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TILING PROBLEMS AND LOGICAL UNSOLVABILITY

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Tiling problems and logical unsolvability

by
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Teëlingsprobleme en logiese onoplosbaarheid

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Opsomming

Die ideaal om 'n universele logiese taal en deduksiereëls te ontwerp waarin enige probleem gestel en opgelos kan word is ten minste so oud soos Leibniz wat 'n 'characteristica universalis' wou skep: 'n universele taal waarin enige wetenskaplike feit uitdrukbaar is. Ook is 'n algemene metode ten doel gestel om die waarheid al dan nie van enige uitdrukking of *formule* in die taal te bepaal.

Die werk van Frege, Russel & Whitehead, Tarski, Gödel en andere het gestalte gegee aan 'n kandidaat vir so 'n universele taal, naamlik **eerste orde logika.**

Die volledigheidsstelling van Gödel wat verseker dat alle geldige formules gevolge is van die aksiomas en deduksiereëls, het die hoop op die universaliteit van die taal en metode versterk.

Volgens Hilbert en sy skool was 'n *volledig* en *konsistente* logiese stelsel wat alle wiskundige beredenering bevat, die enigste sinvolle grondlegging vir wiskunde. Met volledigheid word bedoel dat elke goed gevormde uitdrukking in die taal bewysbaar waar of bewysbaar onwaar is. Konsistentheid beteken dat geen goed gevormde uitdrukking bewysbaar waar en bewysbaar onwaar is nie.

In 1935 het Gödel bewys dat in enige konsistente logiese stelsel met ten minste 'n gegewe minimale uitdrukkingsvermoë, daar altyd goed gevormde formules sal wees wat nie bewysbaar is nie en waarvan die negering van die formule ook nie bewysbaar is nie. Met ander woorde alle sulke logiese stelsels is onvolledig. 'n Oop vraag was of daar 'n metode bestaan om sulke formules te identifiseer.

In 1935 het Church en Turing bewys dat daar geen so 'n metode kan bestaan nie en ons noem dan eerste orde logika *(rekursief) onoplosbaar*, daar is dus geen algoritme wat altyd korrek uitspraak kan gee oor of 'n formule uit die aksiomas en deduksiereëls volg of nie.

Aangesien die klas van alle eerste orde logika formules onoplosbaar is, kan ons vra welke subklasse oplosbaar is en welke nie.

Oorspronklik is subklasse onoplosbaar bewys deur te toon dat enige eerste orde formule ekwivalent is aan 'n formule in die spesifieke subklas, omdat die hele klas formules onoplosbaar is, is die subklas dan ook onoplosbaar. Klasse wat so onoplosbaar bewys is, is reduksieklasse genoem.

In die vroee 1960's het Büchi en Wang gewys dat sekere berekenings of kombinatoriese stelsels deur eenvoudige logiese uitdrukkings beskryf kan word. Die onoplosbaarheid van die probleme buite logika - die Domino probleem en die Halt probleem - beteken dan dat sekere subklasse van logiese formules onoplosbaar is. Die klasse is nie reduksie klasse nie en is 'werklike' subklasse van die logiese stelsel. Die resultate is baie skerper as die verkry met vroeëre metodes.

Subklasse is onoplosbaar bewys deur verskillende mense met uiteenlopende metodes. Die bewyse van onoplosbaarheid is hergiet in 'n homogene vorm deur Lewis,[20], waarmee die tesis grotendeels gemoeid is.

Die eerste paar hoofstukke handel oor die onoplosbaarheid van sekere kombinatoriese probleme, hierdie resultate word dan in latere hoofstukke gebruik om sekere klasse formules van eerste orde logika onoplosbaar te bewys.

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Summary

The goal of finding some universal language and method for stating and solving arbitrary scientific problems has been an ideal since Leibniz who envisioned a "characteristica universalis": a universal language in which any scientific fact could be expressed. In addition, a general method for deciding the truth or falsity of any statement in the language, was envisioned.

Through the work of Frege, Russel & Whitehead, Tarski, Gödel and others this goal of a universal language for formalising mathematical facts, found expression in the development of what is now known as first order logic. The Completeness Theorem of Gödel which states that all valid formulas are deducible, was seen as a justification for the ideals for the universality of the language and method.

According to Hilbert and his school, the only justifiable foundation for the practice of mathematics was a logical system encompassing all of mathematical reasoning which further, was complete and consistent, i.e. all meaningful statements expressible in the language are either provably true or provably false(completeness) and no contradiction can arise in the system(consistency).

In 1931 Gödel showed that in any consistent logical system of some minimal expressivity there are expressions within the language which are not provably true or provably false. An open question was whether there was some method of identifying such expressions.

In 1935 Church and Turing showed that no general method can exist for deciding whether or not a formula is deducible and that first order logic is therefore undecidable, i.e. there is no algorithmic procedure which can always correctly decide whether a formula is valid, i.e. deducible from given axioms and inference rules, or not.

Given that the predicate calculus as a whole is undecidable, one may ask which subclasses of the predicate calculus are decidable and which are not.

Initially, the method which showed some particular class of formulas of first order logic to be undecidable or **unsolvable** was a presentation of an effective method by which any quantificational formula could be reduced to some formula in the particular class. Such classes are called *reduction classes.* The unsolvability of the class then follows from the unsolvability of quantificational theory as a whole. See Turing,[28].

In the early 1960's Büchi and Wang showed how simple computational or combinatorial systems- *Turing Machines or Dominoes-* could be naturally described by simple quantificational formulas.

The unsolvability of these extralogical problems -the dominoe problem or the Halting problem-then implied the unsolvability of certain logical classes, these classes not being reduction classes, that is the classes are actual 'subclasses' of the class of quantificational formulas. These results are much sharper than those obtained by the previous reduction methods.

This work is concerned with proofs that certain classes are undecidable, the results are due to various different people(see chapter 6 for references) but were recast in a homogeneous form by Lewis,[20] which is the main source for this work.

The first few chapters are concerned with proving the unsolvability of various combinatorial problems, these results are then used in later chapters to prove the unsolvability of the classes of formulas.

Chapter 1

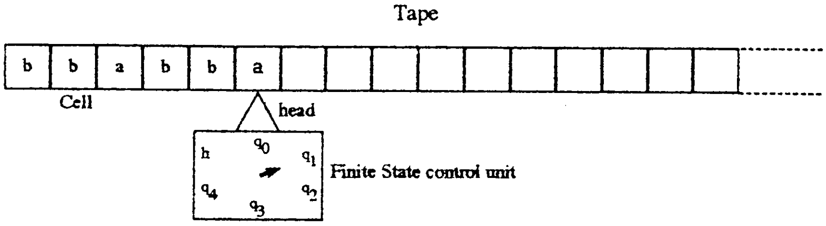
Turing machines and tilings of the plane

In this chapter we use the unsolvability of the halting problem for Turing machines to show that a certain combinatorial problem, the origin- constrained tiling problem, is unsolvable. Most of this chapter is taken from Lewis and Papadimitriou,[21].

1.1 Turing machines

A Turing machine can be imagined as a device consisting of a finite state control unit with a potentially infinite tape serving as input and output medium. A single head communicates between the two.

fig 1(from Odifreddi,[23])



Formally, a Turing machine is a quadruple

$$(K, \Sigma, \delta, s)$$

with

K : A finite set of states with $s \in K$, the initial state. The halt state h is not in K,

Σ : A finite set of symbols, containing the blank symbol #, but not containing the symbols L and R,

δ (the transition function) : A function from $K \times \Sigma$ to $(K \cup h) \times (\Sigma \cup L, R)$.

The transition function is interpreted as follows:

If $\delta(q, a) = (p, b)$ with $q \in K, a \in \Sigma$, then the machine in state q scanning symbol a, will enter state p and

1. if $b \in \Sigma$, rewrite the a as b or

2. if b is L or R, move the head one cell left(L) or right(R).

Since δ is a function the operation of the machines is deterministic and the machine will stop only when it enters the halt-state or attempts to move off the left tape-end, with the consequent absence of a symbol scanned.

A *configuration* of a Turing machine is a complete recording of the relevant information of the computation in a given instant. It takes the form

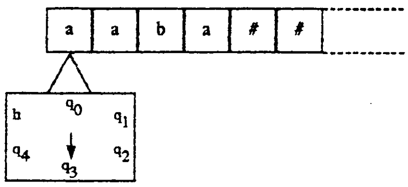
$$K \cup h \times \Sigma^* \times \Sigma \times (\Sigma^*(\Sigma - \# \cup e)$$

where Σ^* is the set of words over Σ and e is the empty string. This specifies the current state, the contents of the tape and the head position. The definition is set up so that the string to the right of the scanned cell does not end with a blank.

Example:

(q_3, e, a, aba) is the following configuration:

fig 2



We now define the relation *yields in one step* for configurations:

Let $\mathcal{M}$ be a Turing machine and let (q_1, w_1, a_1, u_1) and (q_2, w_2, a_2, u_2) be configurations of $\mathcal{M}$.

Then

$$(q_1, w_1, a_1, u_1) \vdash_M (q_2, w_2, a_2, u_2)$$

iff for some

$$b \in \Sigma \cup L, R$$

$$\delta(q_1, a_1) = (q_2, b)$$

and either

1. $b \in \Sigma, w_1 = w_2, u_1 = u_2$ and $a_2 = b$ or
2. $b = L, w_1 = w_2 a_2$, and

   either

   a)$u_2 = a_1 u_1$, if $a_1 \neq \#$ or $u_1 \neq e$, or

   b)$u_2 = e$, if $a_1 = \#$ and $u_1 = e$;

   or
3. $b = R, w_2 = w_1 a_1$ and

   either

   a)$u_1 = a_2 u_2$ or

   b)$u_1 = u_2 = e$ and $a_2 = \#$.

Case 1 corresponds to $\mathcal{M}$ writing a symbol (without the head moving).
Case 2 corresponds to the head moving one cell left, if it moves to the left of blank tape then the blank symbol on the cell just scanned is dropped from the configuration.
Case 3 corresponds to the head moving one cell right, if it moves onto blank tape then a new blank symbol is added as the new scanned symbol.
A *computation* by a Turing machine $\mathcal{M}$ is a sequence of configurations

$$C_0, C_1, \ldots, C_n; n \geq 0$$

such that

$$C_0 \vdash_M C_1 \ldots \vdash_M C_n.$$

We will use the following well-known result, due to Turing,[28]:

Theorem 1.1 (Unsolvability of the halting problem) *There exists no general algorithm which can answer the following class of questions correctly:*

> *Given a Turing-machine $\mathcal{M} = (K, \Sigma, \delta, s)$ and an input string w, does $\mathcal{M}$ halt on input w?*

The result can be strengthened to the case where w is the empty string. This is the form of the unsolvability of the halting problem which we will use.

1.2 Tiling problems

Tiling Problems have the following general form:
We are given:

1) a set X, the space to be tiled
2) a finite set T of representative tiles
3) a spatial relation $R \subset X^k$, the sampling configuration, and
4) a set $Q \subset T^k$.

The **tiling problem** is the problem of determining, given some tiling system $\mathcal{P} = (X, T, R, Q)$, whether or not there is a mapping

$$\tau : X \rightarrow T$$

such that

$$(s_1, s_2, \ldots, s_k) \in R \Rightarrow (\tau s_1, \tau s_2, \ldots, \tau s_k) \in Q$$

Intuitively, whether or not it is possible to *tile* X with (an unlimited supply of) copies of *tiles* in T such that all k-tuples of tiles at sampling configurations R are in some specified set Q of k-tuples of tiles.

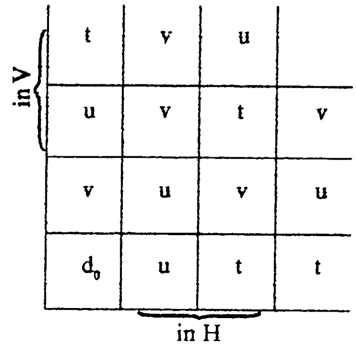
Mappings or *tilings* for which this is true are called *accepted* tilings. The term accepted is descriptive of an agent checking to see whether all k-tuples in R are in the set Q. If some k-tuple in R is not in Q then the tiling is not accepted.

Note that k-tuples in R are the only sets of points of the space which we have access to and that we have no control over the order in which such tuples are visited.

1.3 Tilings of the plane

The tiling problems we discuss in this section have as space to be tiled the first quadrant of the plane-N^2. We imagine the quadrant to be divided into cells, one for each coordinate pair (x, y) with each cell containing a single tile from some set D.
An accepted tiling is one in which the tile at the origin is some fixed tile $d_0 \in D$, all pairs of tiles in horizontally adjacent cells are in some given set $H \subset D^2$ and tiles in vertically adjacent cells are in some given set $V \subset D^2$. Because of the origin-condition such tiling-systems are called origin-constrained tiling problems.

fig 3



Formally:
R consists of three parts R_0, R_H, R_V with

1. $R_0 := (0,0)$
2. $R_H := ((x,y);(x+1,y); x,y \in N$
3. $R_V := ((x,y);(x,y+1); x,y \in N$.
   Q also consists of three parts: d_0, H, V with
4. $d_0 \in D$
5. $H \subset D^2$
6. $V \subset D^2$.

Since R is the same for all such tiling systems, we will specify an origin-constrained tiling system $\mathcal{P}$ by the 4-tuple

$$\mathcal{P} = (D, d_0, H, V).$$

1.4 Unsolvability of the origin-constrained tiling problem

In this section we will show that the tiling problem for origin constrained tiling problems is **unsolvable**. That is, there exists no general algorithm which, given an arbitrary origin-constrained tiling system (D, d_0, H, V) can always correctly decide whether or not an accepted tiling exists for the system.

We show that if there were such an algorithm then the halting problem for Turing machines would be solvable.

1.4.1 Turing machines as tiling systems

Given a Turing machine $\mathcal{M}$ we construct a tiling system $\mathcal{P}_{\mathcal{M}}$ such that an accepted tiling for $\mathcal{P}_{\mathcal{M}}$ represents an infinite computation by $\mathcal{M}$ when started on the leftmost square of the blank tape.

Let

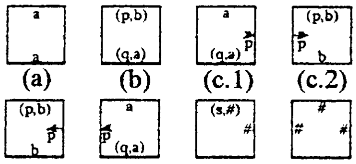
$$\mathcal{M} = (K, \Sigma, \delta, s)$$

then

$$\mathcal{P}_{\mathcal{M}} = (D, d_0, H, V)$$

where D contains the following tiles:

fig 4



(a) (b) (c.1) (c.2)
(d.1) (d.2) (e) (f)

Tiles of type (a)(see fig 4) for each $a \in \Sigma$. These tiles communicate unchanged symbols between consecutive configurations.

Tiles of type (b) for each $a \in \Sigma, q \in K$ such that $\delta(q, a) = (p, b)$ where $p \in K, b \in \Sigma$ These tiles communicate the head position upwards and change the state and scanned symbol appropriately.

Tiles of type (c.1) for each $a \in \Sigma, q \in K$ such that $\delta(q, a) = (p, R)$ where $p \in K$ and tiles of type (c.2) for each $b \in \Sigma$. These tiles communicate head movement one square from left to right and change the state appropriately.

Tiles of type (d.1),(d.2) for analogous situations as the above but with $\delta(q, a) = (p, L)$.

A tile of type (e)(the origin tile). This tile specifies on its vertical edge the initial state and the blank symbol. Its right edge can be matched only by the left edge of tiles oftype (f), which in turn propogate to the right the information that the top edge of every tile in the bottom row is marked with the blank.

Theorem 1.2 *The symbol-sequence between rows k and $k+1$ is exactly the configuration of $\mathcal{M}$ after $k-1$ steps.*

Proof:

The proof is by induction on the computation step:

Basis:Computation step 0:

The last two tiles ensure that the edge between the first and 2nd row of tiles correctly gives the starting configuration.

Induction step:

Assume the tiling correctly represents the computation up to step n: Let

$$(q, a), a \in \Sigma, q \in K \cup halt$$

be the single state-symbol pair in row n.

If the machine halts at step n then we cannot tile the next row. (No tile contains the halt state so there are no tiles with bottom edge $(halt, a)$.)

If the machine does not halt($q \in K$) then the state (possibly)changes and the machine either overwrites a symbol or moves one cell left or one cell right. That is

$$\delta(q, a) = (p, b) \text{ or } (p, R) \text{ or } (p, L)$$

for some

$$p \in K, b \in \Sigma$$

If the first case holds then the pair (p, b) appears above (q, a) in the next row. This follows from the construction of the tile set since the only tile with bottom edge (q, a) has (p, b) as top edge. (δ is a function). No other symbol in row n changes since only the tile directly above or one of the tiles adjacently above the state-symbol tile can change, and the transition function uniquely determines which of these is the case.

If one of the other two cases hold then the only tile with bottom edge (q, a) is a tile of type (c.1) or (d.2), then the only tile allowed to the right(left) of this tile is of type (c.2) or (d.1).

Again no other symbol changes.

So if it exists, then computation step $n + 1$ is correctly represented in each case. This completes the proof.

The tiling simulates the computation of $\mathcal{M}$ perfectly. If the computation is infinite then the whole of the first quadrant is tiled in such a way that the conditions d_0, V and H are satisfied and conversely if the computation halts after k steps then we can only tile $k + 1$ rows. So if the tiling problem were solvable then the halting problem would be solvable.

We therefore have the result:

The origin constrained tiling problem is algorithmically unsolvable.

References

Turing Machines are of course due to Turing,[28].

The tiling problems presented here are essentially dominoe-systems which are due to Wang,[31].

The proof of the unsolvability of the origin-constrained tiling problem presented here is adapted from Lewis and Papadimitriou,[21].

Chapter 2

Linear sampling problems

In this and the next chapter we use the unsolvability of the origin-constrained tiling problem to show that another tiling problem, the linear sampling problem, is unsolvable. All the results and proofs of this chapter are due to Lewis,[20]. The structure and sequence(internal or external) of the proofs have sometimes been changed to facilitate understanding.

2.1 Definition

A linear sampling problem is a tiling problem of the following kind:
the space X to be tiled by some set of tiles T is

$$Z \times (1,2).$$

That is, two disjoint copies of the integers which we consider as two numbered *tapes*.

The configurations of tiles R_0^θ and $R_1; \theta > 0, \theta \in Z$ we may sample are defined by linear equations, hence the name: linear sampling problems.

These conditions R_0^θ and R_1 are defined as follows.

1. **Local Condition**: for $i = 1, 2$, $R_0^\theta((n_1, i) \ldots, (n_\theta, i))$ iff

   $n_{j+1} = n_j + 1; j = 1, \ldots, \theta$ (i.e. we may sample any θ consecutive tiles on either of the two tapes).

2. **Global Condition**: $R_1((n_1, 1), (n_2, 2), (n_3, 1), (n_4, 2))$ iff

   $n_2 - n_1 = n_4 - n_3$ (i.e. we may sample any two pairs of tiles, a pair on each tape, such that the distance between the tiles on the first tape is equal to the distance between the tiles on the second).

Figures 5 and 6 illustrate the forms of the configurations we may sample for $\theta = 4$.

fig 5

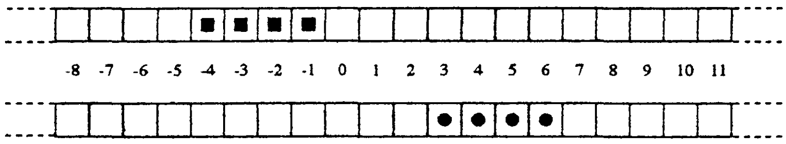
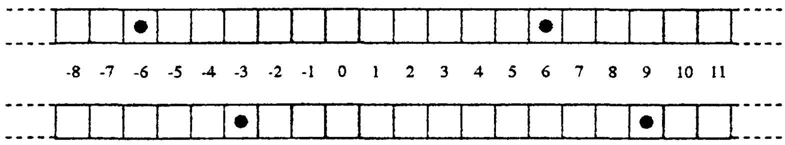


fig 6



It is crucial to note that R_0^θ and R_1 define the only access we have to the tapes, we cannot view any other configurations of tiles. Further, we do not have access to coordinates of cells but only to the tuples of tiles at configurations defined by R_0^θ and R_1.

An **accepted tiling** is one in which all tuples in R_0^θ are in some given subset of $T^\theta(= T \times \ldots \times T; \theta$ times) which we denote by L(for *local*) and all samples in R_1 are in some given subset of T^4 which we denote by G(for *global*).

We will generally denote a linear sampling system by the triple (T, L, G) where T is the set of tiles and L and G are the subsets mentioned above. The value of θ will be seperately specified.

Once again(and this is a crucial point), we have no control over the order in which our samples are taken.

The **linear sampling problem** is the problem of determining, given some linear sampling system, (T, L, G) whether an accepted tiling of the tapes exists or not.

This whole chapter is devoted to showing that the linear sampling problem is unsolvable for any given $\theta \geq 2$. To show this, it suffices to show that the problem is unsolvable for $\theta = 2$ as the following argument shows.

Theorem 2.1 *If the problem is unsolvable for $\theta = 2$, it is unsolvable for all $\theta \geq 2$*

Proof: Given some $\theta = k$, k a natural number, then for each tiling system with $\theta = 2$ there is an 'equivalent' tiling system with $\theta = k$. This is the system with the same global condition and local conditions L as follows: L consists of all

k-tuples such that all consecutive pairs in the k-tuple are in the local condition for the $\theta = 2$ system. Clearly our systems allow exactly the same tilings. So the unsolvability for the $\theta = 2$ system would imply the unsolvability for all systems with $\theta \geq 2$.

Theorem 2.2 *The unsolvability of any $\theta \geq 2$ implies the unsolvability for $\theta =$ 2.*

Proof: Suppose that the problem is unsolvable for some $\theta \geq 2$. We show that this implies the unsolvability for $\theta = 2$.

