Completeness properties in the vector lattice C(X)

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own independent work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Candidate's signature:

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Abstract

In this thesis we study certain vector lattice properties of the space $C(X)$ of continuous functions on a given topological space X. We show that $C(X)$ is always relatively uniformly complete, and characterize those X for which $C(X)$ is Dedekind complete. We characterise the bands and projection bands in $C(X)$, for X a Tychonoff space, and characterize those Tychonoff spaces X for which $C(X)$ has the projection property.

Contents

Introduction

Many of the space of interest in functional analysis are naturally equipped with a partial order which is compatible with linear structure of the space, yielding a partially ordered vector space. For instance, spaces of continuous, real-valued functions on a topological space and spaces of measurable functions on a measure space may be ordered pointwise. A particularly well-studied class of such spaces are the Riesz spaces; i.e. those partially ordered vector spaces which are also lattices (see Section 1.1 for the definition). The function spaces mentioned above are all examples of vector lattices.

There are several notions of completeness for Riesz spaces. Among these, relatively uniform completeness, Dedekind completeness and the projection property are the most important. In this thesis, we study these completeness concepts in the context of the Riesz space $C(X)$ of continuous, real-valued functions on a topological space X. We show that $C(X)$ is always relatively uniformly complete. On the other hand, $C(X)$ fails to be either Dedekind complete or to satisfy the projection property for all "reasonable" topological spaces. We prove that $C(X)$ is Dedekind complete if and only if X is extremally disconnected, if and only if $C(X)$ has the projection property.

The thesis is structured as follows. In Chapter 1 we recall general definitions and results from the theory of Riesz spaces. In particular, Section 1.1 deals with elementary identities and inequalities in Riesz spaces. In Section 1.2 we introduce completeness properties in Riesz spaces: Relatively uniform completeness and Dedekind completeness. Section 1.3 deals with order convergence, and Section 1.4 concerns ideals and bands in Riesz spaces.

In Chapter 2 we consider the vector lattice $C(X)$. In Section 2.1 we collect such preliminary results and definitions as are required for the presentation of the results in the rest of the chapter. In Section 2.2 we show that $C(X)$ is relatively uniformly complete, and in Section 2.3 we characterise those X for which $C(X)$ is Dedekind complete. Finally, we study bands and projection bands in $C(X)$ in Section 2.4.

Chapter 1

Riesz Spaces

1.1 Elementary identities and inequalities

In this section we collect some basic results on Riesz spaces. These can be found in [5, Chapter 1] or [11, Chapter 1].

Definition 1.1.1. Let E be a topological space equipped with the relation $\leq$ with the following properties:

(i) For all $f \in E$ we have that $f \leq f$.

(ii) For all $f, g \in E$, if $f \leq g$ and $g \leq f$ then $f = g$.

(iii) For all $f, g, h \in E$, if $f \leq g$ and $g \leq h$ then $f \leq h$.

Then E is a *partially ordered space.*

Definition 1.1.2. Let E be a partially ordered space. We call E a *lattice* if for every $x, y \in E$, $\{x, y\}$ has a supremum in E, denoted by $x \vee y$, and an infimum in E, denoted by $x \wedge y$.

Definition 1.1.3. Let E be a real vector space.

(i) If E is partially ordered in such a manner that the partial ordering is compatible with the algebraic structure (i.e. if $f, g \in E$, then $f \leq g$ implies that $f + h \leq g + h$ for every $h \in E$, and $0 \leq f$ implies $0 \leq af$ for any $a \in [0, \infty)$), E is called a *partially ordered vector space.*

(ii) E is called a *Riesz space* if E is both a lattice and a partially ordered vector space.

From this point on we will assume that E is always a Riesz space.

Definition 1.1.4. The collection $E^+ = \{u \in E \colon u \geq 0\}$ is called the *positive cone* of E. Furthermore, for any $f \in E$ we define $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, and $|f| = (-f) \vee f$.

Theorem 1.1.5. *Let D be a non-empty subset of E and assume that $f_0 = \sup D$ exists. If $g \in E$, then $f_0 \wedge g = \sup\{f \wedge g \colon f \in D\}$. Similarly if $f_1 = \inf D$ exists, $f_1 \vee g = \sup\{f \vee g \colon f \in D\}$.*

Proof. Note that $f_0 \geq f$ for all $f \in D$ so $f_0 \wedge g \geq f \wedge g$ for all $f \in D$. So $f_0 \wedge g$ is an upper bound for $\{f \wedge g \colon f \in D\}$. Now let $u \in L$ be any upper bound for $\{f \wedge g \colon f \in D\}$. Then it follows that $u \geq f \wedge g = f + g - (f \vee g) \geq f + g - (f_0 \vee g)$ for all $f \in D$. So we have that $u - g + (f_0 \vee g) \geq f$ for all $f \in D$ and hence by the supremum criterion $u - g + (f_0 \vee g) \geq f_0$. Therefore $u \geq f_0 + g - (f_0 \vee g) = f_0 \wedge g$. This holds for all such upper bounds u so $f_0 \wedge g$ is the least upper bound of $\{f \wedge g \colon f \in D\}$.

The second statement is proven in similar fashion. We have $f_1 \leq f$ for all $f \in D$ so $f_1 \vee g \leq f \vee g$ for all $f \in D$. So $f_1 \vee g$ is a lower bound for $\{f \vee g \colon f \in D\}$. Now let $v \in L$ be any lower bound for $\{f \vee g \colon f \in D\}$. Then it follows that $v \leq f \vee g = f + g - (f \wedge g) \leq f + g - (f_1 \wedge g)$ for all $f \in D$. So we have that $v - g + (f_1 \wedge g) \leq f$ for all $f \in D$ and hence by the infimum criterion $v - g + (f_1 \wedge g) \leq f_1$. Therefore $v \leq f_1 + g - (f_1 \wedge g) = f_1 \vee g$. This holds for all such lower bounds v so $f_1 \vee g$ is the greatest lower bound of $\{f \vee g \colon f \in D\}$. □

Theorem 1.1.6. *Let $f, g, h \in E$. The following hold:*

(i) $-(f \vee g) = (-f) \vee (-g)$ *and* $-(f \wedge g) = (-f) \wedge (-g)$.

(ii) $(f + h) \vee (g + h) = (f \vee g) + h$ *and* $(f + h) \wedge (g + h) = (f \wedge g) + h$.

(iii) $(f - g) \vee (f - h) = f - (g \wedge h)$ *and* $(f - g) \wedge (f - h) = f - (g \vee h)$.

(iv) $(af) \vee (ag) = a(f \vee g)$ *and* $(af) \wedge (ag) = a(f \wedge g)$ *for every* $a \in \mathbb{R}^+$.

(v) $(f \vee g) \vee h = f \vee (g \vee h)$ *and* $(f \wedge g) \wedge h = f \wedge (g \wedge h)$.

(vi) $f^+, f^-, |f| \in E^+$; $f^+ \wedge f^- = 0$; *and* $f^+ \vee f^- = |f|$.

(vii) $f = 0$ *if and only if* $|f| = 0$.

(viii) $(f \vee g) + (f \wedge g) = f + g$ *and* $(f \vee g) - (f \wedge g) = |f - g|$.

(ix) $|f + g| \leq |f| + |g|$ *and* $||f| - |g|| \leq |f - g|$.

Proof. (i): First note that $f \leq f \vee g$ and $g \leq f \vee g$. So $-(f \vee g) \leq -f$ and $-(f \vee g) \leq -g$ and hence $-(f \vee g) \leq (-f) \wedge (-g)$. Now consider any $k \in E$ such that $k \leq (-f) \wedge (-g)$. It follows that $f \leq -k$ and $g \leq -k$ and hence $f \vee g \leq -k$. So $k \leq -(f \vee g)$ and therefore $-(f \vee g) = (-f) \wedge (-g)$. In the same way it follows that $-(f \wedge g) = (-f) \vee (-g)$.

(ii): First note that $f + h \leq (f \vee g) + h$ and $g + h \leq (f \vee g) + h$ and hence $(f+h) \vee (g+h) \leq (f \vee g) + h$. Now consider any $k \in E$ such that $f + h \leq k$ and $g + h \leq k$. It follows that $f \leq k - h$ and $g \leq k - h$ and hence $(f \vee g) \leq k - h$. Therefore $(f \vee g) + h \leq k$ and hence $(f + h) \vee (g + h) = (f \vee g) + h$. Using a similar method, it can be shown that $(f + h) \wedge (g + h) = (f \wedge g) + h$.

(iii): Since $-(g \wedge h) = (-g) \wedge (-h)$ and $-(g \vee h) = (-g) \wedge (-h)$ from (i), the result follows from (ii).

(iv): The proof is trivial for $a = 0$. For any $a > 0$, $af \leq a(f \vee g)$ and $ag \leq a(f \vee g)$. Now consider any $k \in E$ such that $af \leq k$ and $ag \leq k$. Then $f \leq \frac{1}{a}k$ and $g \leq \frac{1}{a}k$ and hence $f \vee g \leq \frac{1}{a}k$. It follows $a(f \vee g) \leq k$ so $a(f \vee g) \leq (af) \vee (ag)$ and therefore $a(f \vee g) = (af) \vee (ag)$. In the same way it can be shown that $a(f \wedge g) = (af) \wedge (ag)$.

(v): First note that $f \vee g \leq f \vee (g \vee h)$ since $g \leq g \vee h$. Also $h \leq (f \vee (g \vee h)$ and hence $(f \vee g) \vee h \leq f \vee (g \vee h)$. Now consider any $k \in E$ such that $(f \vee g) \vee h \leq k$. We have that $f \leq f \vee g \leq k$ and $g \leq f \vee g \leq k$. Also $h \leq k$, so $g \vee h \leq k$ and therefore $f \vee (g \vee h) \leq k$. It follows that $f \vee (g \vee h) \leq (f \vee g) \vee h$ and hence $(f \vee g) \vee h = f \vee (g \vee h)$. A similar proof shows that $(f \wedge g) \wedge h = f \wedge (g \wedge h)$.

(vi): Firstly $f^+, f^- \in E^+$ by definition. Also by definition $f \leq |f|$ and $-f \leq |f|$. It follows that $|f| \in E^+$ as $0 \leq 2|f|$. It follows from Theorem 1.1.5 that

$$f^+ \wedge f^- = (f \vee 0) \wedge ((-f) \vee 0) = 0 \wedge (f \vee (-f)) = 0 \wedge |f| = 0.$$

It also follows that

$$f^+ \vee f^- = (f \vee 0) \vee ((-f) \vee 0) = 0 \vee (f \vee (-f)) = 0 \vee |f| = |f|.$$

(vii): If $f = 0$ then $-f = 0$ and hence $|f| = 0$. Now assuming $|f| = 0$, we have that $f \leq 0$ and $-f \leq 0$. So $0 \leq f$ and $f = 0$.

$(viii)$: We have the following:

$$\begin{aligned} f \vee g &= \frac{1}{2}(2(f \vee g)) \\ &= \frac{1}{2}((2f) \vee (2g)) \quad \text{by } (iv) \\ &= \frac{1}{2}((f+g) + (f-g) \vee (g-f)) \quad \text{by } (ii) \\ &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g|. \end{aligned}$$

We also have

$$\begin{aligned} f \wedge g &= \frac{1}{2}(2(f \wedge g)) \\ &= \frac{1}{2}((2f) \wedge (2g)) \quad \text{by } (iv) \\ &= \frac{1}{2}((f+g) - (f-g) \vee (g-f)) \quad \text{by } (ii) \\ &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g|. \end{aligned}$$

Therefore $(f \vee g) + (f \wedge g) = f + g$ and $(f \vee g) - (f \wedge g) = |f - g|$.

(ix): Since $f \leq |f|$, $-f \leq |f|$, $g \leq |g|$ and $-g \leq |g|$, it follows that $f+g \leq |f|+|g|$ and $-(f+g) = -f-g \leq |f|+|g|$. Therefore $|f+g| \leq |f|+|g|$. It now follows that $|f| \leq |f-g| + |g|$ and hence $|f| - |g| \leq |f-g|$. By similar argument we have that $|g| - |f| \leq |g-f| = |f-g|$, and therefore $||f| - |g|| \leq |f-g|$. □

We use these identities to prove the Riesz decomposition property and Birkoff's inequalities.

Theorem 1.1.7. *[Riesz decomposition property] Let $u, z_1, z_2 \in E^+$ be such that $u \leq z_1 + z_2$. Then there exists $u_1, u_2 \in E^+$ satisfying $u = u_1 + u_2$ and $u_1 \leq z_1$ and $u_2 \leq z_2$.*

Proof. Define $u_1 = u \wedge z_1$ and $u_2 = u - u_1$. Clearly $0 \leq u_1 \leq z_1$ and $u = u_1 + u_2$. Also $u_1 \leq u$ so $u_2 \in E^+$. Lastly by Theorem 1.1.6 (ii),

$$u_2 = u - u_1 = u - (u \wedge z_1) = (u - z_1) \vee 0 \leq z_2 \vee 0 = z_2.$$

□

Theorem 1.1.8. *[Birkoff's inequalities] The following inequalities hold for all $x, y, z \in E$:*

(i) $|x \vee z - y \vee z| \leq |x - y|$

(ii) $|x \wedge z - y \wedge z| \leq |x - y|$.

Proof. Firstly we know that $|p - q| = p \vee q - p \wedge q$ for any $p, q \in E$ from the lattice identities established in Theorem 1.1.6. Now substituting p with $x \vee z$ and q with $y \vee z$ we have the following:

$$\begin{aligned}
|x \vee z - y \vee z| &= (x \vee z) \vee (y \vee z) - (x \vee z) \wedge (y \vee z) \\
&= (x \vee (z \vee (y \vee z))) - (z \vee x) \wedge (z \vee y) \\
&= x \vee (y \vee z) - z \vee (x \wedge y) \\
&= (x \vee y) \vee z - (x \wedge y) \vee z.
\end{aligned}$$

Also substituting p with $x \wedge z$ and q with $y \wedge z$ we have the following:

$$\begin{aligned}
|x \wedge z - y \wedge z| &= (x \wedge z) \vee (y \wedge z) - (x \wedge z) \wedge (y \wedge z) \\
&= (z \wedge x) \vee (z \wedge y) - (x \wedge (z \wedge (y \wedge z))) \\
&= z \wedge (x \vee y) - x \wedge (y \wedge z) \\
&= (x \vee y) \wedge z - (x \wedge y) \wedge z.
\end{aligned}$$

Therefore it follows:

$$\begin{aligned}
&|x \vee z - y \vee z| + |x \wedge z - y \wedge z| \\
&= (x \vee y) \vee z - (x \wedge y) \vee z + (x \vee y) \wedge z - (x \wedge y) \wedge z \\
&= ((x \vee y) \vee z + (x \vee y) \wedge z) - ((x \wedge y) \vee z + (x \wedge y) \wedge z) \\
&= ((x \vee y) + z) - ((x \wedge y) + z) \\
&= x \vee y - x \wedge y \\
&= |x - y|.
\end{aligned}$$

Since $|x \vee z - y \vee z|, |x \wedge z - y \wedge z| \in E^+$ we have that $|x \vee z - y \vee z| \leq |x - y|$ and $|x \wedge z - y \wedge z| \leq |x - y|$. □

Notation 1.1.9. For every $x, y \in E$, we write $x \perp y$ whenever $|x| \wedge |y| = 0$.

