Properties and performance of the *c*-chart for attributes data

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Abstract

The effects of parameter estimation are examined for the well-known *c*-chart for attributes data. The exact run length distribution is obtained for Phase II applications, when the true average number of non-conformities, *c*, is unknown, by conditioning on the observed number of non-conformities in a set of reference data (from Phase I). Expressions for various chart performance characteristics, such as the average run length (ARL), the standard deviation of the run length (SDRL) and the median run length (MDRL) are also obtained. Examples show that the actual performance of the chart, both in terms of the false alarm rate (FAR) and the in-control ARL, can be substantially different from what might be expected when *c* is known, in that an exceedingly large number of false alarms are observed, unless the number of inspection units (the size of the reference dataset) used to estimate *c* is very large, much larger than is commonly used or recommended in practice. In addition, the actual FAR and the in-control ARL values can be very different from the nominally expected values such as 0.0027 (or ARL_0=370), particularly when *c* is small, even with large amounts of reference data. A summary and conclusions are offered.

Introduction

In many applications the quality characteristics studied are categorical and the units or items that are inspected are classified simply as 'conforming' (they meet certain specifications) or 'non-conforming' (they do not meet the specifications) with respect to one or more of the specification(s) on some desired characteristics. Such characteristics are often labeled as 'attributes' and the data collected for these attributes are called 'attributes data' (see for example, 5). Statistical process control with attributes data constitutes an important area of research and applications; see 7 for a review.

Two widely used attributes control charts are (1) the *p*-chart (based on the binomial distribution), where one works with the fraction or the proportion of non-conforming or defective items produced or manufactured in a sample of size *n* and (2) the *c*-chart, where one works with the total number of defects or non-conformities in an inspection unit. Montgomery

5 states 'The inspection unit is simply an entity for which it is convenient to keep records. It could be a group of 5 units of product, 10 units of product, and so on.' The *p*-chart has been studied in a recent paper, see 2; the *c*-chart is studied here. It may be noted that control charts have been devised for the total number of non-conformities in an inspection unit as well as for the average number of non-conformities in an inspection unit, the former is considered here. It is assumed that the number of non-conformities in a unit follows a Poisson distribution with an average of *c*, say.

Estimation of process parameters and its effects on a chart's performance are of interest from a practical as well as a theoretical point of view. This has generated a lot of work in the recent control chart literature. Much of this work has focused on variables control charts, *i.e*. control charts for quality characteristics that can be expressed in terms of a numerical measurement on a continuous scale. It is generally known that when (point) estimates are used instead of known parameter values, the operational properties of control charts, for both the in- as well as the out-of-control situations, are substantially affected. For example, the false alarm rate (FAR), which is the probability of a signal being given by the control chart when the process is actually in-control, is often greatly increased, so that the chart gives many more erroneous signals or false alarms, which is clearly undesirable. Moreover, parameter estimation significantly alters the run length distribution of the chart and thus performance characteristics of the chart, such as the average run length (ARL) and the standard deviation of the run length (SDRL) can be highly affected. Several authors have contributed to this area, including Ghosh *et al.* 4, Del Castillo 3, Quesenberry 6 and Chakraborti 1, among others.

This paper focuses on the *c*-chart (for the number of non-conformities in an inspection unit) and examines the effects of parameter estimation. While the *c*-chart is quite easy to apply, in many applications the true average *c* is either unknown or unspecified and thus needs to be estimated from historical or retrospective data (called reference or Phase I data). In order to set the stage, first consider the case where *c* is known.

The *c*-chart for number of non-conformities: standard known

The c-chart with standard known

Suppose that the *number of non-conformities in an inspection unit* in a (production or manufacturing) process follows a Poisson distribution with a true average $c=c_0$, where c_0 is a given value, for example, specified by the management. This situation is referred to as the standard known case (hereafter Case K).

The upper control limit (*UCL*), the lower control limit (*LCL*) and the centerline (*CL*) are

$$\begin{aligned} UCL &= c_0 + 3\sqrt{c_0} \\ CL &= c_0 \qquad\qquad (1) \\ LCL &= c_0 - 3\sqrt{c_0} \end{aligned}$$

and these are typically referred to as 3-sigma control limits. Should these calculations yield a negative value for the *LCL*, the lower control limit is adjusted and set equal to zero, *i.e*. *LCL* = 0.

The chart is typically implemented as follows. Independent inspection units are taken at random at equally spaced time intervals and the number of non-conformities in the ith inspection unit, X_i, is calculated and plotted on the control chart. If one of these points falls on or outside either of the two control limits, a 'signal' or alarm is given and the process is declared out-of-control. A search for assignable causes is typically started next.