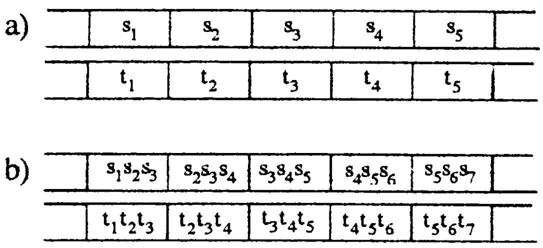
Let $\mathcal{T} = (T, L, G)$ be any l-ary tiling system. Let $\mathcal{T}' = (T', L', G')$ be the following system: $T' = T^l$ and

a)$((s_1, s_2, ..., s_l), (s_{l+1}, ..., s_{2l})) \in L'$ iff $s_{l+1} = s_2, s_{l+2} = s_3$,.... i.e. $s_{l+i} = s_{i+1}$ for $i = 1, ..., l-1$ and $(s_1, ..., s_l), (s_{l+1}, ..., s_{2l}) \in L$,

b)$((s_1, s_2, ..., s_l), ...(s_{3l+1}, ..., s_{4l})) \in G'$ iff $(s_1, s_{l+1}, s_{2l+1}, s_{3l+1}) \in G$.

The conditions are defined in such a way that a tiling accepted by $\mathcal{T}$ codes for a tiling accepted by $\mathcal{T}'$ and conversely. Figure 7 shows a correlated pair of tilings.

fig 7



2.2 Correlating the tapes with the plane

2.2.1 Coordinate correlation

We want to show that the linear sampling problem is unsolvable by using the unsolvability of the origin-constrained tiling problem. We will show the problem unsolvable for $\theta = 4$. To do this we must show that for each origin-constrained tiling system there is a corresponding linear-sampling system with $\theta = 4$ which accepts a tiling iff the origin-constrained tiling system accepts some tiling. We must therefore correlate coordinates of $Z \times (1, 2)$ with coordinates of N^2. There are many ways to do this but our correlation must satisfy some strict conditions if we want the correlation to work. Each coordinate of N^2 must be represented infinitely often and at bounded intervals along at least one of the tapes. It will become clear later why this is a prerequisite.

Notation: *The p_i-order of a whole number n denoted by $\mathcal{O}_{p_i}$, is the highest power of p_i which divides n without a remainder.*

When the value of p_i is fixed, we often write $O(n)$ for $O_{p_i}(n)$.
Example: $\mathcal{O}_2(8) = 3$ since $2^3 = 8$, while $\mathcal{O}_2(10) = 1$.

Choose two numbers p_1 and p_2 which are relatively prime and associate the coordinate (i, j) in the first quadrant with all coordinates n on the plane such that the p_1-order n is i and p_2-order n is j. It is clear why p_1 and p_2 must be relatively prime if we want to represent all tuples $(i, j) \in N^2$. If we correlate coordinates in this way then the stipulations in **2.2.1** are satisfied.

2.2.2 Conditions on the linear sampling system

We must now construct conditions on tilings of the tapes which are equivalent to the conditions on tilings of the plane in that an accepted tiling of the tapes exists iff an accepted tiling of the plane exists. Specifically, the following three conditions must be met for a tiling of the tapes to be accepted.

1. If two cell coordinates have equal p_1- and p_2-orders then the cells must contain the same tile, since all such coordinates are correlated with a single coordinate of N^2.

2. If a cell coordinate has p_1- and p_2-orders zero then the cell must contain a tile allowed at the origin of N^2.

3. If two cell coordinates n_1 and n_2 have p_1- and p_2- orders

   $$(i, j)\&(i, j+1)$$

   $$(i, j)\&(i+1, j); i, j \in N,$$

   then the tiles in the cells must form a pair in V or H respectively. For notational convenience we change H, V to A_1, A_2.

The third conditions correspond to conditions V and H on tilings of the plane.

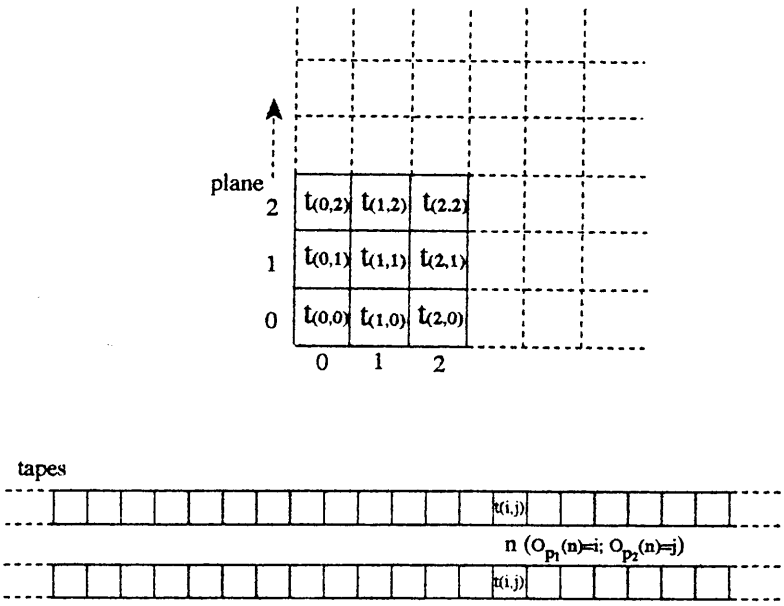
The idea is that these conditions on tilings of the tapes will be equivalent to the conditions on the tilings of the plane and that an accepted tiling of the tapes will code for an accepted tiling of the plane and conversely. Note that in the origin-constrained tiling problem we knew all samples were horizontally or vertically adjacent, in fact these were the only configurations we could sample, in the correlation with the linear sampling problem we must first find the cells corresponding to adjacent cells on the plane before we can place conditions on the tiles in such cells.

We sketch the idea of the mapping between the plane and the tapes.

2.2.3 The idea of the mapping between accepted tilings

Figure 8 illustrates the mapping between the tapes and the plane which we have in mind:

fig 8



Plane to tapes:

For a given accepted tiling τ of the plane, define a tiling Γ of the tapes as follows:

for all $n \in Z$ and for both tapes

$$\Gamma(n) = \tau(i, j)$$

where i, j are the p_1 and p_2 orders of n.

Tapes to plane:

Given an accepted tiling Γ of the tapes, define a tiling τ of the plane as follows

$$\tau(i, j) = \Gamma(n)$$

where n has p_1- and p_2- orders i and j, respectively.

The idea is that accepted tilings of the tapes map to accepted tilings of the plane and conversely. If this is the case then the linear sampling problem will of course be unsolvable.

The correlation between the two tiling systems just sketched is in fact a laborious task and will occupy the rest of this chapter.

2.3 Construction of the conditions

Implicit in conditions 1-3 of paragraph **2.2.2** is the following.

The consistency constraint:

1.1) Identifying all pairs of cell-coordinates with equal p_1- and p_2-orders.

1.2) Forcing all such cells to contain the same tile.

The origin constraint:

2.1) Identifying all cell-coordinates with p_1- and p_2-orders zero.

2.2) Placing the origin constraint on these cells.

The adjacency constraint:

3.1) Identifying pairs of cell coordinates with p_1 and p_2 orders

$$(i,j)\&(i,j+1)$$

$$(i,j)\&(i+1,j); i,j \in N.$$

(These cells of course correspond to cell-pairs which are horizontally or vertically adjacent on the plane.)

3.2) Placing conditions V and H, respectively, on the tile-pairs in such pairs of cells.

2.3.1 Identifying the cells

It is important to note that we do not have access to the coordinates at which samples are taken, we can therefore not correlate coordinates of cells with the tiles appearing in the cells to see for each sample whether 1-3 in **2.2.2** are satisfied or not. In order to identify cells 1.1-3.1 in **2.3** therefore, we must include *within* a cell information about the p_1- and p_2-orders of the coordinate of the cell.

2.3.2 Addressing information

Since the cells have some fixed finite size, a cell cannot contain its coordinate or even its p_1- and p_2-orders. We devote the next section to showing that, somewhat surprisingly, it is enough for a cell $\nu = (n,c)$; $0 < n; c = 1,2$ to contain the numbers $\lambda_{p_1}(n)$ and $\lambda_{p_2}(n)$, where

$$\lambda_{p_i}(n) := rem(np_i^{-O_{p_i}}(n), p_i^2)$$

where rem (l,n) is the remainder of l when divided by n. Note that $\lambda_{p_i}(0)$ can be any element of $\{q \mid 0 < q < p^2; q \neq 0(\text{mod}p)\}$

Informally, $\lambda_{p_i}(n); i = 1,2$ is the result of writing n in p-ary notation, deleting any trailing zeros and taking the last two digits of what is left.

Example

For $n = 1,\ldots,120$: the sequence $\lambda_{10}(n); 1 \leq n \leq 120$ is the following:

01 02 03 04 05 06 07 08 09 01 11 12 13 14 15 16 17 18 19 02 21 22 23 24 25 26 27 28 29 03 31 32 33 34 35 36 37 38 39 04 41 42 43 44 45 46 47 48 49 05 51 52 53 54 55 56 57 58 59 06 61 62 63 64 65 66 67 68 69 07 71 72 73 74 75 76 77 78 79 08 81 82 83 84 85 86 87 88 89 09 91 92 93 94 95 96 97 98 99 01 01 02 03 04 05 06 07 08 09 11 11 12 13 14 15 16 17 18 19 12

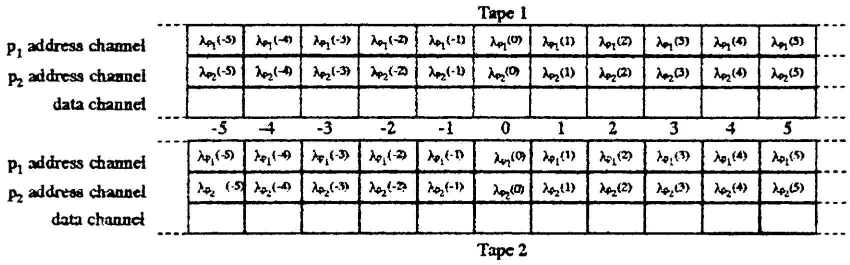
Note that for such a tiling λ:
$\lambda(n+d) = \lambda(n)$ whenever $O(d) \leq O(n) + 2$.
This is known as the **agreement property**.

Example

For $p = 10$, $\lambda(12) = \lambda(12+100) = \lambda(112)$.

To enable a cell to contain this information we divide both tapes into 3 channels as figure 9 illustrates:

fig 9



We imagine that the first and second channel of each cell with coordinate $(n,c); c = 1,2$ on both tapes contains the numbers $\lambda_{p_1}(n)$ and $\lambda_{p_2}(n)$ respectively, i.e. that we have mappings $\lambda_i(\pi_i(n,c)) = \lambda_{p_i}(n); i = 1,2; c = 1,2$ where π_i is the projection onto the i-th channel of a tape. We reserve the third channel for tiles $t \in T$. We will use this information to identify cells satisfying the relations in 1.1, 2.1 and 3.1 in **2.3**.

So if the set of tiles for an origin-constrained tiling problem is T, then the corresponding set of tiles for the corresponding linear sampling system is:

$$S_{p_1} \times S_{p_2} \times T,$$

where

$$Sp_i := \{\lambda_{p_i}(n), n \in Z\} = \{0 < q < p^2 : q \neq 0(\text{mod} p_i)\}.$$

A tiling for which the address channels of cells $\nu_i = (n_i, c); c = 1,2$ contain $\lambda_{p_i}(n); i = 1,2$ in the i-th address channel is called a **perfect** tiling. We also talk of the p_i-address channel being tiled perfectly for $p = p_1$ or p_2. This means that the p-th address channel contains λ_p of the coordinate. Note that a perfect tiling does not mention the tiles in the third channel. We will often work in a fixed address channel and will write $\lambda(n)$ for $\lambda_{p_i}(n)$. Through an abuse of notation, we also sometimes write $\lambda(\nu)$ for $\lambda(n)$ when $\nu = (n,c); c = 1$ or 2.

2.3.3 Weakening the conditions

We can slightly weaken 1.1,2.1 and 3.1 in **2.3** to the following.

Given p_i, $i = 1$ or 2, we must identify:

1. cells with p_i order zero: $i = 1$ or 2,
2. cells with equal p_i order,
3. pairs of cells with p_i order j and $j+1$ respectively, $j \in N$.

We can then combine 1-3 for $i = 1$ and 2 to identify cells satisfying 1.1, 2.1 and 3.1 of **2.3** as follows.
Cells satisfying (1) for $i = 1$ and 2 are exactly the cells in 1.1 of **2.3**.
Cell pairs satisfying (3) for $i = 2$ and (2) for $i = 1$ are exactly the cells with p_i -orders (k,l) & $(k,l+1)$, as in 3.1 of **2.3**. Analogously for pairs with p_i-orders (k,l) & $(k+1,l)$.

2.4 The origin constraint

Let $\nu_i \in Z \times 1,2$.

Assume the p-address channels are tiled perfectly.

Given $p = p_1$ or p_2, we define a relation $Z_p \subset \Sigma_p^4$ on local samples of 4-tuples $(\nu_1, \nu_2, \nu_3, \nu_4)$ such that

$$Z_p(\lambda(\nu_1), \lambda(\nu_2), \lambda(\nu_3), \lambda(\nu_4))$$

iff

$$O_p(\nu_1) = 0.$$

2.4.1 Constructing the relations Z_{p_i}

Given $p = p_1$ or p_2. Let λ_p be a perfect tiling of address channel p.

Let (q_0, q_1, q_2, q_3) be the entries in the p-address channels of a local sample of cells at $(n, n+1, n+2, n+3)$ say, i.e.

$$q_j = \lambda_p(n+j), 0 \leq j \leq 4$$

on either of the two tapes.

Now assume n has p-order zero. Then $n \equiv \lambda_p(n) \neq 0(\text{mod}p)$, since the p-ary expansion of n has no trailing zeros. Denote $\lambda_p(n)$ by $\lambda(n)$, and $O_p(n)$ by $O(n)$.

Set $q = rem(n,p)(= rem(\lambda(n),p)$. There are three possibilities.

1. If $1 \leq q \leq p-3$, then $q_1 \equiv q_0 + 1(\text{mod}p)$ and
$$\lambda(n+2) = q_2 \equiv n+2 \equiv q+2(\text{mod}p).$$
2. If $q = p-2$, then $\lambda(n+1) = q_1 \equiv n+1 \equiv q+1 = p-1(\text{mod}p)$.

3. If $q = p - 1$, then $\lambda(n+2) = q_2 \equiv q + 2 \equiv 1(\text{mod}p)$.
So in 1, 2 and 3: $q_1 \equiv q_0 + 1(\text{mod}p)$ or $q_2 \equiv q_0 + 2(\text{mod}p)$.

Conversely, if (q_0, q_1, q_2, q_3) is a local sample and either $q_1 \equiv q_0 + 1(\text{mod}p)$ or $q_2 \equiv q_0 + 2(\text{mod}p)$, then if we assume that $O(n) \neq 0$, that is $n \equiv 0(\text{mod}p)$, then $\lambda(n+1) \equiv n + 1 \equiv 1(\text{mod}p)$ and $\lambda(n+2) \equiv n + 2 \equiv 2(\text{mod}p)$ so, by our assumption $\lambda(n) \equiv 0(\text{mod}p)$ which is impossible, since $\lambda(n)$ has last digit non-zero for any $n \in Z$, by the definition of L.

2.4.2 Definition of Z_{p_i}

For $p = p_1, p_2$

$$Z_p(\lambda(\nu_1), \lambda(\nu_2), \lambda(\nu_3), \lambda(\nu_4))$$

iff

$$(\nu_1, \nu_2, \nu_3, \nu_4) \in R_0$$

and

$$\lambda(\nu_1) \equiv \lambda(\nu_0) + 1(\text{mod}p)$$

or

$$\lambda(\nu_2) \equiv \lambda(\nu_0) + 2(\text{mod}p).$$

So we have a relation on the address channel of consecutive 4-tuples of tiles in p-address channels of perfect tilings which is satisfied iff the coordinate of the first cell of the tuple has p-order zero. We now use Z_{p_i} to place the origin condition on tilings of the tapes.

2.4.3 The origin condition

If the address channels of a local sample of cells satisfy Z_p for $p = p_1$ and p_2 then the first cell of the tuple must have an origin tile in the data channel.

Formally: given a local sample (s_1, s_2, s_3, s_4) of a perfect tiling:

$$\text{if } Z_{p_i}(\pi_i(s_1), \pi_i(s_2), \pi_i(s_3), \pi_i(s_4)) \text{ for } i = 1 \text{ and } 2, \text{ then}$$

$$\pi_3(s_1) \in D_0.$$

2.5 Properties of the tilings

Note from the sequence in section 2.3.2 that the tiles at cells with order 1, are only determined mod$p$ by the preceding sequence of tiles at cells of order 0, that is, tiles in cells of order 0 are equally close to all tiles in cells of order 1 which are equal mod p. Further, surrounding tiles at coordinates of order 0 give us no information about tiles at cells of order 2, that is, tiles in cells of order 0 are equally close to all tiles in cells of order 2. The following lemma states these results formally.

Theorem 2.3 (Tiling lemma) *Let* $n_1, n_2 \in Z$.

a) If $O(n_1) = O(n_2 - n_1) = O(n_2)$ *then* $\lambda(n_1) + \lambda(n_2 - n_1) \equiv \lambda(n_2)(\mod p^2)$.

b)If $O(n_1) = O(n_2 - n_1) = O(n_2) - 1$ *then*

$$\lambda(n_1) + \lambda(n_2 - n_1) \equiv p\lambda(n_2) \not\equiv 0(\text{mod} p^2).$$

Note that $p\lambda(n_2) \equiv 0(\text{mod} p)$.

c)If $O(n_1) = O(n_2 - n_1) \leq O(n_2) - 2$ *then* $\lambda(n_1) + \lambda(n_2 - n_1) \equiv 0(\text{mod} p^2)$.

We prove part (c) as an example. If $O(n_1) = O(n_2 - n_1) = i$, say, then $n_2 = n_2' p^{i+2}$ for some $n_2' \in Z$ and set $n_1 = n_1' p^i$ for some $n_1' \not\equiv 0(\text{mod} p)$. Then

$\lambda(n_1) + \lambda(n_2 - n_1) \equiv n_1 p^{-i} + (n_2 - n_1)p^{-i} = n_2 p^{-i} = n_2' p^2 \equiv 0(\text{mod} p^2)$.
The proofs of (a) and (b) are analogous.

2.6 The consistency constraint

Notation:

Let $q_i = \lambda(\nu_i); i = 1, 2, 3, 4$ and let $n_i = \pi_1(\nu_i); i = 1, 2, 3, 4$. Further, for $\nu = (n, c)$, set $O(\nu)$=$O(n)$.

We will define relations $E_p, p = p_1, p_2$ on global samples $(\nu_1, \nu_2, \nu_3, \nu_4)$ such that if the p-channel is perfectly tiled and

$$E_p(\lambda(\nu_1), \lambda(\nu_2), \lambda(\nu_3), \lambda(\nu_4)),$$

then

$$O_p(\nu_1) = O_p(\nu_2) = O_p(\nu_3) = O_p(\nu_4).$$

We will define E_p by placing conditions on a global sample which eliminate all possible relations between the p-orders of the cells except $O_p(\nu_i)$ equal for $1 \leq i \leq 4$.

2.6.1 Constructing the relations E_p.

Set $p = p_1$ or p_2. We construct the relation E_p.

Let λ be some perfect p-tiling.

Take any global sample (q_1, q_2, q_3, q_4).

Now by the definition of λ: $\lambda(n_i) \equiv n_i p^{-O(n_i)}(\text{mod} p^2)$

(we get the right hand side by eliminating all trailing zeros).

Let $1 \leq l \leq 4$ be such that n_l has the lowest p-order of n_1, n_2, n_3, n_4.

Set $e_i = O(n_i) - O(n_l)$ for $i = 1, 2, 3, 4$, note that $e_i \geq 0$.

Now, since $n_2 - n_1 = n_4 - n_3$, we have

$$\lambda(n_2)p^{O(n_2)} - \lambda(n_1)p^{O(n_1)} \equiv \lambda(n_4)p^{O(n_4)} - \lambda(n_3)p^{O(n_3)}(\text{mod} p^2).$$

So multiplying through by $p^{-O(n_l)}$ we obtain

$$\lambda(n_2)p^{e2} - \lambda(n_1)p^{e1} \equiv \lambda(n_4)p^{e4} - \lambda(n_3)p^{e3}(\text{mod} p).$$

This is the equation we will work with. Now we know that whatever global sample was taken, it will be the case that $e_i = 0$ for some $1 \leq i \leq 4$.
We want to specify conditions under which $e_i = 0$ for all $1 \leq i \leq 4$. This will imply that the p-orders of all four cell coordinates are the same.
We do this by setting up conditions which rule out each case in which $e_i > 0$ for some i.

By cases:

a)If $e_i > 0$ for exactly 3 different i then $e_l = 0$ by definition and the three terms $\lambda(n_i)p^{e_i}; i \neq l$ are all multiples of p, this means that $\lambda(n_l) \equiv 0(\text{mod} p)$, which is impossible by the definition of λ. It follows that it is impossible for one term of a global sample to have p-order lower than all the rest.

b)If $e_i > 0$ for only one i say $e_k > 0, 1 \leq k \leq 4$, then it follows that $\lambda(n_2) - \lambda(n_1) \not\equiv \lambda(n_4) - \lambda(n_3)(\text{mod} p)$.

So to rule out this possibility we require that

1)$\lambda(n_2) - \lambda(n_1) \equiv \lambda(n_4) - \lambda(n_3)(\text{mod} p)$.

c)If $e_i > 0$ for exactly two i, then by cases: if $e_1, e_2 > 0$ and $e_3 = e_4 = 0$ then $\lambda(n_4) - \lambda(n_3) \equiv 0(\text{mod} p)$.

To rule out this possibility we require that $\lambda(n_4) - \lambda(n_3) \not\equiv 0(\text{mod} p)$ i.e. that

2)$q_4 - q_3 \not\equiv 0(\text{mod} p)$.

By the same method as above, to rule out the possibility that any other two e_i are 0 we require that the following be incongruent to 0 mod$p$:

3)$q_3 - q_1$; 4)$q_1 + q_4$; 5)$q_2 + q_3$; 6)$q_4 - q_2$; 7)$q_2 - q_1$.

