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Control Lyapunov-Barrier Function Based Stochastic Model Predictive Control for COVID-19 Pandemic ⋆

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Abstract: In this paper, a stochastic model predictive control (MPC) is proposed to design a non-pharmacutical policy to control and prevent the COVID-19 pandemic. The system dynamics of COVID-19 is described by a stochastic SEIHR model subject to practical constraints, and the model is proved to be feedback linearizable. A stochastic Control Lyapunov-Barrier Function (CLBF) is constructed for the feedback linearizable system. Constraints on hospitalized individuals are regarded as the unsafe region to construct the corresponding stochastic CLBF. In the proposed stochastic MPC, the stochastic CLBF constraints are applied to improve the overall performance on controlling and preventing the epidemic. Both theoretical proof and simulation results imply that, with the CLBF-based stochastic MPC, the proposed policy is effective in controlling and preventing COVID-19 pandemic.

Keywords: Model predictive control, control Lyapunov-barrier function, stochastic systems, feedback linearizable systems, COVID-19

1. INTRODUCTION

The long lasting COVID-19 pandemic has made adverse impressions and impacts on both individuals and communities worldwide. Recently, due to its prediction and optimization nature, model predictive control (MPC) has been applied in wide range to predict, control and prevent the pandemic. Based on the SIR model, an On-Off social isolation strategy with hard constraints on symptomatic individuals are applied in Morato et al. (2020). In Delavar and Baghbadorani (2022), effects of social distancing, hospitalization, and vaccination rate are considered as three control inputs, and hospitalized and deceased populations are used to estimate other immeasurable states with unscented Kalman filter. In Parino et al. (2021), MPC is applied to devise optimal scheduling of first and second vaccinations. A state observer is designed in Péni and Szederkényi (2021) to estimate other states based on the measured number of hospitalized people, where an output feedback MPC structure is designed in the presence of parameter uncertainties. In Köhler et al. (2021), a robust MPC is developed against model inaccuracies, uncertain state measurements and inexact inputs in COVID-19 model. In She et al. (2022), a learning-based MPC framework is proposed for epidemic mitigation, where the model parameters can be estimated online, and control inputs can be calculated simultaneously. Stochastic settings are adopted in Scarabaggio et al. (2021), such that chance constraints on hospitalized people can be approximated by sample approximation approach.

In the above works on modeling, prevention and control for COVID-19 pandemic, most of them investigate deterministic dynamics with or without hard constraints. Comparatively, researches on stochastic influences on COVID-19 are still relatively rare. For a large scale system, however, stochastic disturbances should be concerned. In stochastic MPC strategies, stochastic Control Lyapunov–Barrier Function (CLBF) can be applied for safety and performance guarantee. When safety is a major design consideration, there are many applications for safety-critical systems, e.g., obstacle avoidance (Romdlony and Jayawardhana, 2016), walking robots (Ames et al., 2019), chemical process control (Wu et al., 2019) and so on. A constructive design of CLBF is presented in Romdlony and Jayawardhana (2016), where Control Lyapunov Function (CLF) and Control Barrier Functions (CBFs) are combined. CLBF can be integrated into MPC framework (Wu et al. (2019)) to improve feasibility and closed-loop performance. The new concept of stochastic CBF is presented in Clark (2021). Application of stochastic CLBF in MPC is proposed in Zheng and Zhu (2022). Some other stochastic MPC applications can be found in Zheng and Zhu (2021).

In this paper, it is supposed that COVID-19 is subject to stochastic disturbances and some practical constraints, and it can be modeled by a stochastic SEIHR (Susceptible-

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Exposed-Infected-Hospitalized-Removed) model. Control and prevention policy is designed within the framework of stochastic MPC, such that both practical constraints and stochastic disturbances can be treated. Our main contributions include that: 1) some practical constraints and stochastic disturbances can be modeled by stochastic CLBF, such that the proposed constrained and stochastic SEIHR model is more realistic, and the proposed policy appears more applicable; 2) the proposed stochastic CLBF can be applied to improve the performance of MPC for the feedback linearizable nonlinear COVID-19 system; and 3) it is proved theoretically and validated numerically that the proposed policy is effective in controlling and preventing COVID-19 pandemic.

The paper is organized as follows. In Section 2, the COVID-19 pendemic is modelled in a stochastic SEIHR system. In Section 3, the stochastic CLBF is proposed, and its construction for feedback linearizable systems is described. In Section 4, simulation examples are provided to demonstrate the effectiveness of the CLBF-based MPC in controlling and preventing COVID-19 pandemic. The paper is concluded in the final section.

