Integration of Multi-State Systems in a series of EPQ models for deteriorating products

by

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Abstract

This thesis contains a collection of problems dealing with the modelling and optimisation of multi-state production systems, hence addressing challenges within the broader category of flexible manufacturing. These systems are often subjected to random degradations, failures, age of machines, human errors, power supply disruptions, or changes in demand.

In literature, many inventory models have been developed under the assumption that the lifetime of systems is infinite, meaning the performance of a system or equipment remains unchanged and is fully usable for satisfying future demand. Some other models have extended this assumption by considering the functioning of systems (or equipment) under binary modelling conditions in which two states are considered: operational state and failure state. However, a growing body of literature is beginning to take into consideration the numerous scenarios that may occur during the lifetime of an equipment. These situations contribute to the multiplicity of the possible states of systems. Such systems are called multi-state systems (MSS). MSS are generally subject to several failure modes, in particular degradation and age of the systems, with various effects on their performance. The operational characteristics of MSS allow them to continue to function; however, they have a reduced level of performance, demonstrating the adaptability and scalability of the equipment. In the literature, techniques to increase the performance of binary systems are often based on strategies including redundancy or preventive maintenance. In the case of multi-state systems (MSS), continuity of service is ensured by reconfiguration.

The objective of this research is to develop models for managing inventory models for deteriorating items in a multi-state manufacturing environment. In many research based on the binary modelling conditions, ensuring the continuity of the production is an important issue. These models assume complete shutdowns of production systems upon failure of manufacturing resources, which can be extremely costly and lead to substantial manufacturing losses. By addressing these limitations that are present in many of the current literature, the models proposed in this thesis are more practical and thus beneficial for operations management practitioners when making decisions involving multi-state systems in manufacturing processes. For such systems, the breakdown or failure of any component only minimally or at least partially disrupts their performance. In this way, the system can continue to provide service with an acceptable level of degradation. The contribution of this thesis is the development of three mathematical models to optimise a series of Economic Production Quantity (EPQ) systems for deteriorating products.

The first model deals with A lot-sizing model for a deteriorating product with shifting production rates, freshness-, price-, and stock-dependent demand with price discounting. The system consists of one machine producing a single type of product. When the component of the machine breaks down, the system is minimally or at least partially disrupted. Thus, it may continue to operate at a rate lower than the initial rate until a specific inventory level is reached. Initially, demand is influenced by its selling price and the level of stock displayed. As freshness declines, demand then depends on the product's freshness condition. As production continues, there is also a shift in production rate over time. To account for declining freshness affecting consumer interest and purchasing behaviour, discounts are applied after a certain period. The optimisation problem was solved using numerical methods and supported by sensitivity analysis to demonstrate its practical implications. However, at this stage, the model does not explore how raw material with imperfect quality could impact this system.

The second scenario presents a two-echelon supply chain inventory model for perishable products, incorporating a shifting production rate, stock-dependent demand rate, and imperfect quality raw material. This novel model extends the classic EPQ as well as the first novel developed in this thesis to account for the use of raw materials with imperfect quality in the production process. Two scenarios are formulated within this framework: one involves selling imperfect raw materials at a discounted price after a screening period, while the other entails keeping imperfect items in stock until they are returned to the supplier at the end of an inventory cycle. Both scenarios consider product deterioration as well as shifts in production rate. Numerical solutions were derived for these scenarios. The findings indicate that maximising profit may involve selling the proportion of imperfect raw material rather than retaining it until a new lot arrives from the supplier. This approach is particularly crucial in manufacturing systems where imperfect products appear in both the raw materials and finished goods. The results were validated through a sensitivity analysis.

The third model expands previous novels by considering the scenario of a production system that continually declines, leading to an increasing rate of defects over time. It takes into consideration various elements including deterioration of finished products, stock levels, product quality, and the influence of corporate social responsibility (CSR). CSR plays a critical role in enhancing the reputation of the company, building customer loyalty, and increasing sales by demonstrating a commitment to ethical practices and societal well-being. The objective of the model presented in this scenario is to identify the optimal inventory level and cycle time that minimise the total cost per cycle. To illustrate the effectiveness of this model, numerical examples are provided along with sensitivity analysis.

The findings show that the profit generated can increase by as much as 14% if manufacturers integrate a setup cost policy and selling price decisions. Extending product shelf life by 60% can increase the net profit by as much as 7%. In another model involving a two-echelon supply chain system, the profit can be increased by as much as 360% and

386%, respectively, if the selling price and the demand enhancement parameter for inventory level increase by 20%. Furthermore, the unit selling price can decrease the total cost by as much as 34%. Operations managers can use all these mechanisms to increase profits in their production systems. Under reasonable conditions, other industrial fields like automotive, mineral processing plants, assembly lines, as well as the production of mechanical components, may also also benefit from the results obtained.

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List of Abbreviations and Acronyms

A	Demand parameter
$AM - GM$	Arithmetic Mean-Geometric Mean
b	Elasticity parameter of the unit selling price
B	Aggregation parameter for some known variables
CSR	Corporate Social Responsibility
C_d	Deterioration cost per unit item
C_{d_p}	Disposal cost per unit item
C_p	Penalty cost per unit lost sale
C_r	Unit cost of raw material
DEL	Dynamic Economic Lot
DC	Disposal Cost
DCF	discounted cash flow
$D(s_p, I(t)), D(I)$	Demand for the product
$d(s_p)$	Price component of the demand for the product
$d_{1,2}$	Proportion of defective units produced
E	Aggregation parameter for some known variables
EOQ	Economic Order Quantity
$ETC(t)$	Expected total cost per unit time
EPQ	Economic Production Quantity
$FIFO$	First-In-First-Out
F_r	raw material ordering cost
G	Fixed setup cost
$G_{1,2}$	Set-up cost associated with stage i
h_p	Inventory carrying cost per item produced per time
h_r	Inventory carrying cost per unit of raw material per time
H_M	Hessian Matrix
HJB	Hamilton-Jacobi-Bellman
$I(t)$	Instantaneous inventory level
ICC	Inventory Carrying Cost
k_1	Initial production rate at the start of the cycle
k_2	Production rate following the shift respectively
$Li \in 1, 2, 3, 4, 5$	Constant of integration
$LIFO$	Last-In-Last-Out
LP	Lost production cost
LSC	Lost Sale Cost
MC	Maintenance cost
MSS	Multi-State-Systems
n	Shelf-life of the product
OFC	Online fulfilment centre
OM	Operations Management

OR	Operations Research
O_rC	Ordering cost
SEA	Seeker evolutionary algorithm
p_{c_1}	Unit production cost at the start of production
p_{c_2}	Unit production cost after the machine's production rate has been scaled down
p_l	Lost production cost per end product.
p_r	Purchase cost of raw material per unit
p_c	Purchase price per unit product
p_{c_1}	Unit production cost at the start
PT	Preservation technology
s_p	Market selling price of the product
s_d	Discounted unit selling price of imperfect finished products
s	Social donation amount per sale
s_r	Discounted unit selling price of imperfect raw material
p_{c_2}	Unit production cost after the machine's production rate has been scaled down
QD_p	Quantity of deteriorated products
Q	Production batch size
Q_s	Quantity of good products sold at a normal price
Q^*	Optimal batch size
q	Proportion of raw materials that is of imperfect quality
q	The inventory remaining at the end of the cycle
T	Cycle time
TC	Average total cost per cycle
TDC	Total deterioration cost
$TDPC$	Total disposal cost
THC	Total holding cost
TR	Average revenue per cycle
TP	Average profit per cycle
TPC	Total production cost
T_I	Length of time during an inventory cycle for which there is stock-out
$t_{\mathrm{i}\in\{1,2,3,4\}}$	Time duration of each phase of the cycle
t_s	Screening period
t_p	Production period
TSC	Total set-up cost
TC	Total cost
USD	United States Dollar
VAT	Value-Added Tax
VMI	Vendor-Managed Inventory
WIP	Work-In-Process
x	Screening rate for imperfect raw material
y	Raw material order size per cycle
ZAR	South African Rand
α	Discount percentage offered on selling price
β	Freshness parameter
$\beta(\tau^*)$	Fraction of demand backordered during a stockout
β_0	Initial fraction of demand backordered during a stockout,
β_M	Maximum fraction of demand backordered over a stock-out interval

δ	Aggregation parameter for some known variables
λ	Rate at which the proportion of defective items increases over time
η_m	Maximum proportion of defective products that can be produced
γ	Demand enhancement parameter for inventory level
λ	Rate at which the proportion of defective items increases over time
μ	Non-negative constant
ω	Rate at which the production rate declines
ϕ	Per unit cost of running the machine independent of the production rate, including labor and energy costs
ψ	Aggregation parameter for some known variables
$\rho_{1,2}$	Aggregation parameters for some known variables
σ	Parameter that reflects the impact of defects on customer demand
τ	Aggregation parameter for some known variables
τ_M^*	Maximum customer's waiting time
τ^*	Time remaining until the next replenishment order is received
$\theta(t)$	Deterioration rate per unit per time
ζ	Increase in unit machining cost due to an increase in the production rate
$\varphi(t)$	Freshness function
ξ, υ	Aggregation parameters for some known variables
Π, Ω	Weibull parameters

Chapter 1

Introduction

1.1 Context

In today's fast-paced and volatile business environment, organisations face numerous challenges when it comes to effectively managing their production and inventory. Managing inventory systems is crucial for any organisation, especially if the organisation deals with perishable products and experiences variable demand. Multi-state production systems, which involve several stages in the production process, add complexity to the inventory management problem. In these systems, the production process is divided into several stages, each with its own production rate and capacity.

The production process is the backbone of any manufacturing industry. It is the process of converting raw materials into finished products. However, the assumptions of constant, smooth production may not always hold. Recently, the manufacturing domain has undergone a major transformation due to the introduction of key enabling technologies required for Industry 4.0 (Sima et al.,2020). To meet clients' customised requirements and enable global manufacturers' personalised production, there is a need to transform current production and process capabilities. This is because recent demands such as shorter product life cycles, higher production rates, increased job complexity, higher product quality, and cost-effectiveness are crucial factors for any manufacturing industry (Lenz et al., 2020). In addition to meeting these requirements, there is a need to improve the system's capabilities and keep it under control from degradation, breakdowns, and external forces.

One of the factors that can cause the production rate in a typical system to be scaled down is the age of the machinery. As machines age, they may become less efficient, and the cost of maintenance and repairs may increase. This can lead to longer production times, increased downtime, a decrease in quality of products and higher costs associated with producing each unit. Another factor that can cause production to be scaled is the machines' usability. Machines that are difficult to operate or maintain may result in longer production times, higher downtime, and increased costs. Changes in market demand are another factor. Changes in market demand or the introduction of new technologies may make existing products less competitive, resulting in declining production rates. Breakdowns of machines can also cause production to decrease. Process degradation is an inherent characteristic that cannot be avoided during manufacturing processes. The degradation rate is a crucial factor in the lifespan of manufacturing systems since the impact of degradation on different types of manufacturing systems can be observed through the severity of their failures (Samala et al., 2021). A degraded machine can affect the quality of manufactured parts (Rivera et al., 2018).

One approach to mitigate the random occurrence of failures is to integrate processes that are highly flexible, configurable, and accessible (Ashraf and Hasan, 2018). Several reliability studies employ binary configuration to model most systems, restricting them to only two states: nominal and complete failure. The performance of such systems is considered to be at the same level during their useful life (operation). Although such modelling has countless practical applications, it is considered insufficient to describe different situations that occur in the life of systems and that can have an impact on their performance (Aven, 1993; Wu and Chan, 2003). In practice, systems and their components can exhibit varying levels of performance. These performance levels can typically be influenced by controlled actions such as fault tolerance, architecture duplication, and safe operation monitoring design to achieve the desired performance. These actions aim to ensure that the system or its component continues to function, even in the presence of faults. This may, however, result in a decrease in the system's performance, but not necessarily its complete failure, making it a reconfigurable system (Soro, 2011). A system that can operate at different levels of performance is considered a multi-state system (MSS) (Levitin and Lisnianski, 2001; Tshinangi et al., 2022).

To thrive and remain competitive, organisations often create strategies that find the right balance between optimising inventory levels, managing production and optimising cost or profit. Traditionally, organisations have employed static, rule-based approaches that unfortunately fail to account for the inherent variability in today's business landscape. Inventory management and production/capacity planning cannot be overstated. Organisations that fail to adapt and develop effective strategies to address challenges risk significant financial losses, loss of customer satisfaction, and diminished market competitiveness. By optimising inventory and production capacity, organisations can minimise costs, improve customer service levels, enhance agility and responsiveness, mitigate risks associated with disruptions, and position themselves for scalable growth. Accordingly, this thesis aims to develop models for managing the production of deteriorating items in multi-state systems.

1.2 Relevance

1.2.1 The importance of inventory control

Inventory control is critical to supply chain management, particularly for companies that produce or distribute deteriorating and perishable products. These products, such as food, pharmaceuticals, and certain chemicals, have a limited shelf life and can quickly become unfit for use or consumption. One of the main challenges of inventory control for deteriorating and perishable products is the need to balance supply and demand in real time. Unlike non-perishable products, the demand for these products can be highly variable and unpredictable, making it challenging to maintain optimal inventory levels. Excess inventory can lead to waste and increased costs, while insufficient inventory can result in stockouts, lost sales, and damage to the company's reputation. To address these challenges, companies must implement inventory control strategies tailored to the unique characteristics of deteriorating and perishable products. Effective inventory control for deteriorating and perishable products also requires a strong focus on supply chain management and logistics. This includes careful monitoring of production and storage conditions, such as temperature, humidity, and light, to ensure that products are maintained at optimal conditions throughout the supply chain. Effective inventory control is essential to minimise waste, reduce costs, and ensure timely delivery of these products to customers.

1.2.2 The importance of Multi-state systems

In many industries, machines' structural, functional, and behavioural performance are critical to ensuring smooth operations and maintaining production schedules. However, Despite meticulous maintenance, machines can encounter issues leading to delays, downtime, and revenue loss. While preventative maintenance and monitoring systems can help identify potential problems before they escalate, predicting when a breakdown might occur is a challenge that is difficult to overcome. The complexity of modern production systems and the machines used are the primary reasons why it is challenging to maintain a smooth production. Many systems consist of various components, each of which may be subject to wear and tear or other forms of damage. These components may fail independently or simultaneously, making it difficult to predict which component will fail and when. A well-maintained machine may be more reliable than an older machine that has not been serviced regularly. However, even a well-maintained machine can still be affected by factors that may be difficult to predict, such as unforeseen environmental factors (extreme temperatures, humidity, and corrosive materials), human factors, and randomness. While techniques such as predictive maintenance can help identify potential issues before they occur, there is no foolproof method to predict when a breakdown will happen. Systems that operate at different levels of performance, even in the presence of faults, play a critical role in ensuring the reliability and efficiency of a supply chain. These systems are designed to operate under different levels of performance, with each level representing a different state of the system. In the event of a fault or failure, the system can transition to a lower level of performance, allowing it to continue operating even in the presence of faults. One of the critical benefits of multi-state systems is their ability to provide different levels of performance based on the severity of the fault or failure. This allows the system to continue operating at a reduced level of performance rather than completely shutting down. By providing a backup mode of operation, multi-state systems can minimise downtime and increase responsiveness in various industries.

1.3 Research gap analysis

Currently, there aren't many known lot-sizing models in the literature for integrated multi-state systems in inventory models for perishable products, which entail re-configurable systems wherein the component or machine used for manufacturing deteriorates and continues to function even in the presence of faults. Various production factors such as flexibility, availability of machines, the state of the equipment and variable setup time have received attention from researchers. However, these studies often overlook the dynamic nature of equipment states in real manufacturing situations. While many researchers have ignored the effect of the degradation of machines, a growing body of literature now considers numerous situations that may occur during the lifetime of the production system. Such systems may be "in a state of control" or "out of control" (Ben-Daya, Hariga,

& Khursheed, 2008). Systems that are deemed "out of control" can incur substantial costs and negatively impact customer demand, as unplanned machine breakdowns lead to shutdowns that may disrupt the production process, especially in a make-to-order environment.

If, from the production point of view, a system is conceived in a way that at the occurrence of any failure, a reconfiguration is undertaken automatically, allowing the degraded or deteriorated machine or any other equipment to be functional, but with a decrease in the service delivered, it refers to as a multi-state system (MSS). Thus, a third state is added to the two previous states, which is referred to as the degraded state. Such models are particularly relevant for mineral processing systems, and power grid control systems (Soro, 2011). Despite the significance of MSS models for industries such as mineral processing, food industries or power grid control systems, the majority of the current MSS models overlook several important features such as price-dependent demand, stock level, freshness-dependent demand, quality control, and shortages. Furthermore, these models do not account for multi echelons in the supply chain, which are prevalent in the food industries as well as mining sectors. Thus, it is essential to explore multi-state systems and multi-echelon supply chain inventory models that are more reflective of real-world inventory systems, given the intricate and multifaceted nature of supply chains.

1.4 Objectives

The primary goal of this thesis is to develop inventory models for managing deteriorating products in a multi-state system. Three inventory models are presented in this thesis, each representing a sub-objective of the thesis. Each sub-objective targets a specific aspect of the overarching aim, which is to develop comprehensive inventory models that better reflect real-world conditions and enhance the efficiency of managing perishable goods. The models are as follows:

- An inventory model with a shifting production rate for perishable products with freshness, price, stock-dependent demand rate, and price discounting;
- A two-echelon supply chain inventory model for perishable products with a shifting production rate, stock-dependent demand rate and imperfect quality raw material;
- An integrated Economic Production Quantity (EPQ) model for deteriorating products with declining production rate, increasing defects, stock and price-dependent demand, and effects of corporate social responsibility activities.

The three models are briefly described next.

1.4.1 Sub-Objective 1: An inventory model with a shifting production rate for perishables products with freshness, price, and stock-dependent demand rate and price discounting

Classical EPQ models make assumptions about demand rate, production quality and the nature of manufactured products that may not accurately reflect real-world conditions. In practice, factors like price fluctuations, machine conditions, and product deterioration can significantly impact demand or production rate. Efficiently managing deteriorating items in inventory is crucial for manufacturers and retailers. Special consideration is needed for accounting for the type and quantity of stocks. Numerous researchers have explored demand dependencies on pricing and have investigated inventory models tailored for perishable items. These models incorporate features such as advanced payment, trade credits, and holding costs that vary linearly with time. Freshness is another crucial factor that influences consumers' behaviour. As such, effective inventory management models for perishable products must not only consider financial aspects but also prioritise maintaining product quality to meet safety standards and enhance customer satisfaction. Thus, this sub-objective introduces the concept of a shifting production rate with demand that is dependent on factors such as freshness, price, and inventory levels.

1.4.2 Sub-Objective 2: A two-echelon supply chain inventory model for perishable products with a shifting production rate, stock-dependent demand rate and imperfect quality raw material

Building on the insights from sub-objective 1, sub-objective 2 extends the focus to a two-echelon supply chain involving both manufacturers and suppliers. Ben-Daya et al.(2008) analysed the effect of a shifting production rate on lot sizing decisions due to speed losses caused by some process deterioration. The cycle starts with a certain production rate, and after a random time, the production rate shifts to a lower rate. They assumed that the product quality was not affected by this shift in production rate. While Ben-Daya et al. (2008) provided valuable insights into lot sizing decisions affected by shifting production rates, their assumption might overlook some important factors, such as the quality and deterioration nature of products manufactured, which may not align with real-world scenarios. The limitations in these assumptions underscore the need for more comprehensive models that account for the impact of shifting production rates on both quantity and product quality. Hence, a two-echelon supply chain inventory model for perishable products with a shifting production rate, stock-dependent demand rate and imperfect quality raw material is proposed. Since the production process requires the use of raw materials, it is assumed that the raw material received contains a proportion of imperfect quality. The model is investigated for the case where the imperfect raw material is kept in inventory after screening and then sold at a salvage value and for a case where the imperfect material is returned to the supplier. This helps to understand how shifting production rates and imperfect raw materials affect decision-making in inventory management.

1.4.3 Sub-Objective 3: An integrated EPQ Model for deteriorating products with declining production rate, increasing defects, stock and price-dependent demand, and effects of corporate social responsibility activities

While the previous models considered the effects of shifting production rates and imperfect raw materials, this sub-objective focuses on the broader production environment. A vast majority of existing models in inventory management predominantly focus on scenarios featuring constant production rates or systems with discrete production rates and a consistent rate of defective items. However, these simplifications may not fully capture the complexity of real-world production environments. Therefore, there is a pressing need to explore the dynamics of a production system characterised by continuous changes, especially concerning both the production rate and the generation of defective items over time. Furthermore, it's noteworthy that, regardless of the state of the production system, some companies strategically invest in the community through practices commonly referred to as Social Corporate Responsibility (CSR). Many researchers emphasise that such initiatives not only contribute to social welfare but also serve as a means for companies to gain a competitive advantage and attract customers who value ethical and socially responsible business practices. This sub-objective integrates and expands upon the previous models by presenting a continuously declining production system with an increasing rate of defective products while considering various factors, including the deterioration of products, stock levels, time, product quality, and CSR impact.

1.5 Thesis organization

Except for the introductory chapter, this thesis consists of four additional chapters, which are structured in the following manner:

Chapter 2 provides a comprehensive review of important inventory models documented in the literature. This review primarily focuses on inventory models with price-dependent demand, stock-dependent demand, time-dependent demand, mixed-dependent demand, imperfect quality of items, perishable items with expiration dates, multi-state systems, inventory models with planned shortages, multi-echelon inventory systems, and models with corporate social responsibility activities.

Building upon the literature review, Chapter 3 delves into the mathematical underpinnings of selected models, forming the theoretical foundation for the development of the novel models that are subsequently presented in Chapter 4.

Chapters 4, 5, and 6 represent the core of this thesis, wherein the primary objective is achieved through the formulation and development of three distinct inventory models. Each model is formulated to represent a specific scenario under specific conditions that commonly arise in the production industry. These conditions encompass factors such as shifting production rates, price-dependent demand, inventory level considerations, freshness-dependent demand, the presence of imperfect quality items, and the existence of expiration dates.

Chapter 7 is the concluding chapter. Here, a concise summary of the findings obtained throughout the thesis is presented, highlighting the valuable contributions this research makes to the existing body of knowledge in the field of inventory management. Finally, the chapter concludes with suggestions for future research avenues in inventory management, specifically focusing on growing items within multi-echelon supply chains.

Chapter 2

Literature Review

2.1 Introduction

Inventory typically constitutes the most substantial portion of the current assets section of a business's balance sheet. As such, any shortcomings in inventory management can exert substantial adverse financial ramifications on the entire organisation. In addition to the financial implications, an inadequately managed inventory system can diminish customer satisfaction levels, thereby exacerbating the financial burden due to reduced customer retention rates. Furthermore, these detrimental consequences extend beyond the focal business, influencing the entire supply chain (Sebatjane, 2020).

Inventory management involves optimizing the quantity of available stock, aiming to strike a balance that prevents two undesirable extremes on a spectrum: overstocking and understocking. When a business overstocks a product, it not only incurs substantial expenses associated with storing the product, in addition to the procurement cost, but it also foregoes potential opportunities that might have arisen from the capital invested in inventory. Conversely, in the case of understocking a product, the business is compelled to reject prospective orders, resulting in not only foregone sales but also diminished levels of customer satisfaction and the loss of potential repeat business in the future (Sebatjane, 2020).

The questions surrounding effective inventory management have received considerable scholarly interest in recent decades. As executives have recognised the financial burdens associated with superfluous stock levels, researchers have sought to minimise excessive inventory while preserving client service standards. While maintaining inventories is sometimes important to satisfy global customer demand Gourdin (2001), management aims to hold only the requisite inventory to achieve this goal. Consistent with this perspective, Chase et al. (2021) define inventory as "the stock of any item or resource used in an organisation". Hence, proper inventory administration necessitates suitable supervisory protocols and directives to routinely evaluate inventory levels and adjust them according to organisational needs (Chase et al., Chase et al. 2021). Moreover, this study acknowledges the significance of replenishment processes and inventory quantity. Furthermore, Pycraft et al. (2010) provide a more comprehensive definition of inventory as "the stored accumulation of material resources in a transformation system". This encompasses manufacturers retaining material stocks, tax bureaus maintaining informational stocks, and amusement parks managing client stocks (Pycraft et al., 2010). The establishment of an efficient inventory management system is an indispensable prerequisite for attaining success in business. Consequently, decisions about the quantity of stock to be ordered/produced, the frequency at which stock is replenished, or products are produced represent pivotal managerial considerations.

2.2 Types of inventories

Stock and Lambert (2001) delineate six major inventory forms and their uses in manufacturing:

- ***Fluctuation inventory***: Fluctuation inventory, also known as safety stock or buffer stock, is a category of inventory that serves as a cushion against uncertainties and fluctuations in demand and supply. Businesses with dynamic demand and with the product life cycle in the growth phase typically require this inventory to manage sudden demands.

- ***Anticipation inventory***: Anticipation inventory, also known as seasonal or speculative inventory, is a category of inventory built up in anticipation of future demand fluctuations or expected events. Unlike fluctuation inventory, which is used to buffer against uncertainties, anticipation inventory is purposefully accumulated based on forecasts, market trends, or known seasonal patterns. Anticipation inventory forecasts demand in seasonal industries that expect to elevate sales in specific periods for specific products like winter clothing or Christmas decorations.

- ***Cycle inventory***: Cycle inventory, also known as lot-size or replenishment inventory, refers to the inventory held to satisfy demand between successive replenishments or production cycles. It arises due to the periodic ordering or production process of goods.

- ***Transportation inventory***: Transportation inventory, also known as In-transit inventory or pipeline inventory, is the stock that has been ordered but has still not been delivered. Work-in-process (WIP) inventory is considered part of this category and is intended for the plant design and layout processes type.

- ***Decoupling inventory***: Decoupling inventory, also known as strategic inventory, is a category of inventory that is strategically positioned at different points within the supply chain to decouple or separate the interdependencies between different stages of production or distribution.

- ***Dead stock***: dead stock, also known as obsolete inventory, refers to unwanted stocks that are not expected to be used for any immediate or long-term purposes. Therefore, additional costs are incurred to store and maintain this inventory. In some cases, the stock may be stored to anticipate an eventual increase in demand or simply because disposal costs are higher than the storage costs. However, customer service is a primary reason that pushes businesses to stock dead inventory so that an occasional buyer can procure them at a salvage price in the future.

Waters (2008) proposed another classification of inventory based on the function of the inventory within the process as follows (shown in Figure 2.1):

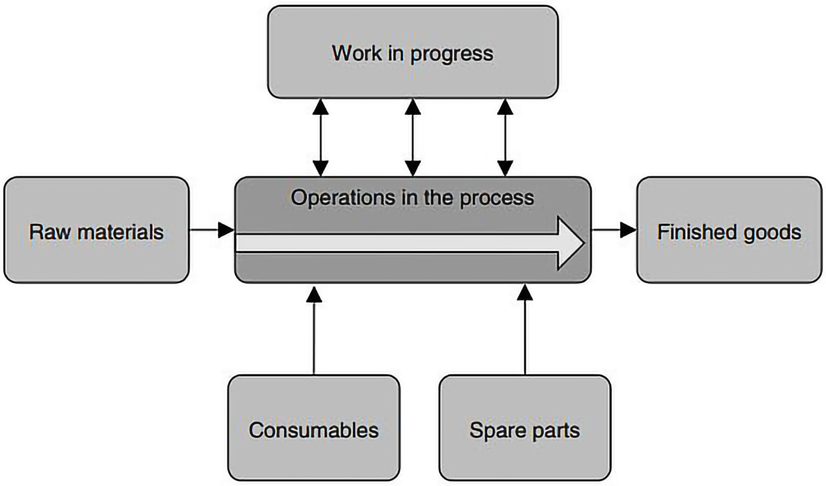


Figure 2.1: Different types of inventory

- **Raw materials**: This inventory type represents the initial stage of the production process. Raw materials are fundamental inputs utilised to create finished products, and they are crucial for commencing the manufacturing cycle.
- **Work in progress**: This inventory signifies the intermediate stage of the production process. Work-in-progress inventories are integral to the production cycle and are still undergoing transformation.
- **Finished goods**: These inventories represent the final stage of the production process. Finished goods are products that are ready for sale and delivery to customers.
- **Spare parts**: These are maintenance-related inventories. Spare parts are not directly used in producing finished goods but are essential for the upkeep and repair of machinery and equipment employed in the production process.
- **Consumables**: Comsumables such as lubricants, cleaning supplies, and office materials are necessary for the operation of the production process, though they do not become part of the final product. Consumables are utilised in maintenance and support activities.

2.2.1 Importance of inventory

Organisations across diverse industries exhibit significant variation in their inventory holdings. While building materials like bricks and sand require sizeable storage spaces with minimal specialised attention, expensive goods such as platinum, gold and diamonds require smaller storage areas with heightened security measures. Perishable goods, such as meat and milk, need special types of storage. Similarly, information can be stored in vast quantities but must facilitate rapid searching, sorting, and retrieval. Despite these distinct characteristics, inventory plays a crucial and indispensable role in every organisation. Without inventory, most operations would be rendered impossible. At a minimum, inventory enables operations to become more efficient and productive. It affects lead times and material availability, thereby impacting customer service, satisfaction, and the perceived value of products. Furthermore, inventory affects operating costs, profitability, return on assets, return on investment, and various other financial performance metrics. Inventory also shapes broader operational facets, such as the optimal size, location, and type of facilities (Waters, 2008).

2.3 Inventory Control

Inventory control refers to organising inventory management procedures to avail customers with goods as required (Wild, 2017). Procurement, manufacturing, storage, and distribution play integral roles by supporting the objectives of a typical organisation (Wild, 2017). Inventory control coordinates these primary operating activities to harmonise supply with projected demand. Effective procurement, manufacturing, storage, and distribution are, thus, imperative for meeting customer needs. By synchronising procurement, production, warehousing and delivery operations, inventory management aims to facilitate the realisation of business objectives. Thus, inventory control regulates the provision of finished goods, spare parts, raw materials, obsolete items, and other necessities (Wild, 2017; Jaber et al., 2009). Logistics, customer services and production significantly rely on inventory control efficiency (Jaber et al., 2009). Efficient inventory control systems offer numerous benefits, including improved customer satisfaction on an annual basis, reduced investment through effective planning and allocation, opportunities for trade purchase discounts, procurement of materials that adhere to product specifications, streamlined purchase and storage processes, as well as optimised production scheduling and reordering. Moreover, efficient control assured proper receipt, transaction and storage procedures for future purposes (Clodfelter, 2022).

2.4 Challenges of Inventory Control

Inventory control faces several challenges that impact customer satisfaction and operational efficiency. Inefficient stock management can result in delivery unavailability, order shortages, lost sales, and bottlenecks, leading to dissatisfied customers. On the other hand, excessive emphasis on customer service can lead to overstocking, tying up excessive capital in inventories (Biswas et al., 2017). Balancing these conflicting objectives is crucial. Managers need to determine the optimal level of customer service while controlling inventory costs. This involves avoiding both overstocking and understocking to ensure materials are available when required in manufacturing operations (Mula et al., 2006). Effective inventory control also contributes to economic efficiency, waste reduction, and minimising losses in the process.

The efficiency of inventory control directly affects the firm's flexibility. Inadequate procedures and strategies can result in undesirable inventory levels, with some items being overstocked while others face stock-outs. Inefficient inventory control leads to increased investment levels or higher ordering costs for excessive inventory or operational compromises in the case of low inventory levels (Mathur, 1994). Proper inventory control involves measures such as ABC analysis, setting inventory holding standards, and determining reorder costs (Mathur, 1994). Furthermore, the primary goal of inventory control is to provide a continuous and timely flow of high-quality and relevant information/materials to enable retailers/suppliers to serve end buyers effectively (Zinn and Charnes, 2005). However, unexpected disruptions like stock-outs render inventory management ineffective, and various factors may contribute to the "Bullwhip Effect".

Material management is another critical aspect of inventory control, encompassing procurement, warehousing, scheduling, inventory control of raw materials, and transportation (Federgruen and Zipkin, 1984). Procurement plays a vital role in operational activities, involving relationships between buyers and suppliers at strategic levels. Determining the required demand relies on factors such as scheduled orders, sales history, marketing initiatives, and customer feedback. Accurate demand forecasting is essential and involves both internal components within the firm and external partners such as suppliers and customers. In logistics, transportation is a significant concern as it directly impacts the movement of products. The choice of transportation methods affects the number of warehouses required for inventory management. Additionally, transportation economies play a role in the accumulation of inventories or raw materials, as purchasing the full load capacity lowers per-unit transportation costs (Jaillet et al., 1997).

Inventory control faces challenges in achieving customer satisfaction, balancing inventory levels, managing the procurement of raw materials, forecasting demand, transportation logistics, and maintaining cost-efficiency. Overcoming these challenges requires effective inventory control systems and strategies that align with the organisation's goals and optimise operational processes. Mathematical models have been developed to offer invaluable guidance to management when making decisions regarding order quantities and replenishment frequencies. The genesis of these models is credited to Harris (1913), who conceptualised the inaugural model, widely recognised as the Economic Order Quantity (EOQ) model. This model was explicitly designed to determine the optimal lot size, referred to as the order quantity and the replenishment frequency, with the primary aim of minimising the costs associated with inventory management. In its most fundamental form, the EOQ model attains this objective by striking a balance between the expenses associated with inventory holding and the fixed costs associated with order placement (Sebatjane, 2020).

2.5 The classic EOQ/ EPQ model

2.5.1 Initial development of the EOQ model

Determining the appropriate quantity of units to order is a crucial factor in supply decisions for all companies. Given this significance, the EOQ model has gained increasing importance over the past century. Its origin can be traced back to Harris, who first introduced the model in 1913. This model operates on the assumption that the optimal order size can minimise certain inventory-related costs. Thus, it becomes essential to consider factors such as holding costs and the trade-off between total ordering size. As described by Schroeder (2007), the EOQ represents the optimal quantity that strikes a balance between minimising inventory holding costs and the costs associated with reordering. Throughout decades of research, the strength of the EOQ model has been widely accepted, serving as a foundation for inventory control practices in subsequent stages of development.

The classic EOQ model stands as the most straightforward inventory control model. It aims to determine a fixed order quantity that minimises the combined costs of holding inventory and placing orders. While procurement cost could likewise be included, it typically does not impact the optimum order quantity unless quantity discounts are considered. At its core, the EOQ model achieves equilibrium between holding cost and ordering cost. As the order quantity increases, holding costs rise while ordering costs decrease, and vice versa. Visual representations are shown in Figures 2.2 and 2.3 to illustrate this dynamic trade-off and how total costs vary to changes in order quantity and inventory levels over time (Sebatjane, 2018). Such illustrations demonstrate the interplay between diverse cost elements the EOQ model seeks to balance.

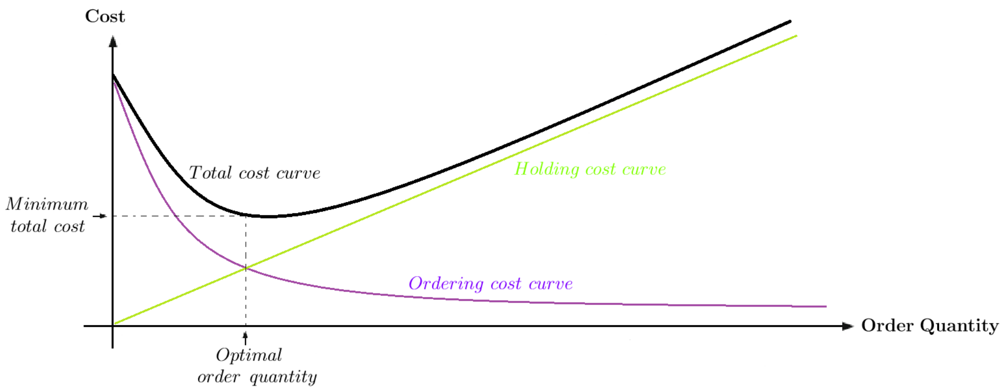
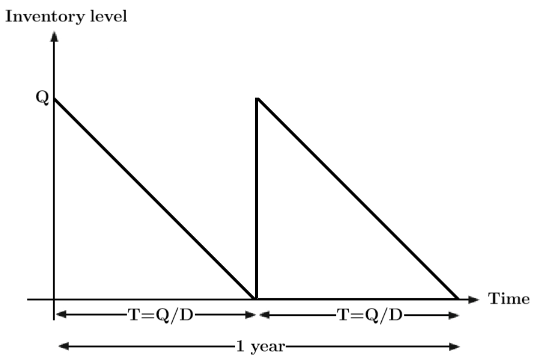
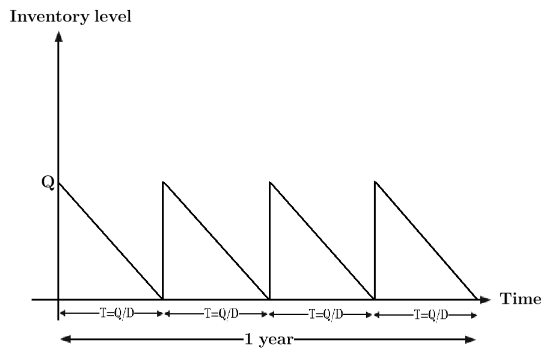


Figure 2.2: Holding cost, ordering cost and total cost as functions of order quantity



(a) Fewer large orders result in higher inventory holding costs and lower setup costs



(b) Numerous small orders result in lower inventory holding costs and higher setup costs

Figure 2.3: Typical inventory system behaviour for the classic EOQ model

The inventory system illustrated in Figure 2.3 focuses on a single type of item and assumes there is no lead time. At the beginning of each inventory planning cycle, an order for Q items is received as a single shipment. Placing an order for Q items incurs an ordering cost of O_rC. The items are consumed at a constant annual rate of D until they are completely depleted by the end of period T. As soon as the order is depleted, a new order for Q items is received. The items are stored in inventory, incurring an annual holding cost of h per item. The model assumes no quantity discounts or shortages. The total cost per unit time, denoted as TC, is calculated as follows:

$$TC = h\left(\frac{Q}{2}\right) + O_rC\left(\frac{D}{Q}\right) \tag{2.1}$$

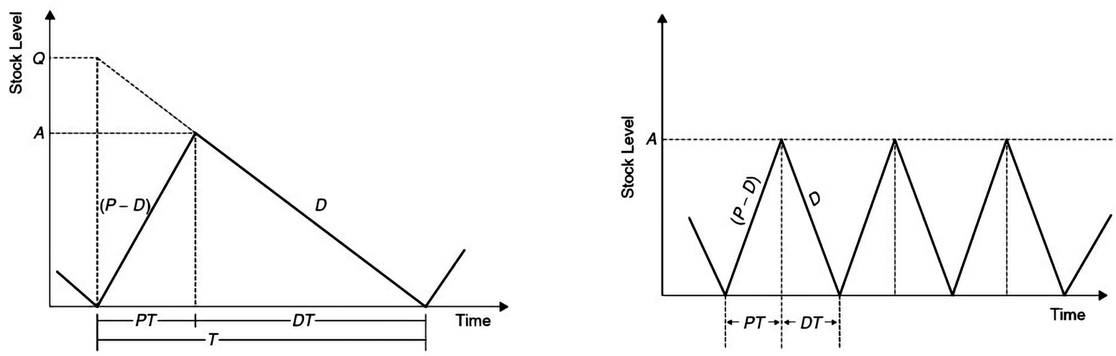
The value of Q which minimises Equation (2.1), denoted by Q^* and referred to as the EOQ, is determined using differential calculus as:

$$Q^* = \sqrt{\frac{2O_rCD}{h}}. \tag{2.2}$$

2.5.2 Initial Development of the EPQ model

As previously stated in Section 2.5.1, the primary objective of the original EOQ model developed by Harris (1913) is to provide managers with guidelines for ordering optimal quantities from suppliers. However, it is widely recognised that the practical application of the EOQ model is predominantly observed in batch manufacturing models, where the requirement is to have all materials available at the time of processing the entire production batch at once. Consequently, an extension of the EOQ model was introduced, known as the Economic Production Quantity (EPQ) model, which governs the optimal batch size. This model, as suggested by Taft (1918), is particularly suitable for manufacturing a single type of product. The units produced are intended to fulfil immediate customer requirements, while the remaining units are produced and stored to meet future customer needs.

The overall approach to optimise this model is the same as the economic order quantity. The derivation of the EOQ model is based on the assumption that the entire replenishment is instantaneous. The EPQ model assumes that replenishment becomes available at a rate of K per unit time, which corresponds to the production rate of the machinery used to manufacture the item. As a result, the sawtooth diagram depicted in Figure (2.3) is modified to that of Figure (2.4).



(a) One stock cycle with a finite replenishment rate

(b) NVariation in stock level with finite replenishment rate

Figure 2.4: Typical inventory system behaviour for the classic EPQ model

The primary change in the derivation is the calculation of the average inventory level, which is now $\frac{Q}{2}(1 - \frac{D}{P})$. The total relevant costs can then be determined by modifying the model represented in Equation (2.1). The total cost per unit time, TC, is:

$$TC = h\left(\frac{Q}{2}\right) \cdot (1 - \frac{D}{P}) + G \cdot \left(\frac{D}{Q}\right) \tag{2.3}$$

With G: Set-up cost.
The value of Q, which minimises Equation (2.3), denoted by Q^* and referred to as the Optimal batch size, is determined using differential calculus as:

$$Q^* = \sqrt{\frac{2 \cdot G \cdot D}{h}} \cdot \sqrt{\frac{P}{P - D}} \tag{2.4}$$

The practical application of Harris(1913)'s and Taft (1918)'s models to real-life inventory systems is limited due to their restrictive assumptions. As a result, researchers have sought to extend the model to address this limitation. These extensions involve either relaxing the original implicit and explicit assumptions or introducing new ones to accommodate diverse practical scenarios (Andriolo et al.,2014). These extended lot-sizing models provide increased utility for practitioners and serve as the basis for the research presented in this thesis. Specifically, the work builds upon models tailored for deteriorating products, products in multi-echelon supply chain systems, products with imperfect quality, perishable products with expiration dates, manufacturing with both shifting production rates and continuous decline, products with price-dependent demand, and products with inventory-level dependent demand.

2.6 Relevant inventory management models

2.6.1 Inventory models for deteriorating items

One of the first areas in which several extensions of the classic EOQ/EPQ model were developed is the area of deterioration of products. The classical inventory models of Harris (1918), rehashed in Taft (1918), assume that the depletion of inventory is due to a constant demand rate only. In real life, decay or deterioration of items is a natural phenomenon. Vegetables, fruits, foods, perfumes, chemicals, pharmaceuticals, radioactive substances and electronic equipment are examples of items that lose value over time through deterioration. Effective management of deteriorating inventory items is critical for manufacturers and retailers. Due to the perishable nature of these items, stockholders monitor the levels of inventory to prevent losses due to spoilage. Careful evaluation of costs associated with deterioration stands out as a defining characteristic of these inventory systems. Managing such products has resulted in significant research within the area of inventory control.

Whitin (1957) was the first to consider the effect of deterioration on fashion items after a prescribed date. Ghare and Schrader (1963) proposed a replenishment policy for an exponentially decaying inventory. Covert and Philip (1973) examined an inventory model for deteriorating items characterized by a variable rate of deterioration, where deterioration refers to decay, damage, or spoilage, rendering the item unusable for its original purpose. A Weibull distribution is employed to represent the probability distribution of the time until deterioration occurs. The Economic Order Quantity (EOQ) formula is derived under the assumption of constant demand, instantaneous delivery, and no shortages. Misra (1975) presented a production lot size model for an inventory system that deals with deteriorating items. The analysis considers both the scenarios of varying and constant rates of deterioration. A numerical method is proposed as a solution approach for the varying rate case since obtaining a straightforward expression for the production lot size proves challenging. On the other hand, for the constant rate of deterioration, an approximate expression is derived to determine the production lot size. Dave and Patel (1981) developed an inventory model for deteriorating items with a deterioration rate, which is a constant fraction of the on-hand inventory. Benkherouf (1995) presented an optimal replenishment policy for a constant deterioration rate with a known and finite planning horizon. Jaggi et al. (2006) proposed an optimal inventory replenishment policy for deteriorating items in the presence of inflationary conditions. The analysis employs a discounted cash flow (DCF) approach, which enables a comprehensive assessment of the financial implications associated with inventory management, including opportunity costs and out-of-pocket expenses. Moreover, the DCF approach allows for a precise consideration of the timing of cash flows within the inventory system. Ouyang et al. (2006) focused on developing a suitable inventory model for non-instantaneous deteriorating items that incorporate a permissible delay in payments to identify an optimal replenishment policy that minimizes the total relevant inventory cost. Srivastava and Gupta (2007) studied an EOQ model for deteriorating items with a constant deterioration rate, with both constant and time-dependent demand rates and no-shortages. Yan et al. (2011) proposed an integrated production-distribution model for a deteriorating item in a two-echelon supply chain. The model considers a restriction on the supplier's production batch size, which must be an integer multiple of the discrete delivery lot quantity to the buyer. Exact cost functions are developed for the supplier, the buyer, and the entire supply chain. These cost functions enable the determination of optimal policies for each entity involved, as well as the optimal policy for the overall integrated supply chain. A procedure is outlined to determine the optimal decisions for the supply chain to minimise the total system cost. Mishra (2013) formulated an inventory model for instantaneously deteriorating items. The model acknowledges the significance of managing the deterioration rate by implementing preservation technology (PT) to address this concern effectively, mitigate the impact of deterioration, and ensure better preservation of goods or assets. Mishra et al. (2013) developed an inventory model for deteriorating items with a time-proportional deterioration rate and time-varying holding cost. Majumder et al. (2015) developed an Economic production quantity model for deteriorating items under a partial trade credit policy. Pal et al. (2014) presented an inventory model for deteriorating items experiencing fluctuating demand in a fuzzy environment and incorporated the impact of inflation in the model. Wu et al. (2016) developed models for deteriorating items having a lifespan within a supplier-retailer-customer chain. Viji and Karthikeyan (2018) proposed a production-inventory model for deteriorating items with three levels of production, and the rate of deterioration follows a two-parameter Weibull distribution. Agi and Soni (2020) presented a deterministic inventory policy for a perishable product subject to both physical deterioration and freshness condition degradation. Sepehri et al. (2021) proposed a sustainable inventory model for deteriorating products with both quality and environmental concerns. Halim et al. (2021) discussed a production inventory model for deteriorating items along with an overtime production opportunity. Jain and Singh (2022) examined the effect of frequent inspections in lot sizing under partial advance payment and deterioration to reduce food wastage due to spoilage. Duary et al. (2022) discussed an inventory problem for deteriorating items that integrates the concepts of advance and delay payment. The model also incorporated the impact of advertisements on products. Lu et al. (2022) examined the implications of various carbon emission policies on the optimal production-inventory decisions for deteriorating items. Mahapatra et al. (2022) investigated three continuous review EOQ models for time-dependent deterioration, considering the utilisation of preservation technology. Initially, a crisp model is formulated, which is then extended into a fuzzy model to accommodate the imprecise nature of demand. Furthermore, the impact of the learning effect is analysed within the fuzzy environment. All models are designed for a finite time horizon and incorporate promotional efforts.Three algorithms are developed to determine the optimal solutions for each of the three models. Rahaman et al. (2022) presented an inventory model for deteriorating inventory in which preservation technology to recover substantial loss of items during production is implemented. Tiwari et al. (2022) explored how the inventory of imperfect quality items is affected by deterioration and trade credit policy. Salas-Navarro et al. (2023) proposed a vendor-managed inventory model for deteriorating items with a three-layer supply chain.

All the aforementioned studies mentioned above on perishable items can be summarised in five categories such as constant, linear, logarithmic, exponential and Weibull functions. Wang, Lin, and Jonas (2011) examined the methodologies employed to model the deterioration process (see Figure 2.5).

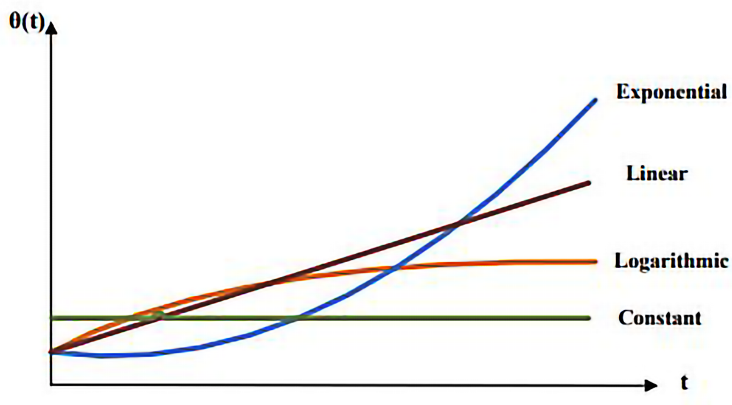


Figure 2.5: Different types of the continuous deterioration functions

- **Constant function**: Many researchers have assumed that the deterioration rate remains constant over time. This assumption aligns well with goods such as oil, alcohol, and certain cosmetic products (Chung and Wee, 2008; Huang and Yao, 2006; Aggarwal and Jaggi, 1995);

- **Linear function**: Other researchers such as Lin and Lin (2006) assumed deterioration rate as $\theta(t) = \theta_1 + \theta_2.\mathrm{t}$ where $0 \leq \theta(t) < 1$ and $0 < \theta_{1,}\theta_2 < 1$. Linear deterioration form is suitable for modelling products that experience a uniform decline in value or quality over time such as machinery, equipment, electronic devices, infrastructure and buildings;

- **Logarithmic function**: This type of deterioration function is well-suited for modelling products that experience a dramatic increase in their deterioration rate during an initial phase, followed by rapid stabilisation. For example, numerous integrated circuit (IC) chip products exhibit an increasing deterioration rate before packaging, which then stabilises afterwards;

- **Exponential function**: Certain dairy products, such as milk, rechargeable batteries, fossil fuels or mineral deposits, may degrade slowly at first, but as the products approach their expiration dates, the deterioration rate accelerates rapidly due to spoilage (Mahata, 2011; Lawrence et al, 2013; Gothi et al. 2017);
- **Weibull function**: Several researchers like Sharmila and Uthayakumar 2016), have assumed that the generalised form of $\theta(t)$ could be represented as a Weibull distribution, where $\theta(t) = \Pi\Omega t^{\Omega-1}, \Pi, \Omega \geq 0$. The Weibull deterioration function is suitable for modelling products with diverse deterioration patterns, including both increasing and decreasing rates over time. The Weibull function can capture a wide range of deterioration behaviours, from constant to exponential, depending on the value of the shape parameter Ω. Specifically, for $\Omega = 1$, the deterioration rate remains constant over time. For $1 < \Omega < 2$, the deterioration follows a logarithmic pattern. Setting $\Omega = 2$ yields a linear deterioration rate, while $\Omega > 2$ results in an exponential deterioration. This flexibility makes the Weibull function a powerful and versatile tool for modelling the deterioration of diverse products.

2.6.2 Inventory models with planned shortage

Shortages have a significant impact on systems that account for time delays in delivery or payment, leading to various effects such as decreased profitability and sales. The occurrence of shortages may result in higher sales losses as customers may opt not to wait for the next replenishment. Furthermore, shortages can affect the selling price, with some items being sold at discounted prices during backlog cycles to retain customers. Researchers often address the question of how much to backlog, how long the backlog cycle should be, and whether the permissible delay in the cycle is affected by the batch size or order quantity.

Many studies in inventory models prioritised the characterisation of optimal policies for perishable products by excluding shortage costs from their models. This simplified approach allowed researchers to focus on understanding the dynamics and implications of certain products within a specific context. While this omission may not accurately capture real-world scenarios, it provided valuable insights into replenishment strategies and pricing decisions of certain products. By omitting shortages, these models lay the groundwork for more comprehensive approaches that incorporate all relevant factors impacting the overall cost of systems under study. These models provided a framework for future research to expand upon by including shortages and developing solution techniques that can handle more complex scenarios. As time progressed, inventory models considering the combined effects of deterioration processes and shortages became more common. However, the basic models without shortages still played a crucial role in advancing the understanding of perishable inventory management from a scientific perspective. It is crucial to determine the optimal batch quantity in situations with shortages. Balancing costs related to shortages, inventory holding, and potential sales loss is essential for finding a profitable and efficient solution.