The inequalities mod$p$ are not all independent but are stated thus for clarity. If conditions 1-7 are met then the coordinates of the sampled 4-tuple will have the same p- order.

2.6.2 Definition of E_{p_i}

So, for $p = p_1$ and p_2 define:

$$E_p(q_1, q_2, q_3, q_4) \text{ iff}$$
$$q_i = \lambda(\nu_i) \text{ and } (\nu_1, \nu_2, \nu_3, \nu_4) \in R_1 \text{ and:}$$
$$q_4 - q_3 \equiv q_2 - q_1(\text{mod} p)$$

and the following are incongruent to 0 mod p:

$$q_2 - q_1, q_4 - q_3, q_3 - q_1, q_1 + q_4, q_2 + q_3, q_4 - q_2.$$

It then follows that if the p-address channel is perfectly tiled, then

$$E_p(q_1, q_2, q_3, q_4) \text{ iff } O_p(n_1) = O_p(n_2) = O_p(n_3) = O_p(n_4).$$

The point of setting conditions E_{p_i} is using them to identify cells with equal p_1- **and** p_2-orders, in order to force such cells to contain the same tile in the data channel.

2.6.3 The consistency condition

Let π_i be the projection function onto the i-th channel of a cell.

Given a global sample (s_1, s_2, s_3, s_4) of a perfect tiling.

$$\text{If } E_{p_i}(\pi_i(s_1), \pi_i(s_2), \pi_i(s_3), \pi_i(s_4)) \text{ for } i = 1 \text{ and } 2 \text{ then}$$

$$\pi_3(s_3) = \pi_3(s_4).$$

2.6.4 The consistency condition effectively imposes the consistency constraint

The gist of the following theorem is that if $n_1, n_2 \in Z$ have equal p_i-orders for $i = 1$ and 2, then we can force the cells at coordinates n_1 and n_2 to contain the same tile in the data channels.

Instead of treating all possible combinations of positions of cells with coordinates n_1 and n_2 (both on the first tape, both on the 2nd tape, one on each tape) seperately, we prove the following theorem.

Theorem 2.4 (E.2) *Given any perfect p_1-tiling λ_1 and any perfect p_2-tiling λ_2 and any given $n_1, n_2 \in Z$ with $O_{p_i}(n_1) = O_{p_i}(n_2)$ for $i = 1, 2$.*

There are $n, m, m_1, m_2 \in Z$ such that $m_1 - m = n_1 - n$, $m_2 - m = n_2 - n$ and for $i = 1, 2; j = 1, 2$:

$E_{p_i}(\lambda_i(m, 1), \lambda_i(m_j, 2), \lambda_i(n, 1), \lambda_i(n_j, 2)$ *and*
$E_{p_i}(\lambda_i(m_j, 1), \lambda_i(m, 2), \lambda_i(n_j, 1), \lambda_i(n, 2)$.

In other words the p_1- and p_2 address channels of the following global samples:

$1)((m, 1), (m_1, 2), (n, 1), (n_1, 2))$, $2)((m, 1), (m_2, 2), (n, 1), (n_2, 2))$
$3)((m_1, 1), (m, 2), (n_1, 1), (n, 2))$, $4)((m_2, 1), (m, 2), (n_2, 1), (n, 2))$

satisfy E_{p_1} and E_{p_2} respectively.

Note that n_1 and n_2 both appear for both tapes.

Clearly, if we can force the tuples in 1-4 to have the same tile in the data channels then **all** cell-coordinates with equal p_i orders for $i = 1$ and 2 will have the same tile in the data channel.

Proof:

We first find n, m, m_1, m_2 which satisfy the above conditions.

Given any n_1, n_2 with equal p_1 and p_2 orders say

$$Op_1(n_i) = e_1, Op_2(n_i) = e_2; i = 1, 2.$$

Then we want n, m to have the same orders so set

$n = Np_1^{e_1}p_2^{e_2}$,
$m = Mp_1^{e_1}p_2^{e_2}$.

Then set $m_1 = m - n + n_1$ and $m_2 = m - n + n_2$.

In order to deal with congruences modp_1 and modp_2 we temporarily introduce N_1, N_2 for N and M_1, M_2 for M. We require N_1 to satisfy the conditions modp_1 for N, and N_2 the conditions modp_2 for N. N is then chosen using the Chinese Remainder Theorem. We do exactly the same for M. For this purpose we take $0 \leq N_i, M_i \leq p_i$.

To ensure that the p_1- and p_2-orders of n, m are e_1 and e_2 respectively, we require that

1)$N_1 \not\equiv 0(\text{mod} p_1), N_2 \not\equiv 0(\text{mod} p_2)$,

2)$M_1 \not\equiv 0(\text{mod} p_1), M_2 \not\equiv 0(\text{mod} p_2)$.

To ensure that m_1 and m_2 have p_1-order e_1 and p_2-order e_2 we require that

3)$M_1 - N_1 + \lambda_1(n_1) \not\equiv 0(\text{mod} p_1), M_2 - N_2 + \lambda_2(n_1) \not\equiv 0(\text{mod} p_2)$,

4)$M_1 - N_1 + \lambda_1(n_2) \not\equiv 0(\text{mod} p_1), M_2 - N_2 + \lambda_2(n_2) \not\equiv 0(\text{mod} p_2)$.

For the tuples 1-4 to satisfy E_{p_1} and E_{p_2} we must have:

$\lambda_1(n_1) - \lambda_1(n) \equiv \lambda_1(m_1) - \lambda_1(m)(\text{mod} p_1)$,

$\lambda_2(n_1) - \lambda_2(n) \equiv \lambda_2(m_1) - \lambda_2(m)(\text{mod} p_2)$,

$\lambda_1(n_2) - \lambda_1(n) \equiv \lambda_1(m_2) - \lambda_2(m)(\text{mod} p_1)$,

$\lambda_2(n_2) - \lambda_2(n) \equiv \lambda_2(m_2) - \lambda_2(m)(\text{mod} p_2)$. We can write these conditions as:

a)$\lambda_i(n_j) - \lambda_i(n) \equiv \lambda_i(m_j) - \lambda_i(m)(\text{mod} p_i); i = 1, 2; j = 1, 2$.

This condition follows directly from condition 1 of E_{p_i} for $i = 1, 2$.

Also each of the following must be incongruent to $0(\text{mod} p_i)$.

For $i = 1, 2; j = 1, 2$:

b)$\lambda_i(m_j) - \lambda_i(m)$, c)$\lambda_i(n_j) - \lambda_i(n)$, d)$\lambda_i(n) - \lambda_i(m)$,

e)$\lambda_i(m) + \lambda_i(n_j)$, f)$\lambda_i(n) + \lambda_i(m_j)$, g)$\lambda_i(n_j) - \lambda_i(m_j)$.

These conditions follow directly from conditions 2-7 of E_{p_i} for $i = 1, 2$.

We must find n, m, m_1, m_2 which satisfy the above.

If we can choose N, M by the Chinese Remainder Theorem so that 1-4 are satisfied with N_i, M_i replaced by N, M then we will have:

$Op_i(n_1) = Op_i(n_2) = Op_i(n) = Op_i(m) = Op_i(m_1) = Op_i(m_2)$.

If, further, n is chosen such that

$Op_i(n) = Op_i(n_1 - n) = Op_i(n_2 - n)[= Op_i(m_1 - m) = Op_i(m_2 - m)]$,

then by 1-4 and a) of the tiling lemma

$\lambda_i(n_j) - \lambda_i(n) \equiv \lambda_i(n_j - n) = \lambda_i(m_j - m) \equiv \lambda_i(m_j) - \lambda_i(m)(\text{mod} p)$.

In other words

$\lambda_i(n_j) - \lambda_i(n) \equiv \lambda_i(m_j) - \lambda_i(m)(\text{mod} p)$, which is condition (a) above.

To obtain this , we have further requirements on N_i:

5)$\lambda_i(n_1) - N_i \not\equiv 0(\text{mod} p_i)$,

6)$\lambda_i(n_2) - N_i \not\equiv 0(\text{mod} p_i)$.

Which will give

$$O(n_1 - n) = O(n_2 - n) = O(n).$$

It will then follow from (a) that

(b)$\lambda_i(m_j) - \lambda_i(m) \not\equiv 0(\text{mod} p_i)$,

(c)$\lambda_i(n_j) - \lambda_i(n) \not\equiv 0(\text{mod} p_i)$,
since they are congruent to $\lambda_i(m_j - m) \not\equiv 0(\text{mod} p_i)$ by the definition.
To satisfy (d) we require that N_i, M_i satisfy:
7)$N_i - M_i \not\equiv 0(\text{mod} p_i)$.
To satisfy (e) we require that M_i satisfy:
8) $M_i + \lambda_i(n_1) \not\equiv 0(\text{mod} p_i)$,
9) $M_i + \lambda_i(n_2) \not\equiv 0(\text{mod} p_i)$.
Finally, (f) and (g) will follow from

$$\lambda_i(n) - \lambda_i(m) \not\equiv 0(\text{mod} p_i), \lambda_i(m) + \lambda_i(n_j) \not\equiv 0(\text{mod} p_i),$$

and

$$\lambda_i(n_j) - \lambda_i(n) - \lambda_i(m_j) + \lambda_i(m) \equiv 0(\text{mod} p_i).$$

It now remains to show that N_i and M_i can be chosen to satisfy the conditions 1-9 and to construct N, M from them.

Conditions 1-9 eliminate at most p_1 choices for (N_1, M_1) and at most p_2 choices for (N_2, M_2). Consider conditions 3 and 5 as examples:

(3) $M_1 - N_1 \equiv -\lambda_1(n_1)(\text{mod} p_1), M_2 - N_2 \equiv -\lambda_2(n_1)(\text{mod} p_2)$. At most one value for N_i is eliminated for fixed M_i, therefore at most p_i pairs (N_i, M_i).

(5) $\lambda_i(n_1) - N_i \not\equiv 0(\text{mod} p_i)$. This eliminates at most one value for N_i, so again at most p_i pairs (N_i, M_i).

In total then, at most $9p_i$ pairs are eliminated for (N_i, M_i).

We will therefore use $p_i \geq 10$ which will enable us to choose pairs (N_1, M_1) and (N_2, M_2) satisfying conditions 1-9. Now given $N_1 \equiv k(\text{mod} p_1), 0 < k \leq p_1$, $N_2 \equiv l(\text{mod} p_2), 0 < l \leq p_2$, with p_1 and p_2 relatively prime. Then by the Chinese Remainder Theorem we can find an N such that $N \equiv k(\text{mod} p_1)$, $N \equiv l(\text{mod} p_2)$.

Therefore N satisfies conditions 1-9 (replacing N_1 and N_2 with N).

Analogously for M. This completes the proof of E.2.

2.7 The adjacency constraint

For the adjacency constraint, we need further relations I_p, for $p = p_1, p_2$ such that for any perfect p-tiling λ:

if $I_p(\lambda(\nu_1), \lambda(\nu_2), \lambda(\nu_3), \lambda(\nu_4))$ for some global sample $(\nu_1, \nu_2, \nu_3, \nu_4)$ then
$Op(\nu_1) = Op(\nu_2) = Op(\nu_3) = Op(\nu_4) - 1$,
i.e. n_4 has p-order one more than n_1, n_2, n_3.

We will then combine E_p and I_p to identify 'adjacent' cells.

2.7.1 Constructing the relations I_{p_i}

Let $p = p_1$ or p_2, and let the p-channel be perfectly tiled.

As in the proof of E_1 let n_l, $1 \leq l \leq 4$ have the lowest p-order amongst n_1, n_2, n_3, n_4 and let $e_i = O(n_i) - O(n_l)$. We will find conditions under which

$$e_1 = e_2 = e_3 = 0 \text{ and } e_4 = 1.$$

Now $n_2 - n_1 = n_4 - n_3$ so
$q_2 p^{O(n_2)} - q_1 p^{O(n_1)} \equiv q_4 p^{O(n_4)} - q_3 p^{O(n_3)} (\bmod p)$, since $q_i p^{(n_i)} \equiv n_i (\bmod p)$.
Multiplying through with $p^{-O(n_l)}$ we again get
i)$q_2 p^{e_2} - q_1 p^{e_1} \equiv q_4 p^{e_4} - q_3 p^{e_3} (\bmod p)$.
If $e_i > 0$ for exactly three i then as before, $q_l \equiv 0 (\bmod p)$, which is impossible.

To rule out the possibility that exactly two e_i are zero we use the same conditions as in E_{p_i}, i.e. we require that the following be incongruent to $0(\bmod p)$:

1)$q_2 - q_1$, 2) $q_4 - q_3$,
3)$q_3 - q_1$, 4) $q_1 + q_4$,
5)$q_2 + q_3$, 6) $q_4 - q_2$.
We want $e_4 > 0$ (specifically e_4=1) and we have from (i) that
$q_4 p_4^e \equiv q_2 p_2^e - q_1 p^e 1 + q_3 p_3^e (\bmod p)$,
we therefore require that
7)$q_2 - q_1 + q_3 \equiv 0 (\bmod p)$
(this also rules out the possibility that all four $e_i = 0$).
We now find conditions on our sample which force the case $e_1 = e_2 = e_3 = 0$, this will mean that $e_4 > 0$.

It follows from condition (7) above that the following are all incongruent to $0(\bmod p)$: $q_3 - q_1, q_2 + q_3, q_2 - q_1$, since $q_i \not\equiv 0 (\bmod p)$.
Now if $e_1 > 0$ and $e_2 = e_3 = e_4 = 0$ then from (i):
$q_2 - q_4 + q_3 \equiv 0 (\bmod p)$, and from (7)
$q_1 - q_2 - q_3 \equiv 0 (\bmod p)$, therefore $q_1 - q_4 \equiv 0 (\bmod p)$.
To eliminate this possibility we require that
8) $q_1 - q_4 \not\equiv 0 (\bmod p)$.
If $e_2 > 0$ and $e_1 = e_3 = e_4 = 0$ then by (i)
$-q_1 - q_4 + q_3 \equiv 0 (\bmod p)$, so also $-q_2 - q_4 \equiv 0 (\bmod p)$,
since we require that $-q_2 \equiv -q_1 + q_3$ by (7).
To avoid this case we require that
9)$q_2 + q_4 \not\equiv 0 (\bmod p)$.
If $e_3 > 0$ and $e_1 = e_2 = e_4 = 0$ then from (i):
$q_2 - q_1 - q_4 \equiv 0 (\bmod p)$, so $-q_3 - q_4 \equiv 0 (\bmod p)$ since (7) implies $-q_3 \equiv q_2 - q_1 (\bmod p)$.
To avoid this case we require that
10)$q_3 + q_4 \not\equiv 0 (\bmod p)$.
So the only remaining possibility is that $e_1 = e_2 = e_3 = 0$ and $e_4 > 0$. We want conditions under which $e_4 = 1$.

If $e_1 = e_2 = e_3 = 0$ and $e_4 \geq 2$ then $O(n_1) = O(n_2) = O(n_3) \leq O(n_4) - 2$, then by (c) of the tiling lemma:

$\lambda(n_3) + \lambda(n_4 - n_3) \equiv 0(\text{mod} p^2)$, but $n_4 - n_3 = n_2 - n_1$ and
$O(n_4 - n_3) = O(n_3) < O(n_4)$ so
$\lambda(n_3) + \lambda(n_4 - n_3) = \lambda(n_3) + \lambda(n_2 - n_1)$
$= \lambda(n_3) + \lambda(n_2) - \lambda(n_1) = q_3 + q_2 - q_1(\text{mod} p^2)$
by (a) of the tiling lemma.
To eliminate this possibility we require that
$q_3 + q_2 - q_1 \not\equiv 0(\text{mod} p^2)$.
This completes the construction.

2.7.2 Definition of I_{p_i}

For $p = p_1$ and p_2, $(\nu_1, \nu_2, \nu_3, \nu_4) \in R_1$ and $q_i = \lambda(\nu_i)$ define:
$I_p(q_1, q_2, q_3, q_4)$ iff

$$q_1 - q_2 - q_3 \equiv 0(\text{mod} p),$$
$$q_1 - q_2 - q_3 \not\equiv 0(\text{mod} p^2)$$

and each of the following is incongruent to $0(\text{mod} p)$

$$q_4 + q_1, q_4 - q_1,$$
$$q_4 + q_2, q_4 - q_2,$$
$$q_4 + q_3, q_4 - q_3.$$

We now combine E_{p_i} and I_{p_i} to set up the adjacency condition.

2.7.3 The adjacency condition

Given a global sample (s_1, s_2, s_3, s_4).

$$\text{If, for } i = 1 \text{ and } 2 \; I_{p_i}(\pi_i(s_1), \pi_i(s_2), \pi_i(s_3), \pi_i(s_4))$$

and

$$E_{p_{3-i}}(\pi_{3-i}(s_1), \pi_{3-i}(s_2), \pi_{3-i}(s_3), \pi_{3-i}(s_4))$$

then

$$(\pi_3(s_3), \pi_3(s_4)) \in A_i.$$

2.7.4 The adjacency condition effectively imposes the adjacency constraint

The gist of the proof of the next theorem is that given any p_i-orders e_i, then there is some global sample of cells which have equal p_i-orders and the fourth cell has p_{3-i}-order one higher than the rest of the cells.

The adjacency condition will then ensure that any accepted tiling will be such that the pair of tiles in the data channels of the third and fourth cells, form a pair in A_{3-i}.

Formally: by I_p: $O_{p_i}(\nu_3) = O_{p_i}(\nu_4) - 1 = e_i$, and by E_p:
$O_{p_{3-i}}(\nu_3) = O_{p_{3-i}}(\nu_4) = e_{3-i}$.

Hence, by the mapping between the tapes and the plane: $\lambda_3(\nu_3) = \tau(e_1, e_2)$ and $\lambda_3(\nu_4) = \tau(e_1', e_2')$ where, if $i = 1$, then $e_1' = e_1 + 1$ and $e_2' = e_2$ and if $i = 2$, then $e_2' = e_2 + 1$ and $e_1' = e_1$. But then by the adjacency condition in **2.7.2**: $(\pi_3(s_3), \pi_3(s_4)) \in A_i$.

Theorem 2.5 ($I.2$) *We prove the following:*

Let $\lambda_i; i = 1, 2$ *be perfect* p_i*-tilings. Take any* $e_1, e_2 \in N$. *Then for* $i = 1, 2$ *there is some global sample:* $(\nu_1, \nu_2, \nu_3, \nu_4); \nu_i \in Z \times 1, 2$

such that

$$O_{p_1}(n_1) = e_1, O_{p_2}(n_1) = e_2 \textit{ and}$$

$$E_{p_i}(\lambda_i(\nu_1), \lambda_i(\nu_2, \lambda_i(\nu_3), \lambda_i(\nu_4))$$

and

$$I_{p_{3-i}}(\lambda_{3-i}(\nu_1), \lambda_{3-i}(\nu_2), \lambda_{3-i}(\nu_3), \lambda_{3-i}(\nu_4)).$$

We do the case $i = 1$, the case $i = 2$ is analogous. Let $n_i = \pi_1(\nu_i)$, then n_1, n_2, n_3, n_4 will have the following form:

$n_1 = kp_1^{e_1}p_2^{e_2}k \not\equiv 0(\text{mod}p_i)$,

$n_2 = lp_1^{e_1}p_2^{e_2}l \not\equiv 0(\text{mod}p_i)$,

$n_3 = mp_1^{e_1}p_2^{e_2}m \not\equiv 0(\text{mod}p_i)$,

$n_4 = np_1^{e_1}p_2^{e_2+1}n \not\equiv 0(\text{mod}p_i)$ with

$n = m + (l - k)$, since $n_2 - n_1 = n_4 - n_3$.

This will ensure that $O_{p_i}(\nu_1) = e_i$.

We treat the condition I_{p_2} first.

We want k, l, m to be such that

$$I_{p_2}(\lambda_2(n_1), \lambda_2(n_2), \lambda_2(n_3), \lambda_2(n_4)),$$

so by the definition of I_p we require that

1)$\lambda_2(n_1) - \lambda_2(n_2) - \lambda_2(n_3) \equiv 0(\text{mod}p_2)$,

2)$\lambda_2(n_1) - \lambda_2(n_2) - \lambda_2(n_3) \not\equiv 0(\text{mod}p_2^2)$ and

3) The following be $\not\equiv 0(\text{mod}p_2)$:

$\lambda_2(n_4) \pm \lambda_2(n_1)$,

$\lambda_2(n_4) \pm \lambda_2(n_2)$,

$\lambda_2(n_4) \pm \lambda_2(n_3)$,

Consider condition (1).

If we can choose k, l, m such that

i)$Op_2(n_3) = Op_2(n_4 - n_3) = Op_2(n_4) - 1$ then by (b) of the tiling lemma $\lambda_2(n_3) + \lambda_2(n_4 - n_3) \equiv 0(\text{mod}p_2)$, but $n_4 - n_3 = n_2 - n_1$ so if we also can get that

ii)$Op_2(n_2 - n_1) = Op_2(n_1) = Op_2(n_3)$ then by (a) of the tiling lemma

$\lambda_2(n_2 - n_1) \equiv \lambda_2(n_2) - \lambda_2(n_1)(\text{mod}p_2^2)$, so

$\lambda_2(n_3) + \lambda_2(n_4 - n_3) = \lambda_2(n_3) + \lambda_2(n_2 - n_1)$

$\equiv \lambda_2(n_3) + \lambda_2(n_2) - \lambda_2(n_1)(\text{mod}p_2^2)$.
This gives

$$\lambda_2(n_3) + \lambda_2(n_2) - \lambda_2(n_1) \equiv 0(\text{mod}p_2).$$

Consider conditions (i) and (ii).
We have that $n_4 - n_3 = np_2^{(e_2+1)}p_1^{e_1} - mp_1^{e_1}p_2^{e_2} = (np_2 - m)(p_1^{e_1}p_2^{e_2})$
$= ([m + l - k]p_2 - m)p_1^{e_1}p_2^{e_2}$. Remember $n = m + l - k$.
So for (i) we require that $(m - (k - l))p_2 - m \not\equiv 0(\text{mod}p_2)$, i.e.
a)$(p_2 - 1)m - (k - l)p_2 \not\equiv 0(\text{mod}p_2)$.
For (ii) $n_2 - n_1 = (k - l)p_1^{e_1}p_2^{e_2}$ so we require that
b)$k - l \not\equiv 0(\text{mod}p_2)$.
Consider condition (2).
Working backwards with the calculation in (ii) above, we have
$\lambda_2(n_1) - \lambda_2(n_2) - \lambda_2(n_3) \equiv -\lambda_2(n_2 - n_1) - \lambda_2(n_3) = -\lambda_2(n_3) - \lambda_2(n_4 - n_3)$
$= -(\lambda_2(n_3) + \lambda_2(n_4 - n_3))(\text{mod}p_2^2)$.
Now $Op_2(n_4) = Op_2(n_3) - 1$, so $\lambda_2(n_3) + \lambda_2(n_4 - n_3) \not\equiv 0(\text{mod}p_2^2)$ which gives $-(\lambda_2(n_3) + \lambda_2(n_4 - n_3)) \not\equiv 0(\text{mod}p_2^2)$.
So if (i) is satisfied then (ii) is also satisfied.