Theorem 1.1.10. *If $x \in E$ then the following hold:*

(*i*) $x = x^+ - x^-$

(*ii*) *If* $x \geq 0$ *then* $x = x^+$ *and if* $x \leq 0$ *then* $-x = x^-$.

(*iii*) $x^+ \perp x^-$.

(*iv*) *If* $x = u - v$ *where* $u \geq 0$,$v \geq 0$ *and* $u \perp v$, *then* $u = x^+$ *and* $v = x^-$.

Proof. The proofs will make use of Theorem 1.1.6. Let $x \in E$.
(*i*):

$$\begin{aligned}
x^+ - x^- &= (x \vee 0) - ((-x) \vee 0) \\
&= (x \vee 0) - ((-x) \vee (-0)) \\
&= (x \vee 0) - (-(x \wedge 0)) \\
&= (x \vee 0) + (x \wedge 0) \\
&= x + 0 \\
&= x.
\end{aligned}$$

(*ii*): If $x \geq 0$ it follows that $x = x \vee 0 = x^+$. Now if $x \leq 0$ then $(-x) \geq 0$. It follows that $-x = (-x) \vee 0 = x^-$.

(*iii*): Since $x^+, x^- \in E^+$, the result follows directly from the fact that $x^+ \wedge x^- = |x^+| \wedge |x^-| = 0$.

(*iv*): Since $u \perp v$, $u \geq 0$ and $v \geq 0$ we have that

$$|u| \wedge |v| = u \wedge v = 0.$$

We are given that $x = u - v$ and hence we have the following:

$$\begin{aligned}
x^+ &= x \vee 0 \\
&= (u - v) \vee (u - u) \\
&= u + (-v) \vee (-u) \\
&= u - v \wedge u \\
&= u.
\end{aligned}$$

We also have the following:

$$\begin{aligned}
x^- &= (-x) \vee 0 \\
&= (v - u) \vee (v - v) \\
&= v + (-v) \vee (-u) \\
&= v - u \wedge v \\
&= v.
\end{aligned}$$

So $x^+ = u$ and $x^- = v$. $\square$

Definition 1.1.11. A Riesz space E is called Archimedean if for every $u \in E^+$,

$$\inf\{n^{-1}u \colon n \in \mathbb{N}\} = 0.$$

Example 1.1.12. Consider the following spaces:

(*i*) The set $\mathbb{R}$ equipped with the standard ordering and vector space structure is Archimedean as a result of the Archimedean property of $\mathbb{R}$.

(*ii*) Consider $\mathbb{R}^2$ equipped with coordinatewise vector space structure, and the relation $\preceq$ defined by $\langle x_1, y_1\rangle \preceq \langle x_2, y_2\rangle$ when either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$. This space is not Archimedean. We can see this as $\langle 0, 5\rangle$ is a lower bound of $\{n^{-1}u \colon n \in \mathbb{N}\}$ for $u = \langle 4, 6\rangle$, but $\langle 0, 0\rangle \preceq \langle 0, 5\rangle$.

(*iii*) For a topological space X, the set of all real-valued continuous functions on X, denoted by $C(X)$, is an Archimedean Riesz space.

For the rest of the document we shall consider E to be an Archimedean Riesz space.

1.2 R.u. completeness and Dedekind completeness

We now recall the relevant completeness properties in Riesz spaces. These can be found in [11, Sections A, B], for instance.

Definition 1.2.1. Let E be a Riesz space, $u \in E$ and (u_n) a sequence in E. We say that (u_n) is *relatively uniformly (r.u.) convergent* to u if there exists $e \in E^+$ such that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ so that if $n \geq N_\varepsilon$ then $|u_n - u| \leq \varepsilon e$.

Definition 1.2.2. Let E be a Riesz space, and (u_n) a sequence in E. We say that (u_n) is *relatively uniformly (r.u.) Cauchy* if there exists $e \in E$ where $e \geq 0$ such that for every $\varepsilon > 0$ there exists $M_\varepsilon \in \mathbb{N}$ so that if $n \geq N_\varepsilon$ then $|u_n - u_m| \leq \varepsilon e$.

Remark 1.2.3. Note the element e referred to in the above two definitions is known as a *regulator* for the sequence (u_n).

Definition 1.2.4. Let E be a Riesz space. Then E is *r.u. complete* if every r.u. Cauchy sequence in E is r.u. convergent in E.

Proposition 1.2.5. *Let E be a Riesz space, and let (u_n) be a r.u. convergent sequence in E. Then (u_n) is r.u. Cauchy.*

Proof. Suppose that (u_n) is r.u. convergent to some $u \in E$, and let $e \in E^+$ be a regulator for the sequence (u_n). Fix any $\varepsilon > 0$ and note that there exists $N_\varepsilon \in \mathbb{N}$ so that $|u_n - u| \leq \frac{\varepsilon}{2} e$ whenever $n \geq N_\varepsilon$. Now assuming that $m, n \in \mathbb{N}$ where $m, n \geq N_\varepsilon$ we have that

$$|u_n - u_m| = |u_n - u + u - u_m| \leq |u_n - u| + |u - u_m| \leq \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e.$$

This holds for all $\varepsilon > 0$, so (u_n) is r.u. Cauchy. □

Definition 1.2.6. We say that E is *Dedekind complete* if for non-empty every subset $A \subset E$, if A is bounded above then $\sup A$ exists in E.

1.3 Order Convergence

An important concept in the theory of Riesz spaces is that of order convergence. In this section we recall the definition of an order convergent net and collect some basic results dealing with order convergence. Several different definitions of order convergence of nets are used in the literature, see for instance [1]. We use the definition of [2].

Notation 1.3.1. For a net (x_α) in E, we write $x_\alpha \downarrow 0$ if (x_α) is a decreasing net with infimum 0.

Definition 1.3.2. For a net (x_α) in E and $x \in E$, we write $x_\alpha \xrightarrow{o} x$ if there exists a net (q_β) satisfying $q_\beta \downarrow 0$ such that for every β there exists α_0 for which if $\alpha \geq \alpha_0$ then $|x_\alpha - x| \leq q_\beta$. We say that (x_α) *converges in order* to x.

The following proposition establishes the uniqueness of order limits in an Archimedean Riesz space if the limit exists.

Proposition 1.3.3. *Let (x_α) be a net in E such that $x_\alpha \xrightarrow{o} x$ and $x_\alpha \xrightarrow{o} y$ for some $x, y \in E$. Then $x = y$.*

Proof. Since $x_\alpha \xrightarrow{o} x$, there exists (q_β) where $q_\beta \downarrow 0$ such that for each β there exists α_0 so that $|x_\alpha - x| \leq q_\beta$ whenever $\alpha \geq \alpha_0$. Also since $x_\alpha \xrightarrow{o} y$, there exists (p_γ) where $p_\gamma \downarrow 0$ such that for each γ there exists α_1 so that $|x_\alpha - x| \leq p_\gamma$ whenever $\alpha \geq \alpha_1$. Consider $r_{\beta,\gamma} = q_\beta + p_\gamma$ and suppose $\alpha \geq \alpha_0$ and $\alpha \geq \alpha_1$. Note that $r_{\beta,\gamma} \downarrow 0$. We have

$$|x - y| \leq |x - x_\alpha| + |x_\alpha - y| \leq q_\beta + p_\gamma = r_{\beta,\gamma}.$$

So $|x - y|$ is a lower bound of the net $(r_{\beta,\gamma})$ as this holds for all β and γ. It follows that $|x - y| = 0$, and $x - y = 0$. Hence $x = y$. □

Proposition 1.3.4. *Suppose (x_α) and (y_α) are nets in E such that $x_\alpha \xrightarrow{o} x$ and $y_\alpha \xrightarrow{o} y$, with $x, y \in E$. Let $\lambda \in \mathbb{R}$. The following holds:*

(i) $x_\alpha + y_\alpha \xrightarrow{o} x + y$.

(ii) $\lambda x_a \xrightarrow{o} \lambda x$.

(iii) $x_\alpha \vee y_\alpha \xrightarrow{o} x \vee y$.

(*iv*) $x_\alpha \wedge y_\alpha \xrightarrow{o} x \wedge y$.

(*v*) ${x_\alpha}^+ \xrightarrow{o} x^+$.

(*vi*) ${x_\alpha}^- \xrightarrow{o} x^-$.

(*vii*) $|x_\alpha| \xrightarrow{o} |x|$.

Proof. (i): There exist nets (q_β) and (p_γ) satisfying $q_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ such that, for all β and γ, there exists α_0, for which $|x_\alpha - x| \leq q_\beta$ holds for every $\alpha \geq \alpha_0$ and there exists α_1 for which $|x_\alpha - x| \leq p_\gamma$ holds for every $\alpha \geq \alpha_1$. Now the net $(r_{\beta,\gamma})$ where $r_{\beta,\gamma} = q_\beta + p_\gamma$ satisfies $r_{\beta,\gamma} \downarrow 0$. Let α_2 be a mutual upper bound for α_0 and α_1. Taking any $\alpha \geq \alpha_2$, we get that

$$|(x_\alpha + y_\alpha) - (x + y)| \leq |x_\alpha - x| + |y_\alpha - y| \leq q_\beta + p_\gamma = r_{\beta,\gamma}.$$

Therefore $x_\alpha + y_\alpha \xrightarrow{o} x + y$.

(ii): For each β there exists α_0 for which $|x_a - x| \leq q_\beta$ holds for every $\alpha \geq \alpha_0$. Now the net $(|\lambda| q_\beta)$ satisfies $|\lambda| q_\beta \downarrow 0$ and taking any $\alpha \geq \alpha_0$, we also have that

$$|\lambda x_\alpha - \lambda x| = |\lambda(x_\alpha - x)| = |\lambda||(x_\alpha - x)| \leq |\lambda| q_\beta.$$

Therefore $\lambda x_a \xrightarrow{o} \lambda x$.

(iii): There exist nets (q_β) and (p_γ) satisfying $q_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ such that for a given β and a given γ there exists α_0 for which $|x_\alpha - x| \leq q_\beta$ holds for every $\alpha \geq \alpha_0$ and there exists α_1 for which $|x_\alpha - x| \leq p_\gamma$ hold for every $\alpha \geq \alpha_1$. The net $(r_{\beta,\gamma})$ where $r_{\beta,\gamma} = q_\beta + p_\gamma$ satisfies $r_{\beta,\gamma} \downarrow 0$. Let α_2 be a mutual upper bound for α_0 and α_1. Taking any $\alpha \geq \alpha_2$, we get that from Birkoff's inequalities, Theorem 1.1.8

$$\begin{aligned}
|(x_\alpha \vee y_\alpha) - (x \vee y)| &= |(x_\alpha \vee y_\alpha) - (x \vee y_\alpha) + (x \vee y_\alpha) - (x \vee y)| \\
&\leq |(x_\alpha \vee y_\alpha) - (x \vee y_\alpha)| + |(y_\alpha \vee x) - (y \vee x)| \\
&\leq |x_\alpha - x| + |y_\alpha - y| \quad \text{(Birkoff's Inequalities)} \\
&\leq q_\beta + p_\gamma \\
&= r_{\beta,\gamma}.
\end{aligned}$$

Therefore $x_\alpha \vee y_\alpha \xrightarrow{o} x \vee y$.

(iv): There exist nets (q_β) and (p_γ) satisfying $q_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ such that for a given β and a given γ there exists α_0 for which $|x_\alpha - x| \leq q_\beta$ holds for every $\alpha \geq \alpha_0$ and there exists α_1 for which $|x_\alpha - x| \leq p_\gamma$ hold for every $\alpha \geq \alpha_1$. The net $(r_{\beta,\gamma})$ where $r_{\beta,\gamma} = q_\beta + p_\gamma$ satisfies $r_{\beta,\gamma} \downarrow 0$. Let α_2 be a mutual upper bound for α_0 and α_1. Taking any $\alpha \geq \alpha_2$, we get that from Birkoff's inequalities, Theorem 1.1.8

$$\begin{aligned}
|(x_\alpha \wedge y_\alpha) - (x \wedge y)| &= |(x_\alpha \wedge y_\alpha) - (x \wedge y_\alpha) + (x \wedge y_\alpha) - (x \wedge y)| \\
&\leq |(x_\alpha \wedge y_\alpha) - (x \wedge y_\alpha)| + |(y_\alpha \wedge x) - (y \wedge x)| \\
&\leq |x_\alpha - x| + |y_\alpha - y| \quad \text{(Birkoff's Inequalities)} \\
&\leq q_\beta + p_\gamma \\
&= r_{\beta,\gamma}.
\end{aligned}$$

Therefore $x_\alpha \wedge y_\alpha \xrightarrow{o} x \wedge y$.

(v): The net (z_α) with $z_\alpha = 0$ for each α is such that $z_\alpha \xrightarrow{o} 0$. So by (iii) we have that

$$x_\alpha{}^+ = x_\alpha \vee 0 = x_\alpha \vee z_\alpha \xrightarrow{o} x \vee 0 = x^+.$$

The proof for (vi) is done similarly, noting the fact that $-x_\alpha \xrightarrow{o} -x$ by (ii).

(vii): We are given that $x_\alpha \xrightarrow{o} x$ and thus $-x_\alpha \xrightarrow{o} -x$ by (ii). It follows from (iii) that

$$|x_\alpha| = (x_\alpha) \vee (-x_\alpha) \xrightarrow{o} x \vee (-x) = |x|.$$

□

Definition 1.3.5. We call $A \subset E$ *order closed* if for every net (x_α) in A such that $x_\alpha \xrightarrow{o} x$ for some $x \in E$, we have $x \in A$.

1.4 Ideals and Bands

This section deals with important classes of subspaces of a Riesz space, namely, ideals and bands. The results presented in this section are based on [3].

Definition 1.4.1. We call a vector subspace G of E a *Riesz subspace* of E if G is closed under the lattice operations inherited from E.

Definition 1.4.2. A subset A of E is called *solid* if $|x| \leq |y|$ with $x \in E$ and $y \in A$ implies that $x \in A$.

Definition 1.4.3. We call a solid vector subspace of E an *order ideal*, or just an ideal for short.

