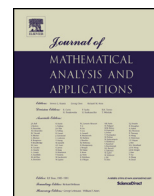




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Differentiable, holomorphic, and analytic functions on complex Φ -algebrasM. Roelands^a, C. Schwanke^{b,*}^a *Mathematical Institute, Leiden University, 2300 RA Leiden, the Netherlands*^b *Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa*

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ABSTRACT

Using the notion of order convergent nets, we develop an order-theoretic approach to differentiable functions on Archimedean complex Φ -algebras. Most notably, we improve the Cauchy-Hadamard formulas for universally complete complex vector lattices given by both authors in a previous paper in order to prove that analytic functions are holomorphic in this abstract setting.

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1. Introduction

The theory of complex analysis on complex Banach spaces enjoys a long, well-known, and fruitful history, which is summarized in the text [11]. This theory generalizes the classical theory of functions of a complex variable by using norms as an abstraction of the ordinary modulus on the complex plane. In this paper we investigate complex analysis via an alternative generalization of the complex modulus: *the modulus on Archimedean complex vector lattices*. This complex vector lattice modulus naturally leads to the notion of order convergence, which provides us with an order-theoretic perspective on complex analysis. In particular, we introduce an order-theoretic approach to complex differentiation on Archimedean complex Φ -algebras.

Specifically, in Section 3 we utilize the notion of order convergent nets in complex vector lattices to develop suitable definitions for order differentiable functions in Definition 3.5. It is then illustrated that order differentiable functions satisfy the classical sum, product, chain, and, when E is uniformly complete, quotient rule. A conception of holomorphic functions in the present setting is also introduced.

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We then introduce in Section 4 the notions of super order differentiable functions and σ -super order differentiable functions as a means to strengthen our results on order continuity, the chain rule, and quotient rule from Section 3.

In Section 6 we apply our theory from Section 3 to power series and show that if a function is analytic in our abstract setting, then it is holomorphic. In order to perform this task, we first improve our Cauchy-Hadamard formulas from [12] in Section 5, which now include spectra of convergence of Ω that are not necessarily order bounded. The key to this improvement is our obtainment of new formulas for the finite and infinite parts of a positive element in the sup-completion, which are introduced in [3]. Our alternative approach perhaps yields simpler formulas than the ones found in [3], which facilitate our proofs for the new and improved Cauchy-Hadamard formulas.

This paper continues our development of complex analysis on complex vector lattices, which began with a study of series and power series on such spaces in [12]. The reader will infer that our order-theoretical approach reveals new insights, such as how the notions of weak order units and projection bands play a compelling role in complex analysis.

2. Preliminaries

The reader is referred to the standard texts [1,10,13,14] for any unexplained terminology or basic results in vector lattice theory. As usual, \mathbb{C} denotes the field of complex numbers, and the set of strictly positive integers is denoted by \mathbb{N} . We also write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ throughout.

An Archimedean (real) vector lattice F is said to be *square mean closed* (see [4, p. 482] or [8, p. 356]) if $\sup\{(\cos \theta)x + (\sin \theta)y : \theta \in [0, 2\pi]\}$ exists in F for every $x, y \in F$, in which case we write

$$x \boxplus y := \sup\{(\cos \theta)x + (\sin \theta)y : \theta \in [0, 2\pi]\}.$$

Given a square mean closed Archimedean (real) vector lattice F , the vector space complexification $F \oplus iF$ is called an *Archimedean complex vector lattice* [8, p. 356–357], and F is called the *real part* of $F \oplus iF$.

Notation 2.1. We denote the real part of an Archimedean complex vector lattice E by F throughout this paper.

The *modulus* on an Archimedean complex vector lattice E is defined by

$$|x + iy| := x \boxplus y \quad (x, y \in F).$$

The *positive cone* E_+ of E is the set of all elements which are invariant under the modulus, that is, $E_+ := \{z \in E : |z| = z\}$. Observe that $E_+ = F_+$.

In [12] it was necessary to confine our theory to the context of universally complete complex vector lattices. Indeed, without the assumption of universal completeness, fundamental results fail to be true. For example, it is shown in [12, Remark 3.7] that the geometric series can fail to converge in order in Archimedean complex vector lattices that are not universally complete. Hence we will focus solely on universally complete vector lattices in Sections 5 and 6, where we study power series. We are able to relax the conditions on E in Sections 3 and 4 to Archimedean complex Φ -algebras, although some instances still require uniform completeness.

If F is an Archimedean (real) Φ -algebra, that is, an Archimedean (real) f -algebra possessing a multiplicative identity, then its multiplication canonically extends to a complex multiplication on $E = F \oplus iF$. The multiplication on E will be indicated by juxtaposition, and we denote the multiplicative identity of E by e throughout. Therefore, E is endowed with a complex Φ -algebra structure and can rightly be called a *complex Φ -algebra*. Note that E is semiprime by [7, Corollary 10.4].

As usual, if $z \in E$ is (multiplicatively) invertible, we denote its multiplicative inverse by z^{-1} . Moreover, we set $z^0 := e$ for all $z \in E$. For more information on complex Φ -algebras we refer the reader to [6].

We next record some useful basic properties of E , which we repeatedly and freely use throughout this paper. Note that real Φ -algebra analogues of statements (i)–(iv) can be found in [13, Section 142].

For all $z, w \in E$:

- (i) $zw = wz$.
- (ii) $|zw| = |z||w|$.
- (iii) $|z| \wedge |w| = 0$ if and only if $zw = 0$.
- (iv) If z is invertible and positive then z^{-1} is positive.

We next turn to the notion of order convergent nets in E .

Definition 2.2. A net $(z_\alpha)_\alpha$ in E converges in order to $z \in E$ (we denote this by writing $z_\alpha \rightarrow z$), if there exists a positive decreasing net $(q_\beta)_\beta$ with $\inf_\beta q_\beta = 0$ (in symbols $q_\beta \downarrow 0$) such that for all β , there exists α_0 for which $|z_\alpha - z| \leq q_\beta$ for all $\alpha \geq \alpha_0$.

We add that a natural analog of Definition 2.2 for sequences is obtained by replacing the nets $(z_\alpha)_\alpha$ and $(q_\beta)_\beta$ with sequences $(z_n)_{n \geq 0}$ and $(q_m)_{m \geq 0}$. We will use the notation $z_n \rightarrow z$ to indicate when a sequence $(z_n)_{n \geq 0}$ in E converges in order to $z \in E$.

3. Differentiation for order convergence

Notation 3.1. E will denote an Archimedean complex Φ -algebra in this section.

Next we extend the classical notion of differentiable functions on \mathbb{C} to the Archimedean complex Φ -algebra setting. The notation $f: \text{dom}(f) \rightarrow E$ is used throughout, where by $\text{dom}(f)$ we indicate the domain of f .

In the classical theory of differentiation, the points at which a function is differentiable are contained in some open set. For the order theoretical analogue, the situation is similar, but with subtle differences, as will be pointed out where necessary. To that end, we start by introducing the following notation.

Notation 3.2. For $z, w \in F$ we write $z \ll w$ to indicate when $w - z$ is a positive invertible element in F .

The order theoretical analogues of open and closed disks are as follows.

Definition 3.3. For an invertible element $r \in E_+$ and $c \in E$ define

$$\overset{\circ}{\Delta}(c, r) := \{z \in E: |z - c| \ll r\},$$

which plays the role of the analogue of an open disk. Moreover, we define

$$\bar{\Delta}(c, r) := \{z \in E: |z - c| \leq r\},$$

which plays the role of the analogue of a closed disk. We say for $c \in E$ and $r \in E_+$ an invertible element that a set of the form $U_c := \overset{\circ}{\Delta}(c, r)$ is an *order open neighborhood* of c . Furthermore, a set $U \subseteq E$ is said to be *order open* if for every point $c \in U$ there is an order open neighborhood of c contained in U .

Remark 3.4. Note that for $c \in E$ and any invertible element $r \in E_+$ the order disk $\overset{\circ}{\Delta}(c, r)$ is order open, which is not the case for the set $\{z \in E: |z - c| < r\}$. Consider for example \mathbb{C}^2 with $r := (1, 1)$ and $c := (0, 0)$. Then $(1, 0) < (1, 1)$, but $\{(z, w) \in \mathbb{C}^2: |(z, w)| < (1, 1)\}$ does not contain an order open neighborhood of $(1, 0)$.

With the analogue of open disks in place, we introduce differentiable functions in the current setting.

Definition 3.5. A function $f: \text{dom}(f) \rightarrow E$ is said to be *order differentiable at* $c \in \text{dom}(f)$ if there exists an order open neighborhood $U_c \subseteq \text{dom}(f)$ of c and an $f_c \in E$ such that, for every $h_\alpha \rightarrow 0$ with $c + h_\alpha \in U_c$ for all α , there is a net $q_\beta \downarrow 0$ with the property that for all β , there is an α_0 for which

$$|f(c + h_\alpha) - f(c) - h_\alpha f_c| \leq |h_\alpha| q_\beta \quad (1)$$

holds whenever $\alpha \geq \alpha_0$. In this case we call U_c a *D-neighborhood* for f at c .

Remark 3.6. In Definition 3.5, the order openness of U_c implies that there exists an invertible element $r \in E_+$ such that $\overset{\circ}{\Delta}(c, r) \subseteq U_c$. Taking $(h_\alpha)_\alpha$ to be the sequence defined by $h_n := \frac{1}{n+1}r$ consisting of positive invertible elements, we have $h_n \rightarrow 0$ and $c + h_n \in U_c$ for every $n \in \mathbb{N}_0$. Thus we can divide both sides of the inequality (1) by $|h_n| = h_n$ to obtain

$$|(f(c + h_n) - f(c))h_n^{-1} - f_c| \leq q_\beta,$$

implying that f_c is unique.

Definition 3.7. We call the unique f_c in Definition 3.5 the *order derivative of f at c* and will from now on denote it by $f'(c)$.

Remark 3.8. In Definition 3.5, the assumption that (1) holds for all $h_\alpha \rightarrow 0$ with $c + h_\alpha \in U_c$ does not imply in general that (1) holds for all $h_\alpha \rightarrow 0$ with $c + h_\alpha \in \text{dom}(f)$, as the following example illustrates.

Consider the universally complete vector lattice of all complex-valued sequences $\mathbb{C}^{\mathbb{N}}$. Define $f: \mathbb{C}^{\mathbb{N}} \rightarrow \mathbb{C}^{\mathbb{N}}$ by

$$f(z_1, z_2, z_3, \dots) := \begin{cases} (z_1, z_2, z_3, \dots) & \text{if } |(z_1, z_2, z_3, \dots)| \ll e \\ (z_2, z_3, z_4, \dots) & \text{if } |(z_1, z_2, z_3, \dots)| \not\ll e. \end{cases}$$

Let $h_\alpha \rightarrow 0$ satisfy $h_\alpha \in \overset{\circ}{\Delta}(0, e)$ for all α . Then for each α ,

$$|f(0 + h_\alpha) - f(0) - eh_\alpha| = 0 \leq |h_\alpha|^2$$

holds. Thus f is order differentiable at 0 and $f'(0) = e$.

However, it is not true that

$$|f(0 + h_\alpha) - f(0) - h_\alpha| \leq |h_\alpha| q_\beta$$

holds for some net $q_\beta \downarrow 0$ in $\mathbb{C}_+^{\mathbb{N}}$ whenever $h_\alpha \in \text{dom}(f)$ for all α and $h_\alpha \rightarrow 0$.