Condition (3) requires that the following be incongruent to $0(\text{mod}p_2)$:
c)$(n \pm k)p_1^{e_1}$, d)$(n \pm l)p_1^{e_1}$,
e)$(n \pm m)p_1^{e_1}$,
by the form of n_1, n_2, n_3, n_4.
We now treat the condition E_{p_1}. By definition of E_p, we must have that
1)$\lambda_1(n_4) - \lambda_1(n_3) \equiv \lambda_1(n_2) - \lambda_1(n_1)(\text{mod}p_1)$,
and that the following are incongruent to $0(\text{mod}p)$:
2)$\lambda_1(n_2) - \lambda_1(n_1)$,
3)$\lambda_1(n_4) - \lambda_1(n_3)$,
4)$\lambda_1(n_3) - \lambda_1(n_1)$,
5)$\lambda_1(n_1) + \lambda_1(n_4)$,
6)$\lambda_1(n_2) + \lambda_1(n_3)$,
7)$\lambda_1(n_4) - \lambda_1(n_2)$.
For (1),(2) and (3) we require that
$\lambda_1(np_1^{e_1}p_2^{(e_2+1)}) - \lambda_1(mp_1^{e_1}p_2^{e_2}) \equiv \lambda_1(lp_1^{e_1}p_2^{e_2}) - \lambda_1(kp_1^{e_1}p_2^{e_2}) \not\equiv 0(\text{mod}p_1)$.
By the definition of $\lambda_1(n)$, to satisfy the above, we require:

$$np_2^{(e_2+1)} - mp_2^{e_2} \equiv lp_2^{e_2} - kp_2^{e_2} \not\equiv 0(\text{mod}p_1).$$

We can rewrite this as
f)$(np_2 - m)p_2^{e_2} \equiv (l - k)p_2^{e_2} \not\equiv 0(\text{mod}p_1)$.
Analogously, (4),(5),(6),and (7) reduce to the conditions:
g)$(m - k)p_2^{e_2} \not\equiv 0(\text{mod}p_1)$,
h)$kp_2^{e_2} + (m + l - k)p_2^{e_2+}1 \not\equiv 0(\text{mod}p_1)$,
i)$(l + m)p_2^{e_2} \not\equiv 0(\text{mod}p_1)$,
j)$(m + l - k)p_2^{e_2} - lp_2^{e_2} \not\equiv 0(\text{mod}p_1)$.

We have so many conditions now, also depending on the values of p_1 and p_2 that instead of showing that not all possible choices are eliminated as was done in E, we give an explicit choice for k, l, m, n for $p_1 = 10, p_2 = 11$:

$$k = 2, l = 4, m = p_i - 2, n = 1,$$

in our case $i = 2$ so $m = 9$, for $i = 1$ we have $m = 8$.

It is routine to check that these choices satisfy the conditions (a) to (j).

This completes the proof of I_2.

2.8 Accepted tilings of the plane map to accepted tilings of the tapes and conversely

Plane to tapes:

Given an origin constrained tiling system $\mathcal{D}$. Let $\tau : N \times N \rightarrow D$ be an accepted tiling of N^2.

We define a tiling of $Z \times 1, 2$:

$$\lambda_3(n, c) = \tau(O_{p_1}(n), O_{p_2}(n)), \text{ for } n \neq 0, \text{ and}$$

$$\lambda_3(0, 1) = \lambda_3(0, 2) \text{ an arbitrary element of } D.$$

Define

$$\lambda(n, c) = (\lambda_1(n, c), \lambda_2(n, c), \lambda_3(n, c)).$$

We show that λ is accepted by $\mathcal{L}_{\mathcal{D}}$.

1)If $\nu_1, \nu_2, \nu_3, \nu_4 \in Z \times 1, 2$ and $Z_p(\lambda_i(\nu_1), \lambda_i(\nu_2), \lambda_i(\nu_3), \lambda_i(\nu_4))$ then $O_p(\nu_1) = 0; i = 1, 2$

By the definition of λ_3 and the fact that $\tau(0, 0) \in D_0$ we have that $\lambda_3(\nu_1) \in D_0$.

That is, (2) of **2.2.2** is satisfied, since if any local sample satisfies Z_{p_1} and Z_{p_2} then the first cell has p_1-order 0 and p_2-order 0 by (Z) so by the mapping λ_3 the tile in the data channel is a tile in D_0.

Further, (1) of **2.2.2** is satisfied since if any global sample satisfies E_{p_1} and E_{p_2} then the p_1- and p_2-orders of all the cells are equal so by the mapping λ_3 they all have the same tile in the data channel.

Finally, (3) of **2.2.2** is satisfied since if any global sample satisfies $E_{p_{3-i}}$ and I_{p_i} then the p_{3-i}-orders of all the cells are equal (specifically those of n_3, n_4) and the p_i-order of n_4 is one more than the p_i-order of the rest.

Then by the mapping λ_3 and the fact that τ satisfies V, H we have that the pair $(\lambda_3(\nu_3), \lambda_3(\nu_3))$ is in A_i.

Tapes to plane:

Suppose $\mathcal{L}_{\mathcal{D}}$ accepts a tiling β. Assume that both address-channels are perfectly tiled. The next chapter will justify the assumption.

Define a tiling of the plane; $\tau : N \times N \longrightarrow D$ as follows:

$$\tau(e_1, e_2) = \pi_3(\lambda(n, c), \text{ where } O_{p_i}(n) = e_i \text{ for } i = 1, 2.$$

Note that n is any integer with p_i-order equal to e_i; $i = 1, 2$.

This is a well defined function by theorem $E.1$.We show that τ is accepted by $\mathcal{D}$.

Origin constraint: the tile at the origin is by the mapping τ the same tile as appears at any ν on the tapes with p_1- and p_2-orders 0. By Z this tile must be in D_0, since the tape tiling is accepted.

Adjacency constraints: let $e_1, e_2 \in N$ and let $i = 1$ or 2. Then by $I.2$ there is some global sample $(\nu_1, \nu_2, \nu_3, \nu_4)$ such that
$O_{p_j}(\nu_1) = e_j$ for $j = 1, 2$, $I_{p_i}(\lambda_i(\nu_1), \lambda_i(\nu_2), \lambda_i(\nu_3), \lambda_(\nu_4))$ and
$E_{p_{3-i}}(\lambda_{3-i}(\nu_1), \lambda_{3-i}(\nu_2), \lambda_{3-i}(\nu_3), \lambda_{3-i}(\nu_4))$.

By I_p: $O_{p_i}(\nu_3) = O_{p_i}(\nu_4) - 1 = e_i$ and by E_p: $O_{p_{3-i}}(\nu_3) = O_{p_{3-i}}(\nu_4) = e_{3-i}$. Hence $\lambda_3(\nu_3) = \tau(e_1, e_2)$ and $\lambda_3(\nu_4) = \tau(e_1', e_2')$ where, if $i = 1$, then $e_1' = e_1 + 1$ and $e_2' = e_2$ and if $i = 2$, then $e_2' = e_2 + 1$ and $e_1' = e_1$. But then by the adjacency constraint: if $i = 1$ then $A_1(\tau(e_1, e_2), \tau(e_1 + 1, e_2)$ or if $i = 2$ then $A_2(\tau(e_1, e_2), \tau(e_1, e_2 + 1)$. So the adjacency constraint is satisfied.

Informally: given any coordinate-pair on the plane, then there are cells one of which has p-orders the coordinate-pair and the other having one p-order the same and the other p-order one more. This pair of cells is forced by the adjacency conditions to have tiles in the data-channels such that the pair of data channel tiles is a pair in A_i. By the mapping to the plane, the horizontally or vertically adjacent pair of tiles, form a pair in A_i.

References:

Linear sampling problems are first described by Aanderaa and Lewis,[2], deriving from the outomata of Aanderaa,[1].

The 1-systems of Aanderaa and Lewis,[2] are single tape versions of the two tape systems of this chapter, which are due to Lewis,[20].

Chapter 3

Less-than-perfect tilings

3.1 Introduction

In the previous section we showed the following: if the address channels of a tiling are perfectly tiled, then we can identify cells with equal p_i-orders, cells with p_i-orders zero and cell pairs with p_i-orders corresponding to vertically or horizontally adjacent cells on the plane. We then set up conditions on the data channels of local and global samples which satisfy $Z_{p_i}, E_{p_i}, I_{p_i}$ and found that these conditions are equivalent to our conditions on tilings of the plane in that a tiling of the plane codes for a tiling of the tapes and conversely.

Our assumption all along was that the address channels contain the correct information, i.e. that the address-channels were perfectly tiled.

We have blithely ignored the fact that if a tiling is accepted, then any translate of the tiling will also be accepted, that is, our sampling conditions are not sensitive to (independent) movement of the two tapes. This follows directly from the sampling conditions, G and L of section **2.1**. A translation of a perfect tiling throws out our correlation between tilings of the tapes and the plane, specifically, the two tapes need no longer have the same entries in cells with the same coordinates.

Figure 10 shows a translation of a perfect tiling, with $k, m \geq 0$.

fig 10

p_1 address channel	$\lambda_{p_1}(m-5)$	$\lambda_{p_1}(m-4)$	$\lambda_{p_1}(m-3)$	$\lambda_{p_1}(m-2)$	$\lambda_{p_1}(m-1)$	$\lambda_{p_1}(m)$	$\lambda_{p_1}(m+1)$	$\lambda_{p_1}(m+2)$	$\lambda_{p_1}(m+3)$	$\lambda_{p_1}(m+4)$	$\lambda_{p_1}(m+5)$
p_2 address channel	$\lambda_{p_2}(m-5)$	$\lambda_{p_2}(m-4)$	$\lambda_{p_2}(m-3)$	$\lambda_{p_2}(m-2)$	$\lambda_{p_2}(m-1)$	$\lambda_{p_2}(m)$	$\lambda_{p_2}(m+1)$	$\lambda_{p_2}(m+2)$	$\lambda_{p_2}(m+3)$	$\lambda_{p_2}(m+4)$	$\lambda_{p_2}(m+5)$
data channel											
	-5	-4	-3	-2	-1	0	1	2	3	4	5
p_1 address channel	$\lambda_{p_1}(-k-5)$	$\lambda_{p_1}(-k-4)$	$\lambda_{p_1}(-k-3)$	$\lambda_{p_1}(-k-2)$	$\lambda_{p_1}(-k-1)$	$\lambda_{p_1}(-k)$	$\lambda_{p_1}(-k+1)$	$\lambda_{p_1}(-k+2)$	$\lambda_{p_1}(-k+3)$	$\lambda_{p_1}(-k+4)$	$\lambda_{p_1}(-k+5)$
p_2 address channel	$\lambda_{p_2}(-k-5)$	$\lambda_{p_2}(-k-4)$	$\lambda_{p_2}(-k-3)$	$\lambda_{p_2}(-k-2)$	$\lambda_{p_2}(-k-1)$	$\lambda_{p_2}(-k)$	$\lambda_{p_2}(-k+1)$	$\lambda_{p_2}(-k+2)$	$\lambda_{p_2}(-k+3)$	$\lambda_{p_2}(-k+4)$	$\lambda_{p_2}(-k+5)$
data channel											

We have gone to great trouble to set up an internal coordinate system for the tapes by 'numbering' the cells, only to find that we cannot keep the tapes still!

Now if a linear sampling system accepts some translate of a perfect tiling then it also accepts a perfect tiling, which as was shown, can be mapped to an accepted tiling of the plane. But what if 'worse' tilings are accepted than translates of perfect tilings i.e. how much can an accepted tiling differ from a perfect tiling? This is a crucial question since the constructions in the first chapter depend on the address channels being perfectly tiled.

In this section we will deal mainly with the address channels, and specifically with the question:

To what extent can the address channels of an accepted tiling differ from the address channels of a perfect tiling, and can an accepted tiling of the tapes in each case be mapped to an accepted tiling of the plane?

If we can no longer assume that address channels are perfectly tiled then the only way to try to ensure this is to place conditions on local and global samples of entries(tiles) in **address channels**.

3.2 Conditions on address channels

3.2.1 The local condition

If we examine the example of the perfect tiling of an address channel for $n = 1, \ldots, 120$ and $p = 10$ in **2.3.2** then it is apparent that sequences of $p-1$ out of p tiles are consecutive numbers. Every p cells the tiling 'jumps'. For cells of order 1 the jump is 'half-predictable' in that the entries in the address-channels of such cells are determined (mod$p$) by the entry in the address-channel of the preceding cell , on the other hand, at cells of order 2 or more we cannot predict the 'jump' at all from the entry in the address channel of the preceding cell.

Note that the tiling lemma is a direct consequence of this fact generalized to arbitrary p-order.

It would therefore be natural to set up as local condition on an address channels, the condition L_p, $p = p_1$ or p_2, which we now define.

Definition of L_{p_i}

Let $p = p_1$ or $p_2; p \geq 10$.

L_p contains the following 4-tuples:

for $0 \leq x \leq p-1$, the tuples

$(xi, x(i+1), x(i+2), x(i+3)$; where $1 \leq i \leq p-4$,

for $0 \leq x < p-1$, the tuples
$(x(p-3), x(p-2), x(p-1), a(x+1))$
$(x(p-2), x(p-1), a(x+1), (x+1)1)$
$(x(p-1), a(x+1), (x+1)1, (x+1)2)$
$(a(x+1), (x+1)1, (x+1)2, (x+1)3)$; where $0 \leq a \leq p-1$,

for $x = p-1$,the tuples
$((p-1)(p-3), (p-1)(p-2), (p-1)(p-1), ab)$
$((p-1)(p-2), (p-1)(p-1), ab, 01)$
$((p-1)(p-1), ab, 01, 02)$
$(ab, 01, 02, 03)$; where $0 \leq a \leq p-1, 0 < b \leq p-1$.

These conditions will ensure that under some translation, cells of order 0 will be perfectly tiled and cells of order 1 will be correct mod$p$ which by the discussion above is the best we can do with the local condition.

To place restrictions on the address channels of cells of p-order 2 or higher, we must place restrictions on **global samples**.

3.2.2 The global condition

Let λ_p be a perfect tiling of address-channel p:

Given $p = p_1$ or p_2, then the tiling lemma implies the following:
if $n_l, n_k \in Z$ then $\lambda_p(n_k - n_l)$ is congruent (modp^2) to one of the following:
1)$\lambda(n_k) - \lambda(n_l)$ if $O(n_l) = O(n_k - n_l) = O(n_k)$
2)$p\lambda(n_k) - \lambda(n_l)$ if $O(n_l) = O(n_k - n_l) = O(n_k) - 1$
3)$\lambda(n_k) - p\lambda(n_l)$ if $O(n2) = O(n_k - n_l) = O(n_l) - 1$
4)$-\lambda(n_l)$ if $O(n_l) = O(n_k - n_l) \leq O(n_k) - 2$
5)$\lambda(n_k)$ if $O(n2) = O(n_k - n_l) = O(n_l) - 1$
6)$[\lambda(n_k) - \lambda(n_l)]/p(\text{mod}p)$ if $O(n_l) = O(n_k) = O(n_k - n_l) - 1$
in this case $\lambda(n_l) \equiv \lambda(n_k)(\text{mod}p)$
7)ab: a, b arbitrary with $0 < b < p; 0 \leq a < p$
if $O(n_l) = O(n_k) \leq O(n_k - n_l) - 2$.
In this case $\lambda(n_l) \equiv \lambda(n_k)(\text{mod}p^2)$.

These are the only possible cases since at least two of $n_k, n_l, n_k - n_l$ must have the same order and if n_k, n_l have the same order then $n_k - n_l$ cannot have a lower order.

Now if (q_1, q_2, q_3, q_4) is some global sample of address channels of a perfect tiling at say n_1, n_2, n_3, n_4, then, since $q_i := \lambda(n_i)$ and $n_2 - n_1 = n_4 - n_3$, we have that

$$\lambda(n_2 - n_1) = \lambda(n_4 - n_3).$$

Therefore, some term in the set

$$\{q_2 - q_1, pq_2 - q_1, q_2 - pq_1, -q_1, q_2\}$$

is congruent modp^2 to some term in the set:

$$\{q_4 - q_3, pq_4 - q_3, q_4 - pq_3, -q_3, q_4\}$$

or

$$q_4 \equiv q_3(\text{mod} p)$$

and some term in the first set is congruent mod$p$ to the term $(q_4 - q_3)/p$,

or

$$q_2 \equiv q_1(\text{mod} p)$$

and some term in the second set is congruent mod$p$ to the term $(q_2 - q_1)/p$,

or

$$q_2 \equiv q_1(\text{mod} p^2) \text{ or } q_3 \equiv q_4(\text{mod} p^2).$$

Conversely if for some arbitrary 4-tuple (q_1, q_2, q_3, q_4) we have any of the above congruences, then we can always find n_1, n_2, n_3, n_4 such that $\lambda(n_i) = q_i$ and $n_2 - n_1 = n_4 - n_3$, by adding multiples of p^2 to the q_i; $1 \leq i \leq 4$.

3.2.3 Definition of G_{p_i}

From the above discussion, we can define the global sampling conditions for p_i-address channels as follows.

For $p = p_1, p_2$: $G_p(q_1, q_2, q_3, q_4)$ iff

some term in the set $\{q_2 - q_1, pq_2 - q_1, q_2 - pq_1, -q_1, q_2\}$ is congruent modp^2 to some term in the set $\{q_4 - q_3, pq_4 - q_3, q_4 - pq_3, -q_3, q_4\}$, or

$q_4 \equiv q_3(\text{mod}p)$ and some term in the first set is congruent mod$p$ to the term $[q_4 - q_3]/p$, or

$q_2 \equiv q_1(\text{mod}p)$ and some term in the 2nd set is congruent mod$p$ to the term $[q_2 - q_1]/p$, or

$q_2 \equiv q_1(\text{mod}p^2)$ or $q_3 \equiv q_4(\text{mod}p^2)$.

3.3 Accepted imperfect tilings

Any other tiling of which all the globally sampled 4-tuples of tiles in p_i-address channels are in the sets G_{p_i} and local samples are in the set L_{p_i}, will be accepted. We of course would like that accepted tilings are at worst, translates of perfect tilings. This hope is forlorn.

We call a tiling of address channels which is not a perfect tiling, an imperfect tiling.

Since we used only the tiling lemma to define condition G_p, any other tiling satisfying the tiling lemma will be accepted. In the next section we construct a tiling which is accepted but far from perfect.

3.3.1 Construction of an imperfect but accepted tiling

For $i = 1$ or 2, let $O(n) = O_{p_i}(n)$ and $\lambda(n) = \lambda_{p_i}(n)$.

By (a) of the tiling lemma: if $O(n_1) = O(n_2) = O(n_2 - n_1)$ then we must have that

$$\lambda(n_1) + \lambda(n_2 - n_1) \equiv \lambda(n_2)(\text{mod}p^2)$$

i.o.w.

$$\lambda(n_2) - \lambda(n_1) \equiv \lambda(n_2 - n_1)(\text{mod}p^2).$$

This will be satisfied if the difference between consecutive cells of the same order is the same $(\text{mod}p^2)$.

Consider the following piece of perfect tiling:

$\lambda_{10}(n)$ of $n = 1, \ldots, 140$

01	02	03	04	05	06	07	08	09	**01**	11	12	13	14	15	16	17	18	19	**02**
21	22	23	24	25	26	27	28	29	**03**	31	32	33	34	35	36	37	38	39	**04**
41	42	43	44	45	46	47	48	49	**05**	51	52	53	54	55	56	57	58	59	**06**
61	62	63	64	65	66	67	68	69	**07**	71	72	73	74	75	76	77	78	79	**08**
81	82	83	84	85	86	87	88	89	**09**	91	92	93	94	95	96	97	98	99	01
01	02	03	04	05	06	07	08	09	**11**	11	12	13	14	15	16	17	18	19	**12**
21	22	23	24	25	26	27	28	29	**13**	31	32	33	34	35	36	37	38	39	**14**

Digits in bold are in cells of order 1. It is apparent that the entries in cells immediately preceding $\lambda(10)$ and $\lambda(110)$ are the same. The local condition cannot force cells of order 1 to be correct modp^2 (we can only force them be correct mod$p$), we can therefore replace the digit-pairs in cells of order 1 with others which equal them mod$p$. If we just ensure that the entries are such that any two consecutive cells of order 1 differ by the same number $(\text{mod}p^2)$, the tiling remains accepted but is no longer perfect.

The first few numbers of order 1 are:
10,20,30,40 having as values for $\lambda(n)$:01 02 03 04.
We can change 01 to 11 to get the sequence
11 22 33 44... (from 11 22 33 44...),
or to 21 to get the sequence
21 42 63 84... (from 21 42 63 84...),
or to 51 to get the sequence
51 02 53 04... (from 51 102 153 204...), and so on.

Each new sequence is the same mod$p$ as the original sequence, we have just replaced 01 by some $k1 (0 \leq k < p)$, multiplied all terms of order 1 with $k1$ and taken the remainder modp^2 to get the new cell entries for cells of order 1.

Formally we set $q(n) = rem((\lambda(n)(kp + 1)), p^2)$ for all n of order 1.