Proposition 1.4.4. *Every order ideal is a Riesz subspace.*

Proof. Let J be an order ideal of E. By definition J is a vector subspace of E, so it remains to be shown that J is closed under the lattice operations inherited from E. Note that for any $z \in J$, $|z| \geq 0$ by Theorem 1.1.6 and hence $-|z| \leq 0 \leq |z|$. So $||z|| = |z| \vee (-|z|) = |z|$, and hence $|z| \in J$ as J is an order ideal. Take any $x, y \in J$. We have that $x+y, x-y, y-x, |x-y| \in J$, as J is an order ideal. We have

$$\begin{aligned}
x \vee y &= \frac{1}{2}(2(x \vee y)) \\
&= \frac{1}{2}((2x) \vee (2y)) \\
&= \frac{1}{2}((x+y) + (x-y) \vee (y-x)) \\
&= \frac{1}{2}(x+y) + \frac{1}{2}|x-y|.
\end{aligned}$$

So $x \vee y \in J$ since J is a vector subspace of E. We also have that $x \wedge y = x \vee y - |x-y|$ as a result of Theorem 1.1.6, and hence $x \wedge y \in J$ as J is a vector subspace of E. □

Definition 1.4.5. An order closed ideal B of E is called a *band*.

Notation 1.4.6. Given $A \subset E$, define

$$A^d := \{x \in E \colon x \perp a \text{ for all } a \in A\}.$$

Proposition 1.4.7. *For every subset A of Riesz space E, A^d is a band of E.*

Proof. Let $x, y \in A^d$ and $c \in \mathbb{R}$. Noting that $|z| \geq 0$ for any $z \in E$, we have that for any $a \in A$

$$0 \leq |x+y| \wedge |a| \leq (|x| + |y|) \wedge |a| = |x| \wedge |a| + |y| \wedge |a| = 0$$

and hence $x + y \in A^d$. If $|c| \leq 1$ then

$$0 \leq |cx| \wedge |a| = |c||x| \wedge |a| \leq |x| \wedge |a| = 0,$$

so $|cx| \wedge |a| = 0$. If $|c| > 1$ then

$$|cx| \wedge |a| \leq |cx| \wedge |c||a| = |c||x| \wedge |c||a| = |c|(|x| \wedge |a|) = 0,$$

so $|cx| \wedge |a| = 0$. It follows that for any $c \in \mathbb{R}$, $cx \in A^d$ and hence A^d is a vector subspace of E. Now suppose $|x| \leq |y|$ for some $x \in E$ and $y \in A^d$. Then for any $a \in A$, $0 \leq |x| \wedge |a| \leq |y| \wedge |a| = 0$. Hence $x \in A^d$ and thus A^d is an order ideal. Now consider any net (x_α) in A such that $x_a \xrightarrow{o} x$ for some $x \in E$ and let $a \in A$. The net $(|x_\alpha|)$ satisfies $|x_\alpha| \xrightarrow{o} |x|$ by Proposition 1.3.4. It follows by Proposition 1.3.4 that $(|x_\alpha| \wedge |a|)$ satisfies $|x_\alpha| \wedge |a| \xrightarrow{o} |x| \wedge |a|$. But $|x_\alpha| \wedge |a| = 0$ for all α and hence $|x_\alpha| \wedge |a| \xrightarrow{\alpha} 0$. By Proposition 1.3.3, $|x| \wedge |a| = 0$ and hence $x \in A^d$. Therefore A^d is a band of E. $\square$

Definition 1.4.8. Given $A \subset E$ we call the smallest band of E that contains A the *band generated by* A and denote it by $B(A)$.

Notation 1.4.9. For any $x \in E^+$, define the set $[0, x] := \{y \in E \colon 0 \leq y \leq x\}$.

Proposition 1.4.10. *If J is an ideal of E, then*

$$B(J) = \{y \in E \colon \textit{there exists } (x_\alpha) \textit{ in } J \textit{ such that } x_\alpha \xrightarrow{o} y\}.$$

Proof. Let $B = \{y \in E \colon \text{there exists } (x_\alpha) \text{ in } J \text{ such that } x_\alpha \xrightarrow{o} y\}$. Note that for any band that contains J will contain B as well. This is because for any $x \in B$ and an existing net in the form of (x_α) in J such that $x_\alpha \xrightarrow{o} x$ we have that any band containing J will contain (x_a) and would hence contain x as well due to the definition of a band. So it suffices to show that B is a band as B is a subset of every band containing J. To this end, let $x, y \in B$

and $c \in \mathbb{R}$. There exists nets (x_α) and (y_α) in J which converge in order to x and y respectively. Since J is an order ideal, it follows from Proposition 1.3.4 that $(x_\alpha + y_\alpha)$ and (cx_α) are nets in J that converge in order to $x+y$ and cx respectively. So $x+y, cx \in B$ and hence B is a vector subspace of E as this holds for all $x, y \in B$ and $c \in \mathbb{R}$. Now suppose $|x| \leq |y|$ for some $x \in E$ and $y \in B$. Then $x \leq |x| \leq |y|$. There exists a net (y_α) in J such that $y_\alpha \xrightarrow{o} y$ and thus by Proposition 1.3.4 we have that $|y_\alpha| \xrightarrow{o} |y|$. So by Proposition 1.3.4 it follows that $x \wedge |y_\alpha| \xrightarrow{o} x \wedge |y| = x$. Taking the net (z_α) where $z_\alpha = x \wedge |y_\alpha|$ for each α, we get a net in J that converges in order to x since J is an ideal. Therefore $x \in B$ and thus B is an order ideal.

Next, let (x_α) be a net in B^+ such that $x_\alpha \xrightarrow{o} x$ for some $x \in E$. By Proposition 1.3.4 we have that $x_\alpha = x_\alpha^+ \xrightarrow{o} x^+$ and hence by Proposition 1.3.3, $x = x^+ \in E^+$. Note that by Proposition 1.3.4, we have that

$$x_\alpha \wedge x \xrightarrow{o} x \wedge x = x.$$

We also have that $x_\alpha \wedge x \leq x$ for each α so x is an upper bound of the net $(x_\alpha \wedge x)$. Suppose $y \in E^+$ is an upper bound of the net $(x_\alpha \wedge x)$. By Proposition 1.3.4,

$$x_\alpha \wedge x = (x_\alpha \wedge x) \wedge y \xrightarrow{o} x \wedge y.$$

Therefore by Proposition 1.3.3, $x = x \wedge y$ and hence $x \leq y$ so that $\sup\{x_\alpha \wedge x \colon \alpha\} = x$.

For each α, we have $x_\alpha \in B$ and hence there exists a net (y_γ^α) in J^+ such that, considering convergence in order with γ as the indexing parameter, $y_\gamma^\alpha \xrightarrow{o} x_\alpha$. Since (x_α) and (y_γ^α) are nets in J^+, considering convergence in order with γ as the indexing parameter, it follows by Proposition 1.3.4 that

$$0 \leq y_\gamma^\alpha \wedge x_\alpha \xrightarrow{o} x_\alpha \wedge x_\alpha = x_\alpha.$$

It follows from Proposition 1.3.4 that

$$0 \leq y_\gamma^\alpha \wedge x_\alpha \wedge x = (y_\gamma^\alpha \wedge x_\alpha) \wedge x \xrightarrow{o} x_\alpha \wedge x.$$

Consider the net (z_γ^α) where $z_\gamma^\alpha = y_\gamma^\alpha \wedge x_\alpha \wedge x$. Note that $z_\alpha^\gamma \in [0, x]$ since $0 \leq z_\alpha^\gamma \leq x$, so x is an upper bound for the net (z_γ^α). Also $0 \leq z_\alpha^\gamma \leq y_\alpha^\gamma$ so since J is an ideal, $z_\alpha^\gamma \in J$ because $|z_\alpha^\gamma| \leq |y_\alpha^\gamma|$ for each γ. So $z_\alpha^\gamma \in J \cap [0, x]$.

Suppose $y \in E^+$ is an upper bound of the net (z_γ^α). Considering convergence in order with γ as the indexing parameter, it follows from Proposition 1.3.4 that $z_\gamma^\alpha = z_\gamma^\alpha \wedge y \xrightarrow{o} (x_\alpha \wedge x) \wedge y$ for each α. So by Proposition

1.3.3, it follows that the nets $x_\alpha \wedge x = (x_\alpha \wedge x) \wedge y$ for each α and hence $(x_\alpha \wedge x)$ and $((x_\alpha \wedge x) \wedge y)$ equal nets in E^+. Considering convergence in order with α as the indexing parameter, it follows from Proposition 1.3.4 that $x_\alpha \wedge x = (x_\alpha \wedge x) \wedge y \xrightarrow{o} x \wedge y$. So by Proposition 1.3.3, $x = x \wedge y \leq y$ and hence $x = \sup\{z_\gamma^\alpha \colon \alpha, \gamma\}$. The set $J \cap [0, x]$ is bounded above by x. Also note that if $y \in E$ is an upper bound of $J \cap [0, x]$ then it is also an upper bound of $\{z_\gamma^\alpha \colon \alpha, \gamma\}$ since $\{z_\gamma^\alpha \colon \alpha, \gamma\}$ and $J \cap [0, x]$ are non-empty sets such that $\{z_\gamma^\alpha \colon \alpha, \gamma\} \subset (J \cap [0, x])$. Therefore $x = \sup(J \cap [0, x])$.

Furthermore, $J \cap [0, x]$ can be considered a net that converges in order to x (taking (y_α) where $y_\alpha = \alpha$ for $\alpha \in J \cap [0, x]$, $y_\alpha \uparrow x$) so by definition of B, $x \in B$ as $J \cap [0, x] \subset J$. Let (x_α) be a net in B such that $x_\alpha \xrightarrow{o} x$ for some $x \in E$. Note that $({x_\alpha}^+)$ and $({x_\alpha}^-)$ are nets in B^+ and $x^+, x^- \in E^+$. From Proposition 1.3.4 we have that ${x_\alpha}^+ \xrightarrow{o} x^+$ and ${x_\alpha}^- \xrightarrow{o} x^-$ so $x^+, x^- \in B^+$ and hence $x = x^+ - x^- \in B$ by Theorem 1.1.10. $\square$

Definition 1.4.11. A Riesz subspace G of E is *order dense* in E if for each $0 \neq x \in E^+$ there exists $y \in G^+$ such that $0 \neq y \leq x$.

Lemma 1.4.12. *Let G be an ideal of E and for any $0 \neq x \in E^+$, define the set $H_x \coloneqq G \cap [0, x]$. Then G is order dense in E if and only if for every $0 \neq x \in E^+$, $x = \sup(H_x)$.*

Proof. Consider any $0 \neq x \in E^+$. First note that $0 \in H_x$ so H_x is non-empty and H_x is bounded above by x. Suppose that $x \leq y$ for any $y \in E^+$ that bounds H_x above i.e. $x = \sup(H_x)$. Since $0 \neq x \in E^+$ and x bounds H_x above, there exists a $y \in H_x$ such that $y \geq 0$ and $0 \neq y$, i.e. $y \in G^+$ and $0 \neq y \leq x$. Since $0 \neq x \in E^+$ is arbitrary, G is order dense in E.

Assume G is order dense in E and suppose that for some $0 \neq x \in E^+$, there exists $y \in E^+$ which is an upper bound of H_x and $x \nleq y$. Choosing $z = x \wedge y$, we have that z bounds H_x above and $0 \neq x - z \in E^+$. So there exists $y' \in G^+$ such that $0 \neq y' \leq x - z \leq x$ and thus $z \leq z + y' \leq x$. It follows that $y' \in H_x$ so $y' \leq z$. If $y' = z$ then $2y' = z + y' \leq x$. Hence $2y' \in H_x$ since $2y' \in G$. Therefore $2z = 2y' \leq z$ so that $z \leq 0$. Since z is an upper bound for H_x, it follows that $H_x = \{0\}$, contradicting the fact that G is order dense in E. There exists no $k \in \mathbb{N}$ such that $z - (k-1)y'$ is an upper bound of H_x but $z - ky' \in E$ is not. This is because if one such $k \in \mathbb{N}$ did exist then there exists $y'' \in H_x$ such that $y'' \nleq z - ky'$. It would then follow that $y' + y'' \nleq z - (k-1)y'$. But $y' + y'' \in H_x$ since $y' + y'' \leq y' + z \leq x$ and $y' + y'' \in G^+$. This contradicts $z - (k-1)y'$ being an upper bound of H_x.

Since $z - 0y'$ is an upper bound of H_x, it follows that $z - y'$ must be an upper bound of H_x. In fact for any $n \in \mathbb{N}$, if $z - ny'$ bounds H_x above then so does $z - (n+1)y'$. Therefore $z - ky'$ bounds H_x above for all $k \in \mathbb{N}$ by induction and hence $y' \leq z - ky'$. Therefore $y' \leq \frac{1}{k+1}z$ for all $k \in \mathbb{N}$ and since $0 \neq y' \in G^+$, this contradicts the definition of an Archimedean Riesz space. Hence $x = sup(H_x)$. □

Lemma 1.4.13. *Let A is an order ideal in E. Then A is order dense in E if and only if $A^d = \{0\}$.*

Proof. Assume A is order dense in E. If $x \in A$ then $x \in A^d$ if and only if $|x| \wedge |x| = |x| = 0$, i.e. $x = 0$. So suppose $x \notin A$. If $x \neq 0$ then $|x| \neq 0$ and $|x| \geq 0$. So there exists $y \in A^+$ such that $0 \neq y \leq |x|$, i.e $|x| \wedge |y| = y \neq 0$ and hence $x \notin A^d$ because $x \not\perp y$. Therefore $A^d = \{0\}$. Now assume $A^d = \{0\}$ and consider any $0 \neq x \in E^+$. We know $x \notin A^d$ so there exists $y \in A$ such that $|x| \wedge |y| \neq 0$. But $|x| \wedge |y| \leq |y|$ so $|x| \wedge |y| \in A$. It follows that

$$0 \neq |x| \wedge |y| = x \wedge |y| \leq x,$$

and hence A is order dense in E. □

Proposition 1.4.14. *If J is an order ideal in E, then $(J^d)^d = B(J)$.*

Proof. Consider any $x \in J$. For any $y \in J^d$ we have that $y \perp x$ hence $x \in (J^d)^d$ and it follows that $J \subset (J^d)^d$. Also by Proposition 1.4.7 we have that $(J^d)^d$ is a band of E that contains J and hence $B(J) \subset (J^d)^d$.

Note that since $J^d \cap (J^d)^d = \{0\}$, J is order dense in $(J^d)^d$ due to Lemma 1.4.13. So suppose $x \in ((J^d)^d)^+$. From the proof of Proposition 1.4.10, the set $H_x = ((J)^d)^d \cap J \cap [0, x] = J \cap [0, x]$ can be expressed as a net in J that converges in order to x since $x = \sup\limits_{y \in (J^d)^d}(H_x)$ by Lemma 1.4.12. Hence $x \in B(J)$ by Proposition 1.4.10. If $x \in ((J^d)^d)$, x^+ and x^- are elements of $((J^d)^d)^+$ and therefore are elements of $B(J)$. Hence by Theorem 1.1.10, $x = x^+ - x^- \in B(J)$. □

Corollary 1.4.15. *Let J be an order ideal in E. The following statements are equivalent:*

(i) *J is order dense in E.*

(ii) $\sup([0, f] \cap J) = f$ *for any* $f \in E^+$.