Properties of the c-chart with standard known

When a plotted point falls outside the control limits, *i.e*. lies on or above the upper control limit, or lies on or below the lower control limit, the process is declared out-of-control. This is usually represented by a signal and such an event is called a signaling event. The complementary event, that is when the plotted point lies within the control limits, is simply referred to as 'no-signal' or a 'non-signaling event'. Thus, the probability of a 'no-signal' on the c-chart is the probability that a plotting or charting statistic X_i plots between the two control limits, with both endpoints excluded. This probability is a function of the true average number of non-conformities in an inspection unit (c) and the specified or the desired average number of non-conformities in an inspection unit (c_0) and is denoted by $\bar{c}$. Assuming that X_i follows a Poisson distribution with parameter c, one can write

$$\begin{aligned}\beta(c_0, c) &= \Pr\left(LCL < X_i < UCL \mid c\right) \\ &= \Pr(X_i < UCL \mid c) - \Pr(X_i \leq LCL \mid c) \\ &= \sum_{j=0}^{b} \frac{e^{-c}c^j}{j!} - \sum_{j=0}^{a} \frac{e^{-c}c^j}{j!}\end{aligned} \qquad (2)$$

where $$b = \begin{cases} [UCL - 1] & \text{if } UCL \text{ is an integer} \\ [UCL] & \text{if } UCL \text{ is not an integer} \end{cases} \qquad (3a)$$

and $$a = \max\{0, [LCL]\} \qquad (3b)$$ and where $[x]$ denotes the largest integer not exceeding x with LCL and UCL as defined in Equation (1). Note that the constants a and b above have been suitably modified to take account of the fact that the Poisson distribution only assigns non-zero probabilities to the non-negative integers. In addition, note that $LCL = c_0 - 3\sqrt{c_0} <$ if and only if c_0<9 and in these cases the lower control limit is set to zero in Equation (3).

The performance of a control chart is typically judged on the basis of its run length distribution. The run length distribution is the probability distribution of the random variable, say N, which denotes the number of inspection units that must be sampled before the first signal is observed on the chart. Assuming that the inspection units are mutually independent and that the probability of a no-signal $\beta(c_0, c)$, given in Equation (2), stays constant over time (that is) the run length distribution is geometric with probability of a signal (success) equal to $1 - \beta(c_0, c)$. This is written as $N \sim Geo(1 - \beta(c_0, c))$. Consequently, the *probability mass* *function* (*pmf*) of *N* is

$$\Pr(N = k) = \beta(c_0, c)^{k-1}\,(1 - \beta\,(c_0, c)) \quad k = 1, 2, \ldots. \qquad (4)$$

and the *cumulative distribution function (cdf)* is

$$\Pr(N \le k) = 1 - \beta(c_0, c)^k \quad k = 1, 2, \ldots \qquad (5)$$

Given that the run length distribution is geometric in Case K, the *average run length* (*ARL*) and the *standard deviation of the run length* (*SDRL*) are just moments of this distribution and are given by

$$ARL(c_0, c) = \frac{1}{1 - \beta(c_0, c)} \qquad (6)$$

and

$$SDRL(c_0, c) = \frac{\sqrt{\beta(c_0, c)}}{1 - \beta(c_0, c)} \qquad (7)$$

respectively.

The in-control average run length (ARL_0), that is, when c=c_0, is found from Equation (6), $ARL_0 = ARL\,(c_0, c_0) = \dfrac{1}{1 - \beta\,(c_0, c_0)}$ whereas the false alarm rate (FAR) or, the probability of a signal when the process is actually in- control (*i.e.* c=c_0), is found from Equation (2), $FAR = 1 - \beta(c_0, c_0)$.

Example

(1) Suppose that the true average is c_0=14 non-conformities in an inspection unit. The 3-sigma control limits for the *c*-chart are: *LCL* = 2.78, *UCL* = 25.22, and *CL* = 14. Using Equation (3), these yield $b = [25.22] = 25$ and a = [2.78] = 2. Then, using Equation (2) it is found that $\beta(14, 14) = 0.9973$, so that the false alarm rate $FAR = 1 - \beta(14, 14) = 0.0027$. Hence, the (in-control) average run length, ARL_0, is equal to $1/0.0027 = 370.4$. On the other hand, when the true average increases from c=14 to c=32 non-conformities in an inspection unit, one finds that $\beta(14, 32) = 0.1228$ and the probability of a signal increases to $1 - \beta(14, 32) = 0.8772$ and the (out-of-control) average run length equals $ARL_1 = 1/0.8772 = 1.14$. This implies that one would expect a false alarm, on average, every 370.4 samples if the process is in-control with an average of c=14, and if it happens that the true average number of non-conformities increases from 14 to 32 (and stays fixed at 32) one would expect to detect such a shift on (approximately) the first sample following the shift.