2. PROBLEM STATEMENT

Since COVID-19 has a long latent period, the basic SIR model in epidemiology is inadequate to describe its dynamics. In this paper, we adopt SEIHR model (Niu et al. (2021)) to model the COVID-19 dynamics. The applied SEIHR model is given by

$$d\begin{bmatrix} S(t) \\ E(t) \\ I(t) \\ H(t) \\ R(t) \end{bmatrix} = \begin{bmatrix} \omega_R R(t) - [\eta E(t) + \alpha I(t)]\, S(t) u \\ -(\beta + \omega_E)\, E(t) + [\eta E(t) + \alpha I(t)]\, S(t) u \\ \beta E(t) - (\gamma + \omega_I)\, I(t) \\ \gamma I(t) - \omega_H H(t) \\ \omega_E E(t) + \omega_I I(t) + \omega_H H(t) - \omega_R R(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ -\sigma I(t) \\ \sigma I(t) \\ 0 \end{bmatrix} dW \tag{1}$$

where the states $S(t)$, $E(t)$, $I(t)$, $H(t)$ and $R(t)$ denote proportions of susceptible, exposed, infectious, hospitalized, and removed individuals, respectively; the input u includes non-pharmaceutical policies to reduce infection rate, e.g., use of face masks, social distancing, isolation, lockdown, etc (Scarabaggio et al., 2021). W denotes a Brownian motion; η and α are transmission rates of exposed and (symptomatic) infected individuals respectively; β is reciprocal of the mean latent period; γ is rate at which infected individuals are hospitalized; ω_E, ω_I, and ω_H denote recovery rates of non-hospitalized exposed, non-hospitalized infected and hospitalized individuals respectively; σ describes random fluctuations from I to H; and ω_R denotes the re-infection rate (Zhu et al., 2021).

We aim at preventing and controlling the pandemic, reducing hospital occupancy, and keeping social economy in a certain prediction horizon, or mathematically,

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} J(\mathbf{x}, \mathbf{u}),$$
$$\text{s.t. } H(t) \le H_m, \mathbf{u} \in U$$

where

$$J(\mathbf{x}, \mathbf{u}) = \sum_{i=0}^{N-1} \mathbf{x}^T(i|k)\, \mathbf{Q}_0 \mathbf{x}(i|k) + \Delta\mathbf{u}^T(i|k)\, \mathbf{R}_1 \Delta\mathbf{u}(i|k) + [1 - \mathbf{u}(i|k)]^T \mathbf{R}_0 [1 - \mathbf{u}(i|k)]$$

and N denotes the prediction horizon; $\cdot(i|k)$ denotes the prediction of time $i + k$ from time k; H_m denotes the maximum occupancy of hospitals; $\mathbf{Q}_0, \mathbf{R}_0, \mathbf{R}_1$ are weighting matrices of predicted states, inputs and increment of inputs (with $\Delta\mathbf{u}(0|k) = \mathbf{u}(k) - \mathbf{u}(k-1)$) respectively. The term $\Delta\mathbf{u}$ is to keep continuities in policy.

Physically, the mortality rate may largely increase if the hospital occupancy exceeds limits (Köhler et al., 2021). Such constraints can be regarded as a hard constraint and unsafe region. Control Lyapunov-Barrier Function based methods can be applied to decrease infected individuals (both symptomatic and asymptomatic) and prevent system trajectory from entering unsafe region D. Another consideration is that, an excessively strict policy would possibly deteriorate the social economy; consequently, the control input should be penalized in the cost function, and it is subject to hard constraints:

$$0 \le u \le 1,$$

where $u = 1$ implies no prevention or control policies are exerted, and $u = 0$ indicates that policy is fairly strict.

3. CLBF-BASED STOCHASTIC MPC FOR FEEDBACK LINEARIZABLE SYSTEMS

In this section, stochastic CLBF is reviewed, an approach to construct stochastic CLBF for feedback linearizable systems is provided. CLBF constraints are integrated into the MPC framework to enhance the overall performance.

3.1 Stochastic Control Lyapunov-Barrier Function

The COVID-19 model (1) satisfies the following form of stochastic nonlinear systems:

$$d\mathbf{x} = (f(\mathbf{x}) + g(\mathbf{x})\, \mathbf{u})\, dt + \sigma(\mathbf{x})\, dW \tag{2}$$

where $\mathbf{x} \in X \in \mathbb{R}^n$ and $\mathbf{u} \in U \in \mathbb{R}^m$ denote states and constrained control inputs respectively; W is an n-dimensional Brownian motion. The functions f, g and σ are locally Lipschitz with proper dimensions. We first introduce notions of control Lyapunov function and control barrier function for stochastic nonlinear affine systems.

Definition 1. A twice differentiable positive-definite function $V : X \to \mathbb{R}$ is called a stochastic control Lyapunov function of (2) if it satisfies

$$\inf_{\mathbf{u} \in U} \mathcal{L}V(\mathbf{x}) < 0, \quad \forall \mathbf{x} \in X, \mathbf{x} \ne 0 \tag{3}$$

where the infinitesimal generator $\mathcal{L}V(\mathbf{x})$ is given by

$$\mathcal{L}V(\mathbf{x}) = L_f V(\mathbf{x}) + L_g V(\mathbf{x})\, \mathbf{u} + \frac{1}{2}\mathrm{tr}\left(\sigma^T(\mathbf{x}) \frac{\partial^2 V}{\partial \mathbf{x}^2} \sigma(\mathbf{x})\right)$$

A stochastic nonlinear system (2) is stochastically asymptotically stabilizable if and only if there exists a stochastic control Lyapunov function satisfying (3). If control action u satisfies $\mathcal{L}V(\mathbf{x}_t) < 0$ all the time, the zero solution $\mathbf{x}_t \equiv 0$ is asymptotically stable in probability.