Some researchers have dealt with shortages as completely back-ordered demand. Luo (1998) proposed an integrated inventory model for deteriorating items with backorders. The study analyzed the impact of marketing strategies on system profitability, specifically pricing and advertising. It determined the optimal production quantity and backorder level to maximise net profit. Das and Maiti(2003) presented a profit maximisation inventory model for a differential item, where units are sold from two shops owned by a single management. Shortages are allowed in both shops, and they are fully backlogged. Hou(2006) proposed a mathematical model to address the inventory management of deteriorating items with shortages completely backlogged. The proposed model expands upon previous research by incorporating factors such as deterioration rate, inflation, and time value of money into the decision-making process over a finite planning horizon. Wee et al. (2007) formulated an efficient inventory model for defective items and shortages. The research assumes that in case of a shortage, customers are willing to wait for new supply and backorders are allowed. This assumption is based on the understanding that customers recognise and accept the occasional unavailability of the item and are prepared to wait for it to be restocked. Najid et al. (2011) proposed a model that combines the problems of capacitated lot sizing with complete shortage cost and determining optimal maintenance cycles.This integration aimed to mitigate the impact of preventable failures and reduce the frequency of unplanned events, incorporating strategic schedules for preventive maintenance actions within specified time windows to improve customer demand fulfilment. Kharde (2012) presented a replenishment policy for planned shortages using the concept of Equivalent Holding Cost (EHC) to optimise the EPQ system. These models are developed under complete backlogging. Pal et al. (2015) examined a production inventory model for deteriorating items with shortages under the impact of inflation. The deterioration rate follows a two-parameter Weibull distribution. Additionally, the model takes into account the effect of inflation on stock shortages within a finite period to address the devaluation of money caused by inflation. The inventory model is solved in a fuzzy environment to account for parameters with vague or imprecise definitions. To find the optimal solution, the authors considered production time and production rate as decision variables in different scenarios. By incorporating a symmetric triangular fuzzy number, they defuzzified the solution using the total lambda integral value. Viji and Karthikeyan (2018) introduced a multi-level economic production inventory model for deteriorating items. The model takes into account the production rate, demand rate, and deterioration rate, which is modelled using a two-parameter Weibull distribution. Their approach allows shortages as well as dynamic adjustments of the production levels during different time periods to optimise stock levels and minimise holding costs while meeting consumer demands. Their objective was to achieve overall cost reduction throughout the inventory cycle while maximising potential profits and customer satisfaction by finding the optimal production time. San-José et al. (2019) developed an inventory model that is deterministic with discrete scheduling. This model takes into account the possibility of backlogged shortages. One of the assumptions made is that both the inventory cycle and stock-out periods should be multiples of a fixed time period called the basic period. Any unfulfilled demand is fully backlogged. The main goal of implementing this inventory management system is to minimise the cost per unit time associated with maintaining and managing inventory levels. The total cost includes holding costs, backlogging costs, and ordering costs. The model involves two important integer decision variables: the number of periods within a cycle and the number of periods for stock-in. A two-dimensional search method is used to find the optimal inventory policy by considering the characteristics of the cost function. This approach aims to calculate the economic order quantity and determine an optimal inventory cycle that minimises total cost per unit time. Numerical examples were provided to illustrate theoretical findings, and sensitivity analysis was performed to assess how changes in specific parameters impact the optimal policy in this deterministic demand pattern with backlogged shortages setting. Overall, this model provides a systematic framework for efficient inventory management while minimising costs. Tiwari et al. (2020) proposed an inventory model for deteriorating items that takes into account shortages and partial delayed payments. The research provides theoretical findings on the optimal replenishment time and the duration needed to deplete all stock, which are then used to compute the most cost-effective ordering and backlogging strategies for retailers. Analytical methods were employed to obtain optimal solutions, while numerical validation was also conducted for the inventory model. Sicilia et al. (2022) proposed an inventory model for multi-item systems and limited storage capacity. The scheduling period or inventory cycle is predetermined and fixed. Initially, an aggregate cycle demand is determined, which is then gradually released into the inventory system following specific power patterns within the cycle. Additionally, shortages are permitted and fully backlogged while considering limited warehouse capacity. Their objective was to determine an optimal inventory policy that maximises expected profit per unit time through an algorithmic approach for calculating optimal inventory levels and maximising expected profit in online and in-shop sales.

The partial backordered state, which is a more generalised form of the completely backordered situation, can be classified into two categories: time-independent and time-dependent models (Sazvar, 2013). In time-independent cases, the number of unfulfilled back-ordered customers does not depend on the waiting time to meet their requirements. Law and Wee (2006), Lo et al. (2007), Hu et al. (2009), and Tshinangi et al. (2022) took into account that a constant percentage of demand was back ordered over the length of a stock-out interval. Pal et al. (2006) developed a deteriorating inventory problem with a demand rate dependent on stock level, selling price and frequency of advertisement, in which partial back ordering is considered. Mishra et al. (2013) developed an inventory model for deteriorating items with a time-proportional deterioration rate, time-dependent linear demand rate and time-varying holding cost under partial backlogging. Yang (2014) investigated an inventory control policy with a stock-dependent demand rate and stock-dependent holding cost rate with partial backordering. Tyagi et al. (2014) presented an inventory model for non-instantaneous deteriorating items with stock-dependent demand and variable holding cost and shortages. Ghiami et al. (2013) investigated a two-echelon supply chain model for deteriorating inventory with stock-dependent demand and partial backlogging in which the retailer's warehouse has a limited capacity. San-José et al. (2022) formulated an inventory system integrating time-dependent demand and partial backordering, which maximized the Return on Inventory.

On the other hand, in models where backorders are partially allowed and depend on time, it is assumed that the number of back-ordered demands during a stock-out cycle decreases as the waiting time to fulfil customer demand increases. This means that customers are more likely to wait if the waiting time is short. Thus, it is assumed that the fraction of demand back ordered during a stock-out (β^*) decreases as the remaining time until replenishment (τ^*). Zhou et al. (2004) presented a model for managing inventory lot-sizing model with waiting-time-dependent backlogging and a lot-size-dependent replenishment cost. This model not only accommodates the conversion of backlogged demands to lost sales but also models this conversion rate through a continuously decreasing function based on the remaining waiting time until the next replenishment delivery. Abad (2003) investigated the lot-sizing problem for a perishable good involving finite production, exponential decay, and partial backordering, along with lost sales. The study considered impatient customers, with the backlogging rate modelled as a negative exponential function of waiting time. Abad (2003) further assumed a first-come-first-served basis for customer service during shortage periods and presented a solution procedure to determine the lot size for maximizing the net profit cycle. However, the study did not account for the shortage cost related to backlogged items and the cost of lost goodwill arising from lost sales in the objective function. Yang (2005) examined partial backlogging in the context of deteriorating items, considering four distinct inventory models. The analysis incorporates both the opportunity cost associated with lost sales and the purchase cost, facilitating a comprehensive comparison among the four alternatives. The backlogging rate is defined as a differentiable and decreasing function of time to ensure generality. Teng et al. (2007) extended Abad's (2003) lot-sizing model by incorporating both the shortage cost for backlogged items and the cost of lost goodwill due to lost sales. Subsequently, they introduced a new modelling approach inspired by Goyal and Giri (2003) to address the lot-sizing inventory problem. Yang et al. (2010) developed an EOQ model that incorporates several important factors, including partial backlogging, inflation and time value of money, and time-varying replenishment cycles and shortage intervals to offer a comprehensive and flexible approach to EOQ modelling. The theoretical framework developed in the paper provides a basis for analysing and optimising inventory management decisions under various real-world scenarios. Sarkar and Sarkar (2013) proposed an inventory model for time-varying deteriorating items with time-varying backlogging rates. Yang and Chang (2013) proposed a two-warehouse partial backlogging inventory model with a permissible delay in payment under inflation. The model assumes that the partial backlogging rate decreases as the waiting time up to the next replenishment increases, and the two warehouses have different deterioration rates. Tiwari et al. (2018) investigated an inventory model for deteriorating items in a two-level partial trade credit setting, considering shortages. The study focused on a supplier-retailer-customer supply chain and incorporated allowable shortages. The study aimed to determine the optimal selling price, replenishment cycle time, and time to reach zero inventory while considering factors such as deteriorating items, credit periods, and shortages. Meena et al. (2022) proposed an inventory model that considers non-instantaneous deteriorating items with delayed payments while also taking into account the visibility of products to diverse customer segments. They derived the model by employing the discount cash flow method to incorporate marketing costs and salvage value for deteriorated units throughout the extended planning horizon marked. The inventory model accounts for shortages which are partially backlogged. The rate at which items are back-ordered during stock-out periods varies based on the time until the next replenishment is received. They conducted numerical examples and sensitivity analysis on key parameters to validate our approach before discussing management implications based on these findings.

As described previously, the generalisation form of a complete backorder situation is a partial backorder state. Studies on partial backorder assumptions can be divided into two categories: time-independent and time-dependent models, as shown in Figure 2.6. In time-independent partial backorder models, the number of unfulfilled demands is independent of the waiting time required to meet the customer's requirements (Law and Wee, 2006; Goyal and Giri,2003). In time-dependent partial backorder models, the number of backordered demands during a stockout cycle is inversely proportional to the time it takes to fulfil the customer's demand. That is, more customers are willing to wait if the wait

time is shorter.
Pentico and Drake (2011) categorised models for time-dependent partial backorders into six different classes, as outlined in Table 2.1.

Table 2.1: Partial backorder forms

Form of $\beta(\tau^*)$	Equation	Range for τ^*
Linear 1	$\beta(\tau^*) = \beta_M - (\beta_M - \beta_0)\left(\frac{\tau^*}{T_1}\right)$	$0 \leq \boldsymbol{\tau^*} \leq \boldsymbol{T}_1$
Linear 2	$\beta(\tau^*) = \beta_M - \left(\frac{\beta_M}{\tau_M^*}\right)(\tau^*)$	$0 \leq \boldsymbol{\tau^*} \leq \boldsymbol{\tau}_M^*$
Rational	$\beta(\tau^*) = \left(\frac{\beta_M}{1+a\tau^*}\right) a > 0$	$0 \leq \boldsymbol{\tau^*}$
Step	$\beta(\tau^*) = 1$	$0 \leq \tau^* \leq \boldsymbol{\tau}_M^*$
Exponential	$\beta(\tau^*) = \beta_M e^{-a\tau^*} a > 0$	$0 \leq \boldsymbol{\tau^*}$
Mixed Exponential*	$\beta\left(\tau_1^*, \tau_2^*\right) = \beta_1 e^{-a_1\tau_1^*} + \beta_2 e^{-a_2\tau_2^*}, \quad \beta_1, \beta_2 \geq 0 \quad \beta_1 + \beta_2 \leq 1 \quad a_1, a_2 > 0$	$0 \leq \boldsymbol{\tau}_1^*, \boldsymbol{\tau}_2^*$

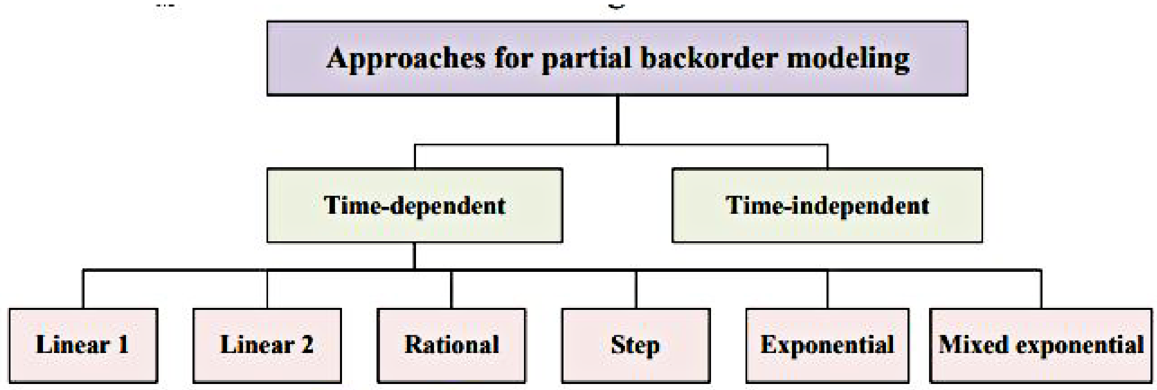


Figure 2.6: Partial backorder forms in the literature (Sazvar, 2013)

- Linear 1: The first form for a non-constant backorder was discussed by Montgomery et al. (1973). In this form, $\beta(\tau^*)$ has an initial value, β_0, at the time the stock-out begins (i.e., when τ^* has its maximum value) and then increases linearly until it reaches a maximum value, β_M, when the new replenishment order is received (when $\tau^* = 0$);

- Exponential: In this model, $\beta(\tau^*)$ is obtained by multiplying β_M with a negative exponential function of τ^*. This exponential form was first proposed by Papachristos and Skouri(2000) and later it was incorporated into the basic EOQ model with partial backlogging by San José et al. (2006);

*It is assumed that there are two kinds of customers.
†β_0: Initial value of $\beta(\tau^*)$
‡β_M: Maximum value of $\beta(\tau^*)$ over a stock-out interval
§T_I: Length of time during an inventory cycle for which there is stock-out
¶τ_M^*: Maximum customer's waiting time
‖:Time remaining until the next replenishment order is received

- Rational: $\beta(\tau^*)$ is β_M divided by a positive linear function of τ^* (San José et al., 2005);

- Linear 2: This form provides a more flexible approach compared to the linear 1 form, as it allows the initial backorder rate β^*, to vary based on the length of the stockout period. In the linear 2 model, the backorder rate is initially zero until a time τ_M^* before the replenishment order arrives, and then it increases linearly until it reaches its maximum value, β_M, at $\tau^* = 0$ which is the time of replenishment. This flexibility can make the linear 2 model more suitable in certain situations, as it captures the impact of the stockout duration on the backorder rate. The linear 2 model was introduced and analysed by San José et al. (2005);

- Step: This form is similar to the linear 2 policy as the backorder parameter $\beta(\tau^*)$ equals to zero until a time τ_M^* before the arrival of the replenishment order. However, it diverges from the linear 2 policy because $\beta(\tau^*)$ immediately increases to one and maintains this value until a new replenishment is received, rather than gradually increasing as in the linear 2. San José et al. (2005) analysed this particular form. The $\beta(\tau^*)$ function differs from the previous ones, exhibiting a U-shaped or dish-shaped pattern rather than a non-decreasing function of τ^*;

- Mixed exponential: The basic assumption behind the mixed exponential model, as examined in Sicilia et al. (2009), is that there are two distinct customer segments. For some customers, the reluctance to backorder decreases as the replenishment time approaches. Conversely, another group exhibits the opposite behaviour, becoming more inclined to backorder if the wait is prolonged. Sicilia et al.(2009) assumed that this might occur if customers believe that a longer wait will ensure they receive higher-quality items or will get a discount. The combination of these two customer types results in a backordering-likelihood function that is the sum of two exponential forms. One exponential form, with a maximum value of β_1, is based on τ^* (referred to as τ_1^*), while the other, with a maximum value of β_2, is based on $T_1 - \tau^*$ (referred to as τ_2^*), representing the time elapsed from the stockout onset until the requirements for the customers that are willing to wait for the next replenishment arrive. The aggregation of these two exponential forms yields a dish-shaped function.

The extensions of the EOQ/EPQ models with planned shortages differed based on the type of shortage function used in each study. However, these studies provided valuable insights into how shortages affect the formulation of the model.

2.6.3 Inventory models with price-dependent demand rates

Another extension of the EOQ/EPQ model concerns scenarios where demand changes with respect to price. These scenarios consider the correlation between consumer demand and product price. In 1974, Ladany and Sternlieb conducted one of the pioneering studies extending the EOQ model to include varying demand rates linked to the product's selling price. Their model investigated how the EOQ interacts with diverse pricing policies to ascertain the order quantity that maximises a company's net profit. Prior to constructing the model, several assumptions were made, including deterministic demand dependent on the selling price, single-batch supply ordering, and a unit cost that decreases either linearly or hyperbolically. Subsequently, Ray et al. (2005) carried out another study that explored the link between product demand and its selling price, integrating this association into the EOQ model. This investigation focused on a scenario involving a company selling a single product utilising mark-up pricing. Additionally, linear and log-linear demand functions were considered in this study. The study demonstrated that, from a profitability perspective, for customers highly sensitive to pricing changes with non-linear demand, managers should avoid substantial price reductions and instead adopt an assertive approach.

The EOQ model was further expanded through a series of research that explored demand as a function of the selling price, making the key assumption that the demand function was deterministic. Notable contributions in this area include the work of Fibich et al. (2003) and Chou and Parlar (2006), who employed a linear deterministic demand function; Jeuland and Shugan (1988) and Agrawal and Ferguson (2007), who utilised a power deterministic demand function; Hanssens and Parsons (1993) and Song et al. (2008), who expressed the deterministic demand function in an exponential form; Chen et al. (2006), who employed a logarithmic deterministic demand function; Chen and Simchi-Levi (2012), who incorporated a logit-based deterministic demand function into their research; Avinadav et al. (2014), who proposed an optimal ordering and pricing policy to maximise profits given a deterministic demand function that is affected by both price and inventory age; Feng et al. (2017) expanded Wu et al. (2016)'s model by considering pricing strategy as an additional factor; Bai et al. (2017) analysed a sustainable supply chain system with deteriorating items, which consists of a manufacturer and a retailer subject to carbon cap-and-trade regulation, and the demand is affected by three endogenous variables: promotional efforts by the retailer, product selling price, and the sustainable level determined by the manufacturer; Khan et al. (2020), who developed mathematical models for perishable items with advance payment, linearly time-dependent holding cost and demand dependent on advertisement and selling price. These studies collectively contributed to a deeper understanding of how variations in selling price influence demand and, subsequently, how this information can be effectively integrated into the EOQ model to optimize inventory management and ordering decisions. Torkaman et al. (2022) introduced a Mixed-Integer Nonlinear Program (MINLP) to address the Production-Routing Problem with Price-Dependent Demand (PRP-PD). Feng et al. (2022) studied pricing and lot-sizing decisions for fresh (or perishable) goods and incorporated multiple payment types, including advance, cash, and credit payments when demand is a generalised downward-sloping curve that depends on unit price, displaying stocks and product age.

The aforementioned models all operated under the assumption that the demand function was deterministic. However, a range of studies took the EOQ/EPQ model a step further by introducing demand functions characterised by random probability distribution functions, as demonstrated by Huang et al. (2013). Federgruen and Heching (1999) extended the EOQ model to encompass demand distribution influenced by the item's price. This extended model was predicated on several assumptions, including the price being dependent on the state of the system being managed and the ability to place a replacement order at the beginning of a given period. Additionally, Petruzzi and Dada (1999) introduced the connection between demand and the selling price into an EOQ model. Their research examined different forms of uncertainty, including additive, multiplicative, and hybrid uncertainties. Their case study revealed that when uncertainty took an additive form, the optimal price did not surpass that obtained from the deterministic model. However, when the uncertainty was in a multiplicative form, the optimal price did not fall below the deterministic model's outcome. Moreover, their model demonstrated that a single-period model could be effectively applied to a multiple-period problem, enhancing the practicality and versatility of this innovative model. Chen and Simchi-Levi's model, introduced in 2004, is yet another model that treated the demand function as stochastic. In this research, demand was regarded as a random variable, with its probability distributions influenced by the product's selling price. Similar to earlier studies, this model involved making pricing and ordering decisions at the beginning of the period and encompassing both fixed and variable costs in the ordering cost. The researchers demonstrated that when the demand model followed an additive pattern, the optimal policy involved managing inventory based on the principle that an order is placed only when the inventory level at the beginning of the period falls below a specific reorder point. Furthermore, in this scenario, the price was best done based on the inventory position at the beginning of the period. Moreover, other stochastic demand models found in the literature included those developed by Kocabıyıkoglu and Popescu (2011) and Phillips (2005). These models collectively enriched our understanding of how stochastic demand can influence inventory management and ordering decisions in various scenarios.

Additional demand models, often referred to as "willingness to pay demand function," operate on the premise that consumers exhibit varying levels of willingness to purchase a product from a company at a specific selling price, which may be less than the maximum price they are prepared to pay for that product (Huang et al., 2013). Among these models, one noteworthy example is the model crafted by Kalish in 1985. Kalish's (1985) model tackled this concept by integrating the uncertainty received from consumer feedback and experiences with the product, adjusting the product's value accordingly.

Moreover, there exist Poisson flow models that delve into the dynamics of customers' purchasing processes and their evolving price preferences. These models take into account shifts in customers' willingness to pay for certain products. Zhao and Zheng (2000) were among the first to introduce an inventory model that considers customers whose reservation price distribution dynamically changes over time, following a non-homogeneous Poisson process. This model also considers changes in the probability function of this distribution as time progresses. An (2003) explored a continuous-time dynamic pricing model where a seller aims to sell a single item within a finite time frame. Customers arrive following a Poisson process, and upon arrival, they purchase if the posted price is lower than their individual reservation price; otherwise, they exit without a purchase. Notably, some customers who have made a purchase may subsequently return the item at an exponential rate, expecting a full refund. The seller's objective is to dynamically adjust the price to maximise the expected total revenue as the sales period concludes by formulating the dynamic pricing problem as a dynamic programming model. In scenarios with more general reservation price distributions, the author presented an approximation of the original model by discretising both time and the permissible price range and then introduced an algorithm to numerically compute the optimal policy within this discrete-time model. Lin (2004) proposed a sequential dynamic pricing model where the seller sells a given stock to a random number of customers. He also formulated the seller's problem as a stochastic dynamic programming model and developed an algorithm to compute the optimal policy. Wen and Chen (2005) established a dynamic pricing model for the sale of a uniform inventory of perishable products over a limited time horizon by putting forth three theorems to elucidate the characteristics of expected revenue and time-based thresholds in a single-product, single-retail store problem. In this model, the authors considered demand in a stochastic Poisson form and assumed that it is price-sensitive. All customers are treated as independent entities, and they purchase if the current price falls below their respective reservation price. Each customer's reservation price signifies the maximum amount they are willing to pay for the product. Typically, these reservation prices exhibit a continuous distribution across the customer population. Furthermore, both the arrival rate of customers and the distribution of reservation prices fluctuate over time. Additionally, the authors assumed that there are no competing firms operating on the Internet, meaning they disregard the impact of external competition. Liu and Yang (2012) focused on the practice of dynamic pricing, specifically on the markup strategy employed by firms. Their model involved using real-time inventory data to determine the most opportune moments to increase sales prices. In this model, demand follows a Poisson process, with the instantaneous rate being the result of two factors: one influenced by the current price and the other by the current time. By employing a combination of mathematical induction and sample-path arguments, they established the optimality of threshold policies. Pang et al. (2015) explored the behaviour of bid prices in revenue management (RM) environments. In RM, the optimal pricing decision depends on consumer valuation and bid price. The study provides a probabilistic characterisation of optimal bid-price processes, showing that they have an upward trend before inventory levels fall to one and then a downward trend. This pattern is driven by the resource scarcity effect and the resource perishability effect. The article also demonstrates how bid prices and consumer valuation interact over time to drive the optimal price process. The findings apply to network RM problems as well. Katehakis et al. (2022) focused on analysing an irreversible dynamic pricing situation, specifically when a firm uses real-time inventory information to determine the optimal time to increase its sales prices. They assume that the aggregate demand is influenced by independent purchase decisions made by a multitude of buyers and that the firm's pricing decisions impact the arrival rate of demand. The study proposes a demand pattern that includes a time-dependent term and a price-dependent factor and establishes the optimality of a threshold policy for both markup and markdown cases. The policy involves price-switching at specific time points based on current price choices and inventory levels. Additionally, this research discusses the concept of the "leapfrog" phenomenon, the challenges of maintaining value function monotonicity, and tests to determine the optimality of arrival patterns. Other studies that adopt the Poisson flow to model the demand function include the work of Bitran and Mondschein (1997), Xu and Hopp (2009), and Cao et al. (2012). These models offer valuable insights into how customers' behaviour and preferences evolve and their impact on revenue.

All the above models operate on the assumption that a single firm exclusively sells the given product. However, in the real world, it is more common to encounter scenarios where multiple competitors are selling similar products. Consequently, several models were developed to address this reality, taking into account the impact of competition on demand in a multi-firm environment. Notable among these models are those devised by Anderson et al. (1992) and Vives (1999). These models consider a linear demand function and incorporate the effects of both the product's price and the prices of competitors' products. It is worth noting that these models place a greater emphasis on the impact of the product's price compared to the influence of competitors' prices. Lu (2019) discussed a mathematical model for revenue management in a competitive market. It combines the stochastic knapsack problem and a non-cooperative game model to understand the rational behaviour of sellers and then establishes a dynamic recursive procedure that incorporates the value and utility functions of the game, leading to the identification of some important structural properties.

Based on the comprehensive review of the diverse price-dependent models mentioned above, it is evident that the methodological approach and underlying assumptions for model development can vary significantly depending on the characteristics of the demand function and the competitive landscape. Furthermore, the broad spectrum of demand functions employed in these models offers a range of advantages and disadvantages for each. This variety of insights is valuable when it comes to selecting the most suitable demand function for developing models in the context of this thesis.

2.6.4 Inventory models with time-dependent demand rates

In addition to investigating how inventory models are influenced by product selling prices and the shortage, previous literature has also explored the effects of time on demand, leading to the development of extended models. This has led to the development of extended models to accommodate such variations. These models primarily focus on goods with finite shelf lives, which experience decay, spoilage, or loss of utility over time. Examples of such products include fish, medicine, vegetables, and airline tickets, all of which start to deteriorate as soon as they are produced. Many research studies have expanded the traditional EOQ/EPQ model to incorporate the impact of time on product demand. The earliest models in this area were developed by Resh, Friedman, and Barbosa (1976) and Donaldson (1977). Resh et al. (1976) extended the EOQ model to incorporate deterministic demand that increases linearly with time. Their model was based on three key assumptions: prompt inventory replenishment based on the required number of items, a well-defined planning horizon, and the inclusion of replenishment, carrying, and shortage costs. The derived model aimed to determine the optimal schedule for inventory replacement to minimize costs when the inventory level reaches zero. The researchers demonstrated that for a given number of required replacements "m," there exists a unique vector of "m" time intervals that minimises the total cost. They also developed an algorithm to determine the optimal value of "m" and the corresponding replacement schedule using the derived mathematical formula. Furthermore, the researchers expanded the scope of the formulated model by considering products with increasing rates of demand and diminishing markets, simultaneously. Donaldson (1977) utilised dynamic programming methods, focusing on the replacement cycle and cycle time to derive the demand.

Another important study that considered time-dependent demand rates is the study by Bose, Goswami, and Chaudhuri (1995). Unlike previous models, this study incorporated shortages, backlogging, inflation and the time value of money. The model was based on several assumptions, including a constant deterioration rate, an infinite replenishment rate, and a finite time horizon with multiple reorder points. The comprehensive model also took into account three types of costs: production cost, carrying cost, and shortage cost. The production cost was influenced by the internal inflation rate, increasing total cost, while the unit purchase price was affected by the external inflation rate.

The carrying cost included both opportunity costs and out-of-pocket expenses unrelated to operations, such as insurance, taxes, and storage.Numerical cases were examined to analyse the effects of shortages on reorder numbers, system cost, and scheduling period. Additionally, a sensitivity analysis provided an understanding of the influence of independent variables on the model's results. Balkhi and Benkherouf (2004) presented an inventory model for deteriorating items with stock-dependent and time-varying demand rates over a finite planning horizon. Panda et al. (2009) developed an inventory model for perishable products with time-varying demand. Maihami and Kamalabadi (2012) introduced a comprehensive joint pricing and inventory control model for non-instantaneous deteriorating items. This model considers a demand function that is influenced by both price and time, allowing shortages and partial backlogging. The primary objective of this model is to determine the optimal selling price, replenishment schedule, and order quantity, all to maximise the overall profit. To facilitate the practical application of the model, the authors provided a straightforward algorithm for finding the optimal solution. Additionally, they included a numerical example to illustrate the model's use in real-world scenarios. Sarkar et al. (2012) looked into the development of an optimal inventory replenishment policy for an item experiencing deterioration with time-quadratic demand, time-proportional deterioration functions, as well as variable replenishment cycles in their model. Additionally, they considered the impact of waiting-time-dependent partial backlogging. Khanra et al. (2013) explored an EOQ model for an item under time-dependent demand characteristics. This model incorporates a quadratic time-dependent demand function that allows for payment delays and presents a mathematical model for determining the optimal order quantity and cost within the inventory system. The authors derived the model under three distinct scenarios and used numerical examples to illustrate the outcomes. Moreover, the authors extended the model's scope to include shortages and introduced trade credit periods as additional variables. The resulting inventory model describes the relationship between demand rate, inventory levels, and time. Two particular cases were examined: one where the buyer accrues interest and another where the buyer doesn't incur interest but earns it during a specific period. Guchhait et al. (2013) presented an EPQ model for damageable items with variable demand rates in which both the inventory carrying cost and the production rate are assumed to be time-dependent. Chowdhury et al. (2015) developed an inventory model for deteriorating items to determine the optimal strategy for a firm that sells a seasonal item over a finite planning time to maximise the firm's expected profit.Chen et al. (2019) focused on the problem of managing a single product inventory system over a finite horizon. The product is perishable and deteriorates over time, with the deterioration rate dependent on the inventory level. Demand for the product is deterministic and influenced by the inventory level, price, and time. The objective is to determine the pricing and replenishment policy that maximises average profit per period. The authors propose a model and algorithm to determine the optimal policy. A numerical example illustrates the model and a sensitivity analysis is conducted. The paper concludes with a summary of the research and suggestions for future work. Sanni et al. (2020) investigated a reverse logistics EOQ model for deteriorating items that address the inventory problem of determining when to order and how much quantity to order with the reverse flow of items in the system to maximise profit using the Karush-Kuhn-Tucker (KKT) conditions. Akan et al. (2021) discussed the optimal pricing and inventory policy for a follower firm in the fashion industry, with the demand for fashion products going through three stages: growth, maturity, and decline. Small retailers, as followers, only have the opportunity to observe the demand levels in the second stage. The study aims to find the optimal pricing and inventory policies considering a single replenishment opportunity and a shrinking market. The interest rate is assumed to be time-dependent, and the article proposes a discrete pricing heuristic using a control theoretic approach. Their findings suggest that a piece-wise constant pricing strategy may be more practical in real-life settings. Khan et al. (2022) discussed two sustainable inventory models for perishable items with limited storage capacity, advanced payment, time-dependent demand and time-varying holding cost under partial backlogging. Aarya et al. (2022) introduced a two-storage production inventory model, where the demand is influenced by both the pricing and the passage of time. The demand rate is shaped by the selling price and the elapsed time. In their analysis, they made an initial assumption that the rate of deterioration remains constant, and for the storage they own (OW), the holding cost varies with time. Unlike the OW, for rented storage (RW), they considered that the deterioration rate is time-dependent while the holding cost remains constant. Recently, Kumar et al. (2023) formulated a framework designed to pinpoint the optimal replenishment time and quantity to effectively manage sales operations while preventing stockouts. Their study integrates the influence of promotional activities and product reliability on customer purchasing behaviour, employing the concept of the Caputo fractional order derivative. The authors engaged in a comprehensive exploration of various inventory models and their respective limitations, underscoring the necessity for a model that accounts for the interplay of time, product reliability, and promotional efforts in shaping demand. To illustrate the practical application of their proposed model, they conducted a case study involving an e-commerce retailer. The results of this case study provide compelling evidence for the advantages of utilising this model in the context of inventory planning. Saranya and Chandrasekaran (2023) presented a depleted demand inventory model with constant deterioration. The demand rate is assumed to be time-dependent, with initial non-zero demand occurring due to advertisements. The research encompasses two distinct replenishment strategies: one without shortages and another with shortages. The primary aim is to ascertain an appropriate replenishment policy that effectively minimises the total inventory cost. The paper meticulously delineates the assumptions, symbols, and formulation of the model. Rukonuzzaman et al. (2023) discussed an inventory planning problem for a company that sells perishable items with a quantity-based discount under time-dependent demand. The study aims to minimise the company's costs by considering the impact of the discount on inventory planning. The authors introduced two solution algorithms for the problem and provided an example of a mango business to verify the inventory procedure. The findings suggest that the company can benefit from a reduced acquisition price by making larger purchases under the discount program.

The extensions of the EOQ/EPQ model that considered time-dependent demand varied depending on the specific demand function used in each study. It is important to note that all of these models focused solely on scenarios where competition is not present, highlighting a limitation of these models. Nonetheless, valuable insights were obtained regarding the impact of various assumptions on the formulation of the model.

2.6.5 Inventory models with stock level-dependent demand rates

Another area of inventory management that has witnessed significant attention focused on a class of inventory models that consider demand rates dependent on the stock or inventory level. These models assume that the presence of retail inventory has a stimulating effect on customer behaviour. It is not uncommon to observe stores displaying large quantities of items as "psychic stock" to encourage sales of specific retail products, as noted by Larson and DeMarais (1990). This phenomenon can also be observed with products that are essentially similar but have slight individual differences. In such cases, higher inventory levels provide customers with a broader selection, increasing the likelihood of making a purchase.

Unlike classical inventory models where demand is treated as an exogenous variable, these inventory models with inventory-level-dependent demand assume that the demand rate is endogenous to the firm and is influenced by the inventory level. Consequently, retailers are incentivised to maintain higher inventory levels, even with increased holding costs, as long as the item remains profitable and the demand is positively correlated with the inventory level. This approach leads to additional sales, improved fill rates, and potentially greater profits. The field of operations management has recognised this motivating effect of inventory on demand and has developed models that incorporate this relationship. Datta and Pal (1990) proposed a deterministic inventory system for deteriorating items with a constant deterioration rate and demand rate that is a linear function of stock level. Chakrabarti and Chaudhuri (1997) presented a replenishment inventory problem for a deteriorating item over a finite horizon with a linear trend in demand rate. Datta and Pal (2001) discussed an inventory model with both price and stock-dependent demand under a finite time horizon. Urban (2005) provided a comprehensive review of inventory models with inventory-level-dependent demand. It explores two types of models: one where the demand rate is a function of the initial inventory level and the other where it is dependent on the instantaneous inventory level. The article demonstrates the equivalence of these two types of models and proposes an alternative approach to sensitivity analysis for these models. The article also discusses the motivating effect of inventory on demand and how higher inventory levels can lead to increased sales and profits. Pal et al. (2006) discussed the impact of deterioration, marketing strategies (such as pricing and advertising), and storage capacity on inventory management. The demand rate depends on various factors, including the displayed stock level, the selling price, and the frequency of advertising across both electronic and print media, as well as through sales representatives. The model allows for shortages, which, if they occur, are partially backlogged at a variable rate, dictated by the waiting time until the arrival of the next replenishment. The model also accounts for transportation costs associated with replenishing goods, and the authors assumed that the showroom or shop has a finite storage capacity. They explored various scenarios and sub-scenarios based on the relative influence of stock level-dependent demand parameters and the storage capacity of the showroom/shop. These scenarios are solved using the GRG (generalised reduced gradient) method and a computational approach. Alfares (2007) presented an inventory model with a stock-level dependent demand rate and a variable holding cost throughout the inventory cycle to discuss how demand for a product can be influenced by internal factors such as price and availability. The author proposed a unique step structure for the holding cost function of perishable items and developed an inventory model with two types of time-dependent holding cost increase functions, such as retroactive holding cost increase and incremental holding cost increase function. The paper by Panda et al. (2008) deals with an EPQ model for seasonal products with stock-dependent demand. Roy et al. (2009) presented a mathematical model for an inventory system that focuses on deteriorating items with uncertain planning horizons and stock-dependent demand. The model assumes a linear relationship between displayed stock and demand. It takes into consideration both fuzzy and random parameters while also accounting for inflation and the time value of money. The time horizon is treated as a random variable following an exponential distribution. The authors developed a genetic algorithm (GA) that implements roulette wheel selection, arithmetic crossover, and random mutation techniques to optimise the total expected profit. When dealing with crisp inflation effects, the GA aims to maximise profits over the planning horizon when dealing with crisp inflation effects. However, when the inflation effect is fuzzy, the total expected profit becomes fuzzy as well. The authors employed the possibility/necessity measure of a fuzzy event to obtain the optimistic/pessimistic return of the expected profit. Sarkar (2012) developed an EOQ model that incorporates a finite replenishment rate and proposed an ideal replenishment policy for scenarios where demand is stock-dependent and permissible delay payment. Lee and Dye (2012) formulated a deteriorating inventory model with stock-dependent demand by allowing preservation technology cost as a decision variable in conjunction with replacement policy. Sakar and Sakar (2013) discussed the EOQ model that focused on deteriorating items with stock-dependent demand. The model incorporates time-varying backlogging and deterioration rates to determine the optimal cycle length for each product, aiming to minimise the expected total cost, including holding, shortage, ordering, deterioration, and opportunity costs.Additionally, necessary and sufficient conditions are presented to demonstrate the existence and uniqueness of the optimal solution. Practical applications of this proposed model are illustrated through numerical examples accompanied by sensitivity analysis and graphical representations. Chakraborty et al. (2015) investigated multi-item integrated production-inventory models involving suppliers and retailers. The models focus on the collaboration between suppliers and retailers dealing with products experiencing a constant rate of deterioration, subject to stock-dependent demand. Several aspects are considered in these models, including the supplier's production cost, retailer procurement cost, and supplier transportation cost. The supplier's production cost is treated as a nonlinear function that depends on the production rate. The retailer's procurement cost is exponentially dependent on the credit period, and the transportation cost is considered a nonlinear function of the quantity purchased. The primary objective of the models developed by Chakraborty et al. (2015) was the determination of credit periods and the total duration of the supply chain cycle that optimised the inventory system, all within the constraints of space and budget limitations. The models are further adapted to function within fuzzy, random, and bifuzzy environments. Several variables, such as ordering cost, procurement cost, retailer selling prices, holding costs, production cost, transportation cost, supplier setup cost, total storage area, and budget, are all considered in fuzzy and bifuzzy environments. Lee et al. (2017) investigated a vendor-managed inventory system with a consignment stock agreement applied to an integrated vendor-buyer system. In this system, the vendor produces a single product in batches and delivers it to the buyer in equal-sized transfer lots. Consignment inventory refers to a situation where a supplier stocks their products in a buyer's warehouse without receiving payment until the products are used or sold. This arrangement can offer several benefits to both the supplier and the buyer, including reduced holding costs, improved cash flow, and increased availability of products. Their research also discussed the different stocking policies that can be employed in consignment agreements, such as the forward stocking policy, which involves pushing inventory forward to the buyer's warehouse as soon as possible, and the backward stocking policy, which involves replenishing inventory based on actual usage or sales. However, the choice of stocking policy depends on various factors, such as the cost structure, demand patterns, and supply chain characteristics. The objective of the models was to maximise the total profit for the coordinated system. Lee et al. (2017)'s research demonstrated that, for any stock-dependent demand, a minimum restocking level on the buyer's sales floor is a more profitable strategy than the traditional run-out replenishment policy. It further demonstrated that when the unit inventory holding cost decreases as the stock moves downstream in the supply chain, the vendor should adopt a forward stocking policy. Pando et al. (2018) analysed an inventory model for deteriorating items with a constant rate of deterioration and stock-dependent demand rate. The cumulative holding cost for items held in stock is defined by a nonlinear function in both the time and stock level. The objective of the model is to maximise the total profit per unit of time. To obtain an approximate optimal solution, the author presented a numerical algorithm. Comparisons are made between the proposed model and models without deterioration. San-José et al. (2018) presented an inventory model for items with demand that depends on price and time. The demand rate is assumed to be a combination of a time-power function and a price-logit function. Mishra et al. (2019) formulated an inventory model for deteriorating items, taking into account a hybrid price-stock-dependent demand. The model considers the trade credit policy and applies the discounted cash flow approach specifically for re-manufactured products. The study aims to identify the optimal cycle time, selling price, and present profit of future cash flows through the application of mathematical modelling, theoretical analysis, case examples, sensitivity analysis, and managerial implications. Chen et al. (2019) presented an inventory model of short life cycle products that deteriorate over time. The study focuses on a finite horizon multi-period setting with deterministic, stock-level-dependent, time-varying, and price-dependent demand to maximise the average profit per period by determining the optimal replenishment and pricing strategy. Their research demonstrates the importance of advertisement in increasing the demand for products and the significance of managing inventory about the expiration date of perishable items. It also highlights the role of advance payment in the relationship between suppliers and retailers. Halim et al. (2021) discussed an EPQ model with a nonlinear price structure and stock-dependent demand while also accounting for the possibility of overtime production opportunities. Cárdenas-Barrón et al. (2020) proposed an economic order quantity (EOQ) model that considers both nonlinear stock-dependent demand and nonlinear holding cost. Mallick et al. (2023) considered the concept of stock-dependent demand within an inventory control system for deteriorating items. Additionally, the model takes into account the increase in market price over time, as well as the inflation in the model. The analysis of the proposed model is conducted within a finite time horizon. The optimal profit is calculated based on the optimal replenishment period for each cycle. As the parameters associated with the system may not always be deterministic, the model introduced a degree of uncertainty. To address this uncertainty, the model is extended to a fuzzy model by considering fuzzy membership functions. The fuzzy model is then defuzzified using the Center of Gravity (COG) method and the fuzzy extension principle. Akhtar et al. (2023) presented an inventory model with price-dependent demand that maximises the retailer's total profit over a finite time horizon.

As seen from the above review of the various models, in real-life scenarios, demand functions are not always constant, linear, nonlinear, price, or time-dependent. Instead, the demand rate for items can fluctuate in the market due to the stock level or the combination of the stock level with a different factor that could influence the demand function.

2.6.6 Inventory models with imperfect quality

Another factor that needs to be considered when developing EOQ/EPQ models is the quality of products. In procurement, imperfect products can be attributed to various stages. One common scenario arises when suppliers are not adequately assessed, resulting in the selection of a supplier incapable of fulfilling the required quantity or quality standards. Moreover, unclear specifications and requirements may also result in items that do not meet their intended purpose or desired level of quality, further exacerbating issues with inventory management. During manufacturing, defects can occur for various reasons, such as equipment malfunction, human error, or external factors, and they can lead to a decrease in the product's quality. When defects occur, the production process may need to be slowed down or stopped to identify and rectify the issue. This can result in additional costs, such as setup costs, inspection costs, and downtime costs, which can impact the overall production. Moreover, defects can lead to increased customer complaints, product returns, and decreased customer loyalty, which can have a negative impact on the company's reputation. Several studies have investigated the effect of declining production rates and defects on inventory management.

Item quality was incorporated into inventory management research by Salameh and Jaber (2000). Unlike previous models, which suggested that no defective items are produced, Salameh and Jaber (2000)'s research considered defective items. To ensure quality control, a 100% screening process is conducted at a rate of x units per unit of time. Both categories, consisting of good and poor-quality items, are subsequently put up for sale. The good quality items are sold continuously throughout the inventory replenishment cycle, while the poor quality items are salvaged. The optimal solution for this model was obtained through mathematical equations derived from closed-form expressions.

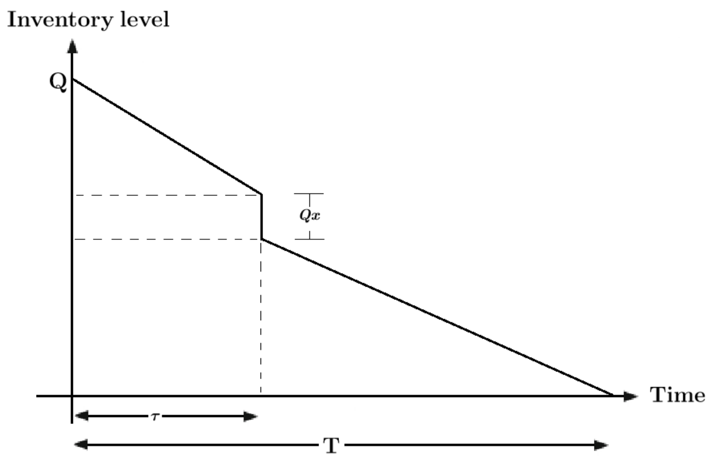


Figure 2.7: Inventory system behaviour for items with imperfect quality

The study proposed by Salameh and Jaber (2000) and depicted in Figure 2.7 incorporates the following key elements:

- The model considers a scenario where a lot size of Y items is delivered instantaneously with a purchasing price of p_c per unit and an ordering cost of OC_r.

- Not all of the items received in each lot are of good quality. To ensure quality control, a 100% screening process is conducted at a rate of x units per unit of time, which separates the good quality items from those of poor quality.
- The lot contains a certain percentage of defective items, denoted as q, which follows a known probability density function, $f(p)$.
- The screening process occurs for the duration t.
- Good quality items are sold for s_p per unit and are demanded at an annual rate of D. Poor quality items are sold as a single batch at a price s_d per unit, which is lower than the price charged for good quality items at the end of the screening period.

2.6.6.1 Extensions made to the classic model for items with imperfect quality

Goyal and Cardenas-Barron (2002) proposed a simplified approach to determine the optimal solution for the inventory model introduced by Salameh and Jaber (2000). They achieved this by modifying the expression for the expected value of the total profit cycle. Unlike Salameh and Jaber, Goyal and Cardenas-Barron separately determined the expected values of revenue and total cost. This approach required less computational effort compared to Salameh and Jaber's method. Huang (2002) extended the model introduced by Salameh and Jaber (2000) by incorporating a vendor-buyer cooperative supply chain relationship. Unlike Salameh and Jaber, who focused on optimising the buyer's costs, Huang aimed to optimise the total costs incurred by both the buyer and the vendor. In this cooperative model, the vendor supplies items to the buyer, who screens them for quality before offering them for sale. The buyer sells only good quality items, while the vendor incurs a warranty cost for the fraction of poor quality items supplied to the buyer. Chan et al. (2003) proposed an inventory model that distinguishes between three groups of items based on their quality: good quality items, poor quality items, and defective items. The distinction between the three groups allowed for a more nuanced treatment of items based on their quality and potential for sale. It takes into account the possibility of selling poor quality items at a reduced price or after reworking them rather than treating them as a single category with no differentiation. This differentiation in the treatment of items enables the development of more accurate and comprehensive inventory management strategies. By considering the different categories separately, the model accounted for the costs and revenues associated with each group, leading to improved decision-making regarding pricing, rework, and rejection of items based on their quality characteristics. Chang (2004) incorporated fuzzy sets theory into the basic EOQ model with imperfect quality items. The primary aim was to identify the order quantity that would yield the maximum total profit, considering the presence of fuzzy inputs within the model. Two distinct scenarios were examined during the application of fuzzy theory. In the first scenario, the fraction of imperfect quality items was treated as a fuzzy variable. The second scenario extended the analysis by assuming fuzziness in both the fraction of imperfect quality items and the demand rate. By incorporating fuzzy sets theory into the EOQ model, Chang aimed to enhance the realism and adaptability of inventory management decisions, effectively addressing the inherent fuzziness and uncertainty associated with the input parameters. Wee et al. (2007) extended Salameh and Jaber (2000)'s model by developing an inventory model for items with imperfect quality items, where shortages were allowed and fully back-ordered. This adjustment implied the company did not incur a cost for lost sales, as all customers were willing to wait for the backordered stock. Jaber et al. (2008) expanded upon the imperfect quality EOQ by incorporating learning effects. The primary distinction between their model and Salameh and Jaber's (2000) lies in the assumption that the fraction of imperfect quality items decreases according to a learning curve. To account for the learning effect, Jaber et al. (2008) assumed that the fraction of imperfect quality items could be represented by an S-shaped logistic learning curve. Glock et al. (2012) considered the impact of sustainability on the production process and provided insights into how manufacturers can balance economic goals with environmental responsibility. They focused on a manufacturer producing a single product in a market where demand is influenced by both price and quality and where the production process is assumed to have an environmental impact, such as emissions or waste or the consumption of non-renewable resources.

Many traditional inventory models assume that warehouses have unlimited capacity. However, in real-world scenarios, retailers may be motivated by factors like temporary price discounts to purchase quantities of goods that surpass their warehouse's limitations. Consequently, these retailers might find it necessary to rent extra warehouses to meet their business requirements. Chung et al. (2009) relaxed this assumption by considering an inventory model that incorporates both imperfect quality and two warehouses to address the practical challenges associated with imperfect quality and the need for additional warehousing capacity. The proposed model and solution approach provides decision-makers with a framework to optimise their inventory management strategies. Chen and Kang (2010) studied a vendor-buyer inventory system for items with imperfect quality. In addition, they assumed that the vendor grants the buyer trade credit financing by allowing the buyer to receive stock and only pay for it at a later stage. The buyer incurs an interest charge for this transaction, which is paid to the vendor. The objective function in their model was the total system costs for both the buyer and the vendor. Khan et al. (2010) expanded upon the model presented in Salameh and Jaber (2000) by introducing two significant modifications. Firstly, they introduced the assumption that the rate of screening for defective items follows a learning curve. Secondly, they incorporated the transfer of knowledge during the learning process when screening transitions from one cycle to the next. This transfer of learning was considered in three scenarios: (i) no transfer of learning, (ii) complete transfer of learning, and (iii) partial transfer of learning. Wahab et al. (2011) extended Salameh and Jaber (2000)'s model to a vendor-buyer supply chain model. While previous models had touched upon similar concepts, this study approached it from three realistic perspectives. Firstly, they examined the situation where both the buyer and vendor operate within the same country - an assumption implicitly made in earlier inventory models for items with imperfect quality. Secondly, they explored a scenario where the buyer and vendor are located in different countries while considering stochastic exchange rates between them. Lastly, they investigated how carbon emission costs impact production and logistics activities involved in fulfilling orders when there is a geographical separation between vendors and buyers across borders. Sana (2011) proposed an integrated production-inventory model for a three-level supply chain involving suppliers, manufacturers, and retailers. The model considers both perfect and imperfect quality items and considers various business strategies, such as optimising the order size of raw materials, production rate, unit production cost, and idle times in different sectors to examine the impact of the collaborative marketing system. Yassine et al. (2012) proposed a modified version of Salameh and Jaber (2000)'s model to address the issue of shipping poor-quality items by exploring two scenarios: aggregation and desegregation. In the aggregation scenario, multiple production runs are combined to form a single batch for shipment. On the other hand, in the desegregation scenario, poor-quality items are assumed to be sold separately during each production cycle. Singh et al. (2013) proposed an inventory model incorporating two distinct warehouses - an owned warehouse/showroom and a rented warehouse. The proposed model also considers the realistic assumption of limited storage capacity in the rented warehouse. Defective units are accounted for as a natural occurrence in production processes, representing imperfect quality production. Additionally, inflation is considered within this model. A solution method is provided to determine the optimal replenishment cycle, production cost, inspection cost, damaged item cost and preservation technology cost to minimise total costs per unit of time. Hsu and Hsu (2013) proposed an inventory model for items with imperfect quality items and sales returns. They also considered errors in the screening process as a factor that increased the likelihood of customers returning items. When a sales return occurs, the retailer incurs additional costs, including item costs, customer refunds, and reverse logistics expenses. Modak et al. (2015) incorporated one of the recent trends in inventory management for items with imperfect quality, namely preventive maintenance. Their model considered a just-in-time manufacturing environment that produces both perfect and imperfect quality items, regardless of the preventive maintenance nature. Preventive maintenance is an integral part of the production process, requiring periodic shutdowns to improve the condition of the production unit to an acceptable level. During these shutdowns, just-in-time buffers for both perfect and imperfect quality items are considered for maintaining normal operations. The duration of preventive maintenance is stochastic, and it is influenced by the production unit's condition. The percentage of imperfect quality items is also subject to randomness. The study aims to determine the optimal just-in-time buffer to minimise the system's running cost, accounting for holding costs of perfect and imperfect quality items, as well as shortage costs for both. Khan et al. (2016) introduced the concept of Vendor-Managed Inventory (VMI) to the inventory model for items with imperfect quality. In this vendor-buyer inventory system, the vendor supplies the buyer with items, not all of which are of good quality. In a VMI agreement, the vendor retains ownership of the stock, which is stored at the buyer's warehouse or store. However, the responsibility for managing and controlling the stock lies with the buyer. The motivation behind their model stemmed from the observed increase in the number of manufacturers and retailers opting for VMI agreements. These agreements enable closer collaboration and coordination between vendors and buyers, resulting in improved inventory management, better stock availability, and reduced stockouts. Nobil and Taleizadeh (2016) constructed a multi-product Economic Production Quantity (EPQ) inventory model for a defective production system on a single machine with a rework process and auction. their focus lay on addressing issues relating to reworking faulty products and determining the optimal cycle length and percentage of reworking every faulty product that would ultimately lead to minimizing total inventory cost. Al-Salamah (2016) proposed an EPQ model that addresses imperfections in both the production process and inspection. The model focuses on determining the optimal lot size for a manufacturer producing items in batches, where batches undergo either destructive or non-destructive acceptance sampling before reaching the market. The model takes into account potential errors in testing, including Type 1 and Type 2 errors. If a lot is rejected, it undergoes a more expensive non-destructive screening stage to categorise items as non-defective, reworkable, or salvage. The expected net profit function considers various components such as primary and secondary market sales, salvage item sales, setup and variable production costs, return cost, rework cost, screening cost, destructive cost, work-in-process, sales items inventory, rework item inventory, and salvage inventory. Kim et al. (2018) explored an improved approach to calculating imperfect items within an integrated inventory model. The model employs a distribution-free approach for lead time demand, aiming to optimise lot size, safety factor, number of shipments, and lead time simultaneously. The backorder rate is influenced by the reduced lead time. In their model, the quality of products is taken into account, and each item is inspected by the buyer. Defective items are returned to the supplier during the delivery of the next lot. Additionally, certain investments are made to enhance product quality and reduce setup costs. De et al. (2018) presented an EPQ model for items with imperfect quality. In the proposed model, the imperfect items that can be reworked are sent back for reprocessing, aiming to improve their quality. On the other hand, the remaining imperfect items that cannot be reworked are sold at a discounted price to recover some value. Furthermore, the model incorporates environmental regulations, and a carbon tax is introduced, assuming that the manufacturer incurs this tax if its production processes generate a specified quantity of carbon emissions. Sebatjane and Adetunji (2019) proposed an inventory system for the growing items. An assumption is made about a fraction of these items being of lower quality than desired. The process involves ordering live newborn items, feeding them until they reach a customer-preferred weight, and then slaughtering them. The slaughtered items undergo screening to separate good quality ones from those of poorer quality before being put on sale. The model aimed to maximise the expected total profit. Guha and Bose (2020) reviewed the article by Al-Salamah (2016). Two major modifications are introduced to address inconsistencies in the inventory computations presented in Al-Salamah's work. First, Guha and Bose's model accounts for defective items sent to the primary market. Second, the model assumes that, for non-destructive testing, the expected proportion of defective items in accepted and rejected lots should differ from that of the production process, a consideration not included in the original article. Ahmed et al. (2021) proposed a new inventory model that integrates partial backordering and multi-delay-in-payments. The model considers the presence of defective items and the possibility that repaired batches may still contain defects. Due to imperfections in the manufacturing system of a global supplier, imperfect items are produced. However, instead of resorting to costly reverse logistics for replacement from a distant manufacturer, the purchaser can utilise local repair stores for minor reworking of these valuable defective goods. Asadkhani et al. (2021) developed an EOQ model with different types of imperfect items, such as salvage, scrap, and reject. Tiwari et al. (2022) examined how managing imperfect quality items, deterioration, and trade credit policies can impact inventory control. A two-warehouse model is developed to analyse deterioration in quality and the use of two-level trade credit. The researchers determine the optimal lot size that maximises total profit per cycle through analytic calculations, utilising differential calculus methods to find the best solution. The results highlight the potential cost savings in the supply chain when combining trade credit policies with deteriorating imperfect quality items.