(iii) $B(J) = E$.

Proof. $(i) \iff (ii)$: This follows from Lemma 1.4.12.

$(i) \iff (iii)$: First note that for any $f \in E$ we have that $|f| \wedge |0| = 0$ and hence $f \perp 0$. Due to Lemma 1.4.13 and Proposition 1.4.14, it follows J is order dense in E if and only if $B(J) = (J^d)^d = \{0\}^d = E$. □

Lemma 1.4.16. *Let J be an order dense Riesz subspace of some Riesz space E and suppose $x \in J$ and (x_α) is a net in J such that $x_\alpha \xrightarrow{o} x$ in J. Then $x_\alpha \xrightarrow{o} x$ in E.*

Proof. There exists a net (q_β) in J satisfying $(q_\beta) \downarrow 0$ in J such that for every β there exists α_0 for which $|x_a - x| \leq q_\beta$ holds for every $\alpha \geq \alpha_0$. Now (q_β) is a net in E so it remains to show that $(q_\beta) \downarrow 0$ in E. Consider any $0 \neq y \in E^+$ and note that there exists $z \in J^+$ such that $0 \neq z \leq y$. It follows that there exists β such that $z \nleq q_\beta$ and hence $y \nleq q_\beta$. Therefore $(q_\beta) \downarrow 0$ in E. □

Notation 1.4.17. Let U and V be vector subspaces of E for which $E = U + V$. If $U \cap V = 0$, then we write $E = U \oplus V$.

Definition 1.4.18. A band B of Riesz space E is called a *projection band* if $B \oplus B^d = E$.

Remark 1.4.19. Let B, C be vector subspaces of Riesz space E.

(i) If $x \in B \cap B^d$ then $|x| \wedge |x| = |x| = 0$ and hence $x = 0$. So $B \cap B^d = \{0\}$ and if B is a band in E then B is a projection band if $B + B^d = E$.

(ii) Let $B \oplus C = E$ and suppose $x_1, x_2 \in B$ and $y_1, y_2 \in C$ such that $z = x_1 + y_1 = x_2 + y_2$ for any $z \in E$. It follows that $x_1 = x_2 + y_2 - y_1 \in B$ and hence $y_2 - y_1 \in B \cap C$. Therefore $y_1 = y_2$ and hence $x_1 = x_2$. So every element of $z \in E$ has a unique decomposition $z = x + y$ where $x \in B$ and $y \in C$.

(iii) If B is a projection band in E then so is B^d since $B^d + (B^d)^d = B^d + B = E$ by Proposition 1.4.14.

Proposition 1.4.20. *Let B be a projection band in E. Then there exists a linear, idempotent map $P\colon E \to B$ such that $0 \leq P(x) \leq x$ for every $x \in E^+$.*

Proof. Define $P\colon E \to B$ where for each $x \in E$, $P(x)$ is the unique element in B such that $x - P(x) \in B^d$. For any $x, y \in E$, $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y - (\alpha P(x) + \beta P(y)) = \alpha(x - P(x)) + \beta(y - P(y)) \in B^d.$$

Since the direct sum decomposition is unique, $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y)$, and hence P is a linear map. It follows that for any $x \in E$, $P(x) - P(P(x)) = P(x - P(x)) \in B \cap B^d = \{0\}$. So $P(P(x)) = P(x)$, and hence P is an idempotent map. Take any $x \in E^+$ and note that $x \leq |P(x)| + |x - P(x)|$ where $|P(x)| \in B$ since B is a band and $||P(x)|| = |P(x)|$. From the Riesz decomposition property Theorem 1.1.7, it follows that there exists unique $p, q \in E^+$ which satisfy $x = p + q$ and $p \leq |P(x)|$ and $q \leq |x - P(x)|$. But B and B^d are bands, so $p \in B$ and $q \in B^d$. Therefore $P(x) = p$ and hence $0 \leq P(x) \leq x$. □

Remark 1.4.21. We refer to P established in Proposition 1.4.20 as a *band projection* onto B.

Lemma 1.4.22. *The following statements are true.*

(i) Let U and V be vector subspaces of E. If $V \subset U^d$ and $E = U \oplus V$, then $V = U^d$ and U and V are projection bands.

(ii) If $P\colon E \to E$ is linear and idempotent and $0 \leq P(x) \leq x$ holds for every $x \in E^+$, then $U := P(E)$ is a projection band, and $I - P$ is the band projection onto U^d.

Proof. (i): Note that for all $x \in U$ and $y \in V$, $x \perp y$ and hence $U \subset V^d$. Letting $x \in U^d$, there exists $y \in U$ and $z \in V$ such that $x = y + z$. Then $y = x - z \in U \cap U^d = \{0\}$. So $y = 0$, $x = z \in V$ and thus $V = U^d$. It can be shown by similar argument that $U = V^d$. By Proposition 1.4.7 it follows that U and V are bands and are hence projection bands since $U \oplus V = E$.

(ii): Let $Q = I - P$ where I is the identity map. Since I and P are linear maps, so is Q. We also have the following due to P being idempotent:

$$\begin{aligned} QQ &= (I - P)(I - P) \\ &= (I - P)I - (I - P)P \\ &= I - P - P + PP \\ &= I - P \\ &= Q. \end{aligned}$$

Therefore Q is an idempotent map. For any $x \in E^+$, since $0 \leq P(x) \leq x$ we have that $Q(x) = x - P(x) \geq 0$ and $Q(x) = x - P(x) \leq x$. Let $U = P(E)$ and $V = Q(E)$. Since P and Q are linear maps, U and V are vector subspaces of E. For any x in E, $x = P(x) + Q(x)$ and hence $U + V = E$. Let $x \in U \cap V$ and note that there exists $y, z \in E$ such that $x = P(y) = Q(z)$. It follows that

$$\begin{aligned} P(y) = z - P(z) &\implies z = P(y) + P(z) \\ &\implies z \in U. \end{aligned}$$

So there exists some $w \in E$ such that $P(w) = z$ and hence $P(z) = P(P(w)) = P(w) = z$. It follows that

$$x = Q(z) = z - P(z) = z - z = 0.$$

So $U \oplus V = E$.
It remains to show that $U \subset V^d$ as it would then follow from (i) that P and Q are band projections onto projection bands U and $V = U^d$ respectively. First we show that U and V are ideals. Consider any $f \in U^+$ and let $0 \leq g \leq f$ for some $g \in E^+$. There exists unique $g_1, h_1 \in U$ and $g_2, h_2 \in V$ such that $g = g_1 + g_2$ and $f - g = h_1 + h_2$, i.e. $P(g) = g_1$, $P(f-g) = h_1$, $Q(g) = g_2$ and $Q(f-g) = h_2$. It follows that $f = g + (f-g) = (g_1 + h_1) + (g_2 + h_2)$ and hence $g_2 + h_2 = f - (g_1 + h_1) \in U \cap V = \{0\}$. Therefore $g_2 + h_2 = 0$. But since $0 \leq g_2 \leq g$ and $0 \leq h_2 \leq f - g$, it follows that $g_2 = h_2 = 0$. Therefore $g = P(g)$ and hence $g \in U$. Now consider any $f \in U$ and $g \in E$ such that $|f| \geq |g|$. Firstly we have that $0 \leq P(|f|) \leq |f|$. Now for any $u \in U$ there exists $x \in E$ such that $u = P(x)$. Therefore $P(u) = P(P(x)) = P(x) = u$ as P is idempotent. Since $f, -f \in U$, we have the following:

$$\begin{aligned} f &= P(f) \\ &\leq P(f) + P(|f| - f) \quad \text{since } f \leq |f| \\ &= P(|f|) \quad \text{since } P \text{ is linear.} \end{aligned}$$

We also have that

$$\begin{aligned} -f &= P(-f) \\ &\leq P(-f) + P(|f| + f) \quad \text{since } -f \leq |f| \\ &= P(|f|) \quad \text{since } P \text{ is linear.} \end{aligned}$$

So $|f| = f \vee (-f) \leq P(|f|)$ and hence $|f| = P(|f|) \in U$. It follows that $0 \leq g^+ \leq |f|$ and $0 \leq g^- \leq |f|$ and hence $g^+, g^- \in U$. So $g = g^+ - g^- \in U$ and hence U is an ideal in E. By similar argument it follows that V is also an ideal in E. Let $y \in V$, and consider any $x \in U$. Since U and V are ideals, $0 \leq |x| \wedge |y| \leq |x|$ and $0 \leq |x| \wedge |y| \leq |y|$, it follows that $|x| \wedge |y| \in U \cap V$ and hence $|x| \wedge |y| = 0$. So $y \in U^d$ and hence $V \subset U^d$. $\square$

Note that the above lemma is also the converse of Proposition 1.4.20.

Definition 1.4.23. A Riesz space E has the *projection property* if every band of E is a projection band.

We will utilise Definition 1.4.23 in Chapter 2.

Chapter 2

The Riesz space C(X)

2.1 Topological Preliminaries

In this section we recall a few concepts from topology that will be used in the sequel. These can be found in any standard text on the subject, for instance [10].

The natural setting for studying spaces of continuous functions is the so-called Tychonoff spaces, see the definition below. This is due to the following remarkable fact, see for instance [6, Section 3.9]: For every topological space X there exists a Tychonoff space Y so that $C(X)$ and $C(Y)$ are isomorphic as rings, hence also as vector lattices.

Definition 2.1.1. A topological space X is called a *Hausdorff space* if for every $x, y \in X$, if $x \neq y$ then there exists disjoint open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.1.2. A topological space X is called a *Tychonoff space* if it is Hausdorff and for every closed set C in X and every $x \notin C$ there exists a continuous function $f\colon X \to \mathbb{R}$ so that $f[C] = \{0\}$ and $f(x) = 1$.

An important property of Tychonoff spaces is that they admit a compactification, as described in the next result. See for instance [10, Section 5.3].

Theorem 2.1.3. *Let X be a Tychonoff space. There exists a compact Hausdorff space βX and a continuous function $\beta\colon X \to \beta X$ with the following properties.*

(i) *$\beta[X]$ is a dense subspace of βX and β is a homeomorphism onto its range.*

(ii) *For every compact Hausdorff space K and any continuous map $f\colon X \to K$ there exists a unique continuous map $\hat{f}\colon \beta X \to K$ so that $\hat{f} \circ \beta = f$.*

(iii) *The pair $(\beta, \beta X)$ is unique up to unique homeomorphism; that is, if Y is a compact Hausdorff space and $\gamma\colon X \to Y$ is a continuous function which satisfy (i) and (ii), then there exists a unique homeomorphism $g\colon \beta X \to Y$ so that $g \circ \beta = \gamma$.*

Definition 2.1.4. Let X be a Tychonoff space. The space βX in Theorem 2.1.3 is called the *Stone-Čech compactification* of X.

Note the following:

(i) We usually suppress the map β, and simply think of X as a subspace of βX.

(ii) Some care must be taken with closures: Given a subset of U of X, we may take the closure of U either in X or in βX. To distinguish between the two operations, we write $\overline{U}^X$ for the closure of U in X, and $\overline{U}^{\beta X}$ for the closure of U in βX.

For a Tychonoff space X, denote by $C_b(X)$ the set of all bounded, continuous real-valued functions on X. We equip this space with the standard norm

$$||u||_\infty = \sup\{|u(x)|\colon x \in X\}, u \in C_b(X).$$

The following result is standard in functional analysis, see for instance [4, Example 1.6].

Theorem 2.1.5. *Let X be a Tychonoff space. Then $C_b(X)$ is a Banach space with respect to the norm $\|\cdot\|_\infty$. In particular, if X is compact then $C(X)$ is a Banach space with respect to $\|\cdot\|_\infty$.*

As is shown in Section 2.3, the following topological property is related to Dedekind completeness of $C(X)$.

Definition 2.1.6. Let X be a topological space. We call X *extremally disconnected* if the closure of every open set in X is open.

Example 2.1.7. Clearly, every discrete space is extremally disconnected. An example of a non-discrete extremally disconnected space is βD, with D any infinite discrete space, see [6, 6M, p. 96].

2.2 Relative uniform completeness of C(X)

In this section we prove that for every Tychonoff space X, the vector lattice $C(X)$ is relatively uniformly complete. This is a well known fact, see for instance [8, Theorem 43.1]. The following lemma is used the prove the result.

Lemma 2.2.1. *(Every continuous function is locally bounded.) Let X be a topological space and let $v \in C(X)$. For every $x \in X$ there exists an open set V containing x and a real number $M > 0$ such that $|v(y)| < M$ for all $y \in V$.*

Proof. Consider any $x \in X$. For any real number $\delta > 0$ the interval $(-\delta, \delta)$ is open in $\mathbb{R}$. Therefore $v^{-1}[(-\delta, \delta)]$ is open in X as $v \in C(X)$. Let $V = v^{-1}[(-(|v(x)| + 1), |v(x)| + 1)]$ and choose $M = |v(x)| + 1$. Then V is open in X and $x \in V$ since $-(|v(x)| + 1) < v(x) < |v(x)| + 1$. By definition of V and M, it follows $|v(y)| < M$ for all $y \in V$. $\square$

Theorem 2.2.2. *Let X be a Tychonoff space. Then $C(X)$ is r.u. complete.*

Proof. Let (u_n) be r.u. Cauchy in $C(X)$ with regulator e'. Consider $e = e' \vee \mathbb{1}$. We claim that e is a regulator for (u_n). To show this, fix any $\varepsilon > 0$. There exists $M_\varepsilon \in \mathbb{N}$ so that $|u_n - u_m| \leq \varepsilon e'$ whenever $m, n \geq M_\varepsilon$. Then $|u_n - u_m| \leq \varepsilon e$ whenever $m, n \geq M_\varepsilon$ because $e' \leq e$ and $\varepsilon > 0$. This holds for all $\varepsilon > 0$ so e is a regulator for (u_n) .

Now consider the sequence $(u_n(x))$ for some arbitrary $x \in X$. Fix any $\varepsilon > 0$; note that $\frac{\varepsilon}{e(x)} > 0$. Since e is a regulator for (u_n), there exists M_ε such that

$$|u_n(x) - u_m(x)| \leq \frac{\varepsilon}{e(x)} e(x) = \varepsilon$$

whenever $m, n \geq M_\varepsilon$. So $(u_n(x))$ is Cauchy and is hence convergent in $\mathbb{R}$ as $\mathbb{R}$ is complete.