To verify, suppose this statement is true. Note that the sequence

$$h_n = (\underbrace{0, \dots, 0}_{n \text{ terms}}, 2, 2, 2, \dots)$$

satisfies $h_n \rightarrow 0$ and $h_n \in \text{dom}(f)$ for all $n \in \mathbb{N}_0$. Moreover, for $n \geq 2$ we have

$$\begin{aligned} |f(0 + h_n) - f(0) - h_n| &= |f(h_n) - h_n| \\ &= |(\underbrace{0, \dots, 0}_{n-1 \text{ terms}}, 2, 2, 2, \dots) - (\underbrace{0, \dots, 0}_n, 2, 2, \dots)| \\ &= |(\underbrace{0, \dots, 0}_{n-1 \text{ terms}}, 2, 0, 0, 0, \dots)|. \end{aligned}$$

Next fix β . Note that for any $n \geq 2$ we have

$$|(\underbrace{0, \dots, 0}_{n-1 \text{ terms}}, 2, 0, 0, 0, \dots)| \leq |(\underbrace{0, \dots, 0}_n, 2, 2, 2, \dots)|q_\beta,$$

which is a contradiction. Thus we will say that f is not *super order differentiable* at 0, a topic which is explored in Section 4.

Remark 3.9. One can see from Remark 3.8 that the function $f: \mathbb{C}^{\mathbb{N}} \rightarrow \mathbb{C}^{\mathbb{N}}$ defined by

$$f(z_1, z_2, z_3, \dots) := (z_2, z_3, z_4, \dots)$$

is not order differentiable at 0. Similarly, one can show that the function $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $f(z, w) := (w, z)$ is not order differentiable at 0 either. This fact displays a fundamental difference between order differentiability and Fréchet differentiability, as f has a Fréchet derivative everywhere on \mathbb{C}^2 .

The following lemma extends the classical result which states that differentiable complex functions are continuous. It will be strengthened to order continuity without restriction in Lemma 4.2 using the notion of super order differentiability.

Lemma 3.10. *If a function $f: \text{dom}(f) \rightarrow E$ has order derivative at $c \in \text{dom}(f)$, then there exists an order open neighborhood U_c of c such that the restriction $f|_{U_c}$ to U_c is order continuous at c .*

Proof. Suppose f has order derivative $f'(c)$ at c . Then there is an order open neighborhood U_c of c such that for $h_\alpha \rightarrow 0$ with $c + h_\alpha \in U_c$ for all α , there exists a net $q_\beta \downarrow 0$ such that for every β there exists an α_0 for which

$$|f(c + h_\alpha) - f(c) - h_\alpha f'(c)| \leq |h_\alpha|q_\beta \quad (\alpha \geq \alpha_0).$$

Let $(z_\alpha)_\alpha$ be a net in U_c converging in order to c . Then the net $h_\alpha := z_\alpha - c$ converges in order to 0.

Moreover, there is a net $p_\gamma \downarrow 0$ such that for all γ there exists α_1 such that $|h_\alpha| \leq p_\gamma$ whenever $\alpha \geq \alpha_1$. For $\alpha_2 \geq \alpha_0, \alpha_1$ we have

$$\begin{aligned} |f(z_\alpha) - f(c)| &\leq |f(c + (z_\alpha - c)) - f(c) - (z_\alpha - c)f'(c)| + |z_\alpha - c||f'(c)| \\ &\leq |z_\alpha - c|q_\beta + |z_\alpha - c||f'(c)| \leq p_\gamma(q_\beta + |f'(c)|). \end{aligned}$$

Hence $f(z_\alpha) \rightarrow f(c)$. \square

We next shift our focus to differentiating the function $z \mapsto z^{-1}$, which is the foundation for the quotient rule. Denote by E^{-1} the set of invertible elements of E .

Remark 3.11. A net $(z_\alpha)_\alpha$ of invertible elements converging in order to an invertible element z has the property that $z_\alpha^{-1} \rightarrow z^{-1}$ if and only if there exists an index α_0 such that $\{|z_\alpha|^{-1} : \alpha \geq \alpha_0\}$ is order bounded. To verify this fact, note that if $\{|z_\alpha|^{-1} : \alpha \geq \alpha_0\}$ is order bounded for some α_0 , then for an upper bound u of $\{|z_\alpha|^{-1} : \alpha \geq \alpha_0\}$, we have

$$|z_\alpha^{-1} - z^{-1}| = |z_\alpha|^{-1} |z|^{-1} |z_\alpha - z| \leq u |z|^{-1} |z_\alpha - z|$$

for all $\alpha \geq \alpha_0$, and so $z_\alpha^{-1} \rightarrow z^{-1}$.

On the other hand, if $z_\alpha^{-1} \rightarrow z^{-1}$, then there is a net $p_\beta \downarrow 0$ such that for any β there is an α_0 such that $|z_\alpha^{-1} - z^{-1}| \leq p_\beta$ for all $\alpha \geq \alpha_0$. Hence $|z_\alpha|^{-1} \leq |z_\alpha^{-1} - z^{-1}| + |z|^{-1} \leq p_\beta + |z|^{-1}$ for all $\alpha \geq \alpha_0$, so $\{|z_\alpha|^{-1} : \alpha \geq \alpha_0\}$ is order bounded.

Lemma 3.12. *If E is uniformly complete, then the set E^{-1} is an order open set. In this case, considering the function $f: E^{-1} \rightarrow E^{-1}$ defined by $f(z) := z^{-1}$ and $c \in E^{-1}$, there exists an order open neighborhood $U_c \subseteq E^{-1}$ of c such that $f|_{U_c}$ is order continuous.*

Proof. Assume that E is uniformly complete. To show that E^{-1} is order open, let $c \in E^{-1}$. If $z \in U_c := \overset{\circ}{\Delta}(c, \frac{1}{2}|c|)$, then $\frac{1}{2}|c| \leq |z|$. Thus $|z|$ is invertible by [7, Theorems 11.1 and 11.4]. Then z is also invertible with $z^{-1} = \bar{z}|z|^{-2}$, where the complex conjugate \bar{z} is defined as usual. It follows that E^{-1} is order open.

Moreover, if $(z_\alpha)_\alpha$ is a net in $\overset{\circ}{\Delta}(c, \frac{1}{2}|c|)$ such that $z_\alpha \rightarrow c$, then $|z_\alpha|^{-1} \leq 2|c|^{-1}$ for every α , so $z_\alpha^{-1} \rightarrow c^{-1}$ as explained in Remark 3.11. Hence $f|_{U_c}$ is order continuous. \square

Remark 3.13. The set E^{-1} is not necessarily order open if E is not uniformly complete. For example, first consider the Archimedean real Φ -algebra $F = PP[0, 1]$ of all continuous piecewise polynomials on $[0, 1]$. Let $F_1 := F$, and for $n \in \mathbb{N}$, let F_{n+1} be the Archimedean real Φ -algebra of all functions on $[0, 1]$ that are of the form

$$f(x) := \sum_{k=1}^m p_k(x) \sqrt{q_k(x)}$$

for some $m \in \mathbb{N}$, $p_k \in F_n$, and $q_k \in (F_n)_+$ ($k \in \{1, \dots, m\}$). Then $G := \bigcup_{n=1}^{\infty} F_n$ is an Archimedean real Φ -algebra. Moreover, if $f, g \in G$, then there exist $n \in \mathbb{N}$ such that $f, g \in F_n$. Since F_n is a Φ -algebra, we have $f^2 + g^2 \in F_n$ as well, and hence $\sqrt{f^2 + g^2} \in F_{n+1}$. It follows that G is square mean closed and so $G_{\mathbb{C}}$ is an Archimedean complex Φ -algebra. Of course, the constant function e with value 1 is invertible in $G_{\mathbb{C}}$. Let $r \in G_+$ be any invertible element and define

$$z(x) := 1 + x \wedge \frac{1}{2}r(x) \quad (x \in [0, 1]).$$

Note that $z \in G$ satisfies $z \in \overset{\circ}{\Delta}(e, r)$. Moreover, since r is positive, invertible, and continuous, there exists a $\delta > 0$ such that $x < \frac{1}{2}r(x)$ for all $x \in [0, \delta)$. Thus $z(x) = 1 + x$ on $[0, \delta)$, and so z is not invertible in G . Hence $G_{\mathbb{C}}^{-1}$ is not order open.

Our next lemma confirms that the map $z \mapsto z^{-1}$ on a uniformly complete E indeed possesses an order derivative at every point in its domain E^{-1} .

Lemma 3.14. *Suppose E is uniformly complete. Define the function $f: E^{-1} \rightarrow E^{-1}$ by $f(z) := z^{-1}$. Then $f'(c) := -c^{-2}$ is the order derivative of f at c for every $c \in E^{-1}$.*

Proof. For $c \in E^{-1}$ we have $U_c := \overset{\circ}{\Delta}(c, \frac{1}{2}|c|) \subseteq E^{-1}$ as seen in the proof of Lemma 3.12. Let $h_\alpha \rightarrow 0$ be such that $c + h_\alpha \in U_c$ for all α . There exists a net $(p_\beta)_\beta$ with $p_\beta \downarrow 0$ such that for every β , there exists an α_0 such that for all $\alpha \geq \alpha_0$ we have $|h_\alpha| \leq p_\beta$. Thus for each β there is an α_0 such that

$$\begin{aligned} |f(c + h_\alpha) - f(c) + h_\alpha c^{-2}| &= | -c^{-2}(c + h_\alpha)^{-1}h_\alpha c + c^{-2}(c + h_\alpha)^{-1}h_\alpha(c + h_\alpha) | \\ &= |h_\alpha|^2 |c|^{-2} |c + h_\alpha|^{-1} \\ &\leq 2|h_\alpha| |c|^{-3} p_\beta \end{aligned}$$

for all $\alpha \geq \alpha_0$. It follows that $f'(c) = -c^{-2}$ at every $c \in E^{-1}$. \square

The following theorem extends the four essential differentiability theorems from classical complex analysis to the Archimedean complex Φ -algebra setting. The proofs of these rules are slight variations of standard arguments. We later present a variation of the chain rule, see Theorem 4.3(iii), where the use of D -neighborhoods is not needed in the assumption.

Theorem 3.15. *Let $f: \text{dom}(f) \rightarrow E$ and $g: \text{dom}(g) \rightarrow E$ be functions. The following hold.*

- (i) (sum rule) *If $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g have order derivatives $f'(c)$ and $g'(c)$ at c , respectively, then $f'(c) + g'(c)$ is the order derivative of the sum map $z \mapsto f(z) + g(z)$ at c .*
- (ii) (product rule) *If $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g have order derivatives $f'(c)$ and $g'(c)$ at c , respectively, then $f'(c)g(c) + f(c)g'(c)$ is the order derivative of the product map $z \mapsto f(z)g(z)$ at c .*
- (iii) (chain rule) *Suppose that f and g are functions so that $\text{range}(g) \subseteq \text{dom}(f)$. Assume that g has order derivative $g'(c)$ at $c \in \text{dom}(g)$ and f has order derivative $f'(g(c))$ at $g(c)$. If there exist D -neighborhoods U_c for g at c and $V_{g(c)}$ for f at $g(c)$ such that $g(U_c) \subseteq V_{g(c)}$, then $f'(g(c))g'(c)$ is the order derivative of the composite map $z \mapsto f(g(z))$ at c .*
- (iv) (quotient rule) *Suppose $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g have order derivatives $f'(c)$ and $g'(c)$ at c , respectively, and $\text{range}(g) \subseteq E^{-1}$. If E is uniformly complete and there exists D -neighborhoods U_c for g at c and $V_{g(c)}$ for $z \mapsto z^{-1}$ at $g(c)$ such that $g(U_c) \subseteq V_{g(c)}$, then $(f'(c)g(c) - f(c)g'(c))g(c)^{-2}$ is the order derivative of the quotient map $z \mapsto f(z)g(z)^{-1}$ at c .*

Proof. (i) If $\overset{\circ}{\Delta}(c, r_1) \subseteq \text{dom}(f)$ and $\overset{\circ}{\Delta}(c, r_2) \subseteq \text{dom}(g)$ with r_1 and r_2 positive and invertible, then by [13, Theorem 142.2(iii)] $r := r_1 \wedge r_2$ is a positive invertible element, and we have $U_c := \overset{\circ}{\Delta}(c, r) \subseteq \text{dom}(f + g) = \text{dom}(f) \cap \text{dom}(g)$. Let $h_\alpha \rightarrow 0$ be such that $c + h_\alpha \in U_c$ for all α . If $q_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ are so that for fixed β and γ there is an α_0 for which

$$|f(c + h_\alpha) - f(c) - h_\alpha f'(c)| \leq |h_\alpha| q_\beta \quad \text{and} \quad |g(c + h_\alpha) - g(c) - h_\alpha g'(c)| \leq |h_\alpha| p_\gamma \quad (\alpha \geq \alpha_0),$$

then

$$|(f + g)(c + h_\alpha) - (f + g)(c) - h_\alpha(f'(c) + g'(c))| \leq |h_\alpha|(q_\beta + p_\gamma).$$

Hence $f'(c) + g'(c)$ is the order derivative of $z \mapsto f(z) + g(z)$ at c .

