Mosquito feeding preference and pyrethroids repellent effect eliminate backward bifurcation in malaria dynamics

Jean C. Kamgang[1] · Berge Tsanou[2,3,4] · Duplex E. Houpa Danga[1] · Jean M. -S. Lubuma[5]

Received: 3 April 2021 / Revised: 25 March 2022 / Accepted: 30 March 2022

Abstract
Pyrethroid-treated bed-nets (PTNs) protect individuals against malaria by blocking and repelling mosquitoes. We develop and analyze a PTNs malaria model that explicitly includes mosquito host choice (also known as feeding/biting preference) and Pyrethroid repellent effect. Our model reveals that mosquito biting/feeding preference on infectious hosts π and repellent effect r drive for the existence of both the endemic equilibrium points and the occurrence or elimination of backward bifurcation. The threshold parameters for the mosquito biting preference on infectious hosts π^* and repellent effect r^* for the occurrence and elimination of backward bifurcation are computed. Moreover, it is shown that, increasing the mosquito host choice rate or decreasing the repellent effect rate, annihilates backward bifurcation, thus facilitating the control of malaria. Furthermore, we prove that the threshold of mosquito biting preference is a monotone increasing function of the repellent effect r. We show that the model exhibits both trans-critical forward bifurcation and backward bifurcation when either the mosquito host choice π crosses a threshold value π_1 or the repellent effect r passes through a threshold repellent rate r_1. Sufficient conditions for the global asymptotic stability of the equilibrium point are derived. On the other hand, it is established that, decreasing the mosquito biting preference or increasing the rate of the

✉ Berge Tsanou
bergetsanou@yahoo.fr

1 Department of Mathematics and Computer Science, University of Ngaoundéré, P.O. Box 454, Ngaoundéré, Cameroon

2 Department of Mathematics and Computer Science, University of Dschang, P.O. Box 67, Dschang, Cameroon

3 Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa

4 UMMISCO, IRD Sorbonne University, 93143 Bondy, France

5 School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa

repellent effect (i.e personal protection) or the combining both actions, decreases the malaria control reproduction number $\mathcal{R}_0$. Finally, the interplay between the bed-nets treated repellent effect and mosquito host choice and its potential on the dynamics of malaria is investigated and illustrated numerically.

Keywords Mosquito biting preference · Repellent effect · Control reproduction number · Backward bifurcation · Global stability · Malaria · Pyrethroid

Mathematics Subject Classification 92B05 · 37N25 · 34D30 · 65L03

1 Introduction

Malaria is a life-threatening disease caused by parasites transmitted to susceptible humans through the bites of infectious female mosquitoes of the genus Anopheles [5]. The data released by the World Health Organization (WHO) indicates that there were about 228 million malaria cases including 405,000 deaths in 2018 (World Health Organization 2019 [39]). Over 40% of the world's population in more than 80 countries and areas is still at risk of contracting malaria [41].

Insecticide-treated nets (ITNs) have proven to be one of the most effective intervention measures against malaria in reducing morbidity and mortality [8, 14, 17, 18, 22, 30]. Furthermore the pyrethroids insecticide are today the only insecticide authorized for the impregnation of bed-nets [37, 38, 42], because of their efficiency, their repulsive effect and their weak toxicity for humans. Interestingly, promising tests have revealed the efficiency of pyrethroids impregnated mosquito bed-nets to limit the contact between human and mosquito [9, 11, 38].

Mathematical models for the transmission of infectious agents have proven to be useful tools in understanding disease dynamics and assessing the impact of mosquito biting preference. Mosquito biting preference, also known as vector bias in malaria is the propensity of mosquitoes for being more attracted to malaria-infected individuals [20]. Several recent existing works [1, 3, 5, 10, 19, 28, 35, 36, 40] on the mathematical modeling for the transmission of malaria focus only on the attractiveness of infectious humans to mosquitoes. Chamchod and Britton [10] proposed a vector-bias term in a malaria transmission model to account for mosquito preference for infectious humans. Abboubakar et al. [1] extended the model of Chamchod and Britton to include exposed mosquitoes (infected with malaria but not yet infectious to humans). In [28], Rivera et al. extended the models of Chamchod and Britton [10] and Abboubakar et al. [1], by proposing a general dynamic of vector-borne diseases that could be useful for modeling malaria, leishmaniasis, dengue and any other vector-host-pathogen interactions. However, none of the above-mentioned models have added the impact of bed nets utilization to prevent infected mosquitoes form biting human hosts and preferably infected humans. The work in [40] dealt with the study of a diffusive malaria model with vector-bias and the main result was the establishment of threshold dynamics for the spatially heterogeneous system in terms of the basic reproduction ratio, as well as the exhibition of a set of sufficient conditions for the global attractiveness of the positive steady state. Kim et al. [19] determined how vector-bias affects the changes in

the dynamics of malaria transmission. They considered two different incidence areas; one for a high transmission area and the other for a low transmission area. Their results showed different dynamical behavior in the two areas and that considering the vector-bias effect in different areas facilitates prediction of the future dynamics. In Aldila et al. [3] proposed a reasonable mathematical modeling introducing mosquito repellent effects and took into account the relationship between its use and the mosquito population dynamics. The work by Wang et al. [36] investigated the impact of bed-net use by formulating a periodic vector-bias malaria model incorporating the juvenile stage of mosquitoes and the use of insecticide treated bed-nets usage. The study in [5] formulated a malaria model which included the enhanced attractiveness of infectious humans to mosquitoes, as a result of host manipulation by malaria parasite, and the human behavior, represented by insecticide-treated bed-nets usage. More recently, Tsanou et al. [31] have explicitly modeled, incorporated and studied the role of excito-repellent/deterrence effect on the long run dynamics of malaria and on its bifurcation analysis. None of the above-mentioned models have investigated the combined influence of excito-repellent effect of pyrethroids treated bed-nets and the mosquito feeding preference on the transmission dynamics of malaria.

Therefore, in this paper, we fill some of modeling gaps mentioned above by formulating and analyzing a pyrethroids impregnated bed-net malaria model that includes both the role of excito-repellent/deterrence effect on the transmission dynamics of malaria, mosquito host choice, as well as malaria prophylactic treatment are incorporated. We further assess the role of mosquito feeding preference on the occurrence and elimination of backward bifurcation phenomenon. In so doing, we extend our previous bed-net malaria model proposed in [31] in two directions: (1) we model the contact rate between mosquitoes and humans is more realistically modeled by a nonlinear function of bed-net usage and repellent effect; (2) we account for the mosquito biting preference for malaria infected individuals. We show that this new contact rate decreases as a sole function of bed-net usage or repellent effect. Additionally and contrary to [31], we explicitly model the mortality rate of mosquitoes by a linear function of both the pyrethroids repellent rate and the bed-net utilization rate. In order to highlight the impacts of pyrethriods repellent effect and the mosquito feeding preference, we perform a bifurcation analysis and find that the occurrence of backward bifurcation depends either on the mosquito biting preference magnitude or on the range of the pyrethroids repellent rate. Global dynamics of the model using Lyapunov–LaSalle and geometric approaches are established for the disease-free equilibrium and the endemic equilibrium, respectively.

The structure of the remaining of the paper is the following. In Sect. 2, we concentrate on the model formulation and basic mathematical properties. Bifurcation analysis of the model with respect to mosquito feeding preference and repellent effect are respectively presented in Sect. 3. Section 4 deals with the global dynamics of the model. In Section 5, we investigate the interplay between pyrethroids treated bed-nets repellent effect and mosquito host choice and its potential impact on the long-term dynamics of the model. Finally, the paper is concluded and discussed in Sect. 6.

2 Model formulation and basic mathematical properties

2.1 Model formulation

The total human population $N_h(t)$ at time t is divided into three different classes: $S_h(t)$, $I_h(t)$ and $R_h(t)$, where, S_h represents the susceptible, I_h the infectious and R_h the recovered and temporary immune individuals. Susceptible humans acquire malaria at rate λ_h after being bitten by an infected mosquito, and progress to the infectious class. After receiving prophylactic treatment at rate u while in I_h class, a proportion ρ, of those who recover from infection without livelong immunity returns to the S_h class, and the remaining proportion, $(1-\rho)$, who progresses to the recovered class R_h due to full immunity. We also assume that humans in the I_h class can recover naturally (without treatment) at rate γ_h to join the recovered claas R_h. Individuals in the R_h class lose their temporary immunity at rate σ and become susceptible to malaria. Infected humans undergo disease-induced and mortality and natural mortality at rates δ and μ_h, respectively.