Define $u \colon X \to \mathbb{R}$ by $u(x) = \lim\limits_{n\to\infty} u_n(x)$, $x \in X$ i.e, u is the pointwise limit of (u_n). Since e is a regulator for (u_n), for every $\varepsilon > 0$ there exists $M_\varepsilon \in \mathbb{N}$ such that $|u_n - u_m| \leq \varepsilon e$ whenever $m, n \in M_\varepsilon$. It therefore follows that

$$|u(x) - u_n(x)| \leq |u(x) - u_m(x)| + |u_m(x) - u_n(x)| \leq |u(x) - u_m(x)| + \varepsilon e(x)$$

for all $x \in X$ and $n, m \geq M_\varepsilon$. Since $\lim_{m\to\infty} |u(x) - u_m(x)| = 0$, it follows that for every $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{N}$ such that

$$|u(x) - u_n(x)| \leq \varepsilon e(x) \tag{2.1}$$

for all $x \in X$ and $n \geq M_\varepsilon$.

We show that $u \in C(X)$. Fix any $x \in X$. Note that since $e \in C(X)$, it follows from Lemma 2.2.1 that there exists an open set $V \subset X$ so that $x \in V$, and a real number $M > 0$ such that $|e(y)| < M$ for all $y \in V$. Let $\varepsilon > 0$. By (2.1) there exists $n_0 \in \mathbb{N}$ such that $|u(z) - u_{n_0}(z)| \leq \frac{\varepsilon}{3M} e(z) \leq \frac{\varepsilon}{3}$ for all $z \in V$. For all $y \in V$,

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_{n_0}(x)| + |u_{n_0}(x) - u_{n_0}(y)| + |u_{n_0}(y) - u(y)| \\ &\leq \frac{2\varepsilon}{3} + |u_{n_0}(x) - u_{n_0}(y)|. \end{aligned}$$

But $u_{n_0} \in C(X)$ so there exists an open set $W \subset X$ containing x such that $|u_{n_0}(x) - u_{n_0}(y)| < \frac{\varepsilon}{3}$ for all $y \in W$. Therefore if $y \in V \cap W$ then $|u(x) - u(y)| < \varepsilon$. Since $V \cap W$ is open and $x \in V \cap W$ it follows that u is continuous at x, hence on X.

So it follows from (2.1) that $|u - u_n| \leq \varepsilon e$ whenever $n \geq M_\varepsilon$, and hence (u_n) converges r.u. to u in $C(X)$.

□

2.3 Dedekind completeness of C(X)

The main results of this section are the following: If a topological space X is extremally disconnected, then $C(X)$ is Dedekind complete, and, conversely, if X is a Tychonoff space so that $C(X)$ is Dedekind complete, then X is extremally disconnected. These results are well-known, see for instance [8, Theorem 43.11]. We follow an approach via semi-continuous functions, which is adapted from [5] where the result is proven for the case of a compact space X.

Notation 2.3.1. For any subset A of a set X, denote by $\mathbb{1}_A\colon X \to \mathbb{R}$ the *characteristic function* of A,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in X \setminus A \end{cases}$$

In cases where no ambiguity can arise, the characteristic function of X, i.e. the constant function with the value 1, is denoted $\mathbb{1}$.

Definition 2.3.2. Let X be a topological space. A function $u\colon X \to \mathbb{R}$ is

(i) *lower semi-continuous* if for every $c \in \mathbb{R}$ the set $u^{-1}[(c, \infty)]$ is open in X.

(ii) *upper semi-continuous* if for every $c \in \mathbb{R}$ the set $u^{-1}[(-\infty, c)]$ is open in X.

Proposition 2.3.3. *Let X be topological space and let $D \subset C(X)$ be non-empty.*

(i) *If $\{u(x)\colon u \in D\}$ is bounded above for every $x \in X$ then the function $v\colon X \to \mathbb{R}$, where $v(x) = \sup\{u(x)\colon u \in D\}$, is lower semi-continuous.*

(ii) *If $\{u(x)\colon u \in D\}$ is bounded below for every $x \in X$ then the function $w\colon X \to \mathbb{R}$, where $w(x) = \inf\{u(x)\colon u \in D\}$, is upper semi-continuous.*

Proof. (i): Assume that $\{u(x)\colon u \in D\}$ is bounded above for every $x \in X$. Fix any $c \in \mathbb{R}$ and consider the set $v^{-1}[(c, \infty)]$. If $x \in v^{-1}[(c, \infty)]$ then $\sup\{u(x)\colon u \in D\} > c$, so there exists $u \in D$ such that $u(x) > c$; that is, $x \in u^{-1}[(c, \infty)]$. Conversely if $u(x) > c$ for some $u \in D$ then $v(x) > c$ (i.e. $x \in v^{-1}[(c, \infty)]$). So $v^{-1}[(c, \infty)] = \cup_{u \in D} u^{-1}[(c, \infty)]$. Since (c, ∞) is open in

$\mathbb{R}$ and $D \subset C(X)$, $u^{-1}[(c,\infty)]$ is open in X for every $u \in D$. It follows that $v^{-1}[(c,\infty)]$ is open in X, being the union of open sets, and hence v is lower semi-continuous.

(ii): Assume that $\{u(x)\colon u \in D\}$ is bounded below for every $x \in X$. Fix any $c \in \mathbb{R}$ and consider the set $w^{-1}[(-\infty,c)]$. If $x \in w^{-1}[(-\infty,c)]$ then $\inf\{u(x)\colon u \in D\} < c$, so there exists $u \in D$ such that $u(x) < c$; that is, $x \in u^{-1}[(-\infty,c)]$. Conversely if $u(x) < c$ for some $u \in D$ then $w(x) < c$ (i.e. $x \in w^{-1}[(-\infty,c)]$). So $w^{-1}[(-\infty,c)] = \cup_{u\in D} u^{-1}[(-\infty,c)]$. Since $(-\infty,c)$ is open in $\mathbb{R}$ and $D \subset C(X)$, $u^{-1}[(-\infty,c)]$ is open in X for every $u \in D$. It follows that $w^{-1}[(-\infty,c)]$ is open in X, being the union of open sets, and hence w is upper semi-continuous. □

Proposition 2.3.4. *Let X be a topological space. A function $u\colon X \to \mathbb{R}$ is continuous if and only if it is both upper semi-continuous and lower semi-continuous.*

Proof. Suppose $u\colon X \to \mathbb{R}$ is continuous. Then for any $c \in \mathbb{R}$, $u^{-1}[(c,\infty)]$ is open in X so u is lower semi-continuous. Also for any $c \in \mathbb{R}$, $u^{-1}[(-\infty,c)]$ is open in X so u is upper semi-continuous. Now suppose $u\colon X \to \mathbb{R}$ is both lower semi-continuous and upper semi-continuous and consider any open interval (a,b). We note that $u^{-1}[(a,\infty)]$ and $u^{-1}[(-\infty,b)]$ are open in X due to u being lower semi-continuous and upper semi-continuous respectively. Since $(a,b) = (a,\infty) \cap (-\infty,b)$ it follows that

$$u^{-1}[(a,b)] = u^{-1}[(a,\infty) \cap (-\infty,b)] = u^{-1}[(a,\infty)] \cap u^{-1}[(-\infty,b)]$$

which is an open set. Hence u is continuous. □

Definition 2.3.5. Let X be a topological space. A function $u\colon X \to \mathbb{R}$ is

(i) *locally bounded from below* if for every $x \in X$ there exists an open set V containing x and a real number m_x so that $m_x \leq u(y)$ for all $y \in V$.

(ii) *locally bounded from above* if for every $x \in X$ there exists an open set V containing x and a real number M_x so that $M_x \geq u(y)$ for all $y \in V$.

Definition 2.3.6. Let X be a topological space. Assume that $u\colon X \to \mathbb{R}$ is locally bounded from below. Define $\underline{u}\colon X \to \mathbb{R}$ as

$$\underline{u}(x) = \sup\{\inf\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}.$$

Definition 2.3.7. Let X be a topological space. Assume that $u\colon X \to \mathbb{R}$ is locally bounded from above. Define $\overline{u}\colon X \to \mathbb{R}$ as

$$\overline{u}(x) = \inf\{\sup\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}.$$

Remark 2.3.8. The functions $\underline{u}$ and $\overline{u}$ introduced in Definitions 2.3.6 and 2.3.7, respectively, are well defined. For a function $u\colon X \to \mathbb{R}$, the condition that u is locally bounded from below, respectively above, ensures that the set $\{\inf\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}$, respectively $\{\sup\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}$, is non-empty. Furthermore, the set $\{\inf\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}$ is bounded above by $u(x)$ while $\{\sup\{u(y)\colon y \in V\}\colon V \text{ open neighbourhood of } x\}$ is bounded below by $u(x)$.

Proposition 2.3.9. *Let X be a topological space and $u\colon X \to \mathbb{R}$. Then the following statements are true.*

(1) If u is locally bounded from below then $\underline{u}$ is lower semi-continuous and $\underline{u} \le u$.

(2) If u is locally bounded from above then $\overline{u}$ is upper semi-continuous and $\overline{u} \ge u$.

Proof. For the proof of (1), let u be locally bounded from below. Fix any $c \in \mathbb{R}$ and suppose $\underline{u}(x) > c$. By definition of the supremum, there exists an open neighbourhood V_x of x such that $\inf\{u(y)\colon y \in V_x\} > c$. For any $y \in V_x$ we have the following,

$$\begin{aligned}\underline{u}(y) &= \sup\{\inf\{u(z)\colon z \in V\}\colon V \ni y \text{ open}\}\\ &\ge \inf\{u(z)\colon z \in V_x\}\\ &> c\end{aligned}$$

and hence $V_x \subset \underline{u}^{-1}[(c,\infty)]$. We have shown that for every $x \in \underline{u}^{-1}[(c,\infty)]$ there exists an open neighbourhood V_x of x such that $V_x \subset \underline{u}^{-1}[(c,\infty)]$. So $\underline{u}^{-1}[(c,\infty)]$ is open. Remark 2.3.8 shows that $\underline{u} \le u$ because for each $x \in X$, $\underline{u}(x)$ is the supremum of $\{\inf\{u(y)\colon y \in V\}\colon\ V \ni x \text{ open}\}$ and $u(x)$ is an upper bound of the same set.

For the proof of (2), let u be locally bounded from above. Fix any $c \in \mathbb{R}$ and suppose $\overline{u}(x) < c$. By definition of the infimum, there exists an open neighbourhood V_x of x such that $\sup\{u(y)\colon y \in V_x\} < c$. For any $y \in V_x$ we have the following,

$$\begin{aligned}\overline{u}(y) &= \inf\{\sup\{u(z)\colon z \in V\}\colon V \ni y \text{ open}\}\\ &\leq \sup\{u(z)\colon z \in V_x\}\\ &< c\end{aligned}$$

and hence $V_x \subset \overline{u}^{-1}[(-\infty, c)]$. We have shown that for every $x \in \overline{u}^{-1}[(-\infty, c)]$ there exists an open neighbourhood V_x of x such that $V_x \subset \overline{u}^{-1}[(-\infty, c)]$. So $\overline{u}^{-1}[(-\infty, c)]$ is open. Remark 2.3.8 shows how $\overline{u} \geq u$ because for each $x \in X$, $\overline{u}(x)$ is the infimum of $\{\sup\{u(y)\colon y \in V\}\colon\ V \ni x \text{ open}\}$ and $u(x)$ is a lower bound of the same set. □

Recall from Definition 2.1.6 that the closure of every non-empty open set in an extremally disconnected topological space is open.

Proposition 2.3.10. *Let X be an extremally disconnected topological space and $u\colon X \to \mathbb{R}$.*

(i) If u is locally bounded from above and lower semi-continuous then $\overline{u}$ is continuous.

(ii) If u is locally bounded from below and upper semi-continuous then $\underline{u}$ is continuous.

Proof. Suppose u both is locally bounded from above and lower semi-continuous. From Proposition 2.3.9 it follows that $\overline{u}$ is upper semi-continuous. We show that $\overline{u}$ is lower semi-continuous. Fix any $c \in \mathbb{R}$ and consider the set $W = \{x \in X\colon \overline{u}(x) > c\}$. For every $\varepsilon > 0$ define $V_\varepsilon = \{x \in X\colon u(x) > c+\varepsilon\}$. Since u is lower semi-continuous, V_ε is open and so is $\overline{V_\varepsilon}$. It follows that $\bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ is open being the union of open sets. We show that $W = \bigcup_{\epsilon>0} \overline{V}_\varepsilon$. Fix any $\varepsilon > 0$ and let $x \in \overline{V_\varepsilon}$. For any open neighbourhood U of x, $U \cap V_\varepsilon \neq \emptyset$, i.e. there exists $y \in U$ such that $u(y) > c+\varepsilon$. By definition of the supremum, $\sup\{u(z)\colon z \in U\} > c + \varepsilon$ and since this holds for all open neighbourhoods U of x, the set $\{\sup\{u(z)\colon z \in U\}\colon U \ni x \text{ open}\}$ is bounded below by $c + \varepsilon$. By definition of the infimum,

$$\overline{u}(x) = \inf\{\sup\{u(z)\colon z \in U\}\colon U \ni x \text{ open}\} \geq c + \varepsilon > c.$$

Therefore $x \in W$ and hence $\overline{V_\varepsilon} \subset W$. Since this holds for all $\varepsilon > 0$, $\bigcup_{\varepsilon>0} \overline{V_\varepsilon} \subset W$.

Let $x \in W$ (i.e. $\overline{u}(x) > c$) and define $\varepsilon_x = \frac{\overline{u}(x)-c}{2}$ so that $\overline{u}(x) > c + \varepsilon_x$. By definition of the infimum, $\sup\{u(y)\colon y \in V\} > c + \varepsilon_x$. So there exists $y_V \in V$ such that $u(y_V) > c + \varepsilon_x$. Therefore, $V \cap V_{\epsilon_x} \neq \emptyset$ for every open neighbourhood V of x. So $x \in \overline{V_{\varepsilon_x}}$ and hence $x \in \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ which means that $W \subset \cup_{\varepsilon>0}\overline{V_\varepsilon}$. So $W = \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ and is therefore open. Since $c \in \mathbb{R}$ is arbitrary, this shows that $\overline{u}$ is lower semi-continuous. By Proposition 2.3.4 $\overline{u}$ is continuous.

Assume that u is locally bounded from below and upper semi-continuous. From Proposition 2.3.9 it follows $\underline{u}$ is lower semi-continuous. We show that $\underline{u}$ is upper semi-continuous. Fix any $c \in \mathbb{R}$ and consider the set $W = \{x \in X\colon \underline{u}(x) < c\}$. For every $\varepsilon > 0$ define $V_\varepsilon = \{x \in X\colon u(x) < c - \varepsilon\}$. Since u is upper semi-continuous, V_ε is open and so is $\overline{V_\varepsilon}$. It follows that $\bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ is open being the union of open sets. We show that $W = \bigcup_{\epsilon>0} \overline{V}_\varepsilon$. Fix any $\varepsilon > 0$ and let $x \in \overline{V_\varepsilon}$. For any open neighbourhood U of x, $U \cap V_\varepsilon \neq \emptyset$, i.e. there exists $y \in U$ such that $u(y) < c - \varepsilon$. By definition of the infimum, $\inf\{u(z)\colon z \in U\} < c - \varepsilon$ and since this holds for all open neighbourhoods U of x, the set $\{\inf\{u(z)\colon z \in U\}\colon U \ni x \text{ open}\}$ is bounded above by $c - \varepsilon$. By definition of the supremum,

$$\underline{u}(x) = \sup\{\inf\{u(z)\colon z \in U\}\colon U \ni x \text{ open}\} \leq c - \varepsilon < c.$$

Therefore $x \in W$ and hence $\overline{V_\varepsilon} \subset W$. Since this holds for all $\varepsilon > 0$, $\cup_{\varepsilon>0}\overline{V_\varepsilon} \subset W$.

