therefore, by Hahn Banach the trace allows a unique extension to the nuclear elements.

Furthermore,

if $u = \sum_{i=1}^{\infty} u_i$, $u_i \in \mathcal{F}_1$ with $\sum_{i=1}^{\infty} \| u_i\| < \infty$ is any nuclear

representation, then

$$tr(u) = tr(\lim_{n\to\infty} \sum_{i=1}^{n} u_i) = \lim_{n\to\infty} \sum_{i=1}^{n} tr(u_i) = \sum_{i=1}^{\infty} tr(u_i). \qquad \square$$

<u>Proposition 6.9</u> ([9], p 131)

Let E *be a Banach space and* $\mathcal{M} = \mathcal{L}(E)$.

If E *or* E^* *has the m.a.p., then* $\mathcal{M}$ *satisfies Property* 6.7.

<u>Proof</u>

Consider any $U \in \mathcal{F}(E)$ and $\epsilon > 0$.

Since $\dim[U(E)] < \infty$ and E has the m.a.p. we can find $X \in \mathcal{F}(E)$ with $\|X\| \leq 1 + \epsilon$ and $Xy = y$ for all $y \in U(E)$ by Lemma 2.39.

Therefore,

$XUx = Ux$ for all $x \in E$ which implies that $XU = U$.

Similarly by assuming E^* to have the m.a.p. we can show that $UX = U$. $\square$

Finally we are going to investigate the quasi-approximation-property of a semi-prime Banach algebra, given below, under which the trace admits an extension to $\mathcal{N}$.

Definition 6.10

A semi-prime Banach algebra $\mathcal{A}$ is said to have the quasi-approximation-property (q.a.p.) if for each minimal idempotent $q \in \mathcal{A}$, the Banach space $\mathcal{A}q$ (resp. $q\mathcal{A}$) has the approximation property.

Theorem 6.11 ([10], p 667)

Suppose $\mathcal{A}$ has the q.a.p.

(a) *Then every $u \in \mathcal{N}$ has a well defined trace.*

(b) *If $u \in \mathcal{N}$ is nilpotent, then $\operatorname{tr}(u) = 0$.*

Proof

Without loss of generality we can assume that $\mathcal{A}q$ has the approximation property for each minimal idempotent $q \in \mathcal{A}$.

(a) Consider any $u \in \mathcal{N}$ and choose a nuclear representation $u = \sum_{i=1}^{\infty} u_i$ with $\sum_{i=1}^{\infty} \|u_i\| < \infty$.

Since $|\operatorname{tr}(u_i)|\|u_i\| = \|\operatorname{tr}(u_i)u_i\| = \|(u_i)^2\| \leq \|u_i\|^2$ for each $i \in \mathbb{N}$, it follows that $\sum_{i=1}^{n} |\operatorname{tr}(u_i)| \leq \sum_{i=1}^{n} \|u_i\| \leq \sum_{i=1}^{\infty} \|u_i\| < \infty$ for each $n \in \mathbb{N}$.

Consequently,

$$\sum_{i=1}^{\infty} |\operatorname{tr}(u_i)| = \lim_{n\to\infty} \sum_{i=1}^{n} |\operatorname{tr}(u_i)| < \infty$$

which implies that $\sum_{i=1}^{\infty} \operatorname{tr}(u_i)$ is absolute convergent.

As in the proof of Theorem 6.6 we show that if $\sum_{i=1}^{\infty} v_i = 0$, $v_i \in \mathcal{F}_1$ with $\sum_{i=1}^{\infty} \|v_i\| < \infty$, it follows that $\sum_{i=1}^{\infty} \operatorname{tr}(v_i) = 0$.

As before, let $A_1,\dots,A_s$ be a disjoint decomposition of $\{1,2,\dots\}$ induced by $\sim$ on $\mathcal{F}_1$. For each fixed k, choose a minimal idempotent $q_k \in \mathcal{F}_1$ such that $q_k \sim v_i$ for each $i \in A_k$. Define an element $w_k \in \mathcal{A}$

and an operator $L_k \in \mathcal{L}(\mathcal{M}_{q_k})$ by

$$w_k := \sum_{i \in A_k} v_i, \quad L_k x := w_k x.$$

Clearly $w_k \in \mathcal{N}$ and we show that $L_k \in \mathcal{N}(\mathcal{M}q_k)$.