(2) Next suppose that c_0=15. The 3-sigma control limits in this case are *LCL* = 3.38, *UCL* = 26.62 and *CL* = 15 and hence a=3 and b=26. Thus, $FAR = 1 - \beta(15, 15) = 0.0035$ and $ARL_0 = 1/0.0035 = 285.71$ and it is seen that there is no guarantee, even in Case K, that the actual false alarm rate and the actual average run length for the 3-sigma *c*-chart will be equal to their nominally expected values of 0.0027 and 370.4, respectively.

The *c*-chart for number of non-conformities: standard unknown

The c-chart with standard unknown

There are situations in practice when the true average number of non-conformities in an inspection unit, *c*, is unknown or unspecified. This can arise, for example, when a new process is started and not much experience and/or data are available. Such a scenario is referred to as the standard unknown case, or simply Case U. In this case, it is common to estimate *c* from a set of data, typically from *m* (mutually) independent inspection units, taken when the process is thought to be in-control. Such a set of data is referred to as reference data or Phase I data, and this phase of the analysis is called the retrospective phase or Phase I. As in Case K, let X_i be the number of non-conformities in the *i*th inspection unit. The average of these *m* numbers,

$$\bar{c} = \frac{\sum_{i=1}^{m} X_i}{m} = \frac{V}{m} \qquad (8)$$

say, is used as a point estimate of *c*. Note that the random variable *V* denotes the total number of non-conformities in the entire set of *m* inspection units obtained in Phase I.

The estimated 3-sigma control limits and the estimated centerline (also sometimes called the trial limits) of the *c*-chart are then given by

$$\begin{aligned} \hat{U}CL &= \bar{c} + 3\sqrt{\bar{c}} \\ \hat{C}L &= \bar{c} \\ \hat{L}CL &= \bar{c} - 3\sqrt{\bar{c}} \end{aligned} \qquad (9$$

and, as in Case K, if the estimated lower control limit is negative it is set to zero, *i.e.* if $\hat{L}CL < 0$ then set $\hat{L}CL = 0$.

The chart is implemented as follows. First, *m* inspection units are taken when the process is thought to be in-control and the number of non-conformities, *i.e.* $X_1, X_2, \ldots, X_m$, from each of the inspection units, is recorded. Then $\bar{c}$ is calculated (using Equation (8)) along with the control limits (using Equation (9)) and the *X*s are plotted on the control chart together with the control limits. If any of the *X*s fall on or outside the estimated control limits, that unit is investigated further and often dropped to calculate a revised estimate for $\bar{c}$. Then, the control limits are recalculated based on the revised $\bar{c}$ and one checks if all of the remaining *X*s plot within those control limits. This phase of the analysis, known as Phase I or the retrospective phase, is continued in an iterative fashion until all the *X*s plot inside the control limits. When that state is achieved, the process is declared in-control and one moves on to the next phase, called Phase II, or the prospective phase, where one monitors new (or future) inspection units to see if the process remains in-control. The data at hand at the end of the Phase I analysis is called in-control or reference data. Phase II control charts are typically designed based on these in-control data and some chart performance criterion, such as the average run length.

In what follows, it is assumed that *m* denotes the final number of inspection units that are used to calculate $\bar{c}$ in Equation (8), which is then used to calculate the control limits in Equation (9).

Properties of the c-chart with standard unknown

The probability of a 'no-signal' on the *c*-chart, denoted β, in this case is a function of the true average number of non-conformities (c), the (final) number of inspection units from Phase I, *i.e.* m as well as the average number of non-conformities in an inspection unit (c_1) in Phase II. In addition, unlike in Case K, the statistical properties of the control chart are affected by the (sampling) variation and randomness in the estimate $\bar{c}$ or, equivalently, the variation in the random variable V. Consequently, one has to account for the variation when determining the charts' properties - in the statistical design as well as in the implementation of the control chart. To understand the impact of estimation, it is convenient to look at the properties of the control chart conditionally, on having observed a particular estimate $\bar{c}$ (or, equivalently V). The unconditional chart properties are then obtained by averaging over the distribution of V. This two-step analysis provides valuable insight into the (specific as well as the overall) effects of parameter estimation on the performance of the *c*-chart in Phase II applications.