(ii) Since $\text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g)$, there is a positive invertible element r such that, by the same argument as in (i), we have $U_c := \overset{\circ}{\Delta}(c, r) \subseteq \text{dom}(fg)$. Let $h_\alpha \rightarrow 0$ be such that $c + h_\alpha \in U_c$ for all α . As $h_\alpha \rightarrow 0$, there is a sequence $q_\beta \downarrow 0$ such that for any β there is an α_0 such that $|h_\alpha| \leq q_\beta$ for all $\alpha \geq \alpha_0$. Let $p_\gamma \downarrow 0$ and $l_\lambda \downarrow 0$ be so that for fixed γ and λ there is an α_1 such that

$$|f(c + h_\alpha) - f(c) - h_\alpha f'(c)| \leq |h_\alpha| p_\gamma \quad \text{and} \quad |g(c + h_\alpha) - g(c) - h_\alpha g'(c)| \leq |h_\alpha| l_\lambda \quad (\alpha \geq \alpha_1).$$

Next let $\alpha_2 \geq \alpha, \alpha_1$. By adding and subtracting $f(c)g(c + h_\alpha)$ and $(f(c + h_\alpha) - f(c))g(c)$ inside the modulus in the first line below, we obtain

$$\begin{aligned} & \left| f(c + h_\alpha)g(c + h_\alpha) - f(c)g(c) - h_\alpha (f'(c)g(c) + f(c)g'(c)) \right| \\ & \leq \left| f(c + h_\alpha) - f(c) \right| |g(c + h_\alpha) - g(c)| + |h_\alpha| p_\gamma |g(c)| + |h_\alpha| l_\lambda |f(c)| \\ & \leq |h_\alpha| ((p_\gamma + |f'(c)|)(|h_\alpha| l_\lambda + |h_\alpha| |g'(c)|) + p_\gamma |g(c)| + l_\lambda |f(c)|) \\ & \leq |h_\alpha| ((p_\gamma + |f'(c)|)(q_\beta (l_\lambda + |g'(c)|) + p_\gamma |g(c)| + l_\lambda |f(c)|) \\ & = |h_\alpha| r_{\beta, \gamma, \lambda} \end{aligned}$$

whenever $\alpha \geq \alpha_2$, and where $r_{\beta, \gamma, \lambda} \downarrow 0$ is the net

$$r_{\beta, \gamma, \lambda} := (p_\gamma + |f'(c)|)(q_\beta (l_\lambda + |g'(c)|) + p_\gamma |g(c)| + l_\lambda |f(c)|).$$

Hence $f'(c)g(c) + f(c)g'(c)$ is the order derivative of $z \mapsto f(z)g(z)$ at c .

(iii) Let $U_c := \mathring{\Delta}(c, r)$ be a D -neighborhood for g at c and $V_{g(c)}$ be a D -neighborhood for f at $g(c)$ such that $g(U_c) \subseteq V_{g(c)}$. Then $U_c \subseteq \text{dom}(g) \cap g^{-1}(\text{dom}(f)) = \text{dom}(f \circ g)$. Let $h_\alpha \rightarrow 0$ be so that $c + h_\alpha \in U_c$ for every α . There exists $q_\beta \downarrow 0$ such that for a given arbitrary β , there exists α_0 such that

$$|g(c + h_\alpha) - g(c) - h_\alpha g'(c)| \leq |h_\alpha| q_\beta \quad (\alpha \geq \alpha_0).$$

Also, the function $g|_{U_c}$ is order continuous by Lemma 3.12 at c and $g(U_c) \subseteq V_c$, so we have $k_\alpha := g(c + h_\alpha) - g(c) \rightarrow 0$ and $k_\alpha + g(c) \in V_{g(c)}$. Thus there exists a $p_\gamma \downarrow 0$ such that for a given arbitrary γ , there exists α_1 for which

$$\left| f(g(c) + k_\alpha) - f(g(c)) - k_\alpha f'(g(c)) \right| \leq |k_\alpha| p_\gamma \quad (\alpha \geq \alpha_1).$$

Hence for all $\alpha \geq \alpha_0, \alpha_1$, we have

$$\begin{aligned} & \left| (f \circ g)(c + h_\alpha) - (f \circ g)(c) - h_\alpha f'(g(c)) g'(c) \right| \\ & = \left| f(g(c) + k_\alpha) - f(g(c)) - h_\alpha f'(g(c)) g'(c) \right| \\ & = \left| f(g(c) + k_\alpha) - f(g(c)) - k_\alpha f'(g(c)) + k_\alpha f'(g(c)) - h_\alpha f'(g(c)) g'(c) \right| \\ & \leq |k_\alpha| p_\gamma + \left| k_\alpha f'(g(c)) - h_\alpha f'(g(c)) g'(c) \right| \\ & = |k_\alpha| p_\gamma + \left| (g(c + h_\alpha) - g(c) - h_\alpha g'(c)) f'(g(c)) \right| \\ & \leq |g(c + h_\alpha) - g(c) - h_\alpha g'(c) + h_\alpha g'(c)| p_\gamma + |f'(g(c))| |h_\alpha| q_\beta \\ & \leq (|h_\alpha| q_\beta + |h_\alpha| |g'(c)|) p_\gamma + |f'(g(c))| |h_\alpha| q_\beta \\ & = |h_\alpha| (q_\beta p_\gamma + |g'(c)| p_\gamma + |f'(g(c))| q_\beta). \end{aligned}$$

Hence $f'(g(c))g'(c)$ is the order derivative of $z \mapsto f(g(z))$ at c .

(iv) Suppose E is uniformly complete. Assume there exist D -neighborhoods U_c for g at c and $V_{g(c)}$ for the map $z \mapsto z^{-1}$ at $g(c)$ such that $g(U_c) \subseteq V_{g(c)}$. There is an invertible element $r \in E_+$ such that $U_c := \overset{\circ}{\Delta}(c, r) \subseteq \text{dom}(f) \cap \text{dom}(g) = \text{dom}(f/g)$, by the same argument used in part (i) and (ii). Let $h_\alpha \rightarrow 0$ be so that $c + h_\alpha \in U_c$ for every α . By Lemma 3.14 and the chain rule (iii) it follows that $-g'(c)g(c)^{-2}$ is the order derivative for $z \mapsto g(z)^{-1}$ at c , and by the product rule (ii) we have that

$$f'(c)g(c)^{-1} - f(c)g'(c)g(c)^{-2} = \left(f'(c)g(c) - f(c)g'(c)\right)g(c)^{-2}$$

is the order derivative of $z \mapsto f(z)g(z)^{-1}$ at c . \square

We conclude this section with the definition of the order theoretical analogue of a holomorphic function.

Definition 3.16. Let $U \subseteq E$ be an order open set. A function $f: U \rightarrow E$ is said to be *holomorphic* on U , if f is order differentiable at every $c \in U$.

An example of a holomorphic function is the map $z \mapsto z^{-1}$ on E^{-1} for uniformly complete E by Lemmas 3.12 and 3.14.

4. Super and σ -super order differentiability

We introduce the concepts of super and σ -super order differentiability in this section as a means to strengthen the conclusions of Lemma 3.12 and Theorem 3.15.

Definition 4.1. We call a function $f: \text{dom}(f) \rightarrow E$ *super order differentiable* at $c \in \text{dom}(f)$ if f is order differentiable at c and the inequality (1) in Definition 3.5 holds for all nets $h_\alpha \rightarrow 0$ such that $c + h_\alpha \in \text{dom}(f)$ for all α .

By following the proof of Lemma 3.10, one readily deduces the lemma below.

Lemma 4.2. *If a function $f: \text{dom}(f) \rightarrow E$ is super order differentiable at $c \in \text{dom}(f)$, then f is order continuous at c .*

Next we provide an analogue of Theorem 3.15(i) – (iii) for super order differentiable functions. Notice that, unlike Theorem 3.15(iii), no assumptions regarding the D -neighborhoods of g and f are required in the super chain rule below. The proof is similar to that of Theorem 3.15(i) – (iii).

Theorem 4.3. *Let $f: \text{dom}(f) \rightarrow E$ and $g: \text{dom}(g) \rightarrow E$ be functions. The following hold.*

- (i) *(super sum rule) If $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g are super order differentiable at c , then $f + g$ is super order differentiable at c .*
- (ii) *(super product rule) If $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g are super order differentiable at c , then fg is super order differentiable at c .*
- (iii) *(super chain rule) Suppose that f and g are functions so that $\text{range}(g) \subseteq \text{dom}(f)$. If g is order differentiable at c and f is super order differentiable at c , then $f \circ g$ is order differentiable at c . If in addition g is super order differentiable at c , then $f \circ g$ is super order differentiable at c .*

At this point, one might suspect that the map $z \mapsto f(z)g(z)^{-1}$ is super order differentiable whenever f and g are. However, this is not the case in general. In fact, the map $z \mapsto f(z)g(z)^{-1}$ can even fail to be order continuous.

Remark 4.4. It is not true in general that the map $z \mapsto z^{-1}$ on an Archimedean complex Φ -algebra is order continuous. To demonstrate this fact, recall from Remark 3.11 that a net $(z_\alpha)_\alpha$ of positive invertible elements converging in order to a positive invertible element z has the property that $z_\alpha^{-1} \rightarrow z^{-1}$ if and only if there exists an index α_0 such that $\{|z_\alpha|^{-1} : \alpha \geq \alpha_0\}$ is order bounded. With this logical equivalence in mind, consider the space $\mathbb{C}^{\mathbb{N}}$ of complex valued sequences. For $(k, l) \in \mathbb{N} \times \mathbb{N}$ define $f_{k,l} \in \mathbb{C}^{\mathbb{N}}$ by

$$f_{k,l}(n) := \begin{cases} 1 & \text{if } 1 \leq n \leq k \\ l^{-1} & \text{for all } n > k \end{cases},$$

and by ordering the elements of $\mathbb{N} \times \mathbb{N}$ coordinatewise, we obtain the net $(f_{k,l})_{(k,l)}$. Furthermore, for $k \in \mathbb{N}$ define $g_k \in \mathbb{C}^{\mathbb{N}}$ by

$$g_k(n) := \begin{cases} 0 & \text{if } 1 \leq n \leq k \\ 1 & \text{for all } n > k \end{cases},$$

and note that for the constant 1 sequence e we have $|f_{k,l} - e| \leq g_k$ for all $l \in \mathbb{N}$, so $f_{k,l} \rightarrow e$ as $g_k \downarrow 0$. It follows that there is no $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ such that $\{f_{k,l}^{-1} : (k, l) \geq (k_0, l_0)\}$ is order bounded, since the subset $\{f_{k_0,l}^{-1} : l \geq l_0\}$ consists of sequences whose values on $n > k_0$ are unbounded. Hence $f_{k,l}^{-1} \not\rightarrow e$ by what was discussed in the first part of this remark.