The total mosquito population $N_v(t)$ at time t , is divided into two mutually exclusive groups: susceptible $S_v(t)$ and infectious $I_v(t)$. Susceptible vectors move to the infected class by acquiring malaria through contact with infected humans at a rate λ_v. The total mortality rate of mosquitoes is μ_v. Although our model formulation follows the classical approach in modeling malaria transmission, we stress that, in the context of pyrethroid-treated bed-nets (PTNs) utilization and mosquito biting preference, we emphasize here on the more realistic descriptions of λ_h, λ_v and μ_v in order to put forward the influences of PTNs effects and vector-bias preference on those forces of infection and on mosquito mortality rate. Thanks to knock-down and lethal effects of PTNs, the female mosquitoes in search of blood meal can die when they come into contact with a pyrethroids treated bed-net. Thus, mimicking [2], we model the additional mortality rate for mosquitoes μ_b induced by the PTNs lethal effect by

$$\mu_b = \mu_{v1} b, \quad 0 \leq b \leq 1,$$

where, the parameter b stands for the proportion of treated bed-net usage, and μ_{v1} is the maximum insecticide-induced death rate of mosquitoes. We note that, thanks to deterrence effect, female mosquitoes questing for blood meal can also experience natural death without coming into contact with a treated bed-net. More precisely, contrary to knock-down and lethal effects, deterrent/dissuasive effect, does not involve any physical contact between the mosquito and the bed-net: the presence of a PTN in a home/bedroom can cause an avoidance or repellent behavior of mosquitoes, who are dissuaded by the smell of the insecticide. Consequently, for each additional search of blood meal on a host, female mosquitoes encounter a risk μ_r of dying when they inhale insecticide. We model this other additional death rate by a linear and increasing function of repellent rate r as follows:

$$\mu_r = Cr, \quad 0 \leq r \leq 1,$$

where, C is a positive constant accounting for the maximum mortality (when $r = 1$) during host searching caused by the insecticide deterrent effect. A similar formulation of μ_r can be found in Birget et al. [4], where the value $C = 0.03$ was assumed. All in all, if μ_{v0} denotes the natural mortality rate, the actual mortality rate of mosquitoes $\mu_v(r, b)$ is therefore given by:

$$\mu_v(r, b) = \mu_{v0} + \mu_b + \mu_r = \mu_{v0} + \mu_{v1}b + Cr, \quad 0 \le b \le 1, \quad 0 \le r \le 1, \quad (1)$$

In order to describe the forces of infection λ_h, λ_v, we follow the approach in [31]. Thus, if θ is the probability that a mosquito targets human hosts, then the probability $P_h(r, b)$ that a mosquito finally bites a human host, and the probability $P_v(r, b)$ that a mosquito initiates a bite indoors on a non-human host or a human host, are respectively:

$$\begin{aligned} P_h(r, b) &= \frac{\theta(1 - rb)}{1 - \theta rb(1 - (\mu_b + \mu_r))}, \quad 0 \le b \le 1, \quad 0 \le r \le 1, \\ P_v(r, b) &= \frac{1 - rb\theta}{1 - \theta rb(1 - (\mu_b + \mu_r))}, \quad 0 \le b \le 1, \quad 0 \le r \le 1. \end{aligned} \quad (2)$$

Mimicking the modeling framework in [5, 10], we assume that mosquitoes bite humans at probability $\mathbf{m}$ if the host is infectious, and probability $\mathbf{n}$ (with $\mathbf{m} > \mathbf{n}$) if he/she is rather susceptible or recovered. Thus, when a mosquito bites a human being, the probability that this person gets infected is given by the ratio between the total bitten infectious humans and the total bitten humans. That is

$$\frac{\mathbf{m}I_h}{\mathbf{m}I_h + \mathbf{n}(S_h + R_h)}.$$

Similarly, the probability that the bitten human being is susceptible or recovered is the ratio between the total bitten susceptible or recovered humans and the total bitten humans. That is

$$\frac{\mathbf{n}(S_h + R_h)}{\mathbf{m}I_h + \mathbf{n}(S_h + R_h)}.$$

The ratio

$$\pi = \frac{\mathbf{m}}{\mathbf{n}},$$

is called the vector-bias parameter (or the feeding/biting preference rate. Since there is preference for mosquitoes to bite more infected humans than susceptible ones, this number is always greater than or equal to unity $\pi > 1$. The case $\pi = 1$ corresponds to equal biting/feeding probability or no biting/feeding preference at all. To summarize, the forces of infection are modeled here by explicit functions of bed-net usage rate (b), insecticide (pyrethroids) repellent rate (r) and the mosquito feeding/biting preference

Table 1 Description and baseline values of parameters in system (4)

Par.	Descriptions	Baseline value	Ref.
Λ_h	Immigration rate for humans	$10^3/(50 \times 365)$	[7]
Λ_v	Immigration rate for mosquitoes	$10^4/21$	[32]
r	Repellent probability/rate of pyrethroids/ITNs treated bed-net	$(0, 1)$	Variable
$\pi = m/n$	Vector-bias parameter or feeding/biting preference rate	> 1	variable
μ_{v1}	Maximum mosquito ITNs-induced death rate	0.5	[21, 25]
μ_h	Natural mortality rate for humans	$1/(50 \times 365)$	[25]
μ_{v0}	Natural mortality rate for mosquitoes	$1/14$	[12, 25]
δ	Disease-induced death rate for infected humans	$1/1000$	[7]
m_v	Human-to-mosquito probability of disease transmission	1	[2]
m_h	Mosquito-to-human probability of disease transmission	0.195	[12, 25]
γ_h	Natural recovery rate of infectious humans	1×10^{-5}	[23]
ρ	Proportion of those who recovered from malaria infection without livelong immunity	$4/10$	[23]
σ	Rate at which immune humans lose recovered-induced immunity	$1/(5 \times 365)$	[7, 23]
b	Proportion of PTNs usage		Variable
θ	Probability that a mosquito targets a human host	1	Assumed
u	Rate of treatment of malaria infection	0.003	Assumed

rate (π) as follows:

$$\begin{aligned} \lambda_h(r, b, \pi) &= \frac{m_h P_h(r, b) I_v}{\pi I_h + S_h + R_h}, \quad 0 \le b \le 1, \quad 0 \le r \le 1, \\ \lambda_v(r, b, \pi) &= \pi \frac{m_v P_v(r, b) I_h}{\pi I_h + S_h + R_h}, \quad 0 \le b \le 1, \quad 0 \le r \le 1. \end{aligned} \tag{3}$$

Let's recall that in (2), the terms $\lambda_h(r, b, \pi)$ and $\lambda_v(r, b, \pi)$ represent the forces of infection for susceptible humans and for susceptible vectors, respectively. The quantity m_h is the transmission probability per bite from infectious mosquitoes to humans and similarly, m_v is the transmission probability per bite from infectious humans to mosquitoes. The model's parameters are gathered in Table 1 and the above descriptions lead to the following system of ordinary differential equations which models dynamics of malaria transmission.

$$\begin{cases} \dot{S_h} &= \Lambda_h + \sigma R_h - \lambda_h(r, b, \pi) S_h + \rho u I_h - \mu_h S_h \\ \dot{I_h} &= \lambda_h(r, b, \pi) S_h - (\mu_h + \gamma_h + \delta + u) I_h \\ \dot{R_h} &= (1 - \rho) u I_h + \gamma_h I_h - (\sigma + \mu_h) R_h \\ \dot{S_v} &= \Lambda_v - \lambda_v(r, b, \pi) S_v - \mu_v(r) S_v \\ \dot{I_v} &= \lambda_v(r, b, \pi) S_v - \mu_v(r) I_v. \end{cases} \tag{4}$$

2.2 Basic mathematical properties

Since system (4) monitors human and mosquito populations, and all the model parameters are non-negative, we prove that the solutions with non-negative initial conditions exist, remain non-negative and are bounded for all time $t \geq 0$. For that, it suffices to establish the following result:

Theorem 1 *The compact set Ω below is positively invariant, attractive and absorbing for System (4).*

$$\Omega = \left\{ (S_h(t), I_h(t), R_h(t), S_v(t), I_v(t)) \in \mathbb{R}^5_+ : 0 \leq N_h(t) \leq \frac{\Lambda_h}{\mu_h}, 0 \leq N_v(t) \leq \frac{\Lambda_v}{\mu_v(r,b)} \right\}.$$

Proof First of all, one should note that the right hand side of (4) is continuously differentiable. Thus, thanks to Cauchy-Lipschitz theorem, for any initial condition (say at time $t_0 = 0$), system has a unique (local) solution defined in an interval of the form $[0, T)$, $T > 0$. Moreover, it is not difficult to prove that such a solution is non-negative if the initial condition is non-negative. Furthermore, we have the relations

$$\dot{N}_h(t) = \Lambda_h - \mu_h N_h(t) - \delta I_h(t), \quad \dot{N}_v(t) = \Lambda_v - \mu_v(r,b) N_v(t). \tag{5}$$

Using Equation (5), it follows from the non-negativity of the solutions and the Gronwall inequality that for all $t \in [0, T)$,

$$\begin{aligned} N_h(t) &\leq \left(N_h(0) - \frac{\Lambda_h}{\mu_h} \right) e^{-\mu_h t} + \frac{\Lambda_h}{\mu_h}, \quad N_v(t) \\ &= \left(N_v(0) - \frac{\Lambda_v}{\mu_v(r,b)} \right) e^{-\mu_v(r,b)t} + \frac{\Lambda_v}{\mu_v(r,b)}. \end{aligned} \tag{6}$$

The *a priori* bounds in (6) allow us to conclude that the local solution is bounded. Indeed, one has

$$N_h(t) \leq \max \left\{ N_h(0) ; \frac{\Lambda_h}{\mu_h} \right\}, \quad N_v(t) \leq \max \left\{ N_v(0) ; \frac{\Lambda_v}{\mu_v(r,b)} \right\}, \ \forall\, t \in [0, T).$$

Moreover, for all $t \in [0, T)$,

$$N_h(0) \leq \frac{\Lambda_h}{\mu_h} \Longrightarrow N_h(t) \leq \frac{\Lambda_h}{\mu_h}, \text{ and } N_v(0) \leq \frac{\Lambda_v}{\mu_v(r,b)} \Longrightarrow N_v(t) \leq \frac{\Lambda_v}{\mu_v(r,b)}.$$

The last implication, together with the inequality in (6) show that Ω is a positively invariant, attractive and absorbing set for System (4) and that every local solution of (4) is actually global in time. The proof is achieved. □

Thanks to Theorem 1 it is sufficient to study System (4) in Ω. Hence the System (4) is biologically and mathematically well-posed [15].