Part (b) of the tiling lemma requires that for a cell of order 2, we must have that

$$\lambda(a) + \lambda(b - a) \equiv p\lambda(b)(\text{mod}p^2), \text{ whenever } O(a) = O(b - a) = 1 = O(b) - 1.$$

This means that when we reach a multiple of $k1$ which is of order 2 namely $(pk1, 2pk1, ..)$, then the digits in the cell of order 2 (with coordinate $mpk1$; $m \neq 0(\bmod p)$) must be congruent $\bmod p^2$ to $(mp-1)k1+k1$, i.e. the 2nd digit of the cell must be m. This means that the digits in cells of order 2 of our tiling must be the same (mod$p$) as the digits in cells of order 2 of a perfect tiling for our tiling to be accepted.

Exactly as was done for order 1 we can multiply all digits in p_i-address channels of cells of order 2 with some $l1$ where $l1$ is in p_i-ary notation.

We can continue in this way for cells of order 2,3,4 and so on. That is for each order $i \geq 1$ we can multiply all digit-pairs in p-address channels with some number $l1, 0 \leq l < p$. We can choose a different $l(0 \leq l < p)$ for each order. In this way we construct an imperfect but accepted tiling.

We call our constructed tiling a **normal tiling**.

3.3.2 Normal tilings

Definition of normal tilings

Formally we can define normal tilings $\alpha(n)$ as follows:

$$\alpha(n) = rem(\lambda(n)(h(O(n))p+1)), p^2)$$

where h is a function: $h : N - \{0\} \rightarrow \{0, 1, .., p-1\}$, and $N - 0$ denotes the set of natural numbers without 0.

In the above tiling $h(O(n))$ was k and l respectively for orders 1 and 2 respectively. Note that a perfect tiling is a normal tiling with $h(i) = 0$ for all i. In a normal tiling consecutive cells of the same order differ by $a1(\text{mod}p^2)$, for some $0 \leq a < p$ instead of by $01(\text{mod}p^2)$ in a perfect tiling.

Further, as is clear from the definition: $\alpha(n) \equiv \lambda(n)(\text{mod}p)$, i.e. a normal tiling differs from a perfect tiling only on the first digits of the double digit numbers in the address-channels of the cells. Note that it follows from the definition that normal tilings satisfy the agreement property.

We now verify that the tiling lemma is satisfied for normal tilings.

3.3.3 Normal tilings satisfy the tiling lemma

If $n_1, n_2 \in Z$ and
1)$O(n_1) = O(n_2 - n_1) = O(n_2)$, then $\alpha(n_1) + \alpha(n_2 - n_1) \equiv \alpha(n_2)(\text{mod}p^2)$,
2)$O(n_1) = O(n_2 - n_1) = O(n_2) - 1$, then $\alpha(n_1) + \alpha(n_2 - n_1) \equiv p\alpha(n_2)(\bmod p^2)$,
3)$O(n_1) = O(n_2 - n_1) \leq O(n_2) - 2$, then $\alpha(n_1) + \alpha(n_2 - n_1) \equiv 0(\text{mod}p^2)$.

These relations are easily checked using the tiling lemma for perfect tilings and the definition of $\alpha(n)$.

We prove (2) as an example:

$\alpha(n_1) + \alpha(n_2 - n_1) \equiv \lambda(n_1)(kp+1) + \lambda(n_2 - n_1)(kp+1)$

$\equiv (kp+1)(\lambda(n_1)+\lambda(n_2-n_1)) \equiv (kp+1)p\lambda(n_2) \equiv kp^2\lambda(n_2)+p\lambda(n_2)$
$\equiv p\lambda(n_2)(\text{mod}p^2)$
$\equiv p\alpha(n_2)(\text{mod}p^2)$ since $\lambda(n_2 \equiv \alpha(n_2)(\text{mod}p)$.

3.3.4 The sets of global samples of address channels of normal tilings and of perfect tilings are the same

Let $n_1, n_2 \in Z$, then $\alpha(n_2 - n_1)$ is congruent $\text{mod}p^2$ to one of the following:
1) $\alpha(n_2) - \alpha(n_1)$ if $O(n_1) = O(n_2 - n_1) = O(n_2)$,
2) $p\alpha(n_2) - \alpha(n_1)$ if $O(n_1) = O(n_2 - n_1) = O(n_2) - 1$,
3) $\alpha(n_2) - p\alpha(n_1)$ if $O(n_2) = O(n_2 - n_1) = O(n_1) - 1$,
4) $-\alpha(n_1)$ if $O(n_1) = O(n_2 - n_1) \leq O(n_2) - 2$,
5) $\alpha(n_2)$ if $O(n_2) = O(n_2 - n_1) \leq O(n_1) - 2$,
6) $[\alpha(n_2) - \alpha(n_1)]/p$ if $O(n_1) = O(n_2) = O(n_2 - n_1) - 1$.
In this case $\alpha(n_1) \equiv \alpha(n_2)(\text{mod}p)$.
7) kl arbitrary, if $O(n_1) = O(n_2) \leq O(n_2 - n_1) - 2$.
In this case $\alpha(n_1) \equiv \alpha(n_2)(\text{mod}p^2)$.

So if two pairs of cells in a normal tiling (a pair on each tape) are equidistant then the same congruences hold as for perfect tilings.

But if such an equality holds for all global samples then these samples are also allowed in a perfect tiling, so all global samples in a normal tiling are also global samples in a perfect tiling.

Conversely, a perfect tiling is a normal tiling with $h(O(n)) = 0$ for all $n \in N$.

So we have that the set of global samples of perfect and normal tilings are the same.

Hence G_p accepts all normal tilings along with perfect tilings.

So we have to deal with the fact that not all acceptable tilings are perfect.

3.3.5 Normal tilings are not too bad

If an accepted tiling is normal the correlation between the plane and the tapes can still be carried out since our definitions of $Z_{p_i}, E_{p_i}, I_{p_i}$ purposely relied only on the tiling lemma and the values $\text{mod}p$ of the entries in the address channels.

That is, if we replace λ with α the constructions of $Z_{p_i}, E_{p_i}, I_{p_i}$ are defined in exactly the same way with *perfect* replaced by *normal* and perfect tilings λ of address-channels by normal tilings α. So for normal tilings the relations $Z_{p_i}, E_{p_i}, I_{p_i}$ still correctly identify cells with the appropriate order relations and $E.2$ and $I.2$ prove that the conditions are effective.

What if there are accepted tilings which are neither perfect nor normal, nor translates of either?

In the next section we show that normal tilings are nearly as bad as it gets.

3.4 Accepted tilings at their worst

Let us examine an arbitrary accepted tiling:

Our local condition on the p_i-address channel L_{p_i} forces cells of p_i-order 0 to be perfectly tiled under some independent translation of both tapes. This is clear from the discussion above.

We will show that given any $n \in N$, that we can shift the tiling so that both address channels are normally tiled for all cells of p_i-order smaller than n. By induction on n it will follow that we can get accepted tilings with arbitrary long normal segments and then König's infinity lemma will imply the existence of an accepted normal tiling, that is, a tiling for which the address channels are tiled normally for all orders.

The fact that the existence of arbitrarily long accepted stretches of tiling by a finite alphabet implies by König's lemma that an accepted tiling exists, is the reason that each coordinate of N^2 must be represented infinitely often and at bounded intervals along at least one of the tapes. Indeed, if this were not the case for some coordinate n we could collect the segments of tiling between representations of the coordinate, and since there is no upper bound in length for such segments there would exist by König's infinity lemma, an accepted tiling with no representation of n at all.

3.4.1 Aligning the address channels

Given $p = p_1$ or p_2, we will find a translation under which cells of p-order less than i have normally tiled p-address channels. We will then use the Chinese Remainder Theorem to show that we can align both address channels on a tape by a single shift of the tape.

Basis for the Induction

Let β be any accepted tiling.

The address channels of cells of order 0 are normal(in fact perfect) under some translation. This follows from the local condition L. Specifically such address channels are correct (mod$p$) under some translation.

Induction hypothesis

Formally the induction hypothesis is as follows:

The tiling β is such that there is some normal tiling α and integers t_1, t_2 such that

if $O(n) < i$ then $\beta(n + t_c, c) = \alpha(n)$ and
if $O(n) = i$, then $\beta(n + t_c, c) \equiv \alpha(n)(\text{mod} p)$.

That is, the accepted tiling β is the same as the normal tiling α for p-order less than i and congruent mod$p$ to the normal tiling α for p-order i.

We must then show that there is some normal tiling α' and integers t'_1, t'_2 such that

$$\text{if } O(n) < i+1, \text{ then } \beta(n + t'_c, c) = \alpha'(n)$$

and

$$\text{if } O(n) = i+1, \text{ then } \beta(n + t'_c, c) \equiv \alpha'(n)(\text{mod}p).$$

Proof:

We first prove the following:

Lemma 3.4.1 *For every $d \in Z$ there is an $s(d)$; $0 < s(d) < p^2; s(d) \equiv \lambda(d)(\text{mod}p)$ such that*

$$\beta(n + d + t_c, c) - \beta(n + t_c, c) \equiv s(d)(\text{mod}p^2); c = 1, 2$$

whenever $O(n) = O(n + d) = i$.

Since the only non-consecutive 4 tuples we can inspect are at positions of the form $((n_1, 1); (n_2, 2); (n_3, 1), (n_4, 2))$ we prove the following as an intermediary step:

Lemma 3.4.2 *If $O(d) = i$ then there is some $r(d)$; $0 < r(d) < p^2; r(d) \equiv \lambda(d)(\text{mod}p)$ such that*

$$\beta(n + d + t_2, 2) - \beta(n + t_1, 1) \equiv r(d)(\text{mod}p^2)$$

whenever $O(n) = O(n + d) = i$.

Proof of lemma 3.4.2:

Given m_1, m_2 such that $O(m_1) = O(m_1 + d) = i = O(m_2) = O(m_2 + d)$. Let n be any number of order i such that $\lambda(n) + \lambda(d) \not\equiv 0(\text{mod}p)$ (so that $O(n + d) = i$), $\lambda(n) - \lambda(m_1) \not\equiv 0(\text{mod}p)$, $\lambda(n) - \lambda(m_2) \not\equiv 0(\text{mod}p)$, $\lambda(n) + \lambda(m_1) + \lambda(d) \not\equiv 0(\text{mod}p)$, $\lambda(n) + \lambda(m_2) + \lambda(d) \not\equiv 0(\text{mod}p)$. We can choose such an n since $p > 6$.

Then

$$(\beta(m_j + t_1, 1); \beta(m_j + d + t_2, 2); \beta(n + t_1, 1); \beta(n + d + t_2, 2)); j = 1, 2$$

is in G_p since β is accepted.

Since $\beta(m + t_c, c) \equiv \lambda(m)(\text{mod}p)$ for all m of order i by the induction hypothesis, it follows from E_p that $O(m_j + t_1) = O(m_j + d + t_2) = O(n + t_1) = O(n + d + t_2) = O(d + t_2 - t_1)$. The last equality follows from the fact that if $O((m_j + d + t_2) - (m_j + t_1)) > O(m_j + t_1)$ then $\beta(m_j + t_1, 1) \equiv \beta(n + t_1, 1)(\text{ mod } p)$ which is contradictory. By (a) of the tiling lemma it follows that

$\beta(m_j + d + t_2, 2) - \beta(m_j + t_1, 1) \equiv \beta(n + d + t_2, 2) - \beta(n + t_1, 1) \bmod p^2$ for $j = 1, 2$. It follows that $\beta(m_1 + d + t_2, 2) - \beta(m_1 + t_1, 1) \equiv \beta(m_2 + d + t_2, 2) - \beta(m_2 + t_1, 1)(\text{mod} p^2)$. Lemma 1.4.2 follows.

Proof of lemma 3.4.1

We have that $\beta(n + d + t_2, 2) - \beta(n + t_1, 1) \equiv r(d)(\text{mod} p^2)$.

Now given n, d, d_1, d_2 such that
$d = d_2 - d_1; O(n) = O(d_1) = O(d_2) = O(d) = O(n - d_1) = O(n + d_2 - d_1) = i$.
Then
$\beta(n + d + t_2, 2) - \beta(n + t_2, 2) = \beta(n + d_2 - d_1 + t_2, 2) - \beta(n + t_2, 2)$
$= \beta(n + d_2 - d_1 + t_2, 2) - \beta(n - d_1 + t_1, 1) - \beta(n + t_2, 2) - \beta(n - d_1 + t_1, 1)$
$= r(d_2) - r(d_1)(\text{mod} p^2)$.
Further, $r(d_2) - r(d_1) \equiv \lambda(d_2) - \lambda(d_1) \equiv \lambda(d)(\bmod p)$ by (a) of the tiling lemma.

Now if $d'_2 - d'_1 = d$ then
$r(d'_2) - r(d'_1)$
$= \beta(n + d'_2 - d'_1 + t_2, 2) - \beta(n - d'_1 + t_1, 1) - (-\beta(n + t_2, 2) - \beta(n - d'_1 + t_1, 1))$
$= \beta(n + d'_2 - d'_1 + t_2, 2) - \beta(n + t_2, 2) = \beta(n + d + t_2, 2) - \beta(n + t_2, 2)$
$= \beta(n + d_2 - d_1 + t_2, 2) - \beta(n + t_2, 2)$.

So $r(d_2) - r(d_1) = r(d'_2) - r(d'_1)$ if $d_2 - d_1 = d'_2 - d'_1$.

Lemma 1.4.1 for the case $c = 2$ follows if we set $s(d) = r(d_2) - r(d_1)$ where $d = d_2 - d_1$.

It also follows for $c = 1$ since given n, d with $O(n) = O(n+d) = i$ we can find a d' with $O(d') = O(n+d') = O(n+d+d') = i$. Then by the result for $c = 2$ we have $\beta(n+d+t_1, 1) - \beta(n+t_1, 1) \equiv \beta(n+d+d'+t_2, 2) - r(d') - (\beta(n+d'+t_2, 2) - r(d'))$ $= \beta(n + d + d' + t_2, 2) - \beta(n + d' + t_2, 2) \equiv s(d)(\text{mod} p^2)$.

This completes the proof of lemma 1.4.1.

It follows from lemma 1.4.1 that $s(kd) \equiv ks(d)(\text{mod} p^2)$
whenever $O(d) = O(kd) = i$.

So A) $\beta(kp^i + t_c, c) \equiv \beta(p^i + t_c, c) + (k - 1)s(p^i)(\text{mod} p^2)$ if $k \not\equiv 0(\text{mod} p)$, i.e. if $O(kp^i) = i$. Since consecutive cells of i order differ by $s(p^i)$, cells of order i which are $(k - 1)p^i$ apart differ by $(k - 1)s(p^i)$.

We want to use lemma 3.4.1 to align the tape for cells of order i, but we do not want to change the cells of order $< i$ which are already correct, this can be done if the next translate has p-order $i + 1$ or larger.

For $c = 1, 2$ we can choose a $k_c \equiv 1(\text{mod} p); 0 < k_c < p^2$ such that:
B) $\beta(k_c p^i + t_c, c) = s(p^i)$.

This follows from (A) and the following : $s(d) \equiv \lambda(d)(\text{mod} p)$ for each d of order i, $(k_c - 1)s(p^i) \equiv 0(\text{mod} p)$, $\beta(p^i + t_c, c) \equiv \lambda(p^i) \equiv s(p^i) \equiv 01(\text{mod} p^2)$ (since $s(d) \equiv \lambda(d)(\text{mod} p)$ for each d of order i). It follows that we can choose k_c so that $(k_c - 1) +$ the 1st digit of $\beta(p^i + t_c, c)$ add up to the 1st digit of $s(p^i)$. This proves (B).

Now let $l_c = (k_c - 1)p^i$, then $O(l_c) \geq i + 1$ for $c = 1, 2$ and

$\beta(l_c + kd + t_c, c) \equiv ks(d)(\text{mod}p^2)$ when $O(d) = i$ and $k \neq 0(\text{mod}p)$.
Therefore $\beta(l_c + n + t_c, c) \equiv \lambda(n)s(p^i)(\text{mod}p^2)$ for each n of order i, since if $O(n) = i$ then $n = ap^i$ for some $a \neq 0(\text{mod}p)$ so
$\beta(l_c+n+t_c, c) = \beta(l_c+ap^i+t_c, c) \equiv as(p^i)(\bmod p^2)$. Now $s(p^i) \equiv 1(\bmod p)$ and $\lambda(n)$ is the last two digits having 2nd digit non-zero of n in p-ary notation so $\lambda(n) \equiv a(\text{mod}p^2)$.
Now let $t'_c = l_c + t_c$ for $c = 1, 2$ and let h be such that
$\alpha(n) \equiv \lambda(n)(1 + ph(O(n))(\text{mod}p^2)$ for $O(n) < i$. There exists such an h by the fact that cells for orders less than i, are normally tiled.
We have that $\beta(n + t'_c, c) = \beta(l_c + n + t_c, c) \equiv \lambda(n)s(p^i)(\text{mod}p^2)$.

To show that the tiling is normal for cells of order i, we want that for $O(n) \leq i$, $\beta(n + t'_c, c) = \lambda(n)(1 + ph'(O(n)))(\text{mod}p^2)$ for some h'.
We therefore want an h' such that $s(p^i) \equiv (1 + ph'(O(n)))(\text{mod}p^2)$.

Define : $h'(j) = h(j)$ if $j \neq i$
$= (s(p^i) - 1)/p$ if $j = i$.
Remember $s(p^i) \equiv 1(\text{mod}p)$.
Define $\alpha'(n)$ as follows:

$$\alpha'(n) \equiv \lambda(n)(1 + ph'(O(n)))(\text{mod}p^2).$$

If $O(n) < i$ we have that
$\beta(n + t'_c, c) = \beta(l_c + n + t_c, c) = \beta(n + t_c, c)$
$= \alpha(n) \equiv \lambda(n)(1 + ph(O(n))$
$= \lambda(n)(1 + ph'(O(n)))$
$\equiv \alpha'(n)(\text{mod}p^2)$,
since $O(l_c) \geq i + 1$ and $O(n + t_c) < i$.

If $O(n) = i$ then $\beta(n + t'_c, c) = \beta(l_c + n + t_c, c)$
$\equiv \lambda(n).s(p^i)$
$\equiv \lambda(n)(1 + ph'(i))$ by definition of $h'(i)$
$= \lambda(n)(1 + ph'(O(n)))$
$\equiv \alpha'(n)(\text{mod}p^2)$.
Therefore, under our translation the accepted tiling has normal p_i-address channels up to order i.

Tiles in cells of p-order $i + 1$ equal $\alpha'(\text{mod}p)$

We now show that if $O(n) = i + 1$ then $\beta(n + t'_c, c) \equiv \alpha'(n)(\text{mod}p)$.
Proof:
Let n be any integer of order $i + 1$. We show that we can find an n' and d such that $n', n' + d, n - d$ are of order i and
$(\beta(n' + t'_1, 1), \beta(n' + d + t'_2, 2), \beta(n - d + t'_1, 1), \beta(n + t'_2, 2))$
satisfies I_p.

For then:

$p\beta(n+t'_2,2) \equiv \beta(n-d+t'_1,1)+\beta(n'+d+t'_2,2)-\beta(n'+t'_1,1)$
$\equiv \alpha'(n-d)+\alpha'(n'+d)-\alpha'(n') \equiv p\alpha'(n)-\alpha'(d)+\alpha'(n')+\alpha'(d)-\alpha'(n')$
$= p\alpha'(n)(\text{mod}p^2)$.

The last step follows by (a) and (b) of the tiling lemma.

We explain the first step. If a global 4-tuple $(\alpha(n_1),\alpha(n_2),\alpha(n_3),\alpha(n_4))$ satisfies I_p then $O(n_1)=O(n_2)=O(n_3)=O(n_4)-1=O(n_2-n_1)$. The last equality follows from the fact that at least two of $O(n_3),O(n_4),O(n_4-n_3)$ must be equal. Now it follows from (b) of the tiling lemma that

$p\alpha(n_4) \equiv \alpha(n_3)+\alpha(n_4-n_3)(\text{mod}p^2)$.

Then by (a) of the tiling lemma

$\alpha(n_4-n_3)=\alpha(n_2-n_1) \equiv \alpha(n_2)-\alpha(n_1)(\text{mod}p^2)$, hence
$p\alpha(n_4) \equiv \alpha(n_3+\alpha(n_2)-\alpha(n_1)(\text{mod}p^2)$.

Thus $\beta(n+t'_2,2) \equiv \alpha'(n)(\text{mod}p)$.

By symmetry, if we consider the 4-tuple

$\beta(n'+d+t'_1,1),\beta(n'+t'_2,2),\beta(n+t'_1,1),\beta(n-d+t'_2,2)$ it also follows that $\beta(n+t'_1,1) \equiv \alpha'(n)(\text{mod}p)$.

We now show that given any n of order $i+1$ that we can find such an n',d.

We will choose n' and d from $\{p^i,2p^i,....(p-1)p^i\}$. Then $O(n')=O(d)=i$ and we want that $n'+d$ and $n-d$ are of order i, i.e. that $\lambda(n')+\lambda(d) \not\equiv 0(\text{mod}p)$. Note that n is of order $i+1$ and d is of order i so $n-d$ is also of order i. For each choice of n' there are only $p-2$ choices for d, a total of $(p-1)(p-2)$ choices for the pair (n',d).

For I we must have the following incongruent to 0 mod p:

$q_4 \pm q_1, q_4 \pm q_2\ , q_4 \pm q_3 \not\equiv 0(\text{mod}p)$.

So $\beta(n+t_2,2)$ must be incongruent mod$p$ to each of the following:

$\pm\beta(n'+t'_1,1),\pm\beta(n'+d+t'_2,2),\pm\beta(n-d+t'_1,1)$.

Two choices for n are eliminated, two for $n'+d$ and two for $n-d$, i.e. for d (n is given). Each choice eliminated for n' disqualifies $p-1$ choices for the pair (n',d), the same is true for each choice eliminated for d. The same follows for $n'+d$, since for each n' one d is disqualified. Therefore, $(p-2)1$ pairs are eliminated.

So $6(p-2)$ choices are eliminated but the total amount of possible pairs is $(p-1)(p-2)$ so if $p-1>6$ i.e. $p \geq 8$ we can find n',d as desired.