Definition 2. A twice differentiable function $B_i : X \to \mathbb{R}$ is called a stochastic (zero) control barrier function, if

$$B_i(\mathbf{x}) > 0, \forall \mathbf{x} \in D_i \quad (4)$$

$$\inf_{\mathbf{u}\in U} \mathcal{L}B_i(\mathbf{x}) \le -B_i(\mathbf{x}), \forall \mathbf{x} \in X \backslash D_i \quad (5)$$

$$\{\mathbf{x} \in X | B_i(\mathbf{x}) \le 0\} \ne \varnothing \quad (6)$$

Similarly, if control $\mathbf{u}$ satisfies $\mathcal{L}B_i(\mathbf{x}_t) \le -B_i(\mathbf{x}_t)$ all the time, then $P\{\mathbf{x} \in X/D_i | \mathbf{x}_0 \in X/D_i\} = 1$. We now give the definition of stochastic control Lyapunov-Barrier function.

Definition 3. Given an unsafe region $D \subseteq X$, if there exists a twice differentiable function $W_c : X \to \mathbb{R}$, which has a minimum at the origin, satisfying

$$W_c(\mathbf{x}) > 0, \forall \mathbf{x} \in D \quad (7)$$

$$\inf_{\mathbf{u}\in U} \mathcal{L}W_c(\mathbf{x}) < 0, \forall \mathbf{x} \in X \backslash (D \cup \{0\}) \quad (8)$$

$$\{\mathbf{x} \in X | W_c(\mathbf{x}) \le 0\} \ne \varnothing \quad (9)$$

then the function $W_c(\mathbf{x})$ is called a stochastic control Lyapunov-Barrier function for (2).

The following theorem shows the main property of stochastic control Lyapunov-Barrier function.

Theorem 4. Given an unsafe region $D \subseteq X$, if there exists a stochastic control Lyapunov-Barrier function $W_c : X \to \mathbb{R}$ for system (2) and if control action $\mathbf{u}_t$ satisfies $\mathcal{L}W_c(\mathbf{x}_t) < 0$ all the time, then $P\{\mathbf{x}_t \notin D\} = 1, \forall \mathbf{x}_0 \in X \backslash D_{relaxed}$ and the zero solution $\mathbf{x}_t \equiv 0$ is asymptotically stable in probability.

3.2 Stochastic CLBF for Feedback Linearizable Systems

In general, CLF and CBF are designed independently and then combined to form a CLBF. Since CLF and CBF have no impact on each other in design procedure, we will first review stochastic CBF and CLBF presented in previous work, and then give a stochastic CLF design method for feedback linearizable systems.

We assume there exists a function $F_{Bi}(\mathbf{x}) \ge 0$ for every unsafe set such that D_i and X_i can be rewritten as level sets of $F_{Bi}(\mathbf{x})$, i.e.,

$$D_i = \{\mathbf{x} | F_{Bi}(\mathbf{x}) - l_{Di} < 0\}, X_i = \{\mathbf{x} | F_{Bi}(\mathbf{x}) - l_{Xi} < 0\}$$

Then (10) is a stochastic CBF if $B_{imin} < 0, B_{imax} > 0, B_{imax} + B_{imin} > 0$.

$$B_i(\mathbf{x}) = \begin{cases} B_{imin} + \dfrac{B_{imax} - B_{imin}}{1 + e^{-\frac{k_{Bi}(l_{Di} - F_{Bi})}{F_{Bi}(l_{Xi} - F_{Bi})}}}, \forall x \in X_i \\ B_{imin}, \forall x \in X \backslash X_i \end{cases} \quad (10)$$

where $k_{Bi}(\mathbf{x}) > 0$ is a parameter that adjusts the shape of CBF. The following proposition shows how to construct a stochastic CLBF if an unconstrained CLF and CBFs exist.

Proposition 5. For a given unsafe region $D = \bigcup_{i=1}^{n_B} D_i \subseteq X$, suppose there exists an unconstrained stochastic control Lyapunov function $V : X \to \mathbb{R}$ and stochastic control barrier functions $B_i : X \to \mathbb{R}, i = 1, 2, \ldots, n_B$ of system (2), such that

$$c_1 \|\mathbf{x}\|^2 \le V(\mathbf{x}) \le c_2 \|\mathbf{x}\|^2, \ \forall \mathbf{x} \in \mathbb{R}^n, c_2 > c_1 > 0 \quad (11)$$

$$\begin{aligned} B_i(\mathbf{x}) &= -\eta_i < 0, \quad \forall \mathbf{x} \in X \backslash X_i \\ B_i(\mathbf{x}) &\ge -\eta_i, \quad \forall \mathbf{x} \in X_i \end{aligned} \quad (12)$$

where X_i are compact and connected sets satisfying $D_i \subseteq X_i$ and $D_{relaxed} \subseteq X_0 = \bigcup_{i=1}^{n_B} X_i \subseteq X$. Then the following function $W_c(\mathbf{x})$ is a stochastic control Lyapunov-barrier function if (8) holds.