In order to focus mainly on the role of PTNs repellent effect and the mosquito feeding preference, without loss of generality, we assume that all the mosquitoes target humans beings and the entire human population is protected by bed-nets which may have lost progressively their repellent power. That is, we assume $\theta = b = 1$ and r is variable. Based on this assumption, the probabilities P_h and P_v coincide and become

$$P(r) = \frac{1-r}{1-r[1-(\mu_{v1}+rC)]}, \quad 0 \le r \le 1, \tag{7}$$

and the corresponding (when $b = 1$) mortality rate of mosquitoes becomes $\mu_v(r, 1) \equiv \mu_v(r)$, where,

$$\mu_v(r) = \mu_{v0} + \mu_{v1} + Cr, \quad 0 \le r \le 1. \tag{8}$$

3 Influence of the feeding preference (π) and repellent rate (r) on the long-term dynamics

To put more emphasis on the role of mosquito feeding preference rate and pyrethroids repellent effect, instead of using the control/basic reproduction number as usual, we investigate hereafter, the long-run and bifurcating behaviors of the system based on the threshold parameters emanating from the vector-bias π and the treated bed net repellent rate r. Nonetheless, direct links with the corresponding thresholds using instead the control reproduction number $\mathcal{R}_0$ (whose expression will be given shortly) of the model can be easily established.

3.1 Effects of mosquito feeding preference (π) on the asymptotic and bifurcation analysis

Proposition 1 *Define the following vector-bias threshold parameter.*

$$\pi_1 = \frac{(\mu_h + \gamma_h + \delta + u)\mu_v^2(r)\Lambda_h}{m_v m_h P(r)^2 \mu_h \Lambda_v}. \tag{9}$$

Then the disease-free equilibrium E_0 is locally asymptotically stable if the mosquito feeding preference $\pi < \pi_1$. It is unstable if $\pi > \pi_1$.

Proof The disease-free equilibrium of the System (4) is

$$E_0 = \left(S_h^0, I_h^0, R_h^0, S_v^0, I_v^0\right) = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, \frac{\Lambda_v}{\mu_v(r)}, 0\right).$$

Before proceeding, we recall that according to the computational method in [33], the reproduction number $\mathcal{R}_0$ of System (4) is defined as the spectral radius of the *next*

generation matrix FV^{-1}, where:

$$F = \begin{pmatrix} 0 & m_h P(r) \\ \pi m_v P(r) \dfrac{\Lambda_v}{\Lambda_h} \dfrac{\mu_h}{\mu_v(r)} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\mu_h + \gamma_h + \delta + u) & 0 \\ 0 & \mu_v(r) \end{pmatrix}.$$

That is,

$$\mathcal{R}_0 = \frac{P(r)\sqrt{\pi}}{\mu_v(r)} \sqrt{\frac{m_v m_h \mu_h \Lambda_v}{(\mu_h + \gamma_h + \delta + u)\Lambda_h}}. \tag{10}$$

Moreover, notice that with the notation in (9), we have

$$\mathcal{R}_0 = \pi/\pi_1.$$

The local asymptotic stability of the disease-free equilibrium E_0 when $\pi < \pi_1$ is established by calculating the characteristic polynomial of the Jacobian of System (4) at E_0 giving by

$$\frac{1}{\pi_1}(x + \mu_h)\,(x + \mu_v(r))\,(x + \mu_h + \sigma) \times \left(x^2 + (\mu_h + \gamma_h + \delta + u + \mu_v(r))\,x + \mu_v(r)(\mu_h + \gamma_h + \delta + u)\,(\pi_1 - \pi)\right).$$

Clearly, whenever $\pi < \pi_1$, all the roots the above equation have negative real parts, and there exists exactly one root with positive real part whenever $\pi > \pi_1$. □

Now, we examine the existence of endemic equilibrium points and perform a bifurcation analysis of the model to gain more insights into the asymptotic dynamics of the model. If an endemic equilibrium of System (4) is denoted by $EE = \left(S_h^*, I_h^*, R_h^*, S_v^*, I_v^*\right)$, then its components satisfy the following relations:

$$S_h^* = \frac{(\mu_h + \gamma_h + \delta + u)\Lambda_h(\mu_h + \sigma)}{\lambda_h^*(r,\pi)\left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h\,(\gamma_h + u(1-p))\right] + (\mu_h + \gamma_h + \delta + u)\mu_h(\mu_h + \sigma)},$$

$$I_h^* = \frac{\lambda_h^*(r,\pi)\Lambda_h(\mu_h + \sigma)}{\lambda_h^*(r,\pi)\left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h\,(\gamma_h + u(1-p))\right] + (\mu_h + \gamma_h + \delta + u)\mu_h(\mu_h + \sigma)},$$

$$R_h^* = \frac{\lambda_h^*(r,\pi)\Lambda_h\,(\gamma_h + u(1-p))}{\lambda_h^*(r,\pi)\left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h\,(\gamma_h + u(1-p))\right] + (\mu_h + \gamma_h + \delta + u)\mu_h(\mu_h + \sigma)},$$

$$S_v^* = \frac{\Lambda_v}{\lambda_v^*(r,\pi) + \mu_v(r)}, \quad I_v^* = \frac{\lambda_v^*(r,\pi)\Lambda_v}{\left(\lambda_v^*(r,\pi) + \mu_v(r)\right)\mu_v(r)}.$$

With,

$$\lambda_v^*(r,\pi) = \frac{\pi m_v P(r)(\mu_h + \sigma)\lambda_h^*(r,\pi)}{\lambda_h^*(r,\pi)\left[\pi(\mu_h + \sigma) + \gamma_h + u(1-p)\right] + (\mu_h + \gamma_h + \delta + u)(\mu_h + \sigma)},$$

and $\lambda_h^*(r,\pi)$ is the positive roots of the quadratic equation

$$B_2[\lambda_h^*(r)]^2 + B_1\lambda_h^*(r) + B_0 = 0. \tag{11}$$

The coefficients B_2, B_1 et B_0 are as follows

$$\begin{cases} B_2 = \mu_v\left[\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right] \\ \quad \left[\pi m_v P(r)(\mu_h+\sigma)+\mu_v\left(\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right)\right] > 0, \\ B_1 = \dfrac{1}{\pi_1}(\mu_h+\gamma_h+\delta+u)\mu_v^2(r)\dfrac{(\mu_h+\sigma)}{\mu_h} \\ \quad \left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right](\pi_1 G(\pi)-\pi), \\ B_0 = \dfrac{1}{\pi_1}(\mu_h+\gamma_h+\delta+u)^2\mu_v^2(r)(\mu_h+\sigma)^2(\pi_1-\pi), \end{cases}$$

with,

$$G(\pi) = \frac{\pi m_v P(r)\mu_h(\mu_h+\sigma)+2\mu_v(r)\mu_h\left[\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right]}{\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]} > 0. \tag{12}$$

Solving for π the equation $G(\pi)=1$ gives the solution

$$\pi^* = \frac{\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)-\mu_h\left(\gamma_h+u(1-p)\right)\right]}{\mu_h(\mu_h+\sigma)\left(m_v P(r)+2\mu_v(r)\right)}. \tag{13}$$

Hereafter, we shall assume that the number π^* is positive so that

$$(\mu_h+\delta)(\mu_h+\sigma) > \mu_h\left(\gamma_h+u(1-p)\right).$$

This latter assumption is actually not a limitation of our work, because Fig. 2 shows that, using the model parameters as in Table 1, the number π^* is always positive, and more importantly, it is greater than one as it should be.

Now, we can state the following result about the existence equilibrium points for System (4). In order to emphasize on the importance of mosquito biting preference for infectious hosts, we use the mosquito host choice rate π as bifurcation parameter to show that the backward bifurcation may occur at the threshold value π_1.

Theorem 2 *Assume that*

$$\begin{aligned} &\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right] \\ &-\mu_h(\mu_h+\sigma)\left(m_v P(r)+2\mu_v(r)\right)\pi_1 > 0. \end{aligned}$$

Then, the equation $\pi_1 G(\pi)=\pi$ of variable π admits a unique solution π_G given by

$$\pi_G = \frac{2\mu_v(r)\mu_h\left[\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right]\pi_1}{\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]-\mu_h(\mu_h+\sigma)\left(m_v P(r)+2\mu_v(r)\right)\pi_1}.$$

If Moreover $G < 1$, *then* $\pi_G < \pi_1$, *and if* $\pi \leq \pi_G$, *then* $B_1 \geq 0$; *else if* $\pi > \pi_G$ *then* $B_1 < 0$.

Proof It is easy to show that π_G solves the equation $\pi_1 G(\pi) = \pi$, and as $G < 1$, we have $\pi_G = \pi_1 G(\pi_G) < \pi_1$. For the proof of the other last statement, we set $f(\pi) = (\pi_1 G(\pi) - \pi)$. By the assumption of the theorem, we have

$$f'(\pi) = \frac{-\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h(\gamma_h+u(1-p))\right]+\mu_h(\mu_h+\sigma)(m_v P(r)+2\mu_v(r))\pi_1}{\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h(\gamma_h+u(1-p))\right]}$$
$$< 0.$$

The conclusion follows because: whenever $\pi \leq \pi_G$, $f(\pi) \geq 0$, otherwise ($\pi > \pi_G$) $f(\pi) < 0$. □

Lemma 1 *Let* $\Delta(\pi) = B_1^2 - 4B_0B_2$, *then there exists a unique number* $\pi^c \in (\pi_G, \pi_1)$ *such that* $\Delta(\pi^c) = 0$.