Conditional properties

The probability of a 'no-signal', conditional on having observed a value of the estimate $\bar{c}$ (or, equivalently, of V) is given by

$$\begin{aligned}\beta(c_1, c, m \mid \bar{c}) &= \Pr(X_i < \hat{U}CL \mid \bar{c}) - P(X_i \leq \hat{L}CL \mid \bar{c}) \\ &= \Pr\left(X_i < \frac{V}{m} + 3\sqrt{\frac{V}{m}} \mid V\right) - \Pr\left(X_i \leq \frac{V}{m} - 3\sqrt{\frac{V}{m}} \mid V\right) \qquad (10) \\ &= \beta(c_1, c, m \mid V)\end{aligned}$$

say, where X_i for $i = m+1, m+2, \ldots$ denotes the number of non-conformities in the ith inspection unit and c_1 is the average number of non-conformities in an inspection unit in the prospective monitoring phase or, in Phase II. The probability $\beta(c_1, c, m \mid V)$ denotes the *conditional probability of no-signal* given an observed value, *i.e.* v of the random variable V. Since, in Phase II, the number of non-conformities in the ith inspection unit X_i has a Poisson distribution with parameter c_1, the *conditional probability of no-signal* in Equation (10) can be equivalently expressed as

$$\beta(c_1, c, m \mid V) = \begin{cases} 0 & \text{if } V = 0 \\ \displaystyle\sum_{j=0}^{d(m,V)} \frac{e^{-c_1} c_1^j}{j!} - \sum_{j=0}^{c(m,V)} \frac{e^{-c_1} c_1^j}{j!} & \text{if } V = 1, 2, 3, \ldots \end{cases} \qquad (11)$$

where
$$d(m, V) = \begin{cases} [\hat{U}CL - 1] & \text{if } \hat{U}CL \text{ is an integer} \\ [\hat{U}CL] & \text{if } \hat{U}CL \text{ is not an integer} \end{cases} \qquad (12a)$$

and $$c(m, V) = \max\{0, [\hat{L}CL]\} \qquad (12b)$$

and as before $[x]$ denotes the largest integer not exceeding x with $\hat{L}CL$ and $\hat{U}CL$ as defined in Equation (9). Note that, $\hat{L}CL < 0$ if and only if $\bar{c} = V/m < 9$. Thus, for $V < 9m$ or, if we observe less than $9m$ non-conformities in the m reference samples, the lower control limit will be negative and is thus set to zero. Also, when one observes no non-conformities in the reference sample, *i.e.* when V=0 (or $\bar{c}$ =0), it makes sense to pause and examine the situation in more detail. Thus, for V=0 the conditional probability of a 'no-signal' is defined to be 0 in Equation (11), so in that case the conditional probability of a signal is 1.

Using Equation (11), the *conditional probability of a false alarm* or the *conditional false alarm rate* (CFAR) is given by

$$CFAR = 1 - \beta(c_1 = c_0, c = c_0, m \mid V) \qquad (13)$$

where c_0 (in this case) is some arbitrary value of c. Although one can examine the CFAR (a typical measure of a control charts' performance), the focus here is primarily the (conditional) run length distribution, *i.e.* the distribution of the run length random variable, say N, which denotes the number of inspection units that must be sampled before the first signal is observed in Phase II given an observed value of the random variable V from Phase I. This can be found using the same conditioning argument as illustrated above. To this end, note that given an observed value of the random variable V, the conditional run length distribution is geometric with the probability of a success (or a signal) equal to $1 - \beta(c_1, c, m \mid V)$. This is because, for a given fixed value of V, the control limits can all be calculated and the analysis can proceed as if the parameter c is known, which is similar to Case K studied in the earlier section, where the run length distribution was seen to be geometric. However, the conditioning changes the probability of success in the geometric distribution.
All properties and characteristics of the conditional run length distribution follow (conveniently) from the well-known properties of the geometric distribution. For example, the *probability mass function* (*pmf*) of the conditional run length distribution is given by

$$\Pr(N = k \mid V) = \beta(c_1, c, m \mid V)^{k-1}(1 - \beta(c_1, c, m \mid V)) \quad k = 1, 2, \ldots \qquad (14)$$

Compare this to Equation (4). *The conditional average run length (CARL) and the conditional standard deviation of the run length (CSDRL)* distribution are given by

$$CARL(c_1, c, m \mid V) = \frac{1}{1 - \beta(c_1, c, m \mid V)} \qquad (15)$$

and

$$CSDRL(c_1, c, m \mid V) = \frac{\sqrt{\beta(c_1, c, m \mid V)}}{1 - \beta(c_1, c, m \mid V)} \qquad (16)$$

respectively. Compare Equations (15) and (16) with Equations (6) and (7) for a better understanding of conditioning.