$$W_c(\mathbf{x}) = V(\mathbf{x}) + \sum_{i=1}^{n_B} \lambda_i B_i(\mathbf{x}) + \kappa \quad (13)$$

where

$$\lambda_i > \frac{c_2 c_{3i} - c_1 c_{4i}}{\eta_i}$$

$$\sum_{j=1, j\ne i}^{n_B} \lambda_j \eta_j - c_1 c_{4i} < \kappa < \sum_{j=1}^{n_B} \lambda_j \eta_j - c_2 c_{3i}$$

$$c_{3i} = \max_{x \in \partial X_i} \|\mathbf{x}\|^2, c_{4i} = \min_{x \in D_i} \|\mathbf{x}\|^2$$

We now consider a kind of feedback linearizable systems. Inspired by CLF design for deterministic feedback linearizable systems in Example 3.7 (Freeman and Kokotovic (2008)), we give a stochastic version by Proposition 6.

Proposition 6. Suppose there exists a diffeomorphism $\mathbf{z} = \phi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ that transforms the system (2) into

$$d\mathbf{z} = \mathbf{A}\mathbf{z}dt + \mathbf{B}[\mathbf{l_0}(\mathbf{z}) + \mathbf{l_1}(\mathbf{z})\mathbf{u}]dt + \mathbf{C}\mathbf{z}dW \quad (14)$$

where the pair $(\mathbf{A}, \mathbf{B})$ is controllable and the functions $\mathbf{l_0}$ and $\mathbf{l_1}$ are continuous with $\mathbf{l_1}$ nonsingular for all $\mathbf{z}$. Then the function $V(\mathbf{x}) = \phi^T(\mathbf{x})\mathbf{P}\phi(\mathbf{x})$ is an unconstrained stochastic CLF for (2), where $\mathbf{P}$ is the positive definite solution to the stochastic algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R^{-1}}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{P}\mathbf{C} + \mathbf{Q} = 0 \quad (15)$$

Proof. Since ϕ is a diffeomorphism, we have

$$\begin{aligned} \mathcal{L}V(\mathbf{x}) =& \mathcal{L}V(\mathbf{z}) \\ =& L_{\mathbf{Az}+\mathbf{Bl_0(z)}}V(\mathbf{z}) + L_{\mathbf{Bl_1(z)}}V(\mathbf{z})\mathbf{u} \\ &+ \frac{1}{2}\mathrm{tr}\left(\mathbf{z}^T\mathbf{C}^T\frac{\partial^2 V(\mathbf{z})}{\partial \mathbf{z}^2}\mathbf{C}\mathbf{z}\right) \\ =& 2\mathbf{z}^T\mathbf{P}\{\mathbf{Az} + \mathbf{B}[\mathbf{l_0}(\mathbf{z}) + \mathbf{l_1}(\mathbf{z})\mathbf{u}]\} \\ &+ \mathrm{tr}\left(\mathbf{z}^T\mathbf{C}^T\mathbf{P}\mathbf{C}\mathbf{z}\right) \\ =& \phi^T(\mathbf{x})\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}^T\mathbf{P}\mathbf{C}\right)\phi(\mathbf{x}) \\ &+ 2\phi^T(\mathbf{x})\mathbf{P}\mathbf{B}[\mathbf{l_0}(\mathbf{z}) + \mathbf{l_1}(\mathbf{z})\mathbf{u}] \\ =& -\phi^T(\mathbf{x})\mathbf{Q}\phi(\mathbf{x}) + \phi^T(\mathbf{x})\mathbf{P}\mathbf{B}\mathbf{R^{-1}}\mathbf{B}^T\mathbf{P}\phi(\mathbf{x}) \\ &+ 2\phi^T(\mathbf{x})\mathbf{P}\mathbf{B}[\mathbf{l_0}(\mathbf{z}) + \mathbf{l_1}(\mathbf{z})\mathbf{u}] \end{aligned}$$

If $L_gV(\mathbf{x}) = 2\phi^T\mathbf{P}\frac{\partial\phi(\mathbf{x})}{\partial\mathbf{x}}\mathbf{g}(\mathbf{x}) = 2\phi^T\mathbf{P}\mathbf{B}\mathbf{l_1}(\phi(\mathbf{x})) = 0$, it follows that $\phi^T(\mathbf{x})\mathbf{P}\mathbf{B} = 0$, therefore

$$\mathcal{L}V(\mathbf{x}) = -\phi^T(\mathbf{x})\mathbf{Q}\phi(\mathbf{x}) < 0$$

and $V(\mathbf{x})$ is an unconstrained stochastic CLF for system (2), which completes the proof.

To solve stochastic algebraic Riccati equation, please refer to a semidefinite programming associated with LMI (Rami and Zhou (2000)) or Newton's method (Wang (2009)).