Despite the lack of order continuity for the map $z \mapsto f(z)g(z)^{-1}$ in general, we prove in Proposition 4.6 that this map, when defined on a universally complete complex vector lattice, is always σ -order continuous. We stress here that universally complete complex vector lattices are Archimedean complex Φ -algebras, as noted in [12].

It is readily checked that any band in a universally complete complex vector lattice E is itself a universally complete complex vector lattice.

Notation 4.5. Suppose E is universally complete. Given $z \in E$, we denote by z^* the multiplicative inverse of z in the band B_z generated by z in E .

Proposition 4.6. *If E is universally complete, then the function $f: E^{-1} \rightarrow E^{-1}$ defined by $f(z) := z^{-1}$ is σ -order continuous.*

Proof. Let $c \in E^{-1}$, and let $(z_n)_{n \geq 0}$ be a sequence in E^{-1} such that $z_n \rightarrow c$. Then there exists a sequence $(p_m)_{m \geq 0}$ with $p_m \downarrow 0$ such that for every $m \geq 0$, there exists an $N \geq 0$ such that for all $n \geq N$ we have $|z_n - c| \leq p_m$. For each $m \geq 1$, let B_m denote the principal band generated by $(|c| - p_m)^+$, and set $B_0 = \{0\}$. Furthermore, let \mathbb{P}_m be the band projection onto B_m , and set $\mathbb{Q}_m := \mathbb{P}_m - \mathbb{P}_{m-1}$. Then for $m < n$ we have

$$0 \leq \mathbb{Q}_m(e) \wedge \mathbb{Q}_n(e) = (\mathbb{P}_m(e) - \mathbb{P}_{m-1}(e)) \wedge (\mathbb{P}_n(e) - \mathbb{P}_{n-1}(e)) \leq \mathbb{P}_{n-1}(e) \wedge (e - \mathbb{P}_{n-1}(e)) = 0,$$

so $(\mathbb{Q}_m)_{m \geq 1}$ is a pairwise disjoint sequence of band projections.

Let $m \in \mathbb{N}$ be arbitrary. Let N be such that $|z_n - c| \leq p_m$ holds for all $n \geq N$. It follows that

$$(|c| - p_m)^+ \leq |z_n|$$

holds for all $n \geq N$, and thus

$$\mathbb{Q}_m|z_n^{-1}| \leq \mathbb{Q}_m((|c| - p_m)^+)^*$$

holds for all $n \geq N$. If we define $u_m := \left(\bigvee_{k \leq N} \mathbb{Q}_m|z_k^{-1}|\right) \vee \mathbb{Q}_m((|c| - p_m)^+)^*$ for each $m \in \mathbb{N}$, it follows that $\mathbb{Q}_m|z_n^{-1}| \leq u_m$ for all $n \geq 0$. Since $(u_m)_{m \geq 1}$ is a pairwise disjoint sequence, $u := \sup_{m \geq 1} u_m$ exists in E_+ as E is universally complete. For every $m \in \mathbb{N}$ and every $n \in \mathbb{N}_0$ we have that $\mathbb{Q}_m|z_n^{-1}| \leq u$, thus

$$|z_n|^{-1} = \lim_{M \rightarrow \infty} \mathbb{P}_M|z_n|^{-1} = \lim_{M \rightarrow \infty} \sum_{m \leq M} \mathbb{Q}_m|z_n|^{-1} = \lim_{M \rightarrow \infty} \sup_{m \leq M} \mathbb{Q}_m|z_n|^{-1} \leq u,$$

for each $n \in \mathbb{N}_0$. Finally, observe that

$$|z_n^{-1} - c^{-1}| = |z_n - c||c^{-1}||z_n^{-1}| \leq |z_n - c||c^{-1}|u,$$

proving that $z_n^{-1} \rightarrow c^{-1}$. \square

The assumption of universal completeness in Proposition 4.6 cannot be relaxed, as the following remark illustrates.

Remark 4.7. Consider the Dedekind complete complex Φ -algebra $\ell_{\mathbb{C}}^{\infty}$. We look at the sequence $(z_k)_{k \geq 0}$ defined by $z_k := f_{k,k}$ as in Remark 4.4. Then $z_k \rightarrow e$ and $z_k^{-1} \in \ell_{\mathbb{C}}^{\infty}$ for every $k \in \mathbb{N}_0$, but $z_k^{-1} \not\rightarrow e$ as no tail of $(z_k^{-1})_{k \geq 0}$ is order bounded. Hence the map $z \mapsto z^{-1}$ is not σ -order continuous on the invertible elements of $\ell_{\mathbb{C}}^{\infty}$.

The σ -order continuity of the map $z \mapsto z^{-1}$ on a universally complete space motivates the following definition.

Definition 4.8. We call a function $f: \text{dom}(f) \rightarrow E$ σ -super order differentiable at c if f is order differentiable at c and the inequality (1) in Definition 3.5 holds for all sequences $h_n \rightarrow 0$ such that $c + h_n \in \text{dom}(f)$ for all $n \geq 0$.

The notion of σ -super order differentiability allows us to recover a super quotient rule. We begin by proving that the map $z \rightarrow z^{-1}$ is σ -super order differentiable on universally complete complex vector lattices.

Lemma 4.9. Let E be universally complete, and define the function $f: E^{-1} \rightarrow E^{-1}$ by $f(z) := z^{-1}$. Then f is σ -super order differentiable at c for every $c \in E^{-1}$.

Proof. For $c \in E^{-1}$, note that $\overset{\circ}{\Delta}(c, \frac{1}{2}|c|) \subseteq E^{-1}$. Indeed, if $z \in E$ such that $|z - c| \ll \frac{1}{2}|c|$, it follows that $\frac{1}{2}|c| \subseteq |z|$. Hence $z \in E^{-1}$. Let $h_n \rightarrow 0$ be such that $c + h_n \in E^{-1}$ for all $n \in \mathbb{N}_0$. Similarly to what was shown in Proposition 4.6, we have that $u := \sup_{n \geq 0} |c + h_n|^{-1}$ exists in E_+ as E is universally complete. There exists a sequence $(p_m)_{m \geq 0}$ with $p_m \downarrow 0$ such that for every $m \geq 0$, there exists an $N \geq 0$ such that for all $n \geq N$ we have $|h_n| \leq p_m$. Thus for $m \geq 0$ there is an $N \geq 0$ such that

$$\begin{aligned} |f(c + h_n) - f(c) + h_n c^{-2}| &= |-c^{-2}(c + h_n)^{-1}h_n c + c^{-2}(c + h_n)^{-1}h_n(c + h_n)| \\ &= |h_n|^2 |c|^{-2} |c + h_n|^{-1} \\ &\leq |h_n| |c|^{-2} u p_m \end{aligned}$$

for all $n \geq N$. It follows that f is σ -super order differentiable at c . \square

Theorem 4.10. (super quotient rule) Let E be universally complete and let $f: \text{dom}(f) \rightarrow E$ and $g: \text{dom}(g) \rightarrow E$ be functions. Suppose $c \in \text{dom}(f) \cap \text{dom}(g)$ and f and g are σ -super order differentiable at c and $\text{range}(g) \subseteq E^{-1}$. Then the quotient map $z \mapsto f(z)g(z)^{-1}$ is σ -super order differentiable at c .

Proof. Let $h_n \rightarrow 0$ be so that $c + h_n \in \text{dom}(f/g)$ for every $n \in \mathbb{N}_0$. By Lemma 4.9 and the super chain rule Theorem 4.3(iii) it follows that the function $z \mapsto g(z)^{-1}$ is σ -super order differentiable at c . It then follows from the super product rule Theorem 4.3(ii) that $z \mapsto f(z)g(z)^{-1}$ is σ -super order differentiable at c . \square

5. Improved Cauchy-Hadamard formulas

We improve our Cauchy-Hadamard formulas from [12] in this section and therefore focus solely on universally complete spaces, as we did in [12].

Notation 5.1. E will denote a universally complete complex vector lattice in this section (with corresponding real part F as already designated).

We next provide some useful basic properties of E , which we use freely throughout the rest of the paper. Note that (ii) follows from (i), which is a consequence of [1, Theorem 2.37]. Together, [1, Theorem 2.44] and [10, Theorem 18.13] imply (iii)–(v). The veracity of (vi) is explained in [12, Remark 3.3].

For all $z, w \in E$ and any order projection \mathbb{P} on E ,

- (i) $\mathbb{P}(zw) = z\mathbb{P}(w)$.
- (ii) $\mathbb{P}(zw) = \mathbb{P}(z)\mathbb{P}(w)$.
- (iii) $\mathbb{P}(|z|) = |\mathbb{P}(z)|$, that is, \mathbb{P} is a complex vector lattice homomorphism.
- (iv) $\mathbb{P}(\sup A) = \sup[\mathbb{P}(A)]$ for all $\emptyset \neq A \subseteq F$ bounded above.
- (v) \mathbb{P} is order continuous.
- (vi) z is invertible if and only if $|z|$ is a weak order unit.

In particular, E is uniformly complete. Thus for every $x \in E_+$, and $n \in \mathbb{N}$, there exists a unique $y \in E_+$ such that $y^n = x$ (see [5, Corollary 6]). This n th root of x is denoted by $x^{1/n}$ in the remainder of this manuscript.

Remark 5.2. For sequences, our definition of order convergence seems different at first glance from the definition given in [14, Ch. 4 S. 10] and used in [12], but it is in fact, equivalent if E is Dedekind complete. Indeed, if $q_m \downarrow 0$ is such that for all $m \geq 0$ there exists $N \geq 0$ so that $|z_n - z| \leq q_m$ whenever $n \geq N$, then for all $m \geq 0$, define $p_m := \sup_{n \geq m} |z_n - z|$, which are well-defined elements in E by Dedekind completeness. Clearly, the sequence $(p_m)_{m \geq 0}$ is downward directed and satisfies $|z_m - z| \leq p_m$ for all $m \geq 0$. By Dedekind completeness once more, let $p := \inf_{m \geq 0} p_m$. For any $k \geq 0$ there is an m_k such that $p_{m_k} \leq q_k$ and so, we have $p \leq \inf_{k \geq 0} p_{m_k} \leq \inf_{k \geq 0} q_k = 0$, showing that $p_m \downarrow 0$. Note that the converse implication clearly holds.

For an order bounded sequence $(x_n)_{n \geq 0}$ in F , we as usual write

$$\limsup_{n \rightarrow \infty} x_n := \inf_{n \geq 0} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n := \sup_{n \geq 0} \inf_{m \geq n} x_m.$$

Observe that $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$ are the order limits of the sequences

$$\left(\sup_{m \geq n} x_m \right)_{n \geq 0} \quad \text{and} \quad \left(\inf_{m \geq n} x_m \right)_{n \geq 0},$$

respectively.