Let $x = mq_k$ for some $m \in \mathcal{M}$. Then it follows by Lemma 4.15 that for each $i \in A_k$

$D_{v_i,q_k} m := v_i m q_k$ is an operator of rank one on $\mathcal{M}$ with

$\|D_{v_i,q_k}\| \leq \|v_i\|\|q_k\|$ and $tr(D_{v_i,q_k}) = tr(v_i)tr(q_k) = tr(v_i)$ since $q_k^2 = q_k$.

Consequently,

$$L_k x = L_k(mq_k) := \sum_{i \in A_k} v_i m q_k = \sum_{i \in A_k} D_{v_i,q_k} m$$

with $\sum_{i \in A_k} \|D_{v_i,q_k}\| \leq \|q_k\| \sum_{i \in A_k} \|v_i\| < \infty$

which implies that L_k is a nuclear operator on $\mathcal{M}q_k$. Since $\mathcal{M}q_k$ has the approximation property, it follows by Definition 2.43 that L_k has a well defined trace given by

$$tr(L_k) = \sum_{i \in A_k} tr(D_{v_i,q_k}) = \sum_{i \in A_k} tr(v_i). \tag{6.1}$$

On the other hand, for all $x \in \mathcal{M}q_k$,

$$0 = (\sum_{i=1}^{\infty} v_i)xw_k = \sum_{i,j \in A_k} v_i x v_j = (\sum_{i \in A_k} v_i)x(\sum_{j \in A_k} v_j) = w_k x w_k.$$

Since $\mathcal{M}$ is semi-prime, it follows that $w_k = 0$, hence $L_k = 0$ which implies that $tr(L_k) = 0$.

Therefore, we have $\sum_{i \in A_k} tr(v_i) = 0$ for each fixed $k=1,\ldots,s$ from which it follows that

$$\sum_{i=1}^{\infty} tr(v_i) = \sum_{k=1}^{s} (\sum_{i \in A_k} tr(v_i)) = 0.$$

(b) If $u \in \mathcal{N}$ is nilpotent, then clearly $L_k \in \mathcal{N}(\mathcal{M}q_k)$ is a nilpotent operator and by Corollary 2.23 we have

$$tr(L_k) = 0 \quad \text{for each fixed } k=1,\ldots,s.$$

But if we choose a nuclear representation $u = \sum_{i=1}^{\infty} u_i,\ u_i \in \mathcal{F}_1$, $\sum_{i=1}^{\infty} \|u_i\| < \infty$ and put

$$w_k := \sum_{i \in A_k} u_i,\ L_k x := w_k x,\ x \in \mathcal{M} q_k,$$

it follows by (6.1) that

$$tr(L_k) = \sum_{i \in A_k} tr(u_i) \quad \text{for each fixed } k.$$

By using (a) we have

$$tr(u) = \sum_{i=1}^{\infty} tr(u_i) = \sum_{k=1}^{s} \left(\sum_{i \in A_k} tr(u_i)\right) = \sum_{k=1}^{s} tr(L_k) = 0. \qquad \square$$

Finally we investigate the conditions under which an algebra has q.a.p.

<u>Lemma 6.12</u> ([13], p 7)

Let $\mathcal{M}$ *be a* C^**-algebra.*

If $f : \mathcal{M} \longrightarrow \mathbb{C}$ *is a positive functional on* $\mathcal{M}$, *then*

$$\langle f, y^* x \rangle = \overline{\langle f, x^* y \rangle}$$

<u>Proof</u>

Consider the following polarisations:

$$4y^* x = (x+y)^*(x+y) + i(x+iy)^*(x+iy) - (x-y)^*(x-y) - i(x-iy)^*(x-iy)$$
$$4x^* y = 4(y^* x)^* = (x+y)^*(x+y) - i(x+iy)^*(x+iy) - (x-y)^*(x-y) + i(x-iy)^*(x-y)$$

Then by Lemma 5.8

$$(x \pm y)^*(x \pm y) \geq 0 \quad \text{and} \quad (x \pm iy)^*(x \pm iy) \geq 0.$$

Consequently,

$$0 \leq \langle f, (x \pm y)^*(x \pm y) \rangle \in \mathbb{R} \quad \text{and} \quad 0 \leq \langle f, (x \pm iy)^*(x \pm iy) \rangle \in \mathbb{R}$$

and by making use of the polarisations it follows directly that

$$4\overline{\langle f, x^* y \rangle} = 4\langle f, y^* x \rangle. \qquad \square$$

<u>Lemma 6.13</u>

Let $\mathcal{M}$ *be a* C^**-algebra.*

If $x \in \mathcal{M}$ *is selfadjoint, then* $\|x^n\| = \|x\|^n$ *for all* $n \in \mathbb{N}$.