2.6.7 Inventory models with freshness condition and expiration dates

The freshness condition of products is another factor that significantly impacts the economic prospects of manufacturers and retailers. The freshness of products, such as meat, tomatoes, or vegetables, is crucial in determining their quality and safety for consumption. Consumers tend to prefer fresh products due to their better taste, texture, and nutritional value. People typically do not buy products when they are close to their expiry date for several reasons. Firstly, expired products are potential health hazards due to the growth of harmful bacteria or the breakdown of active ingredients. This can be particularly concerning in the food industry. Secondly, some expired products like pharmaceutical products may lose their effectiveness or become unsafe to consume, leading to health risks and waste of money. Thirdly, the perception of expiring products being of lower quality or value may also deter consumers from purchasing them. This can be particularly true for products that have a short shelf-life. As a result, manufacturers and retailers may be forced to discount or dispose of them, leading to financial losses and waste. Food and pharmaceutical safety regulations are in place to ensure that products are safe for consumption, and freshness is a critical factor in determining the safety of the products consumed. Fujiwara and Perera (1993) were the first to investigate the impact of utility deterioration, specifically declining freshness, on inventory management for perishable products. They used an exponential penalty cost function to model the deterioration. Cardello and Schutz (2003) analyzed the various factors associated with the freshness condition of food products and their significance in the food industry. Bai and Kendall (2008) presented a model that manages a deteriorating inventory and shelf space of fresh produce in a single period, assuming that the demand rate is deterministic and dependent on the level of inventory displayed and the freshness condition of the item. Piramuthu and Zhou (2013) extended Bai and Kendall's (2008) model for perishable inventory by linking the demand directly to the amount of shelf space allotted to the specific item and its current quality, using auto-ID technology like Radio Frequency Identification (RFID), which includes the necessary sensors that generate information on an item. Wu et al. (2016) investigated an economic order quantity (EOQ) model that takes into account dynamic timing, expiration date, and inventory volume as factors that influence demand rate. Dobson et al. (2017) investigated the inventory management decisions of a retailer selling a single perishable good in a deterministic environment. The model considers consumers' assessment of product quality over time, with the demand rate being influenced by factors such as freshness, expiration date, and stock level. The traditional assumption of zero-ending inventory is challenged since both freshness and stock levels have an impact on demand. Feng et al. (2017) investigated a model for managing inventory of perishable products in which demand is expressed as a multivariate function of price, freshness, and displayed stocks. It is assumed that the age of the product reduces on-hand stocks and decreases the demand rate. In addition, the product becomes unsuitable for consumption or sale after its expiration date. The selling price is considered an endogenous decision variable, impacting both the demand rate and total profit. The model allows for a closeout sale at a markdown price and keeps displayed stocks fresh and plentiful since the demand is freshness-and-stock dependent. This approach relaxes the traditional assumption of zero-ending inventories to non-zero-ending inventories. Banerjee and Agrawal (2017) developed an inventory model that considers a general demand function and general deterioration distribution. The model takes into account the deterioration of an item and its freshness condition. Li and Teng (2018) developed a model that integrates pricing and lot-sizing decisions for retailers selling perishable products. The demand for these products is influenced by factors such as the selling price, reference price, product freshness (as determined by the expiration date), and displayed stock level. They proposed a deterministic model wherein the retailer determines optimal selling prices and ending inventory levels to maximise discounted total profit. Additionally, they explored two distinct scenarios of demand behaviour: loss neutrality and loss aversion. Jansen et al. (2018) studied an inventory model that takes into account both age and closing days constraints. The proposed stochastic multi-item inventory model considers various factors such as total stock capacity limits, positive lead time, periodic inventory control, target customer service level, and mixed FIFO and LIFO issuing policies for perishable items with fixed lifetimes. They also assessed the impact of the closing days constraint using mathematical inventory models for perishable goods and focused on waste reduction and total costs. The methodology used involves developing a new inventory model with a closing days constraint for perishable items. Through a comparative simulation analysis using rolling planning data, they demonstrated how incorporating the closing days constraint can optimize order decisions while minimising waste quantities and costs in grocery stores. Khan et al. (2023) examined two inventory models with shortages to address the challenges for perishable products such as seasonal fruits, meat, poultry, eggs, dairy products, juices, jams, jellies, and other similar items. To better represent the demand for perishable items, a function is used that considers both linear price dependence and non-linear advertisement dependence. Additionally, the rate at which the products deteriorate over time is influenced by their expiration date. Two critical scenarios were highlighted in this research: models without shortages, which assumed sufficient stock levels to consistently meet customer demands while minimising the risk of running out of stock, and models with shortages, which considered the potential risks associated with unexpected factors causing stock-outs. This study aimed to develop effective inventory management for businesses dealing with perishable goods, where optimising pricing, advertising, and stock management strategies becomes crucial.

2.6.8 Lot-sizing models with corporate social responsibility

Customers' demands can also be influenced by external factors such as natural disasters, pandemics, corporate events, international politics and other unexpected events. These events can cause sudden changes in demand, which can be difficult to predict and manage. The current era of globalisation and technological progress has made it challenging for businesses to establish or uphold a competitive edge due to heightened competition. Currently, firms are increasingly recognising their responsibilities towards promoting societal welfare, commonly known as corporate social responsibility (CSR). This refers to the voluntary inclusion of social, environmental, and health-related objectives in their business practices and strategies (Sharma and Kiran, 2013). The strategic and operational implications of CSR in supply chain management have been explored by numerous researchers. Panda et al. (2017) investigated the impacts of CSR on a socially responsible manufacturer-retailer closed-loop supply chain (CLSC) with a focus on two aspects: profit maximisation and social responsibility via product recycling where the manufacturer demonstrates its social responsibility by recycling used products collected from the retailer through the reverse channel. This study focuses on a manufacturer-retailer supply chain scenario where the manufacturer takes into account stakeholder well-being through

CSR activities alongside profit generation. In addition to manufacturing new products, the manufacturer also collects used items through reverse channels, recycles them into new products, and sells these products through forward channels. It is assumed that the manufacturer demonstrates corporate social responsibility by engaging in product recycling, which showcases a higher level of environmental consciousness to stakeholders and shareholders. The profit generated by the socially responsible manufacturer encompasses both pure profits from selling newly manufactured products, as well as benefits resulting from CSR initiatives, including consumer surplus derived from stakeholders and profits stemming from recycled products. Additionally, Panda et al.(2017) explored how revenue-sharing mechanisms and Nash bargaining strategies could be utilized to resolve channel conflicts and distribute surplus profits while also examining the impact of CSR activities. The authors addressed five key inquiries, including how CSR efforts are linked with product recycling, a feasibility assessment for manufacturers to maximise overall welfare through environment-friendly recycling practices, prioritisation of social responsibility to motivate a retailer to fully engage in collecting and recycling used products, in other words, is it feasible for a socially responsible manufacturer to ensure maximum effort in recycling through the retailer effective coordination of such a supply chain using any type of coordination contract. Lastly, investigate the impact of CSR and, subsequently, recycling on the wholesale price and profitability of the manufacturer. Raj et al. (2018) analysed five types of contracts related to the coordination of the supply chain by assuming that the supplier is putting in greening effort and the buyer is putting in CSR efforts. Raza (2018) proposed supply chain coordination schemes for a single manufacturer-retailer supply chain. These schemes involve making decisions regarding pricing, inventory management, and investments in CSR initiatives, considering the uncertain nature of demand that is influenced by both price and CSR investment factors. Modak et al. (2019) considered the impact of selling price and social work donation (SWD) on demand and provided optimal closed-form solutions for three decentralised and a centralised channel structure in a sustainable supply chain model that incorporates CSR activities. Seyedhosseini et al. (2019) examined the impact of a manufacturer's CSR effort on customer price sensitivity in a two-echelon competitive supply chain consisting of a monopolistic manufacturer and two duopolistic retailers. Modak et al. (2020) presented a comprehensive framework and identified future research directions for CSR in supply chain management. Su et al. (2021) examined the impact of two commonly practised corporate social responsibility activities, namely social donations and green industrial development, on a multi-component assembly production system with imperfect processes. The analysis assumes that demand is influenced by both selling price and CSR effectiveness. They formulated the problem as an EPQ model aiming to maximise profits. The decision variables include the duration of each process during production runs, expenses allocated for social donations, and investments in green industrial development. Their findings demonstrated the existence of an optimal solution, which was also proven to be unique. Furthermore, they presented an algorithm for computing the optimal solution. To validate their proposed model's effectiveness, they conducted numerical simulations using a footwear original brand manufacturing company based in Taiwan as a case study. Raj et al. (2021) considered the impact of greening and CSR on suppliers and buyers and compared the full and asymmetric information on different decision alternatives. Four distinct decision alternatives are considered, each reflecting a combination of greening and CSR efforts by the supplier and buyer. These four scenarios include:

- A scenario where the supplier emphasises greening and corporate social responsibility while the buyer is solely focused on maximising profits;

- A scenario in which the supplier prioritises greening efforts while the buyer focuses on CSR endeavours;
- A situation where it is the supplier who emphasises CSR activities while the buyer concentrates on greening initiatives;
- A case wherein both buyer and supplier parties are equally committed to combining their respective greening and CSR efforts with profit maximisation.

After examining various models under different conditions, they then analysed the centralised and decentralised cases with wholesale price contracts and linear two-part tariff contracts and then developed a cut-off policy and evaluated the value of information in situations involving asymmetric information.

It is essential to recognise that the impact of CSR on economic performance typically manifests as gradual improvements, demonstrated through changes in purchase intentions, increased sales figures, enhanced corporate perception, and boosted employee morale. To illustrate this point further, many companies commonly engage in charitable donations to promote sales or strengthen their brand reputation. This approach often involves allocating a portion of the revenue from each purchase to charitable organisations such as foundations supporting children and families, social welfare initiatives, rural education programs, and rare disorders. To enhance customers' intention to purchase, companies can promote certificates of donation obtained from various organisations. Additionally, companies commonly engage in social donations for disadvantaged groups. For instance, these groups may sell products provided by the companies at lower cost prices as a means to support themselves financially (Su et al., 2021).

2.6.9 Inventory models with shifting production rate

The production process is the backbone of any manufacturing industry. This involves the conversion of raw materials into finished products. However, assumptions of constant smooth production may not always hold. Industry 4.0 technologies have recently transformed the manufacturing domain, necessitating a shift in current production capabilities to meet customised client requirements and enable personalised production on a global scale. The requirement for shorter product life cycles, higher production rates, increased job complexity, superior product quality, and cost-effectiveness have become critical for the industry (Lenz et al., 2020). To address these challenges, manufacturing processes must be highly available, flexible, configurable, and accessible (Ashraf and Hasan, 2018). This transformation is crucial, considering the manufacturing process's significant role in shaping the production-inventory system's overall efficiency and effectiveness. Many production systems operate in a multi-state mode, where different machines or processes can function at varying rates. This flexibility allows for adjustments in production levels, optimising resource utilisation, and enhancing responsiveness to changing market conditions. Scaling production becomes essential for manufacturing plants, particularly in response to fluctuations in product demand and seasonal variations. Aligning production output with market requirements helps prevent negative consequences, such as high holding costs due to rapid inventory buildup.

Moreover, scaling production enables businesses to flexibly allocate resources, ensuring efficient utilisation of operational expenses, including labour, energy, and raw materials. Power consumption also influences production scale, with companies adjusting rates in response to power shortages or limited availability. Quality control and process improvements further motivate the need for scaling production and aligning manufacturing processes with evolving industry standards. In essence, the manufacturing industry faces a dynamic landscape, requiring not only technological advancements but also strategic adaptations in production capabilities to meet the challenges and opportunities presented by Industry 4.0. Gupta and Arora (2010) examined a production inventory system that considers alternating production rates caused by market fluctuations and other relevant constraints to satisfy various demand patterns. Bhowmick and Samanta (2011) presented a continuous production control inventory model with variable production rates, allowing for a switch between production rates during the production process to maintain a certain level of manufactured items at the initial stages. Sivashankari and Panayappan (2015) considered a production inventory model that considers deteriorating items and utilizes two different production rates to avoid excessive inventory and enhance consumer satisfaction while maximising profit. Mishra (2018) introduced a three-rate production-inventory model that considers deteriorating products with a demand rate that is dependent on advertising cost and selling price. The production rates are assumed to be finite and proportional to the demand rate. To address production challenges similar to those experienced during the Covid-19 pandemic, Malumfashi et al. (2022) proposed a model for delayed deteriorating items having a two-phase production period and a holding cost function that is linearly increasing with time.

While fluctuating demand, seasonality, or power consumption are common reasons companies adjust production output, another critical factor influencing the decision to shift between production levels is the state of the manufacturing equipment, particularly for systems with degrading or deteriorating machines. Even with stable demand and consistent power supply, issues can arise with equipment during production, leading to process deterioration. Such deterioration poses a major challenge, causing production delays, downtime, missed deadlines, decreased product quality, increased waste, lost business opportunities, higher costs, and decreased efficiency. Researchers have delved into the impacts of manufacturing efficiency, reliability, process availability, and preventive maintenance in recent years (Rahim and Ben-Daya, 2001, Llaurens, 2011). The age of machinery is a significant factor contributing to system degradation and a decline in production efficiency. Ageing machines may become less efficient, with increased maintenance and repair costs, resulting in longer production times, higher downtime, reduced product quality, and elevated production costs per unit. Another factor influencing production decline is the usability of machines. The rate of degradation plays a crucial role in the lifespan of manufacturing systems, impacting the severity of failures and the quality of manufactured parts (Rivera-Gómez et al., 2018, Samala et al., 2021). In manufacturing industries, various factors, including flexibility, machine availability, and equipment state, have gained attention. The dynamic nature of equipment states is a crucial consideration, with a growing literature acknowledging situations that may arise during the lifetime of a production system. Systems may be "in a state of control" or "out of control". While there are tools such as predictive maintenance techniques that can help identify potential issues before they occur, there is no foolproof method to predict when a breakdown will happen. When a system is designed to automatically reconfigure at the occurrence of a failure, enabling the degraded or deteriorated machine to function with reduced service delivery, it is termed a multi-state system (MSS). This introduces a third state, known as the "degraded state", alongside the two previous states, providing a nuanced perspective on the complexity of manufacturing processes. When machines are not functioning optimally, it may become necessary to shift to a lower production rate and avoid further damage or unplanned repairs. By shifting to lower production rates, manufacturing plants can dedicate more attention and resources to identifying and rectifying any issues that may arise within their production lines, thereby ensuring efficient use of resources. The concept of flexible manufacturing systems with unreliable machines, failures and repairs presented in the form of a Markov Process was first introduced by Rishel (1975) and then Olsder and Suri (1980). Schweitzer and Seidmann (1991) introduced the concept of flexibility to discuss the optimisation of the processing rate for a flexible machine in a manufacturing system. Khouja and Mehrez (1994) presented an extension of the EPQ model that incorporates situations where the production rate becomes a decision variable and also accounts for degradation in the quality of the production process that occurs as the production rate increases. Khouja (2005) reformulated some inventory models to allow for adjustments (minor setups) to the manufacturing resource within the production cycle without interrupting the system. These adjustments do not involve performing all activities of a full set-up and incur only a fraction of a full set-up cost and time. Jaber (2006) extended the work of Khouja (2005) by assuming that the setup cost reduces over time because of learning effects and that the rate of defects reduces because the production process benefits from changes eliminating the defects, and thus reduces with every minor setup. Sana et al. (2007) studied a flexible economic manufacturing quantity (EMQ) model with a reduction in the selling price in an imperfect production system. Kenne and Nkeungoue (2008) introduced a homogenous Markov process utilising the hedging point policy for machines that undergo both failures and repairs caused by the age-dependent nature of the machine. Ben-Daya et al. (2008) proposed an EPQ model that accounts for the changes in production rate due to speed losses caused by the deterioration of the process and their impact on lot sizing decisions. Hajej et al. (2012) developed a joint optimisation approach to determine the optimal production plan and maintenance schedule, taking into account the degradation and availability of a single machine that produces a single product to meet a random demand within a finite production horizon. Golmakani and Moakedi (2012) investigated a multi-component repairable system with components that are susceptible to soft failures. These soft failures lead to a decrease in the system's performance and an increase in operating expenses. Sarkar et al. (2014) discussed an EMQ model of an imperfect production process with time-dependent demand subject to machine breakdown. Omar and Yeo (2014) proposed a model for a continuous time-varying inventory system that satisfies the demand for a finished product over a known and finite planning horizon by supplying both new and repaired items under multiple setups. Ben-Salem et al. (2015) addressed the issue of joint production, maintenance, and emission control for an unreliable manufacturing system that is susceptible to degradation, generating harmful emissions during its operations, and may be subject to environmental penalties under the emission cap approach to propose a feedback strategy that can control the production rate, emission rate, and maintenance rate concurrently, to counteract the effects of the system's degradation. Nobil et al. (2016) formulated a multi-machine multi-product economic production quantity (EPQ) problem for an imperfect manufacturing system problem with non-identical machines in a manufacturing system. Koutras et al. (2017) introduced a general model for multi-state deteriorating systems with condition-based preventive maintenance. The model also accounts for imperfect maintenance, including minor or major failures and sudden failures that may be caused by external factors at any stage of deterioration. Zhang et al. (2018) proposed the Wiener-process-based techniques to represent degradation for manufacturing systems. Nobil et al. (2019) investigated the effects of a machine warm-up period on an imperfect production process with the rework to maximise the efficiency and effectiveness of the production machine. Chiu et al. (2019) investigated the optimal runtime for a fabrication system with backordering, service level constraint, stochastic breakdown, and scrap. Ganesan and Uthayakumar (2020) proposed EPQ models for an imperfect manufacturing system considering warm-up production run, shortages during the hybrid maintenance period and partial backordering. Manna et al. (2020) examined a production process that is susceptible to deterioration over time, transitioning from a state of "in-control" to "out-of-control" to model an inventory system that takes into account inspection errors, time-dependent development costs, and price dependent demand. El Cadi et al. (2021) highlighted the challenges faced by researchers and managers in controlling joint production systems due to the interdependence between system states and control actions. The proposed model considers operation-dependent degradations of reliability and quality and employs a make-to-stock production strategy and an age-based preventive maintenance policy to cope with uncertainty. Hadian et al. (2021) discussed a stochastic model for joint planning of maintenance, production and quality control in manufacturing systems. The model considers the possibility of process deterioration, with state transition time following a general distribution.

Another factor that needs to be considered when developing EPQ models with shifting production rates is defects. Defects can occur during the manufacturing process due to various reasons, such as equipment malfunction, human error, or external factors, and they can lead to a decrease in the product's quality. When defects occur, the production process may need to be slowed down or stopped to identify and rectify the issue. This can result in additional costs, such as setup costs, inspection costs, and downtime costs, which can impact the overall production. Moreover, defects can lead to increased customer complaints, product returns, and decreased customer loyalty, which can have a negative impact on the company's reputation. Several studies have investigated the impact of declining production rates and defects on inventory management. Tai (2013) presented two models for economic production quantity (EPQ) for items that are imperfect and subject to deterioration, with a rework process to restore imperfect items. Cheng et al. (2018) developed a model that integrates production, quality control, and maintenance planning in deteriorating processes. To mitigate uncertainties, a safety stock was maintained to prevent stock-outs. The study implemented a 100% inspection policy for quality control and determined the production lot size based on the maintenance plan and quality control policy. Al-Salamah (2019) proposed inventory models for scheduling an imperfect manufacturing process where a certain proportion of items are defective. To accommodate the manufacturer's flexibility in selecting the rework rate and process, the author presented a flexible rework rate approach, where the rework rate can differ from the production rate, and the rework process can be either asynchronous or synchronous. Ye et al. (2020) presented a new competing failure model that investigates the interactions among machine failures, product quality, and inspection process, enabling the characterisation of time-delayed propagation of failure, accumulation of degradation, and dynamics of states in serial automated manufacturing systems (AMSs). Bose and Guha

(2021) discussed the impact of the manufacturer's choice between full inspection and sampling inspection on the economic production lot sizing decision for items with imperfect quality and inspection errors. They proposed an online sampling inspection model and argued that the choice between full inspection and sampling inspection depends on the threshold level of unit inspection cost. Ahmed et al. (2021) discussed the challenges faced in intercontinental trade and economics when a fraction of defective items is received from a global supplier by proposing an inventory model that considers reworking, multi-period delay-in-payments policy, and shortages to maximise profit. Tshinangi et al. (2022) examined a degrading production system for deteriorating items with shifting production rates, imperfect quality, and partial backlogging of demand with lost sales.

2.6.10 Multi-echelon inventory models

The concept of coordinating inventory decisions in a multi-echelon system, which is now commonly associated with supply chain management, can be traced back to the work of Clark and Scarf in 1960. Their model laid the foundation for studying coordination in inventory management. In a multi-echelon supply chain, each stage represents a distinct process involved in the overall flow of items, such as procurement, manufacturing, or transportation. Each stage has the potential to hold inventory or stock of the item being processed at that particular stage. To model a multi-echelon system, a network representation is commonly used. In this network, nodes represent the stages within the supply chain, while arcs depict the precedence relationship between these stages. Multi-echelon systems can be categorized into different types based on their network structures, such as serial (Fig.2.8a), assembly (Fig.2.8b), distribution (Fig.2.8c), general acyclic (Fig.2.8d), and general cyclic systems (Fig.2.8e).

The objective of inventory optimisation in multi-echelon systems is to minimise the overall inventory cost or maximise its overall profit while maintaining the desired service levels for the final customers. This involves determining the optimal inventory allocation across the various stages of the supply chain. Optimising inventory decisions across the multi-echelon supply chain is a challenging task due to the numerous interdependent decision variables and the presence of non-linear functions that govern service levels. The complexity of multi-echelon inventory optimisation is closely tied to the network structure, which is influenced by factors such as the number of stages and the topology of stage connections (Eruguz et al., 2016). Multi-echelon approaches are widely studied in the literature. Goyal (1977) expanded the Clark and Scarf's (1960) model to coordinate a vendor-buyer inventory system, where the vendor resells products to a buyer. The primary objective was to determine the economic lot size for both entities, denoted as the joint economic lot size (JELS), with the overarching goal of minimizing the collective total costs associated with this type of inventory. The formulation of the JELS problem involves a vendor and a buyer engaged in producing and selling a singular item type, aiming to identify the optimal inventory replenishment policy for both entities. Goyal's (1977) model was formulated under certain assumptions, such as an infinite production rate and a lot-for-lot production policy at the vendor. The shipment policy between the vendor and the buyer in Goyal's JELS model is often referred to as the single setup-single delivery (SSSD) policy. Under this policy, the vendor produces a single cycle or lot and delivers the entire lot to the buyer.

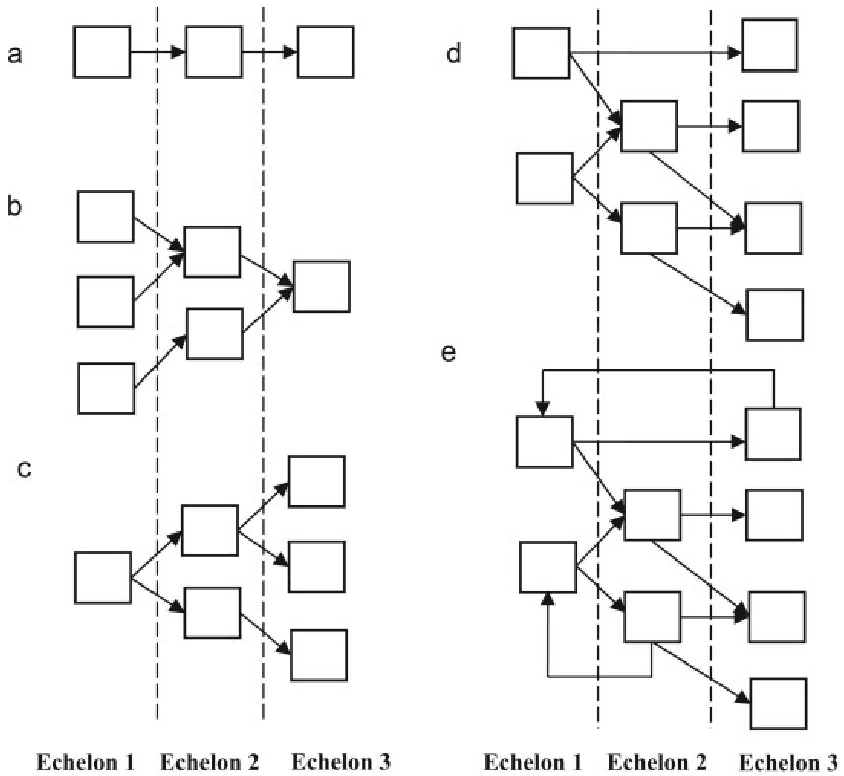


Figure 2.8: Network structures for multi-echelon supply chain systems by Eruguz et al. (2016).

Over the years, Goyal (1977)'s model has undergone several extensions, beginning with Banerjee (1986), who relaxed the assumption of an infinite production rate. Banerjee extended the model to a scenario where the vendor produces items at a specific (i.e., finite) production rate on a lot-for-lot basis. In this case, the vendor generates enough items to meet the demand for the designated period. Goyal (1988) introduced the Single Setup-Multiple Delivery (SSMD) variant of Goyal (1977)'s model, assuming that the vendor produces sufficient items to fulfil the buyer's orders with an integer number of deliveries for each production setup. The SSMD policy involves multiple deliveries from the vendor to the buyer for a single production setup. This policy resulted in lower total system costs, attributed to the smaller lot sizes, leading to reduced holding costs and faster consumption, thereby spending less time in storage. Braun et al. (2003) introduced a Model Predictive Control (MPC) as a strategic approach for multi-product, multi-echelon demand networks, particularly when confronted with uncertainties such as inaccurate estimations of production lead times and demand forecasts. Their primary emphasis lay on achieving robust supply chain management, where robustness would denote the capacity to adeptly handle uncertainties and variations within the system. Rau et al. (2003) developed a multi-echelon inventory model for a deteriorating item to derive an optimal joint total cost from an integrated perspective among the supplier, the producer, and the buyer. The model considered the single supplier, single producer and single buyer. El-Kassar et al. (2012) investigated an Economic Production Quantity model that considers the cost of raw materials essential for production. The model assumes that the supplier's raw materials contain a certain percentage of imperfect quality items. Upon receipt, a screening process with a 100% detection rate for imperfect items is carried out at the start of each inventory cycle. Two scenarios are explored: in one, imperfect items are sold at a discounted price after screening, and in the other scenario, imperfect items are kept in stock until the end of the cycle and then returned to the supplier upon receiving the next order. Zhou et al. (2013) extended Rau et al. (2003)'s model to accommodate multiple suppliers, one producer, and multiple distributors and buyers. Additionally, an algorithm designed by Genetic Algorithm (GA) is used to solve the model. Cárdenas-Barrón et al. (2014) examined the problem of channel coordination in a supply chain consisting of one manufacturer and one retailer. The demand for products in this supply chain is influenced by promotional activities and sales team initiatives. A production-inventory model is formulated, incorporating the procurement cost per unit as a function of the production rate. Different centralised coordinating systems are analysed to address the challenges of channel coordination and meeting demand through promotions. An analytical approach is used to determine optimal production rates, lot sizes, backlogging policies, and sales team initiatives that maximise profits for both the manufacturer and retailer. Priyan and Uthayakumar (2015) studied a two-echelon multi-product multi-constraint product returns inventory model with permissible delay in payments and variable lead time. This study examines a supply chain problem involving a distributor and warehouse. The supply chain includes both serviceable and recoverable parts, with multiple products involved. The distributor faces limitations in terms of space capacity and budget for purchasing all the products. Defective products are returned to the warehouse, where they are recovered into perfect products of equal value to the initially procured ones. The lead time for receiving products from the warehouse to the distributor is a controllable variable with an associated crashing cost. In this system, a portion of product shortages is back-ordered while the remainder is considered lost. To minimise overall costs, an optimisation model is utilised in this research to determine optimal order quantity, lead time adjustments, and total number of deliveries within the system. Ross et al. (2017) introduce a joint three-echelon location inventory model for an industry driven by donation demands. The model considers the presence of a main warehouse, distribution centres, retail stores, and donation-only centres. At each retail store, two classes of products are handled based on the difference between demands and donations received at that specific store from the assigned distribution centre. The proposed model provides a comprehensive solution to determine optimal decisions simultaneously, including the number of open distribution centres, their respective locations, and assignment plans for retailers regarding different types of products. The overall objective is to minimise total annual costs, which encompass facility location expenses along with transportation costs while accounting for inventory holding costs and potential revenue losses due to unmet customer demand. The complexity inherent in this problem led Ross et al. (2017) to explore relaxing constraints through recourse to Lagrangian relaxation. Sarkar et al. (2018) presented a model for a single-vendor multi-buyer supply chain, incorporating a variable production rate and accounting for imperfect product quality. The unit production cost is modelled as a function of the production rate, introducing three distinct production functions to capture the relationship between process quality and production rate. Given the substantial demand from multiple buyers, the lead time demand is treated as a random variable following a normal distribution. The study aims to analyze how the adaptability of the production rate influences both product quality and the overall cost of the supply chain, particularly under a single-setup multiple-delivery policy. The optimisation process employs classical techniques to attain the global optimum solution, and an illustrative algorithm is devised for numerical results. Panda et al. (2019) combined advertisement of the product, price, stock, and credit policy in a two-warehouse inventory model and represented it mathematically. In addition, they added a deteriorating factor to their proposed problem with price- and stock-dependent demand under partial backlogged shortage. Dye (2020) investigated a joint pricing, advertising, and inventory control problem for a firm that sells perishable products with a psychic stock effect to maximise the total profit over the infinite planning horizon. Halat et al. (2021) studied a carbon tax policy in inventory games of multi-echelon supply chains. They analysed four different decision-making structures within the supply chain, namely decentralised, vertical downward cooperation, vertical upward cooperation, and horizontal cooperation, with the main goal of identifying optimal solutions for inventory games under each cooperation scheme, comparing strategies, and assessing the impact of carbon taxes on costs, emissions, and savings resulting from collaboration. Closed-form equilibrium values are computed for various factors, including optimal replenishment cycles, costs, and carbon emissions, using an algebraic method. Other games were resolved using a precise solution approach. Furthermore, an analysis was conducted on how inventory and carbon emissions parameters affected both costs and carbon emissions reductions achieved through cooperative settings in supply chains. In a recent study, Yazdekhasti et al. (2022) introduced an Integer Non-Linear Programming model to optimise a warranty distribution network with a dihedral structure. The WDN consists of two echelons operated by a third-party entity: the first echelon includes a depot repair centre that repairs defective items using support from a high-capacity supplier and follows a continuous review inventory policy, while the second echelon comprises multiple customer support centres with continuous inventory control policy. This two-echelon structure is particularly significant in scenarios like auto parts inventory systems, where the first layer handles repair, replace, and return tasks, while the second layer manages replace and return duties. The paper employs an electric vehicle (EV) battery system as a case study to validate the proposed model. Döngül et al. (2022) presented an integrated location-allocation model that incorporates inventory control decisions in a multi-echelon and multi-period supply chain with uncertain demand. The model focuses on customer importance, as it distinguishes between customers based on their purchasing volumes and their significance to the supply chain. To solve this model, a novel meta-heuristic algorithm called Seeker Evolutionary Algorithm (SEA) is proposed. Comparative results against genetic algorithms and GAMS software show that SEA provides high-quality solutions with acceptable computational efficiency for both small and large-scale scenarios. Gioia and Minner (2023) modelled a network comprising an Online Fulfillment Center (OFC) and physical offline retailers for perishable. Gioia and Minner's research explores the complexity and structure of optimal policies, considering the challenges posed by perishable products and the uncertainties in demand and customer preferences. Additionally, they investigated different extensions to base-stock policies across multi-echelon networks while analysing the impact of potential correlations and imbalances in demand volumes between channels on heuristic approaches. The advantages and disadvantages of such solutions were also identified. To better understand the problem dynamics, they defined state variables along with decision variables or actions that would guide them towards achieving rewards using transition functions for effective decision-making processes within the network configuration where OFC is responsible for order sizes from suppliers as well as acting as a distribution centre for both online and offline channels. Kouki et al. (2023) explored a two-echelon inventory system consisting of a central warehouse and multiple local warehouses with lost sales. The demand at each local warehouse is modelled as a Poisson process, and the stock is controlled using a continuous review base-stock policy. Previous analyses of this type of system have focused on deterministic or exponentially distributed lead times at the central warehouse, deterministic lead times at local warehouses, and approximate performance evaluations. Kouki et al. (2023)'s research expands upon these analyses by considering generally distributed lead times at both the central and local warehouses. They then provided exact closed-form expressions for inventory performance measures in cases where demand is lost if no items are available within specified thresholds. Moreover, they proposed new approximate solutions for situations where waiting time thresholds are smaller than the local warehouse lead-time limits.

2.7 Chapter Summary

Numerous researchers have made significant contributions to the foundations of this thesis, as highlighted in the conducted literature review. The research focuses on exploring various aspects, such as lot-sizing models for deteriorating items, shifting production rates, multi-echelon inventory systems, imperfect quality, expiration dates, price-dependent demand, and inventory level-dependent demand. While the classic EOQ/EPQ model has played a crucial role in these models' development, there are still identified gaps that call for novel models tailored to address practical scenarios encountered in systems with flexible production capabilities.

Particular aspects within the reviewed literature, namely lot-sizing models for inventory with deterioration, lot-sizing models with shifting production rates and those applicable to items within multi-echelon inventory systems, stand out among the rest with ample opportunities for further development. By combining these aspects, there is a promising avenue for developing new models specifically designed for complex, flexible production systems. As such, the unifying theme among the three innovative models presented in this thesis is the consideration of deteriorating items and shifting production rates in inventory.

Other aspects of the literature have been considered in the development of these three novel models. These include lot-sizing models that account for price-dependent demand, imperfect quality, freshness condition, corporate social responsibility, and stock-dependent demand. By integrating these aspects into each model, realistic and practical solutions are formulated specifically for flexible production systems. This adaptation is necessary as pricing decisions, quality control measures, freshness conditions, and stock levels significantly impact how flexible production systems operate. For instance, implementing quality control ensures the integrity of products throughout different supply chain echelons during value-adding processes like processing and packaging. Additionally, managers in retail settings must monitor the expiration dates of products to minimise waste arising from the disposal of expired products that are unfit for sale. The selling price and displayed inventory level of products have been identified as critical factors influencing consumer demand, with demand generally increasing with decreasing prices and rising stock levels on shelves.

Chapter 3

Review of foundational models

3.1 Introduction

This chapter introduces the foundational models that form the basis for the models developed in the subsequent chapters. This review explores and analyses the essential models that have influenced this thesis. Four lot-sizing models previously published in the literature serve as the basis for the three new models introduced in this thesis. These are 1) the Economic production quantity model with a shifting production rate, by Ben-Daya et al.(2008); 2) the inventory model for deteriorating items with freshness and price-dependent demand, by Banerjee and Agrawal (2017); 3) Joint pricing and inventory decisions for perishable products with age, stock, and price-dependent demand rate by Agi and Soni(2020); and 4) EPQ model with imperfect quality raw material, by El-Kassar et al.(2012). Together, these form the foundation of all three models presented in this thesis. The concepts behind these four fundamental models are incorporated with the concepts of the other two foundational models (specifically, models with corporate social responsibility and models with imperfect quality) in formulating the three initial models introduced in this thesis. This chapter provides a concise overview of the mathematical principles governing these three fundamental models.

3.2 Notations

The following notations are used during the development of the models presented in this thesis:

A	Demand parameter
b	Elasticity parameter of the unit selling price
C_d	Deterioration cost per unit item
C_{d_p}	Disposal cost per unit item
C_p	Penalty cost per unit lost sale
c_r	Unit cost of raw material
$D(s_p, I(t)), D(I(t))$	Demand for the product
$d(s_p)$	Price component of the demand for the product
$d_{1,2}$	Proportion of defective units produced

$ETC(t)$	Expected total cost per unit time
F_r	raw material ordering cost
$G_{1,2}$	Set-up cost associated with stage i
h_p	Inventory carrying cost per item produced per time
h_r	Inventory carrying cost per unit of raw material per time
H_M	Hessian Matrix
$I(t)$	Instantaneous inventory level
k_1	Initial production rate at the start of the cycle
k_2	Production rate following the shift in production
ζ	Increase in unit machining cost due to increase in the production rate
p_{c_1}	Unit production cost at the start of production
p_{c_2}	Unit production cost after the machine's production rate has been scaled down
p_l	Lost production cost per end product.
p_r	Purchase cost of raw material per unit
p_c	Purchase price per unit product
Q	Production batch size
Q^*	Optimal batch size
q	Proportion of raw materials that are of imperfect quality
q	The inventory remaining at the end of the cycle
QD_p	Quantity of deteriorated products
ϕ	Per unit cost of running the machine independent of the production rate including labor and energy costs
S_C	Fixed setup cost
s_p	Market selling price of the product
s_d	Discounted unit selling price of imperfect finished products
s	Social donation amount per sale
s_r	Discounted unit selling price of imperfect raw material
y	Raw material order size per cycle
T	Cycle time
TC	Average total cost per cycle
TR	Average revenue per cycle
TP	Average profit per cycle
$t_{\text{i}\in\{1,2,3,4\}}$	Time duration of each phase of the cycle
t	Random time at which the process shifts from a higher to a lower production rate
t_s	Screening period
t_p	Production period
TSC	Total set-up cost
$\theta(t)$	Deterioration rate per unit per time
γ	Demand enhancement parameter for inventory level
$\rho_{1,2}$	Aggregation parameters for some known variables
β	Freshness parameter
α	Discount percentage offered on selling price
n	Shelf-life of the product
x	Screening rate for raw material
μ, η	Non-negative constants
$\rho_{1,2}, \delta, B, E$	Aggregation parameters for some known variables
σ	Parameter that reflects the impact of defects on customer demand
λ	Rate at which the proportion of defective items increases over time
η_m	Maximum proportion of defective products that can be produced

3.3 Primary models

3.3.1 Shifting production rates

Traditional manufacturing systems are based on the belief that the process runs perfectly with durable machinery and equipment, constantly producing high-quality items at a consistent rate. However, this isn't always the case, as facilities and equipment can deteriorate over time due to factors such as ageing, operational stress leading to deformation, regular wear and tear, and exposure to corrosive substances (such as chemicals or environmental elements). These forms of damages can result in reduced production capacity, plant shutdowns due to breakdowns, an increase in defective products being produced or a decrease in market value stemming from design or manufacturing flaws, which ultimately lead to lower quality of finished products. Over time, these damages can result in additional costs, whether direct or indirect. Some manufacturing systems assume that repairing a machine after a breakdown restores it to its original state. However, if this were always true, the systems could operate for an almost unlimited duration, which is nearly impossible. Therefore, it is important to consider the impact of deterioration on manufacturing systems and incorporate quality issues into the analysis.

There has been a growing interest in studying manufacturing systems with machines that may experience deterioration/degradation. Many other systems and their components undergo deterioration during their lifespan due to factors like erosion, vibration, wear, fatigue, or shock. Many production systems and their components are often analysed and represented using different states of deterioration, each reflecting the system or component's stage. In terms of production, a system that automatically undergoes reconfiguration upon failure to allow degraded equipment to remain functional with reduced service is known as a multi-state system (MSS) or degraded system. This introduces a third state known as the degraded state as previously represented states in section 2.6.9. The reliability and functioning of the system depend on the state of its degradation, with some components able to continue providing service at a decreased level of performance. Proper analysis of the degraded mode is often conducted to anticipate any serious failure of the MSS, which includes identifying the component in need of urgent restoration. The evaluation aims to find an optimal balance between different risks and acceptable losses. Once risks are identified, assessed, and prioritised, procedures for managing the degraded mode are put into place.

Shift in production rates in which the production process is restored to the original production rate only at the beginning of the next cycle was introduced by Ben-Daya et al.(2008) through the development of an EPQ model for a type of deterioration observed in machining systems such as turning and milling. For such systems, the production rate is a function of the speed rate, which in turn depends on the tool life. Consequently, any tool failure due to wear out will disrupt the machine operations, resulting in a gradual decrease in the production efficiency. When the production begins, the process starts with a production rate k_1 and after a time t, the production rate shifts to a lower production rate k_2 because of speed losses due to some process deterioration, such that $k_1 \geq k_2$. The new production rate, if it happens, is assumed to be known a priori from experience. The inventory holding cost per unit per time is h_p. The setup cost is G. Lost production during the lower production rate period is assumed to incur some penalty cost, which is represented by p_l. In this model, product quality is not affected by this shift in production rate. During the whole cycle T, the demand D is assumed to be fixed and known. The unit production cost is $\zeta k + \frac{\Phi}{k}$. Two distinct scenarios for the occurrence of the time of the shift can be identified. In scenario 1, the shift takes place during the production run, denoted by $t \in [0, t_\mathrm{p}]$. Meanwhile, in scenario 2, the shift occurs after the production run period, with $t \in [t_\mathrm{p}, \infty]$. The costs per cycle for each of these two scenarios are analysed. The inventory profile is shown in Figure 3.1.

- Scenario 1: The unit cost of production is a function that depends on the production rate k, and it is of the form $\zeta k + \frac{\phi}{k}$. This means the total production cost over the entire cycle is

$$\text{Production cost} = \left(\zeta k_1 + \frac{\phi}{k_1}\right) k_1 t + \left(\zeta k_2 + \frac{\phi}{k_2}\right) k_2 \left(t_\mathrm{p} - t\right) \tag{3.1}$$

The average holding cost per cycle is given by

$$HC = h_p \left[t^2 \left\{\frac{\left(k_1 - k_2\right)^2 - D\left(k_1 - k_2\right)}{2D}\right\} + t_\mathrm{p} t \left\{\frac{k_2\left(k_1 - k_2\right)}{D}\right\} + t_\mathrm{p}^2 \left\{\frac{k_2\left(k_2 - D\right)}{2D}\right\}\right] \tag{3.2}$$

In many practical scenarios, there is a penalty for failing to deliver the promised quantity on time. This can be expressed mathematically as $k_1 t_\mathrm{p} - [k_1 t + k_2 (t_\mathrm{p} - t)]$, or $p_l (k_1 - k_2)(t_\mathrm{p} - t)$. Adding a fixed setup cost, denoted as G per cycle, the total cost per cycle can be expressed as

$$\begin{aligned} TC_1 = & G + \left(\zeta k_1 + \frac{\phi}{k_1}\right) k_1 t + \left(\zeta k_2 + \frac{\phi}{k_2}\right) k_2 \left(t_\mathrm{p} - t\right) \\ & + h_p \left[t^2 \left\{\frac{\left(k_1 - k_2\right)^2 - D\left(k_1 - k_2\right)}{2D}\right\} + t_\mathrm{p} t \left\{\frac{k_2\left(k_1 - k_2\right)}{D}\right\}\right. \\ & \left. + t_\mathrm{p}^2 \left\{\frac{k_2\left(k_2 - D\right)}{2D}\right\}\right] + p_l \left(k_1 - k_2\right)\left(t_\mathrm{p} - t\right) \end{aligned} \tag{3.3}$$

The company's cycle length is

$$T\left(t_\mathrm{p}, t\right) = \frac{k_2 t_\mathrm{p} + \left(k_1 - k_2\right) t}{D} \tag{3.4}$$

- Scenario 2: The same method can be applied to calculate the overall cost per cycle when $t \geq t_\mathrm{p}$. In this scenario, there is no penalty for lost production, and the total cost per cycle is determined by

$$TC_2\left(t_\mathrm{p}\right) = G + t_\mathrm{p}\left(\zeta k_1^2 + \phi\right) + h_p \left\{\frac{k_1\left(k_1 - D\right) t_\mathrm{p}^2}{2D}\right\} \tag{3.5}$$

and the company's cycle length is given by

$$T\left(t_\mathrm{p}\right) = \frac{k_1}{D} t_\mathrm{p} \tag{3.6}$$

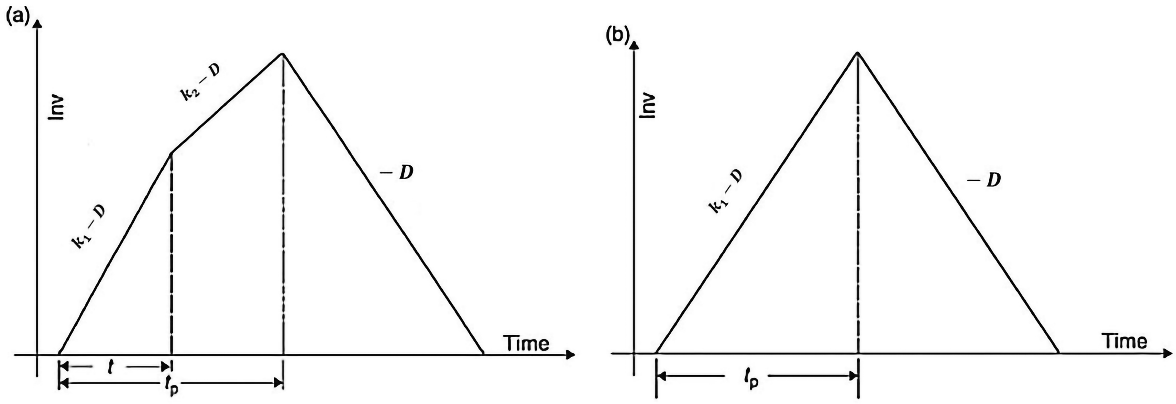


Figure 3.1: Inventory profile with a shift in production rate at time t :(a) $t \in [0, t_\mathrm{p}]$, (b) $t \in [t_\mathrm{p}, \infty]$

The total anticipated cost per cycle is calculated using the probability distribution of shift timings.

$$E\left[TC\left(t_\mathrm{p}\right)\right] = \int_0^{t_\mathrm{p}} TC_1\left(t_\mathrm{p}, t\right) f(t)\mathrm{d}t + \int_{t_\mathrm{p}}^{\infty} TC_2\left(t_\mathrm{p}\right) f(t)\mathrm{d}t \tag{3.7}$$

where $TC_1\left(t_\mathrm{p}, t\right)$ and $TC_2\left(t_\mathrm{p}\right)$ are given by Equations (3.3) and (3.5), respectively. Similarly, the anticipated duration of the cycle is expressed as

$$E\left[T\left(t_\mathrm{p}\right)\right] = \int_0^{t_\mathrm{p}} T\left(t_\mathrm{p}, t\right) f(t)\mathrm{d}t + \int_{t_\mathrm{p}}^{\infty} T\left(t_\mathrm{p}\right) f(t)\mathrm{d}t \tag{3.8}$$

Where $T\left(t_\mathrm{p}, t\right)$ and $T\left(t_\mathrm{p}\right)$ are given by Equations (3.4) and (3.6), respectively.

3.3.2 Multi-echelon inventory systems

The traditional economic production model has been expanded to address the complexities found in inventory management. El-Kassar et al.(2012) examined an EPQ model that accounts for the cost of raw materials needed for production. They introduced a model known as the EPQ model with imperfect quality raw material, which focuses on optimising the inventory replenishment policy in a two-level inventory system involving one supplier and one manufacturer. The supplier is tasked with providing raw materials to the manufacturer while the manufacturer produces the requested products needed for consumption. It is assumed that some of the raw materials obtained from the supplier are items of imperfect quality. Upon receiving the raw materials at the start of each inventory cycle, a thorough screening process is conducted to identify all imperfect quality items. Two different scenarios are considered: one where imperfect quality items are sold at a discounted price after screening, and another where they are kept in inventory until the end of the cycle and then returned to the supplier upon receipt of the new order.

The model starts with the first scenario. In this scenario, a product is being produced at a rate called k, which is greater than the demand rate D. As part of the production process, raw material is required, and a replenishment quantity of y is ordered and received at the start of each production cycle. It is also assumed that the received raw material includes a proportion q of imperfect quality items. The quantity of imperfect quality items within the received raw material is yq, while the remaining items, which are perfect quality, are $y(1-q)$ and are used in producing the finished products. Consequently, the production period is

$$t_p = \frac{y(1-q)}{k} \tag{3.9}$$

and the length of inventory cycle is

$$T = \frac{y(1-q)}{D} \tag{3.10}$$

At the start of the production cycle, there is a screening process with a screening rate x ($x > k$) to ensure all imperfect quality items are identified. The duration of the screening period is

$$t_s = \frac{y}{x} \tag{3.11}$$

During the screening period, only perfect raw materials are utilised in production. As a result, the inventory level of raw materials decreases at a rate k until the end of the screening period. The quantity of raw material remaining at the end of the screening process is

$$y - kt_s = y\left(1 - \frac{k}{x}\right) \tag{3.12}$$

Imperfect items are sold at a reduced price s_r, which is lower than the unit procurement cost of raw material c_r. The inventory level of raw material decreases from $y - Pt_s$ by a quantity qy. The overall remaining quantity of raw material items is $y\left(1 - \frac{k}{x} - q\right)$. The raw material inventory continues to decrease at a constant rate of k until it reaches zero by the end of the production period. The inventory profile of raw materials is illustrated in Figure 3.2.