Proof Suppose $\pi \in (\pi_G, \pi_1)$ thus $\pi > \pi_G$, according to Theorem 2, we have $B_1 < 0$. The derivative of $\Delta(\pi)$ with respect to π gives,

$$\Delta'(\pi) = 2B_1B_1' - 4B_2B_0' > 0, \text{ since } B_1' < 0,\ B_1 < 0,\ B_0' < 0 \text{ and } B_2 > 0.$$

Moreover, as $f(\pi_G) = 0$, we have $\Delta(\pi_G) = -4B_0B_2 < 0$ and $\Delta(\pi_1) = B_1^2 > 0$. Thus, there is a unique $\pi^c \in (\pi_G, \pi_1)$ such that $\Delta(\pi^c) = 0$. This ends the proof. □

Now, we can state the following result about the existence and the number of equilibrium points for (4). In order to emphasize on the importance of mosquito biting preference for infectious hosts, we use the mosquito biting preference parameter π as bifurcation parameter such that the backward bifurcation occurs at the threshold value π_1.

Theorem 3 *: The following statements hold:*

1) *Suppose* $\pi \leq \pi_G$, *then system* (4) *has no endemic equilibrium.*
2) *Suppose* $\pi_G < \pi < \pi_1$ *and:*
   a) *if* $\pi = \pi^c$, *then system* (4) *has a unique endemic equilibrium;*
   b) *if* $\pi < \pi^c$, *then system* (4) *has no endemic equilibrium point;*
   c) *if* $\pi > \pi^c$, *then system* (4) *has two endemic equilibrium points;*
3) *Suppose* $\pi > \pi_1$, *then System* (4) *has a unique endemic equilibrium.*

Proof We have the following equivalence: $\pi = \pi^c \Leftrightarrow \Delta(\pi = \pi^c) = B_1^2 - 4B_0B_2 = 0$. From Theorem 2, if $\pi \leq \pi_G$, then $B_1 \geq 0$. Else if $\pi > \pi_G$ then $B_1 < 0$. Now, if $\pi \leq \pi_G$, then thanks to Theorem 2, we have $B_1 \geq 0$. Moreover, by the Theorem 2, $\pi \leq \pi_G$ implies $\pi < \pi_1$, thus $B_0 > 0$. As $B_1 \geq 0$ and $B_0 > 0$. Thus, Item 1 is established. The conditions $\pi_G < \pi < \pi_1$ and $\pi = \pi^c$, are equivalent to $B_1 < 0$, $B_0 > 0$ and $B_1^2 - 4B_0B_2 = 0$. The former implies that $B_1 < 0$ leading to $\lambda_h^*(r, b) = -B_1/(2B_2) > 0$. This proves Item 2*a*). Notice that $\pi_G < \pi < \pi_1$ is

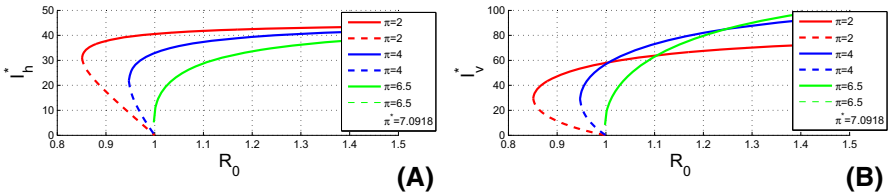

Fig. 1 Bifurcation diagrams using the mosquito biting rate as bifurcation parameter, whenever $r = 0.9$: The remaining parameters are given in Table 1. The solid lines represent stability, the dotted lines represent instability. The computed threshold value for the vector-bias preference is $\pi^* = 7.0918$, such that when $\pi < \pi^*$, backward bifurcation occurs, see Fig. 1. When $\pi \geq \pi^*$ the forward bifurcation occurs. These results illustrate Theorem 4. Moreover, we note that, by increasing the vector feeding preference rate π, backward bifurcation is eliminated whenever $\pi \geq \pi^*$, and the latter alleviates the control of malaria. It further highlights the fact that, increasing the parameter of attractiveness to infectious humans reduces the number of infectious human at the endemic level , see Fig. 1a. On the other hand increasing the parameter of attractiveness to infectious humans increases the infectious vectors at the endemic level, see Fig. 1b

equivalent to $B_1 < 0$ and $B_0 > 0$. For $\pi_G < \pi < \pi_1$, since $\Delta(\pi_G) = -4B_0B_2 < 0$, $\Delta(\pi^c) = 0$ and $\Delta(\pi_1) = B_1^2 > 0$. We get $\Delta(\pi) < 0$ whenever $\pi < \pi^c$, then system (4) has no endemic equilibrium. On the other hand, $\Delta(\pi) > 0$ whenever $\pi^c < \pi < \pi_1$, then System (4) has two endemic equilibrium points, and Items 2*b*) and 2*c*) are established. Suppose $\pi > \pi_1$, then $B_0 < 0$ and Item 3 follows. □

One may notice from this Theorem 3 that, as suspected earlier, the parameter π_1 serves as the bifurcation parameter for the backward bifurcation to occur. Theorem 4 below actually confirms that such bifurcation phenomenon emerges.

Theorem 4 *Let π^* be the number given in* (13)*. Then System* (4) *exhibits at $\pi = \pi_1$ (or equivalently at $\mathcal{R}_0 = 1$) the following bifurcation behavior:* 1- *The backward bifurcation if $\pi < \pi^*$.*

2- *The forward bifurcation if $\pi > \pi^*$.*

Proof See Appendix A.

The results in Theorem 4 are numerically confirmed by Fig. 1 which plots the infected human component $I*_h$ of the endemic equilibrium point versus $\mathcal{R}_0$. Because the illustrations are similar if one chooses to plot any other component of the endemic equilibrium against $\mathcal{R}_0$. □

Remark 1 It is worthwhile that π^* is a mosquito biting preference threshold value for the existence and elimination of the backward bifurcation in Theorem 4. The latter phenomenon has important disease-control implications as it asserts that reducing the biting preference ratio below its threshold is not enough for disease elimination and that further control measures are needed to bring the epidemic under control.

3.2 Effects of repellent rate (r) on the asymptotic behavior and bifurcation analysis

Lemma 2 *Let us consider the reproduction number $\mathcal{R}_0(r)$ as a function of the repellent rate r. If $\mathcal{R}_0(0) > 1$ then there exists a unique number $r_1 \in]0, 1[$, such that $\mathcal{R}_0^2(r_1) = 1$.*

Proof Set $h(r) = R_0^2(r) - 1$, The derivative of h with respect to r gives,

$$h'(r) = 2\mathcal{R}_0(r)\mathcal{R}_0'(r) = \frac{2\pi m_v m_h \mu_h \Lambda_v P(r)}{(\mu_h + \gamma_h + \delta + u)\Lambda_h(\mu_v)^3}\left[\frac{\partial P(r)}{\partial r}\mu_v(r) - CP(r)\right].$$

Direct computations show that

$$\partial P(r)/\partial r = \frac{rC\,(r-2) - \mu_{v1}}{[1 - r(1 - \mu_{v1} - Cr)]^2} \leq 0. \tag{14}$$

We conclude that

$$h'(r) < 0.$$

Thus, h is strictly decreasing in $]0, 1[$. Moreover, $h(0) = R_0^2(0) - 1 > 0$ and $h(1) = -1 < 0$. Therefore, there exists a unique number $r_1 \in]0, 1[$ such that $\mathcal{R}_0^2(r_1) = 1$. This ends the proof. □

Proposition 2 *The disease-free equilibrium E_0 is locally asymptotically stable if the repellent effect $r > r_1$. It is unstable if $r < r_1$.*

Proof The local asymptotic stability of the disease-free equilibrium E_0 when $r > r_1$ is established by showing that the characteristic polynomial of the Jacobian of System (4) evaluated at E_0 is

$$(x + \mu_h)\,(x + \mu_v)\,(x + \mu_h + \sigma)\left(x^2 + (\mu_h + \gamma_h + \delta + u + \mu_v)\,x + \mu_v(\mu_h + \gamma_h + \delta + u)\left(1 - \mathcal{R}_0^2(r)\right)\right).$$

Since $h(r) = \mathcal{R}_0^2(r) - 1 < 0$ is equivalent to $r > r_1$, it is straightforward that, if $r > r_1$, all the roots the above equation have negative real parts, and there exists exactly one root with positive real part whenever $r < r_1$. This ends the proof.