3.3 CLBF-based Stochastic MPC

The proposed CLBF-based Stochastic MPC is designed by

$$(\mathbf{u}^*, \delta^*) = \operatorname*{arg\,min}_{(\mathbf{u}, \delta) \in U \times \mathbb{R}_+^N} J(\mathbf{x}, \mathbf{u}) + \delta^T\mathbf{R}_2\delta \quad (16)$$

subject to

$$\mathbf{x}\left(i+1|k\right)=\left[\mathbf{f}\left(\mathbf{x}\left(i|k\right)\right)+\mathbf{g}\left(\mathbf{x}\left(i|k\right)\right)\mathbf{u}\left(i|k\right)\right]T+\mathbf{x}\left(i|k\right) \tag{17}$$

$$\mathbf{x}\left(0|k\right)=\mathbf{x}(k) \tag{18}$$

$$\mathbf{u}\left(i|k\right)\in U \tag{19}$$

$$\mathcal{L}W_c\left(\mathbf{x}\left(i|k\right),\mathbf{u}\left(i|k\right)\right)<\mathcal{L}W_c\left(\mathbf{x}\left(i|k\right),\Phi_L\left(i|k\right)\right)+\delta\left(i|k\right), \tag{20}$$

for all $\mathbf{x} \in X\setminus(D\cup\{0\})$, where N is the predictive horizon; $\mathbf{x}\left(i|k\right)$ denotes the predicted state of nominal system at $t=t_k+iT$; $\mathbf{u}\left(i|k\right)$ denotes predicted input during $t\in[t_k+iT, t_k+(i+1)T)$. The control input is $\mathbf{u}(t)=\mathbf{u}\left(0|k\right), t\in[t_k, t_k+T)$. Φ_L is the auxiliary Lyapunov controller (Zheng and Zhu, 2022). Slack variable δ is added in (16) to recover feasibility if no feasible solution exists for the original optimization problem (i.e. replace δ with 0).

The infinitesimal generator $\mathcal{L}W_c\left(\mathbf{x}\right)$ can be calculated by

$$\mathcal{L}W_c\left(\mathbf{x}\right)=L_fW_c\left(\mathbf{x}\right)+L_gW_c\left(\mathbf{x}\right)\mathbf{u}+\frac{1}{2}\mathrm{tr}\left(\sigma^T\left(\mathbf{x}\right)\frac{\partial^2W_c}{\partial\mathbf{x}^2}\sigma\left(\mathbf{x}\right)\right).$$

Since $\mathcal{L}W_c\left(\mathbf{x}\right)$ is linear function of $\mathbf{u}$, the feasibility region of original optimization problem can be approximated by

$$X_L=\left\{\mathbf{x}|\,L_fW_c+\sum_{i=1}^{n}L_GW_cu^i+\frac{1}{2}\mathrm{tr}\left(\sigma^T\frac{\partial^2W_c}{\partial\mathbf{x}^2}\sigma\right)<0\right\}$$

where, for $i=1,2,\cdots,m$,

$$L_GW_c\left(\mathbf{x}\right)u^i=\begin{cases}L_gW_c\left(\mathbf{x}\right)u_{max,i}, & \text{if } L_gW_c\left(\mathbf{x}\right)\le 0,\\ L_gW_c\left(\mathbf{x}\right)u_{min,i}, & \text{if } L_gW_c\left(\mathbf{x}\right)>0,\end{cases}$$

and

$$U=\left\{\mathbf{u}|\begin{bmatrix}u_{min1}\\ \vdots\\ u_{minm}\end{bmatrix}\le\mathbf{u}\le\begin{bmatrix}u_{max1}\\ \vdots\\ u_{maxm}\end{bmatrix}\right\}.$$

X_L only combines constraints $\mathbf{u}\left(i|k\right)\in U$ and $\mathcal{L}W_c<0$ at first step in predictive horizon and therefore is an external approximation of feasibility region. However, if weighting matrix $\mathbf{R}_2$ is large enough, solution of (16) is equivalent to original optimization as long as feasible solution exists and optimization (16) is always feasible for $\mathbf{x}\in X_L$.

Proposition 7. If $\mathbf{x}\in X_L$, then the optimization (16) subject to (17)–(20) is feasible, and for a proper sampling time, the overall closed-loop system is ultimately bounded in the mean square.

Proof. Please see Zheng and Zhu (2022) for details.

4. SIMULATION

In (1), the states satisfy $S(t)+E(t)+I(t)+H(t)+R(t)=1$ and the COVID-19 system can be rewritten by

$$d\mathbf{x}=\begin{bmatrix}-\omega_Rx_1+\omega_Rx_2+\omega_Rx_3+\omega_Rx_4\\ -\left(\beta+\omega_E\right)x_2\\ \beta x_2-\left(\gamma+\omega_I\right)x_3\\ \gamma x_3-\omega_Hx_4\end{bmatrix}dt+\begin{bmatrix}\left(\eta x_2+\alpha x_3\right)\left(1-x_1\right)\\ \left(\eta x_2+\alpha x_3\right)\left(1-x_1\right)\\ 0\\ 0\end{bmatrix}udt+\begin{bmatrix}0\\ 0\\ -\sigma x_3\\ \sigma x_3\end{bmatrix}dW, \tag{21}$$