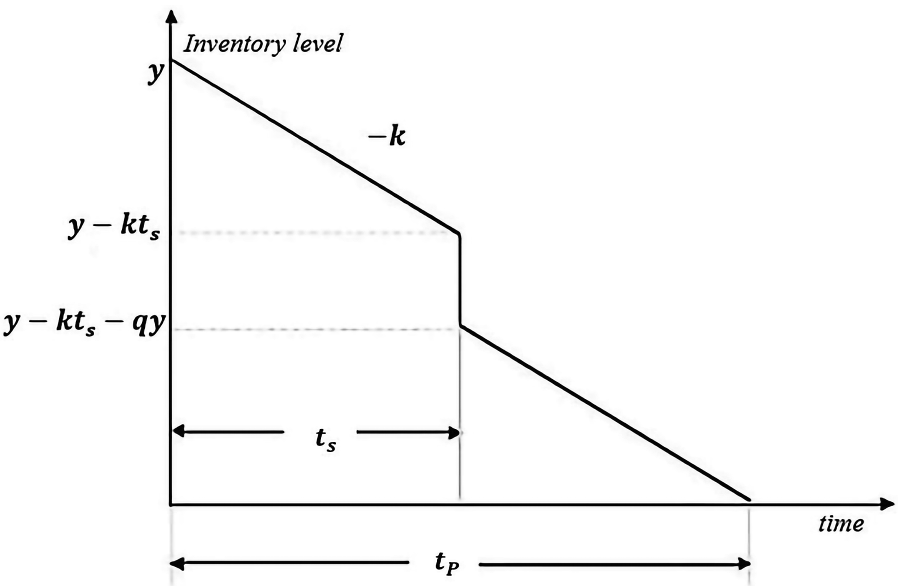


Figure 3.2: Raw material inventory level, imperfect items sold at a discount

In the context of the given scenario, the production of finished products occurs at a constant rate represented by k, while a portion of these products are sold at a rate of

D. Consequently, an inventory of finished products accumulates during the production period at a rate of $k-D$. The inventory level of the finished items is shown in Figure 3.3. As the production progresses, the inventory continues to accumulate until it reaches the maximum level $y_{\max}$. $y_{\max}$ is obtained by multiplying the rate at which inventory is built, $(k-D)$ by the production time, t_P, as expressed in the equation 3.9. This expression leads to the following result

$$y_{\max} = t_P(k-D) = y(1-q)\left(1-\frac{D}{k}\right) \tag{3.13}$$

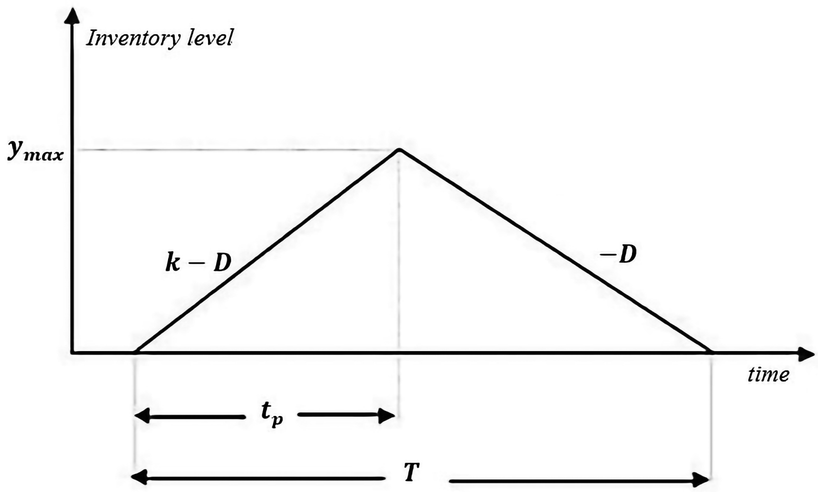


Figure 3.3: Finished product inventory level

The manufacturer incurs a raw material purchasing cost of $c_r y$ at the start of each production cycle, a production cost of $C_p y(1-q)$, an ordering cost of raw material of F_r, and a fixed setup cost of G. The holding cost of raw material, H_r, is calculated as the product of the average inventory of raw material and the holding cost per unit time, h_r. To determine H_r, El-Kassar et al.(2012) multiplied the area under the curve shown in Figure 3.2, which represents the raw material inventory level by h_r and then divided the expression by the cycle length, T. This yielded

$$H_r = y^2\left(\frac{(1-q)^2}{2k}+\frac{q}{x}\right)\frac{1}{T} \tag{3.14}$$

The total holding cost for finished goods, H_p, is the product of the average inventory and the holding cost per unit per unit time h_p. The authors assumed that the holding cost rate per finished good incurred is the aggregate of the two holding costs per unit time, h_P and h_r, and represented the average inventory level of finished goods by $\frac{1}{2}y_{max}$ which is the area under the curve shown in Figure 3.3. Using the expression of the average inventory, they expressed H_p as follows

$$H_p = \frac{y}{2}(1-q)\left(1-\frac{D}{k}\right)(h_p+h_r)\,T \tag{3.15}$$

The joint total inventory cost per cycle $TC(y)$ is the sum of all the costs involved divided by the inventory cycle length $T = y(1-q)/D$. Hence,

$$
\begin{aligned}
TC(y) =& c_r \frac{D}{1-q} + C_p D + (F_r + G) \frac{D}{y(1-q)} + yD \left(\frac{(1-q)}{2k} + \frac{q}{(1-q)x} \right) h_r \\
&+ \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r) + \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r)
\end{aligned} \tag{3.16}
$$

The total sale function $TR(y)$ includes both the sales revenue of the finished products and the discounted imperfect quality raw materials. That is,

$$
TR(y) = s_p D + s_r q D \frac{1}{(1-q)} \tag{3.17}
$$

El-Kassar et al.(2012) 's model was aimed at maximising the total joint inventory management profit TP. The joint total profit per unit time, $TP(y)$, is

$$
\begin{aligned}
TP(y) =& s_p D + s_r q D \frac{1}{(1-q)} - c_r \frac{D}{1-q} - C_p D - (F_r + G) \frac{D}{y(1-q)} \\
&- yD \left(\frac{(1-q)}{2k} + \frac{q}{(1-q)x} \right) h_r - \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r) \\
&- \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r)
\end{aligned} \tag{3.18}
$$

The manufacturer's optimal batch size, y^*, is obtained by setting the first derivative of equation (3.18) with respect to y to zero, resulting in

$$
y^* = \sqrt{\frac{2(F_r + G) D}{(h_p + h_r)\left(1 - \frac{D}{k}\right)(1-q)^2 + D h_r \left(\frac{(1-q)^2}{k} + \frac{2q}{x} \right)}} \tag{3.19}
$$

In the second scenario, the imperfect raw materials are kept until the inventory cycle is completed and then returned to the supplier when the next order is received. The behaviour of the inventory profile of raw materials is shown in Figure 3.4.
The total cost per cycle is identical to that of the first scenario except for the holding cost of the raw material, which in this case is

$$
H_r = y \left(q + \frac{D}{2k}(1-q) \right) h_r T \tag{3.20}
$$

The TC function for the second scenario is

$$
\begin{aligned}
TC(y) =& c_r \frac{D}{1-q} + C_p D + (F_r + G) \frac{D}{y(1-q)} + y \left(q + \frac{D}{2k}(1-q) \right) h_r \\
&+ \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r)
\end{aligned} \tag{3.21}
$$

the total profit function is

$$
\begin{aligned}
TP(y) =& s_p D - c_r D - C_p D - (F_r + G) \frac{D}{y(1-q)} - y \left(q + \frac{D}{2k}(1-q) \right) h_r \\
&- \frac{y}{2}(1-q) \left(1 - \frac{D}{k} \right) (h_p + h_r)
\end{aligned} \tag{3.22}
$$

The batch size is calculated by solving the equation (3.22) using a similar approach to that of the first scenario. Hence

$$
y^* = \sqrt{\frac{2(F_r + G) D}{(h_p + h_r)\left(1 - \frac{D}{k}\right)(1-q)^2 + \left(2q(1-q) + \frac{D}{k}(1-q)^2\right) h_r}} \tag{3.23}
$$

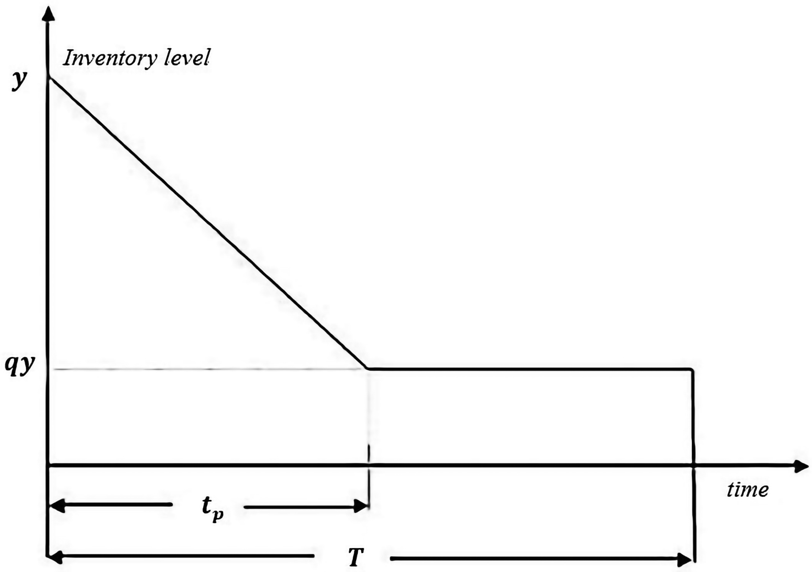


Figure 3.4: Raw material inventory Level, imperfect items returned to supplier

3.3.3 Freshness and time-dependent demand inventory systems

Banerjee and Agrawal (2017) proposed a model for managing deteriorating items that incorporates the impact of product freshness on demand. They suggested that both freshness decline $\varphi(t)$ and deterioration $\theta(t)$ commence after some time t in storage. The graph in Figure 3.5 illustrates the inventory levels throughout a cycle. At time $t = 0$, Q units are replenished and added to the stock immediately. It is assumed that the product is initially in perfect condition. The freshness function $\varphi(t)$ which is a non increasing function of time t, is of the form

$$\varphi(t) \begin{cases} = 1, & \text{if } t < t_1 \\ < 1, & \text{if } t \geq t_1 \end{cases} . \tag{3.24}$$

If there is no decline in freshness, $\varphi(t) = 1$ for $t \in [0, T]$. When the selling price is s_p, the demand rate at time epoch t is given by

$$D(s_p, t) = \begin{cases} f_1(s_p), & \text{if } \varphi(t) = 1 \text{ at epoch } t \\ f_2(s_p, \varphi(t)), & \text{if } \varphi(t) < 1 \text{ at epoch } t \end{cases} , \tag{3.25}$$

where $f_1(s_p)$ is a declining function of s_p and $f_2(s_p, \varphi(t))$ is a declining function of s but increasing function of $\varphi(t)$, with s_p being exogenous and known.

From $(0, t_1)$, inventory decreases solely due to demand, which is influenced by the selling price s_p of the item. After time t_1, the freshness $\varphi(t)$ of the items starts to decline over time. The inventory continues depleting only due to demand, which is now dependent on both selling price s_p and freshness condition $\varphi(t)$. At time t_2, deterioration $\theta(t)$ begins. Inventory depletes due to both demand and deterioration. To stimulate more sales, a discount $\alpha\%$ is offered on the selling price from time $t_3 > t_2$. Inventory reaches zero at time t_4 followed by shortages beginning after this point, resulting in lost sales from unmet demand. An order placement occurs again at time T ($\geq t_4$). Keeping a product in inventory incurs a holding cost of h_p per unit of time. This holding cost applies exclusively to non-deteriorated products, regardless of their freshness level. The cost of deterioration per unit, denoted as c_d, is greater than the purchase price per product p_c. Moreover, the selling price per unit, represented as s_p, is greater than the cost of deterioration per unit.

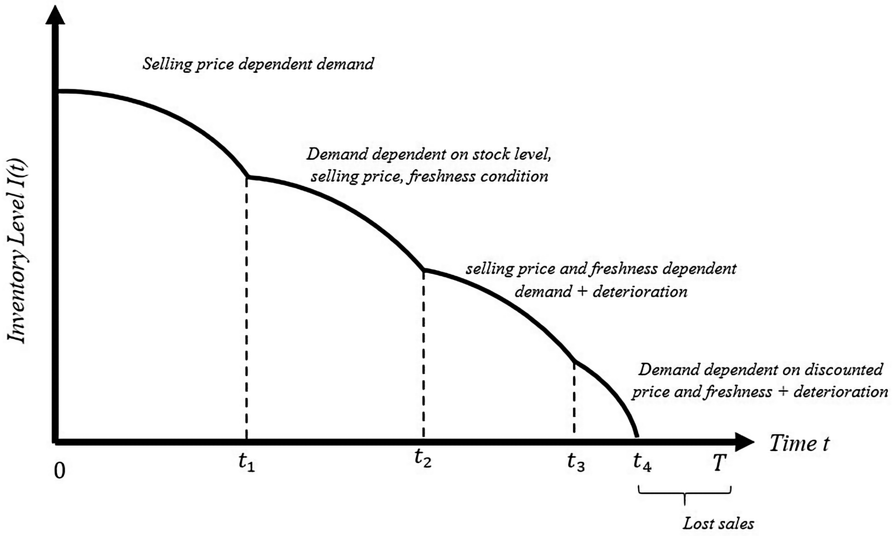


Figure 3.5: Inventory profile for items with freshness dependent demand

Differential equations governing the inventory situations are

$$\begin{aligned}
\frac{dI_1(t)}{dt} &= -f_1(s_p), && 0 \leq t \leq t_1 \\
\frac{dI_2(t)}{dt} &= -f_2(s_p, \varphi(t)), && t_1 \leq t \leq t_2 \\
\frac{dI_3(t)}{dt} &= -f_2(s_p, \varphi(t)) - \theta(t)I_3(t), && t_2 \leq t \leq t_3 \\
\frac{dI_4(t)}{dt} &= -f_2((1-\alpha)s_p, \varphi(t)) - \theta(t)I_4(t), && t_3 \leq t \leq t_4.
\end{aligned} \tag{3.26}$$

The order quantity Q is given by

$$\begin{aligned}
Q =& f_1(s_p)t_1 + \int_{t_1}^{t_2} f_2(s_p, \varphi(x))\, dx \\
&+ \exp\left(-\int_0^{t_2} \theta(x)\, dx\right) \left[\int_{t_2}^{t_d} f_2(s_p, \varphi(x)) \exp\left(\int_0^x \theta(y)\, dy\right)\, dx\right. \\
&\left.+ \int_{t_3}^{t_4} f_2((1-\alpha)s_p, \varphi(x)) \exp\left(\int_0^x \theta(y)\, dy\right)\, dx\right].
\end{aligned} \tag{3.27}$$

The total holding cost over the time period $[0, t_4]$ is

$$THC = h_p \left(\int_0^{t_1} I_1(t)dt + \int_{t_1}^{t_2} I_2(t)dt + \int_{t_2}^{t_3} I_3(t)dt + \int_{t_3}^{t_4} I_4(t)dt\right) \tag{3.28}$$

The total cost of deterioration over the entire cycle is

$$TCD = C_d \left[Q - \left(\int_0^{t_1} f_1(s_p)dt + \int_{t_1}^{t_3} f_2(s_p, \varphi(t))dt + \int_{t_3}^{t_4} f_2((1-\alpha)s_p, \varphi(t))dt \right) \right] \tag{3.29}$$

The cost of lost sales is

$$CLS = l_{sc} \int_{t_4}^{T} f_2((1-\alpha)s_p, \varphi(t))dt \tag{3.30}$$

The revenue TR over the entire cycle is

$$TR = (s_p - p_c) \left[\int_0^{t_1} f_1(s_p)dt + \int_{t_1}^{t_3} f_2(s_p, \varphi(t))dt \right] + [(1-\alpha)s_p - p_c] \int_{t_3}^{t_4} f_2((1-\alpha)s_p, \varphi(t))dt \tag{3.31}$$

The net profit is

$$NP = TR - THC - TCD - CLS \tag{3.32}$$

3.3.4 Inventory level-dependent demand and deterioration

Agi and Soni(2020) proposed a model for optimising pricing and managing inventory of perishable products, where demand is influenced by the on-hand inventory level. Their approach accounts for both physical deterioration and diminishing freshness of the product over time, which leads to reduced demand as the product ages. The model also departs from the traditional assumption of zero-ending inventory, allowing for positive end-of-cycle inventory levels to capture benefits from higher stock levels that stimulate increased demand. Furthermore, they considered that any remaining inventory at the end of the cycle could be sold at a fixed price. In this system, a quantity Q of the product is received by the retailer at time $t = 0$ with selling price s_p set throughout period $[0, T]$. During this time frame, on-hand inventory diminishes at a constant rate θ while simultaneously losing freshness. The demand rate of the product is assumed to be increasing with the current stock level and decreasing with the price and the freshness condition of the product. The inventory level $I(t)$ follows the pattern depicted in Figure 3.6.
Agi and Soni (2020) used the Equation (3.33) to represent the demand function.

$$D(s_p, I(t), t) = \frac{n-t}{n} d(s_p) + \gamma I(t) + \theta I(t) \tag{3.33}$$

The above equation describes the relationship between the demand rate, $D(s_p, I(t), t)$, and the inventory level over time, $I(t)$, where γ represents the demand sensitivity to the current level of inventory, $d(s_p)$ represents the price component of the demand for the product and n represents the shelf-life of the product. Therefore, the inventory level changes with time is governed by the following differential equation

$$\frac{dI(t)}{dt} = -\frac{n-t}{n} d(s_p) - \gamma I(t) - \theta I(t), \quad 0 \le t \le T \tag{3.34}$$

Upon solving equation (3.34), an expression for Q (in terms of q, s_p and T) is obtained, which is then used to compute the company's holding cost.

$$Q = d(s_p) \left(\frac{1}{\gamma + \theta} + \frac{1}{n(\gamma+\theta)^2} \right) \left(e^{(\gamma+\theta)T} - 1 \right) - \frac{d(s_p)}{n(\gamma+\theta)} T e^{(\gamma+\theta)T} + q e^{(\gamma+\theta)T} \tag{3.35}$$

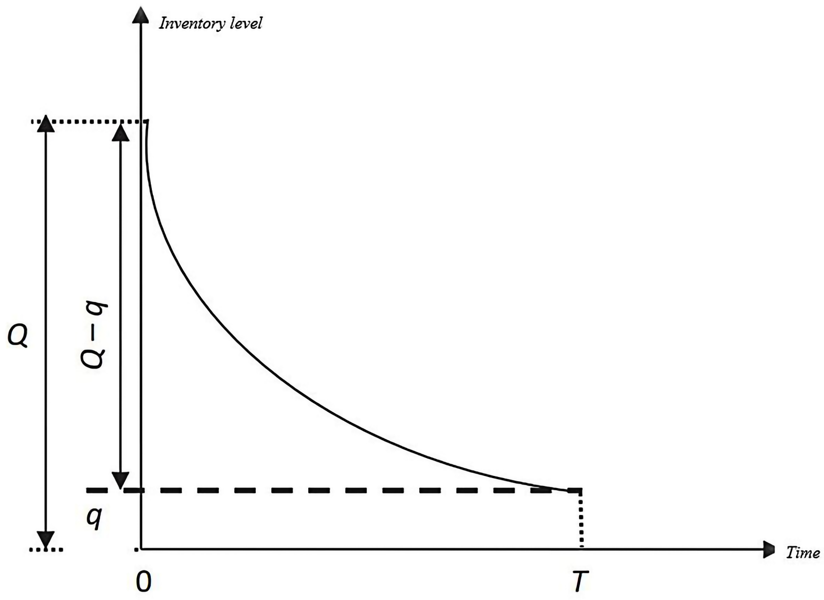


Figure 3.6: Graphical representation of the inventory system with time-dependent demand

The company is charged a holding cost of h_p per unit of time for keeping a single item in stock. Moreover, the company incurs a fixed ordering cost of O_rC whenever an order is placed, and the cost of purchasing each product is c. If the company sells the products at a selling price of s_p per product and salvages some products for s_d per product, then its total profit per unit time, TPU, is

$$TPU = \frac{1}{T}\left\{\begin{array}{l} (s_p - c)d(s_p)T\left(1 - \frac{1}{2n}T\right) - \frac{[s_p\gamma - h_p - c(\gamma+\theta)]}{\gamma+\theta}d(s_p)T \\ + \frac{[s_p\gamma - h_p - c(\gamma+\theta)]}{2n(\gamma+\theta)}d(s_p)T^2 \\ + \frac{[s_p\gamma - h_p - c(\gamma+\theta)]}{\gamma+\theta}\left[d(s_p)\left(\frac{1}{\gamma+\theta} + \frac{1}{n(\gamma+\theta)^2}\right) + q\right] \times \left(e^{(\gamma+\theta)T} - 1\right) \\ - \frac{[s_p\gamma - h_p - c(\gamma+\theta)]}{n(\gamma+\theta)^2}d(s_p)Te^{(\gamma+\theta)T} + (s_p - c)q - O_rC \end{array}\right\} \tag{3.36}$$

Owing to the difficulty associated with obtaining a closed-form solution to equation (3.36), an algorithm was used to find the company's optimal price s_p^*, the optimal cycle time T^*, the optimal order quantity Q^* and the optimal product quantity remaining at the end of the cycle q^*.

3.3.5 Concluding remarks

In this chapter, the mathematical principles underpinning the four fundamental models that form the basis for developing the three original models presented in this thesis are examined. These foundational models encompass scenarios commonly encountered by inventory managers in the industry. In various industries, such as automotive and mineral processing, it is crucial to implement production systems that can adjust production rates due to the ever-changing market conditions and operational variables. For example, in the automotive sector, shifts in consumer demand for vehicles are influenced by economic factors, technological advancements, and seasonal trends. By utilising adaptable production systems that can respond to these changes in demand, automakers can optimise manufacturing output and quickly adapt to shifting market preferences. Similarly, in the electronics and consumer goods industries, where technology and consumer preferences frequently change rapidly, having the ability to modify production rates allows manufacturers to efficiently meet evolving demands and remain competitive within the marketplace.

Moving into heavy industries such as steel and mineral processing, the need for production systems that can shift production rates becomes even more pronounced. Changes in commodity prices, variations in ore quality, and shifts in global economic conditions all influence the profitability and sustainability of mineral processing operations. The ability to modulate production rates in response to these factors allows mineral processing plants to maintain operational efficiency, optimise resource utilisation, and adapt to the complexities of a constantly changing market. Whether driven by market demands, technological advancements, or external economic factors, the adoption of flexible production systems is an essential strategic move across various industries to ensure flexibility and resilience amidst dynamic business environments. In addition to market dynamics and operational variables, the need for production systems with adaptable rates is further emphasised by external factors such as the unpredictability of electricity supply, which includes challenges like load shedding and power outages. Industries ranging from automotive to mineral processing face challenges related to energy availability and reliability. In most African countries, where power interruptions are common, having flexible production systems becomes crucial for mitigating the impact of intermittent energy supply. These systems can intelligently adjust production rates based on power availability, ensuring continuous operations during periods of electricity instability.

Furthermore, the adoption of production systems with the ability to shift production rates is not only influenced by market dynamics, operational variables, and power supply challenges but also due to considerations related to product deterioration and the freshness of consumable goods. In sectors like automotive manufacturing, where just-in-time production practices are common, the ability to control production rates becomes crucial to avoid excess inventory and ensure that vehicles reach consumers in optimal condition. This flexibility is even more vital in industries dealing with perishable products like those in the food and beverage sector. Here, fluctuations in demand and concerns about product shelf life require adaptable production systems that can meet market needs while minimising wastage. Then the mineral processing sector, where extracted materials' quality and characteristics may deteriorate over time. The ability to adjust production rates is essential for preserving finished products. In mining operations, timely ore processing is critical to maintaining mineral quality and preventing degradation that could affect their usability in subsequent applications. By integrating flexibility into production systems, these industries can not only meet market demands, but also tackle the specific issues related to product deterioration and the freshness conditions of consumable goods.

Two fundamental models, namely the model with shifting production rates and the model incorporating deterioration and stock-dependent demand, form a common core in the three original models discussed in the next chapters of this thesis. These specific models collectively provide a practical representation of inventory systems across various industries. Consequently, these foundational models serve as a framework for developing new inventory management models that address shifting production rates, accommodating deterioration and stock-dependent demand.

Chapter 4

A lot-sizing model for a deteriorating product with shifting production rates, freshness-, price-, and stock-dependent demand with price discounting*

4.1 Introduction

4.1.1 Context

In the complex world of supply chain dynamics, managing flexible production systems has become increasingly crucial for industries aiming to enhance their adaptability and agility. This study seeks to explore the intricate relationship between changing production rates and inventory modelling, with a specific focus on the unique challenges posed by perishable products. Unlike traditional production models that assume fixed production rates, flexibility in production systems is essential for improving responsiveness, particularly in industries where adaptability and efficiency are paramount. As industries continue to transition towards more agile and adaptable production systems, this research recognises the growing need to accommodate dynamic production rates within this context. It acknowledges the specific challenges presented by perishable products that require a responsive approach to production. By addressing this requirement for flexibility in handling shifting production rates, the research aims to contribute practical insights into implementing adaptive production models in industries where real-time adjustments are vital for operational efficiency.

4.1.2 Purpose

This section introduces an extended inventory model for deteriorating products. The model builds upon the work of Tshinangi et al.(2022) and Ben-Daya et al.(2008) and expands upon their research by considering the relationship between the concept of shifting

*A modified version of this chapter has been accepted to *International Journal of Mathematical, Engineering and Management Sciences.*

production rates and both the freshness condition and deterioration of products, as proposed by Banerjee and Agrawal (2017). The core purpose of this research is to contribute to the development of an advanced inventory model tailored for flexible production systems, emphasising their crucial role in fostering adaptability and agility. With a specific focus on perishable goods, the study explores how shifts in production rates can seamlessly integrate into inventory models, accounting for variables such as freshness, price dynamics, and stock-dependent demand.

4.1.3 Relevance

The importance of this research goes beyond the traditional scope of managing perishable goods. As highlighted in the literature review, no prior study has specifically explored a production system that combines processes with scalable production rates, producing items of varying quality and subject to deterioration to meet demand influenced by freshness conditions, price, and stock level. Filling this gap in the literature is vital for implementing practical inventory management strategies in industries handling perishable goods. This approach is especially relevant in sectors such as agriculture, food, and pharmaceuticals, where the timely adjustment of production rates can significantly impact product quality, market competitiveness, and overall sustainability. The findings of the study have broader implications beyond just perishable goods; they are positioned to influence a range of industries by offering a framework for integrating flexibility into production systems. This fosters agility and efficiency in response to the increasing trend towards adaptable manufacturing processes. Professionals who grasp the proposed inventory system can navigate the complexities of adapting production rates to meet dynamic demand influenced by price and stock levels while adhering to freshness constraints. As a result, this research contributes not only to theoretical progress but also offers actionable insights for industries seeking efficient and adaptive solutions when facing challenges related to perishable inventories.

4.1.4 Gap analysis

The gap in existing literature addressed in this study is examined in Table 4.1, which presents an analysis of the inventory models from previous research, highlighting the different factors considered and how this chapter contributes to the research on production systems with varying production rates. The current literature review indicates a lack of extensions for deteriorating item inventory models that take into account imperfect systems with shifting production rates, freshness, price- and stock-dependent demand, as well as price discounting.

4.1.5 Organisation

In addition to the introductory section, this chapter has five other sections. In Section 4.2, the nomenclatures and notations necessary for the development of the inventory model are outlined. Sections 4.3 and 4.4 focus on the description and formulation of the inventory model, taking into account the shift in production, freshness, stock and price-dependent demand, and the inclusion of deterioration, discount and imperfect production. In Section 4.5, numerical examples are solved to provide practical illustrations, a sensitivity analysis

Table 4.1: Gap analysis of related works in literature.

	Characteristics of the inventory system						
Authors	EOQ/EPQ models	Production	Imperfect quality	Deterioration	Demand	Discount	Freshness
Ben-Daya et al. (2008)	EPQ	Variable	×	×	Constant	×	×
Bhowmich & Samanta (2012)	EPQ	Variable	×	Exponential distribution	Constant	×	×
Banerjee and Agrawal (2017)	EOQ	×	×	Weibull and Exponential distribution	Price dependent	✓	✓
Viji and Karthikeyan (2018)	EPQ	Variable	×	Weibull distribution	Constant	×	×
Agi and Soni (2020)	EOQ	×	×	Constant	Age, stock, and price dependent	×	✓
Tshinangi et al. (2022)	EPQ	Variable	✓	Exponential distribution	Constant	×	×
Salas-Navarro et al. (2023)	EPQ	Constant	×	Constant	Time dependent	×	×
This paper	EPQ	Variable	✓	Exponential distribution	Age-,price and stock dependent	✓	✓

is conducted, and observations are discussed to provide managerial insights. Finally, Section 4.6 offers a conclusion respectively.

4.2 Notations and assumptions

4.2.1 Notations

The following notations are utilised in this chapter

A	Demand parameter
b	Elasticity parameter of the unit selling price
C_d	Deterioration cost per unit item
C_{d_p}	Disposal cost per unit item
$D(s_p, I(t), \varphi(t)), D(I)$	Demand for the product
$d_{1,2}$	Proportion of defective units produced
$G_{1,2}$	Set-up cost associated with stage i
h_p	Inventory carrying cost per item produced per time
H_M	Hessian Matrix
$I(t)$	Instantaneous inventory level
k_1	Initial production rate at the start of the cycle
k_2	Production rate following the shift respectively
$L_{\mathrm{i}\in\{1,2,3,4,5\}}$	Constant of integration
p_{c_1}	Unit production cost at the start of production
p_{c_2}	Unit production cost after the machine's production rate has been scaled down
QD_p	Quantity of deteriorated products
s_p	Market selling price of the product
T	Cycle time
TC	Average total cost per time
THC	Total holding cost
TDC	Total deterioration cost
TPC	Total production cost
$TDPC$	Total disposal cost
TR	Average revenue per time
TSC	Total set up cost
NP	Average profit per time
$t_{\mathrm{i}\in\{1,2,3,4\}}$	Time duration of each phase of the cycle
$\theta(t)$	Deterioration rate per unit per time
γ	Demand enhancement parameter for inventory level
$\rho_{1,2}$	Aggregation parameters for some known variables
β	Freshness parameter
α	Discount percentage offered on selling price
n	Shelf-life of the product
$\rho_{1,2}$	Aggregation parameters for some known variables
$\varphi(t)$	Freshness function

4.2.2 Assumptions

Some assumptions were made to formulate the mathematical model. These include:

- The inventory procedure is for a single product.
- At the start of the process, a production rate of k_1 is employed. After a time, t_1, the decision maker switches to a lower production rate of k_2.
- Demand is deterministic and dependent on the freshness, price and stock level

$$D\left(s_p, I(t)\right) = \begin{cases} A - bs_p + \gamma I(t) & \text{for } 0 < t \leq t_3 \\ \left(A - bs_p\right)\varphi(t) & \text{for } t_3 < t \leq t_4 \\ \left[A - b(1-\alpha)s_p\right]\varphi(t) & \text{for } t_4 < t \leq t_5 \end{cases} \tag{4.1}$$

  With A, b, and $\gamma \neq 0$

- All products are sold at a unit selling price s_p.
- Some manufactured products are accidentally damaged (or contaminated) and have to be discarded as scrap during each of the two production phases at constant rates d_i with $i \in \{1, 2\}$
- The manufactured products are subject to deterioration. The deterioration function is of the form

$$\theta(t) = \begin{cases} \theta e^{-\theta t}, & \text{for } t > t_3 \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

- The product has a maximum shelf-life, n, beyond which its perceived value is lost, leading to potential financial losses. The freshness decreases linearly from a particular time, $t_3 < n$, following the function, $\varphi(t)$, similar to the model proposed by Banerjee and Agrawal (2017), where

$$\varphi(t) \begin{cases} = 1, & \text{if } t < t_3 \\ = \frac{[n - \beta(t - t_2)]}{n}, & \text{if } t \geq t_3 \end{cases} \tag{4.3}$$

  This implies that while production ends at t_2, products are still considered fresh for a length of time until time t_3, from when its freshness starts to decline until when it is considered unacceptable after the shelf life is reached. There is no decline in quality up until t_3, and the freshness function $\varphi(t)$ will be equal to 1 for $t \in [0, t_3]$.

- There is no rework or replacement of poor-quality products.

4.3 Problem description

Figure 4.1 illustrates the changes in inventory level throughout the cycle. At the beginning of the production cycle, the product is manufactured at a production rate k_1, and inventory is built until it reaches the level I_1 at time t_1. During the interval $[0, t_1]$, the product is considered completely fresh, and the inventory is withdrawn solely due to demand, which depends on both the level of stock displayed and the selling price of the product. In the interval $[0, t_1]$, it is assumed that some manufactured products are damaged and taken away at a rate d_1. At time t_1, the operator scales down the machine and continues the production at a rate k_2 until time t_2, during which the inventory reaches its maximum level, I_2. In the interval $[t_1, t_2]$, inventory continues to be withdrawn due to demand, which still depends on both the stock level and the selling price of the product, while the damaged are taken away at rate d_2. At time t_2, the system stops production. During the interval $[t_2, t_3]$, the inventory continues to deplete due to demand. After t_3, the freshness of the product begins to decline, product deterioration starts, and inventory depletion occurs due to both demand and deterioration. The demand function now depends on both the selling price and the freshness of the product, but no longer on the level of stock displayed. To stimulate demand, a fixed discount of $\alpha\%$ is offered on the selling price starting from time t_4. The inventory level hits zero at time T.

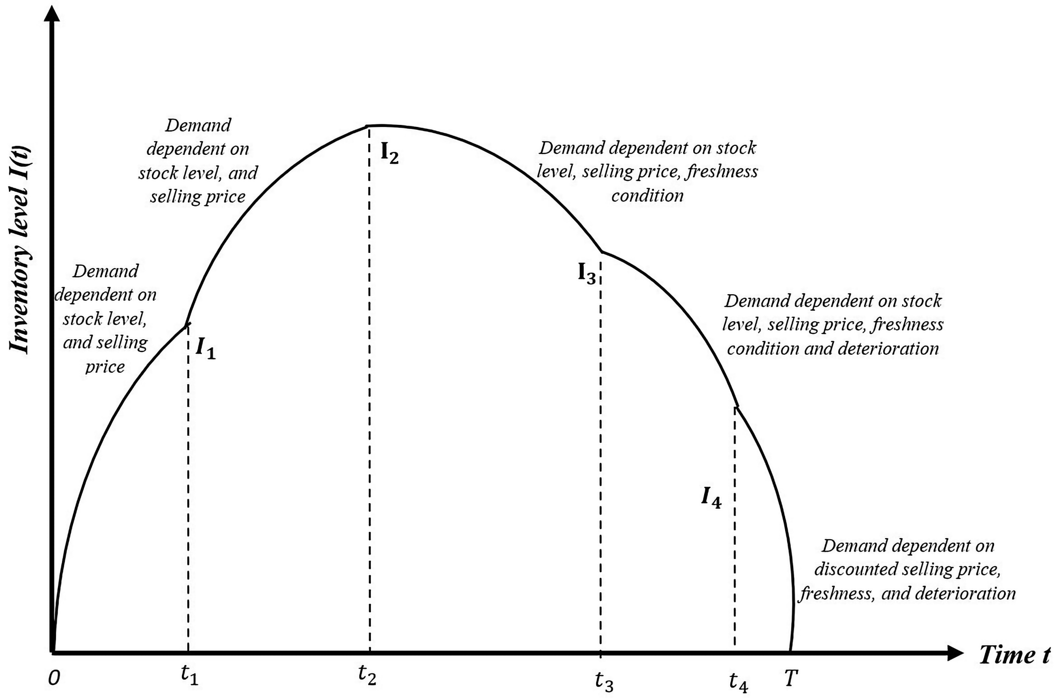


Figure 4.1: Inventory system behaviour for deteriorating products with a shift in production and freshness, price, and stock-dependent demand

4.4 Model formulation

The differential equations that govern the inventory situations in the interval $[0, T]$ are as follows

$$\frac{dI(t)}{dt} = (1 - d_1)\, k_1 - [A - bs_p + \gamma I(t)] \qquad 0 \le t \le t_1 \tag{4.4a}$$
$$\frac{dI(t)}{dt} = (1 - d_2)\, k_2 - [A - bs_p + \gamma I(t)] \qquad t_1 \le t \le t_2 \tag{4.4b}$$
$$\frac{dI(t)}{dt} = -\,[A - bs_p + \gamma I(t)] \qquad t_2 \le t \le t_3 \tag{4.4c}$$
$$\frac{dI(t)}{dt} + \theta I(t) = -\,[A - bs_p]\,\varphi(t) \qquad t_3 \le t \le t_4 \tag{4.4d}$$
$$\frac{dI(t)}{dt} + \theta I(t) = -\,[A - b(1-\alpha)s_p]\,\varphi(t) \qquad t_4 \le t \le T \tag{4.4e}$$

The model is developed under the following conditions $I(0) = 0$, $I(t_1) = I_1$, $I(t_2) = I_2$, $I(t_3) = I_3$ and $(t_4) = I_4$. Solving Equation (4.4a), yields

$$I(t) = \frac{\lfloor (1 - d_1)\, k_1 - A + bs_p \rfloor}{\gamma} + L_1 e^{-\gamma t} \tag{4.5}$$

From Equation (4.5), under the boundary condition, $I(0) = 0$ the following is obtained

$$I(t) = \frac{\rho_1}{\gamma}\left[1 - e^{-\gamma t}\right] \quad 0 \le t \le t_1 \tag{4.6}$$

With: $(1 - d_1)\, k_1 - A + bs_p = \rho_1$

Linearising the exponential terms containing t in Equation (4.6) by using Taylor's series expansion for $e^{-\gamma t}$ leads to the following

$$e^{-\gamma t} = \sum_{m=1}^{\infty} \frac{(-1)^m \gamma^m t^m}{m!} = 1 - \frac{\gamma t}{1} + \frac{\gamma^2 t^2}{2!} - \frac{\gamma^3 t^3}{3!} + \frac{\gamma^4 t^4}{4!} \approx 1 - \gamma t \tag{4.7}$$

Substituting Equation (4.7) into Equation (4.6) yields

$$I(t) = \rho_1 t \quad 0 \le t \le t_1 \tag{4.8}$$

Solving differential Equation (4.4b) yields

$$I(t) = \frac{[(1 - d_2)\, k_2 - A + bs_p]}{\gamma} + L_2 e^{-\gamma t} \tag{4.9}$$

From Equation (4.9) under the boundary condition, $I(t_1) = I_1$, one gets

$$I(t) = \frac{\rho_2}{\gamma} + \left[I_1 - \frac{\rho_2}{\gamma}\right] e^{-\gamma(t - t_1)} \quad t_1 \le t \le t_2 \tag{4.10}$$

With: $(1 - d_2)\, k_2 - A + bs_p = \rho_2$

Utilising Taylor's series expansion for $e^{-\gamma(t-t_1)}$ to linearise the exponential terms near $t = t_1$ in Equation (4.10) yields

$$I(t) = I_1 - (\gamma I_1 - \rho_2)(t - t_1) \quad t_1 \leq t \leq t_2 \tag{4.11}$$

The solution of the differential Equation (4.4c) is

$$I(t) = -\frac{(A - bs_p)}{\gamma} + L_3 e^{-\gamma t} \tag{4.12}$$

From Equation (4.12) under the boundary condition, $I(t_2) = I_2$, we obtain

$$I(t) = -\frac{A - bs_p}{\gamma} + \left[I_2 + \frac{A - bs_p}{\gamma}\right] e^{-\gamma(t-t_2)} \quad t_2 \leq t \leq t_3 \tag{4.13}$$

Again, linearising the exponential term $e^{-\gamma(t-t_2)}$ to near $t = t_2$ in Equation (4.13) utilising Taylor's series expansion yields

$$I(t) = I_2 - (\gamma I_2 + A - bs_p)(t - t_2) \qquad t_2 \leq t \leq t_3 \tag{4.14}$$

solving the differential Equation (4.4d), leads to

$$I(t) = -(A - bs_p)\left[\frac{[n - \beta(t - t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] + L_4 e^{-\theta t} \tag{4.15}$$

On solving Equation (4.15) under the boundary condition $I(t_3) = I_3$, the following is obtained

$$I(t) = -(A - bs_p)\left[\frac{\varphi(t)}{\theta} + \frac{\beta}{n\theta^2}\right] + \left[I_3 + (A - bs_p)\left[\frac{\varphi(t_3)}{\theta} + \frac{\beta}{n\theta^2}\right]\right] e^{-\theta(t-t_3)} \quad t_3 \leq t \leq t_4 \tag{4.16}$$

With: $\frac{[n-\beta(t_3-t_2)]}{n} = \varphi(t_3)$

Linearising the exponential terms $e^{-\theta(t-t_3)}$ in Equation (4.16) by applying Taylor's series expansion near $t = t_3$, results in the following

$$I(t) = I_3 - \theta\{I_3 + (A - bs_p)\varphi(t_3)\}(t - t_3) \quad t_3 \leq t \leq t_4 \tag{4.17}$$

The solution to differential Equation (4.4e) is

$$I(t) = -[A - b(1-\alpha)s_p]\left[\frac{[n - \beta(t - t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] + L_5 e^{-\theta t} \tag{4.18}$$

Using the boundary condition, $I(t_4) = I_4$ in Equation (4.18), we obtain

$$\begin{aligned} I(t) = &-[A - b(1-\alpha)s_p]\left[\frac{[n - \beta(t - t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] \\ &+ \left[I_4 + [A - b(1-\alpha)s_p]\left[\frac{\varphi(t_4)}{\theta} + \frac{\beta}{n\theta^2}\right]\right] e^{-\theta(t-t_4)} \quad t_4 \leq t \leq T \end{aligned} \tag{4.19}$$

With: $\frac{[n-\beta(t_4-t_2)]}{n} = \varphi(t_4)$

Linearising the exponential term $e^{-\gamma(t-t_4)}$ to near $t = t_4$ in Equation (4.19) utilising Taylor's series expansion yields

$$I(t) = I_4 - \theta\{I_4 + [A - b(1-\alpha)s_p]\varphi(t_4)\}(t - t_4) \quad t_4 \leq t \leq T \tag{4.20}$$

4.4.1 Manufacturer's cost components

The behaviour of the manufacturer's processed inventory level is depicted in Figure 4.1. The manufacturer's total cost function is the sum of the setup, holding, deterioration, production, and disposal costs.

4.4.1.1 Manufacturer's set up cost

This production model incorporates variable production rates across the production cycle, and each shift is associated with its unique setup cost denoted as SUC_i for each shift i. The total setup cost is expressed as

$$TSC = \sum_{i=1}^{2} G_i \tag{4.21}$$

4.4.1.2 Manufacturer's inventory holding cost

The manufacturer's cost for holding inventory per unit time, THC, is determined as the product of the cumulative holding inventory at different time intervals within the overall cycle and the cost of holding a single unit per time unit (h_p). The holding cost incurred by the manufacturer is thus

$$THC = h_p \left[\int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt + \int_{t_2}^{t_3} I(t)dt + \int_{t_3}^{t_4} I(t)dt + \int_{t_4}^{T} I(t)dt \right] \tag{4.22}$$

$$\int_0^{t_1} I(t)dt = \frac{1}{2}\rho_1 t_1^2 \tag{4.23a}$$
$$= \frac{1}{2} I_1 t_1 \tag{4.23b}$$

$$\int_{t_1}^{t_2} I(t)dt = I_1 \left(t_2 - t_1\right) - \frac{1}{2}\left(\gamma I_1 - \rho_2\right)\left(t_2 - t_1\right)^2 \tag{4.23c}$$
$$= \frac{1}{2}\left(I_1 + I_2\right)\left(t_2 - t_1\right) \tag{4.23d}$$

$$\int_{t_2}^{t_3} I(t)dt = I_2 \left(t_3 - t_2\right) - \frac{1}{2}\left(\gamma I_2 + A - bs_p\right)\left(t_3 - t_2\right)^2 \tag{4.23e}$$
$$= \frac{1}{2}\left(I_2 + I_3\right)\left(t_3 - t_2\right) \tag{4.23f}$$

$$\int_{t_3}^{t_4} I(t)dt = \left[I_3 - \frac{1}{2}\theta \left[I_3 + \left(A - bs_p\right)\varphi\left(t_3\right)\right]\left(t_4 - t_3\right)\right]\left(t_4 - t_3\right) \tag{4.23g}$$
$$= \frac{1}{2}\left(I_3 + I_4\right)\left(t_4 - t_3\right) \tag{4.23h}$$

$$\int_{t_4}^{T} I(t)dt = I_4 \left(T - t_4\right) - \frac{1}{2}\theta \left[I_4 + \left[A - b(1-\alpha)s_p\right]\varphi\left(t_4\right)\right]\left(T - t_4\right)^2 \tag{4.23i}$$

$$= \frac{1}{2}\left(T - t_4\right) I_4 \tag{4.23j}$$

The inventory carrying cost during the entire cycle $[0, T]$ can be obtained by summing up equations (4.23b), (4.23d), (4.23f), (4.23h) and (4.23j) as

$$THC = \frac{1}{2} h_p \Big[I_1 t_2 + I_2 \left(t_3 - t_1\right) + I_3 \left(t_4 - t_2\right) + I_4 \left(T - t_3\right) \Big] \tag{4.24}$$

4.4.1.3 Manufacturer's deterioration cost

The manufacturer's cost for deteriorating inventory, TCD, is determined as the product of the net quantity of deteriorated products and the unit cost of a deteriorating product (C_d). The net quantity of deteriorated products over the entire cycle is given by QD_p

$$QD_p = \left[\int_{t_3}^{t_4} I(t)dt + \int_{t_4}^{T} I(t)dt\right] \tag{4.25}$$

The expression in Equation (4.25) is derived by considering the changes in inventory over different phases of the provided model. The inventory is built up during the time interval from 0 to t_2 until until it reaches I_2. The net quantity of deteriorated products is zero at the end of this interval. This is because the model starts with an empty inventory, and no deterioration occurs since the products are considered fresh, and deterioration only starts when products start losing their freshness. The same approach was used by Banerjee and Agrawal (2017) in their model for perishable products.

$$\int_{t_3}^{t_4} I(t)dt = \int_{t_3}^{t_4} \left[I_3 - \theta \Big(I_3 + \left(A - bS_p\right)\varphi\left(t_3\right) \Big) \left(t - t_3\right) \right] dt \tag{4.26a}$$

$$= I_3 \left(t_4 - t_3\right) - \frac{1}{2}\theta \left\{I_3 + \left(A - bS_p\right)\varphi\left(t_3\right)\right\}\left(t_4 - t_3\right)^2 \tag{4.26b}$$

$$\tag{4.26c}$$

$$\int_{t_4}^{T} I(t)dt = \int_{t_4}^{T} \Big[I_4 - \theta \left[I_4 + \left[A - b(1-\alpha)S_p\right]\varphi\left(t_4\right)\right]\left(t - t_4\right) \Big] dt \tag{4.27a}$$

$$= \frac{1}{2} \Big[I_4 + I_4 - \theta \Big[I_4 + \left[A - b(1-\alpha)S_p\right]\varphi\left(t_4\right) \Big] \left(T - t_4\right) \Big] \left(T - t_4\right) \tag{4.27b}$$

$$\tag{4.27c}$$

Hence, the net quantity is

$$QD_p = I_3 \left(t_4 - t_3\right) - \frac{1}{2}\theta \left\{I_3 + \left(A - bS_p\right)\varphi\left(t_3\right)\right\}\left(t_4 - t_3\right)^2 + \frac{1}{2} \Big[I_4 + I_4 - \theta \Big[I_4 + \left[A - b(1-\alpha)S_p\right]\varphi\left(t_4\right) \Big] \left(T - t_4\right) \Big] \left(T - t_4\right) \tag{4.28}$$

$$QD_p = \frac{1}{2}\left(I_3 + I_4\right)\left(t_4 - t_3\right) + \frac{1}{2}\left(T - t_4\right) I_4 \tag{4.29}$$

The cost of deterioration is therefore given by

$$TCD = C_d\theta \left\{\frac{1}{2}\left(I_3 + I_4\right)\left(t_4 - t_3\right) + \frac{1}{2}\left(T - t_4\right) I_4\right\} \tag{4.30}$$

4.4.1.4 Manufacturer's production cost

The manufacturer's production is:

$$TPC = p_{c1}\int_0^{t_1} k_1 dt + p_{c2}\int_{t_1}^{t_2} k_2 dt \tag{4.31}$$

$$TPC = p_{c1}k_1t_1 + p_{c2}k_2\left(t_2 - t_1\right) \tag{4.32}$$

4.4.1.5 Manufacturer's disposal cost

The manufacturer's disposal cost is dependent on the production rate. This cost is incurred to manage defective products produced during the manufacturing process. It is the product of the disposal cost per product (c_{dp}) and the integral of the total quantity of defective produced produced over $[0, t_2]$. Thus

$$TDPC = c_{dp}\left[\int_0^{t_1} d_1k_1 dt + p_{c2}\int_{t_1}^{t_2} d_2k_2 dt\right] \tag{4.33}$$

$$TDPC = c_{dp}\Big[d_1k_1t_1 + d_2k_2\left(t_2 - t_1\right)\Big] \tag{4.34}$$

4.4.2 Manufacturer's total cost per time

In order to determine the manufacturer's average total cost incurred, TC, Equations (4.21), (4.24), (4.30), (4.32), and (4.34) are summed, and this leads to

$$\begin{aligned} TC = & \frac{1}{T}\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{2} G_i + \frac{1}{2}h_P\Big[I_1t_2 + I_2\left(t_3 - t_1\right) + I_3\left(t_4 - t_2\right) + I_4\left(T - t_3\right)\Big] \\ + \left(p_{c1} + d_1c_{dp}\right)k_1t_1 + \left(p_{c2} + d_2c_{dp}\right)k_2\left(t_2 - t_1\right) \end{array}\right\} \\ & + \frac{C_d\theta}{T}\left\{\frac{1}{2}\left(I_3 + I_4\right)\left(t_4 - t_3\right) + \frac{1}{2}\left(T - t_4\right)I_4\right\} \end{aligned} \tag{4.35}$$

4.4.3 Manufacturer's total revenue per time

The manufacturer's total revenue is obtained by multiplying the total quantity of products sold by the unit selling price (s_p).

$$TR = \frac{s_p}{T}\left\{\begin{array}{l} \displaystyle\int_0^{t_1}\left[A - bs_p + \gamma I(t)\right]dt + \int_{t_1}^{t_2}\left[A - bs_p + \gamma I(t)\right]dt + \int_{t_2}^{t_3}\left[A - bs_p + \gamma I(t)\right]dt \\ \displaystyle + \int_{t_3}^{t_4}\left[A - bs_p\right]\varphi(t)dt + (1-\alpha)\int_{t_4}^{T}\left[A - b(1-\alpha)s_p\right]\varphi(t)dt \end{array}\right\} \tag{4.36}$$

$$\int_0^{t_1} [A - bs_p + \gamma I(t)]\, dt = (A - bs_p)\, t_1 + \frac{1}{2}\gamma\rho_1 t_1^2 \tag{4.37a}$$

$$= \left[(A - bs_p) + \frac{1}{2}\gamma I_1\right] t_1 \tag{4.37b}$$

$$\int_{t_1}^{t_2} [A - bs_p + \gamma I(t)]\, dt = (A - bs_p)(t_2 - t_1) + \gamma I_1 (t_2 - t_1) + \frac{1}{2}\gamma(\rho_2 - \gamma I_1)(t_2 - t_1)^2 \tag{4.38a}$$

$$= \left[(A - bs_p) + \frac{1}{2}\gamma(I_1 + I_2)\right](t_2 - t_1) \tag{4.38b}$$

$$\int_{t_2}^{t_3} [A - bs_p + \gamma I(t)]\, dt = (A - bs_p)(t_3 - t_2) + \gamma I_2 (t_3 - t_2) - \frac{1}{2}\gamma(\gamma I_2 + A - bs_p)(t_3 - t_2)^2 \tag{4.39a}$$

$$= \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right](A - bs_p + \gamma I_2)(t_3 - t_2) \tag{4.39b}$$

$$\int_{t_3}^{t_4} [A - bs_p]\, \varphi(t) dt = \left[\frac{2n(t_4 - t_3) - \beta(t_4^2 - t_3^2) + 2\beta t_2 (t_4 - t_3)}{2n}\right](A - bs_p) \tag{4.40a}$$

$$= \left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3)(A - bs_p) \tag{4.40b}$$

$$\int_{t_4}^{T} [A - b(1-\alpha)s_p]\, \varphi(t) dt = \left[\frac{2n(T - t_4) - \beta(T^2 - t_4^2) + 2\beta t_2 (T - t_4)}{2n}\right][A - b(1-\alpha)s_p] \tag{4.41a}$$

$$= \left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)[A - b(1-\alpha)s_p] \tag{4.41b}$$

Hence, the average total revenue is expressed as

$$TR = \frac{s_p}{T}\left\{\begin{array}{l} \left[(A - bs_p) + \frac{1}{2}\gamma I_1\right] t_1 + \left[(A - bs_p) + \frac{1}{2}\gamma(I_1 + I_2)\right](t_2 - t_1) \\ + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right](A - bs_p + \gamma I_2)(t_3 - t_2) \\ + \left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3)(A - bs_p) \\ + (1-\alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)[A - b(1-\alpha)s_p] \end{array}\right\} \tag{4.42}$$

4.4.4 Manufacturer's profit per time

The manufacturer's profit, $NP(s_p, T)$, is obtained by subtracting Equation (4.35) from Equation (4.42). Thus

$$
\begin{aligned}
NP(s_p,T) =& \frac{s_p}{T}\left\{\begin{aligned}&\left[(A-bs_p)+\frac{1}{2}\gamma I_1\right]t_1+\left[(A-bs_p)+\frac{1}{2}\gamma\left(I_1+I_2\right)\right]\left(t_2-t_1\right)\\
&+\left[1-\frac{1}{2}\gamma\left(t_3-t_2\right)\right]\left(A-bs_p+\gamma I_2\right)\left(t_3-t_2\right)\\
&+\left[\frac{2n-\beta\left(t_4+t_3\right)+2\beta t_2}{2n}\right]\left(t_4-t_3\right)\left(A-bs_p\right)\\
&+(1-\alpha)\left[\frac{2n-\beta\left(T+t_4\right)+2\beta t_2}{2n}\right]\left(T-t_4\right)\left[A-b(1-\alpha)s_p\right]\end{aligned}\right\}\\
&-\frac{1}{T}\left\{\begin{aligned}&\sum_{i=1}^{2}G_i+\frac{1}{2}h_P\left[I_1t_2+I_2\left(t_3-t_1\right)+I_3\left(t_4-t_2\right)+I_4\left(T-t_3\right)\right]\\
&+\left(p_{c1}+d_1c_{dp}\right)k_1t_1+\left(p_{c2}+d_2c_{dp}\right)k_2\left(t_2-t_1\right)\end{aligned}\right\}\\
&-\frac{C_d\theta}{T}\left\{\frac{1}{2}\left(I_3+I_4\right)\left(t_4-t_3\right)+\frac{1}{2}\left(T-t_4\right)I_4\right\}
\end{aligned}
\tag{4.43}
$$

4.5 Solution

The optimisation problem addressed in this chapter is as follows

$$
\mathrm{Max}_{s_p,T}\left\{NP\left(s_p,T\right)\right\} \tag{4.44}
$$

The objective function is maximising the total profit per unit time, NP, which consists of: the total revenue (TR), the disposal cost ($TDPC$), the production cost (TPC), the deterioration cost (TCD), the holding cost (THC) and the setup cost TSC.