Let us recall that the coefficients B_2, B_1 and B_0 of Eq. 11 take the form

$$\begin{cases} B_2 = \mu_v\left[\pi(\mu_h + \sigma) + \gamma_h + u(1-p)\right] \\ \quad \left[\pi m_v P(r)(\mu_h + \sigma) + \mu_v\left(\pi(\mu_h + \sigma) + \gamma_h + u(1-p)\right)\right] > 0, \\ B_1 = (\mu_h + \gamma_h + \delta + u)\mu_v^2(r)\dfrac{(\mu_h + \sigma)}{\mu_h} \\ \quad \left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h\left(\gamma_h + u(1-p)\right)\right]\left(G(r) - \mathcal{R}_0^2(r)\right), \\ B_0 = (\mu_h + \gamma_h + \delta + u)^2\mu_v^2(r)(\mu_h + \sigma)^2\left(1 - \mathcal{R}_0^2(r)\right), \end{cases}$$

with,

$$G(r) = \frac{\pi m_v P(r)\mu_h(\mu_h+\sigma) + 2\mu_v(r)\mu_h\left[\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right]}{\mu_v(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]}. \quad (15)$$

□

Theorem 5 *Assume that*

$$m_v\mu_h(\mu_h+\sigma)(\mu_h+\gamma_h+\delta+u)\Lambda_h - C\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right] < 0, \text{ and } G(0)-\mathcal{R}_0^2(0) < 0.$$

Then, the equation $\left(G(r)-\mathcal{R}_0^2(r)\right)=0$ *has a unique solution* r_G *in the interval* $(0,1)$. *Moreover, one has:*

(i) *If* $\left(G(r_1)-\mathcal{R}_0^2(r_1)\right)<0$, *then* $r_G > r_1$.
(ii) *If* $r \geq r_G$, *then* $B_1 \geq 0$.
(iii) *If* $r < r_G$ *then* $B_1 < 0$.

Proof Set $H(r) = \left(G(r)-\mathcal{R}_0^2(r)\right)$, The derivative of H with respect to r gives,

$$H'(r) = \frac{m_v\mu_h(\mu_h+\sigma)(\mu_h+\gamma_h+\delta+u)\Lambda_h - C\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]}{(\mu_h+\gamma_h+\delta+u)\Lambda_h\mu_v^2(r)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]} \times\left[\frac{\partial P(r)}{\partial r}\mu_v(r) - CP(r)\right].$$

By the assumption of the theorem 5 and the inequation 14, we have $H'(r) > 0$. Thus, H is strictly increasing in $]0,1[$. Moreover, $H(0) = G(0)-\mathcal{R}_0^2(0) < 0$ and $H(1) = \frac{2\mu_v(1)\mu_h\left[\pi(\mu_h+\sigma)+\gamma_h+u(1-p)\right]}{\mu_v(1)\left[(\mu_h+\delta)(\mu_h+\sigma)+\mu_h\left(\gamma_h+u(1-p)\right)\right]} > 0$. Therefore, there exists a unique number $r_G \in (0,1)$ such that $\left(G(r)-\mathcal{R}_0^2(r)\right)=0$. □

The proofs of (i), (ii) and (iii) follow readily because it suffices to see that the following three statements hold, respectively. (i) If $H(r_1) < 0$, then $r_1 < r_G$. (ii) $r \geq r_G$ implies $H(r) \geq 0$ and $B_1 \geq 0$. (iii) Whenever $r < r_G$, we have $B_1 < 0$.

Since $B_2 > 0$, the number of positive roots of Equation (11) (and of course the number of endemic equilibrium of system (4)) are determined by the sign of B_0 and B_1 which, in turn depend on the values of $\mathcal{R}_0$ and G. Let $\Delta(r) = B_1^2 - 4B_0B_2$ denote the discriminant of (11).

Lemma 3 *Let* $\Delta(r) = B_1^2 - 4B_0B_2$. *By the assumption of the theorem 5, then there exists a unique number* $r^c \in (r_1, r_G)$ *such that* $\Delta(r^c) = 0$.

Proof Suppose $r \in (r_1, r_G)$ thus $r < r_G$, according to the theorem 5, we have $B_1 < 0$. As $H'(r) > 0$, we get $B_1'(r) > 0$. The derivative of $\Delta(r)$ with respect to r gives,

$$\Delta'(r) = 2B_1B_1' - 4B_2B_0' < 0, \text{ since } B_1' > 0,\ B_1 < 0,\ B_0' > 0 \text{ and } B_2 > 0.$$

Moreover, $\Delta(r_G) = -4B_0B_2 < 0$ and $\Delta(r_1) = B_1^2 > 0$. Thus, there is a unique $r^c \in (r_1, r_G)$ such that $\Delta(r^c) = 0$. This ends the proof. □

We can now state the result about the existence and the number of equilibrium points for the system (4). In order to emphasize on the importance of the repellent effect, we use the repellent effect parameter r as bifurcation parameter such that the backward bifurcation occurs at the threshold value r_1.

Theorem 6 *: The following statements hold.*

1) *Suppose $r \geq r_G$, then system (4) has no endemic equilibrium.*
2) *Suppose $r_1 < r < r_G$ and:*
   a) *if $r = r^c$, then system (4) has a unique endemic equilibrium;*
   b) *if $r > r^c$, then system (4) has no endemic equilibrium point;*
   c) *if $r < r^c$, then system (4) has two endemic equilibrium points;*
3) *Suppose $r \leq r_1$, then system (4) has a unique endemic equilibrium.*

Proof We have the following equivalence: $r = r^c \Leftrightarrow \Delta(r = r^c) = B_1^2 - 4B_0B_2 = 0$. According to the theorem 5, If $r < r_G$, then $B_1 < 0$ and if $r \geq r_G$, then $B_1 \geq 0$. If $r < r_1$, according to the proof of lemme 2, we have $h(r) > 0$, thus $B_0 < 0$ and if $r \geq r_1$, then $B_0 \geq 0$.

1) If $r \geq r_G$, then we have $B_1 \geq 0$. Moreover, $r \geq r_G$, implies $r > r_1$, thus $B_0 > 0$. As $B_1 \geq 0$ and $B_0 > 0$. Thus $1.a)$ is established.
2) a) The conditions $r_1 < r < r_G$ and $r = r^c$, are equivalent to $B_1 < 0$, $B_0 > 0$ and $B_1^2 - 4B_0B_2 = 0$. The former implies that $B_1 < 0$ leading to $\lambda_h^*(r, b) = -B_1/(2B_2) > 0$. This proves of $1.b)i)$.
2) b) and c) $r_1 < r < r_G$ is equivalent to $B_1 < 0$ and $B_0 > 0$. For $r_1 < r < r_G$, since $\Delta(r_G) = -4B_0B_2 < 0$, $\Delta(r^c) = 0$ and $\Delta(r_1) = B_1^2 > 0$. We get $\Delta(r) < 0$ whenever $r > r^c$, then system (4) has no endemic equilibrium. On the other hand, $\Delta(r) > 0$ whenever $r_1 < r < r^c$, then System (4) has two endemic equilibrium points, and $2)b)$ and $c)$ are established.
3) If $r = r_1$, then $r < r_G$. Thus, we get $B_0 = 0$ and $B_1 < 0$. For $r < r_1$, we get $B_0 < 0$. Therefore $1.c)$ is proven.

Let's now consider π^* as a function of the repellent rate r. This is

$$\pi^* := \pi^*(r) = \frac{\mu_v(r)(\mu_h + \delta)(\mu_h + \sigma) - \mu_h\mu_v(r)\,(\gamma_h + u(1-p))}{\mu_h(\mu_h + \sigma)\,(m_v P(r) + 2\mu_v(r))}.$$

It is not difficult to prove that π^* is monotone increasing on the interval [0,1]. In fact,

$$\frac{\partial \pi^*}{\partial r} = \frac{1}{\mu_h(\mu_h + \sigma)} \frac{m_v\left(CP(r) - \mu_v(r)\dfrac{\partial P(r)}{\partial r}\right)\left[(\mu_h + \delta)(\mu_h + \sigma) - \mu_h\,(\gamma_h + u(1-p))\right]}{(m_v P(r) + 2\mu_v(r))^2}.$$

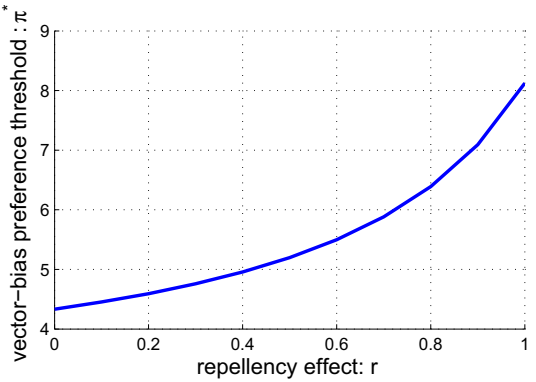

Fig. 2 The threshold value for the mosquito biting preference π^* versus the repellent rate r. All parameters are given in Table 1 . We see that $\pi^* > 0$, furthermore we note that as the repellent effect increases, the threshold value for the mosquito biting preference π^* increases and vice versa

Using the expression of π^* in Equation (13) yields

$$\frac{\partial \pi^*}{\partial r} = \frac{1}{\mu_v(r)} \frac{m_v \left(C P(r) - \mu_v(r) \dfrac{\partial P(r)}{\partial r} \right) \pi^*}{(m_v P(r) + 2\mu_v(r))}.$$

On the other hand, we have

$$\frac{\partial P(r)}{\partial r} = \frac{rC\,(r-2) - \mu_{v1}}{[1 - r(1 - \mu_{v1} - Cr)]^2} \leq 0.$$

Thus $\partial \pi^*/\partial r > 0$. Now, we denote its lower and upper bounds respectively by

$$\pi_L^* = \frac{(\mu_{v0} + \mu_{v1})(\mu_h + \delta)(\mu_h + \sigma) - \mu_h(\mu_{v0} + \mu_{v1})\,(\gamma_h + u(1-p))}{\mu_h(\mu_h + \sigma)\,(m_v + 2(\mu_{v0} + \mu_{v1}))},$$

and

$$\pi_U^* = \frac{(\mu_{v0} + \mu_{v1} + C)(\mu_h + \delta)(\mu_h + \sigma) - \mu_h \mu_v(1)\,(\gamma_h + u(1-p))}{2\mu_h(\mu_h + \sigma)\,(\mu_{v0} + \mu_{v1} + C)}.$$

The monotone behavior of π^* as the function of the repellent effect rate r is illustrated in Fig. 2 □

Theorem 7 *: Assume that the feeding preference rate satisfies the relation:* $\max\left\{1, \pi_L^*\right\} < \pi < \pi_U^*$. *Then there exists a unique threshold value of the repellent effect* r^* *in* $(0, 1)$ *such that* $G(r^*) = 1$.