Let $x \in W$ (i.e. $\underline{u}(x) < c$) and define $\varepsilon_x = \frac{\overline{u}(x)-c}{2}$ so that $\underline{u}(x) < c - \varepsilon_x$. By definition of the supremum, $\inf\{u(y)\colon y \in V\} < c - \varepsilon_x$. So there exists $y_V \in V$ such that $u(y_V) < c - \varepsilon_x$. Therefore, $V \cap V_{\epsilon_x} \neq \emptyset$ for every open neighbourhood V of x. So $x \in \overline{V_{\varepsilon_x}}$ and hence $x \in \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ which means that $W \subset \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$. So $W = \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$ and is therefore open. Since $c \in \mathbb{R}$ is arbitrary, this shows that $\underline{u}$ is upper semi-continuous. By Proposition 2.3.4 $\underline{u}$ is continuous. □

Proposition 2.3.11. *Let X be a topological space and let $u\colon X \to \mathbb{R}$. The following statements are true:*

(i) If u is upper semi-continuous then u is locally bounded from above and $u = \overline{u}$.

(ii) If u is lower semi-continuous then u is locally bounded from below and $u = \underline{u}$.

(*iii*) *If u is continuous then u is locally bounded from above, locally bounded from below and $u = \overline{u} = \underline{u}$*

Proof. (i): Assume u is upper semi-continuous. Fix any $x \in X$ and choose $V = u^{-1}[(-\infty, u(x)+1)]$. Since u is upper semi-continuous, it follows that V is an open neighbourhood of x in X. Choosing $M_x = u(x) + 1$, we have that $u(y) \leq M_x$ for all $y \in V$ and thus u is locally bounded from above. It follows from Proposition 2.3.9 that $u \leq \overline{u}$. Fix any $\varepsilon > 0$ and note that for any $x \in X$, $V_\varepsilon = u^{-1}[(-\infty, u(x) + \frac{\varepsilon}{2})]$ is an open neighbourhood of x. It follows that

$$\overline{u}(x) \leq \sup\{u(y) \colon y \in V_\varepsilon\} \leq u(x) + \frac{\varepsilon}{2} < u(x) + \varepsilon.$$

So $\overline{u}(x) < u(x) + \varepsilon$ for any $x \in X$ and taking $\varepsilon \to 0$ we have that $\overline{u}(x) \leq u(x)$. Therefore $\overline{u} \leq u$ and hence $\overline{u} = u$.

(ii): Assume u is lower semi-continuous. Fix any $x \in X$ and choose $V = u^{-1}[(u(x)-1, \infty)]$. Since u is lower semi-continuous, it follows that V is an open neighbourhood of x in X. Choosing $m_x = u(x) - 1$, we have that $u(y) \geq m_x$ for all $y \in V$. It follows from Proposition 2.3.9 that $u \geq \underline{u}$. Fix any $\varepsilon > 0$ and note that for any $x \in X$, $V_\varepsilon = u^{-1}[(u(x) - \frac{\varepsilon}{2}, \infty)]$ is an open neighbourhood of x. It follows that

$$\underline{u}(x) \geq \inf\{u(y) \colon y \in V_\varepsilon\} \geq u(x) - \frac{\varepsilon}{2} > u(x) - \varepsilon.$$

So $\underline{u}(x) > u(x) - \varepsilon$ for any $x \in X$ and taking $\varepsilon \to 0$ we have that $\underline{u}(x) \geq u(x)$. Therefore $\underline{u} \leq u'$ and hence $\underline{u} = u$.

(iii): Assume u is continuous. By Proposition 2.3.4 u is both upper semi-continuous and lower semi-continuous, and hence by (i) and (ii), it follows that u is locally bounded above, locally bounded from below, and $u = \overline{u} = \underline{u}$. □

Proposition 2.3.12. *Let X be a topological space and let $u \colon X \to \mathbb{R}$ and $v \colon X \to \mathbb{R}$ such that $u \leq v$. The following statements are true:*

(*i*) *If u is locally bounded from below then v is locally bounded from below. Furthermore, $\underline{u} \leq \underline{v}$.*

(*ii*) *If v is locally bounded from above then u is locally bounded from above. Furthermore, $\overline{u} \leq \overline{v}$.*

Proof. (i): Suppose u is locally bounded from below. Fix any $x \in X$. There exists an open set V containing x and a real number m_x such that $m_x \leq u(y)$ for every $y \in V$. It follows that $m_x \leq u(y) \leq v(y)$ for every $y \in V$, and since this holds for all $x \in X$, v is locally bounded from below. Also for every open neighbourhood V of x, $u(y) \leq v(y)$ for all $y \in V$. So $\inf\{u(y)\colon y \in V\} \leq \inf\{v(y)\colon y \in V\}$ and hence $\underline{u}(x) \leq \underline{v}(x)$ for each $x \in X$ since the previous inequality holds for all open neighbourhoods V of x. Therefore $\underline{u} \leq \underline{v}$.

(ii): Suppose v is locally bounded from above. Fix any $x \in X$. There exists an open set V containing x and a real number M_x such that $M_x \geq v(y)$ for every $y \in V$. It follows that $M_x \geq v(y) \geq u(y)$ for every $y \in V$, and since this holds for all $x \in X$, u is locally bounded from above. Also for every open neighbourhood V of x, $u(y) \leq v(y)$ for all $y \in V$. So $\sup\{u(y)\colon y \in V\} \leq \sup\{v(y)\colon y \in V\}$ and hence $\overline{u}(x) \leq \overline{v}(x)$ for each $x \in X$ since the previous inequality holds for all open neighbourhoods V of x. Therefore $\overline{u} \leq \overline{v}$.

□

We now have enough tools to show for certain topological spaces X that $C(X)$ is Dedekind complete if X is extremally disconnected.

Theorem 2.3.13. *Let X be a Tychonoff space. If $C(X)$ is Dedekind complete then X is extremally disconnected.*

Proof. Assume $C(X)$ is Dedekind complete. Let U be a non-empty open subset in X. We claim that for every $x \in U$ there exists a function $u_x \in C(X)$ so that $u_x \in [0, \mathbb{1}]$ ($\mathbb{1} \in C(X)^+$), $u_x(x) = 1$, and $u_x(y) = 0$ for all $y \in X \setminus U$. Note that $X \setminus U$ is closed in X. Since X is a Tychonoff space it follows that for every $x \in U$ there exists $u_x \in C(X)$ so that $0 \leq u_x \leq \mathbb{1}$, $u_x(x) = 1$ and $u_x(y) = 0$ for all $y \in X \setminus U$. Define $D = \{u \in [0, \mathbb{1}];$ for all $x \in X \setminus U,\ u(x) = 0\}$. For any given $x \in U$, the function u_x belongs to D, so D is non-empty. Since $u \leq \mathbb{1}$ for any $u \in D$ and $\mathbb{1} \in C(X)$, it follows that D has an upper bound in $C(X)$. It follows that $v = \sup D$ exists in $C(X)$ because $C(X)$ is Dedekind complete.

Consider the closure $\overline{U}$ of U. We show that $v(x) \geq 1$ for all $x \in \overline{U}$. For any $x \in U$, there exists $u_x \in D$ such that $u_x(x) = 1$. Since $v = \sup D$, $u \leq v$ and hence $1 = u_x(x) \leq v(x)$. Now since $v \in C(X)$ it follows that $v(x) \geq 1$ for all $x \in U$. Indeed, because v is continuous, $v^{-1}[[1, \infty)]$ is closed in X. Since $U \subseteq v^{-1}[[1, \infty)]$ it follows that $\overline{U} \subseteq v^{-1}[[1, \infty)]$ so that $v(x) \geq 1$ for all $x \in \overline{U}$.

Consider any $x \in X \setminus \overline{U}$. Then $v(x) \geq 0$ because $0 \leq u \leq v$ for all $u \in D$. X is a Tychonoff space so there exists $w \in C(X)$ so that $0 \leq w$, $w(x) = 0$ and $w(y) = 1$ for all $y \in \overline{U}$. For all $y \in U$, $w(y) = 1$, and for all $y \in X \setminus U$ we have $w(y) \geq 0$. Therefore w is an upper bound for D. Since $v = \sup D$, $v \leq w$ and hence $v(x) \leq w(x) = 0$. So it follows that $v(x) = 0$ for all $x \in X \setminus \overline{U}$.
The interval $(\frac{1}{2}, \infty)$ is open in $\mathbb{R}$ and $\overline{U} = v^{-1}[(\frac{1}{2}, \infty)]$ so $\overline{U}$ is an open set in X. Since this holds for all non-empty open sets $U \subset X$ we conclude that X is extremally disconnected. $\square$

The converse of Theorem 2.3.13 is also true, where X need not be Tychonoff.

Theorem 2.3.14. *Let X be an extremally disconnected topological space. Then $C(X)$ is Dedekind complete.*

Proof. Suppose $D \subset C(X)$ is non-empty and is bounded above in $C(X)$. Let u' be an upper bound for D in $C(X)$. Define $v \colon X \to \mathbb{R}$ as $v(x) = \sup\{u(x) \colon u \in D\}$, $x \in X$. We note that v is well-defined as for each $x \in X$, $\{u(x) \colon u \in D\}$ is non-empty and is bounded above in $\mathbb{R}$ by $u'(x)$. For each $x \in X$, $v(x) \leq u'(x)$ and hence $v \leq u'$. By Proposition 2.3.11, u' is locally bounded above so by Proposition 2.3.12, v is locally bounded above and hence $\overline{v}$ is defined. Since D is bounded above in $C(X)$, it follows that $\{u(x) \colon u \in D\}$ is bounded above in $\mathbb{R}$ for every $x \in X$. By Proposition 2.3.3, v is lower semi-continuous and thus by Proposition 2.3.10, $\overline{v}$ is continuous. It follows from Proposition 2.3.9 that $\overline{v} \geq v$, so $\overline{v} \geq u$ for all $u \in D$ and hence $\overline{v}$ is an upper bound of D. For any upper bound u' of D in $C(X)$, we know from Proposition 2.3.11 that u' is locally bounded from above and $\overline{u'} = u'$. Also $v \leq u'$ so by Proposition 2.3.12, $\overline{v} \leq \overline{u'} = u'$. Thus we can conclude that $\overline{v}$ is the supremum of D in $C(X)$. $\square$

2.4 Characterising bands and projection bands

In this section we study ideals and bands in the Riesz space $C(X)$. We first give a description of the norm closed ideals in $C(X)$ for X a compact Hausdorff space. We then proceed to characterize bands and projection bands in $C(X)$, also in the case of a general Tychonoff space. These results are well known in the compact case, see for instance [9, p. 57 - 58], and were recently generalized to Tychonoff spaces, see [7].

Notation 2.4.1. Given a topological space X and $f \in C(X)$ and $F \subset X$, we write $f|_F \equiv 0$ whenever $f(x) = 0$ for all $x \in F$.

Notation 2.4.2. Given a topological space X and $F \subset X$, we set

$$J_F(X) = \{f \in C(X) \colon f|_F \equiv 0\}.$$

Lemma 2.4.3. *Let X be a topological space and let F be a subset of X. Then $J = J_F(X)$ is an order ideal.*

Proof. Suppose that $J = J_F(X)$. For any $f, g \in J$ and $\alpha \in \mathbb{R}$, $(f+g)|_F \equiv 0$ and $(\alpha f) \equiv 0$. Therefore $f + g, \alpha f \in J$ and hence J is a vector subspace of $C(X)$. In order to show that J is an order ideal, it remains to show that J is solid. So let $|f| \leq |g|$ where $f \in C(X)$ and $g \in J$. We have that $g|_F \equiv 0$ so for any $x \in F$ we have that $|f(x)| \leq |g(x)| = 0$, so $f(x) = 0$ for all $x \in F$ as $|f(x)| \geq 0$, by definition. Therefore $f|_F \equiv 0$, i.e. $f \in J$, and hence J is solid. $\square$

Theorem 2.4.4. *If X is a compact Hausdorff space and J a subspace of $C(X)$, then J is a closed order ideal if and only if $J = J_F(X)$ for some closed set F of X.*

Proof. Suppose that $J = J_F(X)$ for some closed set F of X. It follows by Lemma 2.4.3 that J is an order ideal. Since X is a compact Hausdorff space we have that $C(X)$ is a Banach space. So consider any sequence (f_n) in J such that (f_n) converges to some $f \in E$ in norm. Then f is the pointwise limit of (f_n) and hence for any $\varepsilon > 0$ and any $x \in F$, there exists $N \in \mathbb{N}$ such that $|f(x)| = |f(x) - f_N(x)| < \varepsilon$ since $f_N|_F \equiv 0$. So $f(x) = 0$ for any $x \in F$ and hence $f \in J$ implying that J is closed.

Now suppose that J is a closed order ideal and consider the set

$$F \coloneqq \{t \in X \colon f(t) = 0 \text{ for all } f \in J\}.$$

We know $J_F(X)$ is an order ideal from Lemma 2.4.3. Now for any $f \in J$, we have by definition of F that $f|_F \equiv 0$. Thus $f \in J_F(X)$ and hence $J \subset J_F(X)$. Also note that if $t \in F$ then $t \in f^{-1}[\{0\}]$ for all $f \in J$. Conversely, if $t \in f^{-1}[\{0\}]$ for all $f \in J$ then $f(t) = 0$ for all $f \in J$ so $t \in F$. The singleton set $\{0\}$ is closed in $\mathbb{R}$ and hence $f^{-1}[\{0\}]$ is closed in X for any $f \in J$ (since $J \subset C(X)$). So $F = \cap_{f\in J} f^{-1}[\{0\}]$ is closed in X being the intersection of closed sets.