The in-control properties of the conditional run length distribution can be obtained by substituting c_1=c=c_0 in Equations (14) through (16). For example, the in-control conditional average run length is found from Equation (15), and is given by

$$CARL(c_0, c_0, m \mid V) = \frac{1}{1 - \beta(c_0, c_0, m \mid V)} \qquad (17)$$

Unconditional properties

Typically, users of the *c*-chart would have their own estimates of c ($\bar{c}$ or V) in their own applications, based on their data, and that estimate would determine the performance of the chart in that specific application, as illustrated by the conditional chart properties described above. However, the overall performance of the control chart is also of interest, which shows how the chart performs when all possible estimates, from all possible data samples, are taken into consideration. This performance would be the same for all users and requires studying the unconditional properties of the chart, which can be obtained from the conditional properties by 'averaging' over the distribution of the estimator V. For example, using Equation (11) and the fact that V has a Poisson distribution with mean mc, the unconditional cumulative distribution function of the run length is obtained as

$$\Pr(N \leq k) = 1 - \sum_{v=0}^{\infty} (\beta(c_1, c, m \mid v))^k \frac{e^{-mc}(mc)^v}{v!}, \quad k = 1, 2, \ldots \tag{18}$$

This may be compared with Equation (5) in Case K.

The in-control unconditional cumulative distribution function can be obtained from Equation (18) by letting $c_1=c=c_0$. Various other characteristics of the unconditional run length distribution can be found similarly. For example, the *unconditional false alarm rate* (*UFAR*) is found from Equation (13)

$$\begin{aligned} UFAR &= \sum_{v=0}^{\infty} CFAR \frac{e^{-mc}(mc)^v}{v!} \\ &= \sum_{v=0}^{\infty} (1 - \beta(c_1 = c_0, c = c_0, m \mid v)) \frac{e^{-mc}(mc)^v}{v!} \\ &= 1 - \sum_{v=0}^{\infty} \beta(c_1 = c_0, c = c_0, m \mid v) \frac{e^{-mc}(mc)^v}{v!} \end{aligned} \tag{19}$$

In addition, the *unconditional average run length* (*UARL*) and the *unconditional variance of the run length* (*UVARRL*), in general, are

$$UARL(c_1, c, m) = \sum_{v=0}^{\infty} \left(\frac{1}{1 - \beta(c_1, c, m \mid v)} \right) \frac{e^{-mc}(mc)^v}{v!} \tag{20}$$

and

$$\begin{aligned} UVARRL(c_1, c, m) &= \sum_{v=0}^{\infty} \left(\frac{\beta(c_1, c, m \mid v)}{(1 - \beta(c_1, c, m \mid v))^2} \right) \frac{e^{-mc}(mc)^v}{v!} \\ &+ \sum_{v=0}^{\infty} \left(\frac{1}{1 - \beta(c_1, c, m \mid v)} \right)^2 \frac{e^{-mc}(mc)^v}{v!} \\ &- \left(\sum_{v=0}^{\infty} \left(\frac{1}{1 - \beta(c_1, c, m \mid v)} \right) \frac{e^{-mc}(mc)^v}{v!} \right)^2 \end{aligned} \tag{21}$$

respectively. Thus, all desired characteristics of the unconditional run length distribution can be obtained by calculating the expectation of the conditional characteristics with respect to the distribution of *V*.

Note that the in-control *unconditional average run length* (1) ($UARL_0$) and the in-control *unconditional variance of the run length* ($UVARRL_0$) can be obtained from Equations (20) and (21), respectively, by substituting c_1=c=c_0.

Percentiles and other moments of the unconditional run length distribution can be found using a similar conditioning argument. For example, using Equation (18), the *unconditional median run length* (*UMDRL*) is found to be the smallest integer *k* such that

$$P(N \leq k) \geq \frac{1}{2} \qquad (22)$$

Both the conditional and the unconditional performance characteristics provide interesting insights into the performance of the chart.

Example

To illustrate, consider Example 6.3 in Montgomery [5, p. 310] about the quality control of manufactured printed circuit boards. A total of 26 successive inspection units (samples), each consisting of 100 individual units of product, were obtained in Phase I to estimate *c*. However, it was found that units 6 and 20 were out-of-control and were therefore eliminated. The revised control limits, to be used for prospective monitoring in Phase II, were calculated using the remaining *m*=24 inspection units, with the number of non-conformities in each inspection unit shown in Table 6.7 of Montgomery [5, p. 311]. Theoretically, the variable *V*, the total number of non-conformities in the 24 inspection units, could take on any positive integer value (including zero) *i.e.* $V \in \{0, 1, 2, 3, \ldots\}$. However, for the given (Phase I) data, it was found that *V*=472. Hence, using Equation (8), the average number of non-conformities per inspection unit is estimated as $\bar{c} = (472/24) = 19.67$, so that the *estimated* 3-sigma control limits from Equation (9) are $\hat{UCL} = 32.97$, $\hat{CL} = 19.67$ and $\hat{LCL} = 6.36$. Using Equation (12a), these yield *c*(24, 472)=6 and *d*(24, 472)=32.