Proof Let $F(r) = G(r) - 1$, we recall that

$$G(r) = \frac{\pi m_v P(r) \mu_h(\mu_h + \sigma) + 2\mu_v(r)\mu_h \left[\pi(\mu_h + \sigma) + \gamma_h + u(1-p)\right]}{\mu_v(r) \left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h\,(\gamma_h + u(1-p))\right]}.$$

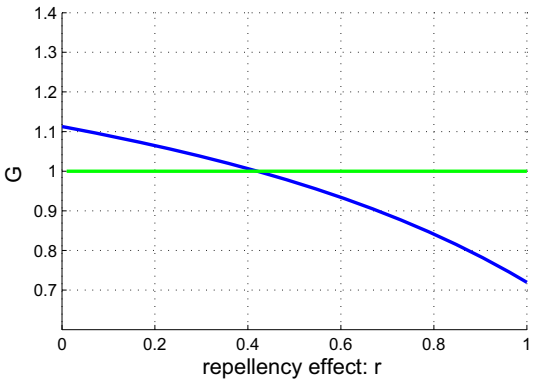

Fig. 3 The function $G(r)$ versus the repellent effect r. The mosquito feeding preference $\pi = 5$ is chosen such that it lies between $\pi_L^* = 4.3325$ and $\pi_U^* = 8.1235$, while all the other parameters are given as in Table 1. In this case, there exists a unique threshold value of the repellent effect $r^* = 0.4194$ in $]0, 1[$ which solves the equation $G(r^*) = 1$

Thus

$$F'(r) = \frac{\pi m_v \mu_h (\mu_h + \sigma)[\mu_v \dfrac{\partial P(r)}{\partial r} - C P(r)]}{\mu_v^2 \left[(\mu_h + \delta)(\mu_h + \sigma) + \mu_h (\gamma_h + u(1-p))\right]}.$$

Direct computations show that

$$\frac{\partial P(r)}{\partial r} = \frac{rC\,(r-2) - \mu_{v1}}{[1 - r(1 - \mu_{v1} - Cr)]^2} \leq 0. \tag{16}$$

We conclude that F is a decreasing function of repellent effect r. Moreover $\max\left\{1, \pi_L^*\right\} < \pi$ is equivalent to $F(0) > 0$, and $\pi_U^* > \pi$ is equivalent to $F(1) < 0$. Thus there is a unique r^* in $(0, 1)$ such that $F(r^*) = G(r^*) - 1 = 0$.

This achieves the proof. This result stated in Theorem 7 is numerically confirmed by Fig. 3. □

The following theorem summarizes the role that repellent effect plays on the bifurcation of system (4) for a suitable range of the mosquito feeding rate π.

Theorem 8 *Suppose* $\max\left\{1, \pi_L^*\right\} < \pi < \pi_U^*$. *Then System* (4) *exhibits at* $\mathcal{R}_0 = 1$*:*
1- *The backward bifurcation whenever* $r > r^*$.
2- *The forward bifurcation whenever* $r < r^*$.

Proof See Appendix B.

The results in theorem 8 are numerically confirmed by Fig. 4. In this figure, we have chosen to plot the infected human component I_h^* of the endemic equilibrium point versus $\mathcal{R}_0$, but the illustration is similar if one chooses to plot any other component of the endemic equilibrium against $\mathcal{R}_0$. □

4 Global stability of the equilibrium points

In this section, the global asymptotic stability of equilibrium points are proven under certain condition, using Lyapunov–LaSalle techniques and the second additive com-

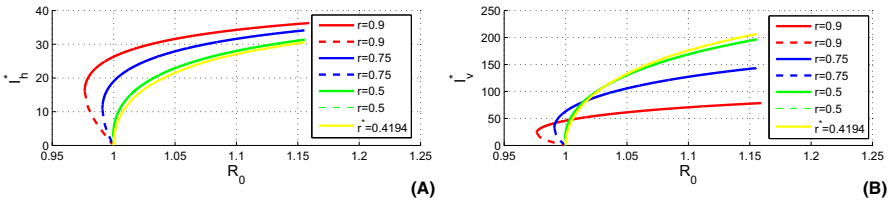

Fig. 4 Illustration of the bifurcation diagrams using the repellent effect as bifurcation parameter: The parameters are given in Table 1. The solid lines represent stability, the dotted lines represent instability. The calculation of π_U^* and π_L^* give $\pi_L^* = 4.3325$ and $\pi_U^* = 8.1235$. $\pi = 5$, thus we have $\pi_L^* < \pi < \pi_U^*$, therefore there exists the threshold of repellent rate. The computation of the threshold value for the repellent effect is $r^* = 0.4194$, such that for $r > r^*$, the backward bifurcation occurs, see Fig. 4. When $r \leq r^*$ the forward bifurcation occurs. These results illustrate theorem 8. Moreover, we note that, by decreasing the repellent effect value, backward bifurcation is eliminated as long as $r \leq r^*$, making malaria easier to be controlled. It further shows that, decreasing the repellent effect reduces the infectious population at the endemic level , see Fig. 4**a**. On the other hand decreasing the repellent effect increases the infectious vectors at the endemic level, see Fig. 4**b**

pound matrix method. This method is usually applied to three-dimensional systems. We expand its application to four-dimensional systems.

Theorem 9 *The disease-free equilibrium point E_0 is globally asymptotically stable (GAS) when $\pi \leq \pi_1 \mu_h^2/(\mu_h + \delta)^2$.*

Proof To prove that statement, Lyapunov–LaSalle techniques are used in Ω by considering the following Lyapunov function:

$$L = L(S_h, I_h, R_h, S_v, I_v) = \mu_v \mu_h I_h + m_h(\mu_h + \delta)P(r)I_v + \mu_v(r)\mu_h \times \left(S_h - S_h^0 \ln S_h\right) + m_h(\mu_h + \delta)P(r)\left(S_v - S_v^0 \ln S_v\right).$$

Direct, yet simple computations show that

$$\begin{aligned}\frac{dL}{dt} = &-\frac{\mu_v \mu_h^2 (S_h - S_h^0)^2}{S_h} - \frac{m_h P(r)(\mu_h + \delta)\mu_v(r)(S_v - S_v^0)^2}{S_v} \\ &+ \left[\frac{\mu_v \mu_h m_h P(r) S_h^0}{\pi I_h + S_h + R_h} - m_h P(r)(\mu_h + \delta)\mu_v(r)\right] I_v + \\ &\times \left[\frac{m_h P(r)(\mu_h + \delta)\pi m_v P_v S_v^0}{\pi I_h + S_h + R_h} - \mu_v \mu_h(\mu_h + \delta + \gamma_h + u)\right] \\ &\times I_h - \mu_v(r)\mu_h \rho u \left(\frac{S_h^0 - S_h}{S_h}\right) I_h - \mu_v(r)\mu_h \sigma \left(\frac{S_h^0 - S_h}{S_h}\right) R_h.\end{aligned}$$

Next, since $dN_h/dt \geq \Lambda_h - (\mu_h + \delta)N_h$, the application of Gronwall lemma yields a lower bound for N_h as follows: there is a T such that

$$N_h \geq \frac{\Lambda_h}{\mu_h + \delta}, \text{ for all } t \geq T.$$

Using the above lower bound for N_h, straightforward calculations and careful rearrangements lead to

$$\frac{dL}{dt} \leq -\frac{\mu_v(r)\mu_h^2(S_h - S_h^0)^2}{S_h} - \frac{m_h P(r)(\mu_h + \delta)\mu_v(r)(S_v - S_v^0)^2}{S_v} + \frac{(\mu_h + \delta)^2 m_v m_h P(r)P(r)\Lambda_v}{\Lambda_h \mu_v(r)} \left(\pi - \frac{\mu_h^2}{(\mu_h + \delta)^2}\pi_1\right) I_h,$$

for all $t \geq T$. Obviously, we have $\dot{L} \leq 0$ when $\pi \leq \pi_1 \mu_h^2/(\mu_h + \delta)^2$. Moreover it is easy to see that the largest invariant subset contained in the set

$$\mathcal{E} = \left\{(S_h, I_h, R_h, S_v, I_v) \in \Omega / \dot{L} = 0\right\}$$

is the disease-free equilibrium E_0. Thus the LaSalle Invariance Principle applies easily to L and E_0 is globally attractive in Ω. Hence, E_0 is GAS. □

Theorem 10 *The endemic equilibrium point EE of System (4) is globally asymptotically stable provided:*

$$\pi > \pi_1, \quad \frac{\Lambda_h}{2m_h P(r)} > \frac{\Lambda_v}{\mu_v(r)},$$

and

$$\mu_h > \max\left\{\sigma, \quad \gamma_h + u(1-\rho) + \max\{\mu_v(r), \delta + \gamma_h + u\} - \sigma, \right.$$
$$\left. \frac{(5\mu_v(r) + \sigma)\mu_v(r)\Lambda_h + 2m_h P(r)\Lambda_v \delta}{\mu_v(r)\Lambda_h - 2m_h P(r)\Lambda_v}\right\}.$$

The proof of the GAS of the endemic equilibrium EE uses the following instrumental result.