This completes the proof.

A useful image for the alignment of the address channels order by order is the following : imagine having to align some copy C of $\lambda(n); n \in Z$ for $p=10$ with the sequence $\lambda(n); n \in Z$, without the ability to 'see' C globally.
It is impossible to know globally where on C you find yourself since sequences of arbitrary length repeat at different places, for example, the sequence

01 02 03 04 05 06 07 08 09 01 11 12 13 14 15 16 17 18 19 02

...
81 82 83 84 85 86 87 88 89 09 91 92 93 94 95 96 97 98 99 01
appears in cells with coordinates
$1, \ldots, 100$, also $1001, \ldots, 1100$ also $2001, \ldots, 2100$, etc.

The best one can do is align C for cells order 0 and as soon as you come across a discrepancy with a cell of order 1, align C so that cells of order 1 are correct and continue in this way for higher and higher orders.

In general this alignment of C will not terminate since E and I can only help us to align the cells one order up from those which are already aligned so there could always be some higher order of which the cells are not yet correctly aligned.

3.4.2. Combining the alignments

It follows from the lengthy proof above that we can for p_1-and p_2-address channels seperately, translate the channels of an accepted tiling in such a way that the $p_1(p_2)$ address channels are normally tiled up to an arbitrary large $p_1(p_2)$ order.

We now combine the translations for the p_1 address channels and p_2 address channels to show that if we have an accepted tiling then we can shift the tapes to get both the first and 2nd address channels normal up to any given order. Remember that shifting the tapes does not spoil the acceptability of a tiling.

By the infinity lemma(König) it will follow that the system accepts some normal tiling since it accepts arbitrary long normal segments.

We use the Chinese Remainder Theorem:

Let β be any accepted tiling and β_i the projection function onto the ith address channel. Given any positive integer r, we show that β has a normally addressed segment of length $2r + 1$.

Let $i + 1$ be the smallest integer larger than $log_{p_1}(r)$.

If $0 < |n| \leq r$ then $Op_1(n) < i + 1$ and $Op_2(n) < i + 1$ (we set $p_1 < p_2$).

Now by the discussion in **3.4.1**, for p_1 and p_2, there are integers $t_{11}, t_{12}, t_{21}, t_{22}$ such that for some normal tiling α

$\beta_1(n + t_{11}, 1) = \alpha_1(n), (p_1$ order 1st tape),
$\beta_1(n + t_{12}, 2) = \alpha_1(n), (p_1$ order 2nd tape),
$\beta_2(n + t_{21}, 1) = \alpha_2(n), (p_2$ order 1st tape),
$\beta_2(n + t_{22}, 2) = \alpha_2(n), (p_2$ order 2nd tape).

Now by the Chinese Remainder Theorem there are integers t_1, t_2 such that
$t_1 \equiv t_{11} (\mathrm{mod} p_1^{i+2})$,
$t_1 \equiv t_{21} (\mathrm{mod} p_2^{i+2})$,
$t_2 \equiv t_{12} (\mathrm{mod} p_1^{i+2})$,
$t_2 \equiv t_{22} (\mathrm{mod} p_2^{i+2})$.

Let $s_{11} = t_1 - t_{11}, s_{12} = t_2 - t_{12}, s_{21} = t_1 - t_{21}, s_{22} = t_2 - t_{22}$, then

$Op_1(s_{11}) > i+1, Op_1(s_{12}) > i+1; Op_2(s_{21}) > i+1, Op_2(s_{22}) > i+1$
and so by the agreement property, if $O_{p_i}(n) \leq i+1$, then:
$\beta_1(n+t_1,1) = \alpha_1(n,1)$,
$\beta_1(n+t_2,2) = \alpha_1(n,2)$,
$\beta_2(n+t_1,1) = \alpha_2(n,1)$,
$\beta_2(n+t_2,2) = \alpha_2(n,2)$.
The following pair of sequences are normally addressed:
$S_1 = (\beta(t_1-r,1), \beta(t_1-r+1,1), ..., \beta(t_1+r,1))$,
$S_2 = (\beta(t_2-r,2), \beta(t_2-r+1,2), ..., \beta(t_2+r,2))$,
since if $0 < |n| \leq r$ then
$O_{p_1}(n) < i+1$ and $O_{p_2}(n) < i+1$ hence
$\beta_k(n+t_c,c) = \alpha_k(n)$ for $k=1,2; c=1,2$.

This combines the alignment for p_1-and p_2-orders into a single shifting of the tapes.

3.4.3 Summary

For any accepted tiling we have that given any order i, we can move the two tapes so that the address channels up to order i are tiled normally. When we align the address channels for order $i+1$ we move the tapes by some multiple of p^{i+2} which leaves the the address channels of cells with order smaller than $i+1$ unchanged.

In this inductive way we obtain for each $l \in N$ an accepted tiling with address channels normal up to order l, König's infinity lemma then implies that there is an accepted tiling with address channels tiled normally for all orders of p_1 and p_2.

This justifies the mapping from accepted tilings of the tapes to accepted tilings of the plane defined at the end of chapter 1, since if any tiling is accepted then a tiling with normal address channels is accepted and for normal address channels we can use $Z_{p_i}, E_{p_i}, I_{p_i}$ to place conditions on a tiling of the tapes which are equivalent to conditions on the tilings of the plane.

References:

The first part of this chapter is an explanation (not given in Lewis,[20]) for the shift from *perfect* tilings to *normal* tilings in the proof in Lewis,[20]. The proof that any accepted tiling can be transformed into a normal tiling is the *admissibility lemma* in Lewis,[20]:p37-42. The *admissible* and *i-admissible* tilings of Lewis,[20]:p29 are not explicitly defined in this presentation of the proof.

The sets L_{p_i} and G_{p_i} which have been defined explicitly here, are defined in Lewis,[20]: p23-24 essentially as the *sets of local and global samples of normal tilings*, which does not seem to be a very helpful definition.

Lewis,[20] constructs the relations $Z_{p_i}, E_{p_i}, I_{p_i}$ for normal tilings, while they are presented here as constructed for perfect tilings, and it is then just mentioned that the same constructions go through for normal tilings, these constructions are exactly those in Lewis,[20] and this presentation is solely to facilitate understanding of the development from *perfect* tilings to *normal* tilings.

Chapter 4

A three channel version of the linear sampling problem

4.1 Introduction

In this chapter we prove a slightly different form of the linear sampling problem to be unsolvable. This is necessary for the application to first order logic. The constructions are from Lewis,[20].

4.2 The three channel version

We want to demonstrate the unsolvability of the linear sampling problem for systems $\mathcal{L}^\theta; \theta > 0$ where

$$X = Z \times 1, 2, 3$$

that is, the space to be tiled is three disjoint copies of the integers or three disjoint *tapes*.

Our local sampling conditions are the same as for the previous system, except that we now have three tapes, that is, we may sample any θ consecutive cells on any of the three tapes.

The global sampling configurations, however, become *triples* of tiles:

$$(n_1, n_2, n_3) \text{ with } n_3 = n_2 - n_1.$$

Further, we must show that the tiling problem for such systems is unsolvable even if cells on the third tape may contain only either a 0 or a 1 , and no local condition is imposed on the third tape.

To clarify the idea of the three tape system we eventually construct, we first sketch an interim three-tape system.

4.3 The interim system

Let T be a two tape system, with $|T| = q$, and consider the following three tape system:
the first two tapes contain tiles $t \in T$, from the original system.
We now consider the third tape.

Set up some bijection f between pairs of tiles from T and numbers $0, 1, 2, \ldots, q^2-1$. That is, $f(a, b) \in \{0, 1, 2, .., q^2-1\}$ for each pair $(a, b) : a, b \in T$. Let each cell on the third tape be divided into q^2 channels and let each channel contain either a 0 or a 1.

For each pair of tiles let $\pi_l(n) = 1$ indicate that the pair of tiles (a, b) with $f(a, b) = l$ appears distance n apart, with tile a on the first tape, b on the second. Further, let $\pi_l(n) = 0$ indicate that the pair does not appear distance n apart anywhere on the first two tapes(one on each tape). That is, the l-th channel of cell n on the third tape indicates whether or not the pair of tiles referred to by l appears distance n apart.

Let the global sampling configurations of our three-tape system be triples of coordinates : $(n_1, n_2, n_2 - n_1)$.

Now say we take a global sample of tiles from some $n_1, n_2; n_2 > n_1$ and from the k-th channel of $n_3 = n_2 - n_1$. Let the tiles a and b appear at n_1, n_2 on the first and second tape respectively, and let $f(c, d) = k$. Now, a 1 in channel k of $n_2 - n_1$ would indicate that the pair of tiles (c, d) appear distance $n_2 - n_1$ apart somewhere on the first two tapes. Disallowing a 1 in this position makes it impossible for the pair (c, d) to appear distance $n_2 - n_1$ apart and consequently for (a, b) and (c, d) to both appear $n_2 - n_1$ apart.

In general, disallowing the triple $(a, b, 1)$ where the 1 is read in channel k, makes it impossible for (a, b) and (c, d) to appear equal distance l apart, for any $l \in N$.

In this way we can place conditions on the three tape system which only allow certain pairs of tiles to appear equal distance apart on the first two tapes. This is of course exactly what we did in the original two-tape systems.

4.4 The final system

Since our tiling system must contain a single digit (1 or 0) in each cell of the third tape, we cannot divide the third tape into channels as in the interim system, instead we let q^2 adjacent cells on the third tape play the role of these q^2 channels.

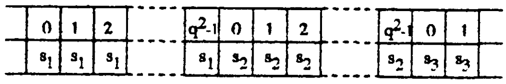
To carry through the construction in this form we must repeat each tile on the first two tapes q^2 times, then a 1 or a 0 in the cell at $dq^2 + k$ on the third tape will indicate whether or not the pair $f^{-1}(k)$ appears distance dq^2 apart one on each of the first two tapes. Note that tiles appearing distance dq^2 apart in this system correspond to tiles appearing distance d apart in the interim system.

The idea is exactly the same as the interim construction.

4.5 Conditions on the three-tape system

An accepted tiling of $Z \times \{1,2,3\}$ will be a 'stretched' version of an accepted tiling of $Z \times \{1,2\}$ in that the tiling of $Z \times \{1,2,3\}$ contains q^2 repeats of each tile in $Z \times \{1,2\}$, with the numbers $0,\ldots,q^2-1$ written above the q^2 repeats, as figure 11 illustrates:

fig 11



We fix some correspondence between $\{0,..q^2-1\}$ and the q^2 possible pairs of tiles in T.

4.5.1 The correspondence

We will show the problem unsolvable for $\theta = 2$, the proof for the case for $\theta = i; i \geq 2$ is analogous to that for the two-tape system, at the beginning of chapter 2.

Given a $Z \times \{1,2\}$ tiling system, $\mathcal{L} := (T, L, G)$ we set up the conditions for a corresponding $Z \times \{1,2,3\}$ system $\mathcal{L}' = (T', L', G')$.

Let $T' = (\{0,1,2,\ldots,q^2-1)\} \times T$ and let L' and G' be as follows.

The local condition

Let (s_1, s_2) and $(s_2, s_3) \in L$, then for each $j : 0 \leq j \leq q^2-1$,
$L' \subset T^\theta$ contains:

$$((j, s_1, (j+1, s_1), \ldots, (q^2-1, s_1), (0, s_2), \ldots, (q^2-1, s_2), (0, s_3), \ldots, (j-1, s_3))$$

These are just the θ-tuples which would appear if we stretched out the tiling of $Z \times \{1,2\}$ as in figure 11.

The global condition

a) For any s_1, s_2 and $j_1, j_2; 0 \leq j_1 \leq j_2 \leq q^2-1$ with $f(j_2-j_1) = (s_1, s_2)$:
i)$((j_1, s_1), (j_2, s_2), 0) \notin G'$,
ii)$((j_1, s_1), (j_2, s_2), 1) \in G'$ iff (s_1, s_2, s_1, s_2) in G,
(this is of course always the case if $s_1, s_2 \in T$).

This condition ensures that, if two tiles s_1, s_2 appear dq^2+k apart and $f(k) = (s_1, s_2)$ then a 1 appears at dq^2+k on the third tape.

We now place restrictions on which pairs may appear equal distances apart.

b) For any $s_1, s_2 \in T$, and any j_1, j_2; $0 \leq j_1 \leq j_2 \leq q^2 - 1$
if $f(j_2 - j_1) = (s_3, s_4) \neq (s_1, s_2)$:

i)$((j_1, s_1), (j_2, s_2), 0) \in G'$ and

ii)$((j_1, s_1), (j_2, s_2), 1)$ in G' iff (s_1, s_2, s_3, s_4) in G.

c) For any $s_1, s_2 \in T$, and any j_1, j_2; $0 \leq j_2 < j_1 \leq q^2 - 1$ and for $e = 0, 1$:

$(j_1, s_1, j_2, s_2) \in G'$.

Condition (b) is the condition corresponding to the global condition of $\mathcal{L}$. Condition (c) just means that we do not place conditions on pairs for which $j_2 < j_1$. Condition (b) on pairs for which $j_1 \leq j_2$ suffices.

It should be intuitively clear that the systems $\mathcal{L}$ and $\mathcal{L}'$ are equivalent. For a formal proof in the form of a mapping from accepted tilings of $\mathcal{L}$ to accepted tilings of $\mathcal{L}'$ and conversely, see Lewis,[20]:p47-48.

From this the unsolvability of this form of the linear sampling problem follows. We can strengthen the result to the following.

The linear sampling problem for three tapes is unsolvable, even if we restrict ourselves to the class of tiling-systems which are such that, if some tiling is accepted then a tiling with a 0 at coordinate zero on the third tape is also accepted. The following argument sketches the reason:

Given a three-tape tiling system $\mathcal{L} = (T, L, G)$ then we construct a system $\mathcal{L}' = (T', L', G')$ with $T' = T \cup \overline{T}$ where $\overline{T} := \{\overline{t} | t \in T\}$. Consider the tiles $\overline{t}$ as the 'complements' of tiles t. Set $\overline{0} = 1$ and $\overline{1} = 0$. We then construct as new local and global conditions:

$$L' = L \cup \{(\overline{s}, \overline{t}) | (s, t) \in L\}$$

$$G' = G \cup \{(\overline{s}, \overline{t}, 1 - e) | (s, t, e) \in G\}.$$

This means that the tilings accepted by $\mathcal{L}'$ are the tilings accepted by $\mathcal{L}$ along with those tilings where each tile is replaced with its complement. One of the tilings must have a 0 at coordinate zero on the third tape.

References: This presentation is essentially that of Lewis,[20] except for the interim system constructed here.

Chapter 5

First order logic

This chapter, taken mainly from Lewis and Papadimitriou,[21], and Chang and Li,[6], is a very informal survey of the results and concepts needed for the proofs in the next few chapters.

5.1 Language

5.1.1 Alphabet

The formulas of first order logic which we will work with are certain strings of the following symbols:

- quantifiers: $\forall$ and $\exists$
- logical connectives: $\vee$ and $\neg$;

  $\leftrightarrow, \wedge, \rightarrow$ are constructed from these in the usual way.
- a countably infinite number of variables: $x, y, z \ldots$
- a countably infinite number of k-place predicate signs for each $k \geq 0$: $P, Q, R \ldots$
- a countably infinite number of k-place function signs for each $k \geq 0$: $f, g, h \ldots$ (0-place function signs are *constants*)
- parenthesis ().

(Note that in first order logic quantification must be over elements of the domain, and not over functions or predicates.)

5.1.2 Terms

Terms are recursively defined as follows:

constants and variables are terms and if $t_1, t_2, \ldots, t_n$ are terms and f is an n-place function sign then $f(t_1, t_2, \ldots, t_n)$ is a term.

We write $f^2(t)$ for $f(f(t))$ and so on.

5.1.3 Formulas

Formulas are recursively defined as follows:

if P is an n-place predicate symbol, and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an *atomic* formula. Formulas are constructed from atomic formulas using the logical connectives, quantifiers and parenthesis. We will sometimes omit certain parenthesis and write for example, $Pxf(x)$ for $P(x, f(x))$.

If $F_1, .., F_n$ are formulas then $(F_1 \vee \ldots \vee F_n)$ is a formula, called the *disjunction* of $F_1, \ldots, F_n$ and $(F_1 \wedge \ldots \wedge F_n)$ is a formula, called the *conjunction* of $F_1, \ldots, F_n$.

5.2 Interpretations and models

An *interpretation* $\mathcal{I}$ of a formula F in first order logic consists of a non-empty domain D, and an assignment of 'values' to each constant, function symbol and predicate symbol occuring in F, as follows.

1. To each constant, we assign an element in D.
2. To each n-place function symbol, $n > 0$ we assign a mapping from D^n to D, where $D^n := \{(x_1, \ldots, x_n) \mid x_1 \in D, x_2 \in D, \ldots, x_n \in D\}$.
3. To each n-place predicate symbol P, we assign an n-ary relation on D.

We consider an n-ary relation as a subset S of D^n, interpreted as the n-tuples in D for which P is true. We can consider this as a mapping from D^n to $\{T, F\}$: $P(t_1, t_2)$ is mapped to T(true) iff t_1 and t_2 are mapped to elements d_1, d_2 in D such that $(d_1, d_2) \in S$. An interpretation under which a formula is true, is a *model* for the formula.

5.2.1 Free and bound variables

Roughly speaking, an occurence of a variable is free, if no quantifier applies to it, otherwise the occurence is bound, consider the formula $Q(x) \vee \forall x P(x)$. The first occurence of x is free and the second is bound.

Formulas containing free occurences of variables cannot be evaluated to be true or false. All the formulas we consider will not contain any free occurences of variables, such formulas are called *closed*.

5.2.2 Truth values of formulas

For a given interpretation, truth values of closed formulas are recursively defined in the following way: given the truth values of F, G then the truth value of $F \vee G, F \wedge G, \neg F$ are determined as follows.

- If either G or F is true then the formula $G \vee F$ is true otherwise it's false.
- If both F and G are true then the formula $F \wedge G$ is true, otherwise false.
- $\neg F$ is true iff F is false.
- $\forall x F$ is true if F is true for all replacements of x by elements of D, otherwise it's false.
- $\exists x F$ is true if F is true for at least one element of D, otherwise it's false.

Example

Let P be a two place predicate sign and f a 1-place function sign, and let F be the formula $\forall x P x f(x)$.

Now let $D := N = \{1, 2, 3...\}$; $P := \{(m, n) : m, n \in N, m < n\}$; f be the successor function: $f(n) = n + 1; n \in N$.

Then F is true since every number is less than its successor. Therefore the formula is true under this interpretation.

5.2.3 Satisfiability of a formula

A formula F is *satisfiable* iff there exists an interpretation I such that F is true under I. If this is the case then we call I a *model* for F.

The **satisfiability problem** is the problem of deciding, given a formula or a class of formulas, whether the formulas are satisfiable or not.

Two formulas are *equivalent* if they have the same truth values under the same interpretations.

The problem of determining whether or not a formula is satisfiable by trying to see whether it has a model or not is clearly problematic since there are in general an infinite number of possible domains and interpretations.

5.2.4 Rectified formulas

A formula is *rectified* if no variable occurs both free and bound, and there is at most one occurence of a quantifier with any particular variable. Any formula can be transformed into an equivalent rectified formula by renaming variables, a rectified form of the formula $(Q(x) \vee \forall x P(x, x))$ would be $(Q(x) \vee \forall y P(y, y))$.

5.2.5 Prenex formulas

If a formula F is of the form $q_1v_1q_2v_2...q_nv_nF^M$ where each q_i is either $\forall$ or $\exists$ and F^M contains no quantifiers, then F is said to be in *prenex form* and $q_1v_1..q_nv_n$ is called the *prefix* while F^M is called the *matrix*.

It is well known that any formula in first order logic can be transformed into an equivalent rectified formula in prenex form.

Example

The formula $\exists z\forall y\exists w((Q(z) \wedge (P(z,y) \vee \neg Q(w)) \wedge R(x)$
is a rectified prenex form of the formula
$(\exists x(Q(x) \wedge (\forall y P(x,y) \vee \neg\forall y Q(y)) \wedge R(x)$.

For a formula F in rectified prenex form, the variables of F are in one-to-one correspondence with the occurences of quantifiers.

5.2.6 x-and y-variables

We call the variables of a formula F governed by existential quantifiers, *x-variables* and the variables governed by universal quantifiers, *y-variables*.

5.2.7 The functional form of a formula

Consider the formula:

$F = \exists x_1\forall y_1\forall y_2\exists x_2\forall y_3\exists x_3F^M(x_1,y_1,y_2,x_2,y_3,x_3)$.

Now $F^M(x_1,y_1,y_2,x_2,y_3,x_3)$ is the matrix of F and contains the variables x_1,y_1,y_2,x_2,y_3,x_3.

The *functional form* of this formula would be:

$$\forall y_1\forall y_2\forall y_3F^M(a,y_1,y_2,f(y_1,y_2),y_3,g(y_1,y_2,y_3)).$$

Where we replace the x-variables as follows: x_1 with some constant a, x_2 with the term $f(y_1,y_2)$ and x_3 with the term $g(y_1,y_2,y_3)$.

This form of the formula makes explicit the following: the value of x_1 should be independent of the values of any of the other variables, we must therefore be able to assign some constant value to x_1 to make the formula true.The value of the second x-variable, x_2, depends on the values of y_1 and y_2 and we therefore replace x_2 with the term $f(y_1,y_2)$ to indicate the dependence. In the same way x_3 is replaced with the term $g(y_1,y_2,y_3)$ to indicate that the choice of x_3 depends on the values of y_1,y_2 and y_3.

In this way the functional form reflects the fact that it must be possible to choose values for x-variables, corresponding to replacements of y-variables with elements from the domain, in such a way that the matrix F^M comes out true in each case.

In general, if F is a closed rectified formula then the functional form F^* of F, is obtained by replacing in the matrix F^M each x-variable x_i of F by the term $f_{x_i}(y_{i_1}, ...y_{i_n})$ where $n \geq 0$ is the number of y-variables $(y_{i_1}, ...y_{i_n})$ governing x_i in F, if x_i is governed by no y-variable then we replace x_i with a 0-place function sign or *constant*. The new function signs we introduce are called *indicial function signs* and the terms $f_{x_i}(y_{i_1}, ...y_{i_n})$ that replace x_i are called *indicial terms* for x_i in F.