4.5.1 Determination of the decision variables

Two important principles must be met by the production system being examined. It is essential that within an economical domain, increasing the values of decision variables should not lead to a decrease in overall profit, thus minimising the system's inefficiency. This principle emphasises a crucial factor referred to as average profit denoted as $NP\left(s_p,T\right)$, which corresponds to the initial partial derivative or gradient of the total profit function. This optimisation problem could be fractioned into two sub-problems. The first one is the optimisation with respect to s_p, and the second problem is the optimisation with respect to T. The following is obtained

$$
\begin{aligned}
\frac{\partial NP\left(s_p,T\right)}{\partial s_p} = \frac{1}{T} & \left\{ \begin{array}{l} \left[(A-bs_p)+\frac{1}{2}\gamma I_1\right]t_1+\left[(A-bs_p)+\frac{1}{2}\gamma\left(I_1+I_2\right)\right]\left(t_2-t_1\right) \\ +\left[1-\frac{1}{2}\gamma\left(t_3-t_2\right)\right]\left(A-bs_p+\gamma I_2\right)\left(t_3-t_2\right) \\ +\left[\frac{2n-\beta\left(t_4+t_3\right)+2\beta t_2}{2n}\right]\left(t_4-t_3\right)\left(A-bs_p\right) \\ +\left(1-\alpha\right)\left[\frac{2n-\beta\left(T+t_4\right)+2\beta t_2}{2n}\right]\left(T-t_4\right)\left[A-b(1-\alpha)s_p\right] \end{array} \right\} \\
+\frac{s_p}{T} & \left\{ \begin{array}{l} \left[-b+\frac{1}{2}\gamma\frac{\partial I_1}{\partial s_p}\right]t_1+\left[-b+\frac{1}{2}\gamma\left(\frac{\partial I_1}{\partial s_p}+\frac{\partial I_2}{\partial s_p}\right)\right]\left(t_2-t_1\right) \\ +\left[1-\frac{1}{2}\gamma\left(t_3-t_2\right)\right]\left(-b+\gamma\frac{\partial I_2}{\partial s_p}\right)\left(t_3-t_2\right) \\ -b\left[\frac{2n-\beta\left(t_4+t_3\right)+2\beta t_2}{2n}\right]\left(t_4-t_3\right) \\ -b(1-\alpha)(1-\alpha)\left[\frac{2n-\beta\left(T+t_4\right)+2\beta t_2}{2n}\right]\left(T-t_4\right) \end{array} \right\} \\
-\frac{1}{T} & \left\{\frac{1}{2}h_P\left[\frac{\partial I_1}{\partial s_p}t_2+\frac{\partial I_2}{\partial s_p}\left(t_3-t_1\right)+\frac{\partial I_3}{\partial s_p}\left(t_4-t_2\right)+\frac{\partial I_4}{\partial s_p}\left(T-t_3\right)\right]\quad\right\} \\
-\frac{C_d\theta}{2T} & \left\{\frac{\partial I_3}{\partial S_p}\left(t_4-t_3\right)+\frac{\partial I_4}{\partial S_p}\left(T-t_3\right)\right\}
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
\frac{\partial NP\left(s_p,T\right)}{\partial T} =& \frac{s_p}{T}\left\{ \begin{array}{l} -\frac{\beta}{2n}(1-\alpha)\left(T-t_4\right)\left[A-b(1-\alpha)s_p\right] \\ +\left(1-\alpha\right)\left[\frac{2n-\beta\left(T+t_4\right)+2\beta t_2}{2n}\right]\left[A-b(1-\alpha)s_p\right] \end{array} \right\}-\frac{1}{2T}h_P I_4 \\
& -\frac{C_d\theta}{2T}I_4-\frac{1}{T}NP\left(s_p,T\right)
\end{aligned}
\tag{4.46}
$$

with

$$I_1 = \rho_1 t_1 \tag{4.47a}$$

$$\frac{\partial I_1}{\partial s_p} = b t_1 \tag{4.47b}$$

$$I_2 = I_1 - (\gamma I_1 - \rho_2)(t_2 - t_1) \tag{4.47c}$$

$$\frac{\partial I_2}{\partial s_p} = \frac{\partial I_1}{\partial s_p} - (\gamma \frac{\partial I_1}{\partial s_p} - b)(t_2 - t_1) \tag{4.47d}$$

$$I_3 = I_2 - (\gamma I_2 + A - b s_p)(t_3 - t_2) \tag{4.47e}$$

$$\frac{\partial I_3}{\partial s_p} = \frac{\partial I_2}{\partial s_p} - (\gamma \frac{\partial I_2}{\partial s_p} - b)(t_3 - t_2) \tag{4.47f}$$

$$I_4 = I_3 - \theta \left\{ I_3 + (A - b s_p)\varphi(t_3) \right\} (t_4 - t_3) \tag{4.47g}$$

$$\frac{\partial I_4}{\partial s_p} = \frac{\partial I_3}{\partial s_p} - \theta \left[\frac{\partial I_3}{\partial s_p} - b\varphi\left(t_3\right) \right] (t_4 - t_3) \tag{4.47h}$$

$$I_5 = I_4 - \theta \left\{ I_4 + \left[A - b(1-\alpha) s_p \right] \varphi\left(t_4\right) \right\} (T - t_4) \tag{4.47i}$$

, The optimal solution to the proposed inventory system, which has $NP(s_p, T)$, represented by Equation (4.43) as the objective function and s_p and T as decision variables, is determined by setting the partial derivatives in Equations (4.45) and (4.46) to zero

$$\frac{\partial NP\left(s_p, T\right)}{\partial s_p} = 0 \tag{4.48}$$

$$\frac{\partial NP\left(s_p, T\right)}{\partial T} = 0 \tag{4.49}$$

4.5.2 Optimality condition

The second profit function principle is that there exists a feasible region for which the Hessian matrix is positive (semi)definite or surely nonnegative, to be more precise. However, the functions in equations (4.45) and (4.46) are highly nonlinear, making it challenging to derive a closed-form analytical proof. Nonetheless, it is possible to numerically demonstrate the concavity of the profit function by establishing its positive (semi)definiteness using the Hessian matrix presented in Equation (4.50) while also satisfying the conditions presented in Equations (4.51) and (4.52)

$$H_M = \begin{bmatrix} \frac{\partial^2 NP(s_p,T)}{\partial s_p^2} & \frac{\partial^2 NP(s_p,T)}{\partial s_p \partial T} \\ \\ \frac{\partial^2 NP(s_p,T)}{\partial T \partial s_p} & \frac{\partial^2 NP(s_p,T)}{\partial T^2} \end{bmatrix} \geq 0 \tag{4.50}$$

$$\frac{\partial^2 NP\left(s_p, T\right)}{\partial s_p^2} \leq 0 \tag{4.51}$$

$$\frac{\partial^2 NP\left(s_p, T\right)}{\partial T^2} \leq 0 \tag{4.52}$$

Taking the second-order derivatives of $NP\left(s_p, T\right)$ in (4.43), we found

$$\begin{aligned}\frac{\partial^2 NP(s_p,T)}{\partial s_p^2} = \frac{2}{T}\Bigg\{&\left[-b+\frac{1}{2}\gamma\frac{\partial I_1}{\partial s_p}\right]t_1+\left[-b+\frac{1}{2}\gamma\left(\frac{\partial I_1}{\partial s_p}+\frac{\partial I_2}{\partial s_p}\right)\right](t_2-t_1)\\ &+\left[1-\frac{1}{2}\gamma\left(t_3-t_2\right)\right]\left(-b+\gamma\frac{\partial I_2}{\partial s_p}\right)(t_3-t_2)\\ &-b\left[\frac{2n-\beta\left(t_4+t_3\right)+2\beta t_2}{2n}\right](t_4-t_3)\\ &-b(1-\alpha)^2\left[\frac{2n-\beta\left(T+t_4\right)+2\beta t_2}{2n}\right](T-t_4)\Bigg\}\end{aligned} \tag{4.53}$$

$$\begin{aligned}\frac{\partial^2 NP(s_p,T)}{\partial s_p\partial T} =\frac{\partial^2 NP(s_p,T)}{\partial T\partial s_p} = \frac{1}{T}&\left\{\begin{array}{l}-\dfrac{\beta}{2n}(1-\alpha)\left(T-t_4\right)\left[A-b(1-\alpha)s_p\right]\\ +(1-\alpha)\left[\dfrac{2n-\beta(T+t_4)+2\beta t_2}{2n}\right]\left[A-b(1-\alpha)s_p\right]\end{array}\right\}\\ &+\frac{s_p}{T}\left\{\frac{\beta}{2n}b(1-\alpha)^2\left(T-t_4\right)-b(1-\alpha)^2\left[\frac{2n-\beta(T+t_4)+2\beta t_2}{2n}\right]\right\}-\frac{h_P}{2T}\frac{\partial I_4}{\partial s_p}\\ &-\frac{C_d}{2T}\frac{\partial I_4}{\partial s_p}-\frac{1}{T}\frac{\partial NP(s_p,T)}{\partial s_p}\end{aligned} \tag{4.54}$$

$$\frac{\partial^2 NP\left(s_p,T\right)}{\partial T^2} = -\frac{\beta}{nT}\left[s_p(1-\alpha)\right]\left[A-b(1-\alpha)s_p\right]-\frac{2}{T}\frac{\partial NP\left(s_p,T\right)}{\partial T} \tag{4.55}$$

4.5.3 Numerical results

A numerical experiment is provided to demonstrate the use of the proposed model. The following parameter values were selected, guided by values suggested from some previous models and examples presented in the literature:

Demand parameter, $A = 40$ units/day

Demand enhancement parameter for inventory level, $\gamma = 0.8$

Deterioration cost, $C_d = \$\ 0.8$/unit

Deterioration rate per unit per time, $\theta(t) = 0.004$

Discount percentage offered on selling price, $\alpha = 20\%$

Disposal cost per unit item, $C_{d_p} = \$\ 0.5$/unit

Elasticity parameter of the unit selling price, $b = 0.2$

Fixed setup cost during the first production cycle, $G_1 =\$\ 657$/setup

Fixed setup cost during the second production cycle, $G_2 = \$\ 947$/setup

Fraction of defective products during the first production cycle, $d_1 = 0.08$

Fraction of defective products during the second production cycle, $d_2 = 0.1$

Freshness parameter, $\beta = 0.6$

Initial production rate at the start of the cycle, $k_1 = 150$ units/day

Inventory carrying cost per item produced per time, $h_p = \$\ 0.092$/unit/day

Production cost at the start of production, $p_{c_1} = \$\ 0.54$/unit

Production cost at the start of production, $p_{c_2} =\$\ 0.7$/unit

Production rate following the shift, $k_2 = 130$ units/day

Production time, $t_1 = 1$ day

Production time, $t_2 = 2$ days

Consumption time, $t_3 = 3$ days

Consumption time, $t_4 = 6$ days

Shelf-life of the product, $n = 40$ days

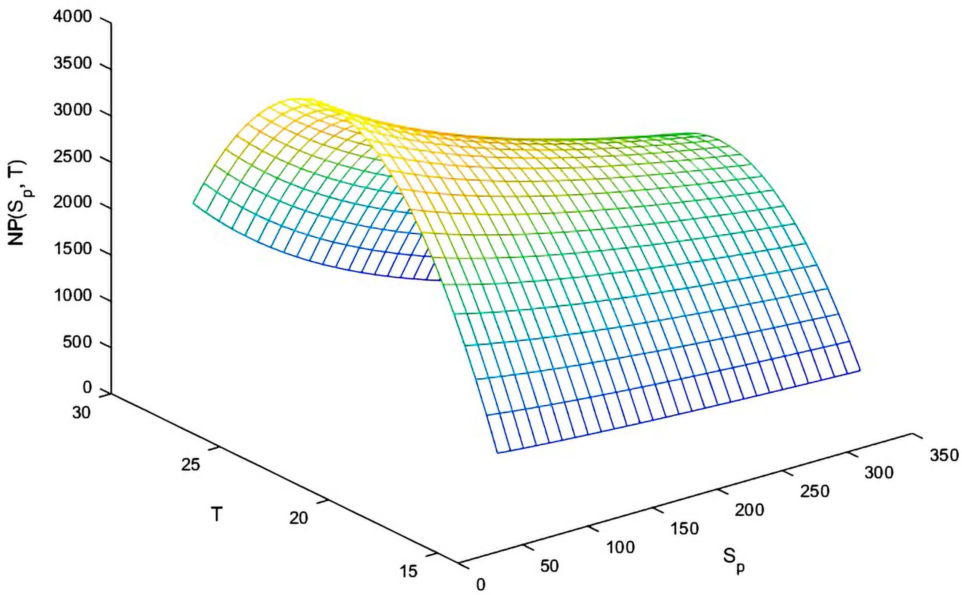


Figure 4.2: Graphical observation of NP against s_p and T

The optimization is performed using the Newton-Raphson method, implemented using MATLAB software due to the complexity of the equations. Solving Equations (4.48) and (4.49), we obtain the optimal values for the product's price $s_p^* = \$167.6/$ unit, and the optimal length of the cycle $\mathrm{T}^* = 21.5$ days. Subsequently, we calculate the average total profit $\mathrm{NP}\left(\mathrm{s_p^*},\ \mathrm{T}^*\right) = \$2977$ per day. The concavity of the profit function can be observed in Figure 4.2 for the values of the selling price ranging between 30 and 310 and the cycle

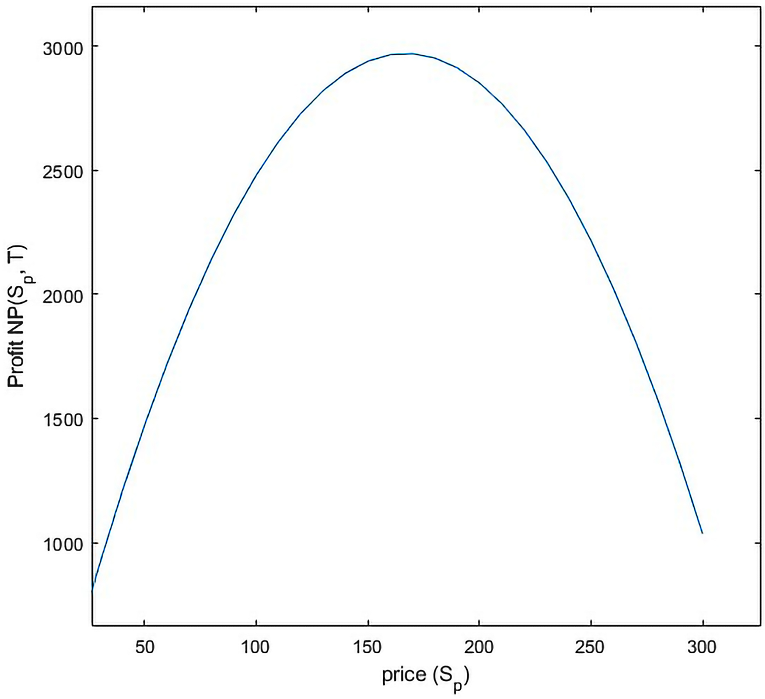


Figure 4.3: Graphical representation of NP with respect to s_p

time ranging between 14 and 28. Through numerical calculus using various values of s_{p}, ranging from 20 to 300, and it is evident from Figure 4.3 that the profit function per unit time, NP $(s_{\mathrm{p}}, \mathrm{T})$, is strictly concave versus the selling price.

4.5.4 Sensitivity analysis

Based on the aforementioned example, sensitivity analysis is conducted by changing the value of one parameter while keeping the other parameters constant. This sensitivity analysis is conducted to assess the impact of varying individual parameters on the optimal values of the unit selling price, cycle time, and maximum profit. This is achieved by adjusting parameter values within the range of -20% to +60% at intervals of 20%, and the changes in optimal values are summarised in Tables 2, 3 and 4 and Figures 4, 5 and 6 for the impacts of the changes on the optimal cycle time, selling price and net profit respectively

Figure 4.4 shows the effect of changing parameter values on the cycle time. The cycle time T^* is highly sensitive with respect to shape parameters b, γ and β; production rates k_1 and k_2; setup cost parameters G_1, G_2; time parameters t_1, t_2, t_3, t_4; the discount rate α; moderately sensitive to unit production costs p_{c1}, p_{c2}; inventory holding cost h_p; defective rates d_1, d_2; and the shelf-life period n. The cycle time T^* is insensitive with respect to the parameter disposal cost C_{dp}; deteriorating cost C_d and the deterioration rate θ. The analysis reveals that, for a higher values of $b, h_p, p_{c1}, p_{c2}, G_1, G_2, t_3, t_4$ and n, the model suggests decreasing the cycle time. On the other hand, the model suggests increasing the cycle time T^* for a higher production rate k_1 and k_2, time parameter t_1, shape parameters γ and β and the discount rate α.

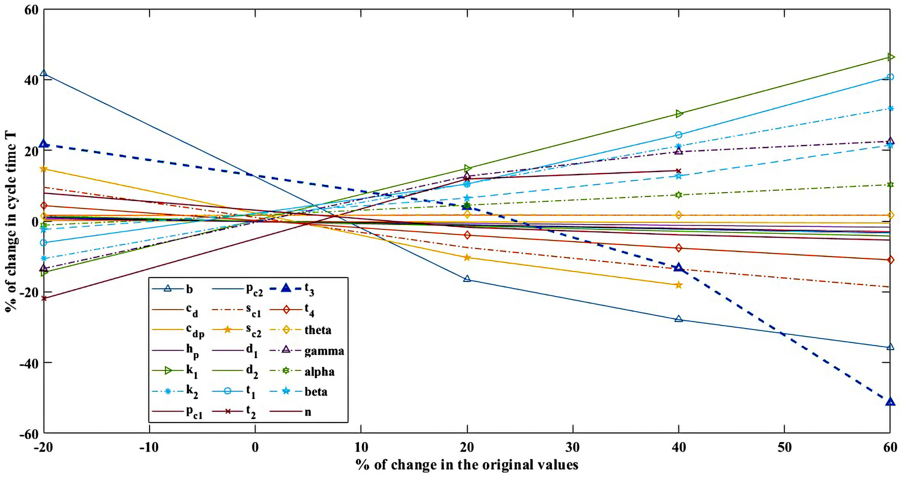


Figure 4.4: Effect of changing parameters on T

From Figure 4.5, it can be seen that the selling price s_p^* is highly sensitive with respect to shape parameters b, time parameter t_3 and discount rate α; moderately sensitive to production rates k_1 and k_2, unit production costs p_{c1} and p_{c2}, setup cost parameters G_1, and G_2, time parameters t_1, t_2 and t_4. The sensitivity analysis indicates that changes in $C_d, C_{dp}, h_p, \theta, \gamma, \beta, d_1, d_2$, and n have a relatively minor impact on the unit selling price s_p^* as compared to other factors. Therefore, the unit selling price s_p^* appears to be insensitive to variations in these parameters. The parameters $k_1 k_2, p_{c1}, p_{c2}, G_1, G_2, t_1, t_2, t_3$, and the discount rate α have an impact on the optimal selling price s_p^* in positive way. For instance, when the values of the mentioned parameters increase, the optimal selling price also increases. The discount parameter α and time parameter t_3 have the greatest impact on the optimal selling price s_p^*. On the other hand, the shape parameter of demand b and the time parameter t_4 have the opposite impact on the optimal selling price. In other words, the optimal selling price decreases with the increase in values of the mentioned parameters. The shape parameter b has the greatest effect on the optimal selling price.

The total profit per NP^* is highly sensitive with respect to changes in setup cost parameters G_1, G_2, shape parameter β and the shelf-life period n (Figure 4.6). The total profit per NP^* is moderately sensitive to changes in parameter γ, disposal cost C_{dp}, inventory holding cost h_p; unit production costs p_{c1}, p_{c2}, time parameters t_1, t_2, t_3 and t_4. The total profit per NP^* exhibits a strong positive sensitivity to changes in setup cost parameters G_1, G_2, disposal cost C_{dp}, inventory holding cost h_p; unit production costs p_{c1}, p_{c2}, time parameters t_2, t_4 and the shelf-life period n. Among these parameters, the setup cost parameters, the shelf live period n and time parameter t_4 have the greatest impact on the optimal profit. NP^* is highly sensitive in a negative to changes in shape parameters b, γ and β, production rates k_1, k_2, time t_1 and t_3. Among these parameters, shape parameters γ and β, and time parameter t_1 have the greatest impact on the optimal profit per cycle. The change in C_d, α, and θ has a minimal effect on the net profit NP^*, indicating that these parameters have a relatively insignificant influence on revenue when compared to other factors that affect profitability.

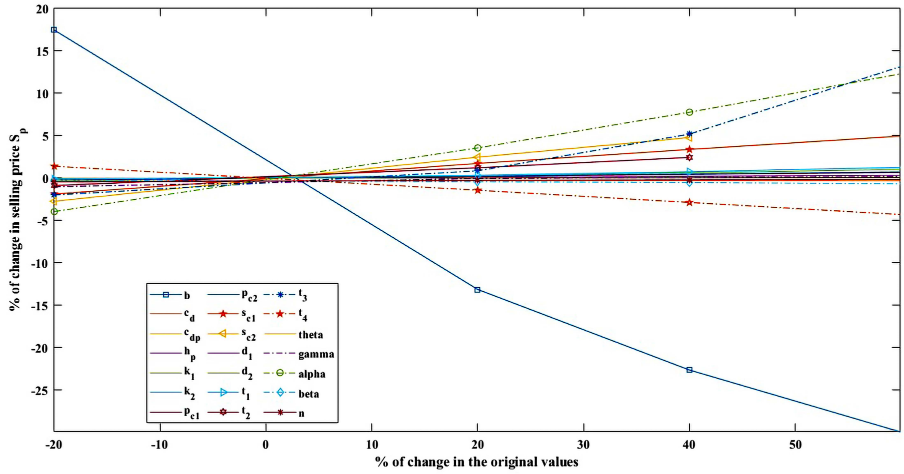


Figure 4.5: Effect of changing parameters on s_p

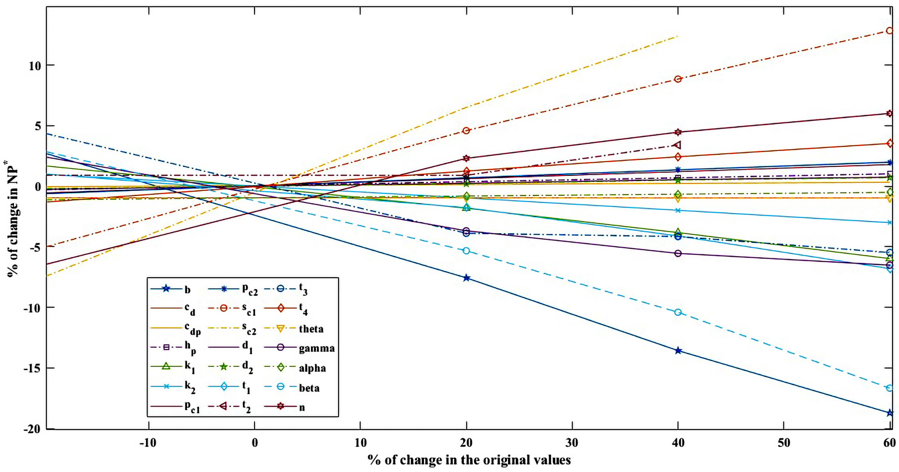


Figure 4.6: Effect of changing parameters on N_P

4.5.5 Managerial implications

The findings from the sensitivity analysis provide valuable suggestions to managers and decision-makers for enhancing the total profit. These suggestions aim to optimise various factors and improve overall profitability.

- It is important to note that the demand elasticity factor, b, is an important parameter in determining the net profit, the selling price, and the cycle time. This may provide the managers with the necessary alternative optima, especially in situations where there may be operating constraints on possible practical values. For instance, market forces may impose limitations on the achievable cycle time, necessitating managers to operate within a specific range of values. It can be observed from Figure 4.4 from the changes in the gradients that the changes in cycle time are more sensitive to a decrease in demand elasticity than its increase. Hence, the manager may appear to have some leeway in allowing an increase in this value. However, seeing the implications on the net profit function in Figure 4.6 discourages this assumption.

- The setup cost parameters G_1 and G_2 play a crucial role in shaping the cycle time and the net profit. When the setup costs increase, the manufacturer should optimize the profit margin by taking some action to offset the increase in total cost. Analysis of the provided tables reveals a notable decrease in cycle time, which proves to be beneficial in optimizing the revenue. This reduction in cycle time helps mitigate holding costs, particularly in production systems where storage costs, deterioration, or obsolescence of inventory are significant concerns. Additionally, the manufacturer should consider increasing the selling price per product as the G_1 and G_2 increase to maintain their profit margin. The model suggests smaller, well-justified price increases s_p^* may be more acceptable to customers. To protect profitability, a sustainable pricing strategy must consider the long-term viability of the business. By adjusting the selling price s_p^* to cover increased setup costs, manufacturers can ensure that the business remains financially stable.

- The sensitivity analysis shows that an increase in the time from which price discount is offered, t_4, results in a moderate decrease in the cycle time. The model then suggests a moderate increase in selling price to result in a moderate increase in the optimal Net profit NP^*. At first glance, this behaviour seems peculiar. This suggests that as the combined effects of both product deterioration and the freshness function kick in, the demand starts to drop, and spoilage starts to increase, and there is the need to promptly introduce demand stimulants, especially because of the impact of the freshness function. The extra sale will then compress the cycle time since the quantity produced is already fixed, and this can mitigate the overall effect of the holding cost and deteriorated quantity, amongst others, thereby raising profit. On the other hand, in a typical production system, there can be a trade-off between the production time (t_2) and the cycle time (T^*). When t_2 increases, it is often associated with producing a larger quantity of items within each cycle. As a result, T^* may also increase to accommodate the production of these additional items. However, it is important to note that an increase in production time does not necessarily have a negative impact on the overall profitability of the manufacturer. It can contribute to higher revenue earned per cycle. To ensure that the revenue outweighs the total cost incurred, managers need to carefully balance the production time, selling price, and cost factors. Increasing the selling price $\left(s_p^*\right)$ can be one strategy to offset the potential increase in costs associated with longer cycle times and larger inventories. By adjusting the selling price strategically, managers can aim to maintain a competitive advantage while covering the additional costs incurred due to increased production and inventory.

- Based on an observation derived from sensitivity analysis, it has been noted that any change in the freshness parameter (β) necessitates careful consideration to strike a balance between cycle time T^* and the total profit NP^*. As β increases, the product freshness drops, and the demand drops correspondingly. Consequently, since products are already made, the cycle time lengthens, which increases the quantity that deteriorates, makes the time the discounted product is sold longer, increases the holding cost, and subsequently depletes the profit. It can be seen from Figure 4.6 that this factor has one of the greatest impacts on profit, and as the cycle time increases further, comparing its slope to that of the demand elasticity factor, it can quickly become the most important cause of profitability decline.

Table 4.2: Results from the sensitivity analysis

	% change	Cycle time T		Price		Profit per time	
		days	% change	USD	% change	USD	% change
Base		**21.50**		**$ 167.60**		**$2,977**	
b	-20	30.47	41.7%	197.4	17.8%	3087.15	3.7%
	+20	17.93	-16.6%	146.1	-12.8%	2780.52	-6.6%
	+40	15.50	-27.9%	130.2	-22.3%	2601.90	-12.6%
	+60	13.80	-35.8%	118.0	-29.6%	2447.09	-17.8%
c_d	-20	21.50	0.0%	167.6	0.0%	2977.00	0.0%
	+20	21.50	0.0%	167.6	0.0%	2977.00	0.0%
	+40	21.50	0.0%	167.6	0.0%	2977.00	0.0%
	+60	21.50	0.0%	167.6	0.0%	2977.00	0.0%
c_{d_p}	-20	21.52	0.1%	168.1	0.3%	3003.79	0.9%
	+20	21.46	-0.2%	168.3	0.4%	3009.75	1.1%
	+40	21.41	-0.4%	168.3	0.4%	3012.72	1.2%
	+60	21.39	-0.5%	168.3	0.4%	3015.70	1.3%
h_p	-20	21.61	0.5%	167.9	0.2%	2997.84	0.7%
	+20	21.37	-0.6%	168.4	0.5%	3015.70	1.3%
	+40	21.24	-1.2%	168.6	0.6%	3027.61	1.7%
	+60	21.11	-1.8%	168.8	0.7%	3036.54	2.0%
k_1	-20	18.38	-14.5%	167.6	0.0%	3057.38	2.7%
	+20	24.70	14.9%	168.6	0.6%	2950.21	-0.9%
	+40	28.01	30.3%	169.3	1.0%	2890.67	-2.9%
	+60	31.48	46.4%	169.9	1.4%	2828.15	-5.0%
k_2	-20	19.22	-10.6%	167.6	0.0%	3036.54	2.0%
	+20	23.78	10.6%	168.8	0.7%	2977.00	0.0%
	+40	26.06	21.2%	169.4	1.1%	2947.23	-1.0%
	+60	28.34	31.8%	170.1	1.5%	2914.48	-2.1%
p_{c_1}	-20	21.72	1.0%	167.8	0.1%	2988.91	0.4%
	+20	21.26	-1.1%	168.6	0.6%	3024.63	1.6%
	+40	21.05	-2.1%	168.9	0.8%	3042.49	2.2%
	+60	20.86	-3.0%	169.3	1.0%	3057.38	2.7%
p_{c_2}	-20	21.74	1.1%	167.8	0.1%	2985.93	0.3%
	+20	21.24	-1.2%	168.6	0.6%	3027.61	1.7%
	+40	21.01	-2.3%	168.9	0.8%	3045.47	2.3%
	+60	20.77	-3.4%	169.4	1.1%	3066.31	3.0%
G_1	-20	23.56	9.6%	165.1	-1.5%	2857.92	-4.0%
	+20	19.89	-7.5%	171.1	2.1%	3143.71	5.6%
	+40	18.58	-13.6%	173.8	3.7%	3268.75	9.8%
	+60	17.48	-18.7%	176.5	5.3%	3387.83	13.8%
G_2	-20	24.68	14.8%	163.6	-2.4%	2780.52	-6.6%
	+20	19.29	-10.3%	172.3	2.8%	3200.28	7.5%
	+40	17.61	-18.1%	176.1	5.1%	3375.92	13.4%
	+60	*	*	*	*	*	*

Table 4.3: Results from the sensitivity analysis

	% change	Cycle time T		Price		Profit per time	
		days	% change	USD	% change	USD	% change
Base		**21.5**		**$167.60**		**$2,977**	
d_1	-20	21.69	0.9%	167.6	0.0%	2968.07	-0.3%
	+20	21.20	-1.4%	167.6	0.0%	2982.95	0.2%
	+40	21.01	-2.3%	167.6	0.0%	2991.89	0.5%
	+60	20.58	-4.3%	167.6	0.0%	2997.84	0.7%
d_2	-20	21.80	1.4%	167.6	0.0%	2971.05	-0.2%
	+20	21.20	-1.4%	167.6	0.0%	2982.95	0.2%
	+40	20.88	-2.9%	167.6	0.0%	2991.89	0.5%
	+60	20.60	-4.2%	167.6	0.0%	2997.84	0.7%
t_1	-20	20.19	-6.1%	167.9	0.2%	3036.54	2.0%
	+20	23.76	10.5%	168.6	0.6%	2953.18	-0.8%
	+40	26.75	24.4%	169.4	1.1%	2881.74	-3.2%
	+60	30.27	40.8%	170.3	1.6%	2801.36	-5.9%
t_2	-20	16.79	-21.9%	166.6	-0.6%	3033.56	1.9%
	+20	24.06	11.9%	170.3	1.6%	3033.56	1.9%
	+40	24.57	14.3%	172.3	2.8%	3110.97	4.5%
	+60	*	*	*	*	*	*
t_3	-20	26.17	21.7%	164.8	-1.7%	3137.76	5.4%
	+20	22.36	4.0%	169.6	1.2%	2890.67	-2.9%
	+40	18.64	-13.3%	176.8	5.5%	2887.69	-3.0%
	+60	10.47	-51.3%	190.2	13.5%	2843.04	-4.5%
t_4	-20	22.42	4.3%	170.4	1.7%	2965.09	-0.4%
	+20	20.64	-4.0%	165.8	-1.1%	3042.49	2.2%
	+40	19.84	-7.7%	163.4	-2.5%	3078.22	3.4%
	+60	19.14	-11.0%	160.9	-4.0%	3110.97	4.5%
θ	-20	21.46	-0.2%	167.6	0.0%	2977.00	0.0%
	+20	21.54	0.2%	167.6	0.0%	2977.00	0.0%
	+40	21.54	0.2%	167.6	0.0%	2977.00	0.0%
	+60	21.56	0.3%	167.6	0.0%	2977.00	0.0%
γ	-20	18.61	-13.4%	166.4	-0.7%	3078.22	3.4%
	+20	24.22	12.7%	168.1	0.3%	2896.62	-2.7%
	+40	25.71	19.6%	168.3	0.4%	2840.06	-4.6%
	+60	26.35	22.5%	168.4	0.5%	2810.29	-5.6%
α	-20	21.25	-1.2%	161.6	-3.6%	2971.05	-0.2%
	20	22.46	4.5%	174.1	3.9%	2982.95	0.2%
	40	23.09	7.4%	181.2	8.1%	2985.93	0.3%
	60	23.71	10.3%	188.7	12.6%	2991.89	0.5%
β	-20	20.99	-2.4%	167.8	0.1%	3093.10	3.9%
	+20	22.90	6.5%	167.4	-0.1%	2846.01	-4.4%
	+40	24.24	12.8%	167.3	-0.2%	2694.19	-9.5%
	+60	26.11	21.5%	167.1	-0.3%	2509.61	-15.7%

Table 4.4: Results from the sensitivity analysis

	% change	**Cycle time T**		**Price**		**Profit per time**	
		days	% change	USD	% change	USD	% change
Base		**21.5**		**\$167.60**		**\$2,977**	
n	-20	23.20	7.9%	167.4	-0.1%	2810.29	-5.6%
	+20	21.11	-1.8%	167.8	0.1%	3073.16	3.2%
	+40	20.66	-3.9%	167.8	0.1%	3137.76	5.4%
	+60	20.34	-5.4%	167.9	0.2%	3185.39	7.0%

- It is interesting to see that changes in the shelf life, n, do not have a significant impact on the optimal cycle time or on the optimal selling price. For instance, a 60% increase in the product life decreases the cycle time by just about 5 % and the selling price by less than 0.5%. However, it increases the profit moderately, by up to 7 %. This may be because longer shelf life dampens the effect of declining freshness (which reduces demand), and decreases the cycle time (shortens the length of discount period on fresh products, since t_4 is fixed), hence, improving profit.

4.6 Conclusion

Companies are actively seeking effective inventory strategies for deteriorating products. While deteriorated products are unsellable, products that gradually lose freshness experience a decrease in demand as they age, although they can still be sold. Many previous models on inventory models for deteriorating products often assume that the demand remains unchanged regardless of the freshness level of the product. In reality, demand is affected by freshness. Another assumption is that the production rate is constant in a manufacturing system. This can be problematic as it doesn't consider the variability in and the complexity of manufacturing processes. In reality, production shifts from one rate to another due to a variety of factors, such as variability of demand, machine breakdowns, material shortages, energy constraints, and quality issues. To effectively respond to changes in demand or unforeseen disruptions in the production process, many organizations implement flexible production systems for better efficiencies and profitability.

This study examined an EPQ model with alternating production rates and price-dependent demand while employing price discounting as a strategy to optimise profit. Additionally, we modified the inventory model to incorporate deteriorating products, where demand is not only influenced by the unit selling price but also by factors such as the level of stock displayed and the freshness condition of the product. The primary objective is to determine both the selling price and the inventory cycle time that maximises the profit. The objective function of this model is highly non-linear, making it challenging to find an analytical solution, hence, solved numerically. We presented a numerical example and then conducted sensitivity analyses to gain managerial insights based on changes in parameter values. It is not always economically viable for the decision maker to sell products at the regular price, particularly when the production costs per unit or the setup costs increase. In such cases, decision-makers can reduce their cycle time as much as possible and increase their unit selling price to protect their profit. Furthermore, when faced with an increase in discount, the analysis suggests that in such situations, the decision maker may respond by increasing their price and cycle time to maintain the profit. However, it is important for managers to consider market dynamics and customer behaviour when contemplating cycle time and price adjustments. Adjustments in cycle time or price may be more effective in preserving profitability without compromising the overall profitability of the production system.

Future research can expand this model in several ways, such as considering non-linear shortages, inflation, incremental discount facilities, prepayment instalments on pricing, and other factors. Furthermore, one could explore models with demand dependent on advertisement and selling price, nonlinear stock-dependent holding cost, non-instantaneous deterioration, and preservation technology, as well as introducing various credit policies (single-level, two-level, partial, credit risk customers, etc.). The model can also be extended to stochastic or fuzzy models.

Chapter 5

A two-echelon supply chain inventory model for perishable products with a shifting production rate, stock-dependent demand rate and imperfect quality raw material †

5.1 Introduction

5.1.1 Context

El-Kassar et al. (2012) examined an Economic Production Quantity model that accounts for the cost of raw materials required for production. The model assumes that a certain proportion of imperfect quality items are present in the supplier's raw materials, which undergo a screening process at the beginning of each inventory cycle. However, the model has limitations as it does not accurately represent real-life inventory systems due to several underlying assumptions, four of which will be discussed in this section. The first assumption is that after receiving an order of raw materials at the start of a production cycle, products are manufactured at fixed costs throughout the entire cycle. In many practical environments, production costs are rarely fixed throughout the entire production cycle. Various factors, such as changes in raw material prices, labor costs, and other operational expenses, can contribute to changes in the overall production cost. Additionally, unforeseen events or disruptions can impact the cost structure, making it variable rather than fixed. The second assumption is that demand remains constant, which does not account for the dynamic nature of market conditions. Demand for products can fluctuate due to factors such as stock, seasonality, price, quality, market trends, economic conditions, or shifts in consumer preferences. The third assumption is that the production rate is fixed and that all products are of perfect quality; this is not always the case, especially in today's business environment, where managing flexible production systems has become increasingly crucial for industries aiming to enhance their adaptability and agility. The fourth assumption is that there is no degradation. In various industries, products have a limited shelf life or experience gradual deterioration over time, which can

†A modified version of this chapter will be submitted to the journal of *mathematics*.

be influenced by factors such as expiration dates, quality, or technological obsolescence.

5.1.2 Purpose

This study aims to consider the implications of stock displayed, shift in production, and quality of raw material on the inventory management policies of a two-echelon supply chain of deteriorating products. To this end, an integrated model for managing inventory in a two-echelon supply chain for deteriorating products is proposed. The two echelons correspond to the supplier, supplying raw materials, and a manufacturer that operates downstream, transforming these inputs into finished products. Flexibility in production is an essential characteristic in inventory control. Two distinct inventory models for deteriorating products are developed under imperfect production, considering the combination of imperfect raw materials and the concept of shifting production rate. In both models, demand for the product is dependent on the current stock level. In the first model, the imperfect raw materials are sold at a discounted price at the end of the screening period, whereas in the second one, imperfect items are kept in stock until the end of the inventory cycle and then returned to the supplier.

5.1.3 Relevance

This research holds significant relevance in today's field of supply chain management, especially for industries facing challenges related to managing deteriorating products. In the current dynamic business landscape, characterised by rapid technological advancements and evolving consumer demands, the adaptability and resilience of supply chains are crucial. Table 5.1 presents the gap in the existing literature that this study aims to address. It provides an analysis of inventory models from previous research, specifically focusing on the various factors that have been considered. This chapter focuses on the impact of the level of stock displayed, production shifts, and raw material quality within a two-echelon supply chain to provide practical insights that can significantly improve inventory management strategies. By comprehending the complexities associated with imperfect raw materials and changes in production rates, businesses can refine their approaches to minimise risks, optimise profits, and ensure efficient resource utilisation. The proposed inventory management model offers a practical framework for industries dealing with imperfect raw materials and variability in demand alongside non-reliable production systems. Ultimately, the study contributes to and expands existing literature on inventory management for deteriorating products with stock-dependent demand in two distinct ways: First is its consideration of simultaneous quantity deterioration, imperfect raw material, and stock-level dependent demand along with an increasing defective rate of finished products which has not been done before. Secondly, it combines all these factors in a two-echelon supply chain.

Table 5.1: Comparison of the study with related works in literature.

Authors	**Characteristics of the inventory system**						
	Inventory Model	**Production Stage**	**Imperfect Quality**	**Echelon**	**Demand Function**	**Setup Cost/Ordering Cost**	**Production Cost**
Sana et al. (2006)	EMQ	Single	Salvage	Single	Price Dependent	Fixed	Variable
Panda et al. (2008)	EPQ	Single	NA	Single	Stock Dependent	Fixed	Variable
Ben-Daya et al. (2008)	EPQ	Multi	NA	Single	CST	Fixed	CST
El-Kassar et al. (2012)	EOQ/EPQ	Single	Salvage	Multi	CST	Fixed	CST
Singh & Pattnayak (2013)	EOQ	NA	NA	Single	Time Dependent	Fixed	NA
Avinadav et al. (2014)	EOQ	NA	NA	Single	Price and Age-Dependent	Fixed	NA
Omar & Yeo (2014)	EOQ/EPQ	Multi	NA	Multi	CST	Variable	CST
Sarkar et al. (2014)	EMQ	Single	Rework	Single	Price and Time-Dependent	Fixed	Variable
Yang (2014)	EOQ	NA	NA	Single	Stock Dependent	Fixed	NA
This paper	**EPQ**	**Multi**	**Salvage**	**Multi**	**Time Dependent**	**Variable**	**Variable**

5.1.4 Organisation

Apart from the introductory part, this particular section consists of five additional subsections. Before delving into the model development presented in Subsection 5.3, the assumptions used in this chapter are outlined in Subsection 5.2. Subsection 5.4 provides numerical results and a demonstration of the model's optimality conditions, while Subsection 5.5 presents sensitivity analysis, aiming to investigate the influence of changes in important parameter values on decision variables to understand their impact on the profit comprehensively. The section concludes with Subsection 5.6, which consists of concluding remarks and suggestions for future research.

5.2 Notations and assumptions

5.2.1 Notations

The following notations are used during the development of the models presented in this thesis:

A	Demand parameter
ψ	Aggregation parameter for some known variables
c_d	Deterioration cost per item
c_r	Unit cost of raw material
$D[I(t)]$	Demand for the product
$d_{1,2}$	Proportion of defective units produced
τ	Aggregation parameter for some known variables
F_r	raw material ordering cost
$G_{1,2}$	Fixed set-up cost associated with stage i
h_p	Inventory carrying cost per item produced per time
h_r	Inventory carrying cost per unit of raw material per time
H_M	Hessian Matrix
$I(t)$	Instantaneous inventory level
k_1	Initial production rate at the start of the cycle
k_2	Production rate following the shift respectively
ζ	Increase in unit machining cost due to increase in the production rate
p_{c_1}	Unit production cost at the start of production
p_{c_2}	Unit production cost after the machine's production rate has been scaled down
p_l	Lost production cost per product
p_c	Purchase price per product
Q	Production batch size
Q_s	Quantity of good products sold at a normal price
Q^*	Optimal batch size
QD_p	Quantity of deteriorated products
q	Proportion of raw materials that are of imperfect quality
ϕ	Per unit cost of running the machine independent of the production rate including labor and energy costs
s_p	Market selling price of the product

s_d	Discounted unit selling price of imperfect finished products
s_r	Discounted unit selling price of imperfect raw material
y	Raw material order size per cycle
T	Cycle time
$t_{\text{i}\in\{1,2\}}$	Time duration of each phase of the cycle
t_s	Screening period
$TCRM$	Total purchase cost of raw material
THR	Total carrying cost of raw material
THC	Total carrying cost of finished products
TDC	Total deterioration cost
TPC	Total production cost
TSC	Total set-up cost
LP	Lost production cost
TR	Average revenue per time
TC	Average total cost per cycle
TR	Average revenue per cycle
TP	Average profit per cycle
$\theta(t)$	Deterioration rate per unit per time
γ	Demand enhancement parameter for inventory level
$\rho_{1,2}, \xi, \upsilon$	Aggregation parameters for some known variables
x	Screening rate for imperfect raw material
y	Raw material order size

5.2.2 Assumptions

Several assumptions are made to model the proposed inventory system. These assumptions include:

- A single type of product is considered.
- Deterioration is observed on manufactured products only.
- The quality of all items produced does not always meet the quality standard; therefore, a proportion d_i is considered to be defective in each stage of the production cycle.
- At the start of the process, a production rate of k_1 is employed. After a time, t_1, the decision maker switches to a lower production rate of k_2.
- The demand rate is dependent on the level of stock displayed and is of the form

  $$D(I) = A + \gamma I(t) \tag{5.1}$$

  Where $A \geq 0$ is the base demand rate, independent of the inventory level, $\gamma > 0$ is the demand enhancement parameter for inventory level $I(t) > 0$

- The production cost per unit is of the form

  $$p_{ki} = p_{ci} + \frac{\phi}{k_i} + \zeta k_i \tag{5.2}$$

  where p_{ci}, ϕ and ζ are nonnegative constants. The cost of production is a combination of the following factors (Panda et al., 2007):

1. p_{ci} is the fixed cost per unit produced, independent of the batch size
2. The factor $\frac{\phi}{k_i}$ indicates that there is a cost associated with the economy of scale in the batch that affects the unit fixed cost of production. As the production rate decreases, some costs, like labour, energy, etc., increase.
3. The factor ζk_i is associated with machine and technology costs and is proportional to the production rate.

- All the good products are sold at a unit selling price s_p.
- Process deterioration occurs in the production run period.
- The changeover cost and time from k_1 to k_2 is assumed to be negligible.
- The discounted unit selling price of imperfect raw material (s_r) is always greater than the unit purchasing cost of raw material C_r.
- Some manufactured products are of imperfect quality and have to be discounted as a batch at a discounted price at the end of the cycle at a unit selling price s_r.
- The manufactured products are subject to deterioration. The deterioration function is of the form

$$\theta(t) = \begin{cases} \theta e^{-\theta t}, & \text{for } t > 0 \\ 0, & \text{otherwise} \end{cases} \tag{5.3}$$

- It is assumed that the raw material doesn't deteriorate but contains a proportion q that is considered to be of imperfect quality.
- There is no rework or replacement of poor quality products since it is handled by using in-house capacity.