The conditional false alarm rate (CFAR) and the conditional average run length (CARL)

For the given (observed) value of *V*=472 one can investigate the charts' performance using the conditional properties. Using Equations (11) and (13), and assuming that $c_1 = c = 20 = c_0$, the *conditional false alarm rate* (*CFAR*), *i.e*. the false alarm rate given *V*=472, is found to be equal to 0.004983. The *CFAR* is approximately 72% larger than the value of 0.0029 one would have obtained in Case K for c_0=20, and is 85% higher than the nominal value 0.0027. Note that this is true even though the estimated average number of non-conformities in an inspection unit $(\bar{c} = 19.67)$ is within $(19.67 - 20/\sqrt{20}) = -0.07$ standard deviation units of the true average number of non-conformities per inspection unit (c_0=20).

In Figure 1, the conditional false alarm rate (*CFAR*) is plotted as a function of the estimated average number of non-conformities, $\bar{c}$, observed from an in-control reference sample in Phase I. It is seen that unless $\bar{c}$ is 'close' to the *true* average number of non-conformities, which is 20 in this case, the conditional false alarm rate (*CFAR*) can be either excessively large or remarkably small compared to the Case K value (0.0029) indicated by the dotted line. For example, suppose that instead of *V*=472 one observed *V* = 600, then the estimated average number of non-conformities per inspection unit is $\bar{c} = 600/24 = 25$ and subsequently, using Equation (13), the conditional false alarm rate is 0.01086, which is approximately 275% larger than the Case K value (0.0029) and is 303% larger than the nominal 0.0027. Similarly, if one observed *V*=528, the estimated average $\bar{c} = (528/24) = 22$ and the conditional false alarm rate is 0.001201, which is approximately 59% less than the Case K value and is 55% *less* than the nominal value (0.0027).

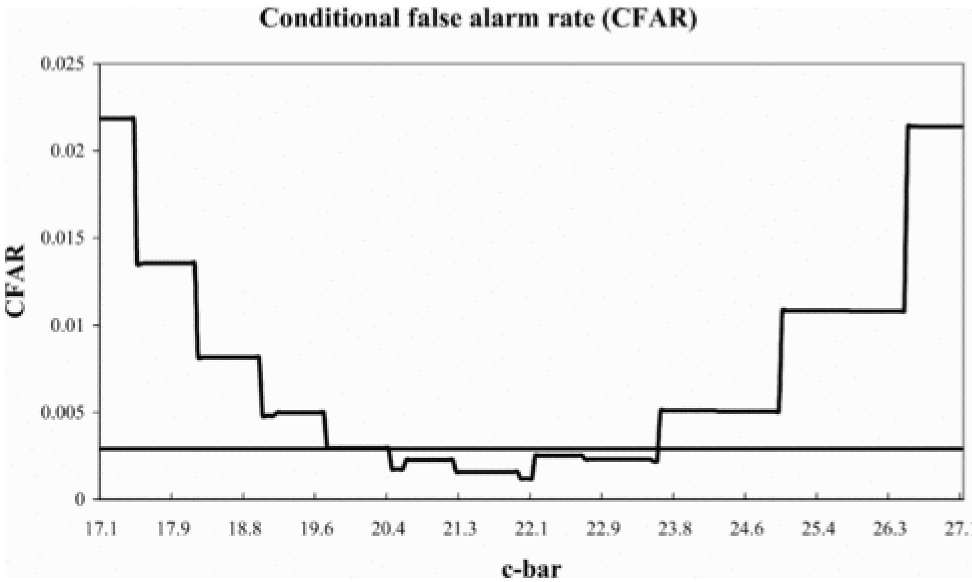

Figure 1. Conditional false alarm rate (*CFAR*) as a function of $\bar{c}$observed from an in-control reference sample.