Proof

Clearly $\|x^{2^n}\| = \|x\|^{2^n}$, $n \in \mathbb{N}$. Therefore,

$$\|x\|^4 = \|x^4\| \leq \|x^3\|\|x\| \leq \|x\|^3\|x\| = \|x\|^4$$

which implies that $\|x^3\| = \|x\|^3$.

By continuing in this way, the result follows directly. □

Theorem 6.14 ([10], p 666)

(a) *Let* $\mathcal{M} = \mathcal{L}(E)$. *Then*

$\mathcal{M}$ *possesses q.a.p. if and only if* E *or* E^* *has the approximation property.*

(b) *A commutative Banach algebra* $\mathcal{M}$ *possesses q.a.p.*

(c) *A* C^**-algebra* $\mathcal{M}$ *has the q.a.p.*

Proof

(a) Consider any minimal idempotent $Q \in \mathcal{M}$.

We show that $\mathcal{M}Q \cong E$ (resp. $Q\mathcal{M} \cong E^*$).

Clearly Q is a projection of rank one by Gelfand Mazur and Theorem 2.9.

Put $Q(E) := E_1$. Choose $0 \neq y_0 \in E_1$ with $\|y_0\| = 1$ such that

$E_1 = \overline{\text{span}\{y_0\}}$. Define a mapping $\Phi: \mathcal{M}Q \longrightarrow E$ by

$$\Phi(TQ) := Ty_0, \quad T \in \mathcal{M}.$$

Clearly Φ is well defined since for any $T,S \in \mathcal{M}$ with $TQ = SQ$ we have $TQy_0 = SQy_0$, i.e. $Ty_0 = Sy_0$. Φ is also linear and isometric, since

$$\|TQ\| = \sup_{\|x\|\leq 1} \|T(Qx)\| = \sup_{|\lambda_x|\leq 1} \|T(\lambda_x y_0)\| = \sup_{|\lambda_x|\leq 1} |\lambda_x|\|Ty_0\|$$

$$= \|Ty_0\| = \|\Phi(TQ)\|.$$

This implies that Φ is one to one. We still have to show that Φ is onto. Let $x \in E$ be any fixed element and define a mapping

$$T_x y_0 := x.$$

Extend T_x linearly to E_1 by $T_x(\lambda y_0) = \lambda_x$. Obviously T_x is linear and bounded since

$$\|T_x\| = \sup_{|\lambda|\leq 1} \|T_x(\lambda y_0)\| = \sup_{|\lambda|\leq 1} |\lambda|\|x\| = \|x\|.$$

By Hahn Banach we can extend T_x to E, i.e. $T_x \in \mathcal{M}$. Furthermore,

$$\Phi(T_x Q) = T_x y_0 = x.$$

(b) Let $\mathcal{A}$ be a commutative Banach algebra and $q \in \mathcal{A}$ minimal idempotent. Then $\mathcal{A}q$ is isomorphic to $\mathbb{C}$ by Proposition 3.12 and consequently has the approximation property.

(c) Let $\mathcal{A}$ be a C^*-algebra and $q \in \mathcal{A}$ minimal idempotent. We show that $\mathcal{A}q$ is a Hilbert space. Put

$$(x,y) := \langle \Gamma_{qq^*}, y^*x \rangle, \quad x,y \in \mathcal{A}q.$$

Clearly (x,y) is well defined and we show that it defines an inner product on $\mathcal{A}q$. By Theorem 5.9(e) Γ_{qq^*} is a positive functional on $\mathcal{A}$ and by Lemma 5.8 we have $(x,x) = \langle \Gamma_{qq^*}, x^*x \rangle \geq 0$. By using Lemma 6.12 it follows that $(x,y) = \overline{(y,x)}$. It is also obvious that $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$.