We proceed with some notes on the sup-completion F^s of F , a notion introduced in [9, Section 1]. Using the terminology from [9, Definition 1.1], a *cone* $(C, +)$ is a commutative monoid, that is closed under a nonnegative real number scalar multiplication, for which the following hold:

- (i) $\alpha(x + y) = \alpha x + \alpha y \quad (\alpha \in [0, \infty), x, y \in C)$,
- (ii) $(\alpha + \beta)x = \alpha x + \beta x \quad (\alpha, \beta \in [0, \infty), x \in C)$,
- (iii) $\alpha(\beta x) = (\alpha\beta)x \quad (\alpha, \beta \in [0, \infty), x \in C)$,
- (iv) $1x = x \quad (x \in C)$, and
- (v) $0x = 0 \quad (x \in C)$.

In (v) above, the first 0 denotes the real number zero, while the second 0 designates the identity element of $(C, +)$.

Following [9], we will from now on denote a cone $(C, +)$ by C , for short. Given a cone C , we denote as in [9] the set

$$C_0 := \{x \in C : x \text{ possesses an inverse under } +\}.$$

The nonnegative real number scalar multiplication restricted to C_0 can be extended to all of the real numbers by defining

$$\alpha x := -\alpha(-x) \quad (\alpha \in (-\infty, 0), x \in C_0).$$

It is readily checked that C_0 is a vector space under $+$ and this expanded scalar multiplication.

A cone C possessing a partial ordering \leq is called an *ordered cone* if the following hold:

- (vi) $x \leq y$ implies $x + z \leq y + z \quad (x, y, z \in C)$, and
- (vii) $x \leq y$ implies $\alpha x \leq \alpha y \quad (\alpha \in [0, \infty), x, y \in C)$.

An ordered cone that is a lattice with respect to its partial ordering is called a *lattice cone*. A lattice cone C is called *Dedekind complete* if every nonempty subset of C which is bounded above (respectively, bounded below) possesses a supremum (respectively, infimum) in C .

Proposition 5.3. [9, Theorem 1.4] *There exists an essentially unique cone F^s (called the sup-completion of F) for which the following hold.*

- (i) F^s is Dedekind complete.
- (ii) $F = (F^s)_0$ with coinciding algebraic and order structures.
- (iii) For each $y \in F^s$, we have

$$y = \sup\{x \in F : x \leq y\}.$$

- (iv) $z + (x \wedge y) = (z + x) \wedge (z + y) \quad (x \in F, y, z \in F^s)$.
- (v) $\{y \in F^s : \text{there exists } x \in F \text{ such that } y \leq x\} \subseteq F$.
- (vi) F^s has a largest element.
- (vii) For any $\emptyset \neq A, B \subseteq F^s$ for which $\sup A = \sup B$, we have

$$\sup_{a \in A} (a \wedge x) = \sup_{b \in B} (b \wedge x)$$

for any $x \in F$.

As noted in [3, (P6)], the following immediate consequence of Proposition 5.3(vii) holds.

Proposition 5.4. *For any $\emptyset \neq A \subseteq F^s$ and $x \in F$, we have*

$$\sup_{a \in A} (a \wedge x) = (\sup A) \wedge x.$$

A salient observation for our purposes is that Proposition 5.3(i) and (vi) together imply that $\sup A$ exists in F^s for any nonempty $A \subseteq F^s$.

We utilize the following proposition from [9, Theorem 1.4] and [3, Remark 10].

Proposition 5.5. *For F^s the following hold.*

- (i) F^s is a distributive lattice.
- (ii) For any projection band B of F , we have that $F^s = B^s \times (B^d)^s$.

Next we begin to enhance our Cauchy-Hadamard formulas originally given in [12] to be able to consider unbounded spectra of convergence. For this task, we rely on the sup-completion of F . By upgrading our Cauchy-Hadamard formulas, we are able to prove that analytic functions on E are holomorphic. We begin with some relevant definitions. For more information on series and power series on universally complete complex vector lattices, we refer the reader to [12].

Definition 5.6. We say that a series $\sum_{n=0}^{\infty} a_n$ converges in order if the sequence of partial sums $(\sum_{n=0}^m a_n)_{m \geq 0}$ converges in order in E . If $\sum_{n=0}^{\infty} |a_n|$ converges in order, then we say that $\sum_{n=0}^{\infty} a_n$ converges absolutely in order.

Definition 5.7. A power series on E , centered at $c \in E$, is of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$, where $a_n \in E$ for every $n \in \mathbb{N}_0$ and z is a variable in E . We say that a power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges uniformly in order on a region $D \subseteq E$ if there exists a sequence $p_m \downarrow 0$ such that for every $m \in \mathbb{N}_0$, there exists a $K \in \mathbb{N}_0$ such that

$$\sup_{z \in D} \left| \sum_{n=0}^k a_n(z-c)^n - \sum_{n=0}^{\infty} a_n(z-c)^n \right| \leq p_m$$

holds for every $k \geq K$.

Definition 5.8. Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series. We define the spectrum of convergence of the power series as

$$\Omega := \left\{ r \in E_+ : \sum_{n=0}^{\infty} a_n(z-c)^n \text{ converges uniformly in order on } \bar{\Delta}(c, r) \right\}.$$

We also define $\rho := \sup \Omega$ to be the radius of convergence of the power series, which exists in the sup-completion F^s of F .

We next provide a series of remarks, definitions, and results which we use to improve the Cauchy-Hadamard formulas given in [12, Theorem 3.11].

Remark 5.9. By [2, Proposition 3], any band projection $\mathbb{P} : F \rightarrow F$ can be extended to a left-order continuous, additive, and positively homogeneous map $\bar{\mathbb{P}} : F^s \rightarrow F^s$ via the formula

$$\bar{\mathbb{P}}(y) := \sup\{\mathbb{P}(x) : x \in F, x \leq y\} \quad (y \in F^s).$$

Given $0 \leq u \in F^s$, we define as in [2] a map $\mathbb{P}_u : F \rightarrow F$ by

$$\mathbb{P}_u(x) := \sup_{n \geq 1} \{x \wedge nu\}$$

for all $x \in F_+$ and then

$$\mathbb{P}_u(x) := \mathbb{P}_u(x^+) - \mathbb{P}_u(x^-)$$

for all $x \in F$. By [2, Lemma 4], the map \mathbb{P}_u is a band projection on F ; in fact, $\mathbb{P}_u = \mathbb{P}_{\mathbb{P}_u(e)}$. We denote the corresponding band $B_{\mathbb{P}_u(e)}$ by B_u , for short. Recall that B_u is itself a (real) universally complete vector lattice.

Lemma 5.10. *Let $0 \leq u \in F^s$. Then $\bar{\mathbb{P}}_u(u) = u$ and $u \in B_u^s$.*

Proof. We first note that

$$\begin{aligned} \bar{\mathbb{P}}_u(u) &= \sup\{\mathbb{P}_u(x) : x \in F, x \leq u\} \\ &= \sup\left\{\sup_{n \geq 1} \{x \wedge nu\} : x \in F, x \leq u\right\} \\ &= \sup\{x : x \in F, x \leq u\} \\ &= u. \end{aligned}$$

Secondly, let $(x_\alpha)_\alpha$ be a net in F such that $x_\alpha \uparrow u$. From the left-order continuity of $\bar{\mathbb{P}}_u$ and Proposition 5.3(iii) we have $\mathbb{P}_u(x_\alpha) = \bar{\mathbb{P}}_u(x_\alpha) \uparrow \bar{\mathbb{P}}_u(u) = u$. Since $\mathbb{P}_u(x_\alpha) \in B_u$ for all α , we conclude that $u \in B_u^s$ by Proposition 5.5(ii). \square

Next we provide an alternative mechanism for dividing positive elements of a sup-completion into finite and infinite parts to that found in [3]. As stated in the introduction, this new approach perhaps yields simpler formulas, which facilitate many of the proofs in this section.

Definition 5.11. For $0 \leq u \in F^s$, we define

$$u_\infty := \inf_{n \geq 1} n^{-1}u$$

and

$$u_{\mathcal{F}} := \sup\{x - x \wedge u_\infty : x \in F_+, x \leq u\}.$$

Note that $x \wedge u_\infty \in F$ for every $x \in F$ by Proposition 5.3(v). Ergo the element $u_{\mathcal{F}}$ is well-defined in F^s .

The following lemma is required for our proof of Theorem 5.13. It is an immediate consequence of the definition of u_∞ .

Lemma 5.12. *Given $0 \leq u \in F^s$ and $m \in \mathbb{N}$, we have $mu_\infty = u_\infty$.*

Our next result represents an alternative way of breaking positive elements of a sup-completion into disjoint finite and infinite parts to that found in [3].

Theorem 5.13. *Let $0 \leq u \in F^s$. Then the following hold.*

- (i) $u = u_{\mathcal{F}} + u_{\infty}$.
- (ii) $u_{\mathcal{F}} \wedge u_{\infty} = 0$.
- (iii) $u \in F$ if and only if $u_{\infty} = 0$.
- (iv) $u_{\mathcal{F}} \in F_+$.

Proof. (i) Note that

$$\begin{aligned} u_{\mathcal{F}} + u_{\infty} &= \sup\{x - x \wedge u_{\infty} : x \in F_+, x \leq u\} + u_{\infty} \\ &\geq \sup\{x + u_{\infty} - x \wedge u_{\infty} : x \in F_+, x \leq u\} \\ &\geq \sup\{x : x \in F_+, x \leq u\} \\ &= u, \end{aligned}$$

where the last equality follows from Proposition 5.3(iii).

Moreover, we have

$$u_{\mathcal{F}} + u_{\infty} \leq u + u_{\infty} = u + \inf_{n \geq 1} n^{-1}u \leq \inf_{n \geq 1} (u + n^{-1}u) = \inf_{n \geq 1} (1 + n^{-1})u = u.$$

Thus (i) holds.

(ii) If $x \in F_+$ and $x \leq u$, then by Proposition 5.3(iv) and Lemma 5.12, we have

$$\begin{aligned} (x - x \wedge u_{\infty}) \wedge u_{\infty} &= x \wedge (u_{\infty} + x \wedge u_{\infty}) - x \wedge u_{\infty} \\ &= x \wedge ((x + u_{\infty}) \wedge 2u_{\infty}) - x \wedge u_{\infty} \\ &= x \wedge ((x + u_{\infty}) \wedge u_{\infty}) - x \wedge u_{\infty} \\ &= x \wedge u_{\infty} - x \wedge u_{\infty} \\ &= 0. \end{aligned}$$

Then by Proposition 5.4 and the string of equalities above, we have

$$\begin{aligned} 0 &\leq u_{\mathcal{F}} \wedge (u_{\infty} \wedge e) \\ &= (\sup\{x - x \wedge u_{\infty} : x \in F_+, x \leq u\}) \wedge (u_{\infty} \wedge e) \\ &= \sup\{(x - x \wedge u_{\infty}) \wedge (u_{\infty} \wedge e) : x \in F_+, x \leq u\} \\ &\leq \sup\{(x - x \wedge u_{\infty}) \wedge u_{\infty} : x \in F_+, x \leq u\} \\ &= 0. \end{aligned}$$

Thus $u_{\mathcal{F}} \wedge u_{\infty} \wedge e = 0$. But since e is a positive invertible element, we have $u_{\mathcal{F}} \wedge u_{\infty} = 0$, as claimed.

(iii) It is evident that if $u \in F$, then $u_{\infty} = 0$, since E is Archimedean.