In addition, also note that when *V*=472 (as in the current situation) and using Equation (17), one finds the in-control conditional average run length to be 1/0.004983 = 200.68, which is 41% less than the Case K value of 339.72 and is 46% less than the nominal value 370.4. Therefore, the 3-sigma control chart incorrectly signals more often than expected. Also, for the two other selected values of *V*, *i.e*. when *V*=600 (or $\bar{c}$ =25) or *V*=528 (or $\bar{c}$ =22), the in-control conditional average run length values are 92.04 and 832.3, respectively. Thus, compared to the Case K value of 339.72, the control chart signals about 3.6 times more or 2.5 times less than would be expected if the standard was in fact known. Only when $V \in \{473, 490\}$ is (or, equivalently when $\bar{c} \in [19.71; 20.42])$ the conditional probability of no signal (obtained from Equations (11) and (12a)) approximately equal to the probability of no signal in Case K (obtained from Equations (2) and (3)). For any value of *V* outside this range, the performance of the *c*-chart, as measured by the false alarm rate (*FAR*) and the average run length (*ARL*), considerably degraded.

The unconditional false alarm rate (UFAR) and the unconditional average run length (UARL)

Finally, using Equations (19) and (20), and averaging over all the possible values and the corresponding probabilities of *V*, which follows a Poisson distribution with mean 480 (*i.e*. 24× 20), the unconditional false alarm rate (*UFAR*) is found to be 0.0039 and the in-control unconditional average run length ($UARL_0$) is found to be 335.30. Thus, the *UFAR* is 20% less than the *CFAR* of 0.0049 and the $UARL_0$ is 67% larger than the *CARL* of 200.68.

However, with regard to the unconditional chart properties, note that the in-control unconditional average run length ($UARL_0$) is 1.3% less than the in-control average run length of 339.72 one would have obtained in Case K for c_0=20 and the unconditional false alarm rate *UFAR* is 34% larger than the *FAR* of 0.0039 obtainable in Case K.

To help understand and illustrate the impact of parameter estimation on the properties of the *c*-chart, Tables 1 and 2 display values of the in-control unconditional average run length ($UARL_0$) and the unconditional false alarm rate (*UFAR*) for different values of the number of inspection units (m) in the reference sample and the *true* average number of non-conformities per inspection unit (c), together with the corresponding values for Case K - given in the last row.

Table 1. The in-control unconditional average run length ($UARL_0$) for different values of the number of inspection units in the reference sample, m, and the true average number of non-conformities in an inspection unit, c.

	c							
m	1	2	4	6	8	10	20	50
5	2.51	6.54	38.49	166.91	436.17	399.00	303.41	256.36
10	2.58	6.82	40.34	162.21	370.41	378.91	330.91	294.22
15	2.61	6.88	41.04	159.53	326.93	356.59	333.40	311.55
20	2.62	6.93	41.48	157.90	315.32	353.51	338.79	320.40
25	2.63	6.94	41.74	156.49	298.67	343.85	336.93	326.23
30	2.63	6.96	41.78	155.76	290.10	333.52	334.53	330.79
50	2.63	6.99	42.22	154.09	276.24	322.48	335.16	338.50
100	2.64	7.03	42.40	154.12	261.79	308.18	334.20	345.56
200	2.64	7.08	42.45	156.83	252.11	295.09	333.51	349.26
300	2.64	7.11	42.47	159.17	248.62	289.87	333.44	351.37
500	2.64	7.13	42.48	161.92	247.04	286.59	334.73	356.15
1000	2.64	7.15	42.50	163.55	246.70	285.75	338.07	367.04
Case K	2.58	7.15	37.81	163.74	246.70	285.74	339.72	396.70
% difference	2.33	0.00	12.40	−0.12	0.00	0.00	−0.46	−7.48

Note: % difference = $\{(UARL_0 \text{ when } m = 1000)/(ARL_0 \text{ of Case K}) - 1\} \times 100$.

Table 2. The unconditional false alarm rate (*UFAR*) for different values of the number of inspection units in the reference sample, *m*, and the true average number of non-conformities in an inspection unit, *c*.

m	c = 1	2	4	6	8	10	20	50
5	0.4067	0.1603	0.0325	0.0136	0.0104	0.0095	0.0078	0.0068
10	0.3901	0.1485	0.0272	0.0097	0.0069	0.0062	0.0052	0.0046
15	0.3845	0.1463	0.0259	0.0087	0.0060	0.0053	0.0045	0.0040
20	0.3824	0.1448	0.0252	0.0082	0.0054	0.0048	0.0041	0.0037
25	0.3813	0.1446	0.0248	0.0079	0.0052	0.0045	0.0039	0.0035
30	0.3807	0.1439	0.0247	0.0077	0.0050	0.0044	0.0038	0.0034
50	0.3799	0.1434	0.0241	0.0073	0.0047	0.0040	0.0035	0.0032
100	0.3796	0.1424	0.0239	0.0070	0.0044	0.0038	0.0033	0.0030
200	0.3795	0.1413	0.0239	0.0066	0.0042	0.0037	0.0032	0.0029
300	0.3794	0.1407	0.0238	0.0064	0.0041	0.0036	0.0031	0.0029
500	0.3794	0.1402	0.0238	0.0062	0.0041	0.0035	0.0030	0.0029
1000	0.3793	0.1399	0.0238	0.0061	0.0041	0.0035	0.0030	0.0028
Case K	0.3869	0.1399	0.0264	0.0061	0.0041	0.0035	0.0029	0.0025
% difference	−1.96	0.00	−9.85	0.00	0.00	0.00	3.44	1.12