We now prove that $(x,x)^{\frac{1}{2}} = \|x\|$. Let $x = mq$, $m \in \mathcal{A}$. Then

$$\|xqq^*\|^2 = \|qq^*x^*xqq^*\| = \langle \Gamma_{qq^*}, x^*x \rangle \|qq^*\| = (x,x)\|qq^*\| = (x,x)\|q\|^2$$

and by using Lemma 6.13 we have

$$\begin{aligned}(|\langle \Gamma_q, q^* \rangle| \|q\|)^2 &= (\|qq^*q\|)^2 = \|(qq^*q)^*(qq^*q)\| \\ &= \|q^*qq^*qq^*q\| \\ &= \|q^*q\|^3 \\ &= \|q\|^6.\end{aligned}$$

Hence $|\langle \Gamma_q, q^* \rangle| = \|q\|^2$ from which it follows that

$$\frac{\|xqq^*\|}{\|q\|} \leq \|xq\| = \frac{\|xqq^*q\|}{|\langle \Gamma_q, q^* \rangle|} \leq \frac{\|xqq^*\|}{\|q\|},$$

i.e. $\|xqq^*\| = \|xq\|\|q\|$.

Consequently, $(x,x) = \dfrac{\|xq\|^2\|q\|^2}{\|q\|^2} = \|(mq)q\|^2 = \|x\|^2$, hence $\mathcal{A}q$ is an inner product space. Since $\mathcal{A}q$ is complete, it follows by Proposition 2.36 that $\mathcal{A}q$ has the approximation property. □

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THE TRACE OF NUCLEAR ELEMENTS IN BANACH ALGEBRAS

by

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SUMMARY

The classical Ascoli's theorem has proved to be of great interest to many mathematicians and has been the object of many modifications and generalisations.

K Vala [14] studied compact and finite elements in a Banach algebra, giving a definition which generalises a theorem in operator theory which states that the mapping $:T \longrightarrow ATC$ on the Banach algebra of operators on a Banach space E is compact (of finite rank), if and only if both mappings A and C are compact (finite rank) operators on E. In this paper a different definition for finite (in particular one-dimensional) elements in a Banach algebra, due to J Puhl [10], is given, generalising the following theorems in operator theory:

(i) An operator $T \neq 0$ on a Banach space E is of rank one if and only if there exists a non-zero functional r_T on the Banach algebra of operators on E such that $TRT = \langle r_T, R \rangle T$ for all operators R.

(ii) T is of finite rank if and only if it can be written as a finite sum of operators of rank one. It is shown that the two different definitions for finite elements, given by Vala and Puhl respectively, coincide.

Since most of the results throughout the paper require the Banach algebra to be semi-prime, a condition which is equivalent for this concept is proved. A well defined trace for one-dimensional elements is introduced,

provided the Banach algebra is semi-prime. The trace of finite elements is also defined and the results are analogous to those of finite rank operators.

Furthermore, the spectrum of a one-dimensional element is shown to consist of exactly two elements and that of a finite element to be finite, by using the same result which is proved to be valid for finite rank operators on a Banach space E.

We also prove that if the Banach algebra is semi-prime, the one-dimensional elements and the minimal left (right) ideals are in one to one correspondence. Furthermore, the socle of a semi-prime algebra always exists and equals the class of all finite elements.

Nuclear elements are defined in a natural way and a well defined nuclear norm is introduced, which dominates the norm on the Banach algebra. It is shown that if the Banach algebra fulfils certain conditions, the trace can be extended to these elements.

However, it is shown that the definition for nuclear elements, given by Vala, implies that of Puhl, but the converse is not necessarily true (even in C^*-algebras). The spectrum of a nuclear element is shown to be at most countable, with zero the only point of accumulation.

THE TRACE OF NUCLEAR ELEMENTS IN BANACH ALGEBRAS

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SAMEVATTING

Die klassieke stelling van Ascoli het tot dusver groot belangstelling deur baie wiskundiges geniet en was die onderwerp van verskeie wysigings en veralgemenings.