Table 1. Newton's method to solve stochastic algebraic Riccati equation

Algorithm 1
1. Select $\mathbf{P}_0, \varepsilon_0, \varepsilon_1$ such that $\mathbf{A}_1$ is stable
2. Let $\mathbf{L}_1=-\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_0, \mathbf{A}_1=\mathbf{A}+\mathbf{B}\mathbf{L}_1$
3. Solve the mixed-type Lyapunov equations $\mathbf{A}_1^{\mathbf{T}}\mathbf{X}+\mathbf{X}\mathbf{A}_1+\mathbf{C}^{\mathbf{T}}\mathbf{X}\mathbf{C}=-\mathbf{Q}-\mathbf{L}_1^{\mathbf{T}}\mathbf{R}\mathbf{L}_1$
3.1. Let $\alpha_1=\frac{2\theta_1}{\left(2\lVert A_1\rVert_2+\lVert C\rVert_2^2\right)}, \theta_1\ge 1$
3.2. Compute $\mathbf{r}_1=-\mathbf{Q}-\mathbf{L}_1^{\mathbf{T}}\mathbf{L}_1-\mathbf{A}_1^{\mathbf{T}}\mathbf{P}_0-\mathbf{P}_0\mathbf{A}_1-\mathbf{C}^{\mathbf{T}}\mathbf{P}_0\mathbf{C}$
3.3. If $\lVert\mathbf{r}_1\rVert_F<\varepsilon_1$, turn to 4; else, $\mathbf{P}_0=\mathbf{P}_0+\alpha_1\left(\mathbf{r}_1\mathbf{A}_1^{\mathbf{T}}+\mathbf{A}_1\mathbf{r}_1+\mathbf{C}^{\mathbf{T}}\mathbf{r}_1\mathbf{C}\right)$, turn to 3.2
4. Compute $\mathbf{r}_0=\mathbf{A}^{\mathbf{T}}\mathbf{P}_0+\mathbf{P}_0\mathbf{A}-\mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_0+\mathbf{C}^{\mathbf{T}}\mathbf{P}_0\mathbf{C}+\mathbf{Q}$
5. If $\lVert\mathbf{r}_0\rVert_F<\varepsilon_0$, $\mathbf{P}=\mathbf{P}_0$, end; else, turn to 2

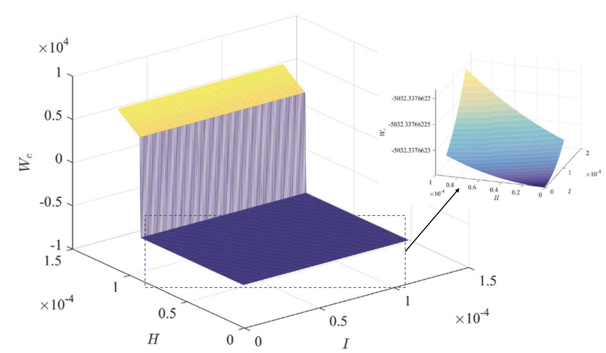

Fig. 1. $W_c(x)$ in the proposed stochastic CLBF with $S=1$, and $E=0$.

where $\mathbf{x}=[x_1\ x_2\ x_3\ x_4]^T=[1-S(t)\ E(t)\ I(t)\ H(t)]^T$. The model (21) is feedback linearizable, and a diffeomorphism $\mathbf{z}=\phi\left(\mathbf{x}\right)$ exists, such that the dynamics of z is linear. For the SEIHR model, all states are physically nonnegative. Therefore l_1 becomes zero only when $z_2=z_3=0$. This means the pandemic ends and can be excluded. The matrices in (15) are given by

$$\mathbf{A}=\begin{bmatrix}-\omega_R & \omega_R & \omega_R & \omega_R\\ 0 & -\left(\beta+\omega_E\right) & 0 & 0\\ 0 & \beta & -\left(\gamma+\omega_I\right) & 0\\ 0 & 0 & \gamma & -\omega_H\end{bmatrix},\mathbf{B}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-\sigma&0\\0&0&\sigma&0\end{bmatrix},\ l_0=0,\ l_1=\left(\eta z_2+\alpha z_3\right)\left(1-z_1\right)$$

and $V\left(\mathbf{x}\right)=\phi^T\left(\mathbf{x}\right)\mathbf{P}\phi\left(\mathbf{x}\right)=\mathbf{x}^T\mathbf{P}\mathbf{x}$ is an unconstrained CLF if $\mathbf{P}$ is the symmetric positive definite solution to (15). We use numerical method in Wang (2009) to solve (15). The algorithm is summarized in Table 1.

Let $\alpha=0.46$, $\beta=0.14$, $\gamma=0.7$, $\eta=0.338$, $\omega_E=\omega_I=\omega_H=0.1$, $\sigma=0.3$, $\omega_R=0.005$, $\mathbf{Q}=\mathbf{I_4}$, $\mathbf{R}=100$, $\mathbf{P_0}=diag\left(3,1,1,1\right)$, $\varepsilon_0=\varepsilon_1=10^{-12}$, the numerical solution of stochastic algebraic Riccati equation is calculated by

$$\mathbf{P}=\begin{bmatrix}10.3037 & -0.8329 & -0.4510 & -0.4480\\ -0.8329 & 2.8648 & 1.5168 & 1.4886\\ -0.4510 & 1.5168 & 4.0251 & 3.8120\\ -0.4480 & 1.4886 & 3.8120 & 4.9235\end{bmatrix}$$