5.3 Problem description

5.3.1 First scenario's Formulation

In this scenario, it is assumed that the system starts production at a rate of k_1 (Figure 5.1). During the first part of the cycle, inventory accumulates at a rate $(1 - d_1)\, k_1 - A - \gamma I(t) - \theta I(t)$ while the imperfect quality items accumulate at the rate $d_1 k_1$ and are sold at a discounted price as a single batch at the end of the cycle. An order of size y of raw materials is assumed to be placed and received prior to the start of the cycle because the production process requires input materials. However, the raw material delivered by the supplier for production contains a percentage q that is assumed to be of poor quality. Therefore, a quantity yq is deemed to be of imperfect quality and only $y(1 - q)$ units of raw materials received are used during the production cycle. It is also assumed that once the order is received, a 100% screening process is conducted at a rate x, where $x > k_1$. Therefore, the length of the screening period is:

$$t_s = \frac{y}{x} \tag{5.4}$$

Throughout the screening process, materials of perfect quality are separated from imperfect ones, and only perfect materials are used to produce items that are used to satisfy

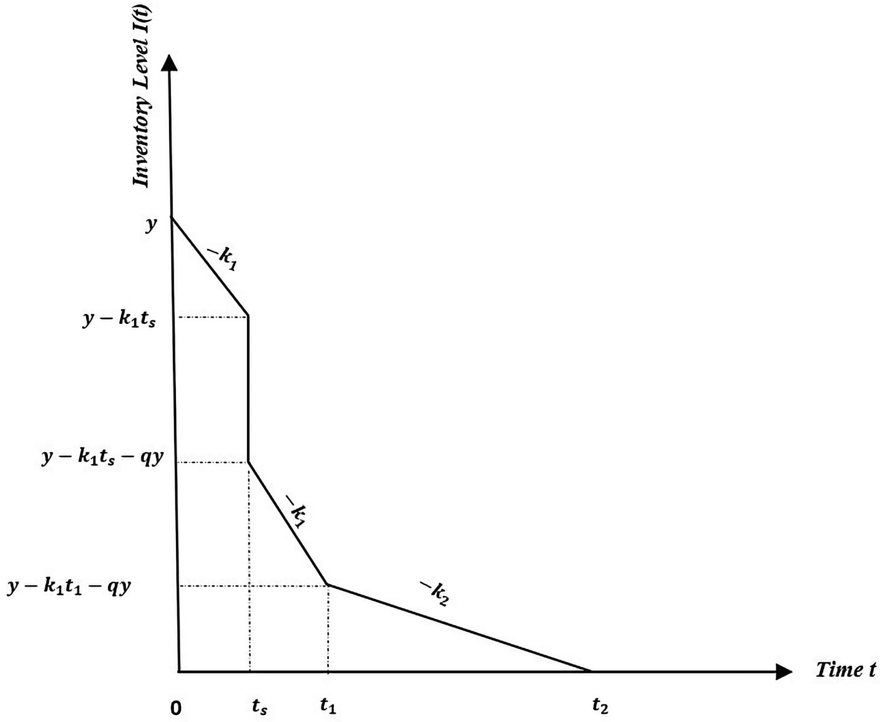


Figure 5.1: Raw material inventory level with imperfect items sold at a discount

the demand. Therefore, the stock level of raw material used for production is depleted at a rate of k_1 until the end of the screening cycle. When the screening process stops, the quantity of raw material reaches a level of

$$y - k_1 t_s = y\left(1 - \frac{k_1}{x}\right) \tag{5.5}$$

At this time, the imperfect raw materials are separated from the perfect ones and sold as a single batch at a discounted price s_r. Thus, the level of raw material in Equation (5.5) drops further by a quantity qy. The quantity of raw material left is

$$y - k_1 t_s - qy = y\left(1 - \frac{k_1}{x} - q\right) \tag{5.6}$$

Moreover, the production continues at the rate k_1 until the end of the first production cycle (Figure 5.2). At time t_1, the raw material level in Equation (5.6) drops by a quantity

$$k_1\left(t_1 - t_s\right) \tag{5.7}$$

Subtracting Equation (5.7) from Equation (5.6) results in

$$y - k_1 t_s - qy - k_1\left(t_1 - t_s\right) = y(1 - q) - k_1 t_1 \tag{5.8}$$

At this time during the cycle, the number of perfect quality items produced reaches a level of inventory I_1. However, it is assumed that the production rate switches over at t_1 (Figure 5.2), just after reaching the inventory level I_1, and the system is automatically reconfigured to continue to be operational but at a lower production rate, k_2 ($k_2 < k_1$) to ensure the continuity of the production. The raw material in Equation (5.8) continues

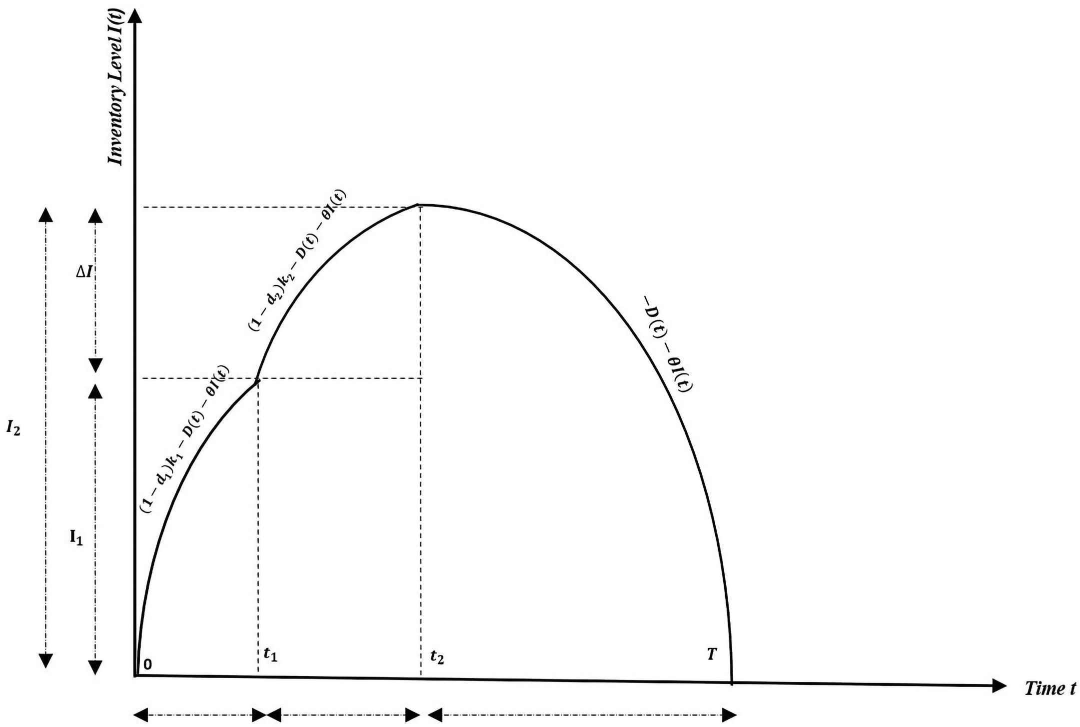


Figure 5.2: Inventory profile with a shift in production rate and stock dependent demand

to decrease at a rate k_2 until it reaches zero at the end of production cycle t_2, and the inventory of perfect items produced accumulates at the rate $(1 - d_2)\, k_2 - A - \gamma I(t) - \theta I(t)$ until a level I_2 is reached. During this period, the quantity of imperfect quality items accumulates at the rate $d_2 k_2$, which ends at time t_2. The production is then stopped at t_2. The cycle then repeats itself after time T.

The differential equations that represent the state of the production system in the interval $[0, T]$ are given by

$$\frac{dI(t)}{dt} + \theta I(t) = (1 - d_1)\, k_1 - A - \gamma I(t) \qquad 0 \leq t \leq t_1 \tag{5.9a}$$

$$\frac{dI(t)}{dt} + \theta I(t) = (1 - d_2)\, k_2 - A - \gamma I(t) \qquad t_1 \leq t \leq t_2 \tag{5.9b}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -A - \gamma I(t) \qquad t_2 \leq t \leq t_3 \tag{5.9c}$$

Solving Equation (5.9a), one can obtain

$$I(t) = \left[\frac{(1 - d_1)\, k_1}{\theta + \gamma} - \frac{A}{\theta + \gamma}\right] + L_1 e^{-(\theta+\gamma)t} \tag{5.10}$$

From Equation (5.10) under the boundary condition, $I(0) = 0$, the following is obtained

$$L_1 = \frac{A}{\theta + \gamma} - \frac{(1 - d_1)\, k_1}{\theta + \gamma} \tag{5.11}$$

Substituting Equation (5.11) into Equation (5.10) results in

$$I(t) = \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[1-e^{-(\theta+\gamma)t}\right] \quad 0 \le t \le t_1 \tag{5.12}$$

Solving Equation (5.9b) leads to

$$I(t) = \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] + L_2 e^{-(\theta+\gamma)t} \tag{5.13}$$

Using the boundary condition, $I(t_1) = I_1$ in Equation (5.13), leads to

$$L_2 = \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\} e^{(\theta+\gamma)t_1} \tag{5.14}$$

Substituting Equation (5.14) into Equation (5.13) results in

$$I(t) = \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] + \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\} e^{-(\theta+\gamma)(t-t_1)} \quad t_1 \le t \le t_2 \tag{5.15}$$

Solving Equation (5.9c), leads to

$$I(t) = -\frac{A}{\theta+\gamma} + L_3 e^{-(\theta+\gamma)t} \tag{5.16}$$

From Equation (5.16) under the boundary condition, $I(t_2) = I_2$, the following is obtained

$$L_3 = \left(I_2 + \frac{A}{\theta+\gamma}\right) e^{(\theta+\gamma)t_2} \tag{5.17}$$

Substituting Equation (5.17) into Equation (5.16) leads to

$$I(t) = -\frac{A}{\theta+\gamma} + \left[I_2 + \frac{A}{\theta+\gamma}\right] e^{-(\theta+\gamma)(t-t_2)} \quad t_2 \le t \le T \tag{5.18}$$

From Equation (5.12), the inventory level, $I(t_1)$, at time t_1, can be described by the following equation

$$I\left(t_1\right) = \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[1-e^{-(\theta+\gamma)t_1}\right] \tag{5.19}$$

On the other hand

$$I\left(t_1\right) = I_1 \tag{5.20}$$

Equating Equations (19) and (20) leads to

$$I_1 = \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[1-e^{-(\theta+\gamma)t_1}\right] \tag{5.21}$$

Dividing both sides of Equation (5.21) by $\left[\frac{(1-d_1)k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]$ leads to

$$\frac{(\theta+\gamma)I_1}{\left[(1-d_1)\,k_1 - A\right]} = 1 - e^{-(\theta+\gamma)t_1} \tag{5.22}$$

$$e^{-(\theta+\gamma)t_1} = 1 - \frac{(\theta+\gamma)I_1}{[(1-d)k_1 - A]} \tag{5.23}$$

Solving for t_1 in Equation (5.23) results in

$$\ln e^{-(\theta+\gamma)t_1} = \ln\left[1 - \frac{(\theta+\gamma)I_1}{[(1-d_1)\,k_1 - A]}\right] \tag{5.24}$$

$$t_1 = -\frac{1}{(\theta+\gamma)}\ln\left[1 - \frac{(\theta+\gamma)I_1}{(1-d_1)\,k_1 - A}\right] \tag{5.25}$$

From Taylor's series expansion, and the assumption that $(\theta+\gamma)^2 \ll 1$ (neglecting higher powers of $(\theta+\gamma)$, the expansion of the logarithmic function of Equation (5.25) leads to the following approximation

$$\begin{aligned} t_1 &= -\frac{1}{\theta+\gamma}\left[-\frac{(\theta+\gamma)I_1}{(1-d_1)\,k_1 - A} - \frac{(\theta+\gamma)^2 I_1^2}{2\left[(1-d_1)\,k_1 - A\right]^2}\right] && (5.26a)\\ &= \frac{I_1}{(1-d_1)\,k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d_1)\,k_1 - A\right]^2} && (5.26b)\\ &= \frac{I_1}{\rho_1} + \frac{(\theta+\gamma)I_1^2}{2\rho_1^2} && (5.26c)\\ &\approx \frac{I_1}{\rho_1} && (5.26d) \end{aligned}$$

With

$$(1-d_1)\,k_1 - A = \rho_1 \tag{5.27}$$

Thus, t_1 can be written in terms of I_1 and so, t_1 is not a decision variable.

From Equation (5.15), under the boundary condition $I(t_2) = I_2$, one can obtain

$$I_2 = \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] + \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}e^{-(\theta+\gamma)(t_2-t_1)} \tag{5.28}$$

Subtracting $\left[\frac{(1-d_2)k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]$ from both sides of Equation (5.28) leads to

$$I_2 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] = \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}e^{-(\theta+\gamma)(t_2-t_1)} \tag{5.29}$$

Dividing both sides of Equation (5.29) by $\left\{I_1 - \left[\frac{(1-d_2)k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}$ leads to

$$\frac{\frac{(\theta+\gamma)I_2-(1-d_2)k_2+A}{\theta+\gamma}}{\frac{(\theta+\gamma)I_1-(1-d_2)k_2+A}{\theta+\gamma}} = e^{-(\theta+\gamma)(t_2-t_1)} \tag{5.30}$$

$$\frac{\frac{(\theta+\gamma)I_1-(1-d_2)k_2+A}{-(1-d_2)k_2+A}}{\frac{(\theta+\gamma)I_2-(1-d_2)k_2+A}{-(1-d_2)k_2+A}} = e^{(\theta+\gamma)(t_2-t_1)} \tag{5.31}$$

$$e^{(\theta+\gamma)(t_2-t_1)} = \left[1 - \frac{(\theta+\gamma)I_1}{(1-d_2)\,k_2 - A}\right]\left[1 - \frac{(\theta+\gamma)I_2}{(1-d_2)\,k_2 - A}\right]^{-1} \tag{5.32}$$

Solving for t_2 in Equation (5.32) results in

$$\ln e^{(\theta+\gamma)(t_2-t_1)} = \ln\left[1 - \frac{(\theta+\gamma)I_1}{(1-d_2)\,k_2 - A}\right]\left[1 - \frac{(\theta+\gamma)I_2}{(1-d_2)\,k_2 - A}\right]^{-1} \tag{5.33}$$

$$t_2 - t_1 = \frac{1}{\theta+\gamma}\ln\left[1 - \frac{(\theta+\gamma)I_1}{(1-d_2)\,k_2 - A}\right] - \frac{1}{\theta+\gamma}\ln\left[1 - \frac{(\theta+\gamma)I_2}{(1-d_2)\,k_2 - A}\right] \tag{5.34}$$

From Taylor's series expansion, and the assumption $(\theta+\gamma)^2 \ll 1$ (neglecting higher powers of $(\theta+\gamma)$, the expansion of the logarithmic function of Equation (54) leads to the following approximation

$$\begin{aligned} t_2 - t_1 = {} & \frac{1}{\theta+\gamma}\left[-\frac{(\theta+\gamma)I_1}{(1-d_2)\,k_2 - A} - \frac{(\theta+\gamma)^2 I_1^2}{2\left[(1-d_2)\,k_2 - A\right]^2}\right] \\ & - \frac{1}{\theta+\gamma}\left[-\frac{(\theta+\gamma)I_2}{(1-d_2)\,k_2 - A} - \frac{(\theta+\gamma)^2 I_2^2}{2\left[(1-d_2)\,k_2 - A\right]^2}\right] \end{aligned} \tag{5.35}$$

$$t_2 - t_1 = \frac{I_2 - I_1}{(1-d_2)\,k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]^2} \tag{5.36a}$$

$$t_2 = \frac{I_2 - I_1}{\rho_2} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\rho_2^2} + \frac{I_1}{\rho_1} + \frac{(\theta+\gamma)I_1^2}{2\rho_1^2} \tag{5.36b}$$

$$t_2 \approx \frac{I_2 - I_1}{\rho_2} + \frac{I_1}{\rho_1} \tag{5.36c}$$

With

$$(1-d_2)\,k_2 - A = \rho_2 \tag{5.37}$$

Thus, t_2 can be written in terms of I_1 and I_2. Therefore, t_2 is not a decision variable. Applying the boundary condition $I(T) = 0$ in Equation (5.18), results in

$$0 = -\frac{A}{\theta+\gamma} + \left[I_2 + \frac{A}{\theta+\gamma}\right] e^{-(\theta+\gamma)(T-t_2)} \tag{5.38}$$

Factoring out $\left[I_2 + \frac{A}{\theta+\gamma}\right]$ from Equation (5.38), and solving for T yields

$$\frac{\frac{A}{\theta+\gamma}}{\left[\frac{A+(\theta+\gamma)I_2}{\theta+\gamma}\right]} = e^{-(\theta+\gamma)(T-t_2)} \tag{5.39}$$

$$T - t_2 = \frac{1}{(\theta+\gamma)}\ln\left[\frac{A+(\theta+\gamma)I_2}{A}\right] \tag{5.40}$$

For small values of $(\theta+\gamma)$ and using Taylor series to approximate Equation (5.40) yields

$$T - t_2 \approx \frac{1}{(\theta+\gamma)}\left[\frac{(\theta+\gamma)}{A}I_2 - \frac{(\theta+\gamma)^2}{2A^2}I_2^2\right] \tag{5.41}$$

Substituting Equation (5.36b) into Equation (5.41) leads

$$
\begin{aligned}
T &= \frac{I_2}{A} - \frac{(\theta+\gamma)I_2^2}{2A^2} + \frac{I_2 - I_1}{\rho_2} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\rho_2^2} + \frac{I_1}{\rho_1} + \frac{(\theta+\gamma)I_1^2}{2\rho_1^2} && (5.42a)\\
&\approx \frac{I_2}{A} + \frac{I_2 - I_1}{\rho_2} + \frac{I_1}{\rho_1} && (5.42b)\\
&\approx \left(\frac{1}{A} + \frac{1}{\rho_2}\right) I_2 + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) I_1 && (5.42c)\\
&\approx \psi I_2 + \tau I_1 && (5.42d)
\end{aligned}
$$

with

$$
\begin{aligned}
\left(\frac{1}{A} + \frac{1}{\rho_2}\right) &= \psi && (5.43a)\\
\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) &= \tau && (5.43b)
\end{aligned}
$$

Extracting I_2 from Equation (5.42d), yields

$$
\begin{aligned}
I_2 &= \frac{T - \tau I_1}{\psi} && (5.44a)\\
&= \frac{T - \left(\frac{\rho_2 - \rho_1}{\rho_1 \rho_2}\right) I_1}{\frac{\rho_2 + A}{A\rho_2}} && (5.44b)\\
&= \frac{\rho_1\rho_2 T + (\rho_1 - \rho_2) I_1}{\rho_1 \rho_2} \frac{A\rho_2}{\rho_2 + A} && (5.44c)\\
&= \frac{A\rho_1\rho_2}{\rho_1(\rho_2 + A)} T + \frac{A(\rho_1 - \rho_2)}{\rho_1(\rho_2 + A)} I_1 && (5.44d)\\
&= \xi T + \upsilon I_1 && (5.44e)
\end{aligned}
$$

with

$$
\begin{aligned}
\xi &= \frac{A\rho_1\rho_2}{\rho_1(\rho_2 + A)} && (5.45a)\\
\upsilon &= \frac{A(\rho_1 - \rho_2)}{\rho_1(\rho_2 + A)} && (5.45b)
\end{aligned}
$$

I_2 can be written in terms of I_1 and T . Therefore, I_2 is not a decision variable.

5.3.2 Manufacturer's cost components

To determine the optimal quantities, the total cost per cycle is first calculated by summing the following costs: ordering cost, deterioration cost, production cost, setup cost for production, inventory holding cost, lost sales cost, lost production cost, shortage cost, and purchasing cost.

5.3.2.1 Manufacturer's ordering cost of raw material

The ordering cost of raw materials is considered fixed and represented by

$$
OC_r = F_r \qquad (5.46)
$$

5.3.2.2 Manufacturer's inventory holding cost of raw material

The holding cost of raw material is the product of the average inventory, and the holding cost per unit of raw material per unit time (h_r). To find the average inventory of raw materials, the area in Figure 5.1 is divided by the cycle length. The graph in Figure 5.1 was decomposed into two trapezoids and a triangle, and then the area of each trapezoid was calculated and summed up to find the total area. Hence, the total area representing the inventory of raw materials is given by

$$\frac{y+(y-k_1t_s)}{2}t_s+\frac{[(y-k_1t_s-qy)+(y-k_1t_1-qy)]}{2}(t_1-t_s) + \frac{(y-k_1t_1-qy)}{2}(t_2-t_1) \tag{5.47a}$$

$$=\frac{[2qy+k_1t_1]}{2}t_s+\frac{[(y-qy-k_1t_s)]}{2}t_1+\frac{(y-qy-k_1t_1)}{2}t_2 \tag{5.47b}$$

The quantity of raw material required for the exact production in each cycle is given by

$$y=k_1t_1+k_2(t_2-t_1)+qy \tag{5.48a}$$

$$=\frac{k_1t_1+k_2(t_2-t_1)}{1-q} \tag{5.48b}$$

Substituting Equations (5.4) and (5.48b) into Equation (5.47b), yields

$$\frac{q\left[k_1t_1+k_2(t_2-t_1)\right]^2}{x(1-q)^2}+\frac{(k_1-k_2)t_1^2+k_2t_2^2}{2} \tag{5.49}$$

Substituting Equations (5.26d) and (5.36c) into Equation (5.49), leads to

$$\frac{q}{x(1-q)^2}\left[k_1\frac{I_1}{\rho_1}+k_2\frac{I_2-I_1}{\rho_2}\right]^2+\frac{1}{2}(k_1-k_2)\left(\frac{I_1}{\rho_1}\right)^2+\frac{1}{2}k_2\left(\frac{I_2-I_1}{\rho_2}+\frac{I_1}{\rho_1}\right)^2 \tag{5.50}$$

Hence, the inventory holding cost of raw materials is given by

$$THR=h_r\left\{\frac{q}{x(1-q)^2}\left[k_1\frac{I_1}{\rho_1}+k_2\frac{I_2-I_1}{\rho_2}\right]^2+\frac{1}{2}(k_1-k_2)\left(\frac{I_1}{\rho_1}\right)^2+\frac{1}{2}k_2\left(\frac{I_2-I_1}{\rho_2}+\frac{I_1}{\rho_1}\right)^2\right\} \tag{5.51}$$

5.3.2.3 Manufacturer's purchasing cost of raw material

To calculate the cost of procuring raw materials, $TCRM$, the unit cost c_r is multiplied by the quantity y required for one production cycle.

$$TCRM=C_r\times y \tag{5.52}$$

Substituting Equation (5.48b) into Equation (5.52) leads to

$$TCRM=\frac{C_r}{1-q}\left[k_1\frac{I_1}{\rho_1}+k_2\frac{I_2-I_1}{\rho_2}\right] \tag{5.53}$$

5.3.2.4 Manufacturer's set up

The setup cost in the manufacturing process varies over different intervals, denoted as SC, with specific values assigned during distinct time periods. In the initial interval, from 0 to t_1, the setup cost is represented by G_1, reflecting the corresponding requirements and expenses during that phase. Subsequently, in the interval t_1 to t_2, the setup cost transitions to G_2, capturing the effect of switching from production rate k_1 to k_2

$$SC = \begin{cases} G_1 & 0 \leq t \leq t_1 \\ G_2 & t_1 \leq t \leq t_2 \end{cases} \tag{5.54}$$

Therefore, the total setup cost is given by the following

$$TSC = G_1 + G_2 \tag{5.55}$$

5.3.2.5 Manufacturer's inventory holding cost

The manufacturer incurs a holding cost, THC, which is the product of the holding inventory carried throughout the production-consumption cycle and the cost of holding a single unit per time unit (h_p). Thus

$$THC = h_p \left[\int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt + \int_{t_2}^{T} I(t)dt \right] \tag{5.56}$$

$$\int_0^{t_1} I(t)dt = \int_0^{t_1} \left\{ \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left[1 - e^{-(\theta+\gamma)t} \right] \right\} dt \tag{5.57a}$$

$$= \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \int_0^{t_1} \left[1 - e^{-(\theta+\gamma)t} \right] dt \tag{5.57b}$$

$$= \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left(t_1 + \frac{1}{\theta+\gamma} e^{-(\theta+\gamma)t_1} \right) \tag{5.57c}$$

$$= \left[\frac{(1-d_1)\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left(t_1 + \frac{1}{\theta+\gamma} e^{-(\theta+\gamma)t_1} \right) - \frac{1}{\theta+\gamma} \left[\frac{(1-d)k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \tag{5.57d}$$

$$= \frac{1}{\theta+\gamma} \left[(1-d_1)\,k_1 - A \right] \left[t_1 + \frac{1}{\theta+\gamma} \left[e^{-(\theta+\gamma)t_1} - 1 \right] \right] \tag{5.57e}$$

Multiplying both sides of Equation (5.23) by $\frac{-1}{(\theta+\gamma)}$ leads to

$$-\frac{I_1}{\left[(1-d_1)\,k_1 - A\right]} = \frac{1}{(\theta+\gamma)} \left[-1 + e^{-(\theta+\gamma)t_1} \right] \tag{5.58}$$

Substituting Equations (5.26b) and (5.58) into Equation (5.57e), leads to

$$\begin{aligned} = \frac{1}{\theta+\gamma} \left[(1-d_1)\,k_1 - A \right] \Bigg[& \frac{I_1}{(1-d_1)\,k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d_1)\,k_1 - A\right]^2} \\ & - \frac{I_1}{\left[(1-d_1)\,k_1 - A\right]} \Bigg] \end{aligned} \tag{5.59}$$

$$= \frac{I_1^2}{2\left[(1-d_1)\,k_1 - A\right]} \tag{5.60a}$$

$$= \frac{I_1^2}{2\rho_1} \tag{5.60b}$$

$$\int_{t_1}^{t_2} I(t)dt = \int_{t_1}^{t_2} \left\{ \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] + \left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t-t_1)} \right\} dt \tag{5.61}$$

$$\begin{aligned} &= \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right] t_2 - \frac{1}{\theta+\gamma}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &\quad - \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right] t_1 \\ &\quad + \frac{1}{\theta+\gamma}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_1-t_1)} \end{aligned} \tag{5.62}$$

$$\begin{aligned} &= \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right] t_2 - \frac{1}{\theta+\gamma}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &\quad - \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right] t_1 + \frac{1}{\theta+\gamma}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_1-t_1)} \end{aligned} \tag{5.63}$$

$$\begin{aligned} &= \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right](t_2-t_1) - \frac{1}{\theta+\gamma}\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \\ &\quad - \frac{1}{\theta+\gamma}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} + \frac{1}{\theta+\gamma} I_1 \end{aligned} \tag{5.64}$$

Multiplying both sides of Equation (5.28) by $-\frac{1}{\theta+\gamma}$ results in

$$\begin{aligned} &-\frac{1}{(\theta+\gamma)}\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] - \frac{1}{(\theta+\gamma)}\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &= -\frac{1}{\theta+\gamma} I_2 \end{aligned} \tag{5.65}$$

Substituting Equations (5.36a) and (5.65) into Equation (5.64), yields

$$= \frac{1}{\theta+\gamma}\left[(1-d_2)\,k_2 - A\right]\left[\frac{I_2 - I_1}{(1-d_2)\,k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]^2} \right] - \frac{1}{\theta+\gamma} I_2 + \frac{1}{\theta+\gamma} I_1 \tag{5.66a}$$

$$= \frac{\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]} \tag{5.66b}$$

$$= \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} \tag{5.66c}$$

$$\int_{t_2}^{T} I(t)dt = \int_{t_2}^{T} \left[-\frac{A}{\theta+\gamma} + \left[I_2 + \frac{A}{\theta+\gamma} \right] e^{-(\theta+\gamma)(t-t_2)} \right] dt \tag{5.67}$$

$$= -\frac{A}{\theta+\gamma}(T-t_2) - \frac{1}{\theta+\gamma}\left[I_2 + \frac{A}{\theta+\gamma} \right] e^{-\theta(T-t_2)} + \frac{1}{\theta+\gamma}\left[I_2 + \frac{A}{\theta+\gamma} \right] \tag{5.68}$$

$$= -\frac{A}{\theta+\gamma}(T-t_2) - \frac{1}{\theta+\gamma}\left[-\frac{A}{\theta+\gamma} + \left(I_2 + \frac{A}{\theta+\gamma} \right) e^{-(\theta+\gamma)(T-t_2)} \right] + \frac{1}{\theta+\gamma} I_2 \tag{5.69}$$

Using Equation (5.38) to simplify Equation (5.69) leads to

$$-\frac{A}{\theta+\gamma}(t_3-t_2) - \frac{1}{\theta+\gamma}0 + \frac{1}{\theta+\gamma}I_2 \tag{5.70a}$$

$$= -\frac{A}{\theta+\gamma}\left[\frac{I_2}{A} - \frac{(\theta+\gamma)I_2^2}{2A^2} \right] + \frac{1}{\theta+\gamma}I_2 \tag{5.70b}$$

$$= \frac{I_2^2}{2A} \tag{5.70c}$$

Summing up Equations (5.60b), (5.66c), and (5.70c) results in the following expression for the total inventory carrying cost over the period $[0, T]$

$$THC = h_p \left[\frac{I_1^2}{2\rho_1} + \frac{(I_2^2 - I_1^2)}{2\rho_2} + \frac{I_2^2}{2A} \right] \tag{5.71}$$

5.3.2.6 Manufacturer's deterioration cost

The total number of deteriorated items over the time interval $[0, T]$ is obtained by integrating the deterioration function over the interval $[0, T]$. Therefore

$$QD_p = \int_{0}^{t_1} \left[(1-d_1) k_1 - A - \gamma I(t) \right] dt + \int_{t_1}^{t_2} \left[(1-d_2) k_2 - A - \gamma I(t) \right] dt - \int_{t_2}^{T} [A + \gamma I(t)] dt \tag{5.72}$$

In order to evaluate the total quantity of deteriorated products (QD_p), one proceeds by solving each integral in Equation (5.73).

$$\int_{0}^{t_1} \left[(1-d_1) k_1 - A - \gamma I(t) \right] dt$$
$$= \int_{0}^{t_1} \left\{ (1-d_1) k_1 - A - \gamma \left[\frac{(1-d_1) k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left[1 - e^{-(\theta+\gamma)t} \right] \right\} dt \tag{5.73}$$

$$\begin{aligned} &= [(1-d_1) k_1 - A] \left[\frac{I_1}{(1-d)k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d)k_1 - A\right]^2} \right] \\ &\quad - \gamma \left[\frac{(1-d_1) k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left[t_1 + \frac{1}{\theta+\gamma} e^{-(\theta+\gamma)t_1} \right] \\ &\quad + \gamma \left[\frac{(1-d_1) k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \left[\frac{1}{\theta+\gamma} \right] \end{aligned} \tag{5.74}$$

$$\begin{aligned} &= \left[(1-d_1)\,k_1 - A\right]\left[\frac{I_1}{(1-d)k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d)k_1 - A\right]^2}\right] \\ &- \frac{\gamma}{\theta+\gamma}\left[(1-d_1)\,k_1 - A\right]\left\{t_1 + \frac{1}{\theta+\gamma}\left[e^{-(\theta+\gamma)t_1} - 1\right]\right\} \end{aligned} \tag{5.75}$$

Substituting Equations (5.26b) and (5.57) into equation (5.74), leads to

$$\begin{aligned} &\left[(1-d_1)\,k_1 - A\right]\left[\frac{I_1}{(1-d_1)k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d_1)k_1 - A\right]^2}\right] \\ &- \frac{\gamma}{\theta+\gamma}\left[(1-d_1)\,k_1 - A\right]\left[\frac{I_1}{(1-d_1)\,k_1 - A} + \frac{(\theta+\gamma)I_1^2}{2\left[(1-d_1)\,k_1 - A\right]^2} - \frac{I_1}{\left[(1-d_1)\,k_1 - A\right]}\right] \end{aligned} \tag{5.76a}$$

$$= I_1 + \frac{\theta I_1^2}{2\left[(1-d_1)\,k_1 - A\right]} \tag{5.76b}$$

$$\begin{aligned} &\int_{t_1}^{t_2}\left[(1-d_1)\,k_2 - A - \gamma I(t)\right]dt \\ &\quad = \int_{t_1}^{t_2}\left[(1-d_2)\,k_2 - A\right]dt \\ &\quad - \gamma\int_{t_1}^{t_2}\left\{\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] + \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}e^{-(\theta+\gamma)(t-t_1)}\right\}dt \end{aligned} \tag{5.77}$$

$$\begin{aligned} &= \int_{t_1}^{t_2}\left[(1-d_2)\,k_2 - A\right]dt - \gamma\int_{t_1}^{t_2}\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]dt \\ &- \gamma\int_{t_1}^{t_2}\left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}e^{-(\theta+\gamma)(t-t_1)}dt \end{aligned} \tag{5.78}$$

Equation (5.78) can be integrated separately as follow

$$\int_{t_1}^{t_2}\left[(1-d_2)\,k_2 - A\right]dt = \left[(1-d_2)\,k_2 - A\right](t_2 - t_1) \tag{5.79}$$

Substituting Equation (5.36a), yields

$$\left[(1-d_2)\,k_2 - A\right]\left[\frac{I_2 - I_1}{(1-d)k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d)k_2 - A\right]^2}\right] \tag{5.80a}$$

$$= \left[I_2 - I_1 + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]}\right] \tag{5.80b}$$

$$\int_{t_1}^{t_2}\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]dt = \frac{1}{(\theta+\gamma)}\left[(1-d_2)\,k_2 - A\right](t_2 - t_1) \tag{5.81}$$

Substituting Equation (5.36a) into Equation (5.81), yields

$$\frac{1}{(\theta+\gamma)}\left[(1-d_2)\,k_2 - A\right]\left[\frac{I_2 - I_1}{(1-d_2)\,k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]^2}\right] \tag{5.82a}$$

$$= \frac{1}{(\theta+\gamma)}\left(I_2 - I_1\right) + \frac{\left(I_2^2 - I_1^2\right)}{2\left[(1-d_2)\,k_2 - A\right]} \tag{5.82b}$$

$$\int_{t_1}^{t_2} \left[\left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t-t_1)} \right] dt \tag{5.83}$$

$$\begin{aligned} &= -\frac{1}{(\theta+\gamma)} \left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &\quad + \frac{1}{(\theta+\gamma)} \left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_1-t_1)} \end{aligned} \tag{5.84a}$$

$$\begin{aligned} &= -\frac{1}{(\theta+\gamma)} \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] - \frac{1}{(\theta+\gamma)} \left\{ I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma} \right] \right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &\quad + \frac{1}{(\theta+\gamma)} I_1 \end{aligned} \tag{5.84b}$$

Substituting Equation (5.65) into Equation (5.84b) yields

$$-\frac{(I_2 - I_1)}{\theta+\gamma} \tag{5.85}$$

Combining Equations (5.80b), (5.82b), and (5.85) yields the revised form of Equation Equation (5.78), which can be expressed as

$$\left[I_2 - I_1 + \frac{(\theta+\gamma)\,(I_2^2 - I_1^2)}{2\,[(1-d_2)\,k_2 - A]} \right] - \gamma \left[\frac{I_2 - I_1}{(\theta+\gamma)} + \frac{(I_2^2 - I_1^2)}{2\,[(1-d_2)\,k_2 - A]} - \frac{I_2 - I_1}{\theta+\gamma} \right] \tag{5.86a}$$

$$= I_2 - I_1 + \frac{\theta\,(I_2^2 - I_1^2)}{2\,[(1-d_2)\,k_2 - A]} \tag{5.86b}$$

$$\int_{t_2}^{T} [A + \gamma I(t)] dt = \int_{t_2}^{T} A dt + \gamma \int_{t_2}^{T} \left[-\frac{A}{\theta+\gamma} + \left(I_2 + \frac{A}{\theta+\gamma} \right) e^{-(\theta+\gamma)(t-t_2)} \right] dt \tag{5.87}$$

$$= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} - \frac{\gamma A}{\theta+\gamma}(T-t_2) - \frac{\gamma}{\theta+\gamma} \left[-\frac{A}{(\theta+\gamma)} + \left(I_2 + \frac{A}{\theta+\gamma} \right) e^{-(\theta+\gamma)(T-t_2)} \right] + \frac{\gamma}{\theta+\gamma} I_2 \tag{5.88}$$

Using Equations (5.38) and (5.41) to simplify Equation (5.88) leads to

$$= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} - \frac{\gamma A}{\theta+\gamma} \left[\frac{I_2}{A} - \frac{(\theta+\gamma)I_2^2}{2A^2} \right] + \frac{\gamma}{\theta+\gamma} I_2 \tag{5.89a}$$

$$= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} + \gamma \frac{I_2^2}{2A} \tag{5.89b}$$

$$= I_2 - \frac{\theta I_2^2}{2A} \tag{5.89c}$$

Summing up Equations (5.76b), (5.86b), and (5.89c) yields the total quantity of deteriorated products throughout the entire cycle. This is represented by

$$QD_p = I_1 + \frac{\theta I_1^2}{2\,[(1-d_1)\,k_1 - A]} + I_2 - I_1 + \frac{\theta\,(I_2^2 - I_1^2)}{2\,[(1-d_2)\,k_2 - A]} - I_2 + \frac{\theta I_2^2}{2A} \tag{5.90a}$$

$$= \frac{\theta I_1^2}{2\,[(1-d_1)\,k_1 - A]} + \frac{\theta\,(I_2^2 - I_1^2)}{2\,[(1-d_2)\,k_2 - A]} + \frac{\theta I_2^2}{2A} \tag{5.90b}$$

$$= \frac{\theta I_1^2}{2\rho_1} + \frac{\theta\,(I_2^2 - I_1^2)}{2\rho_2} + \frac{\theta I_2^2}{2A} \tag{5.90c}$$

Hence, the manufacturer's cost for deteriorating inventory, TCD, is obtained by multiplying the quantity of deteriorated products QD_p by the unit cost of a deteriorated unit (c_d)

$$TCD = C_d \left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2A} \right] \tag{5.91}$$

5.3.2.7 Manufacturer's production cost

The total production cost over the period $[0, T]$ is given by

$$\begin{aligned} TPC =& \frac{1}{(1-q)} \left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1 \right] k_1 t_1 + \frac{1}{(1-q)} \left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2 \right] k_2 \left(t_2 - t_1\right) & \text{(5.92a)} \\ =& \frac{1}{(1-q)} \left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1 \right] k_1 \left(\frac{I_1}{\rho_1} \right) + \frac{1}{(1-q)} \left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2 \right] k_2 \left(\frac{I_2 - I_1}{\rho_2} \right) & \text{(5.92b)} \end{aligned}$$

5.3.2.8 Manufacturer's lost production cost

In many practical cases, a penalty is incurred if the producer fails to deliver the agreed quantity in time. The lost production cost captures the penalty involved due to the above reasons, and it represents the opportunity cost for not producing the planned quantity, $k_1 t_2$. Mathematically, it can be written as (Ben- Daya et al., 2008)

$$\begin{aligned} LP &= p_l \left(k_1 - k_2\right) \left(t_2 - t_1\right) \\ &= p_l \left(k_1 - k_2\right) \left(\frac{I_2 - I_1}{\rho_2} \right) \end{aligned} \tag{5.93}$$

5.3.2.9 Total cost rate

The total cost rate is the sum of all the costs incurred at each of the two echelons. Therefore, the total cost per unit time, CT, is the sum of Equations (5.46), (5.51), (5.53), (5.55), (5.71), (5.91), (5.92b), and (5.93) dived by the cycle time T. The mathematical expression of the total cost rate is thus

$$TC_1 = \frac{1}{T} \left\{ \begin{aligned} & \frac{1}{(1-q)} \left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1 \right] k_1 \left(\frac{I_1}{\rho_1} \right) + \frac{1}{(1-q)} \left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2 \right] k_2 \left(\frac{I_2 - I_1}{\rho_2} \right) \\ & + p_l \left(k_1 - k_2\right) \left(\frac{I_2 - I_1}{\rho_2} \right) + h_p \left[\frac{I_1^2}{2\rho_1} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{I_2^2}{2A} \right] \\ & + c_d \left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2A} \right] + G_1 + G_2 + \frac{c_r}{1-q} \left[k_1 \left(\frac{I_1}{\rho_1} \right) + k_2 \left(\frac{I_2 - I_1}{\rho_2} \right) \right] \\ & + \frac{q h_r}{x(1-q)^2} \left[k_1 \left(\frac{I_1}{\rho_1} \right) + k_2 \left(\frac{I_2 - I_1}{\rho_2} \right) \right]^2 \\ & + \frac{1}{2} h_r \left(k_1 - k_2\right) \left(\frac{I_1}{\rho_1} \right)^2 + \frac{1}{2} h_r k_2 \left(\frac{I_2 - I_1}{\rho_2} + \frac{I_1}{\rho_1} \right)^2 + F_r \end{aligned} \right\} \tag{5.94}$$

5.3.2.10 Manufacturer's total revenue per time

The total revenue function (TR) represents the sum of revenue from sales of finished products with perfect quality and discounted sales of imperfect items produced, as well as imperfect raw materials. That is

$$TR_1 = s_p Q_s + s_d(1-q)\left[d_1 k_1 t_1 + d_2 k_2 (t_2 - t_1)\right] + s_r q \left[\frac{k_1 t_1 + k_2 (t_2 - t_1)}{1-q}\right] \tag{5.95}$$

With Q_s, representing the total quantity of products of good quality sold, which is obtained by integrating the demand function, $D[I(t)]$, over the specified time intervals $[0, t_1]$, $[t_1, t_2]$, and $[t_2, T]$ respectively. Hence, the quantity Q_s is

$$Q_s = \int_0^{t_1} D[I(t)]dt + \int_{t_1}^{t_2} D[I(t)]dt + \int_{t_2}^{T} D[I(t)]dt \tag{5.96}$$

Integrating each component of Equation (5.96) separately yields

$$\int_0^{t_1} D[I(t)]dt = \int_0^{t_1} \left[A + \gamma\left[\frac{(1-d_1))\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[1 - e^{-(\theta+\gamma)t}\right]\right] dt \tag{5.97}$$

$$\begin{aligned} &= At_1 + \gamma\left[\frac{(1-d_1))\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[t_1 + \frac{1}{(\theta+\gamma)}e^{-(\theta+\gamma)t_1}\right] \\ &\quad -\gamma\left[\frac{(1-d_1))\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[\frac{1}{(\theta+\gamma)}\right] \end{aligned} \tag{5.98}$$

$$= At_1 + \gamma\left[\frac{(1-d_1))\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[t_1 + \frac{1}{(\theta+\gamma)}\left[-1 + e^{-(\theta+\gamma)t_1}\right]\right] \tag{5.99}$$

Substituting Equations (5.22) and (5.26b) into equation (5.99) leads to

$$\begin{aligned} &= A\left[\frac{l_1}{(1-d)k_1 - A}\right] + \gamma\left[\frac{(1-d_1))\,k_1}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right] \\ &\qquad \times \left[\frac{l_1}{(1-d)k_1 - A} + \frac{(\theta+\gamma)l_1^2}{2\left[(1-d)k_1 - A\right]^2} - \frac{l_1}{\left[(1-d_1)\,k_1 - A\right]}\right] \end{aligned} \tag{5.100}$$

$$= A\left(\frac{I_1}{\rho_1}\right) + \frac{\gamma I_1^2}{2\rho_1} \tag{5.101}$$

The solution of the second integral is

$$\begin{aligned} \int_{t_1}^{t_2} D[I(t)]dt = \int_{t_1}^{t_2} &\left[A + \gamma\left\{\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right.\right. \\ &\left.\left. + \left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\} e^{-(\theta+\gamma)(t-t_1)}\right\}\right] dt. \end{aligned} \tag{5.102}$$

$$\begin{aligned} &= A\,(t_2 - t_1) + \gamma\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right](t_2 - t_1) \\ &\quad - \frac{\gamma}{\theta+\gamma}\left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\} e^{-(\theta+\gamma)(t_2-t_1)} \\ &\quad + \frac{\gamma}{\theta+\gamma}\left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\}. \end{aligned} \tag{5.103}$$

By substituting Equation (5.36c) in Equation (5.103), one can get the folowing

$$= A\left[\frac{I_2 - I_1}{(1-d)k_2 - A}\right] + \gamma\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[\frac{I_2 - I_1}{(1-d)k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d)k_2 - A\right]^2}\right]$$
$$- \frac{\gamma}{\theta+\gamma}\left\{I_1 - \left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\right\} e^{-(\theta+\gamma)(t_2-t_1)} - \frac{\gamma}{\theta+\gamma}\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]$$
$$+ \gamma\frac{I_1}{\theta+\gamma} \tag{5.104}$$

Substituting Equation (5.65) into Equation ((5.104) results in

$$= A\left[\frac{I_2 - I_1}{(1-d)k_2 - A}\right] + \gamma\left[\frac{(1-d_2)\,k_2}{\theta+\gamma} - \frac{A}{\theta+\gamma}\right]\left[\frac{I_2 - I_1}{(1-d)k_2 - A} + \frac{(\theta+\gamma)\left(I_2^2 - I_1^2\right)}{2\left[(1-d)k_2 - A\right]^2}\right]$$
$$- \frac{\gamma I_2}{\theta+\gamma} + \frac{\gamma I_1}{\theta+\gamma} \tag{5.105}$$

$$= A\left[\frac{I_2 - I_1}{(1-d)k_2 - A}\right] + \gamma\left[\frac{(I_2 - I_1)}{\theta+\gamma} + \frac{\left(I_2^2 - I_1^2\right)}{2\left[(1-d)k_2 - A\right]}\right] - \gamma\frac{(I_2 - I_1)}{\theta+\gamma} \tag{5.106}$$

$$= A\left(\frac{I_2 - I_1}{\rho_2}\right) + \frac{\gamma\left(I_2^2 - I_1^2\right)}{2\rho_2} \tag{5.107}$$

Lastly, the remaining quantity of good quantity sold over $[t_2, T]$, which is represented by the third integral, yields

$$\int_{t_2}^{T} D[I(t)]dt = \int_{t_2}^{T} A dt + \gamma\int_{t_2}^{T}\left[-\frac{A}{\theta+\gamma} + \left(I_2 + \frac{A}{\theta+\gamma}\right)e^{-(\theta+\gamma)(t-t_2)}\right]dt \tag{5.108}$$

$$\begin{aligned}= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} &+ \gamma\left[-\frac{A}{\theta+\gamma}T - \frac{1}{\theta+\gamma}\left(I_2 + \frac{A}{\theta+\gamma}\right)e^{-(\theta+\gamma)(T-t_2)}\right] \\ &-\gamma\left[-\frac{A}{\theta+\gamma}t_2 - \frac{1}{\theta+\gamma}\left(I_2 + \frac{A}{\theta+\gamma}\right)\right]\end{aligned} \tag{5.109}$$

$$= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} - \frac{\gamma A}{\theta+\gamma}(T - t_2) - \frac{\gamma}{\theta+\gamma}\left[-\frac{A}{(\theta+\gamma)} + \left(I_2 + \frac{A}{\theta+\gamma}\right)e^{-(\theta+\gamma)(T-t_2)}\right]$$
$$+\frac{\gamma}{\theta+\gamma}I_2 \tag{5.110}$$

Substituting Equation ((5.38) into Equation (5.110) leads to

$$I_2 - \frac{(\theta+\gamma)I_2^2}{2A} - \frac{\gamma A}{\theta+\gamma}(T - t_2) + \frac{\gamma}{\theta+\gamma}I_2 \tag{5.111}$$

$$= I_2 - \frac{(\theta+\gamma)I_2^2}{2A} - \frac{\gamma A}{\theta+\gamma}\left[\frac{I_2}{A} - \frac{(\theta+\gamma)I_2^2}{2A^2}\right] + \frac{\gamma}{\theta+\gamma}I_2 \tag{5.112}$$

$$= I_2 - \frac{\theta I_2^2}{2A} \tag{5.113}$$

Q_s is then obtained by grouping together equations (5.101), (5.107), and (5.113)

$$Q_s = A\left(\frac{I_1}{\rho_1}\right) + \frac{\gamma I_1^2}{2\rho_1} + A\left(\frac{I_2 - I_1}{\rho_2}\right) + \frac{\gamma\left(I_2^2 - I_1^2\right)}{2\rho_2} + I_2 - \frac{\theta I_2^2}{2A} \tag{5.114}$$

Therefore, the total revenue is

$$\begin{aligned} TR_1 = & \left[s_pA + k_1 s_d(1-q)d_1 + \frac{k_1 s_r q}{1-q}\right]\left(\frac{I_1}{\rho_1}\right) + \left[s_pA + k_2 s_d(1-q)d_2 + \frac{k_2 s_r q}{1-q}\right]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ & + s_p\left[\frac{\gamma I_1^2}{2\rho_1} - \frac{\gamma I_1^2}{2\rho_2} + I_2 + \left(\frac{\gamma}{2\rho_2} - \frac{\theta}{2A}\right) I_2^2\right] \end{aligned} \tag{5.115}$$

Dividing the above equation by T, the following expression representing the average revenue per cycle is obtained

$$\begin{aligned} TR_1 = & \frac{1}{T}\left[s_pA + k_1 s_d(1-q)d_1 + \frac{k_1 s_r q}{1-q}\right]\left(\frac{I_1}{\rho_1}\right) + \frac{1}{T}\left[s_pA + k_2 s_d(1-q)d_2 + \frac{k_2 s_r q}{1-q}\right]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ & + \frac{s_p}{T}\left[\frac{\gamma I_1^2}{2\rho_1} - \frac{\gamma I_1^2}{2\rho_2} + I_2 + \left(\frac{\gamma}{2\rho_2} - \frac{\theta}{2A}\right) I_2^2\right] \end{aligned} \tag{5.116}$$

$$TR_1 = \frac{1}{T}\left\{\begin{aligned} & \left[s_pA + k_1 s_d(1-q)d_1 + \frac{k_1 s_r q}{1-q}\right]\left(\frac{I_1}{\rho_1}\right) + \left[s_pA + k_2 s_d(1-q)d_2 + \frac{k_2 s_r q}{1-q}\right]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ & + s_p\left[\frac{\gamma I_1^2}{2\rho_1} - \frac{\gamma I_1^2}{2\rho_2} + I_2 + \left(\frac{\gamma}{2\rho_2} - \frac{\theta}{2A}\right) I_2^2\right] \end{aligned}\right\} \tag{5.117}$$

5.3.2.11 Manufacturer's profit rate

The profit rate is calculated by subtracting the total cost per time from the total revenue generated. That is

$$TP = TR - CT \tag{5.118}$$

$$\begin{aligned} TP_1 = & \frac{1}{T}\left\{\begin{aligned} & \left[s_pA + k_1 s_d(1-q)d_1 + \frac{k_1 s_r q}{1-q}\right]\left(\frac{I_1}{\rho_1}\right) + \left[s_pA + k_2 s_d(1-q)d_2 + \frac{k_2 s_r q}{1-q}\right]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ & + s_p\left[\frac{\gamma I_1^2}{2\rho_1} - \frac{\gamma I_1^2}{2\rho_2} + I_2 + \left(\frac{\gamma}{2\rho_2} - \frac{\theta}{2A}\right) I_2^2\right] \end{aligned}\right\} \\ & - \frac{1}{T}\left\{\begin{aligned} & \frac{1}{(1-q)}\left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1\right] k_1\left(\frac{I_1}{\rho_1}\right) + \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right] k_2\left(\frac{I_2 - I_1}{\rho_2}\right) \\ & + p_l\left(k_1 - k_2\right)\left(\frac{I_2 - I_1}{\rho_2}\right) + h_p\left[\frac{I_1^2}{2\rho_1} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{I_2^2}{2A}\right] \\ & + c_d\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2A}\right] + G_1 + G_2 + \frac{c_r}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right] \\ & + \frac{q h_r}{x(1-q)^2}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right]^2 \\ & + \frac{1}{2}h_r\left(k_1 - k_2\right)\left(\frac{I_1}{\rho_1}\right)^2 + \frac{1}{2}h_r k_2\left(\frac{I_2 - I_1}{\rho_2} + \frac{I_1}{\rho_1}\right)^2 + F_r \end{aligned}\right\} \end{aligned} \tag{5.119}$$

5.3.3 Second scenario's Formulation

In this scenario, imperfect raw materials that have been screened are retained until the end of the cycle and returned to the supplier upon arrival of the next order. The inventory profile of raw materials is shown in Figure 5.3.