Next suppose $u \notin F$; i.e. $u \in F^s \setminus F$. Let $n \in \mathbb{N}$, and take $z \in F_+, z \leq n^{-1}u$. Then

$$\begin{aligned} n^{-1}u &\geq \bar{\mathbb{P}}_{ne < u}(n^{-1}u) = \sup\{\mathbb{P}_{ne < u}(y) : y \in F_+, y \leq n^{-1}u\} \\ &= \sup\left\{\sup_{m \geq 1}\{y \wedge m(u - ne)^+\} : y \in F_+, y \leq n^{-1}u\right\} \end{aligned}$$

$$\begin{aligned}
 &\geq \sup \left\{ \sup_{m \geq 1} \{y \wedge m(nz - ne)^+\} : y \in F_+, y \leq z \right\} \\
 &= \sup \left\{ \sup_{m \geq 1} \{y \wedge mn(z - e)^+\} : y \in F_+, y \leq z \right\} \\
 &\geq \sup \left\{ \sup_{m \geq 1} \{y \wedge m(z - e)^+\} : y \in F_+, y \leq z \right\} \\
 &= \sup \{ \mathbb{P}_{e < z}(y) : y \in F_+, y \leq z \} = \mathbb{P}_{e < z}(z) \\
 &\geq \mathbb{P}_{e < z}(e).
 \end{aligned}$$

Thus $n^{-1}u \geq \sup \{ \mathbb{P}_{e < z}(e) : z \in F_+, z \leq n^{-1}u \}$. Furthermore, we have

$$\begin{aligned}
 \sup \{ \mathbb{P}_{e < z}(e) : z \in F_+, z \leq n^{-1}u \} &= \sup \left\{ \sup_{m \geq 1} \{e \wedge m(z - e)^+\} : z \in F_+, z \leq n^{-1}u \right\} \\
 &= \sup_{m \geq 1} \left(\sup \{e \wedge m(z - e)^+ : z \in F_+, z \leq n^{-1}u \} \right) \\
 &= \sup_{m \geq 1} e \wedge m(n^{-1}u - e)^+ \\
 &= \sup_{m \geq 1} e \wedge mn^{-1}(u - ne)^+ \\
 &= \sup_{m \geq 1} e \wedge m(u - ne)^+ \\
 &= \mathbb{P}_{ne < u}(e).
 \end{aligned}$$

Hence $n^{-1}u \geq \mathbb{P}_{ne < u}(e)$. We thus obtain

$$u_\infty \geq \inf_{n \geq 1} \mathbb{P}_{ne < u}(e).$$

It follows from our assumption $u \in F^s \setminus F$ and [2, Theorem 14] that

$$\inf_{n \geq 1} \mathbb{P}_{ne < u}(e) > 0.$$

Therefore, $u_\infty \neq 0$.

(iv) We use proof by contradiction. To this end, suppose that $u_{\mathcal{F}} \in F^s \setminus F$. By part (iii) of this theorem, we have $(u_{\mathcal{F}})_\infty > 0$. Thus

$$u_{\mathcal{F}} \wedge u_\infty \geq (u_{\mathcal{F}})_\infty \wedge u_\infty = \left(\inf_{n \geq 1} n^{-1}u_{\mathcal{F}} \right) \wedge \left(\inf_{n \geq 1} n^{-1}u \right) = \inf_{n \geq 1} n^{-1}u_{\mathcal{F}} = (u_{\mathcal{F}})_\infty > 0,$$

a contradiction with part (ii) of this theorem. Thus $u_{\mathcal{F}} \in F$ and hence $u_{\mathcal{F}} \in F_+$, which establishes (iv). \square

The finite and infinite parts of elements in the sup-completion, as was shown in Theorem 5.13, yield the corresponding threefold band decomposition of F .

Theorem 5.14. For $0 \leq u \in F^s$, we have

$$F = B_{u_{\mathcal{F}}} \oplus B_{u_\infty} \oplus B_u^d.$$

Proof. Let $0 \leq u \in F^s$. We first show that $B_{u_{\mathcal{F}}} \cap B_{u_{\infty}} = \{0\}$. To this end, suppose $x \in B_{u_{\mathcal{F}}} \cap B_{u_{\infty}}$, with $x \geq 0$. Then $x = \mathbb{P}_{u_{\mathcal{F}}}(x) = \mathbb{P}_{u_{\infty}}(x)$. Using Lemma 5.12, Theorem 5.13(ii), and Proposition 5.4, we obtain

$$\begin{aligned} 0 \leq x &= x \wedge x = \mathbb{P}_{u_{\mathcal{F}}}(x) \wedge \mathbb{P}_{u_{\infty}}(x) = \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} \wedge \sup_{n \geq 1} \{x \wedge nu_{\infty}\} \\ &= \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} \wedge (x \wedge u_{\infty}) = \sup_{n \geq 1} \{(x \wedge nu_{\mathcal{F}}) \wedge (x \wedge u_{\infty})\} \\ &\leq \sup_{n \geq 1} \{n(u_{\mathcal{F}} \wedge u_{\infty})\} = 0. \end{aligned}$$

Hence $x = 0$. It follows that $B_{u_{\mathcal{F}}} \cap B_{u_{\infty}} = \{0\}$.

We next show that $B_u = B_{u_{\mathcal{F}}} + B_{u_{\infty}}$, first handling the inclusion $B_{u_{\mathcal{F}}} + B_{u_{\infty}} \subseteq B_u$. To this end, let $x + y \in B_{u_{\mathcal{F}}} + B_{u_{\infty}}$ with $x, y \geq 0$. Then

$$\begin{aligned} x + y &= \mathbb{P}_{u_{\mathcal{F}}}(x) + \mathbb{P}_{u_{\infty}}(y) = \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} + \sup_{m \geq 1} \{y \wedge mu_{\infty}\} \\ &\leq \sup_{n \geq 1} \{x \wedge nu\} + \sup_{m \geq 1} \{y \wedge mu\} \leq 2\mathbb{P}_u(x \vee y), \end{aligned}$$

and so $x + y \in B_u$.

For the reverse inclusion, note that, from Theorem 5.13(i) and [9, Lemma 1.3], we have

$$u = u_{\mathcal{F}} + u_{\infty} = u_{\mathcal{F}} \vee u_{\infty} + u_{\mathcal{F}} \wedge u_{\infty}.$$

It thus follows from Theorem 5.13(ii) that $u = u_{\mathcal{F}} \vee u_{\infty}$. Hence for any $0 \leq x \in B_u$, we have from Lemma 5.12, Theorem 5.13(ii), Proposition 5.3(ii), and Proposition 5.5 that

$$\begin{aligned} x &= \mathbb{P}_u(x) = \sup_{n \geq 1} \{x \wedge nu\} = \sup_{n \geq 1} \{x \wedge (nu_{\mathcal{F}} + nu_{\infty})\} = \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}} + x \wedge nu_{\infty}\} \\ &= \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}} + x \wedge u_{\infty}\} = \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} + x \wedge u_{\infty} = \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} + \sup_{m \geq 1} \{x \wedge mu_{\infty}\} \\ &= \mathbb{P}_{u_{\mathcal{F}}}(x) + \mathbb{P}_{u_{\infty}}(x). \end{aligned}$$

It hence follows that $B_u = B_{u_{\mathcal{F}}} + B_{u_{\infty}}$. We therefore obtain

$$F = B_u \oplus B_u^d = B_{u_{\mathcal{F}}} \oplus B_{u_{\infty}} \oplus B_u^d. \quad \square$$

We are almost ready to present our new and improved Cauchy-Hadamard formulas but require some lemmas first.

Lemma 5.15. *Let $0 \leq u \in F^s$. Then $\bar{\mathbb{P}}_{u_{\mathcal{F}}}(u) = u_{\mathcal{F}}$.*

Proof. First note that $\bar{\mathbb{P}}_{u_{\mathcal{F}}}(u) \geq \bar{\mathbb{P}}_{u_{\mathcal{F}}}(u_{\mathcal{F}}) = u_{\mathcal{F}}$ by Lemma 5.10. Moreover, using Proposition 5.3(iv) and Theorem 5.13(ii), we have

$$\begin{aligned} \bar{\mathbb{P}}_{u_{\mathcal{F}}}(u) &= \sup \{ \mathbb{P}_{u_{\mathcal{F}}}(x) : x \in F_+, x \leq u \} = \sup \left\{ \sup_{n \geq 1} \{x \wedge nu_{\mathcal{F}}\} : x \in F_+, x \leq u \right\} \\ &\leq \sup_{n \geq 1} \{u \wedge nu_{\mathcal{F}}\} = \sup_{n \geq 1} \{(u_{\mathcal{F}} + u_{\infty}) \wedge nu_{\mathcal{F}}\} = \sup_{n \geq 1} \{u_{\mathcal{F}} + u_{\infty} \wedge (n-1)u_{\mathcal{F}}\} \\ &= u_{\mathcal{F}}, \end{aligned}$$

which proves the lemma. \square

As in [9], we set $y^+ := y \vee 0$ for $y \in F^s$.

Lemma 5.16. *If $0 \leq u \in F^s$, then u_∞ is the largest element of $B_{u_\infty}^s$.*

Proof. Let $y \in B_{u_\infty}^s$. Then using Lemma 5.12 in the fourth equality below, we obtain

$$\begin{aligned} y &\leq y^+ = \bar{\mathbb{P}}_{u_\infty}(y^+) = \sup\{\mathbb{P}_{u_\infty}(x) : x \in F_+, x \leq y^+\} \\ &= \sup\left\{\sup_{n \geq 1}\{x \wedge nu_\infty\} : x \in F_+, x \leq y^+\right\} = \sup\{x \wedge u_\infty : x \in F_+, x \leq y^+\} \leq u_\infty. \quad \square \end{aligned}$$

We next make use of the multiplication on F^s introduced by Azouzi and Nasri in [2, Section 3.2]. This extended multiplication is defined for $0 \leq x, y \in F^s$ as

$$xy := \sup\{vw : v, w \in F_+, v \leq x, w \leq y\}.$$

The following useful lemma from [2] will also be applied in our proofs.

Lemma 5.17. ([2, Lemma 23]) *Let $(x_\alpha)_\alpha, (y_\beta)_\beta$ be nets in F_+^s such that $x_\alpha \uparrow x$ and $y_\beta \uparrow y$ for $0 \leq x, y \in F^s$. Then $x_\alpha y_\alpha \uparrow xy$.*

As an immediate consequence of Lemma 5.17, we have the following.

Lemma 5.18. *Let $(y_\alpha)_\alpha$ be a net in F_+^s such that $y_\alpha \uparrow y$ for $0 \leq y \in F^s$. If $0 \leq x, u \in F^s$ and $xy_\alpha \leq u$ holds for every α , then $xy \leq u$.*

In order to formulate a Cauchy-Hadamard formula which considers unbounded spectra of convergence, we need a notion of a generalized inverse for positive elements in the sup-completion. We would like to remind the reader that for $z \in E$ the notation z^* refers to the multiplicative inverse of z in the band B_z .

Definition 5.19. Let $x \in F_+^s$. The *generalized inverse* of x in F_+^s , denoted by x^{-1} , is defined by $x^{-1} := (x_{\mathcal{F}})^* + \infty_x^d$, where ∞_x^d denotes the largest element of $(B_x^d)^s$.

Theorem 5.20 (Cauchy-Hadamard). *Let $\sum_{n=0}^\infty a_n(z - c)^n$ be a power series on E . Also let $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$, which exists in F^s . The following identities hold.*

- (i) $B_L^d = B_{\rho_\infty}$.
- (ii) $B_{L_\infty} = B_\rho^d$.
- (iii) $(L_{\mathcal{F}})^* = \rho_{\mathcal{F}}$.

In particular, we have that $L^{-1} = \rho$ for the generalized inverse of L in F^s .