We can construct the stochastic CLBF in Section 3 by

Table 2. Parameters in stochastic CLBF

CBF	$B_{min} = -10$	$B_{max} = 15$
	$k_{Bi} = a_i cos\left(k_i \frac{F_{Bi}}{l_{Xi}}\right) + \frac{1}{2}a_i + b_i$	$F_{Bi}(\mathbf{x}) = \frac{k_F}{H}$
	$a_i = 10^4$	$b_i = 0.1$
	$k_i = \frac{2}{3}\pi$	$k_F = 1$
	$l_{Di} = 10^4$	$l_{Xi} = 1.1 \times 10^4$
CLBF	$c_1 = \lambda_{min}(\mathbf{P}) = 0.6233$	
	$c_2 = \lambda_{max}(\mathbf{P}) = 10.7792$	
	$c_{3i} = \max_{\mathbf{x} \in \partial X_i} \|\mathbf{x}\|^2 = 3 + \frac{k_F^2}{l_{Xi}^2}$	
	$c_{4i} = \min_{\mathbf{x} \in D_i} \|\mathbf{x}\|^2 = \frac{k_F^2}{l_{Di}^2}$	
	$\eta_i = -B_{min\_i}$	$K_{\lambda i} = 1000$
	$\lambda_i = \frac{c_2 c_{3i} - c_1 c_{4i}}{\eta_i} + K_{\lambda i}$	
	$\kappa = \frac{1}{2}\left[\max_i\left(\sum_{j=1, j\neq i}^{n_B} \lambda_j \eta_j - c_1 c_{4i}\right) + \sum_{i=1}^{n_B} \lambda_i \eta_i\right]$	
	$-\frac{1}{2}\left[c_2 \max_i (c_{3i})\right]$	

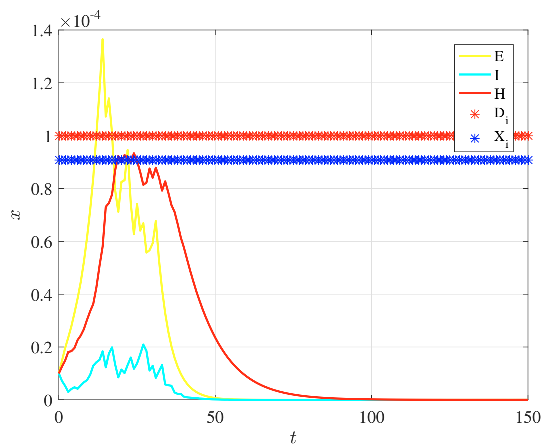

Fig. 2. Trajectories of $E(t)$, $I(t)$, and $H(t)$. Unsafe regions are calculated by $D_i = \{\mathbf{x} | F_{Bi}(\mathbf{x}) - l_{Di} < 0\}$ and correspondingly $X_i = \{\mathbf{x} | F_{Bi}(\mathbf{x}) - l_{Xi} < 0\}$.

$$W_c(\mathbf{x}) = V(\mathbf{x}) + \sum_{i=1}^{n_B} \lambda_i B_i(\mathbf{x}) + \kappa = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

$$+ \sum_{i=1}^{n_B} \lambda_i \left[d_{Xi} \frac{B_{imax} - B_{imin}}{1 + e^{-\frac{k_{Bi}(l_{Di} - F_{Bi})}{F_{Bi}(l_{Xi} - F_{Bi})}}} + B_{imin} \right] + \kappa$$

where

$$d_{Xi} = \begin{cases} 1, \forall x \in X_i \\ 0, \forall x \in X \backslash X_i \end{cases}$$

Parameters of CLBF and their values are listed in Table 2, and $W_c(x)$ is illustrated by Fig. 1. In the auxiliary Lyapunov controller Φ_L, we set $\rho = 0.1$.

Simulations in this section are based on CasADi in Andersson et al. (2019). The continuous-time system dynamics (1) is discretized via Euler–Maruyama method with $T_0 = 0.05$, and the nominal system to calculate MPC is discretized via Euler discretization with $T = 1$. The MPC is processed with predictive horizon $N = 20$, sampling interval $T = 1$ and weighing matrices $\mathbf{Q_0} = \mathbf{0}$, $\mathbf{R_0} = \mathbf{0.1}$, $\mathbf{R_1} = \mathbf{0}$, $\mathbf{R_2} = 10^{14} \times diag(10, 1, 1, \ldots, 1)$. The unit of time is one day. The initial condition is supposed to be

$$\mathbf{X}_0 = \left(6 \times 10^{-4}, 10^{-5}, 10^{-5}, 10^{-5}, 5.7 \times 10^{-4}\right),$$

and simulation results are shown in Figs. 2 and 3.

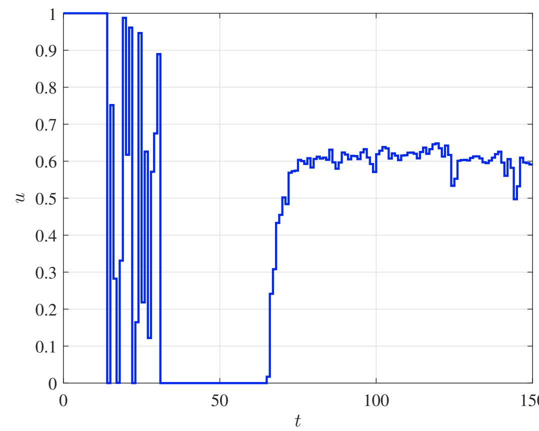

Fig. 3. Control inputs: control actions start from $t = 13$.