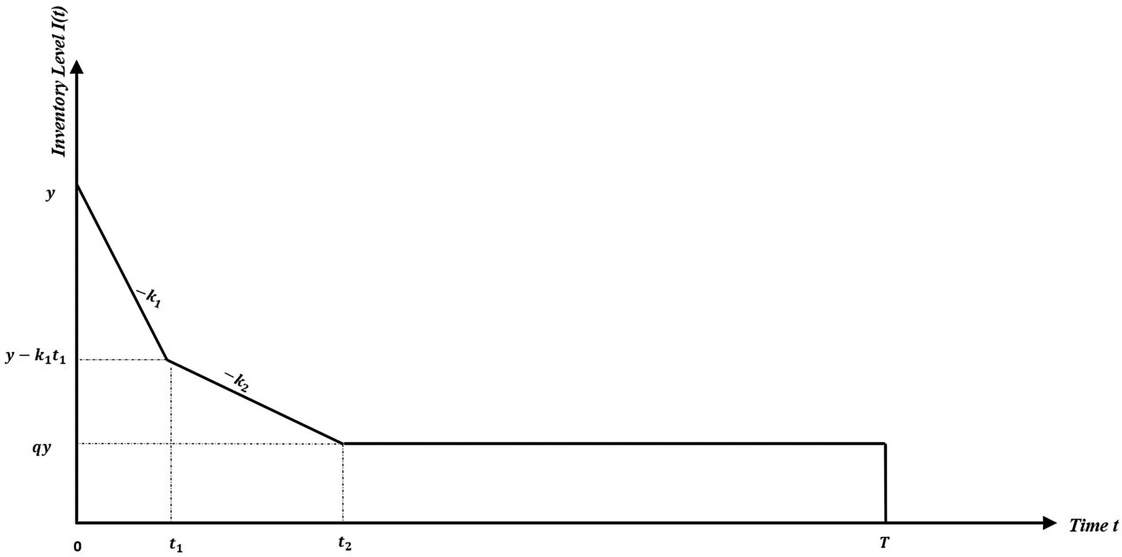


Figure 5.3: Raw material inventory level with imperfect items returned to supplier

The total cost per cycle is identical to that of the first scenario except that the holding costs of the raw material in the two scenarios are different. The area under the curve representing the inventory level in Figure 5.3 is

$$\frac{[y-(y-k_1t_1)]\,t_1}{2}+[(y-k_1t_1)-qy]\,\frac{(t_1+t_2)}{2}+qyT=\frac{k_1t_1^2}{2}+[y-qy-k_1t_1]\,\frac{t_2}{2}+qyT \tag{5.120}$$

Substituting Equations (5.48a) and (5.48b) into Equation (5.120), we obtain

$$\frac{k_1t_1^2}{2}+\frac{k_2\,(t_2-t_1)\,(t_1+t_2)}{2}+q\left[\frac{k_1t_1+k_2\,(t_2-t_1)}{1-q}\right]T \tag{5.121}$$

5.3.3.1 Manufacturer's inventory holding cost of raw material

The inventory cost of holding raw material is calculated by multiplying the unit holding cost of the raw material by the area represented in Figure 5.3, resulting in the total holding cost of raw material. Hence the total holding cost of raw material is

$$THR=h_r\left\{\frac{k_1t_1^2}{2}+\frac{k_2\,(t_2-t_1)\,(t_1+t_2)}{2}+q\left[\frac{k_1t_1+k_2\,(t_2-t_1)}{1-q}\right]T\right\} \tag{5.122}$$

Substituting Equations (5.26d), and (5.36c) into equation (122) leads to

$$HR=h_r\left\{\frac{k_1}{2}\left(\frac{I_1}{\rho_1}\right)^2+\frac{k_2}{2}\left(\frac{I_2-I_1}{\rho_2}\right)\left(\frac{I_2-I_1}{\rho_2}+\frac{2I_1}{\rho_1}\right)+\frac{q}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right)+k_2\left(\frac{I_2-I_1}{\rho_2}\right)\right]T\right\} \tag{5.123}$$

where

$$t_2 + t_1 = \frac{I_2 - I_1}{\rho_2} + \frac{2I_1}{\rho_1} \tag{5.124}$$

5.3.4 Total cost rate

The mathematical expression for the average total cost incurred, TC, is obtained by combining Equations (5.46), (5.53), (5.55), (5.71), (5.91), (5.92b), (5.93), and (5.123) divided by the cycle time T. Hence, the total cost per time is

$$TC_2 = \frac{1}{T}\left\{\begin{array}{l} \frac{1}{(1-q)}\left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1\right] k_1\left(\frac{I_1}{\rho_1}\right) + \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right] k_2\left(\frac{I_2 - I_1}{\rho_2}\right) \\ + p_l\left(k_1 - k_2\right)\left(\frac{I_2 - I_1}{\rho_2}\right) + h_p\left[\frac{I_1^2}{2\rho_1} + \frac{(I_2^2 - I_1^2)}{2\rho_2} + \frac{I_2^2}{2A}\right] \\ + c_d\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2A}\right] + G_1 + G_2 + \frac{c_r}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right] \\ + \frac{h_r k_1}{2}\left(\frac{I_1}{\rho_1}\right)^2 + \frac{h_r k_2}{2}\left(\frac{I_2 - I_1}{\rho_2}\right)\left(\frac{I_2 - I_1}{\rho_2} + \frac{2I_1}{\rho_1}\right) \\ + \frac{h_r q}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right] T + F_r \end{array}\right\} \tag{5.125}$$

5.3.5 Manufacturer's total revenue per time

Since a portion of the scanned raw material is of poor quality and the decision-maker chooses to retain it until the end of the cycle and return it to the supplier rather than selling it and recovering a portion of the cost of purchasing raw material, the mathematical expression of the total revenue per time for this new scenario is thus defined as follows

$$TR_2 = s_p Q_s + s_d(1-q)\left[d_1 k_1 t_1 + d_2 k_2\left(t_2 - t_1\right)\right] \tag{5.126}$$

$$TR_2 = \frac{1}{T}\left\{\begin{array}{l} \left[s_p A + k_1 s_d(1-q) d_1\right]\left(\frac{I_1}{\rho_1}\right) + \left[s_p A + k_2 s_d(1-q) d_2\right]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ + s_p\left[\frac{\gamma I_1^2}{2\rho_1} + \frac{\gamma\left(I_2^2 - I_1^2\right)}{2\rho_2} + I_2 - \frac{\theta I_2^2}{2A}\right] \end{array}\right\} \tag{5.127}$$

5.3.6 Manufacturer's profit rate

The profit rate generated is the difference between the revenue rate and the total cost rate across the entire system. The mathematical formulation of the proposed scenario of the profit rate is thus

$$TP_2 = \frac{1}{T}\left\{\begin{aligned} &[s_pA + k_1 s_d(1-q)d_1]\left(\frac{I_1}{\rho_1}\right) + [s_pA + k_2 s_d(1-q)d_2]\left(\frac{I_2}{\rho_2} - \frac{I_1}{\rho_2}\right) \\ &+ s_p\left[\frac{\gamma I_1^2}{2\rho_1} + \frac{\gamma\left(I_2^2 - I_1^2\right)}{2\rho_2} + I_2 - \frac{\theta I_2^2}{2A}\right] \end{aligned}\right\}$$
$$- \frac{1}{T}\left\{\begin{aligned} &\frac{1}{(1-q)}\left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1\right]k_1\left(\frac{I_1}{\rho_1}\right) + \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right]k_2\left(\frac{I_2 - I_1}{\rho_2}\right) \\ &+ p_l\left(k_1 - k_2\right)\left(\frac{I_2 - I_1}{\rho_2}\right) + h_p\left[\frac{I_1^2}{2\rho_1} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{I_2^2}{2A}\right] \\ &+ c_d\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2A}\right] + G_1 + G_2 + \frac{c_r}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right] \\ &+ \frac{h_r k_1}{2}\left(\frac{I_1}{\rho_1}\right)^2 + \frac{h_r k_2}{2}\left(\frac{I_2 - I_1}{\rho_2}\right)\left(\frac{I_2 - I_1}{\rho_2} + \frac{2I_1}{\rho_1}\right) \\ &+ \frac{h_r q}{1-q}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2 - I_1}{\rho_2}\right)\right]T + F_r \end{aligned}\right\} \tag{5.128}$$

5.4 Solution

5.4.1 Determination of the decision variables

The aim is to determine the optimum values of I_1 and T so as to maximise the total profit of the inventory system. The optimum values of I_1 and T for the maximum profit are the solutions of the equations

scenario 1.

$$\frac{\partial(TP)}{\partial I_1} = 0 \tag{5.129}$$

$$\frac{\partial(TP)}{\partial T} = 0 \tag{5.130}$$

Where

$$
\begin{aligned}
\frac{\partial(TP_1)}{\partial I_1} = \frac{1}{T} & \left\{ \begin{array}{l} \left[s_pA + k_1s_d(1-q)d_1 + \frac{k_1s_rq}{1-q}\right]\frac{1}{\rho_1} + \left[s_pA + k_2s_d(1-q)d_2 + \frac{k_2s_rq}{1-q}\right]\frac{(\upsilon-1)}{\rho_2} \\ + \frac{s_p}{T}\left[\frac{\gamma I_1}{\rho_1} - \frac{\gamma I_1}{\rho_2} + \upsilon + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)I_2\upsilon\right] \end{array} \right\} \\
- \frac{1}{T} & \left\{ \begin{array}{l} \frac{1}{(1-q)}\left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1\right]k_1\frac{1}{\rho_1} + \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right]k_2\frac{(\upsilon-1)}{\rho_2} \\ + p_l(k_1-k_2)\frac{(\upsilon-1)}{\rho_2} + (h + C_d\theta)\left[\frac{I_1}{\rho_1} + \frac{(I_2\upsilon - I_1)}{\rho_2} + \frac{I_2\upsilon}{A}\right] \\ + \frac{C_r}{1-q}\left[k_1\frac{1}{\rho_1} + k_2\frac{(\upsilon-1)}{\rho_2}\right] \\ + \frac{2qh_r}{x(1-q)^2}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2-I_1}{\rho_2}\right)\right]\left[k_1\frac{1}{\rho_1} + k_2\frac{(\upsilon-1)}{\rho_2}\right] \\ + h_r(k_1-k_2)\left(\frac{I_1}{\rho_1}\right)\left(\frac{1}{\rho_1}\right) + h_rk_2\left(\frac{I_2-I_1}{\rho_2} + \frac{I_1}{\rho_1}\right)\left(\frac{(\upsilon-1)}{\rho_2} + \frac{1}{\rho_1}\right) \end{array} \right\}
\end{aligned} \tag{5.131}
$$

$$
\begin{aligned}
\frac{\partial(TP_1)}{\partial T} = & \frac{1}{T}\left\{ \begin{array}{l} \left[s_pA + k_2s_d(1-q)d_2 + \frac{k_2s_rq}{1-q}\right]\frac{\xi}{\rho_2} \\ + s_p\left[\xi + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)I_2\xi\right] \end{array} \right\} \\
& - \frac{1}{T}\left\{ \begin{array}{l} \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right]k_2\left(\frac{\xi}{\rho_2}\right) + p_l(k_1-k_2)\left(\frac{\xi}{\rho_2}\right) \\ + (h + C_d\theta)\left(\frac{I_2\xi}{\rho_2} + \frac{I_2\xi}{A}\right) + \frac{C_r}{1-q}\left(k_2\frac{\xi}{\rho_2}\right) \\ + \frac{2qh_r}{x(1-q)^2}\left[k_1\left(\frac{I_1}{\rho_1}\right) + k_2\left(\frac{I_2-I_1}{\rho_2}\right)\right]\left(k_2\frac{\xi}{\rho_2}\right) \\ + h_rk_2\left(\frac{I_2-I_1}{\rho_2} + \frac{I_1}{\rho_1}\right)\left(\frac{\xi}{\rho_2}\right) \end{array} \right\} - \frac{1}{T}(TP_1)
\end{aligned} \tag{5.132}
$$

Taking the second derivatives with respect to I_1 and T in the Equations (5.131) and (5.132), we have

$$
\begin{aligned}
\frac{\partial^2(TP_1)}{\partial I_1^2} = & \frac{s_p}{T}\left\{\frac{\gamma}{\rho_1} - \frac{\gamma}{\rho_2} + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\upsilon^2 \right\} \\
& - \frac{1}{T}\left\{ \begin{array}{l} (h + C_d\theta)\left[\frac{1}{\rho_1} + \frac{(\upsilon^2-1)}{\rho_2} + \frac{\upsilon^2}{A}\right] + \frac{2qh_r}{x(1-q)^2}\left[k_1\frac{1}{\rho_1} + k_2\frac{(\upsilon-1)}{\rho_2}\right]^2 \\ + h_r(k_1-k_2)\left(\frac{1}{\rho_1}\right)^2 + h_rk_2\left[\frac{(\upsilon-1)}{\rho_2} + \frac{1}{\rho_1}\right]^2 \end{array} \right\}
\end{aligned} \tag{5.133}
$$

$$\frac{\partial^2(TP_1)}{\partial I_1 \partial T} = \frac{\partial^2(TP_1)}{\partial T \partial I_1} = \frac{s_p}{T}\left\{\left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\xi\upsilon\right\} - \frac{1}{T}\left\{\begin{aligned} &(h + C_d\theta)\left(\frac{\xi\upsilon}{\rho_2} + \frac{\xi\upsilon}{A}\right) \\ &+ \frac{2qh_r}{x(1-q)^2}k_2\left(\frac{\xi}{\rho_2}\right)\left[k_1\frac{1}{\rho_1} + k_2\frac{(\upsilon - 1)}{\rho_2}\right] \\ &+ h_r k_2\left(\frac{\xi}{\rho_2}\right)\left(\frac{(\upsilon-1)}{\rho_2} + \frac{1}{\rho_1}\right)\end{aligned}\right\} - \frac{1}{T}\frac{\partial(TP_1)}{\partial I_1} \tag{5.134}$$

$$\frac{\partial^2(TP_1)}{\partial T^2} = \frac{s_p}{T}\left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\xi^2 - \frac{1}{T}\left\{(h + C_d\theta)\left[\frac{\xi^2}{\rho_2} + \frac{\xi^2}{A}\right] + \frac{2qh_r}{x(1-q)^2}\left[k_2\frac{\xi}{\rho_2}\right]^2 + h_r k_2\left[\left(\frac{\xi}{\rho_2}\right)^2\right]\right\} - \frac{2}{T}\frac{\partial(TP_1)}{\partial T} \tag{5.135}$$

scenario 2.

Similar to scenario 1, under the conditions (5.129) and (5.130), the first and second partial derivatives of Equation (5.128) with respect to I_1 and T are

$$\frac{\partial(TP_2)}{\partial I_1} = \frac{1}{T}\left\{\begin{aligned} &[s_pA + k_1 s_d(1-q)d_1]\frac{1}{\rho_1} + [s_pA + k_2 s_d(1-q)d_2]\frac{(\upsilon-1)}{\rho_2} \\ &+ \frac{s_p}{T}\left[\frac{\gamma I_1}{\rho_1} - \frac{\gamma I_1}{\rho_2} + \upsilon + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)I_2\upsilon\right]\end{aligned}\right\}$$
$$- \frac{1}{T}\left\{\begin{aligned} &\frac{1}{(1-q)}\left[p_{c1} + \frac{\phi}{k_1} + \zeta k_1\right]k_1\frac{1}{\rho_1} + \frac{1}{(1-q)}\left[p_{c2} + \frac{\phi}{k_2} + \zeta k_2\right]k_2\frac{(\upsilon-1)}{\rho_2} \\ &+ p_l(k_1 - k_2)\frac{(\upsilon-1)}{\rho_2} + (h + C_d\theta)\left[\frac{I_1}{\rho_1} + \frac{(I_2\upsilon - I_1)}{\rho_2} + \frac{I_2\upsilon}{A}\right] \\ &+ \frac{C_r}{1-q}\left[k_1\frac{1}{\rho_1} + k_2\frac{(\upsilon-1)}{\rho_2}\right] + h_r k_1\left(\frac{I_1}{\rho_1}\right)\left(\frac{1}{\rho_1}\right) \\ &+ \frac{h_r k_2}{2}\left(\frac{\upsilon-1}{\rho_2}\right)\left(\frac{I_2 - I_1}{\rho_2} + \frac{2I_1}{\rho_1}\right) + \frac{h_r k_2}{2}\left(\frac{I_2 - I_1}{\rho_2}\right)\left(\frac{\upsilon-1}{\rho_2} + \frac{2}{\rho_1}\right) \\ &+ \frac{h_r q}{1-q}\left[k_1\left(\frac{1}{\rho_1}\right) + k_2\left(\frac{\upsilon-1}{\rho_2}\right)\right]T\end{aligned}\right\} \tag{5.136}$$

$$
\begin{aligned}
\frac{\partial(TP_2)}{\partial T} =& \frac{1}{T}\left\{[s_pA + k_2 s_d(1-q)d_2]\,\frac{\xi}{\rho_2} + s_p\left[\xi + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right) I_2\xi\right]\right\} \\
& - \frac{1}{T}\left\{\begin{array}{l}
\dfrac{1}{(1-q)}\left[p_{c2} + \dfrac{\phi}{k_2} + \zeta k_2\right] k_2\left(\dfrac{\xi}{\rho_2}\right) + p_l\,(k_1-k_2)\left(\dfrac{\xi}{\rho_2}\right) \\
+ (h + C_d\theta)\left(\dfrac{I_2\xi}{\rho_2} + \dfrac{I_2\xi}{A}\right) + \dfrac{C_r}{1-q}\left(k_2\dfrac{\xi}{\rho_2}\right) \\
+ \dfrac{h_r k_2}{2}\left(\dfrac{\xi}{\rho_2}\right)\left(\dfrac{I_2-I_1}{\rho_2} + \dfrac{2I_1}{\rho_1}\right) + \dfrac{h_r k_2}{2}\left(\dfrac{I_2-I_1}{\rho_2}\right)\left(\dfrac{\xi}{\rho_2}\right) \\
+ \dfrac{h_r q}{1-q}k_2\left(\dfrac{\xi}{\rho_2}\right)T + \dfrac{h_r q}{1-q}\left[k_1\left(\dfrac{I_1}{\rho_1}\right) + k_2\left(\dfrac{I_2-I_1}{\rho_2}\right)\right]
\end{array}\right\} - \frac{1}{T}(TP_2)
\end{aligned}
\tag{5.137}
$$

$$
\begin{aligned}
\frac{\partial^2(TP_2)}{\partial I_1^2} =& \frac{s_p}{T}\left\{\frac{\gamma}{\rho_1} - \frac{\gamma}{\rho_2} + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\upsilon^2\right\} \\
& - \frac{1}{T}\left\{\begin{array}{l}
(h + C_d\theta)\left[\dfrac{1}{\rho_1} + \dfrac{(\upsilon^2-1)}{\rho_2} + \dfrac{\upsilon^2}{A}\right] + h_r k_1\left(\dfrac{1}{\rho_1}\right)^2 \\
+ \dfrac{h_r k_2}{2}\left(\dfrac{\upsilon-1}{\rho_2}\right)\left(\dfrac{\upsilon-1}{\rho_2} + \dfrac{2}{\rho_1}\right) + \dfrac{h_r k_2}{2}\left(\dfrac{\upsilon-1}{\rho_2}\right)\left(\dfrac{\upsilon-1}{\rho_2} + \dfrac{2}{\rho_1}\right)
\end{array}\right\}
\end{aligned}
\tag{5.138}
$$

$$
\begin{aligned}
\frac{\partial^2(TP_2)}{\partial I_1\partial T} = \frac{\partial^2(TP_2)}{\partial T\partial I_1} =& \frac{s_p}{T}\left\{\left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\xi\upsilon\right\} \\
& - \frac{1}{T}\left\{\begin{array}{l}
(h + C_d\theta)\left[\dfrac{\xi\upsilon}{\rho_2} + \dfrac{\xi\upsilon}{A}\right] + \dfrac{h_r k_2}{2}\left(\dfrac{\upsilon-1}{\rho_2}\right)\left(\dfrac{\xi}{\rho_2}\right) \\
+ \dfrac{h_r k_2}{2}\left(\dfrac{\xi}{\rho_2}\right)\left(\dfrac{\upsilon-1}{\rho_2} + \dfrac{2}{\rho_1}\right) \\
+ \dfrac{h_r q}{1-q}\left[k_1\left(\dfrac{1}{\rho_1}\right) + k_2\left(\dfrac{\upsilon-1}{\rho_2}\right)\right]
\end{array}\right\} - \frac{1}{T}\frac{\partial(TP_2)}{\partial I_1}
\end{aligned}
\tag{5.139}
$$

$$
\begin{aligned}
\frac{\partial^2(TP_2)}{\partial T^2} = \frac{s_p}{T}\left[\xi + \left(\frac{\gamma}{\rho_2} - \frac{\theta}{A}\right)\xi^2\right] - \frac{1}{T}\left\{(h + C_d\theta)\left(\frac{\xi^2}{\rho_2} + \frac{\xi^2}{A}\right) + h_r k_2\left(\frac{\xi}{\rho_2}\right)^2 + \frac{2h_r q}{1-q}k_2\left(\frac{\xi}{\rho_2}\right)\right\} \\
- \frac{2}{T}\frac{\partial(TP_2)}{\partial T}
\end{aligned}
\tag{5.140}
$$

5.4.2 Optimality condition

Now, the problem is to determine the optimal values of I_1 and T, which maximise the profit function, TP. Since TP is a function of two variables I_1 and T, where both are continuous variables, therefore, for optimal value of I_1 and T, the sufficient conditions have to be met

$$\frac{\partial^2(TP)}{\partial I_1^2} \leq 0 \tag{5.141}$$

$$\frac{\partial^2(TP)}{\partial T^2} \leq 0 \tag{5.142}$$

$$H_M = \begin{bmatrix} \frac{\partial^2(TP)}{\partial I_1^2} & \frac{\partial^2(TP)}{\partial I_1 \partial T} \\ \\ \frac{\partial^2(TP)}{\partial T \partial I_1} & \frac{\partial^2(TP)}{\partial T^2} \end{bmatrix} \geq 0 \tag{5.143}$$

which are then verified numerically due to the complexity of the equations. The nature of the objective function is also shown graphically in Figure 5.4.

5.4.3 Numerical results

A numerical example is presented to illustrate the use of the models developed in this chapter. Consider a production process where the daily production rates k_1 and k_2 are 85 and 55 units per hour, respectively. A supplier orders raw materials for production, with 10% of the items received being defective. Screening for imperfect quality raw material items is conducted at a rate of 1500 items per hour. The ordering cost for the raw material is \$1000, and the setup costs, G_1 and G_2 are \$1000 and \$1500. The holding cost of raw material is \$0.12 per unit per hour, while the holding cost of one unit of the finished product is \$0.15 per unit per hour. The purchasing cost of one item of raw material is \$5. The cost of a deteriorated unit product is \$1.5, and the lost production cost for not producing the planned quantity is estimated to be \$1.1 per product. The selling price of products of perfect quality is \$66 per unit product, and the selling price of defective products produced is \$55 per product. The manufacturer may sell imperfect quality items screened at a price of \$3 at the end of the screening period, or they may keep the items in stock and return them when the next order arrives. A comparison of the two scenarios is conducted to determine the optimal order policy. The other parameters of the problem are: $A = 40$, $\gamma = 0.02$, $d_1 = 0.08, d_2 = 0.06, p_{c1} = \$1, p_{c2} = \$1.4, \phi = \$100$, and $\zeta = 0.05$. Upon evaluating the policy in which imperfect quality items are sold, the optimal inventory level I_1 and optimal cycle time T were determined using equations (5.131) and (5.132) as 11778 units and 21480 hours, respectively. The optimal number of products manufactured, Q^*, is 88667 units, with an optimal order quantity y^* of 98520 units of raw materials procured. The initial production period is denoted as $t_1 = 243$ hours, while the entire production period is represented by $t_2 = 1480$ hours. The maximum stock level attained, denoted as I_2, amounts to 26224 units. The total cost per time is calculated as \$7297.1 per hour, while the total revenue per hour is \$7560. Consequently, the maximum total profit per hour is determined to be \$267.

In contrast, when the imperfect quality items are returned, the optimal inventory level I_1 and optimal cycle time T are derived from equations (5.136) and (5.137) as 26349 units and 47023 hours respectively. The optimal number of items produced during the production cycle is found to be 201359, with an optimal order quantity y^* of 223733 units of raw material. The first production period, denoted as t_1, amounts to 690 hours, while the entire production period t_2 is calculated as 3285 hours. The total inventory carried during this scenario, I_2, is determined to be 56712 units. The total cost per

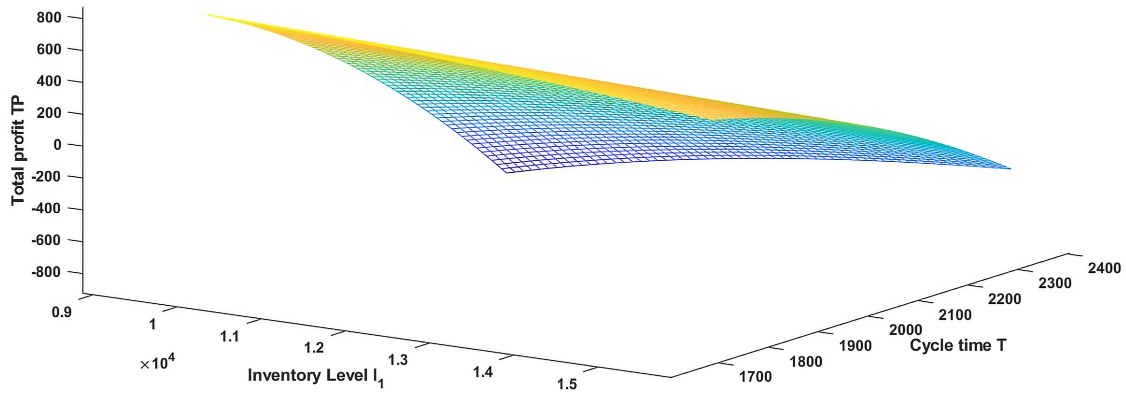


Figure 5.4: Graph of the net profit function of the two echelons supply chain per quantity, per time

hour is assessed as \$17426, whereas the total revenue per hour is \$13017.8. The total profit per time is estimated to be -\$4438.4. In the case where imperfect quality items are returned, the negative maximum total profit per cycle indicates that the overall financial outcome is unfavourable. This means that the costs of managing such a system outweigh the potential revenue generated from the sale. Based on the analysis above, the optimal operating policy suggests selling the imperfect raw material instead of returning it to the supplier when the subsequent order is received.

5.4.4 Sensitivity analysis and managerial implications

A sensitivity analysis was performed to assess the impact of various parameters. The findings of the sensitivity analysis are presented in Tables 5.2 and 5.3. Based on these results, the following inferences can be drawn

- The optimal values of the initial stock level (I_1) and the cycle time (T) are insensitive with respect to $c_r, d_1, F_r, G_1, G_2, pc_1, pc_2, p_l, q, s_r, x$ and ζ. However, the initial stock level is less sensitive with respect to c_d and s_d. On the other hand, the cycle time (T) is less sensitive with respect to k_1, c_d, ϕ and s_d. The profit per time is insensitive with respect to $d_1, F_r, G_1, G_2, pc_1, pc_2, p_l, s_d$ and s_r.

- The profit per time is sensitive to changes in $k_1, k_2, s_p, h_r, \theta, \gamma, c_r, c_d, h_p, d_2, pc_2, \phi, \zeta, x$, and q. However, the most significant changes are observed with respect to k_1, k_2, s_p, θ and γ. Specifically, the selling price (s_p) has the greatest positive impact on increasing the total profit. Although production/holding costs increase with higher inventory levels at higher s_p, the revenue impact outweighs this effect due to a slower scaling up of production costs compared to revenue based on price increment alone. On the other hand, the production parameter k_2 has the greatest negative impact on the total profit. Note that a higher value of k_2 causes a decrease in profitability. The model indicates that faster production capacities do not necessarily result in higher revenues. It also indicates that having consistently high values of k_2 can negatively impact profit outcomes.

Table 5.2: Sensitivity analysis for various inventory model parameters

	% change	**inventory level I_1** units	% change	**Inventory level I_2** units	% change	**Production time t_1** hours	% change	**production cycle t_2** hours	% change	**Cycle time T** hours	% change	**Profit per time TP** USD	% change
Base		**11753**		**26224**		**243**		**1480**		**2135**		**263**	
k_1	-20	14582.11	24%	29007.14	10%	301.2832	24%	1534.192	3%	2259.37	6%	-191.732	-173%
	-10	12553.18	6.58%	26973.54	2.65%	259.3632	6.58%	1491.873	0.62%	2166.211	1.24%	162.6069	-38%
	10	11384.68	-3.34%	25975.65	-1.15%	235.2207	-3.34%	1482.312	-0.03%	2131.703	-0.37%	294.5775	12%
	20	11153.08	-5.30%	25828.15	-1.71%	230.4356	-5.30%	1484.715	0.14%	2130.419	-0.43%	298.4331	13%
k_2	-20	*	*	*	*	*	*	*	*	*	*	*	*
	-10	*	*	*	*	*	*	*	*	*	*	*	*
	10	6992.566	-41%	11542.7	-56%	144.4745	-41%	414.1918	-72%	702.7593	-67%	-675.994	-357%
	20	7939.372	-33%	9480.577	-64%	164.0366	-33%	233.9643	-84%	470.9787	-78%	-2066.26	-885%
c_r	-20	11778.59	0.01%	26278.62	0.0%	243.3593	0.0%	1482.679	0.0%	2139.644	0.0%	309.1318	17%
	-10	11778.17	0.00%	26278.38	0.0%	243.3506	0.0%	1482.685	0.0%	2139.645	0.0%	286.1318	9%
	10	11777.33	0.00%	26277.9	0.0%	243.3333	0.0%	1482.698	0.0%	2139.645	0.0%	240.1315	-9%
	20	11776.91	-0.01%	26277.66	0.0%	243.3247	0.0%	1482.704	0.0%	2139.646	0.0%	217.1313	-17%
c_d	-20	12560.81	6.65%	28619.59	8.9%	259.5209	6.6%	1632.066	10.1%	2347.556	9.7%	332.3135	26%
	-10	12149.34	3.16%	27389.46	4.2%	251.0195	3.2%	1553.594	4.8%	2238.331	4.6%	297.3031	13%
	10	11440.49	-2.86%	25269.08	-3.8%	236.3738	-2.9%	1418.304	-4.3%	2050.031	-4.2%	229.7022	-13%
	20	11133.01	-5.47%	24348.67	-7.3%	230.0209	-5.5%	1359.564	-8.3%	1968.281	-8.0%	196.9337	-25%
h_p	-20	14111.37	19.81%	33252.38	26.5%	291.5573	19.8%	1927.54	30.0%	2758.85	28.9%	443.6964	69%
	-10	12783.57	8.54%	29285.39	11.4%	264.1233	8.5%	1674.535	12.9%	2406.67	12.5%	350.1691	33%
	10	10989.23	-6.69%	23918.11	-9.0%	227.0503	-6.7%	1332.083	-10.2%	1930.035	-9.8%	180.7757	-31%
	20	10354.39	-12.09%	22015.4	-16.2%	213.9336	-12.1%	1210.602	-18.4%	1760.987	-17.7%	101.9627	-61%
h_r	-20	15202.28	29.08%	40717.03	54.9%	314.0968	29.1%	2494.844	68.3%	3512.77	64.2%	697.6982	165%
	-10	12970.21	10.12%	31321.42	19.2%	267.9796	10.1%	1836.459	23.9%	2619.495	22.4%	463.1355	76%
	10	11026.91	-6.38%	23091.06	-12.1%	227.8286	-6.4%	1258.953	-15.1%	1836.229	-14.2%	85.87859	-67%
	20	10505.74	-10.80%	20873.85	-20.6%	217.0607	-10.8%	1103.224	-25.6%	1625.071	-24.0%	-74.5856	-128%
d_1	-20	11707.02	-0.60%	26258.1	-0.1%	241.8806	-0.6%	1485.562	0.2%	2142.015	0.1%	267.4338	2%
	-10	11741.63	-0.31%	26267.85	0.0%	242.5957	-0.3%	1484.153	0.1%	2140.849	0.1%	265.4548	1%
	10	11815.49	0.32%	26289.01	0.0%	244.1216	0.3%	1481.175	-0.1%	2138.4	-0.1%	260.4416	-1%
	20	11854.94	0.66%	26300.5	0.1%	244.9369	0.7%	1479.6	-0.2%	2137.113	-0.1%	257.3593	-2%
d_2	-20	10000.19	-15.09%	21753.38	-17.2%	206.6156	-15.1%	1157.521	-21.9%	1701.355	-20.5%	189.4274	-28%
	-10	10794.09	-8.35%	23796.32	-9.4%	223.0185	-8.4%	1303.835	-12.1%	1898.743	-11.3%	226.5295	-14%
	10	13016.02	10.51%	29346.12	11.7%	268.926	10.5%	1705.171	15.0%	2438.824	14.0%	299.908	14%
	20	14605.94	24.01%	33219.99	26.4%	301.7756	24.0%	1987.831	34.1%	2818.33	31.7%	338.0191	28%
F_r	-20	11797.8	0.17%	26350.6	0.3%	243.756	0.2%	1487.58	0.3%	2146.35	0.3%	265.848	1.0%
	-10	11797.9	0.17%	26351.1	0.3%	243.759	0.2%	1487.62	0.3%	2146.4	0.3%	265.777	1.0%
	10	11798.2	0.17%	26352.1	0.3%	243.765	0.2%	1487.69	0.3%	2146.49	0.3%	259.542	-1.4%
	20	11798.4	0.18%	26352.7	0.3%	243.768	0.2%	1487.72	0.3%	2146.54	0.3%	259.471	-1.4%
G_1	-20	11797.8	0.17%	26350.6	0.3%	243.756	0.2%	1487.58	0.3%	2146.35	0.3%	265.848	1.0%
	-10	11797.9	0.17%	26351.1	0.3%	243.759	0.2%	1487.62	0.3%	2146.4	0.3%	265.777	1.0%
	10	11798.2	0.17%	26352.1	0.3%	243.765	0.2%	1487.69	0.3%	2146.49	0.3%	259.542	-1.4%
	20	11798.4	0.18%	26352.7	0.3%	243.768	0.2%	1487.72	0.3%	2146.54	0.3%	259.471	-1.4%
G_2	-20	11797.8	0.17%	26350.6	0.3%	243.756	0.2%	1487.58	0.3%	2146.35	0.3%	265.848	1.0%
	-10	11797.9	0.17%	26351.1	0.3%	243.759	0.2%	1487.62	0.3%	2146.4	0.3%	265.777	1.0%
	10	11798.2	0.17%	26352.1	0.3%	243.765	0.2%	1487.69	0.3%	2146.49	0.3%	259.542	-1.4%
	20	11798.4	0.18%	26352.7	0.3%	243.768	0.2%	1487.72	0.3%	2146.54	0.3%	259.471	-1.4%
q	-20	11801.16	0.20%	26350.87	0.3%	243.8256	0.2%	1487.391	0.3%	2146.162	0.3%	275.7799	5%
	-10	11789.62	0.10%	26315.16	0.1%	243.5872	0.1%	1485.086	0.2%	2142.965	0.2%	269.5561	2%
	10	11765.54	-0.10%	26239.75	-0.1%	243.0897	-0.1%	1480.201	-0.2%	2136.195	-0.2%	256.4988	-3%
	20	11752.97	-0.21%	26199.91	-0.3%	242.83	-0.2%	1477.611	-0.3%	2132.609	-0.3%	249.6495	-5%

- The parameter γ for demand enhancement has the second greatest impact on profit over time. Like the selling price (s_p), this parameter significantly affects the manufacturer's EPQ. This indicates that more products would need to be produced, along with increased raw material procurement and hold more inventory for both raw materials and finished products; however, the effect on profit is positive because an increase in γ makes demand more sensitive to changes in inventory levels. A higher value of γ indicates a greater responsiveness of demand, meaning that customers are strongly influenced by product availability or scarcity in inventory. With a more responsive demand driven by an increased γ, the system can better adapt to fluctuating inventories and seize revenue opportunities by aligning supply with demand effectively.

Table 5.3: Sensitivity analysis for various inventory model parameters

	% change	**inventory level I_1** units	% change	**Inventory level I_2** units	% change	**Production time t_1** hours	% change	**production cycle t_2** hours	% change	**Cycle time T** hours	% change	**Profit per time TP** USD	% change
Base		**11753**		**26224**		**243**		**1480**		**2135**		**263**	
p_{c1}	-20	11806	0.24%	26356.2	0.3%	243.925	0.2%	1487.53	0.3%	2146.44	0.3%	265.848	1.0%
	-10	11806	0.24%	26356.2	0.3%	243.925	0.2%	1487.53	0.3%	2146.44	0.3%	265.777	1.0%
	10	11794.2	0.14%	26349.3	0.3%	243.681	0.1%	1487.71	0.3%	2146.45	0.3%	259.142	-1.5%
	20	11790.2	0.11%	26347.1	0.3%	243.599	0.1%	1487.77	0.3%	2146.45	0.3%	258.719	-1.7%
p_{c2}	-20	11767	-0.09%	26271.9	0.0%	243.119	-0.1%	1482.86	0.0%	2139.65	0.0%	274.828	4.4%
	-10	11772.4	-0.05%	26275	0.0%	243.231	0.0%	1482.77	0.0%	2139.65	0.0%	268.979	2.2%
	10	11783.1	0.05%	26281.3	0.0%	243.453	0.0%	1482.61	0.0%	2139.64	0.0%	257.285	-2.2%
	20	11788.5	0.09%	26284.4	0.0%	243.565	0.1%	1482.52	0.0%	2139.63	0.0%	251.44	-4.4%
s_p	-20	9167.77	-22.16%	16928.1	-35.6%	189.417	-22.2%	852.694	-42.5%	1275.9	-40.4%	-420.38	-259.8%
	-10	10183	-13.54%	20591.9	-21.6%	210.392	-13.5%	1100.04	-25.8%	1614.84	-24.5%	-99.521	-137.8%
	10	14691.8	24.74%	36606.2	39.3%	303.549	24.7%	2176.58	46.8%	3091.73	44.5%	682.532	159.4%
	20	21968.9	86.53%	62333	137.2%	453.903	86.5%	3903.83	163.3%	5462.15	155.3%	1212.21	360.7%
s_d	-20	11637.1	-1.19%	25985.2	-1.1%	240.436	-1.2%	1466.77	-1.1%	2116.4	-1.1%	259.313	-1.5%
	-10	11707.4	-0.60%	26131.7	-0.6%	241.889	-0.6%	1474.73	-0.5%	2128.02	-0.5%	261.223	-0.7%
	10	11848.1	0.60%	26424.6	0.6%	244.795	0.6%	1490.65	0.5%	2151.27	0.5%	265.04	0.7%
	20	11918.4	1.19%	26571.1	1.1%	246.248	1.2%	1498.61	1.1%	2162.89	1.1%	266.947	1.4%
s_r	-20	11805.3	-0.21%	26333.6	-0.2%	243.91	-0.2%	1485.65	-0.2%	2143.99	-0.2%	401.515	-0.1%
	-10	11817.8	-0.11%	26360.6	-0.1%	244.169	-0.1%	1487.15	-0.1%	2146.17	-0.1%	401.706	0.0%
	10	11842.8	0.11%	26414.7	0.1%	244.685	0.1%	1490.15	0.1%	2150.52	0.1%	402.089	0.0%
	20	11855.2	0.21%	26441.7	0.2%	244.943	0.2%	1491.65	0.2%	2152.69	0.2%	402.281	0.1%
l_p	-20	11773.6	-0.04%	26275.7	0.0%	243.256	0.0%	1482.76	0.0%	2139.65	0.0%	267.642	1.7%
	-10	11775.7	-0.02%	26276.9	0.0%	243.299	0.0%	1482.72	0.0%	2139.65	0.0%	265.387	0.9%
	10	11779.8	0.02%	26279.3	0.0%	243.385	0.0%	1482.66	0.0%	2139.64	0.0%	260.877	-0.9%
	20	11781.9	0.04%	26280.5	0.0%	243.428	0.0%	1482.63	0.0%	2139.64	0.0%	258.622	-1.7%
θ	-20	*	*	*	*	*	*	*	*	*	*	*	*
	-10	22851.6	94.02%	56752.9	116.0%	472.141	94.0%	3369.69	127.3%	4788.51	123.8%	710.588	170.1%
	10	8598.83	-26.99%	17756.5	-32.4%	177.662	-27.0%	960.373	-35.2%	1404.29	-34.4%	-17.366	-106.6%
	20	7029.82	-40.31%	13678.7	-47.9%	145.244	-40.3%	713.522	-51.9%	1055.49	-50.7%	-242.64	-192.2%
γ	-20	7777	-33.97%	13187.8	-49.8%	160.682	-34.0%	623.144	-58.0%	952.839	-55.5%	-785.54	-398.5%
	-10	8796.42	-25.31%	16787.8	-36.1%	181.744	-25.3%	864.77	-41.7%	1284.47	-40.0%	-280.77	-206.7%
	10	41931.2	256.02%	118668	351.6%	866.346	256.0%	7425.05	400.8%	10391.7	385.7%	1279.35	386.2%
	20	*	*	*	*	*	*	*	*	*	*	*	*
η	-20	11790	0.10%	26285.2	0.0%	243.595	0.1%	1482.5	0.0%	2139.63	0.0%	289.704	10.1%
	-10	11783.9	0.05%	26281.7	0.0%	243.469	0.1%	1482.6	0.0%	2139.64	0.0%	276.418	5.0%
	10	11771.6	-0.05%	26274.6	0.0%	243.215	-0.1%	1482.79	0.0%	2139.65	0.0%	249.847	-5.0%
	20	11765.5	-0.10%	26271	0.0%	243.088	-0.1%	1482.88	0.0%	2139.65	0.0%	236.562	-10.1%
μ	-20	11726.6	-0.43%	26017.4	-1.0%	242.284	-0.4%	1463.72	-1.3%	2114.16	-1.2%	287.429	9.2%
	-10	11752.1	-0.22%	26147.4	-0.5%	242.811	-0.2%	1473.18	-0.6%	2126.87	-0.6%	275.332	4.6%
	10	11803.6	0.22%	26409.6	0.5%	243.877	0.2%	1492.25	0.6%	2152.49	0.6%	250.829	-4.7%
	20	11829.7	0.44%	26541.8	1.0%	244.416	0.4%	1501.86	1.3%	2165.4	1.2%	238.426	-9.4%
x	-20	11747.5	-0.26%	26169	-0.4%	242.718	-0.3%	1475.32	-0.5%	2129.54	-0.5%	259.278	-1.5%
	-10	11764.3	-0.11%	26229.5	-0.2%	243.064	-0.1%	1479.4	-0.2%	2135.14	-0.2%	261.418	-0.7%
	10	11788.8	0.09%	26318.1	0.2%	243.571	0.1%	1485.39	0.2%	2143.35	0.2%	264.535	0.5%
	20	11798.1	0.17%	26351.6	0.3%	243.762	0.2%	1487.65	0.3%	2146.44	0.3%	265.706	1.0%

- The initial stock level I_1 is sensitive to changes in $k_1, k_2, h_r, h_p, d_2, s_p, \gamma$ and θ. The initial stock level is highly sensitive in a positive way with respect to the demand enhancement parameter γ. This shows the fact that if γ increases, then the customers' demand also increases, and so, the manufacturer needs to produce and store a large quantity of products to handle customers' high demand. More inventory is justified to capitalise on this demand-enhancing effect. On the other hand, I_1 is highly sensitive in a negative way with respect to the changes in the deterioration parameter of the products produced (θ). For a higher value of θ, the initial inventory level I_1 decreases significantly and then, the manufacturer needs to store a small number of products to avoid the higher carrying cost for products that deteriorate faster and that might be unusable.

- The cycle time (T) time is sensitive to changes in $k_1, k_2, s_p, s_d, h_r, \theta, \gamma, c_d, h_p, d_2, \phi$, and ϕ. However, the most significant changes are observed with respect to in $k_2, s_p, h_r, \theta, \gamma, h_p, d_2$, and ϕ. and θ. Specifically, the demand enhancement parameter (γ) increases the cycle time quite drastically. The increase in cycle time with an increase in γ is likely due to the adjustments made in the production cycle to accommodate changes in demand dynamics caused by varying γ. The manufacturer might be adapting the production strategy to respond to the changing demand landscape influenced by γ, and this adaptation appears to have the side effect of increasing the cycle time. On the other hand, the production parameter k_2 has the greatest negative impact on the cycle time (T). Note that the decrease in cycle time associated with changes in k_2 may indicate a complex relationship between manufacturing efficiency, inventory control, financial performance and customer demand. This highlights the intricate interplay of various factors, such as cost structures and customers within the production process or potential issues in aligning production with actual demand.

- The parameter θ has the second negative impact on the cycle time T. Decreasing the cycle time when θ increases acts as a strategic response to the challenges posed by high deterioration rates. Deteriorating products are at a higher risk of quality degradation over time. A shorter cycle time helps maintain the quality of products, ensuring that they reach consumers in a state that meets quality standards. The model suggests that small cycle times are not just preferred but also a practical strategy for effectively managing goods with a high rate of deterioration.

5.5 Conclusion

The model presented in this chapter extends the classic economic production quantity (EPQ) model to the case where raw materials with imperfect quality items are used during the production process. In this model, we considered both deterioration of products and a flexible production process. This later effect is captured by using the concept of shifts in production rate. In addition, the traditional assumption of constant demand is relaxed to a stock-dependent demand function, reflecting the impact of current inventory levels on demand. As the authors have shown, the quantity purchased by customers may be affected by the quantity of stock displayed. Two scenarios were considered in this chapter. The optimal operating policy was derived by maximising the total profit per unit of time. The uniqueness of the optimal solutions was demonstrated through a numerical example and the sensitivity analysis of the model was analysed. One of the unique contributions of this model is considering the effect of shifts in production rate in conjunction with the ordering of imperfect raw material in a two-echelon supply chain. The analysis demonstrated that, at the optimum, the higher the production rate is, the lower the inventory level and the cycle time, and the lower the profit. From a practical point of view, this means that an increase in production rate may boost the number of units produced but could also lead to higher operational costs and decrease the profit quite substantially. Another important characteristic of this model is that it incorporates defective products into an EPQ model during shifts in production rates.

In this regard, the results show how a high rate of defective products would result in higher inventory levels, longer inventory cycle times and a moderate increase in profit. This counter-intuitive outcome of an increase in defective products contributing to an increase in profit is rooted in the strategic decision to salvage and sell defective products at a discount. Salvages, when carefully implemented, can enhance overall revenue, offset losses, and improve profitability.

This chapter contributes to the existing research on managing perishable product inventory with demand dependent on stock levels. However, there are limitations in the model presented that suggest opportunities for further exploration and expansion. For instance, in this model, the rate of imperfect products is modelled by a constant parameter that is proportional to the rate of production. A possible extension of the model could be to adopt other forms of defective rates, such as the exponential increase over time. Another possible interesting extension could be to use the probability density function to capture the percentage of imperfect raw material contained in the lot size. We could consider using a multiplicative model of stock-level components instead of the additive. The model may be made even more realistic by introducing elements such as stochastic timing of shifts, probabilistic production rates, and uncertain duration for production by modelling these factors as random variables with probability distributions. Other extensions could include incorporating freshness degradation, non-instantaneous deteriorating products, and introducing discount schemes. Additionally, future research could explore competitors' pricing strategies' impact on profit-maximisation.

Chapter 6

An integrated EPQ Model for deteriorating products with declining production rate, increasing defects, stock and price-dependent demand, and effects of corporate social responsibility activities

6.1 Introduction

6.1.1 Context

In the dynamic field of inventory modelling, many approaches often treat production rates as either constant or discrete functions, with defect rates frequently overlooked or considered constants. However, in reality, production systems undergo continuous changes, and defective rates can exhibit dynamic patterns tied to production rates over time. At the heart of inventory models lies the crucial factor of demand, serving as the driving force behind inventory levels and guiding managerial decisions on production and inventory. Despite many models relying on a constant demand rate, this assumption rarely applies because of several factors such as price, stock level, quality, consumer preferences, seasonal changes, market trends, and competitors' actions. Furthermore, customer demand can be influenced by external factors such as natural disasters, pandemics, international politics and corporate events. Recognising these factors, organisations increasingly acknowledge their responsibilities towards promoting societal welfare, commonly known as corporate social responsibility (CSR). This involves voluntarily integrating social, environmental, and health-related objectives into business practices. A practical example involves companies contributing a portion of sales revenue to charitable organisations, fostering goodwill and bolstering their reputation, ultimately leading to improved profitability.

6.1.2 Purpose

This study aims to address the research gap identified in the literature regarding continuously declining production systems, particularly focusing on deteriorating products. Previous studies have extensively examined isolated components within manufacturing systems, leaving a need for deeper exploration of systems characterised by declining production rates and increasing defective rates over time. To this end, an integrated EPQ model is proposed to account for the impact of Corporate Social Responsibility (CSR) activities. The demand function is modelled as a function of various factors, including selling price, stock levels, quality, and CSR contributions. The problem is formulated as a cost minimisation scheme with the inventory level I_m as the sole decision variable.

6.1.3 Relevance

Typical models for production systems overlook speed losses and equipment failures. However, conceptualising the manufacturing process as a series of distinct unit operations has emphasised the need to address issues related to manufacturing. While previous studies have focused on ideal production scenarios with consistently high-quality outputs, manufactured goods often have defects due to human error, machinery malfunctions, mishandling, or imperfect raw material. Traditionally, system reliability studies have employed binary modelling (operational state vs complete failure). Nevertheless, an increasing body of literature now considers multiple scenarios throughout the lifespan of systems. Multi-Production Systems (MPS) and Multi-State Systems (MSS) are two types of systems with distinct characteristics. MPS usually start with a low production rate before increasing to minimize average holding costs by maintaining smaller stock levels for longer periods and larger stocks for shorter periods. On the other hand, a MSS is a system designed to automatically reconfigure upon any failure, enabling the degraded machine or equipment to remain operational but deliver services at reduced a performance level. The continuity of these systems relies heavily on the state of the manufacturing system, where the breakdown of any component minimally or partially affects their performance.
The proposed inventory model addresses several essential aspects of manufacturing systems, including factors such as the degradation behaviour of a typical process represented by declining production rates and the incorporation of corporate social responsibility among various influential elements affecting demand, deterioration, and imperfect quality. Manufacturing systems, like any other system, are susceptible to degradation. Ensuring high-quality output remains crucial at the other end of the production spectrum. The model acknowledges the complex nature of demand, deterioration, and imperfect quality while integrating CSR principles to align with societal and environmental objectives.

6.1.4 Organisation

The remainder of the chapter is organised as follows: Section 6.2 provides an overview of the assumptions and notation used throughout the study, while Section 6.3 provides a description of the production system and the inventory control policies to enhance our understanding of the problem. Section 6.4 presents a numerical example and sensitivity analyses to illustrate the model's features. Finally, Section 6.5 concludes the paper and suggests directions for future research.

Table 6.1: Characteristics of inventory control models in the literature

Author(s)	Type of model	Deterioration of products		Demand patterns								Imperfect quality	Nature of production
		Constant	Variable	Constant	Price-	Stock-	Age-	Quality-	Time-	CSR-impact-	Other considerations		
Ben-Daya et al. (2008)	EPQ			✓									variable, discrete, and declining
Sarkar (2012)	EOQ/EPQ					✓					Delay in payments with the production of defective items within the cycle time	Exponential	Constant
Glock et al. (2012)	EPQ				✓			✓		✓	CSR activities, with emission and scrap		
Tai (2013)	EPQ	✓		✓				✓		✓		Constant	Constant
Avinadav et al. (2014)	EOQ				✓				✓				
Bai et al. (2017)	EPQ	✓			✓				✓	✓	Two-echelon sustainable supply chain system, Demand is a multi-variable function of promotional effort, sustainable level, selling price and time	Constant	
Cheng et al. (2018)	EPQ			✓								Increasing function of deterioration level of the system	Constant
Li and Teng (2018)	EOQ				✓	✓	✓				Demand also depends on the reference price		
San-José et al. (2018)	EOQ				✓				✓				
Raza (2018)	EOQ				✓					✓	CSR, Supply chain coordination under a revenue-sharing contract		
Seyedhosseini et al. (2019)	two Echelons				✓					✓	Competitive supply chain coordination		
Modak et al. (2019)	EPQ				✓					✓	CLSC with emissions tax and demand uncertainty		
Al-Salamah (2019)	EPQ			✓						✓		Constant	Constant
San-José et al. (2019)	EOQ								✓		Shortages are allowed		
Cárdenas-Barrón et al. (2020)	EOQ					✓					Trade credit and shortages		
Dye (2020)	EOQ				✓	✓			✓		Advertising goodwill-dependent on demand		
Su et al. (2021)	EOQ				✓					✓	Social donations and green industrial development, and imperfect products	Constant	
Feng et al. (2022)	EOQ				✓	✓	✓			✓	An advance-cash-credit (ACC) payment scheme is considered		
Khan et al. (2022)	EOQ		✓		✓				✓		Hybrid cash-advance Payment is integrated as well as non-instantaneous deterioration, backordering and non-terminating situations		
Tshinangi et al. (2022)	EPQ		✓	✓							Shortages are allowed	Increasing function of the production	variable, discrete, and declining
This Model	EPQ		✓		✓	✓		✓		✓	Social donation impact	Increasing function of time	Variable, continuous declining

6.2 Notations and assumptions

6.2.1 Notations

The following notations are utilised in this chapter

A	Demand parameter
b	Elasticity parameter of the unit selling price
c_d	Deterioration cost per unit item
c_{d_p}	Disposal cost per unit item
$D(s_p, I(t), s)$	Demand for the product
G	Set-up cost
h_p	Inventory carrying cost per item produced per time
$I(t)$	Instantaneous inventory level
k_0	Initial production rate at the start of the cycle
$k(t)$	Declining production function
$L_{\text{i}\in\{1,2\}}$	Constant of integration
LP	Lost production cost
MC	Maintenance cost
p_c	Unit production cost
p_l	Lost production cost per product
s	Social donation amount per sale
s_p	Market selling price of the product
T	Cycle time
TC	Average total cost per time
THC	Total holding cost
TDC	Total deterioration cost
TPC	Total production cost
$TDPC$	Total disposal cost
t_1	Time duration of production cycle
$\theta(t)$	Deterioration rate
ω	Rate at which the production rate declines
λ	Rate at which the proportion of defective items increases over time
σ	Elasticity parameter that reflects the impact of defects on customer demand
η_m	Maximum proportion of defective products that can be produced
γ	Demand enhancement parameter for inventory level
q	Elasticity factor of quality of stock

δ	Elasticity factor of social donation amount
ψ	Aggregation parameter for some known variables
μ	Aggregation parameters for some known variables
ξ	Constant

6.2.2 Assumptions

This section is based on the following assumptions

- The inventory procedure is appropriate for a single product.