Proof. Towards proving (i), let $r_\alpha \uparrow \rho$, where $r_\alpha \in \Omega$ for every α . Fixing α , we have that $r_\alpha \in \Omega$ implies $\sum_{n=0}^\infty a_n r_\alpha^n$ converges in order. Thus by [12, Theorem 3.6(ii)], we have

$$\limsup_{n \rightarrow \infty} (|a_n|^{1/n} r_\alpha) \leq e.$$

Let $(x_n)_{n \geq 0}$ be a sequence in E_+ and set $x := \limsup_{n \rightarrow \infty} x_n$. Note that x is well-defined in F^s , and assume that $y \in E_+$. It follows from Lemma 5.17 and [3, Lemma 24(iii)], which states that

$$x(y \vee z) = (xy) \vee (xz) \quad (0 \leq x, y, z \in F^s),$$

that

$$xy = \limsup_{n \rightarrow \infty} (x_n y). \quad (2)$$

We thus obtain

$$Lr_\alpha = \limsup_{n \rightarrow \infty} (|a_n|^{1/n} r_\alpha) \leq e.$$

It follows from Lemma 5.18 that $L\rho \leq e$, and thus $L\rho_\infty \leq e$. Then by Lemma 5.12, we have $nL\rho_\infty \leq e$ for each $n \in \mathbb{N}$, hence $L\rho_\infty = 0$ since F is Archimedean. It follows that $L \wedge \rho_\infty = 0$. Thus if $x \in B_{\rho_\infty}$, then by Lemma 5.16 we have

$$|x| \wedge L \leq \rho_\infty \wedge L = 0,$$

so $x \in B_L^d$.

Next let $0 \leq r \in B_L^d$. Then $\limsup_{n \rightarrow \infty} (|a_n|^{1/n} mr) = mLr = 0 \ll e$ for all $m \in \mathbb{N}$, so by [12, Theorem 3.6(i)] we have $mr \in \Omega$ for each $m \in \mathbb{N}$. Then $mr \leq \rho$ and hence $r \leq \frac{1}{m}\rho$ for every $m \in \mathbb{N}$. Thus we get $r \leq \rho_\infty$. Hence $r \in B_{\rho_\infty}$ and (i) holds.

We proceed with the proof of (ii). For this task, let $r \in \Omega$. Then as argued in (i) we have $Lr \leq e$. It follows that $L_\infty r \leq e$, and thus $L_\infty \rho \leq e$ holds by Lemma 5.18. By Lemma 5.12, we have that

$$nL_\infty \rho \leq e$$

holds for all $n \in \mathbb{N}$. This implies that $L_\infty \rho = 0$, which in turn yields $L_\infty \wedge \rho = 0$. Thus if $x \in B_{L_\infty}$, then by Lemma 5.16 we have

$$|x| \wedge \rho \leq L_\infty \wedge \rho = 0.$$

Hence we get $x \in B_\rho^d$.

Next, note that by Theorem 5.14 and (iii) we have $B_\rho^d \perp B_L^d$. It follows from Theorem 5.14 that $B_\rho^d \subseteq B_{L_{\mathcal{F}}} \oplus B_{L_\infty}$. We will show that $B_\rho^d \subseteq B_{L_\infty}$ by illustrating that $B_\rho^d \cap B_{L_{\mathcal{F}}} = \{0\}$. To this end, suppose that $B := B_\rho^d \cap B_{L_{\mathcal{F}}}$ is a nontrivial band, and let \mathbb{P} be its associated order projection. Fix $0 < \epsilon < 1$. Since $\epsilon L_{\mathcal{F}}^*$ is a positive invertible element in $B_{L_{\mathcal{F}}}$, we have that $x := \mathbb{P}(\epsilon L_{\mathcal{F}}^*)$ is a positive invertible element in B . From $x \in B_{L_{\mathcal{F}}}$, we have $x \wedge L_\infty = 0$ by Theorem 5.13(ii). Thus from (2) we have

$$\limsup_{n \rightarrow \infty} (|a_n|^{1/n} x) = Lx = L_{\mathcal{F}}x = L_{\mathcal{F}}\mathbb{P}(\epsilon L_{\mathcal{F}}^*) = \epsilon \mathbb{P}(e) \ll e.$$

Hence by [12, Theorem 3.6(i)] we get that $x \in \Omega$, and so $x \in B_\rho$. We thus have that $x = 0$. This is a contradiction since we above deduced that x was a positive invertible element in the nontrivial band B . Hence $B = \{0\}$, and so $B_\rho^d \subseteq B_{L_\infty}$. This proves (ii).

We conclude this proof by verifying (iii). To this end, note that by Lemma 5.15 and the left-order continuity of $\bar{\mathbb{P}}_{L_{\mathcal{F}}}$ we have

$$L_{\mathcal{F}} = \bar{\mathbb{P}}_{L_{\mathcal{F}}}(L) = \limsup_{n \rightarrow \infty} \mathbb{P}_{L_{\mathcal{F}}}(|a_n|)^{1/n}.$$

Setting $b_n := \mathbb{P}_{L_{\mathcal{F}}}(a_n)$ for every $n \in \mathbb{N}$, we have that $(b_n)_{n \geq 1}$ is order bounded and

$$L_{\mathcal{F}} = \limsup_{n \rightarrow \infty} |b_n|^{1/n}.$$

Then by [12, Theorem 3.11(i)], we obtain

$$(L_{\mathcal{F}})^* = \sup_{r \in \Omega} \mathbb{P}_{L_{\mathcal{F}}}(r) = \bar{\mathbb{P}}_{L_{\mathcal{F}}}(\rho).$$

Using Theorem 5.14, we have the following two band decompositions for F

$$F = B_{L_{\mathcal{F}}} \oplus B_{L_{\infty}} \oplus B_L^d = B_{\rho_{\mathcal{F}}} \oplus B_{\rho_{\infty}} \oplus B_{\rho}^d,$$

and by (i) and (ii) we have $B_{L_{\mathcal{F}}} = B_{\rho_{\mathcal{F}}}$. Thus by Lemma 5.15 we obtain

$$(L_{\mathcal{F}})^* = \sup_{r \in \Omega} \mathbb{P}_{L_{\mathcal{F}}}(r) = \bar{\mathbb{P}}_{L_{\mathcal{F}}}(\rho) = \bar{\mathbb{P}}_{\rho_{\mathcal{F}}}(\rho) = \rho_{\mathcal{F}},$$

proving (iii).

The generalized inverse L^{-1} of L in F^s is given by $(L_{\mathcal{F}})^* + \infty_L^d$, where ∞_L^d is the largest element in $(B_L^d)^s$. By part (iii) we have that $(L_{\mathcal{F}})^* = \rho_{\mathcal{F}}$, and by part (i), we have that ∞_L^d is the largest element in $B_{\rho_{\infty}}^s$, which is ρ_{∞} by Lemma 5.16. We can now conclude that $L^{-1} = \rho_{\mathcal{F}} + \rho_{\infty} = \rho$. \square

6. Differentiation of power series

We apply our theory of order differentiable functions to power series in this section.

Notation 6.1. Throughout the rest of the paper, E denotes a universally complete complex vector lattice.

Theorem 6.2. Let $\sum_{n=0}^{\infty} a_n(z - c)^n$ be a power series on E with spectrum of convergence Ω_1 . Then for the power series $\sum_{n=1}^{\infty} n a_n(z - c)^{n-1}$ with spectrum of convergence Ω_2 the corresponding radii of convergence ρ_1 and ρ_2 are equal. In particular, Ω_1 is order bounded in F if and only if Ω_2 is order bounded in F .

Proof. Let $0 < \epsilon < 1$, and let $r \in \Omega_1$. Then the series $\sum_{n=0}^{\infty} a_n(\epsilon r)^n$ converges absolutely in order by [12, Theorem 3.6(i)], since

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n}(\epsilon r) \leq \epsilon e \ll e.$$

Furthermore, we have $(\epsilon^2 r)(\epsilon r)^* \ll e$ and

$$\limsup_{n \rightarrow \infty} n^{1/n}(\epsilon^2 r)(\epsilon r)^* \ll e,$$

so it follows that the series

$$\sum_{n=0}^{\infty} n((\epsilon^2 r)(\epsilon r)^*)^n$$

also converges absolutely in order by [12, Theorem 3.6(i)]. Hence there is an $R \in E_+$ such that

$$n((\epsilon^2 r)(\epsilon r)^*)^n \leq R$$

for all $n \geq 0$. Thus

$$\begin{aligned} n|a_n|(\epsilon^2 r)^{n-1} &= n|a_n|(\epsilon^2 r)^n (\epsilon^2 r)^* = \epsilon^{-1} n|a_n|(\epsilon^2 r)^n (\epsilon r)^* = \epsilon^{-1} n|a_n|(\epsilon^2 r)^n ((\epsilon r)^*)^n (\epsilon r)^{n-1} \\ &= \epsilon^{-1} n|a_n|(\epsilon^2 r)^n ((\epsilon r)^*)^n (\epsilon r)^n (\epsilon r)^* \leq \epsilon^{-2} r^* R|a_n|(\epsilon r)^n \end{aligned}$$

and so $\sum_{n=1}^\infty na_n(\epsilon^2 r)^{n-1}$ converges absolutely in order by comparison to $\sum_{n=0}^\infty a_n(\epsilon r)^n$, which yields $\epsilon^2 r \in \Omega_2$.

On the other hand, if $r \in \Omega_2$, then similarly we find that $\sum_{n=1}^\infty na_n(\epsilon r)^{n-1}$ converges absolutely in order. Since

$$|a_n|(\epsilon r)^n \leq n|a_n|(\epsilon r)^{n-1}(\epsilon r)$$

for all $n \geq 1$, it follows that $\sum_{n=0}^\infty a_n(\epsilon r)^n$ converges absolutely in order, hence $\epsilon r \in \Omega_1$.

Next, for any $r \in \Omega_2$ we have shown that $\epsilon r \leq \rho_1$, and by letting $\epsilon \rightarrow 1$, we conclude that $r \leq \rho_1$. Hence $\rho_2 \leq \rho_1$. Moreover, if $r \in \Omega_1$, we have that $\epsilon^2 r \leq \rho_2$ which yields $r \leq \rho_2$ and hence $\rho_1 \leq \rho_2$. Therefore, $\rho_1 = \rho_2$. \square

The idea in the following lemma is used to naturally extend the definition of $\overset{\circ}{\Delta}(c, r)$ where $r \in F^s$.