As can be seen from Figs. 2 and 3, as long as CLBF constraints are not violated (except no feasible solution exists and constraints should be relaxed), interventions are unnecessary, and control input remains $u = 1$. It can be seen that control actions are capable of keeping the pandemic to a low level. The proportion of hospitalized are always maintained under 10^{-4}. After $t = 13$, due to the sharp increase of system states, control actions are then exerted. At this stage, CLBF constraints are activated, and the proportion of hospitalized are constrained within the given boundaries. After $t = 31$, the control action u is calculated to be 0, implying an entire lockdown, such that system states are forced to decrease. Finally, after $t = 64$, a reopening restores, and the control action u keeps approximately 0.6 to satisfy CLBF constraints.

Suppose that control actions may be implemented with possible delays. A more realistic situation is to keep control actions constant over N_ω time steps, i.e.

$$\mathbf{u}(N_\omega l | k) = \cdots = \mathbf{u}(N_\omega l + N_\omega - 1 | k)$$

for all $l = 0, \cdots, \frac{N}{N_\omega} - 1$. Moreover, we set $\mathbf{Q_0} = 10^6 \times \mathbf{I_5}$, $\mathbf{R_0} = \mathbf{0.1}$, $\mathbf{R_1} = \mathbf{0.05}$, $\mathbf{R_2} = 10^{20} \times diag(10, 1, 1, \ldots, 1)$ to avoid excessive variations in control actions. The simulation is operated 20 times under the same settings, and the results are displayed in Fig. 4. It can be seen that half of the pandemic can be prevented by CLF constraints at start, while others can be restrained by CBF constraints. Note that several sample trajectories may enter the unsafe region, and then leave instantly. The reason is that, with slack variables, unboundedness of disturbances and the sample-and-hold implementation of MPC may violate some of the constraints with a low probability, especially when additional hard input constraints are included. It is noted that $\mathcal{L}B_i(\mathbf{x})$ is a function of I and H, and $S = 1$ can be assigned, such that the feasible region can be displayed in Fig. 5. It is suggested from Fig. 5 that, the feasibility region X_L increases with H. Overall, it can be claimed that the proposed CLBF-based MPC is effective in designing the prevention and control policy for COVID-19 pandemic.

5. CONCLUSION

A CLBF-based stochastic MPC is proposed to design the policy to prevent and control COVID-19 pandemic. The objective is to reduce the pandemic into small scale in

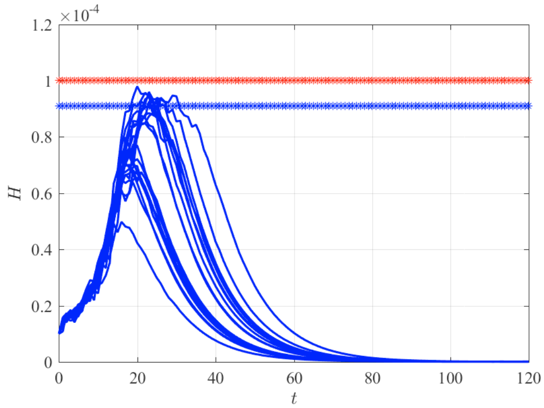

Fig. 4. 20 simulations with same conditions. The red and blue star lines denote boundaries of D_i and X_i, respectively. The initial condition is $\mathbf{X}_0 = (10^{-4}, 10^{-5}, 10^{-5}, 10^{-5}, 7 \times 10^{-5})$. Control actions are exerted since $t = 10$ with $N_\omega = 4$ and $N = 20$.

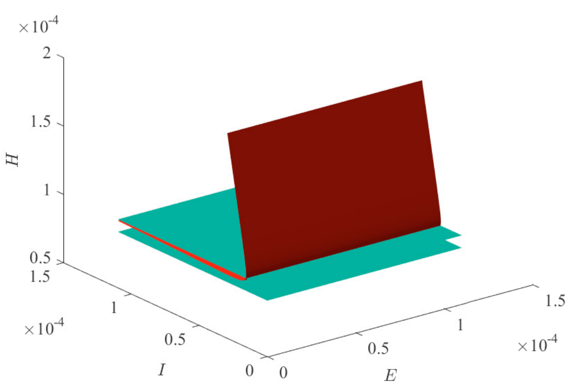

Fig. 5. The green surfaces are boundaries of D_i and X_i. The red surface is the boundary of X_L. The region on the right and below the red surface is X_L.

case of stochastic disturbances and practical constraints. The Stochastic CLBF is designed to model constraints with stochastic disturbances, and it is applied in MPC to enhance the overall performance. It is proved that, the CLBF-based stochastic MPC is feasible, and states of the closed-loop system can be reduced into small scale. Simulations illustrate that the proposed policy is effective in preventing and controlling COVID-19 pandemic.

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