Note: % difference = $\{(UFAR \text{ when } m = 1000)/(FAR \text{ of Case K}) - 1\} \times 100\%$.

For example, when c=10 and m=25 one finds that the *UFAR* = 0.0045 (which is approximately 29% higher than the value of 0.0035 for Case K and approximately 67% higher than the nominal value of 0.0027) and that the $UARL_0$ = 343.85 (which is approximately 20% higher than the value of 285.74 for Case K and approximately 7% less than the nominal value of 370.4). Thus, when the reference sample from Phase I contains m=25 inspection units, the control chart will (on average) signal more often than would be case if the true average number of non-conformities in an inspection unit was in fact known or what is generally expected. However, when one looks at the percentage difference between the *UFAR* and/or the $UARL_0$ values when m=1000 inspection units are used to estimate the unknown standard and the values of the *FAR* and/or the ARL_0 from Case K (denoted by percentage difference), there is almost no discrepancy, suggesting that (on average) the control chart would perform as in Case K when so much data are available.

However, note that neither the in-control unconditional average run length, *i.e.* $UARL_0$, nor the unconditional false alarm rate, *i.e.* UFAR, converges to the nominally expected values of 370.4 and 0.0027, respectively. In fact, they are far off from the nominal values for most *m* and *c* values, particularly when *c* is less than 10 and *m* is less than 1000. This should be a reason for concern for the practitioner. In addition, it may be noted that the *ARL* and the *FAR* values in Case K can also be quite different from their respective nominal values.

For example, for c=6 and m=25, the $UARL_0$ = 156.49, which is 57.7% less than the nominal value of 370.4, whereas the *UFAR* = 0.0079, which is 192.59% larger than the nominal value of 0.0027. Even for m=1000, the $UARL_0$ and the *UFAR* values equal 163.55 and 0.0061, respectively which are close to their Case K values but not the nominally expected values. Thus, the usual *c*-chart with an estimated process parameter will incorrectly signal much frequently than if the parameter *c* had in fact been known, and even more frequently than is nominally expected. In addition, it is seen that the in-control unconditional average run length, *i.e.* $UARL_0$ is no longer equal to the reciprocal of the unconditional false alarm rate, *i.e.* $1/UFAR$.

From Tables 1 and 2, it appears that one needs at least 300 to 500 inspection units in the reference dataset to estimate the unknown standard to ensure that the *c*-chart performs as well as when the standard is in fact known. However, that can still be quite different from what is nominally expected, *i.e.* $ARL_0 = 370.4$ (or $FAR = 0.0027$). For the *c*-chart with unknown standard to perform as is nominally expected, one will need to adjust the control limits, by solving for the charting constant k for a fixed $ARL_0 = 370.4$ using the formulas given in the paper. This would mean either widening or contracting the limits depending on what k turns out to be relative to 3.

Summary and conclusions

Both the false alarm rate (*FAR*) and the average run length (*ARL*) of the *c*-chart are greatly affected by the estimation of the true unknown average number of non-conformities in an inspection unit c. Examples demonstrate that (1) a large amount of Phase I or reference data is needed before the in-control unconditional average run length ($UARL_0$) can be 'near' its corresponding value of Case K, when c is known; (2) even if more data are available, neither the $UARL_0$ nor the *UFAR* will necessarily be equal to the commonly used nominally expected values (primarily due to the discreteness of the underlying distribution); and (3) the unconditional *FAR* is not equal to the reciprocal of the unconditional *ARL* (and vice versa). It is seen that the common practice (and advice) of using 20 to 25 inspection units is simply not enough. If a large amount of reference data is not available or if one needs the chart to perform at a typically used nominal in-control average run length value, one would need to adjust the control limits by finding the desired value of the charting constant $k>0$ so that k-sigma limits could be used (instead of the usual 3-sigma control limits as is typically done in routine applications), which would lead to a desired chart performance characteristic, such as a specified in-control *ARL* value, either from Case K or some typical value such as 370.4. To this end the exact formulae derived in this paper is useful.

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