Lemma 6.3. *Let G be a uniformly complete Archimedean complex Φ -algebra, $r \in G_+$ be an invertible element, and $c \in G$. Then $\overset{\circ}{\Delta}(c, r)$ is the largest order open set in $\bar{\Delta}(c, r)$.*

Proof. Let $z \in \overset{\circ}{\Delta}(c, r)$. Then $s := \frac{1}{2}(r - |z - c|)$ is a positive invertible element and if $y \in \overset{\circ}{\Delta}(z, s)$, it follows that $r - |y - c| \geq r - |y - z| - |z - c| \geq r - s - |z - c| = \frac{1}{2}(r - |z - c|)$, which is positive and invertible by [7, Theorem 11.1, Theorem 11.4], hence $\overset{\circ}{\Delta}(z, s) \subseteq \overset{\circ}{\Delta}(c, r)$ and $\overset{\circ}{\Delta}(c, r)$ is an order open set. Suppose $z \in \bar{\Delta}(c, r) \setminus \overset{\circ}{\Delta}(c, r)$. Furthermore, suppose that there exist an invertible element $s \in G_+$ such that $\overset{\circ}{\Delta}(z, s) \subseteq \bar{\Delta}(c, r)$. It follows that $z + \frac{1}{2}\lambda s \in \overset{\circ}{\Delta}(z, s)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, so that

$$|z - c + \frac{1}{2}\lambda s| \leq r \tag{3}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. By the Kakutani representation theorem and then complexifying, there is a compact Hausdorff space K such that the ideal generated by r is lattice isomorphic to $C_{\mathbb{C}}(K)$, where r corresponds to the constant function $\mathbf{1}$. If we identify $f, g \in C_{\mathbb{C}}(K)$ with $z - c$ and s , then there is $x \in K$ and $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| = 1$ such that $\lambda_0 f(x) = 1$. This now yields

$$|f + \frac{1}{2}\overline{\lambda_0}g|(x) = |\overline{\lambda_0}(\lambda_0 f + \frac{1}{2}g)|(x) = |\lambda_0 f + \frac{1}{2}g|(x) = \lambda_0 f(x) + \frac{1}{2}g(x) > 1,$$

which contradicts (3). Hence, z is not an order interior point of $\bar{\Delta}(c, r)$ and $\overset{\circ}{\Delta}(c, r)$ is the largest order open set contained in $\bar{\Delta}(c, r)$. \square

Notation 6.4. For $z, w \in F$ we use the notation $z \ll_w w$ when $w - z$ is a positive invertible element in B_w . Furthermore, for $y \in F^s$ and $x \in F$, by $x \ll y$ we mean that both $x \leq y$ and $\mathbb{P}_{y_{\mathcal{F}}}(x) \ll_{y_{\mathcal{F}}} y_{\mathcal{F}}$. In addition, if y dominates a positive invertible element in F , the corresponding order disk $\overset{\circ}{\Delta}(c, y)$ will denote the largest order open set in E that is contained in

$$\bar{\Delta}(c, y) := \{x \in E : |x - c| \leq y\} = \{x \in E : \mathbb{P}_{y_{\mathcal{F}}}(|x - c|) \leq y_{\mathcal{F}}\} + B_{y_\infty}.$$

So, similarly to Lemma 6.3, this set is defined by

$$\overset{\circ}{\Delta}(c, y) := \{x \in E : \mathbb{P}_{y_{\mathcal{F}}}(|x - c|) \ll_{y_{\mathcal{F}}} y_{\mathcal{F}}\} + B_{y_{\infty}},$$

and it is readily verified that this set is order open.

If we want to think of a power series as a function, then we need to specify the domain, which necessarily must consist of all the $z \in E$ for which the power series converges in order.

Proposition 6.5. *Let $\sum_{n=0}^{\infty} a_n(z - c)^n$ be a power series on E . Then the domain of the function $f(z) := \sum_{n=0}^{\infty} a_n(z - c)^n$ satisfies $\overset{\circ}{\Delta}(c, \rho) \subseteq \text{dom}(f) \subseteq \bar{\Delta}(c, \rho)$. Furthermore, if ρ dominates a positive invertible element, then $\overset{\circ}{\Delta}(c, \rho)$ is the largest order open set contained in $\text{dom}(f)$.*

Proof. Let $z \in E$ be so that $|z - c| \ll \rho$. Then $\mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) \ll_{\rho_{\mathcal{F}}} \rho_{\mathcal{F}}$ and we can write $|z - c| = \mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) + \mathbb{P}_{\rho_{\infty}}(|z - c|)$. Let $L := \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$. Note that L is well-defined in F^s . It follows from Lemma 5.17 and [3, Lemma 24(iii)], which states that

$$x(y \vee z) = (xy) \vee (xz) \quad (0 \leq x, y, z \in F^s),$$

that $L|z - c| = \limsup_{n \rightarrow \infty} (|a_n|^{\frac{1}{n}}|z - c|)$. From Theorem 5.20 we obtain

$$\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}|z - c| = L|z - c| = L\mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) = L_{\mathcal{F}}\mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) \ll e,$$

so that $f(z)$ converges absolutely in order by [12, Theorem 3.6(i)]. Hence $z \in \text{dom}(f)$. If $z \in E$ is such that $\sum_{n=0}^{\infty} a_n(z - c)^n$ converges in order, then $\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}|z - c| \leq e$ by [12, Theorem 3.6(ii)]. It follows that $L|z - c| = \limsup_{n \rightarrow \infty} (|a_n|^{\frac{1}{n}}|z - c|) \leq e$.

Since F is Archimedean and $L = L_{\mathcal{F}} + L_{\infty}$, we find that $L_{\infty}|z - c| = 0$ by Lemma 5.12, so that $\mathbb{P}_{L_{\infty}}(|z - c|) = 0$, and $L_{\mathcal{F}}|z - c| \leq e$. By Theorem 5.20 and by Theorem 5.14, it follows that $\mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) \leq \rho_{\mathcal{F}}$, and

$$\begin{aligned} |z - c| &= \mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) + \mathbb{P}_{\rho_{\infty}}(|z - c|) + \mathbb{P}_{\rho}^d(|z - c|) = \mathbb{P}_{\rho_{\mathcal{F}}}(|z - c|) + \mathbb{P}_{\rho_{\infty}}(|z - c|) \\ &\leq \rho_{\mathcal{F}} + \rho_{\infty} = \rho, \end{aligned}$$

as ρ_{∞} is the largest element in $(B_{\rho_{\infty}})^s$ by Lemma 5.16. Hence $z \in \bar{\Delta}(c, \rho)$.

Suppose that ρ dominates a positive invertible element. Since $\overset{\circ}{\Delta}(c, \rho)$ is the largest order open set contained in $\bar{\Delta}(c, \rho)$, it must therefore also be the largest order open set in $\text{dom}(f)$ by an argument analogous to the proof of Lemma 6.3. \square

The remainder of this paper is devoted to proving that analytic functions are also holomorphic in our present setting.

Definition 6.6. Let $f: \text{dom}(f) \rightarrow E$ and let $U \subseteq \text{dom}(f)$ be an order open set. Then f is said to be *analytic* on U if for every $c \in U$ there is a positive invertible element r such that $\bar{\Delta}(c, r) \subseteq U$ and there exists a power series $\sum_{n=0}^{\infty} a_n(z - c)^n$ on E that converges uniformly in order to $f(z)$ on $\bar{\Delta}(c, r)$.

In the next theorem we show when power series on universally complete complex vector lattices are order differentiable and consequently infinitely order differentiable. Part of our proof is an adaptation of a classical argument to the present setting.

Theorem 6.7. Let $f(z) := \sum_{n=0}^{\infty} a_n(z - c)^n$ and $g(z) := \sum_{n=1}^{\infty} na_n(z - c)^{n-1}$ be functions defined by the respective power series with equal radius of convergence ρ by Theorem 6.2. Then f is order differentiable at z_0 if and only if ρ dominates a positive invertible element and $z_0 \in \overset{\circ}{\Delta}(c, \rho)$. In this case we have $f'(z_0) = g(z_0)$.

Proof. Suppose f is order differentiable at z_0 . Then there is a positive invertible element s such that $\overset{\circ}{\Delta}(z_0, s) \subseteq \text{dom}(f)$. But then it follows from

$$\frac{1}{2}s = |\frac{1}{2}s| \leq |z_0 - c + \frac{1}{2}s| + |z_0 - c| \leq 2\rho$$

that ρ dominates a positive invertible element. By Proposition 6.5 we have that $\overset{\circ}{\Delta}(z_0, s) \subseteq \overset{\circ}{\Delta}(c, \rho)$, so that $z_0 \in \overset{\circ}{\Delta}(c, \rho)$.

Suppose that ρ dominates a positive invertible element and $z_0 \in \overset{\circ}{\Delta}(c, \rho)$. Let r be a positive invertible element such that $r \ll \rho$ and $z_0 \in \overset{\circ}{\Delta}(c, r)$. Let $h_\alpha \rightarrow 0$ be such that $z_0 + h_\alpha \in \overset{\circ}{\Delta}(c, r)$, then it follows from [12, Theorem 3.6(i)] and Theorem 5.20 that $f(r + c)$ converges absolutely in order. Next for $n \geq 1$ define

$$f_n(z) := \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z - c)^{n-1-j} (z_0 - c)^j$$

on $\overset{\circ}{\Delta}(c, r)$. Note that $f_n(z)$ is well defined since

$$\sum_{j=0}^{n-1} |z - c|^{n-1-j} |z_0 - c|^j \leq nr^{n-1},$$

and $g(r + c)$ converges absolutely in order since $\rho_g = \rho_f$ by Theorem 6.2. Furthermore, we have that $f_n(z_0) = g(z_0)$ and as

$$\sum_{j=0}^{n-1} (z - c)^{n-j} (z_0 - c)^j - \sum_{j=0}^{n-1} (z - c)^{n-j-1} (z_0 - c)^{j+1} = (z - c)^n - (z_0 - c)^n, \tag{4}$$

we have

$$\begin{aligned} (z - z_0)f_n(z) &= (z - c)f_n(z) - (z_0 - c)f_n(z) \\ &= \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z - c)^{n-j} (z_0 - c)^j - \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z - c)^{n-j-1} (z_0 - c)^{j+1} \\ &= f(z) - f(z_0). \end{aligned}$$

Using the equation in (4) again, we find that

$$\begin{aligned} f_n(z) - g(z_0) &= \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z - c)^{n-1-j} (z_0 - c)^j - \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z_0 - c)^{n-1-j} (z_0 - c)^j \\ &= \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-1} (z_0 - c)^j \left((z - c)^{n-1-j} - (z_0 - c)^{n-1-j} \right) \\ &= \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-2} (z_0 - c)^j \left((z - c)^{n-1-j} - (z_0 - c)^{n-1-j} \right). \end{aligned}$$

Since

$$\left((z - c) - (z_0 - c) \right) \sum_{k=0}^{n-2-j} (z - c)^{n-2-j-k} (z_0 - c)^k = (z - c)^{n-1-j} - (z_0 - c)^{n-1-j},$$

we find that

$$f_n(z) - g(z_0) = (z - z_0) \sum_{m=0}^{\infty} a_m \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} (z - c)^{n-2-j-k} (z_0 - c)^{k+j}$$

and

$$\begin{aligned} \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} |z - c|^{n-2-j-k} |z_0 - c|^{k+j} &\leq \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} r^{n-2} = \sum_{j=0}^{n-2} (n - 1 - j) r^{n-2} \\ &= \frac{1}{2} n(n - 1) r^{n-2}. \end{aligned}$$

Since $h_\alpha \rightarrow 0$, there is a net $q_\beta \downarrow 0$ such that for all β there is an α_0 such that $|h_\alpha| \leq q_\beta$ whenever $\alpha \geq \alpha_0$. Then for β and such an α_0 , it follows by combining all of the above established identities that

$$\begin{aligned} |f(z_0 + h_\alpha) - f(z_0) - h_\alpha g(z_0)| &= |h_\alpha f_n(z_0 + h_\alpha) - h_\alpha g(z_0)| = |h_\alpha| |f_n(z_0 + h_\alpha) - g(z_0)| \\ &\leq \frac{1}{2} |h_\alpha| q_\beta \sum_{k=0}^{\infty} |a_k| k(k - 1) r^{k-2} \end{aligned}$$

whenever $\alpha \geq \alpha_0$. Note that this series converges by applying Theorem 6.2 twice. Thus f is order differentiable at z_0 where the derivative is $g(z_0)$. \square

We conclude this paper with the following result, which is a consequence of Theorem 6.7.

Theorem 6.8. *Let $f: \text{dom}(f) \rightarrow E$, and suppose that $U \subseteq \text{dom}(f)$ is order open. If f is analytic on U , then f is holomorphic on U .*

Proof. Let $c \in U$. Then there is a positive invertible element $r \in E$ and a power series $\sum_{k=1}^{\infty} a_k (z - c)^k$ on E such that $\sum_{k=1}^{\infty} a_k (z - c)^k$ converges uniformly to $f(z)$ on $\bar{\Delta}(c, r) \subseteq U$. It follows that $r \leq \rho$, and by Theorem 6.7, the function f is order differentiable at c . Hence f is holomorphic on U . \square

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