The functional form F^* of a formula F is a quantifier free formula with free variables exactly the y-variables of F.
We can algorithmically transform any formula F into it's functional form F^*.

5.2.8 F is satisfiable $\leftrightarrow$ F^* is satisfiable

We sketch the proof.
If F is satisfiable under some interpretation then for each replacement of y-variables with elements from the domain, we must be able to choose values for the x-variables so that the matrix turns out true, we can then expand the interpretation to include the function sign f_x and define $f_x(y_1, .., y_n)$ as the value which makes F true.

On the other hand, if the functional form F^* is satisfiable then there is some interpretation in some domain D for which F^* is true, in particular the element in D denoted by $f_x(y_1, ..., y_n)$, but then F will also be true if we choose for x in F the element denoted by $f_x(y_1, ..., y_n)$.

5.2.9 Herbrand domains

Given a formula in functional form F, define the set of terms $D(F)$ as follows.

- If F contains no function sign, then $1 \in D(F)$.
- If $t_1, .., t_n \in D(F)$; $n \geq 0$ and f is any n-place function sign(indicial or not) occurring in F, then $f(t_1..t_n) \in D(F)$.

This set of terms is the *Herbrand domain.* The Herbrand domain is to be considered as a set of terms **without denotation** i.e. the terms refer to themselves and do not have a 'meaning' in some external structure, for example: $f(f^{n(1)})$ is just the term $f^{n+1}(1)$ and nothing else. We say that the Herbrand domain is a *syntactic* domain.

Note that nothing in the definition of satisfiability rules out using a set of terms themselves as a domain for interpretation of a formula.

5.2.10 Herbrand expansion

The *Herbrand expansion* of a formula is the set of all formulas:
$F\{y_1/t_1, .., y_n/t_n\}$ where $y_1, .., y_n$ are all the y-variables of F and $t_1, .., t_n$ are any terms in $D(F)$. We call the terms $t_n; n \geq 0$ the substituents of the $y_n; n \geq 0$.

Example:
the formula:

$$\forall y \exists x Pyx \wedge \exists z \forall w \neg Pwz$$

has functional form:

$$\forall y Pyf(y) \wedge \forall w \neg Pwa,$$

the matrix of the functional form being

$$Pyf(y) \wedge \neg Pwa.$$

The Herbrand expansion is the following sequence of formulas:

$$Paf(a) \wedge \neg Paa, Pf(a)f(f(a)) \wedge \neg Paa,$$

$$Paf(a) \wedge \neg Pf(a)a, Pf(a)f(f(a)) \wedge \neg Pf(a)a,$$

and so on.

Each formula of the sequence is called a Herbrand instance. The first formula is obtained by substituting a for both y and w, the second by substituting $f(a)$ for y and a for w, and so on.

The atomic formulas $Paf(a), \neg Paa, Pf(a)f(f(a)), \neg Paa$ appear together in the first Herbrand instance, we call such atomic formulas *directly related.*

Analogously for the formulas $Paf(a), \neg Pf(a)a, Pf(a)f(f(a)), \neg Pf(a)a$ which appear together in the second Herbrand instance.

Each instance of the Herbrand expansion is viewed as a formula of propositional logic. In general a formula can have any finite number of indicial function signs for different x-variables and some finite number $k > 0$ of atomic formulas. Each Herbrand instance consists of atomic formulas, logical connectives and parenthesis.
Note that the Herbrand expansion is a listable, countable set since $t_1, .., t_n$ are all in the (countable) Herbrand domain.

We are now ready for the result which we will use in the following chapters.

5.2.11 The functional form of a formula is satisfiable iff the Herbrand expansion is truth functionally consistent

We sketch the proof. A formula in functional form is satisfied under some interpretation iff, for each substitution from the domain for the y-variables, the indicial terms corresponding to the values, make the matrix true.

So to be satisfied the matrix must be true for arbitrary replacements of y-variables and corresponding indicial terms. Now given any formula and an interpretation $\mathcal{I}$ in some domain M for which it is satisfied, we show that the Herbrand expansion is truth-functionally consistent:

define the following mapping from the Herbrand domain to the domain $\mathcal{I}$:

- the constants in F map to anything in the domain M,
- the term $f(t_1,..,t_n), t_1,..,t_n \in D(F)$ maps to $f^{\mathcal{I}}((t_1^{\mathcal{I}}),(t_2^{\mathcal{I}}),..,(t_n^{\mathcal{I}})) \in M$.

Now define the predicate sets for $D(F)$:
if P is an n-place predicate sign:

$$(t_1,..,t_n) \in P_D \leftrightarrow ((t_1^I),..,(t_n^I) \in P^I.$$

It is clear that the matrix is verified, whichever terms in the Herbrand domain we replace into the matrix, since the matrix is verified for arbitrary replacements of elements of M for free variables and corresponding value for indicial terms.

Now, if we have a model as above then we have a truth assignment:

$$P(t_1..t_n) \text{ true iff}((t_1^{\mathcal{I}}),..,(t_n^{\mathcal{I}}) \in P^{\mathcal{I}}.$$

This assignment is consistent since $P^{\mathcal{I}}$ consists of a fixed set of n-tuples. The assignment verifies the matrix of F whenever we substitute terms from $D(F)$. In other words the Herbrand expansion is verified in a truth-functionally consistent fashion.

Conversely, say that there is some truth assignment $\mathcal{A}$ which verifies the entire expansion, that is, a non-contradictory truth assignment for all predicate letters for all applicable tuples from $D(F)$. Then we can define a model for F as follows:

$$P := \{(t_1,..t_n) : Pt_1..t_n \text{ true under} \mathcal{A}\}$$

A satisfied Herbrand expansion is therefore in effect an exhaustive check that the formula is true for all substitutions for free variables and corresponding indicial terms from the Herbrand domain and is therefore a verification that F is satisfied.

The problem of satisfiability thus shifts from a 'Platonic', 'existence' point of view to an (in general) infinite combinatorial problem, i.e. **is it possible to assign truth values to the atomic formulas of the Herbrand expansion, in a non-contradictory way, such that each instance of the Herbrand expansion is simultaneously verified?**

The relation between the Halting problem and the satisfiability problem for formulas is clear from the above. A formula is satisfiable iff all the Herbrand instances can be verified without any inconsistent truth value assignments. It is conceivable that such a check may never terminate and in fact the next section demonstrates the unsolvability of the satisfiability problem for certain classes of formulas of first order logic.

References:

Versions of the expansion theorem were given by Skolem, Herbrand and Gödel.

Chapter 6

Introduction to chapters 7 and 8

One of the ways of classifying formulas of first order logic is by the **form of the prefix** of the formula in prenix-form.

We can, for example, consider the class of first order formulas with prefix

$$\forall y_1 \exists x_1 \forall y_2.$$

The names of the variables are unimportant and we therefore refer to this class simply as

$$\forall\exists\forall.$$

We will use the superscript * to denote classes in which no restrictions are placed on the (finite) number of quantifiers of a certain type at a certain position in the prefix.

Example
The class of formulas of which the prefix contains any finite number of universal quantifiers in succession followed by a single existential quantifier is denoted by $\forall^*\exists$

Another way of classifying formulas of first order logic is by the **number of predicate letters of each degree** occuring in the formulas. For example, we refer to the class of formulas with one monadic predicate letter and two dyadic predicate letters as

$$(1, 2)$$

In general, the k-th element of the tuple denotes the number of predicate letters of order k.

For example, in the class $(1, 2)$, there is one predicate letter of order 1, two of order two and none of higher order. When no restriction is placed on the (finite) number of predicate letters of a certain order we write ∞ in the place of the number.

Example
$(\infty, 1)$ is the class of formulas with any finite number of monadic predicate letters and one dyadic predicate letter.

We can combine the two classifications and, for example, denote by

$$[\forall\exists\forall(\infty, 1)]$$

the class of formulas in the intersection of the classes

$$\forall\exists\forall \text{ and } (\infty, 1).$$

6.1 The Satisfiability problem for the classification

The satisfiability problem for the following classes in this classification were shown to be solvable:

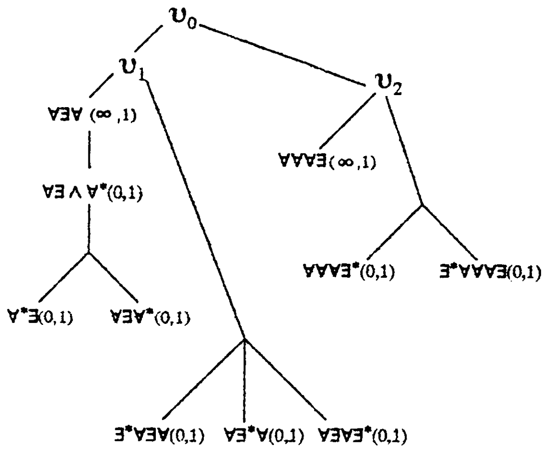
the monadic class (∞)(Löwenheim 1915),
the class $\exists^*\forall^*$(Bernays and Schönfinkel(1928)),
the class $\exists^*\forall\exists^*$(Ackermann(1928), Skolem(1928), Herbrand(1931)),
the class $\exists^*\forall\forall\exists^*$(Gödel(1932,1933), Kalmár(1933), Schütte(1934)).

Given the listed proofs of solvability, the satisfiability problem for this classification of formulas reduces naturally to the following nine classes: (proven unsolvable by the names in brackets)

$[\forall\exists\forall(\infty, 1)]$ (Kahr(1962)),
$[\forall\forall\exists(\infty, 1)]$ (Surányi(1959)),
$[\forall^*\exists(0, 1)]$ (Gurevich 1966b),
$[\forall\exists\forall^*(0, 1)]$ (Denton 1963),
$[\forall\forall\exists^*(0, 1)]$ (Kalmar and Surányi(1947)),
$[\exists^*\forall\forall\exists(0, 1)]$ (Surányi(1959)),
$[\exists^*\forall\exists\forall(0, 1)]$ (Surányi(1959)),
$[\forall\exists^*\forall(0, 1)]$ (Kostryko (1964)),
$[\forall\exists\forall\exists^*(0, 1)]$ (Gurevich (1966a,1966b).
These references are from Lewis[20]:91-93.
Lewis,[20] uses the unsolvability of the linear sampling problem to demonstrate the unsolvability of these classes in a uniform fashion.

Figure 12 illustrates the sequence and dependences of the proofs of Lewis.

fig 12



As an illustration of the method we will show in chapter 7 that the class V_0(to be defined) is unsolvable by using the unsolvability of the linear sampling problem. This proof is due to Lewis.

In chapter 8 we show V_1 unsolvable by showing that for each formula $F \in V_0$, we can construct a formula in $G \in V_1$ of which the Herbrand expansion contains an encoded copy of the Herbrand expansion of F. The proof we present is a version of the proof of Lewis without using his notion of bigraph-machinery.(see Lewis,[20]:p80-86)

We give indications how V_2 is shown unsolvable, the proof of which is very much the same as the proof for V_1. The interested reader is referred to Lewis' monograph for the proofs of the unsolvability of the rest of the diagram.

References: This chapter is essentially from Lewis,[20].

Chapter 7

The linear sampling problem and first order logic

All the results and constructions of this chapter are from Lewis,[20].

Let $\mathcal{V}_0$ be the class of all closed, rectified, prenex formulas:

$$F = \forall y_1 \forall y_2 F^M$$

where F^M contains no function signs but f, and only the predicate letters P(dyadic) and $R_0, R_1, \ldots$ (monadic) and such that:

1. the only atomic formulas of F are $Py_1y_2, Pf(y_1)f(y_2)$ and $R_iy_1, R_iy_2, R_if(y_1); (i \geq 0)$, and
2. if F is satisfiable at all, then F is also verified under some assignment which falsifies Ptt for each $t \in D(F)$.

In other words there is an assignment which gives the value false to all the atomic formulas of the Herbrand expansion of the form $Pf^n(1)f^n(1); n \in (0, 1, 2, 3....)$ and verifies the whole sequence $E(F)$.

In this section we use the unsolvability of the linear sampling problem to show the following.

The satisfiability problem for the class of formulas $\mathcal{V}_0$, is unsolvable.

We show that $\mathcal{V}_0$ is unsolvable by showing that for each linear sampling system $\mathcal{L} = (T, L, G)$ there is some formula in $\mathcal{V}_0$ which is satisfiable iff there exists a tiling acceptable to $\mathcal{L}$. We do this by constructing a formula in $\mathcal{V}_0$ which under a translation states that there is an accepted tiling for $\mathcal{L}$.

7.1 The formula

Let F be the following formula:

$$\forall y_1 \forall y_2 (M_1 \wedge M_2 \wedge M_3 \wedge M_4),$$

where the M_i are as follows (∇ denotes 'exclusive or'):

M_1 : $Py_1y_2 \leftrightarrow Pf(y_1)f(y_2)$,

M_2 : $\nabla_{i=0}^{q-1} R_i y_1 \wedge \nabla_{i=q}^{2q-1} R_i y_1$,

M_3 : $\vee_{(i,j)\in L}(R_i y_1 \wedge R_j f(y_1) \bigwedge \vee_{(i,j)\in L}(R_{q+i} y_1 \wedge R_{q+j} f(y_1))$,

M_4 : $\vee_{(i,j,e)\in G}(R_i y_1 \wedge R_{q+j} y_2 \wedge H_e)$.

Where H_e is Py_1y_2 if $e = 1$ and $\neg Py_1y_2$ if $e = 0$.

7.2 The translation

We associate truth assignments $\mathcal{A}$ on atomic formulas in the Herbrand expansion and tilings T of $\mathcal{T}$ in the following way.

- For $i = 1, \ldots, q-1 : \mathcal{A} \models R_i f_(^n 1)$ iff $\tau(n, 1) = i$. That is, $R_i f^n(1)$ is true under $\mathcal{A}$ iff the tile at n on the 1st tape is i.
- For $i = q, \ldots, 2q-1 : \mathcal{A} \models R_i f^n(1)$ iff $\tau(n, 2) = i - q$. That is, $R_i f^n(1)$ is true under $\mathcal{A}$ iff the tile at n on the 2nd tape is i and
- $\mathcal{A} \models Pf^m(1)f^n(1)$ iff $\tau(n-m, 3) = 1$. That is, $Pf^m(1)f^n(1)$ is true under $\mathcal{A}$ iff the tile at $n - m$ on the third tape is a 1.

Under this correlation $M_1 - M_4$ says the following about τ:

- T_1: The truth value of $Pf^m(1)f^n(1)$ depends only on $n - m$. This follows from M_1 by induction.
- T_2: $\nabla_{i=0}^{q-1} \tau(n, 1) = i \wedge \nabla_{i=q}^{2q-1} \tau(n, 2) = i - q)$. Each cell of the first two tapes contains exactly one tile.

  If y_1 has substituent $f^m(1)$, then $f(y_1)$ has substituent $f^{m+1}(1)$ and M_3 states:
- T_3: $(\bigvee_{(i,j)\in L}(\tau(m, 1) = i \wedge \tau(m+1, 1) = j)) \bigwedge$
  $(\bigvee_{(i,j)\in L}(\tau(m, 2) = i \wedge \tau(m+1, 2) = j))$
  This is exactly the local condition that any two consecutive tiles on either of the first two tapes form a pair in L.
- T_4:$\vee_{\{(i,j,e)\}\in G}(\tau(m, 1) = i \wedge \tau(n, 2) = j \wedge \tau(n-m, 3) = e)$.
  This states that, if we inspect 3 cells, one on each tape, which satisfy our sampling condition, then the tiles form a triple $(i, j, e) \in G$.

7.3 F is satisfiable iff an acceptable tiling exists for $\mathcal{T}$

We now verify that our correlation of truth assignment and tiling is such that a tiling exists iff a truth assignment verifying the Herbrand expansion exists.

Tiling to Truth Assignment

Let τ be a tiling accepted by $\mathcal{L} = (T, L, G)$.

Any two consecutive tiles on either of the two tapes form a pair in L and any 3-tuple at $n_1, n_2, n_2 - n_1$ on the 1st, 2nd and 3rd tape respectively form a triple from G.

Take an arbitrary Herbrand instance of F, say

$$F^*[y_1/f^m(1), y_2/f^n(1)]; n, m \in N,$$

we show that this instance is verified:

Replacing the occurrences of y_1, y_2 in M_1 to M_4 with $f^m(1)$ and $f^n(1)$ respectively, we get:

M_1: $Pf^m(1)f^n(1) \leftrightarrow Pf^{(m+1)}(1)f^{(n+1)}(1)),$

M_2: $(\nabla_{i=0}^{q-1}\tau(m,1) = i) \wedge (\nabla_{i=q}^{2q-1}\tau(m,2) = i - q),$

M_3: $(\vee_{(i,j)\in L}(\tau(m,1) = i \wedge \tau(m+1,1) = j) \bigwedge$
$(\vee_{(i,j)\in L}(\tau(m,2) = i \wedge \tau(m+1,2) = j).$

M_4: $\vee_{(i,j,e)\in G}(\tau(m,1) = i \wedge \tau(m,2) = j \wedge H_e).$
Where H_e is $Pf^m(1)f^n(1)$ if $e = 1$ and $\neg Pf^m(1)f^n(1)$ if $e = 0$.

Since $\tau(n-m,3)$ contains only one tile and by the truth assignment we have that $Pf^{m'}(1)f^{n'}(1)$ has the same truth value for all m', n' such that $m' - n' = m - n$, in particular for m, n and $m+1, n+1$, M_1 is verified.
Since the tiling τ is accepted, each cell of the first two tapes contains exactly one symbol so by the truth assignment M_2 is verified.
Since $\tau(m,1)$ and $\tau(m+1,1)$ form some pair from L we have by the truth assignment that $R_i f^m(1), R_j f^{m+1}(1)$ are true for some pair $(i,j) \in L$, so this Herbrand instance of M_3 is verified.

We have that any triple of tiles sampled at some $n_1, n_2, n_2 - n_1$, (on 1st, 2nd, 3rd tapes respectively), is in G. The disjunct of M_4 which is verified is

$$i = \tau(m,1), j = \tau(n,2), e = \tau(n-m,3),$$

since $\tau(m,1) = i$ implies $R_i f^m(1)$; $\tau(n,2) = j$ implies $R_{(q+j)} f^n(1)$ and $e = \tau(n-m,3)$ implies $\neg Pf^m(1)f^n(1) = 0$ or 1, depending on the value of e.

So we have that

$$R_i f^m(1) \wedge R_{(q+j)} f^n(1) \wedge \neg P f^m(1) f^n(1)$$

for $i = \tau(m,1); j = \tau(n,2)$ and $e = \tau(n-m,3)$.

So this Herbrand instance of M_4 is verified.

Therefore, each Herbrand instance is verified and F is satisfiable.

Further, if $\tau(0,3) = 0$ then $\mathcal{A} \models \neg P f^m(1) f^m(1)$.

Truth assignment to tiling

Let $\mathcal{A}$ be the truth assignment which verifies the Herbrand expansion, then since
$\mathcal{A}(Pf^m(1)f^n(1)) = \mathcal{A}(Pf^{m+1}(1)f^{n+1}(1))$ for all $m,n \in N$ it is also true that
$\mathcal{A}(Pf^m(1)f^n(1)) = \mathcal{A}(Pf^{m+p}(1)f^{n+p}(1))$ for all $m,n,p \in N$,
so we can define unambiguously a mapping:

$$\tau_3 : Z \longrightarrow 0,1$$

as follows

$$\tau_3(n-m) = \begin{cases} 1 & \text{if } \mathcal{A} \models Pf^m(1)f^n(1), \\ 0 & \text{if } \mathcal{A} \models \neg Pf^m(1)f^n(1) \text{ for all } m,n \in N. \end{cases}$$

Note that $n + p - (m + p) = n - m$.

Further, since $\mathcal{A}$ verifies every Herbrand instance of M_2, we can define mappings:
$\tau_1, \tau_2 : N \longrightarrow T = \{0, ..., q-1\}$ by
$\tau_1(n) = i$ iff $\mathcal{A} \models R_i f^n(1)$
$\tau_2(n) = i$ iff $\mathcal{A} \models R_{(q+i)} f^n(1)$

We can combine the above three mappings into one by setting:

$$\tau : (N \times \{1,2\}) \cup (Z \times \{3\}) \longrightarrow T$$

with

$$\tau(m,i) = \tau_i(m).$$

Since $\mathcal{A}$ verifies every Herbrand instance of M_3:
$(\tau(m,i), \tau(m+1,i)) \in L$ for $i = 1,2,3$ and for all m such that $\tau(m,i)$ and $\tau(m+1,i)$ are defined. And since $\mathcal{A}$ verifies every Herbrand instance of M_4:
$(\tau(m,1), \tau(n,2), \tau(n-m,3)) \in G$ for all m,n such that $\tau(m,1)$ and $\tau(n,2)$ are defined i.e where $m,n \in N$. ($\tau(n-m,3)$ is always defined).

So we have a tiling from 1 onwards.

We can extend this tiling to negative n by shifting the tiling to the left as follows:

$$\tau^{(p)}(n,i) = \begin{cases} \tau(n+p,i) & \text{if } i = 1,2 \\ \tau(n,i) & \text{if } i = 3. \end{cases}$$

Now $\tau^{(p)}$ fulfills the conditions for acceptability in the domain

$\{-p, -p+1, ...\} \times \{1,2\} \cup Z \times \{3\}$.

This means that we can find arbitrarily long acceptable pieces so by König's infinity lemma there is a tiling of $Z \times \{1,2,3\}$ accepted by $\mathcal{L}$. Specifically, since we have an infinite number of arbitrary long stretches of tiling, and for each $n \in N$ the segment $(-n, -n+1, \ldots, 0, 1, 2, \ldots, n)$ is one of a finite number of segments, there must exist a tiling of the whole of Z.