Let $0 \neq f \in J_F(X)^+$, let $\varepsilon > 0$ and let $A := \{t \in X\colon f(t) \geq \varepsilon\}$. Since f is continuous, $A = f^{-1}[[\varepsilon, \infty)]$ is closed in X as $[\varepsilon, \infty)$ is closed in $\mathbb{R}$. Also since X is a compact Hausdorff space we have that A is also compact in X. If $t \in A$ then $f(t) \geq \varepsilon > 0$ so $A \cap F = \varnothing$ as $f|_F \equiv 0$. Thus for every $s \in A$ there exists $g_s \in J$ with $g_s(s) > 0$. Define $h_s = g_s^+$ so that $h_s \geq 0$. For any given $s \in A$, the set $\{t \in X\colon h_s(t) > 0\} = h_s^{-1}[(0, \infty)]$ is open in X as $(0, \infty)$ is open in $\mathbb{R}$ and $h_s \in C(X)$. The collection $\{t \in X\colon h_s(t) > 0\}_{s\in A}$ is an open cover for A as $s \in \{t \in X\colon h_s(t) > 0\}$ for every $s \in A$. Since A is compact there exists $t_1, t_2, ..., t_r \in A$ such that $A \subset \bigcup_{i=1}^{r} \{t \in X\colon h_{t_i}(t) > 0\}_{s\in A}$.

Define $g := \vee_{i=1}^{r} h_{t_i}$. Note that $h_{t_i} \in J$ for each $i = 1, 2, ..., r$ and hence for each $i = 1, 2, ..., r$, $h_{t_i}(x) = 0$ for all $x \in F$. So $g(x) = 0$ for all $x \in F$ and it follows that $g \in J$. Now let $\delta > 0$ be such that $g(t) \geq \delta$ for all $t \in A$ and define $h = f \wedge (||f||\delta^{-1}g)$. Note $f \in J_F(X)$ while $||f||\delta^{-1}g \in J$ because J is a subspace of $C(X)$. So since $h \leq f$, $h \leq ||f||\delta^{-1}g$, $0 \leq f \wedge ||f||\delta^{-1}g$ while J and $J_F(X)$ are both order ideals, $h \in J \cap J_F(X)$. We also have that $0 \leq h \leq f$ while for every $t \in A$, $||f||\delta^{-1}g(t) \geq ||f|| \geq f(t)$. So we must have that $h(t) = f(t)$ for $t \in A$. For any $t \in X \setminus A$ we have that $0 \leq h(t) \leq f(t) < \varepsilon$, while for any $t \in A$ we have that $h(t) - f(t) = 0$. So $||h - f|| \leq \varepsilon$.

It follows that $f \in J^+$ and hence $J_F(X)^+ \subset J^+$. Now for any $f \in J_F(X)$ we have that $f^+, f^- \in J_F(X)^+ \subset J^+$. By Theorem 1.1.10 (i), $f = f^+ - f^-$ and hence $f \in J$, and it follows that $J_F(X) \subset J$. Therefore $J = J_F(X)$. $\square$

From this result, we can explore properties of such closed sets in X when characterising special closed ideals such as bands in $C(X)$.

Lemma 2.4.5. *Let X be a compact Hausdorff space, and let B be a closed ideal in $C(X)$. Let F be the closed set such that $J_F(X) = B$ by Theorem 2.4.4. Then $B^d = J_{\overline{X\setminus F}}(X)$.*

Proof. Let $f \in J_{\overline{X\setminus F}}(X)$ and consider any $g \in B$. Note that for any $x \in F$,

$$|f(x)| \wedge |g(x)| = 0 \wedge |g(x)| = 0.$$

We also have that for any $x \in X \setminus F$, $x \in \overline{X \setminus F}$ and hence

$$|f(x)| \wedge |g(x)| = |f(x)| \wedge 0 = 0.$$

Therefore $|f| \wedge |g| = 0$ and so it follows that $f \in B^d$. Hence $J_{\overline{X \setminus F}}(X) \subset B^d$. Let $f \in B^d$ and consider any $t \notin F$. The sets $\{t\}$ and F and disjoint closed sets in X. So by Urysohn's Lemma there exists $g_t \in C(X)$ such that $g_t(t) = 1$ and $g_t(s) = 0$ for all $s \in F$. We have that $g_t|_F \equiv 0$ and hence $g_t \perp f$ since $g_t \in J_F(X)$. For each $t \in X \setminus F$, $f(t) = 0$ because $|g_t| \wedge |f| = 0$ holds. Indeed, because f is continuous, $f^{-1}[\{0\}]$ is closed in X. Since $X \setminus F \subset f^{-1}[\{0\}]$ it follows that $\overline{X \setminus F} \subset f^{-1}[\{0\}]$ and therefore $f|_{\overline{X \setminus F}} \equiv 0$, i.e. $f \in J_{\overline{X \setminus F}}(X)$. So $B^d \subset J_{\overline{X \setminus F}}(X)$ and therefore $B^d = J_{\overline{X \setminus F}}(X)$. □

Lemma 2.4.6. *Let X be a compact Hausdorff space, let (f_n) be a sequence in $C(X)$ and let $f \in C(X)$. If $f_n \to f$ in norm, then $f_n \xrightarrow{o} f$.*

Proof. It can be shown that $f_n \xrightarrow{o} f$ treating (f_n) as a net. To see this fact, note that since $f_n \to f$ for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $||f_n - f|| \leq \varepsilon$ whenever $n \geq N_0$. This implies that for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $|f_n - f| \leq ||f_n - f||\mathbb{1} \leq \varepsilon\mathbb{1}$ whenever $n \geq N_0$. The sequence of continuous functions (q_m) where $q_m = \frac{1}{m}\mathbb{1}$ for $m \in \mathbb{N}$ serves as a net that satisfies $q_m \downarrow 0$. Therefore for every $m \in \mathbb{N}$, since $\frac{1}{m} > 0$ for each $m \in \mathbb{N}$, there exists $N_0 \in \mathbb{N}$ for which $|f_n - f| \leq q_m$ holds for every $n \geq N_0$, i.e. $f_n \xrightarrow{o} f$. □

Lemma 2.4.7. *Let X be a compact Hausdorff space and let B be a band in $C(X)$. Then B is closed.*

Proof. Assuming B is a band, $B = B(B)$. Let (f_n) be a sequence in B such that $f_n \to f$ in norm with $f \in C(X)$. It follows from Lemma 2.4.6 that $f_n \xrightarrow{o} f$. By Proposition 1.4.10, $f \in B$ and hence B is closed. □

Theorem 2.4.8. *Let X be a compact Hausdorff space, and let B be a closed ideal in $C(X)$. Let F be the closed set such that $J_F(X) = B$ by Theorem 2.4.4. Then B is a band if and only if F is the closure of an open set.*

Proof. Assume B is a band. By Proposition 1.4.14, $(B^d)^d = B(B) = B$. Let $U = X \setminus \overline{X \setminus F}$ and note that U is open. By Lemma 2.4.5, $B^d = J_{\overline{X \setminus F}}(X)$. Since B^d is a band by Proposition 1.4.7, B^d is a closed ideal by Lemma 2.4.7. So by Lemma 2.4.5 again we get that $(B^d)^d = J_{\overline{X \setminus \overline{X \setminus F}}}(X) = J_{\overline{U}}(X)$.

Therefore $J_F(X) = J_{\overline{U}}(X)$ and thus it follows that $\overline{U} = F$. To see this, suppose $\overline{U} \setminus F \neq \varnothing$ and let $x \in \overline{U} \setminus F$. By Urysohn's Lemma there exists $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$. It follows that $f \in J_F(X)$ but $f \notin J_{\overline{U}}(X)$. A similar proof can be done to show that $F \setminus \overline{U} = \varnothing$.

Assume there exists some open set U such that $\overline{U} = F$. Since $X \setminus U$ is a closed set, consider $J = J_{X\setminus U}(X)$. Since J is a closed order ideal as a result of Lemma 2.4.3 and Theorem 2.4.4, it follows by Lemma 2.4.5,

$$J^d = J_{\overline{X\setminus(X\setminus U)}}(X) = J_{\overline{U}}(X) = J_F(X) = B,$$

and hence B is a band by Proposition 1.4.7. $\square$

Theorem 2.4.9. *Let X be a compact Hausdorff space, and let B be a closed ideal in $C(X)$. Let F be the closed set such that $J_F(X) = B$ by Theorem 2.4.4. Then B is a projection band if and only if $B = J_F(X)$ for some clopen set $F \subset X$.*

Proof. Assume F is clopen and let $f \in C(X)$. Consider the function $\mathbb{1}_{X\setminus F} \cdot f$, and let U be an open set in $\mathbb{R}$. Note that $X \setminus F$ is also clopen. If $0 \notin U$ then $(\mathbb{1}_{X\setminus F} \cdot f)^{-1}[U] = f^{-1}[U]$ which is open in X as $f \in C(X)$. If $0 \in U$ then $(\mathbb{1}_{X\setminus F} \cdot f)^{-1}[U] = f^{-1}[U] \cup (X \setminus F)$ and is open being a union of open sets. Therefore $(\mathbb{1}_{X\setminus F} \cdot f) \in C(X)$. Furthermore, $(\mathbb{1}_{X\setminus F} \cdot f)|_F \equiv 0$ so $\mathbb{1}_{X\setminus F} \cdot f \in B$.

Define $P\colon C(X) \to B$ by $P(f) = (\mathbb{1}_{X\setminus F} \cdot f)$. Take any $f, g \in C(X)$ and $\alpha, \beta \in \mathbb{R}$. We have the following:

$$\begin{aligned}
P(\alpha f + \beta g) &= (\mathbb{1}_{X\setminus F} \cdot (\alpha f + \beta g)) \\
&= (\mathbb{1}_{X\setminus F} \cdot (\alpha f)) + (\mathbb{1}_{X\setminus F} \cdot (\beta g)) \\
&= \alpha(\mathbb{1}_{X\setminus F} \cdot f) + \beta(\mathbb{1}_{X\setminus F} \cdot g) \\
&= \alpha P(f) + \beta P(g).
\end{aligned}$$

So P is a linear map. We also have that for each $f \in C(X)$,

$$\begin{aligned}
P(P(f)) &= (\mathbb{1}_{X\setminus F} \cdot (\mathbb{1}_{X\setminus F} \cdot f)) \\
&= (\mathbb{1}_{X\setminus F} \cdot \mathbb{1}_{X\setminus F}) \cdot f \\
&= (\mathbb{1}_{X\setminus F} \cdot f) \\
&= P(f).
\end{aligned}$$

So P is idempotent. Take $f \in C(X)^+$. Since $0 \leq (\mathbb{1}_{X \setminus F} \cdot f) \leq f$, we have that $0 \leq P(f) \leq f$. By Lemma 1.4.22, B is a projection band with P being a band projection onto B.

Suppose conversely that B be a projection band. Let P be a band projection onto B as a result of Proposition 1.4.20. Consider $P \circ \mathbb{1}$ and note that $P \circ \mathbb{1}(x) = 0$ for any $x \in F$ since $B = J_F(X)$. By Lemma 1.4.22, $I - P$ is a band projection on B^d. Therefore $(I-P) \circ \mathbb{1} = \mathbb{1} - P \circ \mathbb{1}$ and hence for any $x \in X \setminus F$, $((I-P)\circ \mathbb{1})(x) = \mathbb{1}(x) - (P \circ \mathbb{1})(x) = 0$ since $B^d = J_{\overline{X \setminus F}}(X)$ by Lemma 2.4.5. It follows that $(P \circ \mathbb{1})(x) = 1$ and hence $F = P^{-1}[\{1\}] = P^{-1}[(-\frac{1}{2}, \frac{1}{2})]$ is clopen. $\square$

Theorem 2.4.10. *Let X be a compact Hausdorff space. For a closed vector subspace B of $C(X)$, B is a band if and only if $B = J_F(X)$ for some $F \subset X$ which is the closure of some open set.*

Proof. Assume B is a band. Then B is an ideal and hence by Theorem 2.4.4, $B = J_F(X)$ for some closed set F. By Theorem 2.4.8 $F = \overline{U}$ for some open set U.

Assume $B = J_F(X)$ for some $F \subset X$ where $F = \overline{U}$ for some open set U. F is closed in X so by Theorem 2.4.4 B is an ideal in $C(X)$. Therefore by Theorem 2.4.8, B is a band in $C(X)$. $\square$

Theorem 2.4.11. *Let X be a compact Hausdorff space. For a closed vector subspace B of $C(X)$, we have that B is a projection band if and only if $B = J_F(X)$ for some $F \subset X$ which is clopen.*

Proof. Assume B is a projection band. Then B is an ideal and hence by Theorem 2.4.4, $B = J_F(X)$ for some closed set F. By Theorem 2.4.9, F is clopen in X.

Assume $B = J_F(X)$ for some clopen set F. By Theorem 2.4.4, B is an ideal in $C(X)$. Therefore, by Theorem 2.4.9, B is a projection band. $\square$

Note that Theorem 2.4.10 and Theorem 2.4.11 are extensions to Theorem 2.4.8 and Theorem 2.4.9 respectively, where B is a closed vector space rather than a closed order ideal.

The above results have been proven when X is a compact Hausdorff space. However it can also be shown that they also apply when X is a Tychonoff space.

Proposition 2.4.12. *Let X be a Tychonoff space. Then $C_b(X)$ is an order dense ideal in $C(X)$.*

Proof. Under the lattice operations inherited in $C(X)$, for any $f, g \in C_b(X)$ and $a, b \in \mathbb{R}$ we have that $af + bg, f \vee g, f \wedge g \in C_b(X)$ so $C_b(X)$ is a Riesz subspace of $C(X)$. Consider any $f \in C(X)$, and $g \in C_b(X)$ such that $|f| \leq |g|$. Since $g \in C_b(X)$, there exists $M > 0$ such that $||g||_\infty \leq M$. But $\sup\{|f(x)|\colon x \in X\} \leq ||g||_\infty \leq M$ and hence $f \in C_b(X)$ and $C_b(X)$ is an ideal. Take any $0 \neq f \in C(X)^+$ and choose $g = f \wedge \mathbb{1}$. Since $0 \neq g \in C_b(X)$ ($\sup\{|g(x)|\colon x \in X\} \leq 1$) and $g \leq f$, $C_b(X)$ is an order dense ideal in $C(X)$. $\square$

Proposition 2.4.13. *Let X be a Tychonoff space. The map $T_\beta\colon C(\beta X) \ni f \mapsto f \circ \beta \in C_b(X)$ is an isometric lattice isomorphism onto $C_b(X)$.*

Proof. Firstly let $f, g \in C(\beta X)$, $a, b \in \mathbb{R}$. We have the following:

$$\begin{aligned} T_\beta(af + bg) &= (af + bg) \circ \beta \\ &= (af) \circ \beta + (bg) \circ \beta \\ &= a(f \circ \beta) + b(g \circ \beta) \\ &= aT_\beta(f) + bT_\beta(g). \end{aligned}$$

So T_β is a linear map. We also can show that T_β preserves lattice structure.

$$T_\beta(f \vee g) = (f \vee g) \circ \beta = (f \circ \beta) \vee (g \circ \beta) = T_\beta(f) \vee T_\beta(g),$$

$$T_\beta(f \wedge g) = (f \wedge g) \circ \beta = (f \circ \beta) \wedge (g \circ \beta) = T_\beta(f) \wedge T_\beta(g).$$