Theorem 11 (Vidyasagar [34], Theorem 3.1). *Consider the following $\mathcal{C}^1$ system:*

$$\begin{cases} \dfrac{dx}{dt} = f(x) & x \in \mathbb{R}^n \\ \dfrac{dy}{dt} = g(x, y) & x \in \mathbb{R}^m, \end{cases} \tag{17}$$

with an equilibrium point (x^*, y^*) ie., $f(x^*) = 0$ and $g(x^*, y^*) = 0$.

If x^* is globally asymptotically stable (GAS) in $\mathbb{R}^n$ for the system $dx/dt = f(x)$, and if y^* is (GAS) in $\mathbb{R}^m$ for the system $dy/dt = g(x^*, y)$ then (x^*, y^*) is (locally) asymptotically stable for (17). Moreover, if all the trajectories of (17) are forward bounded, then (x^*, y^*) is GAS for (17).

Now, we are going to show that System (4) has the triangular structure in Theorem 11.

Let $N_v(t) = S_v(t) + I_v(t)$. We have $dN_v/dt = \Lambda_v - \mu_v(r)N_v$, so that System (4) is equivalent to

$$\begin{cases} \dfrac{dN_v}{dt} = \Lambda_v - \mu_v(r)N_v \\ \dfrac{dS_h}{dt} = \Lambda_h + \sigma R_h - \lambda_h(r,\pi)S_h + \rho u I_h - \mu_h S_h \\ \dfrac{dI_h}{dt} = \lambda_h(r,\pi)S_h - (\mu_h + \gamma_h + \delta + u)I_h \\ \dfrac{dR_h}{dt} = (1-\rho)uI_h + \gamma_h I_h - (\sigma + \mu_h)R_h \\ \dfrac{dI_v}{dt} = \lambda_v(r)\,(N_v(t) - I_v(t)) - \mu_v(r)I_v. \end{cases} \tag{18}$$

We have $dN_v/dt = \Lambda_v - \mu_v(r)N_v$. Therefore, $N_v(t) \longrightarrow \Lambda_v/\mu_v(r)$ as $t \longrightarrow +\infty$. Thus, equilibrium $N_v^* = \Lambda_v/\mu_v(r,b)$ is globally asymptotically stable (GAS) in $\mathbb{R}$ for the system $dN_v/dt = \Lambda_v - \mu_v(r)N_v$. Note that the endemic equilibrium point for System (4) translates to the endemic equilibrium point for System (18) which we denote by $\widehat{EE} = (N_v^*, S_h^*, I_h^*, R_h^*, I_v^*)$. Therefore, the proof of the GAS EE for System (4) is equivalent to the proof of the GAS of $\widehat{EE}$ for System (18). To achieve the latter, if we set

$$x = N_v, \quad y = (S_h, I_h, R_h, I_v),$$

then System (18) takes the desirable triangular form in Theorem 11, with

$$f(x) = \Lambda_v - \mu_v(r)x, \quad g(x,y) = \begin{pmatrix} \Lambda_h + \sigma R_h - \lambda_h(r,\pi)S_h + \rho u I_h - \mu_h S_h \\ \lambda_h(r,\pi)S_h - (\mu_h + \gamma_h + \delta + u)I_h \\ (1-\rho)uI_h + \gamma_h I_h - (\sigma + \mu_h)R_h \\ \lambda_v(r,\pi)\,(N_v(t) - I_v(t)) - \mu_v(r)I_v \end{pmatrix}.$$

Moreover, since $x^* = N_v^*$ is GAS for $dx/dt = f(x)$, the GAS of $\widehat{EE}$ for $\mathcal{R}_0 > 1$ will be established as long as the GAS of the endemic equilibrium $\widehat{EE} = (S_h^*, I_h^*, R_h^*, I_v^*)$ of the following reduced System (19) (corresponding to $dy/dt = g(x^*, y)$ holds.

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h + \sigma R_h - \lambda_h(r,\pi)S_h + \rho u I_h - \mu_h S_h \\ \dfrac{dI_h}{dt} = \lambda_h(r,\pi)S_h - (\mu_h + \gamma_h + \delta + u)I_h \\ \dfrac{dR_h}{dt} = (1-\rho)uI_h + \gamma_h I_h - (\sigma + \mu_h)R_h \\ \dfrac{dI_v}{dt} = \lambda_v(r,\pi)\left(N_v^* - I_v(t)\right) - \mu_v(r)I_v. \end{cases} \tag{19}$$

One can easily show that the set,

$$\Omega_0 = \left\{(S_h(t), I_h(t), R_h(t), I_v(t)) \in \mathbb{R}^4_+ : S_h(t) + I_h(t) + R_h(t) \leq \frac{\Lambda_h}{\mu_h}, I_v(t) \leq \frac{\Lambda_v}{\mu_v(r)}\right\},$$

is positively invariant for the flow generated by (19) and that, thanks to Theorem (3), $\widetilde{EE} = (S_h^*, I_h^*, R_h^*, I_v^*)$ is the unique endemic equilibrium for (19) in the interior of Ω_0 when $\mathcal{R}_0 > 1$. On the other hand, since all the trajectories of (18) are forward bounded, then according to the Theorem 11, the GAS of $\widetilde{EE}$ for System (18) is complete if the GAS of $\widetilde{EE}$ for System (19) is established. This GAS of $\widetilde{EE}$ for System (19) is done in the following theorem.

Theorem 12 *Under the assumptions of Theorem 10, the endemic equilibrium point $\widetilde{EE}$ of System (19) is GAS in the interior of Ω_0.*

Proof See Appendix A. □

Remark 2 From the details of the proof of Theorem 12 in Appendix A, it is worth noticing that a different choice of the matrix P in (22) will lead to different sufficient conditions for the global stability of the endemic equilibrium of System (19). Therefore, the suitable statement of Theorem 12 is linked to the corresponding suitable choice of P.

5 Interplay between mosquito feeding preference and pyrethroids repellent effect on disease dynamics

5.1 Impact of mosquito feeding preference and pyrethroids repellent effect on the reproduction number

Here, we assess both theoretically and numerically the role of PTNs usage and mosquito biting preference for infectious host on the control reproduction number $\mathcal{R}_0$.

Theorem 13 *The use of PTNs has a positive impacts on malaria control by decreasing its reproduction number $\mathcal{R}_0$. Furthermore, the control reproduction number is a linear increasing function of mosquito biting preference for infectious hosts.*

Proof For $\mathcal{R}_0 \neq 0$, we have $P(r) \neq 0$, and

$$\frac{\partial \mathcal{R}_0}{\partial r} = \frac{-\pi m_v m_h \mu_h \Lambda_v P(r)}{(\mu_h + \gamma_h + \delta + u)\Lambda_h \mathcal{R}_0 (\mu_v)^3}\left[\frac{rC\,(r-2) - \mu_{v1}}{[1 - r(1 - \mu_{v1} - Cr)]^2} + CP(r)\right] < 0.$$

Furthermore, if $\mathcal{R}_0 = 0$, then $P(r) = 0$ and the disease is absent in the population. In the case where, $\mathcal{R}_0 > 0$, it is an increasing function of π in the sense that:

$$\frac{\partial \mathcal{R}_0}{\partial \pi} = \frac{1}{2\mathcal{R}_0}\frac{m_v m_h P(r)^2}{(\mu_h + \gamma_h + \delta + u)}\frac{\mu_h}{(\mu_v(r))^2}\frac{\Lambda_v}{\Lambda_h} > 0.$$

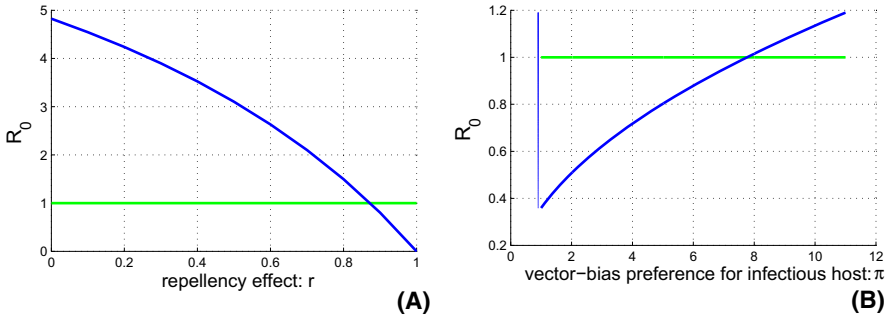

Fig. 5 **a** Relationship between the control reproduction number $\mathcal{R}_0$ and the repellent effect (r), for the chosen parameter $u = 0.06$. **b** Illustration of the curve of the control reproduction number versus mosquito biting preference for infectious host (π), for the chosen parameter $u = 0.06$. The remaining parameters are given in Table 1. **a** The minimum level of repellent effect required to contain malaria disease is $r = 0.87$. We note that as the repellent effect decreases, $\mathcal{R}_0$ increases and vice versa. **b** We note that as the mosquito biting preference for infectious host increases, $\mathcal{R}_0$ increases and vice versa. The maximum level of mosquito biting preference for infectious host required to contain malaria disease is $\pi = 0.78$.