- The deterioration rate of the products is time-dependent and of the form

$$\theta(t) = \theta e^{-\theta t} \tag{6.1}$$

- Neither replacement nor repair is permitted for deteriorated products.

- Su et al. (2021) proposed a demand pattern that incorporates various factors, including selling price, social donation amount, and investment in green industrial development. In this paper, the demand rate is formulated as a function of selling price, stock level, quality of stock and social donation amount

$$D\left(s_p, I(t), s\right) = A - bs_p + \gamma I(t) + qI(t) + \delta s, \tag{6.2}$$

  where A represents the market potential, b represents the elasticity factor of selling price, γ represents the elasticity factor of stock level, q represents the elasticity factor in quality of stock, and δ represents the elasticity factor of social donation amount.

- The degradation behaviour of the process is modelled by a declining production rate $k(t)$ that is considered a continuous function of the form

$$k(t) = k_0 e^{-\omega t} \tag{6.3}$$

  and not as a discrete function proposed by Ben-Daya et al. (2008) and Tshinangi et al. (2022), with k_0 as the initial production rate and ω as the rate at which the production rate declines over time.

- Considering that the production system continuously deteriorates, the defective pattern proposed by Tshinangi et al. (2022) is extended by assuming that the function that captures the proportion of defective items produced is a continuous function of time that increases over time. This is represented by the formula

$$\eta(t) = \left(1 - e^{-\lambda t}\right) \eta_m \tag{6.4}$$

  where λ is the rate parameter that determines the slope of the exponential function, and η_m is the maximum proportion of defective items that can be produced.

- There is no rework or replacement of poor-quality products.

6.3 Problem description

In this section, a production system is set to produce a single product at a rate, $k(t)$ to satisfy a demand that is dependent on the selling price, stock level, quality, and social donation amount dependent ($k(t) > D\left(s_p, I(t), s\right)$). However, the system is subject to wear and degradation, leading to a decrease in the production rate over time. As time increases, the production rate decreases, and the number of defective products increases until production is stopped at time t_1. Once production is stopped, the remaining inventory is depleted at the rate $D\left(s_p, l(t), s\right)$ until the stock level reaches zero. The time duration from where the process stops production until the inventory level reaches 0 is defined as the consumption cycle during which the machine also undergoes maintenance.

After the maintenance is completed, the machine resumes production at the rate $k(t)$. Furthermore, the inventory is not only depleted by the demand but also by the product's natural deterioration over time. Although products are inspected immediately after production, errors and inconsistent inspection result in some defective products being passed on to customers and subsequently returned to the manufacturer. This problem has significant implications for the company, increasing costs due to waste and decreasing customer satisfaction due to the production of low-quality products being passed to customers. The proposed model aims to optimise the total cost per unit time, with the decision variables being the maximum inventory (I_m) and the cycle time (T). The production process is illustrated in Figure 6.1.

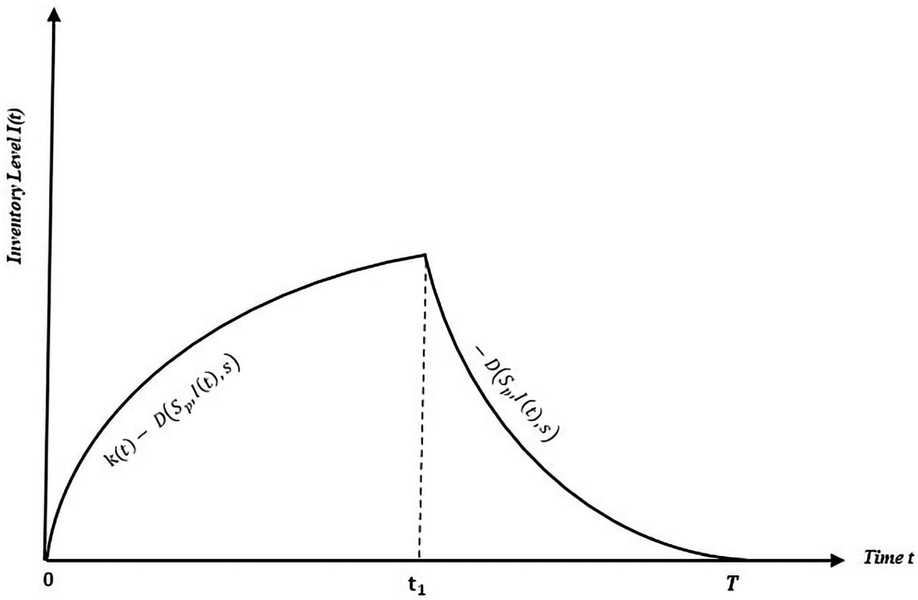


Figure 6.1: Inventory profile of a manufacturing system with a decline in production

6.4 Model formulation

The differential equations that describe the inventory situations outlined above within the time interval $[0, T]$ are

$$\frac{dI(t)}{dt} + \theta e^{\theta t} I(t) = \left[1 - \left(1 - e^{-\lambda t)}\right) \eta_m\right] k_0 e^{-\omega t} \\ - \left[A - bs_p + \gamma I(t) + q(1-\sigma) I(t) + \delta s\right] \quad 0 \le t \le t_1 \tag{6.5}$$

$$\frac{dI(t)}{dt} + \theta e^{\theta t} I(t) = -\left[A - bs_p + \gamma I(t) + q(1-\sigma) I(t) + \delta s\right] \quad t_1 \le t \le T \tag{6.6}$$

With the boundary conditions $I(t) = 0$ at $t = 0$, $I(t) = I_m$ at $t = t_1$, and $I(t) = 0$ at $t = T$.

Linearising $\theta(t)$ by using Taylor's series expansion leads to the following

$$\theta e^{\theta t} = \theta \sum_{m=1}^{\infty} \frac{\theta^m t^m}{m!} = \theta \left[1 + \frac{\theta t}{1} + \frac{\theta^2 t^2}{2!} + \frac{\theta^3 t^3}{3!}\right] \approx \theta \tag{6.7}$$

With $\theta \ll 1$

The solution to Equation (6.5) is

$$\begin{aligned} I(t) = &\frac{(1-\eta_m)\, k_0}{[\theta+\gamma+q(1-\sigma)-\omega]} e^{-\omega t} + \frac{\eta_m k_0}{[\theta+\gamma+q(1-\sigma)-\omega-\lambda]} e^{-(\omega+\lambda)t} \\ &- \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} + L_1 e^{-[\theta+\gamma+q(1-\sigma)]t} \quad 0 \le t \le t_1 \end{aligned} \tag{6.8}$$

From Equation (6.8) under the boundary condition $I(t) = 0$ at $t = 0$, the following is obtained

$$L_1 = -\frac{(1-\eta_m)\, k_0}{[\theta+\gamma+q(1-\sigma)-\omega]} - \frac{\eta_m k_0}{[\theta+\gamma+q(1-\sigma)-\omega-\lambda]} + \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \tag{6.9}$$

Substituting Equation (6.9) back into Equation (6.8) leads to the following

$$\begin{aligned} I(t) =&\frac{(1-\eta_m)\, k_0}{[\theta+\gamma+q(1-\sigma)-\omega]} \left[e^{-\omega t} - e^{-[\theta+\gamma+q(1-\sigma)]t}\right] \\ &+ \frac{\eta_m k_0}{[\theta+\gamma+q(1-\sigma)-\omega-\lambda]} \left[e^{-(\omega+\lambda)t} - e^{-[\theta+\gamma+q(1-\sigma)]t}\right] \\ &- \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[1 - e^{-[\theta+\gamma+q(1-\sigma)]t}\right] \quad 0 \le t \le t_1 \end{aligned} \tag{6.10}$$

Solving Equation (6.6), the following result is obtained

$$I(t) = -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} + L_2 e^{-[\theta+\gamma+q(1-\sigma)]t} \tag{6.11}$$

From Equation (6.11) under the boundary condition $I(t) = I_{max}$ at $t = t_1$, we get

$$L_2 = I_m e^{[\theta+\gamma+q(1-\sigma)]t_1} + \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} e^{[\theta+\gamma+q(1-\sigma)]t_1} \tag{6.12}$$

Substituting Equation (6.12) into (6.11) yields

$$I(t) = -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[1 - e^{-[\theta+\gamma+q(1-\sigma)](t-t_1)}\right] + I_m e^{-[\theta+\gamma+q(1-\sigma)](t-t_1)} \quad t_1 \le t \le T \tag{6.13}$$

Applying the continuity condition from Equations (6.10) and (6.13) at time $t = t_1$, the production cycle time t_1 can be determined as follows

$$\begin{aligned} \frac{(1-\eta_m)\, k_0}{\mu} \left[e^{-\omega t_1} - e^{-\upsilon t_1}\right] + \frac{\eta_m k_0}{\psi} \left[e^{-(\omega+\lambda)t_1} - e^{-\upsilon t_1}\right] - \frac{[A-bs_p+\delta s]}{\upsilon} \left[1 - e^{-\upsilon t_1}\right] \\ = I_m \end{aligned} \tag{6.14}$$

with

$$[\theta+\gamma+q(1-\sigma)] = \upsilon \tag{6.15a}$$
$$[\theta+\gamma+q(1-\sigma)-\omega] = \mu \tag{6.15b}$$
$$[\theta+\gamma+q(1-\sigma)-\omega-\lambda] = \psi \tag{6.15c}$$

Linearising the exponential terms containing t_1 in Equation (6.14) by using Taylor's series expansion for e^x, we get

$$e^{-\omega t_1} = \sum_{\xi=1}^{\infty} \frac{(-1)^{\xi} t_1^{\xi}}{\xi!} = \left[1 - \frac{\omega t_1}{1} + \frac{\omega^2 t_1^2}{2!}\right] \approx 1 - \omega t_1 \tag{6.16}$$

$$e^{-\upsilon t_1} = \sum_{\xi=1}^{\infty} \frac{(-1)^{\xi} t_1^{\xi}}{\xi!} = \left[1 - \frac{\upsilon t_1}{1} + \frac{\upsilon^2 t_1^2}{2!}\right] \approx 1 - \upsilon t_1 \tag{6.17}$$

$$e^{-(\omega+\lambda)t} = \sum_{\xi=1}^{\infty} \frac{(-1)^{\xi} t^{\xi}}{\xi!} = \left[1 - \frac{(\omega+\lambda) t_1}{1} + \frac{(\omega+\lambda)^2 t_1^2}{2!}\right] \approx 1 - (\omega+\lambda) t_1 \tag{6.18}$$

Substituting Equations (6.16)-(6.18) into (6.14) yields

$$\begin{gathered}\frac{(1-\eta_m)\, k_0}{\mu}\left[1 - \omega t_1 - 1 + \upsilon t_1\right] + \frac{\eta_m k_0}{\psi}\left[1 - (\omega+\lambda)t_1 - 1 + \upsilon t_1\right] \\ -\frac{[A - b s_p + \delta s]}{\upsilon}\left[1 - 1 + \upsilon t_1\right] \approx I_m\end{gathered} \tag{6.19}$$

$$(1-\eta_m)\, k_0 t_1 + \eta_m k_0 t_1 - [A - b s_p + \delta s]\, t_1 \approx I_m \tag{6.20}$$

$$t_1 \approx \frac{I_m}{k_0 - [A - b s_p + \delta s]} \tag{6.21}$$

Thus, t_1 can be written in terms of I_m. Therefore, t_1 is not a decision variable.

Solving Equation (6.13) while satisfying the boundary condition $I(T) = 0$ yields the following result

$$\begin{gathered}0 = -\frac{[A - b s_p + \delta s]}{[\theta + \gamma + q(1-\sigma)]} + \frac{[A - b s_p + \delta s]}{[\theta + \gamma + q(1-\sigma)]} e^{-[\theta+\gamma+q(1-\sigma)](T-t_1)} \\ + I_m e^{-[\theta+\gamma+q(1-\sigma)](T-t_1)}\end{gathered} \tag{6.22}$$

$$e^{[\theta+\gamma+q(1-\sigma)](T-t_1)} = 1 + \frac{[\theta + \gamma + q(1-\sigma)] I_m}{[A - b s_p + \delta s]} \tag{6.23}$$

$$T - t_1 = \frac{1}{[\theta + \gamma + q(1-\sigma)]} \ln\left[1 + \frac{[\theta + \gamma + q(1-\sigma)] I_m}{[A - b s_p + \delta s]}\right] \tag{6.24}$$

Linearising the logarithmic term containing I_m in Equation (6.24) leads to the following

$$T - t_1 = \frac{1}{[\theta + \gamma + q(1-\sigma)]}\left[\frac{[\theta + \gamma + q(1-\sigma)]}{[A - b s_p + \delta s]} I_m - \frac{[\theta + \gamma + q(1-\sigma)]^2}{2\,[A - b s_p + \delta s]^2} I_m^2\right] \tag{6.25}$$

$$T \approx t_1 + \frac{1}{[A - b s_p + \delta s]} I_m \tag{6.26}$$

Thus, T can be written in terms of I_m. Therefore, T is not a decision variable.

6.4.1 Cost components

6.4.1.1 Inventory holding cost

The total holding cost, denoted as THC, is determined by multiplying the area under the inventory profile by the holding cost per unit per time. Therefore

$$THC = h_p \left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt \right] \tag{6.27}$$

Solving each integral in Equation (6.27) leads

$$\begin{aligned} \int_0^{t_1} I(t)dt = \int_0^{t_1} \Bigg[& \frac{(1-\eta_m)\,k_0}{[\theta+\gamma+q(1-\sigma)-\omega]} \left[e^{-\omega t} - e^{-[\theta+\gamma+q(1-\sigma)]t} \right] \\ & + \frac{\eta_m k_0}{[\theta+\gamma+q(1-\sigma)-\omega-\lambda]} \left[e^{-(\omega+\lambda)t} - e^{-[\theta+\gamma+q(1-\sigma)]t} \right] \\ & - \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[1 - e^{-[\theta+\gamma+q(1-\sigma)]t} \right] \Bigg] dt \end{aligned} \tag{6.28}$$

$$\begin{aligned} = & \frac{(1-\eta_m)\,k_0}{[\theta+\gamma+q(1-\sigma)-\omega]} \left[-\frac{1}{\omega} e^{-\omega t_1} + \frac{1}{[\theta+\gamma+q(1-\sigma)]} e^{-[\theta+\gamma+q(1-\sigma)]t_1} \right] \\ & + \frac{\eta_m k_0}{[\theta+\gamma+q(1-\sigma)-\omega-\lambda]} \left[-\frac{1}{(\omega+\lambda)} e^{-(\omega+\lambda)t_1} + \frac{1}{[\theta+\gamma+q(1-\sigma)]} e^{-[\theta+\gamma+q(1-\sigma)]t_1} \right] \\ & - \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} [t_1 + \frac{1}{[\theta+\gamma+q(1-\sigma)]} e^{-[\theta+\gamma+q(1-\sigma)]t_1]} \end{aligned} \tag{6.29}$$

$$\begin{aligned} \int_{t_1}^{T} I(t)dt = \int_{t_1}^{T} \Bigg[& -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[1 - e^{-[\theta+\gamma+q(1-\sigma)](t-t_1)} \right] \\ & + I_m e^{-[\theta+\gamma+q(1-\sigma)](t-t_1)} \Bigg] dt \end{aligned} \tag{6.30}$$

$$\begin{aligned} = -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} (T-t_1) - \frac{1}{[\theta+\gamma+q(1-\sigma)]} \Bigg[& -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \\ & + \frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[e^{-[\theta+\gamma+q(1-\sigma)](T-t_1)} \right] + I_m e^{-[\theta+\gamma+q(1-\sigma)](T-t_1)} \Bigg] \\ & + \frac{1}{[\theta+\gamma+q(1-\sigma)]} I_m \end{aligned} \tag{6.31}$$

Recalling that Equation (6.22) = 0, Equation (6.31) becomes

$$-\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} (T-t_1) + \frac{1}{[\theta+\gamma+q(1-\sigma)]} I_m \tag{6.32}$$

Substituting expression (6.26) into (6.32) leads to

$$\begin{aligned} -\frac{[A-bs_p+\delta s]}{[\theta+\gamma+q(1-\sigma)]} \left[\frac{[\theta+\gamma+q(1-\sigma)]}{[A-bs_p+\delta s]} I_m \right] + \frac{1}{[\theta+\gamma+q(1-\sigma)]} I_m \\ = \left[\frac{1}{[\theta+\gamma+q(1-\sigma)]} - 1 \right] I_m \end{aligned} \tag{6.33}$$

Therefore, the total holding cost is

$$THC = h_p \left\{ \frac{(1-\eta_m)\,k_0}{\mu} \left[-\frac{1}{\omega} e^{-\omega t_1} + \frac{1}{\upsilon} e^{-\upsilon t_1} \right] + \frac{\eta_m k_0}{\psi} \left[-\frac{1}{\omega+\lambda} e^{-(\omega+\lambda)t_1} + \frac{1}{\upsilon} e^{-\upsilon t_1} \right] \right. \\ \left. - \frac{[A - bs_p + \delta s]}{\upsilon} \left[t_1 + \frac{1}{\upsilon} e^{-\upsilon t_1} \right] + \left[\frac{1}{\upsilon} - 1 \right] I_m \right\} \tag{6.34}$$

Linearising Equation (6.34), we get

$$THC = h_p \left\{ \begin{array}{l} \dfrac{(1-\eta_m)\,k_0}{\mu} \left[\left(\dfrac{1}{\upsilon} - \dfrac{1}{\omega} \right) + \left(\dfrac{\upsilon}{2} - \dfrac{\omega}{2} \right) t_1^2 \right] \\ + \dfrac{\eta_m k_0}{\psi} \left[\left(\dfrac{1}{\upsilon} - \dfrac{1}{\omega+\lambda} \right) + \left(\dfrac{\upsilon}{2} - \dfrac{(\omega+\lambda)}{2} \right) t_1^2 \right] \\ - \dfrac{[A - bs_p + \delta s]}{\upsilon} \left[\dfrac{1}{\upsilon} + \dfrac{\upsilon t_1^2}{2} \right] + \left[\dfrac{1}{\upsilon} - 1 \right] I_m \end{array} \right\} \tag{6.35}$$

6.4.1.2 Deterioration cost

The deterioration cost, denoted by TDC, is the cost associated with the loss of value of the inventory over time. This is equal to the unit cost incurred per unit of inventory that deteriorates at the rate θ over the cycle $[0, T]$. Therefore,

$$TCD = c_d \theta \left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt \right] \tag{6.36}$$

$$TCD = c_d \theta \left\{ \frac{(1-\eta_m)\,k_0}{\mu} \left[\left(\frac{1}{\upsilon} - \frac{1}{\omega} \right) + \left(\frac{\upsilon}{2} - \frac{\omega}{2} \right) t_1^2 \right] \right. \\ + \frac{\eta_m k_0}{\psi} \left[\left(\frac{1}{\upsilon} - \frac{1}{\omega+\lambda} \right) + \left(\frac{\upsilon}{2} - \frac{(\omega+\lambda)}{2} \right) t_1^2 \right] - \frac{[A - bs_p + \delta s]}{\upsilon} \left[\frac{1}{\upsilon} + \frac{\upsilon t_1^2}{2} \right] \\ \left. + \left[\frac{1}{\upsilon} - 1 \right] I_m \right\} \tag{6.37}$$

6.4.1.3 Production cost

The manufacturer's total production cost, TPC, is computed by multiplying the unit production cost by the total production over the cycle $[0, t_1]$, that is,

$$TPC = p_c \int_0^{t_1} k_0 e^{-\omega t} dt \tag{6.38}$$

$$TPC = -\frac{p_c}{\omega} k_0 e^{-\omega t_1} + \frac{p_c}{\omega} k_0 \approx p_c k_0 \left[t_1 - \frac{\omega t_1^2}{2} \right] \tag{6.39}$$

6.4.1.4 Lost production cost

The manufacturer's lost production cost, LP, represents the opportunity cost for not producing the planned quantity, $k_0 t_1$. Mathematically, it can be written as

$$LP = p_l \int_0^{t_1} \left[k_0 - k_0 e^{-\omega t} \right] dt \approx p_l k_0 \frac{\omega t_1^2}{2} \tag{6.40}$$

6.4.1.5 Disposal cost

The disposal cost is computed by multiplying the cost associated with disposing of a product with the total number of defective products manufactured, that is

$$DSC = c_{dp}\left[\int_0^{t_1}\left(1-e^{-\lambda t)}\right)\eta_m k_0 e^{-\omega t}dt\right] \approx c_{dp}\left[\left(\frac{1}{\lambda+\omega}-\frac{1}{\omega}\right)\eta_m k_0 + \left[\frac{(\lambda+\omega)}{2}-\frac{\omega}{2}\right]\eta_m k_0 t_1^2\right] \tag{6.41}$$

6.4.2 Total cost per time

The total cost per unit time, TC , is computed by summing Equations (6.35), (6.37), (6.39), (6.40) and (6.41), hence

$$TC = \frac{1}{T}\left\{\begin{array}{l} G + M_T + (h_p+\theta c_d)\left\{\dfrac{(1-\eta_m)\,k_0}{\mu}\left[\left(\dfrac{1}{\upsilon}-\dfrac{1}{\omega}\right)+\left(\dfrac{\upsilon}{2}-\dfrac{\omega}{2}\right)t_1^2\right]\right. \\ +\dfrac{\eta_m k_0}{\psi}\left[\left(\dfrac{1}{\upsilon}-\dfrac{1}{\omega+\lambda}\right)+\left(\dfrac{\upsilon}{2}-\dfrac{(\omega+\lambda)}{2}\right)t_1^2\right]-\dfrac{[A-bs_p+\delta s]}{\upsilon}\left(\dfrac{1}{\upsilon}+\dfrac{\upsilon t_1^2}{2}\right) \\ \left.+\left(\dfrac{1}{\upsilon}-1\right)I_m\right\}+p_c k_0\left(t_1-\dfrac{\omega t_1^2}{2}\right)+p_l k_0\dfrac{\omega t_1^2}{2} \\ +c_{dp}\left[\left(\dfrac{1}{\lambda+\omega}-\dfrac{1}{\omega}\right)\eta_m k_0+\left(\dfrac{(\lambda+\omega)}{2}-\dfrac{\omega}{2}\right)\eta_m k_0 t_1^2\right] \end{array}\right\} \tag{6.42}$$

The first term in Equation (6.42) is the fixed set-up cost, which is the cost incurred when preparing the process for production. The second term is the fixed cost of maintaining the system (M_T), which the manufacturer incurs at the beginning of each cycle. The third term is the manufacturer's holding cost plus the deterioration cost per cycle as determined in Equations (6.35) and (6.37). The fourth and fifth terms are the production cost and lost production cost, respectively. The last term is the disposal cost per cycle, as determined in Equation (6.41).

6.5 Solution

6.5.1 Determination of the decision variables

There is one decision variable in the cost function of this model, denoted as I_m. The optimal solution to the proposed inventory system is determined by setting the first-order partial derivative of the objective function with respect to I_m to zero.

$$\begin{aligned} \frac{\partial TC}{\partial I_m} =& \frac{1}{T}\left\{(h_p+\theta c_d)\left[t_1+\left(\frac{1}{\upsilon}-1\right)\right]+\frac{p_c k_0}{[k_0-(A-bs_p+\delta s)]}\right. \\ &\left.+(-p_c k_0\omega+p_l k_0\omega+c_{dp}\lambda\eta_m k_0)\frac{t_1}{[k_0-(A-bs_p+\delta s)]}\right\} \\ &-\frac{1}{T}\left[\frac{k_0}{[k_0-(A-bs_p+\delta s)]\,(A-bs_p+\delta s)}\right]TC = 0 \end{aligned} \tag{6.43}$$

6.5.2 Optimality condition

The problem is to determine the optimal value of I_m, which minimises the total cost function TC. Since TC is a function of one variable I_m, with I_m a continuous variable, for an optimal value of I_m, the necessary condition has to be met

$$\frac{\partial^2(TC)}{\partial I_m^2} \geq 0 \tag{6.44}$$

Taking the second derivatives with respect to I_m in the Equation (6.43), we have

$$\begin{aligned}\frac{\partial^2 TC}{\partial I_m^2} =& \frac{1}{T}\left\{\frac{(h_p+\theta c_d)}{k_0-(A-bs_p+\delta s)}\right.\\ &\left.+(-p_c k_0\omega+p_l k_0\omega+c_{dp}\lambda\eta_m k_0)\left[\frac{1}{k_0-(A-bs_p+\delta s)}\right]^2\right\}\\ &-\frac{2}{T}\left[\frac{k_0}{[k_0-(A-bs_p+\delta s)](A-bs_p+\delta s)}\right]\frac{\partial TC}{\partial I_m}\end{aligned} \tag{6.45}$$

6.5.2.1 Numerical example

To illustrate the use of the model developed above, we consider the following input parameters

Demand parameter, $A = 70$

Elasticity parameter of the unit selling price, $b = 0.6$

Deterioration cost per unit item, $c_d = \$1.5/$ item

Disposal cost per unit item, $_{dp} = \$0.8/$ item

Set-up cost, $G = \$12500$

Inventory carrying cost per item produced per time, $h_p = \$0.5$ per item/per hour

Initial production rate at the start of the cycle, $k_0 = 150$ items/hour

Maintenance cost, $M_T = \$5000$

Unit production cost, $p_c = \$10$ per unit

Lost production cost per product, $p_l = \$14/$ item

Social donation amount per sale, $s = \$20$

Market selling price of the product, $s_p = \$80/$ item

Deterioration rate, $\theta = 0.004$

Rate at which the production rate declines, $\omega = 0.025$

Rate at which the proportion of defective items increases over time, $\lambda = 0.035$

Elasticity parameter that reflects the impact of defects on customer demand, $\sigma = 0.12$

Maximum proportion of defective products that can be produced, $\eta_m = 0.35$

Demand enhancement parameter for inventory level, $\gamma = 0.8$

Elasticity factor of quality of stock, 0.95

Elasticity factor of social donation amount, $\delta = 0.9$

The model yields the following results

Table 6.2: Summary of the results from the numerical example

Variable	Units	Quantity
I_m^*	items	2246
t_1^*	hour	20
T^*	hour	76
TC^*	\$/hour	787

Based on the results of the numerical example, which are summarised in Table 6.2, the company should keep a maximum inventory level of 2246 units. Assuming that time is measured in hours, the production time should take 20 hours. The batch should be produced after every 76 hours, which is the duration of the cycle. The manufacturer will incur an average cost of \$787 per hour. Figures 6.2 and 6.3 show the response of the average total cost function to different inventory levels.

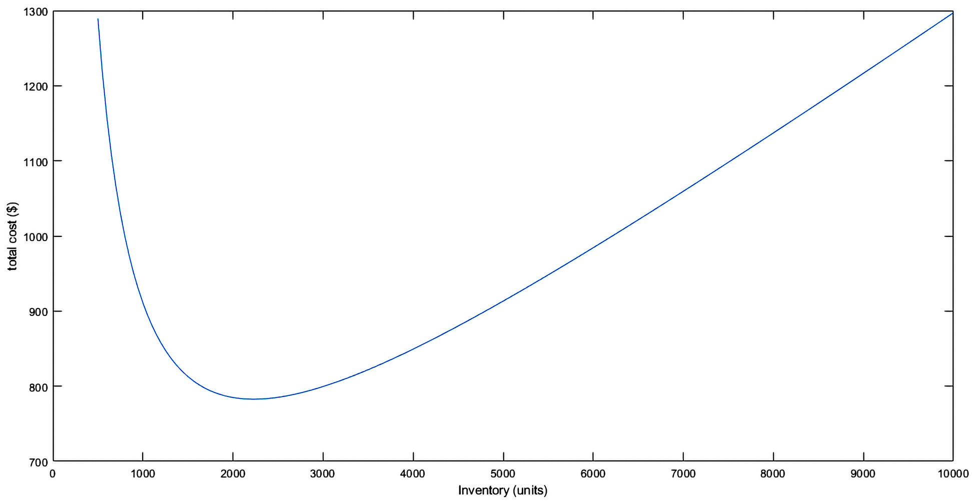


Figure 6.2: Graph of total cost per unit time versus inventory level I_m

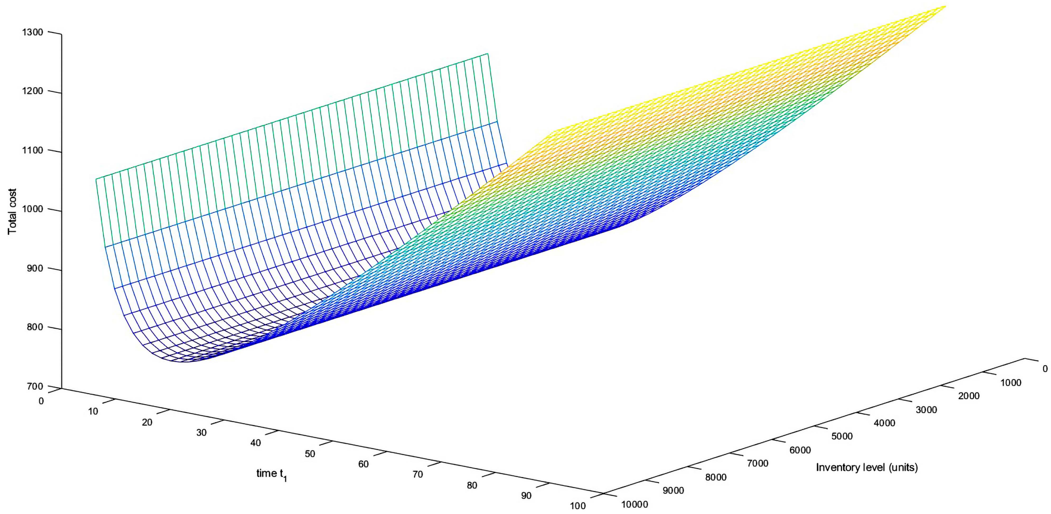


Figure 6.3: Graph of total cost per unit time, production time t_1 and inventory level I_m

Figure 6.2 depicts the total cost per time with varying values of the inventory I_m. The graphical representations presented in Figure indicate that the total cost per time, TC, is strictly convex. Figure 6.3 further illustrates the convexity of the total cost per time when varying values of inventory I_m and production time t_1. The total cost per unit time decreases with the t_1 until it reaches the minimum at the optimal I_m and t_1.

6.5.2.2 Sensitivity analysis

To gain deeper insights into the behaviour of the model, we conducted a sensitivity analysis where we varied several of the model parameters and analysed their effects on the decision variable and the total cost of the manufacturer. These analyses have been implemented by altering each input factor in relative steps of $10\%(-30\%, -20\%, -10\%, +10\%, +20\%, +30\%\ )$ and keeping static the all the other input factors at a time. The findings of these experiments are summarised in Table 6.3.

- The manufacturer's optimal inventory level I_m^* is highly sensitive to A, k_0, h_p, G and p_c. The manufacturer's optimal inventory level I_m^* is most sensitive to k_0. A 30% increase in k_0 result in a 17% increase in I_m^*. If the manufacturer has a huge capacity, obviously, more items would need to be produced in a short space of time. The manufacturer's optimal inventory level I_m^* is moderately sensitive to b, M_T, p_l, s_p, s and δ. Changes to all other inputs did not affect I_m^*.
- The manufacturer's production time t_1^* is highly sensitive to k_0, h_p, G and p_c. The manufacturer's production time t_1^* is most sensitive to k_0. A 30% decrease in t_1^* result in a 30.5% increase in t_1^*. t_1^* was moderately sensitive to A, b, M_T, p_l and s_p. Changes to all other inputs had no effect on t^*.

- The manufacturer's optimal cycle time T^* is highly sensitive to A, b, and s_p. The demand parameter A had the greatest effect on T. A 30% decrease in A resulted in a 95.57% increase in T^*, and A 30% increase in A resulted in a 29% decrease in T^*. This suggests that When there is an increase in the initial demand A, the manufacturer adjusts their production and inventory management accordingly to meet the growing customer demand. This can be observed by the increase in production time t_1. Longer production cycles t_1 allow the manufacturer to produce larger quantities to meet the rising demand. The inventory level I_m decreases with the increase in A. This decrease in I_m can be attributed to the fact that products are consumed at a faster rate. optimal cycle time T^* was moderately sensitive to k_0, h_p, G, p_l, s and δ. The optimal cycle time T^* was not affected by changes to all the remaining input parameters.

- The manufacturer's optimal total cost per time TC^* is highly sensitive to A, b and s_p. Among these parameters, the demand parameter, A, has the greatest impact on the optimal total cost. TC^* is moderately sensitive to k_0, h_p, G, p_c, p_l, s and δ. The optimal total cost TC^* was not affected by changes to all the remaining input parameters.

Table 6.3: *Sensitivity analysis for various inventory model parameters*

	% of change	inventory level I_m units	%change	Time t_1 hours	%change	Cycle time T hours	%change	Total cost TC \$	% change
Base		**2246**		**20**		**76**		**787**	
A	-30%	2497.552	11.2%	18.674	-6.6%	149.3932	96.6%	386.5744	-50.9%
	-20%	2416.247	7.6%	19.086	-4.6%	111.5832	46.8%	523.355	-33.5%
	-10%	2332.471	3.9%	19.528	-2.4%	89.946	18.4%	656.8302	-16.5%
	10%	2156.609	-4.0%	20.51	2.6%	66.3252	-12.7%	913.6283	16.1%
	20%	2063.849	-8.1%	21.06	5.3%	59.28	-22.0%	1036.715	31.7%
	30%	1967.721	-12.4%	21.656	8.3%	53.96	-29.0%	1155.946	46.9%
b	-30%	2058.459	-8.4%	21.092	5.5%	58.9304	-22.5%	1043.641	32.6%
	-20%	2122.47	-5.5%	20.708	3.5%	63.46	-16.5%	959.8252	22.0%
	-10%	2185.133	-2.7%	20.344	1.7%	69.0308	-9.2%	874.2783	11.1%
	10%	2305.519	2.7%	19.672	-1.6%	84.9452	11.8%	698.1477	-11.3%
	20%	2363.915	5.3%	19.36	-3.2%	96.8012	27.4%	607.6427	-22.8%
	30%	2420.963	7.8%	19.062	-4.7%	113.1792	48.9%	515.6424	-34.5%
k_0	-30%	1732.115	-22.9%	26.102	30.5%	69.4336	-8.6%	846.1037	7.5%
	-20%	1924.597	-14.3%	23.564	17.8%	71.6376	-5.7%	825.0908	4.8%
	-10%	2094.17	-6.8%	21.592	8.0%	73.8492	-2.8%	805.2584	2.3%
	10%	2383.455	6.1%	18.676	-6.6%	78.0672	2.7%	770.1582	-2.1%
	20%	2508.557	11.7%	17.552	-12.2%	80.0356	5.3%	754.6543	-4.1%
	30%	2623.328	16.8%	16.578	-17.1%	81.8976	7.8%	740.4096	-5.9%
c_d	-30%	2249.594	0.2%	20.032	0.2%	76.1216	0.2%	786.5278	-0.1%
	-20%	2248.246	0.1%	20.02	0.1%	76.076	0.1%	786.6852	0.0%
	-10%	2247.123	0.1%	20.01	0.1%	76.038	0.1%	786.8426	0.0%
	10%	2244.877	-0.1%	19.99	-0.1%	75.962	-0.1%	787.1574	0.0%
	20%	2243.754	-0.1%	19.98	-0.1%	75.924	-0.1%	787.3148	0.0%
	30%	2242.631	-0.2%	19.97	-0.2%	75.886	-0.2%	787.4722	0.1%
c_{dp}	-30%	2274.749	1.3%	20.256	1.3%	76.9728	1.3%	789.5971	0.3%
	-20%	2265.316	0.9%	20.172	0.9%	76.6536	0.9%	788.7314	0.2%
	-10%	2255.658	0.4%	20.086	0.4%	76.3268	0.4%	787.8657	0.1%
	10%	2236.342	-0.4%	19.914	-0.4%	75.6732	-0.4%	786.1343	-0.1%
	20%	2226.684	-0.9%	19.828	-0.9%	75.3464	-0.9%	785.1899	-0.2%
	30%	2217.251	-1.3%	19.744	-1.3%	75.0272	-1.3%	784.3242	-0.3%
h_p	-30%	2594.579	15.5%	23.104	15.5%	87.7952	15.5%	745.8399	-5.2%
	-20%	2463.413	9.7%	21.936	9.7%	83.3568	9.7%	760.4781	-3.4%
	-10%	2348.193	4.6%	20.91	4.6%	79.458	4.6%	774.0932	-1.6%
	10%	2154.363	-4.1%	19.184	-4.1%	72.8992	-4.1%	799.1198	1.5%
	20%	2071.71	-7.8%	18.448	-7.8%	70.1024	-7.8%	810.61	3.0%
	30%	1996.469	-11.1%	17.778	-11.1%	67.5564	-11.1%	821.4706	4.4%
G	-30%	1945.934	-13.4%	17.328	-13.4%	65.8464	-13.4%	734.5071	-6.7%
	-20%	2050.823	-8.7%	18.262	-8.7%	69.3956	-8.7%	752.8442	-4.3%
	-10%	2150.545	-4.3%	19.15	-4.3%	72.77	-4.3%	770.3156	-2.1%
	10%	2337.412	4.1%	20.814	4.1%	79.0932	4.1%	802.9761	2.0%
	20%	2425.455	8.0%	21.598	8.0%	82.0724	8.0%	818.4013	4.0%
	30%	2510.354	11.8%	22.354	11.8%	84.9452	11.8%	833.2756	5.9%
M_T	-30%	2131.005	-5.1%	18.976	-5.1%	72.1088	-5.1%	766.9315	-2.6%
	-20%	2170.085	-3.4%	19.324	-3.4%	73.4312	-3.4%	773.6997	-1.7%
	-10%	2208.267	-1.7%	19.664	-1.7%	74.7232	-1.7%	780.3892	-0.8%
	10%	2283.059	1.7%	20.33	1.7%	77.254	1.7%	793.4534	0.8%
	20%	2319.444	3.3%	20.654	3.3%	78.4852	3.3%	799.8281	1.6%
	30%	2355.38	4.9%	20.974	4.9%	79.7012	4.9%	806.1241	2.4%
p_l	-30%	2089.005	-7.0%	18.602	-7.0%	70.6876	-7.0%	696.495	-11.5%
	-20%	2137.518	-4.8%	19.034	-4.8%	72.3292	-4.8%	726.9519	-7.6%
	-10%	2189.85	-2.5%	19.5	-2.5%	74.1	-2.5%	757.094	-3.8%
	10%	2306.867	2.7%	20.542	2.7%	78.0596	2.7%	816.6699	3.8%
	20%	2372.674	5.6%	21.128	5.6%	80.2864	5.6%	846.025	7.5%
	30%	2444.771	8.9%	21.77	8.9%	82.726	8.9%	875.0653	11.2%
p_c	-30%	2540.451	13.1%	22.622	13.1%	85.9636	13.1%	741.5114	-5.8%
	-20%	2429.723	8.2%	21.636	8.2%	82.2168	8.2%	757.2514	-3.8%
	-10%	2332.471	3.9%	20.77	3.9%	78.926	3.9%	772.4405	-1.9%
	10%	2168.513	-3.5%	19.31	-3.5%	73.378	-3.5%	801.0086	1.8%
	20%	2098.438	-6.6%	18.686	-6.6%	71.0068	-6.6%	814.6237	3.5%
	30%	2034.876	-9.4%	18.12	-9.4%	68.856	-9.4%	827.7666	5.2%

Table 6.4: Sensitivity analysis for various inventory model parameters

	% of change	inventory level I_m units	%change	Time t_1 hours	%change	Cycle time T hours	%change	Total cost TC \$	% change
Base		**2246**		**20**		**76**		**787**	
s_p	-30%	2058.459	-8.4%	21.092	5.5%	58.9304	-22.5%	1043.641	32.6%
	-20%	2122.47	-5.5%	20.708	3.5%	63.46	-16.5%	959.8252	22.0%
	-10%	2185.133	-2.7%	20.344	1.7%	69.0308	-9.2%	874.2783	11.1%
	10%	2305.519	2.7%	19.672	-1.6%	84.9452	11.8%	698.1477	-11.3%
	20%	2363.915	5.3%	19.36	-3.2%	96.8012	27.4%	607.6427	-22.8%
	30%	2420.963	7.8%	19.062	-4.7%	113.1792	48.9%	515.6424	-34.5%
s	-30%	2312.931	3.0%	19.632	-1.8%	86.2448	13.5%	686.8936	-12.7%
	-20%	2290.92	2.0%	19.752	-1.2%	82.4828	8.5%	720.4985	-8.5%
	-10%	2268.46	1.0%	19.876	-0.6%	79.0856	4.1%	753.8673	-4.2%
	10%	2223.315	-1.0%	20.128	0.6%	73.188	-3.7%	819.8966	4.2%
	20%	2200.406	-2.0%	20.256	1.3%	70.6192	-7.1%	852.5571	8.3%
	30%	2177.272	-3.1%	20.39	2.0%	68.2632	-10.2%	882.9353	12.2%
θ	-30%	2249.369	0.2%	20.03	0.2%	76.114	0.2%	786.5278	-0.1%
	-20%	2248.246	0.1%	20.02	0.1%	76.076	0.1%	786.6852	0.0%
	-10%	2247.123	0.1%	20.01	0.1%	76.038	0.1%	786.8426	0.0%
	10%	2244.877	-0.1%	19.99	-0.1%	75.962	-0.1%	787.1574	0.0%
	20%	2243.754	-0.1%	19.98	-0.1%	75.924	-0.1%	787.3148	0.0%
	30%	2242.631	-0.2%	19.97	-0.2%	75.886	-0.2%	787.4722	0.1%
ω	-30%	2227.134	-0.8%	19.832	-0.8%	75.3616	-0.8%	759.3763	-3.5%
	-20%	2241.508	-0.2%	19.96	-0.2%	75.848	-0.2%	769.9221	-2.2%
	-10%	2246.898	0.0%	20.008	0.0%	76.0304	0.0%	778.9726	-1.0%
	10%	2241.059	-0.2%	19.956	-0.2%	75.8328	-0.2%	794.3191	0.9%
	20%	2233.198	-0.6%	19.886	-0.6%	75.5668	-0.6%	801.0086	1.8%
	30%	2223.315	-1.0%	19.798	-1.0%	75.2324	-1.0%	807.3046	2.6%
γ	-30%	2226.909	-0.9%	19.83	-0.9%	75.354	-0.9%	785.1899	-0.2%
	-20%	2233.872	-0.5%	19.892	-0.5%	75.5896	-0.5%	785.8982	-0.1%
	-10%	2240.385	-0.3%	19.95	-0.3%	75.81	-0.3%	786.4491	-0.1%
	10%	2251.166	0.2%	20.046	0.2%	76.1748	0.2%	787.4722	0.1%
	20%	2255.882	0.4%	20.088	0.4%	76.3344	0.4%	787.9444	0.1%
	30%	2260.15	0.6%	20.126	0.6%	76.4788	0.6%	788.3379	0.2%
q	-30%	2225.786	-0.9%	19.82	-0.9%	75.316	-0.9%	785.1112	-0.2%
	-20%	2233.422	-0.6%	19.888	-0.6%	75.5744	-0.6%	785.8195	-0.2%
	-10%	2239.936	-0.3%	19.946	-0.3%	75.7948	-0.3%	786.4491	-0.1%
	10%	*	*	*	*	*	*	*	*
	20%	*	*	*	*	*	*	*	*
	30%	*	*	*	*	*	*	*	*
σ	-30%	2248.246	0.1%	20.02	0.1%	76.076	0.1%	787.2361	0.0%
	-20%	2247.572	0.1%	20.014	0.1%	76.0532	0.1%	787.1574	0.0%
	-10%	2246.674	0.0%	20.006	0.0%	76.0228	0.0%	787.0787	0.0%
	10%	2245.326	0.0%	19.994	0.0%	75.9772	0.0%	786.9213	0.0%
	20%	2244.428	-0.1%	19.986	-0.1%	75.9468	-0.1%	786.8426	0.0%
	30%	2243.529	-0.1%	19.978	-0.1%	75.9164	-0.1%	786.7639	0.0%
λ	-30%	2259.701	0.6%	20.122	0.6%	76.4636	0.6%	787	0.0%
	-20%	2254.759	0.4%	20.078	0.4%	76.2964	0.4%	786.9213	0.0%
	-10%	2250.267	0.2%	20.038	0.2%	76.1444	0.2%	786.9213	0.0%
	10%	2241.957	-0.2%	19.964	-0.2%	75.8632	-0.2%	787.0787	0.0%
	20%	2238.139	-0.4%	19.93	-0.4%	75.734	-0.4%	787.2361	0.0%
	30%	2234.321	-0.5%	19.896	-0.5%	75.6048	-0.5%	787.3935	0.1%
η_m	-30%	2266.439	0.9%	20.182	0.9%	76.6916	0.9%	788.1805	0.2%
	-20%	2259.476	0.6%	20.12	0.6%	76.456	0.6%	787.787	0.1%
	-10%	2252.738	0.3%	20.06	0.3%	76.228	0.3%	787.3935	0.1%
	10%	2239.262	-0.3%	19.94	-0.3%	75.772	-0.3%	786.6065	-0.1%
	20%	2232.524	-0.6%	19.88	-0.6%	75.544	-0.6%	786.213	-0.1%
	30%	2225.786	-0.9%	19.82	-0.9%	75.316	-0.9%	785.8195	-0.2%
δ	-30%	2312.931	3.0%	19.632	-1.8%	86.2448	13.5%	688.4676	-12.5%
	-20%	2290.695	2.0%	19.752	-1.2%	82.4828	8.5%	721.6003	-8.3%
	-10%	2268.46	1.0%	19.874	-0.6%	79.0856	4.1%	754.4182	-4.1%
	10%	2223.315	-1.0%	20.128	0.6%	73.188	-3.7%	819.267	4.1%
	20%	2200.406	-2.0%	20.258	1.3%	70.6192	-7.1%	851.2979	8.2%
	30%	2177.272	-3.1%	20.39	2.0%	68.2632	-10.2%	882.9353	12.2%

6.6 Conclusion

This section deals with a continuously declining production system for deteriorating items under social donation. The inventory model is designed for a single product, where neither replacement nor repair is permitted for deteriorated and imperfect products. The degraded behaviour of the process is modelled by a continuously declining production rate as a function of time. The constant rate of the defective products manufactured is extended to include a continuous function that increases over time. Defective products are assumed to be scrapped, and customers' returns are accounted for by introducing a parameter to quantify the impact of defects on customer demand. Furthermore, the demand rate is formulated as a function of selling price, stock level, quality of stock, and social donation amount. The theoretical results derived in this section prove the existence and uniqueness of the optimal solution for the given problem, strengthening the model's validity and practical applicability. Looking ahead, future research can extend this inventory model by incorporating an integrated joint inventory approach, considering both manufacturer and retailer perspectives. Moreover, investigating the impact of carbon emissions at the manufacturer's facility and considering scenarios with partial backlogging would further enhance the model's realism and relevance to real-world manufacturing scenarios. This perspective aligns with the growing interest in sustainability and green practices in the manufacturing industry.

Chapter 7

Conclusion

7.1 Summary

The primary aim of this thesis was to develop inventory models for managing deteriorating products in a multi-state system. Three lot sizing models were developed, each addressing a different aspect of the research objective. The multi-state structure of the models, the flexibility of the production systems, and the fact that a vast majority of products are subject to deterioration make these models good representations for large-scale production systems in various industries. To enhance their practicality, each of the models accounts for specific characteristics of production systems, such as stock-dependent demand, price-dependent demand, freshness-dependent demand, product quality, and multi-echelon. Apart from addressing gaps in existing literature, the developed inventory models hold significant practical value for professionals in operations and supply chain management within the manufacturing environment.

7.2 Contributions to knowledge

Chapter 4: An inventory model with a shifting production rate, for perishables products with freshness, price, and stock-dependent demand rate and price discounting

The first model proposed an inventory system with a shifting production rate and other product characteristics such as product deterioration with a limited life span and product demand that is dependent on the stock level, the state of freshness of the product, and the selling price. The product also needed to be discounted as it got close to the expiry date to boost demand and prevent wastage beyond its life span. The inventory system was formulated mathematically to determine the optimal selling price and cycle time that maximises the net profit. The presence of freshness conditions on products means that if the product's freshness drops, the demand drops correspondingly. Consequently, since products are already made, the cycle time lengthens, which increases the quantity, which deteriorates and subsequently depletes the profit. This finding should motivate production and operations managers to pay attention to the shelf life of the products they manufacture and ensure to strike a balance between cycle time and the freshness of the products to maintain their profit margin.

Chapter 5: A two-echelon supply chain inventory model for perishable prod-

ucts with a shifting production rate, stock-dependent demand rate and imperfect quality raw material
The second model considered a two-echelon inventory system that considered a shift in production rate and imperfect raw materials. The model integrated the cost of raw materials necessary for production and considered the presence of imperfect quality items within the acquired raw materials. Upon the immediate receipt of raw materials, a 100% screening process was implemented to identify imperfect quality items. By combining imperfect raw materials and shifting production rates, two different sub-models for deteriorating products were formulated under imperfect production and a demand function that was dependent on the stock level. In the first sub-model, imperfect raw materials were sold at a discounted price at the end of the screening period, whereas in the second one, imperfect items were kept in stock until the end of the inventory cycle and then returned to the supplier. Numerical illustrations demonstrated that selling the imperfect raw material after the screening process was beneficial, as opposed to returning it to the supplier at the end of the cycle.

Chapter 6 An integrated EPQ Model for deteriorating products with declining production rate, increasing defects, stock and price-dependent demand, and effects of corporate social responsibility activities
The last model proposed an inventory system in which the production declined continuously with time while considering various factors, including the stock level, product quality, Corporate Social Responsibility impacts (CSR), and deterioration of end products. It is assumed that the defective function is a continuous and increasing function and that the demand increases with the social donation amount. The main aim of the inventory model developed in this chapter was to find the ideal inventory level that minimised the manufacturer's total cost per unit of time. To illustrate the effectiveness of the proposed inventory model, this chapter included a numerical example. Through the analysis of the model, it was found that investing in CSR led to a decrease in inventory. This indicates CSR has a considerable impact on inventory management as it may increase consumer trust and loyalty, and this presents operations managers with opportunities to reduce excess inventory through better management practices.

7.3 Suggestions for future research

This research work addressed the challenge of managing unreliable production systems with continuous quality control. In this thesis, various production policies were developed. However, there is still room for further research expansion. For instance, extending the models to accommodate popular extensions, such as shortages and permissible delays in payment. Additionally, future research could explore the integration of learning effects in the screening process.

The model with shifting production rates can be extended to include multiple products and multiple machines. The assumption that the facility has only one machine and produces one product type may not necessarily hold. Furthermore, the model can be extended to include other popular EPQ extensions such as time discounting, trade credit financing, inflation, and environment policies, to name a few.

All three models developed in this thesis assumed that production is deterministic, as well as the time at which the production shifts. Further investigation could involve integrating probabilistic or stochastic processes to represent uncertainties related to shift occurrences to enhance the model's ability to reflect actual production system dynamics more realistically. This expansion would increase the model's relevance by better aligning it with the unpredictable nature of production processes in practical environments.

The models developed in this thesis were derived under deterministic demand conditions. This assumption can be relaxed to accommodate stochastic demand, leading to more accurate representations of real-world inventory systems.

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