To show T_β is an isomorphism, consider any $f, g \in C(\beta X)$ such that $f \neq g$ and note that $(f-g)^{-1}[\mathbb{R} \setminus \{0\}]$ is a non-empty open set in βX. By Theorem 2.1.3 (i), $(f-g)^{-1}[\mathbb{R} \setminus \{0\}] \cap \beta[X] \neq \varnothing$ so there exists $x \in X$ such that $f(\beta(x)) \neq g(\beta(x))$. Therefore $T_\beta(f) \neq T_\beta(g)$ and thus T_β is injective. For any $h \in C_b(X)$, $\overline{h[X]}$ is a closed, bounded subset of $\mathbb{R}$ and is hence compact in $\mathbb{R}$. We can thus define $h'\colon X \to H$, $h'(x) = h(x)$ where $H = \overline{h[X]}$ is a compact Hausdorff space due to being a subset of $\mathbb{R}$. By Theorem 2.1.3 (ii), there exists a unique continuous map $\hat{h'}\colon \beta X \to H$ such that $\hat{h'} \circ \beta = h'$. Choosing $\hat{h} \in C(\beta X)$ such that $\hat{h}(x) = \hat{h}'(x)$, it follows that $T_\beta(\hat{h}) = h$ which shows T_β is surjective and is thus an isomorphism between $C(\beta X)$ and $C_b(X)$. Finally to prove the map is isometric, for any $f \in C(\beta X)$:

$$\begin{aligned} ||T_\beta(f)||_\infty &= \sup\{|f(\beta(x))|\colon x \in X\} \\ &= \sup\{|f(y)|\colon y \in \beta[X]\} \\ &= \sup(|f|[\beta[X]]) \\ &= \sup(\overline{|f|[\beta[X]]}) \quad \text{since } |f|[\beta[X]] \subset \mathbb{R}. \end{aligned}$$

We also have that $||f||_\infty = \sup\{|f(x)| : x \in \beta X\} = \sup(|f|[\beta X])$ and $|f|[\beta[X]] \subset |f|[\beta X] \subset \overline{|f|[\beta[X]]}$ as $\beta X = \overline{\beta[X]}^{\beta X}$ and $f \in C(\beta X)$. Therefore

$$\sup(|f|[\beta[X]]) \leq \sup(|f|[\beta X]) \leq \sup(\overline{|f|[\beta[X]]}).$$

It follows that $||T_\beta(f)||_\infty = ||f||_\infty$ and hence T_β is an isometric lattice isomorphism. $\square$

Proposition 2.4.14. *Let X be a Tychonoff space, and let B be a band in $C(X)$. Let $B_b = B \cap C_b(X)$. The following statements are true.*

(i) B_b is a band in $C_b(X)$.

(ii) If B is a projection band in $C(X)$ then B_b is a projection band in $C_b(X)$.

Proof. Let $f \in C_b(X)$, $g \in B_b$ such that $|f| \leq |g|$. Since $f \in C(X)$ and $g \in B$, by virtue of B being a band in $C(X)$, $f \in B$ and hence $f \in B_b$ which shows that B_b is an ideal in $C_b(X)$. Let (f_α) be a net in B_b and let $f \in C_b(X)$ such that $f_\alpha \xrightarrow{o} f$ in $C_b(X)$. It follows that (f_α) is a net in B, $f \in C(X)$ and $f_\alpha \xrightarrow{o} f$ in $C(X)$ by Lemma 1.4.16. So $f \in B \cap C_b(X) = B_b$ by virtue of B being a band in $C(X)$ hence B_b is a band in $C_b(X)$.

Suppose B is a projection band in $C(X)$. By Proposition 1.4.20 there exists a band projection $P : C(X) \to B$ on B. Let P_b be the restriction of P onto $C_b(X)$ and note that P_b is a linear idempotent map such that $0 \leq P_b(f) \leq f$ for any $f \in C_b(X)^+$. Note that $P_b(f) \in C_b(X)^+$ for all $f \in C_b(X)^+$ since $\sup\{P_b f[X]\} \leq ||f||_\infty$. For any $f \in B_b$, we have that $f \in B$ so there exists $u \in C(X)$ such that $f = P(u)$. Since $f \in C_b(X)$ it follows that

$$f = P(u) = P(P(u)) = P(f) = P_b(f) \in P_b(C_b(X)).$$

So $B_b \subset P_b(C_b(X))$. Also for any $f \in C_b(X)$, we have that $P_b(f) = P_b(f^+) - P_b(f^-) \in C_b(X)$ and $P_b(f) = P(f) \in B$. So $P_b(C_b(X)) \subset B_b$ and hence $B_b = P_b(C_b(X))$. Therefore B_b is a projection band by Lemma 1.4.22. $\square$

Corollary 2.4.15. *Let X be a Tychonoff space, and let B be a band in $C(X)$. Let $B_b = B \cap C_b(X)$. The following statements are true.*

(i) $T_\beta^{-1}[B_b]$ is a band in $C(\beta X)$.

(ii) If B is a projection band in $C(X)$ then $T_\beta^{-1}[B_b]$ is a projection band in $C(\beta X)$.

Proof. Let $f \in C(\beta X)$, $g \in T_\beta^{-1}[B_b]$ such that $|f| \leq |g|$. Since T_β is an lattice isomorphism by Proposition 2.4.13 it follows that $T_\beta(f) \in C_b(X)$, $T_\beta(g) \in B_b$ and $|T_\beta(f)| \leq |T_\beta(g)|$. By Proposition 2.4.14 B_b is a band in $C_b(X)$ and hence $T_\beta(f) \in B_b$. So $f \in T_\beta^{-1}[B_b]$ and therefore $T_\beta^{-1}[B_b]$ is an ideal in $C(\beta X)$. Let (f_α) be a net in $T_\beta^{-1}[B_b]$ such that $f_\alpha \xrightarrow{o} f$ for some $f \in C(\beta X)$. There exists a net (q_γ) satisfying $q_\gamma \downarrow 0$ such that for any γ there exists α_0 such that $|f_\alpha - f| \leq q_\gamma$ holds for every $\alpha \geq \alpha_0$. It follows that $(T_\beta(f_\alpha))$ is a net in B_b and $T_\beta(f) \in C_b(X)$. Since T_β is an isometric lattice isomorphism by Proposition 2.4.13, it follows that the net $(T_\beta(q_\gamma))$ satisfies $(T_\beta(q_\gamma)) \downarrow 0$ such that for any γ there exists α_0 such that $|T_\beta(f_\alpha) - T_\beta(f)| \leq T_\beta(q_\gamma)$ for any $\alpha \leq \alpha_0$. So $T_\beta(f_\alpha) \xrightarrow{o} T_\beta(f)$ and thus by Proposition 2.4.14, $T_\beta(f) \in B_b$ since B_b is a band. So $f \in T_\beta^{-1}[B_b]$ and thus $T_\beta^{-1}[B_b]$ is a band in $C(\beta X)$.

Suppose B is a projection band in $C(X)$, then B_b is a projection band in $C_b(X)$ by Proposition 2.4.14. Consider any $h \in C(\beta X)$ and note that $T_\beta(h) \in C_b(X)$. For a set $A \subset C_b(X)$, to differentiate from A^d on $C(X)$ we define $A^\perp := \{f \in C_b(X) \colon f \perp g \text{ for all } g \in A\}$. Since B_b is a projection band in $C_b(X)$ it follows from Proposition 2.4.13 that T_β is a lattice isomorphism and hence there exists unique $f \in T_\beta^{-1}[B_b]$ and $g \in T_\beta^{-1}[(B_b)^\perp]$ such that $T_\beta(h) = T_\beta(f) + T_\beta(g)$ and hence $h = f + g$. Since $T_\beta(f) \perp T_\beta(g)$ we have that $f \perp g$ (i.e. $g \in (T_\beta^{-1}[B_b])^d$). So $T_\beta^{-1}[B_b] \oplus (T_\beta^{-1}[B_b])^d = C(\beta X)$ and thus $T_\beta^{-1}[B_b]$ is a projection band in $C(\beta X)$. □

Theorem 2.4.16. *Let X be a Tychonoff space, and F be a closed subset of X which is the closure (in X) of an open subset of X. Then $J_F(X)$ is a band in $C(X)$.*

Proof. This proof is very similar to the case where X is a compact Hausdorff space in Theorem 2.4.8. Set $B = J_F(X)$ and suppose U is an open set in X such that $\overline{U}^X = F$. Note that $X \setminus U$ is closed and consider $J = J_{X \setminus U}(X)$. Taking the proof of Lemma 2.4.5 except that Urysohn's Lemma is replaced with the fact that X is Tychonoff, it follows that

$$J^d = J_{\overline{X \setminus (X \setminus U)}}(X) = J_{\overline{U}}(X) = J_F(X) = B,$$

and hence B is a band by Proposition 1.4.7. □

Theorem 2.4.17. *Let X be a Tychonoff space and F a clopen subset of X. Then $J_F(X)$ is a projection band in $C(X)$.*

Proof. This proof is very similar to the case where X is a compact Hausdorff space in Theorem 2.4.9. Set $B = J_F(X)$ and define $P = \mathbb{1}_{X\setminus F}$. The result follows in the same method as in Theorem 2.4.9 where P is shown to be a band projection onto B. $\square$

Theorem 2.4.18. *Let X be a Tychonoff space and let B be a band in $C(X)$. There exists an open subset U of X so that, if $F = \overline{U}^X$, then $B = J_F(X)$.*

Proof. From Corollary 2.4.15, $T_\beta^{-1}[B_b]$ is a band in $C(\beta X)$. Since βX is a compact Hausdorff space, it follows by Theorem 2.4.8 that there exists an open subset V of βX so that $T_\beta^{-1}[B_b] = J_G(\beta X)$, with $G = \overline{V}^{\beta X}$. Let $U = X \cap V$ and $F = \overline{U}^X$. Note that $X \cap G$ is a closed set in X that contains $X \cap V = U$ since X is a subspace of βX. So $F \subset X \cap G$. Since F is closed in X and X is a subspace of βX, there exists a closed set C in βX such that $F = X \cap C \subset C$. It follows that $U \subset C$ and hence $X \cap \overline{U}^{\beta X} \subset F$. Consider any $x \in G$ and let O be an open neighbourhood of x in βX. We have that $O \cap V$ is a non-empty open set in βX and hence

$$O \cap U = O \cap (X \cap V) = X \cap (O \cap V) \neq \varnothing$$

because $\overline{X}^{\beta X} = \beta X$ as a result of our suppression of β. So $G \subset \overline{U}^{\beta X}$ and hence $X \cap G \subset X \cap \overline{U}^{\beta X} \subset F$ i.e. $F = X \cap G$. We show that $B = J_F(X)$ as $\beta^{-1}(G) = X \cap G$ by the suppression of β. First note that due to Proposition 2.4.13,

$$\begin{aligned} B_b &= T_\beta(T_\beta^{-1}(B_b)) \\ &= T_\beta(J_G(\beta X)) \\ &= \{T_\beta(f)\colon f \in J_G(\beta X)\} \\ &= \{f \circ \beta\colon f \circ \beta|_{\beta^{-1}[G]} \equiv 0\} \\ &= \{g \in C_b(X)\colon g|_F \equiv 0\}. \end{aligned}$$

Consider any $f \in B^+$ and note that $f \wedge \mathbb{1} \in B_b$ since B is a band and $f \wedge \mathbb{1} \in C_b(X)$. It follows that for any $x \in F$, we have that $\sup\{f(x), 1\} = 0$ and hence $f(x) = 0$. So $B^+ \subset J_F(X)^+$ and hence $B \in J_F(X)$ since $f = f^+ - f^- \in J_F(X)$ for any $f \in B$. Let $g \in B^d$ and consider any $t \notin F$. Since X is a Tychonoff space, there exists $f_t \in C(X) \cap [0, \mathbb{1}]$ such that $f_t(t) = 1$ and $f_t(s) = 0$ for all $s \in F$. So $f_t|_F \equiv 0$ and $f_t \in C_b(X)$ and therefore $f_t \in B_b \subset B$. It follows that $f_t \wedge g$ and hence $g(t) = 0$ for any $t \notin F$. Now for any $f \in J_F(X)$, we have that $|f| \wedge |g| = 0$ and this holds for all $g \in B^d$. So $f \in (B^d)^d = B(B) = B$ by Proposition 1.4.14 and thus $J_F(X) \subset B$ and we are done. $\square$

Theorem 2.4.19. *Let X be a Tychonoff space and let B be a projection band in $C(X)$. There exists an clopen subset F of X so that $B = J_F(X)$.*

Proof. From Corollary 2.4.15, $T_\beta^{-1}[B_b]$ is a band in $C(\beta X)$. Since βX is a compact Hausdorff space, it follows by Theorem 2.4.9 that there exists an clopen subset G in βX so that $T_\beta^{-1}[B_b] = J_G(\beta X)$. Let $F = X \cap G$ and note that since X is a subspace of βX, F must be clopen in X. Therefore $B = J_F(X)$ following the exact same method used in Theorem 2.4.18. $\square$

Theorem 2.4.20. *Let X be a Tychonoff space. Then $C(X)$ has the projection property if and only if X is extremally disconnected.*

Proof. Let B be a band of $C(X)$ and assume that X is extremally disconnected. By Theorem 2.4.18 there exists an open set U in X so that $B = J_{\overline{U}^X}(X)$. But $\overline{U}^X$ is clopen as X is extremally disconnected. Hence B is a projection band by Theorem 2.4.17. Now assume that $C(X)$ has the projection property and let U be an open subset of X. By Theorem 2.4.16, we have that $J_{\overline{U}^X}(X)$ is a band of $C(X)$ and hence a projection band. Therefore $J_{\overline{U}^X}(X) = J_F(X)$ for some clopen set F by Theorem 2.4.19. It thus follows that $\overline{U}^X = F$ and hence $\overline{U}^X$ is open. To see this, suppose $\overline{U}^X \setminus F \neq \varnothing$ and let $x \in \overline{U}^X \setminus F$. Since X is a Tychonoff space, there exists $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$. It follows that $f \in J_F(X)$ but $f \notin J_{\overline{U}^X}(X)$, which contradicts that $J_{\overline{U}^X}(X) = J_F(X)$. So $\overline{U}^X \setminus F = \varnothing$ and by similar argument, we have that $F \setminus \overline{U}^X = \varnothing$. $\square$

Corollary 2.4.21. *Let X be a Tychonoff space. The following statements are equivalent:*

(i) *$C(X)$ is Dedekind complete.*

(ii) *$C(X)$ has the projection property.*

Proof. $(i) \implies (ii)$: Suppose $C(X)$ is Dedekind complete. From Theorem 2.3.13 it follows that X is extremally disconnected and hence $C(X)$ has the projection property by Theorem 2.4.20.

$(ii) \implies (i)$: Suppose $C(X)$ has the projection property. From Theorem 2.4.20 it follows that X is extremally disconnected and hence $C(X)$ is Dedekind complete by Theorem 2.3.14. $\square$

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