K Vala [14] het 'n studie gemaak van kompakte en eindige elemente in 'n Banach algebra en 'n definisie gegee wat 'n veralgemening is van die stelling wat beweer dat die afbeelding $:T \longrightarrow ATC$ op die Banach algebra van operatore op 'n Banach-ruimte E kompak (van eindige rang) is, as en slegs as beide afbeeldings A en C kompakte (eindige rang) operatore op E is. In hierdie verhandeling word 'n ander definisie vir eindige (in die besonder een-dimensionele) elemente in 'n Banach algebra, geinisieer deur J Puhl [10], gegee, en wat veralgemenings is van die volgende stellings:

(i) 'n Operator $0 \neq T$ op 'n Banach-ruimte E is van rang een as en slegs as daar 'n nie-nul funksionaal r_T op die Banach algebra van operatore op E bestaan so dat $TRT = \langle r_T, R\rangle T$ vir alle operatore R.

(ii) T is van eindige rang as en slegs as dit geskryf kan word as 'n eindige som van operatore van rang een.

Daar word bewys dat die twee verskillende definisies vir eindige elemente, deur Vala en Puhl onderskeidelik gegee, harmonieer.

Aangesien die oorgrote meerderheid resultate wat in die verhandeling bewys word, vereis dat die Banach algebra semi-priem is, word 'n ekwivalente voorwaarde vir hierdie begrip bewys. 'n Goed gedefinieerde spoor vir

een-dimensionele elemente word gegee, mits die Banach algebra semi-priem is. Ook word die spoor van eindige elemente gedefinieer en die resultate is analoog aan die van eindige rang operatore.

Verder word aangetoon dat die spektrum van 'n een-dimensionele element uit presies twee elemente bestaan en die van 'n eindige element eindig is, deur gebruikmaking van dieselfde resultaat wat vir eindige rang operatore bewys word.

Ons bewys ook dat as die Banach algebra semi-priem is, daar 'n een-een-duidige verband bestaan tussen die een-dimensionele elemente en minimale linkse (regse) ideale. Verder bestaan die voetstuk van 'n semi-priem algebra altyd en is gelyk aan die klas van alle eindige elemente.

Nukleere elemente word op natuurlike wyse gedefinieer en 'n goed gedefinieerde nukleere norm, wat die norm op die Banach algebra domineer, word gegee. Daar word bewys dat as die Banach algebra aan sekere voorwaardes voldoen, die spoor uitgebrei kan word na hierdie elemente.

Daar word egter bewys dat die definisie van Vala vir nukleere elemente die van Puhl impliseer, maar dat die omgekeerde nie noodwendig geld nie (selfs in C^*-algebras). Ons bewys dat die spektrum van 'n nukleere elemente hoogstens aftelbaar is, met nul as enigste verdigtingspunt.

THE TRACE OF NUCLEAR ELEMENTS IN BANACH ALGEBRAS

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SUMMARY

K Vala defined compact (finite) elements in a Banach algebra. A different definition for finite elements in a Banach algebra is given in this treatise and we show that the two different definitions coincide. A well defined trace for finite elements is introduced, provided the Banach algebra is semi-prime. The one-dimensional elements of a semi-prime Banach algebra are shown to be in one to one correspondence with the minimal left (right) ideals. Furthermore, it is shown that the socle of a semi-prime algebra equals the class of all finite elements.

Nuclear elements are then defined, and if the Banach algebra fulfils certain conditions, the trace can be extended to these elements. However, for nuclear elements, the definition given by Vala implies the latter definition, but the converse does not necessarily hold.

THE TRACE OF NUCLEAR ELEMENTS IN BANACH ALGEBRAS

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SAMEVATTING

K Vala het kompakte (eindige) elemente in 'n Banach algebra gedefinieer. 'n Ander definisie vir eindige elemente in 'n Banach algebra word in die verhandeling gegee en ons toon aan dat die twee verskillende definisies harmonieer. 'n Goed gedefinieerde spoor vir eindige elemente word gegee, mits die Banach algebra semi-priem is. Daar word aangetoon dat die een-dimensionele elemente van 'n semi-priem Banach algebra in een-eenduidige verband met die minimale linkse (regse) ideale is. Verder word aangetoon dat die voetstuk van 'n semi-priem algebra gelyk is aan die klas van alle eindige elemente.

Nukleêre elemente word dan gedefinieer, en as die Banach algebra aan sekere voorwaardes voldoen, kan die spoor uitgebrei word na hierdie elemente. Die definisie gegee deur Vala vir nukleêre elemente impliseer laasgenoemde definisie, maar die omgekeerde is nie noodwendig waar nie.