Thus we have that $\mathcal{L}$ accepts a tiling iff F is satisfiable, but if $\mathcal{L}$ accepts a tiling, then it also accepts a tiling with $\tau(0,3) = 0$ and then we have a truth assignment $\mathcal{A}$ verifying $E(F)$ such that $\mathcal{A} \models \neg Ptt$, for each $t \in D(F)$. So if F is at all satisfiable then $E(F)$ is verifiable with $Pf^m(1)f^m(1)$ false for all $m \geq 0$.

If the satisfiability problem of this subclass of V_0 was solvable then we could for each three-tape linear sampling problem $\mathcal{L}$ of the type discussed in chapter 4, determine whether or not $\mathcal{L}$ accepts a tiling by determining whether or not the corresponding formula $F_{\mathcal{L}}$ is satisfiable, therefore this subclass of V_0 is unsolvable and hence so is V_0.

References:

The constructions and proofs in this chapter are from Lewis,[20].

Chapter 8

The unsolvability of V_1

This chapter is essentially a version of the proof in Lewis,[20], but without using bigraph machinery,and including our false starts, in the hope of clarifying the ideas.

In this section we use the unsolvability of V_0 to demonstrate the unsolvability of the class V_1, defined as the class of formulas G of the form

$$\forall y_1 \exists x \forall y_2 G^M$$

where F contains only the atomic formulas:

$$Py_1y_2, Py_2x, R_iy1, R_iy_2 \text{ and } R_ix; i \in N.$$

The class V_1 in Lewis,[20] differs slightly from the class discussed here.

We show that, given any $F \in V_0$, that we can construct a $G \in V_1$ which is satisfiable iff F is. This will imply the unsolvability of V_1.

8.1 The correlation

Given an arbitrary $F \in V_0$, let us **as a first attempt** associate with F the following formula:

$$G_1 := \forall y_1 \exists x \forall y_2 F^M\{Pf(y_1)f(y_2)/Py_2x, R_if(y_1)/R_ix; 0 \le i \le q-1\}.$$

That is we replace occurrences in F of $Pf(y_1)f(y_2)$ by Py_2x and $R_if(y_1)$ by R_ix. Further, we replace the prefix $\forall y_1 \forall y_2$ by $\forall y_1 \exists x \forall y_2$. We leave the rest of F unchanged. Note that $G_1 \in V_1$ and we have made very few changes to F in order to change it to a formula $G_1 \in V_1$.

In trying to show the satisfiability problem for different formulas, equivalent, we will work directly with the Herbrand expansions of the respective formulas. That is, we will try to show that the expansion of G_1 is satisfiable iff the expansion of F is.

8.2 Comparing the Herbrand expansions of F and G_1

G_1 has as Herbrand expansion $E(G_1)$, the following:
$F^M\{Py_1y_2/Pf^m(1)f^n(1),\ \underline{Pf(y_1)f(y_2)/Pf^n(1)f^{m+1}(1)}\ ,$
$R_iy_1/Rif^m(1), R_iy_2/R_if^n(1),\ \underline{R_if(y_1)/R_if^{m+1}(1)}\ ;$
$0 \leq i \leq q-1; n, m \geq 0\}$,
while F has as Herbrand expansion $E(F)$, the following:
$F^M\{Py_1y_2/Pf^m(1)f^n(1),\ \underline{Pf(y_1)f(y_2)/Pf^{n+1}(1)f^{m+1}(1)}\ ,$
$R_iy_1/R_if^m(1), R_iy_2/R_if^n(1),\ \underline{R_if(y_1)/R_if^{m+1}(1)}\ ;$
$0 \leq i \leq q-1; n, m \geq 0\}$.

The underlined denote substituents affected by the change from F to G_1. Remember that $Pf(y_1)f(y_2)$ is replaced in G_1 by Py_2x hence the substituent $Pf^n(1)f^{m+1}(1)$ for $Pf(y_1)f(y_2)$ in G_1. Although $R_if(y_1)$ is replaced by R_ix in G_1, x has indicial term $f(y_1)$ since $f(y_1)$ is the indicial term corresponding to x in F. Therefore $R_if(y_1)$ has the same substituent $R_if^{m+1}(1)$ in G_1 as in F.

8.2.1 The direct relations

The following pairs of atomic formulas involving the dyadic predicate letter P, are directly related in $E(F)$:

for all $m, n \in N$

$$Pf^m(1)f^n(1)\&Pf^{m+1}(1)f^{n+1}(1)$$

$$Pf^{m+1}(1)f^{n+1}(1)\&Pf^{m+2}(1)f^{n+2}(1)$$

and so on.

On the other hand, in $E(G)$ the following pairs are directly related: for all $m, n \in N$

$$Pf^m(1)f^n(1)\&Pf^n(1)f^{m+1}(1)$$

$$Pf^n(1)f^{m+1}(1)\&Pf^{m+1}(1)f^{n+1}(1)$$

and so on.

It is apparent that the pairs of atomic formulas directly related in $E(G)$ differ from the pairs directly related in $E(F)$.

In $E(G)$ the formula $Pf^n(1)f^{m+1}(1)$ 'seperates' the pair

$$Pf^m(1)f^n(1); Pf^{m+1}(1)f^{n+1}(1)$$

which are directly related in $E(F)$.

Recall that to satisfy a Herbrand instance of $E(F)$ we must assign truth values to the atomic formulas which makes each Herbrand instance true as a formula of the propositional calculus. Now, in $E(F)$ an atomic formula $Pf^m(1)f^n(1)$ appears both for Py_1y_2 and for $Pf(y_1)f(y_2)$ so we must assign to it a truth value which verifies both instances in which it appears. In $E(G)$ the truth value we assign to an atomic formula $Pf^m(1)f^n(1)$ will depend on the truth value of the atomic formula $Pf^n(1)f^{m+1}(1)$ appearing with it in the same Herbrand instance. We therefore cannot independently assign truth values to these two atomic formulas. The same is true for pairs $Pf^m(1)f^n(1); Pf^{m+1}(1)f^n(1)$ in $E(F)$.

Because of the different dependences there is no clear way to correlate truth assignments between $E(F)$ and $E(G)$.

We could try to simulate the direct relations of $E(F)$ by building into a new formula $G \in V_1$ that certain pairs of atomic formulas in the expansion of G have the same truth value, specifically the pairs $Pf^n(1)f^{m+1}(1); Pf^{m+1}(1)f^{n+1}(1)$, since it is $Pf^n(1)f^{m+1}(1)$ which seperates the pairs directly related in $E(F)$.

So given some Herbrand instance of F, say

$$F^M\{Py_1y_2/Pf^m(1)f^n(1),\ Pf(y_1)f(y_2)/Pf^{n+1}(1)f^{m+1}(1), \\ R_iy_1/R_if^m(1), R_iy_2/R_if^n(1),\ R_if(y_1)/R_if^{m+1}(1)\} \qquad m,n \geq 0.$$

We want a formula G, which has as corresponding part of $E(G)$ the following sequence:

$$F^M\{Py_1y_2/Pf^m(1)f^n(1), Pf(y_1)f(y_2)/Pf^n(1)f^{m+1}(1) \\ R_iy_1/R_if^m(1), R_iy_2/R_if^n(1)\ R_if(y_1)/R_if^{m+1}(1)\}$$

$$Pf^n(1)f^{m+1}(1) \leftrightarrow Pf^{m+1}(1)f^{n+1}(1)$$

$$F^M\{Py_1y_2/Pf^{m+1}(1)f^{n+1}(1), Pf(y_1)f(y_2)/Pf^{n+1}(1)f^{m+2}(1) \\ R_iy_1/R_if^{m+1}(1), R_iy_2/R_if^{n+1}(1)\ R_if(y_1)/R_if^{m+2}(1)\}$$

$$Pf^{n+1}(1)f^{m+2}(1) \leftrightarrow Pf^{m+2}(1)f^{n+2}(1).$$

That is, as before, Py_1y_2 has substituent $Pf^m(1)f^n(1)$ and $Pf(y_1)f(y_2)$ has substituent $Pf^n(1)f^{m+1}(1)$, however, the pair of formulas

$$Pf^n(1)f^{m+1}(1); Pf^{m+1}(1)f^{n+1}(1)$$

do not appear in F^M, but are forced to have **the same truth value**. The next pair to appear in F^M is

$$Pf^{m+1}(1)f^{n+1}(1); Pf^{n+1}(1)f^{m+2}(1)$$

and so on.

In this way pairs $Pf^m(1)f^n(1); Pf^{m+1}(1)f^{n+1}(1)$ become in effect directly related in the sense that the atomic formula $Pf^n(1)f^{m+1}(1)$ appearing with $Pf^m(1)f^n(1)$ is forced to have the same truth value as $Pf^{m+1}(1)f^{n+1}(1)$.

At first glance this Herbrand expansion seems equivalent to $E(F)$ in that the direct relations of $E(F)$ seem to be mirrored in this expansion, unfortunately, if we examine the part of $E(G)$ with y_1 having substituent $f^{n-1}(1)$ and y_2 having substituent $f^m(1)$ we get as part of the expansion:

$$Pf^m(1)f^n(1) \leftrightarrow Pf^n(1)f^{m+1}(1)$$

which together with

$$Pf^n(1)f^{m+1}(1) \leftrightarrow Pf^{m+1}(1)f^{n+1}(1)$$

forces the pair

$$Pf^m(1)f^n(1); Pf^{m+1}(1)f^{n+1}(1)$$

to have the same truth value.

This no longer reflects the direct relations of $E(F)$, where such pairs are not forced to have the same truth value, so the two expansions are not equivalent.

8.3 A refinement

The problem arises because any formula $Pf^k(1)f^l(1); k, l \geq 0$ is a substituent for both Py_1y_2 and Py_2x in the constructed expansion. We therefore modify our initial approach by correlating terms $f^m(1)$ in $E(F)$ with terms $f^{2m}(1)$ in $E(G)$. This means that an atomic formula $Pf^m(1)f^n(1)$ in $E(F)$ is correlated with the atomic formula $Pf^{2m}(1)f^{2n}(1)$ in $E(G)$ and an atomic formula $R_if^m(1)$ in $E(F)$ is correlated with the formula $R_if^{2m}(1)$ in $E(G)$.

We want the directly related pairs in $E(G)$, between formulas corresponding to directly related pairs $Pf^{2m}(1)f^{2n}(1); Pf^{2m+2}(1)f^{2n+2}(1)$ in $E(F)$ to have the same truth value.

8.3.1 The monadic predicate formulas

In the Herbrand instances of $E(F)$ with y_1, y_2 having substituents $f^m(1)$ and $f^n(1)$ respectively, the following monadic formulas are directly related:

$$R_if^m(1); R_if^n(1); R_if^{m+1}(1).$$

The last formula appears for $R_if(y_1)$.

If $R_if^m(1)$ appears for R_iy_1 in $E(F)$ then $R_if^{m+1}(1)$ appears for $R_if(y_1)$.

This Herbrand instance is correlated with the Herbrand instance in $E(G)$ where y_1, y_2 have substituents $f^{2m}(1)$ and $f^{2n}(1)$ respectively.

If $R_i y_1$ is replaced by $R_i f^{2m}(1)$ in $E(G)$ it appears in the same instance as $R_i f^{2m+1}(1)$ (which appears for $R_i x$), but in $E(F)$, $R_i f^m(1)$ and $R_i f^{m+1}(1)$ appear together and these two correspond to $R_i f^{2m}(1)$ and $R_i f^{2m+2}(1)$ in $E(G)$.

So to mimic the direct relations of monadic formulas in $E(F)$, $E(G)$ must contain the formula:

$$R_i f^{2m+1}(1) \leftrightarrow R_i f^{2m+2}(1)$$

which in effect will cause $R_i f^{2m}(1)$ and $R_i f^{2m+2}(1)$ to be directly related for the same reason as in the dyadic case.

So for an arbitrary part of $E(F)$, say:

$$F^M\{Py_1y_2/Pf^m(1)f^n(1), Pf(y_1)f(y_2)/Pf^{n+1}(1)f^{m+1}(1)$$

$$R_iy_1/R_if^m(1), R_iy_2/R_if^n(1)R_if(y_1)/R_if^{m+1}(1); 0 \leq i \leq q-1\}$$

$$F^M\{Py_1y_2/Pf^{m+1}(1)f^{n+1}(1), Pf(y_1)f(y_2)/Pf^{n+2}(1)f^{m+2}(1)$$

$$R_iy_1/R_if^{m+1}(1), R_iy_2/Rif^{n+1}(1)R_if(y_1)/R_if^{m+2}(1); 0 \leq i \leq q-1\}$$

we want as corresponding part of $E(G)$:

$$F^M\{Py_1y_2/Pf^{2m}(1)f^{2n}(1), Pf(y_1)f(y_2)/Pf^{2n}(1)f^{2m+1}(1)$$

$$R_iy_1/R_if^{2m}(1), R_iy_2/R_if^{2n}(1)R_if(y_1)/R_if^{2m+1}(1); 0 \leq i \leq q-1; n, m \geq 0\}$$

$$Pf^{2n}(1)f^{2m+1}(1) \leftrightarrow Pf^{2m+1}(1)f^{2n+1}(1)$$

$$Pf^{2m+1}(1)f^{2n+1}(1) \leftrightarrow Pf^{2n+1}(1)f^{2m+2}(1)$$

$$Pf^{2n+1}(1)f^{2(m+1)}(1) \leftrightarrow Pf^{2(m+1)}(1)f^{2(n+1)}(1)$$

$$R_if^{2m+1}(1) \leftrightarrow R_if^{2(m+1)}(1)$$

$$F^M\{Py_1y_2/Pf^{2(m+1)}(1)f^{2(n+1)}(1), Pf(y_1)f(y_2)/Pf^{2(n+1)}(1)f^{2(m+1)+1}(1)$$

$$R_iy_1/R_if^{2(m+1)}(1), R_iy_2/R_if^{2(n+1)}(1)R_if(y_1)/R_if^{2(m+1)+1}(1); 0 \leq i \leq q-1; n, m \geq 0\}$$

$$Pf^{2(n+1)}(1)f^{2(m+1)+1}(1) \leftrightarrow Pf^{2(m+1)+1}(1)f^{2(n+1)+1}(1)$$

$$Pf^{2(m+1)+1}(1)f^{2(n+1)+1}(1) \leftrightarrow Pf^{2(n+1)+1}(1)f^{2(m+1)+2}(1)$$

$$Pf^{2(n+1)+1}(1)f^{2(m+2)}(1) \leftrightarrow Pf^{2(m+2)}(1)f^{2(n+2)}(1)$$
$$R_i f^{2m+1}(1) \leftrightarrow R_i f^{2(m+1)}(1)$$

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That there is no overlap of truth assignments on the atomic formulas, follows directly from the fact that the sets:

$$\{(2n, 2m+1); (2m+1, 2n+1); (2n+1, 2m+2); (2m+2, 2n+2)\}$$

and

$$\{(2l, 2k+1); (2k+1, 2l+1); (2l+1, 2k+2); (2k+2, 2l+2)\}$$

are disjoint if $(m, n) \neq (k, l)$.

We now know what we would like $E(G)$ to look like, we have yet to show, however, that we can construct a $G \in V_1$ with expansion $E(G)$ as above.

8.4 The construction of G

Since only terms of the form $f^{2k}(1)$ were substituents for y_1, y_2 in the matrix F^M, we must distinguish between such terms which we will call *even* and terms of the form $f^{2k+1}(1)$ which we call *odd*. To do this we introduce into G two new monadic predicate signs R_q and R_{q+1} (recall that F contains the monadic predicate signs $R_0, \ldots, R_{q-1}$). We then construct the following formula as part of G^M, the matrix of G: (Remember that the prefix of G must be $\forall y_1 \exists x \forall y_2$.)

$$G_1 : ((R_q y_1 \wedge R_{q+1} x \wedge \neg R_{q+1} y_1) \vee (R_{q+1} y_1 \wedge R_q x \wedge \neg R_q y_1))$$

Let y_1 have substituent $f^m(1)$, then G_1 states that:
$(R_q f^m(1), \neg R_{q+1} f^m(1)$ and $R_{q+1} f^{m+1}(1))$ or
$(R_{q+1} f^m(1), \neg R_q f^m(1)$ and $R_q f^{m+1}(1))$.

That is, consecutive terms cannot both satisfy R_q or both satisfy R_{q+1} and any term $f^m(1)$ must satisfy either of the two but not both.

The class of terms $\{f^k(1); k \geq 0\}$ is therefore split into two disjoint classes, those satisfying R_q and those satisfying R_{q+1}, we call these classes *even* and *odd* respectively.

We now use this division of terms to construct the following subformula as the next part of G:

$$G_2 : R_q y_1 \wedge R_q y_2 \rightarrow F^M\{Pf(y_1)f(y_2)/Py_2x; R_i f(y_1)/R_i x, 0 \leq i \leq q-1\},$$

the other terms having the usual substituents.

This means that, if for $m, n \in N$, $R_q f^m(1)$ and $R_q f^n(1)$, then they are substituents for y_1, y_2 in F^M. Note that this implies that both m, n are of the form $2k+1$ or both of the form $2k$ for some $k \in N$.

The 'intervening' atomic formulas are forced to have the same truth value by the next subformula:

$$G_3 : (R_{q+1}y_1 \wedge R_{q+1}y_2) \rightarrow (Py_1y_2 \leftrightarrow Py_2x)$$

$$(R_q y_1 \wedge R_{q+1}y_2) \rightarrow (Py_1y_2 \leftrightarrow Py_2x)$$

$$(R_{q+1}y_1 \wedge R_q y_2) \rightarrow (Py_1y_2 \leftrightarrow Py_2x).$$

So if either $f^m(1)$ or $f^n(1)$ does not satisfy R_q then the pair of formulas

$$Pf^m(1)f^n(1); Pf^n(1)f^{m+1}(1)$$

are forced to have the same truth value.

Analogously for the monadic predicate letters $R_i, 0 \leq i \leq q-1$, we have the subformula:

$$G_4 : R_{q+1}y_1 \rightarrow \wedge_{i=0}^{q-1}(R_i y_1 \leftrightarrow R_i x).$$

The complete G is then

$$\forall y_1 \exists x \forall y_2 (G_1 \wedge G_2 \wedge G_3 \wedge G_4).$$

8.5 $E(F)$ is satisfiable iff $E(G)$ is satisfiable

$E(F) \Rightarrow E(G)$:

Say there is some truth assignment $\mathcal{A}$ on the atomic formulas

$$R_i f^m(1), Pf^m(1)f^n(1)); m, n \geq 0; 0 \leq i \leq q-1$$

such that the entire expansion $E(F)$ of F, is verified.
We construct a truth assignment $\mathcal{B}$ which verifies $E(G)$.
Set $\mathcal{B}(R_q f^0(1)) = true$. This means that all even terms $f^{2m}(1)$ satisfy R_q and all odd terms satisfy R_{q+1}. Now, any dyadic atomic formula $Pf^m(1)f^n(1)$ in $E(F)$, determines the truth value of the following atomic formulas in $E(G)$:

$$\{Pf^{2m}(1)f^{2n}(1), Pf^{2n-1}(1)f^{2m}(1), Pf^{2m-1}(1)f^{2n-1}(1), Pf^{2n-2}(1)f^{2m-1}(1)\}.$$

That is, set $\mathcal{B}(Pf^{2m}(1)f^{2n}(1)) = \mathcal{B}(Pf^{2n-1}(1)f^{2m}(1)) = \mathcal{B}(Pf^{2m-1}(1)f^{2n-1}(1))$
$= \mathcal{B}(Pf^{2n-2}(1)f^{2m-1}(1)) = \mathcal{A}(Pf^m(1)f^n(1)).$

For $n = 0$, or $m = 0$ of course, certain of the above atomic formulas are undefined.
Analogously, for monadic atomic formula, set
$\mathcal{B}(R_i f^{2m}(1)) = \mathcal{B}(R_i f^{2m-1}(1)) = \mathcal{A}(Rf^m(1); 0 \leq i \leq q-1$.

From the discussion of the correlation between $E(F)$ and $E(G)$, in **8.3.1** and **8.4** it is clear that this assignment verifies $E(G)$.

Now for the converse:
$E(G) \Rightarrow E(F)$:

If we have a truth assignment $\mathcal{B}$ which verifies $E(G)$ then this assignment assigns R_q true to all odd terms $f^{2k+1}(1)$ or to all even terms $f^{2k}(1), k \geq 0$.
In the first case we have the inverse assignment of the above i.e. for all $m, n \in N$ set:

$$\mathcal{A}(Pf^m(1)f^n(1)) = \mathcal{B}(Pf^{2m}(1)f^{2n}(1))$$

and for $0 \leq i \leq q-1$ set:

$$\mathcal{A}(R_i f^m(1)) = \mathcal{B}(Rif^{2m}(1))$$

In the second case odd terms are substituents for y_1, y_2 in F^M and all *such* instances are verified.
So set:

$$\mathcal{A}(Pf^m(1)f^n(1)) = \mathcal{B}(Pf^{2m+1}(1)f^{2n+1}(1))$$

and for $0 \leq i \leq q-1$ set:

$$\mathcal{A}(R_i f^m(1)) = \mathcal{B}(R_i f^{2m+1}(1).$$

In either case $E(F)$ is verified if $E(G)$ is.
We therefore have the result that any formula F in V_0 is satisfiable iff a corresponding G in V_1 is satisfiable. To each F in V_0 there corresponds such a G. Therefore the class V_1 is unsolvable.

8.5.1 Remarks on the unsolvability proof for V_2

The proof that the class V_2 in fig.12 of chapter 6 is unsolvable is analogous to the proof for V_1 except for the following: for an arbitrary formula $F \in V_0$ a formula H is constructed in which **three** terms of the Herbrand domain of H are correlated with each term in the Herbrand domain of $F \in V_0$ (instead of two, as was the case here). This is necessary since it is slightly more elaborate to build into the formula H that $E(H)$ contains an encoded copy of $E(F)$, than it was to build it into G (as was done above). This leads to an interim unsolvable class from which the unsolvability of V_2 follows.

References: The proof presented here is essentially that of Lewis,[20], except that the proof here is by direct comparisons of the Herbrand expansions of the formulas. Lewis uses *bigraph* structures, (see p80-86 in Lewis,[20]) for the proof.

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