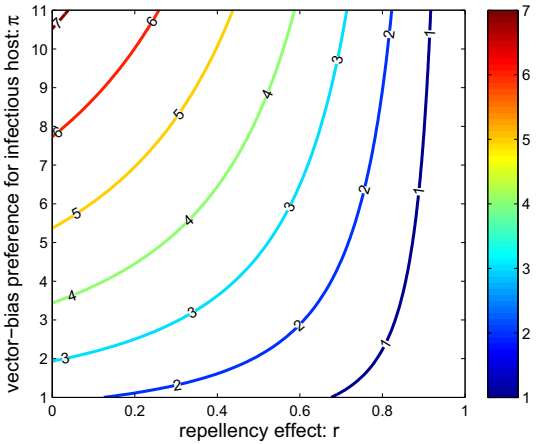

Fig. 6 Contour plot of the control reproduction number showing the potential impact of the interplay between pyrethroids treated bed-nets repellent effect and mosquito host choice on epidemic outbreak of malaria. $u = 0.06$. The remaining parameters are given in Table 1

These results are illustrated in the Figs. 5 and 6. □

We additionally assess the impact of parameters of the bed-net Model (4) on the control reproduction number $\mathcal{R}_0$, by computing the elasticity indexes of $\mathcal{R}_0$ with respect to parameter values given in Table 1. According to the approach proposed in [7, 23, 24, 26], the elasticity index of $\mathcal{R}_0$ with respect to a parameter p, where p is any of the parameters in Table 1 reflected in the expression of $\mathcal{R}_0$, is given by

$$\frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}.$$

Since these indexes quantify the ratio of relative changes on $\mathcal{R}_0$ in response to corresponding changes in the parameters, they can identify critical parameters for disease control. This approach states that the control reproduction number is most sensitive to

Table 2 Elasticity indexes of the control reproduction number $\mathcal{R}_0$.

Parameter	μ_{v1}	r	Λ_h	Λ_v	m_v	γ_h
Elasticity	−0.9423	−0.6943	−0.5	0.5	0.5	−0.000008
Parameter	m_h	μ_h	π	u	μ_{v0}	δ
Elasticity	0.5	0.5	0.5	−0.4991	−0.1523	−0.0008

the parameter with the largest elasticity index value and least sensitive to the parameter with the smallest elasticity index value. Table 2 displays the elasticity indexes of $\mathcal{R}_0$ to the 12 (twelve) parameters, arranged in decreasing magnitude order and hence decreasing sensitivity. As expected, the control reproduction number is most sensitive to the maximum mosquito ITNs-induced death rate μ_{v1} with an elasticity index of -0.9423. It is also highly sensitive to the probability that a vector is repelled by the insecticide (or mechanically blocked by the net) r. Qualitatively, $\mathcal{R}_0$ decreases by 9.423% for an increase in maximum mosquito ITNs-induced death rate of 10%, $\mathcal{R}_0$ is reduced by 6.943% when the repellent effect r is increased by 10% and a 10% increase the mosquito biting preference for infectious host, increases $\mathcal{R}_0$ by 5.000%. The public health implication of these investigation is that the use of insecticide-treated nets with high repellent rate and lethal effects, and of course vector control are important steps to take for the control of malaria.

5.2 Interplay between pyrethroids treated bed-nets repellent effect and mosquito host choice on malaria outcome

We define $f(r)$ so that $\mathcal{R}_0 < 1$ is equivalent to $\pi < f(r)$, thus

$$f(r) = \frac{\mu_v^2(r)(\mu_h + \gamma_h + \delta + u)\Lambda_h}{m_v m_h P(r)^2 \mu_h \Lambda_v}.$$

For any couple (r, π) verifying $\pi < f(r)$, we have $\mathcal{R}_0 < 1$. Thus , if $\pi \geq \pi^*$, the disease disappears in the population. If this requirement is not met, additional control strategies (such as indoor residual spraying and human treatment) might be needed to reduce malaria burden and control the disease. On the contrary, for any couple (r, π) verifying $\pi > f(r)$, we have $\mathcal{R}_0 > 1$, and malaria will persist in the population.

On the other hand, one can observe in Fig. 6 that, for a fix value of repellent effect, the reproduction number $\mathcal{R}_0$ increases when the mosquito biting preference increases. For a fix value of mosquito biting preference, we also note that as the repellent effect decreases, $\mathcal{R}_0$ increases and vice versa. Moreover, if the repellent effect rate $r \in [0, 0.67[$, irrespective of the value of the mosquito biting preference, $\mathcal{R}_0$ will never be brought below one, and consequently, malaria will never be eliminated. On the other hand, for any couple (r, π) in the set $]0.92, 1] \times [1, +\infty[$, $\mathcal{R}_0$ is less that the unity, and consequently, malaria can be controlled. All these investigations show the interplay between π and r and its impact on the epidemic outbreak potential of malaria. One could investigate further on the potential impacts of the interplay between π and r

on the long-term dynamics of the model by studying (for example) the infectious component I_h^* of the endemic equilibrium as a function of π and r, but due to the high complexity of that function I_h^* , we've skipped it and concentrate only on the control reproduction number $\mathcal{R}_0$.

6 Conclusion and discussions

For the best of the authors' knowledge, this is the first research work which models and analyzes a mathematical model of malaria dynamics that combines a pyrethroids treated bed-nets repellent effect with mosquito host choice. We have proposed and analyzed a dynamical PTNs model for malaria transmission, in which mosquito biting preference for infectious host are explicitly incorporated and their role on the long run of the malaria evolution is investigated. The results obtained have revealed the existence of a backward bifurcation for certain parameter values which implies that the reduction of mosquito host choice π below its threshold value π^* or the increase of the repellent effect r above its threshold value r^* is not enough to mitigate malaria evolution. Therefore, additional control strategies such as indoor residual spraying and treatment might be necessary to reduce malaria burden and control the disease. Moreover, the occurrence of a backward bifurcation has been shown to depend on the range of the mosquito biting preference for infectious host and repellent effect. Precisely, we have computed the mosquito biting preference for infectious host threshold value π^* and repellent effect threshold value r^* necessary to study the existence of both the endemic equilibrium points and backward bifurcation when the mosquito host choice $\pi < \pi_1$ or the repellent effect $r > r_1$.

We have shown that the control reproduction number is highly sensitive to the maximum mosquito ITNs-induced death rate μ_{v1} and the repellent parameter (r). We have also proved that the control reproduction number $\mathcal{R}_0$ increases when the mosquito biting preference for infectious hosts increases. The global asymptotic stability of equilibrium points has been established under certain conditions, using Lyapunov–LaSalle techniques and a geometric approach. We have demonstrated that the PTNs utilization (repellent effect) has a positive impact in reducing the control reproduction number. This allowed us to conclude that PTNs utilization decreases the reproduction number, thus reducing the disease burden and helping to control malaria. We have shown that malaria eradication might be more difficult to be achieved, if the mosquito biting preference for infectious host is less than the mosquito biting preference threshold value or if the repellent effect is greatest than the repellent effect threshold value. As a whole, the following results summarize the theoretical and numerical analyses of our model:

(i) The incorporation of mosquito host choice in a dynamical PTNs model for malaria transmission is one of the causes of backward bifurcation, which prevents the classical requirement that, bringing the control reproduction number below unity is sufficient to control malaria.
(ii) If the mosquito biting preference for infectious host threshold value $\pi^* > 0$, then π^* is a increasing function of the repellent probability r.

(iii) By increasing the vector feeding preference value (or decreasing the repellent effect), backward bifurcation is eliminated as soon as $\pi \geq \pi^*$ (or $r \leq r^*$), making the control of malaria easier to be achieved.
(iv) Increasing the mosquito biting preference parameter to infectious humans or decreasing the repellent effect, reduces the number of infectious humans at the endemic level of the disease.

It is worth noticing that among the many researches and experiments in the field, Ogoma et al. [27] conducted an experiment to evaluate the effects of two pyrethroids insecticides (Transfluthrin and Metofluthrin coils), by showing a scenario where 100 mosquitoes approach a house, deterrence comes into play in the first instance and only approximately 62 and 70 mosquitoes enter the house with Transfluthrin and Metofluthrin coils respectively. After mosquitoes were repelled and exited the house, 35 and 39 remained inside the house with Transfluthrin and Metofluthrin coils respectively. Of those, approximately 1 and 3 mosquitoes managed to acquire a blood meal. The authors concluded that through deterrence, irritancy or excito-repellency and feeding inhibition of pyrethroid coils, more than 97% of the mosquitoes would be prevented from contacting humans inside houses before mortality is even considered. Although this experiment and few others could be enough to demonstrate and pave the way to the estimation of repellent effect of pyrethroids and feeding behavior of mosquitoes, more experiments are still needed to actually quantify these effects for better implementation of this research work. Based on this, we emphasize that the values of π and r used in this work, though chosen in the relevant ranges, were purely for illustrative purposes. Thus, we recommend both the vector feeding preference threshold values π_1 and π^*, and the repellent effect thresholds values r_1 and r^* could be quantified empirically in laboratory so that more practicability could be expected from this research work.

Appendices

Appendix A: Proof of Theorem 4

For simplicity, denote $t_h = m_h P(r)$, $t_v = m_v P(r)$. Then, using t_h as the bifurcation parameter, and solving for t_h the equation $\pi_1(t_h) = \pi$ yields

$$t_h := t_h^* = \frac{(\mu_h + \gamma_h + \delta + u)(\mu_v(r, b))^2 \Lambda_h}{\pi t_v \mu_h \Lambda_v}. \tag{20}$$

The Jacobian matrix of System (4) evaluated at the DFE E_0 can then be rewritten as

$$J(E_0) = \begin{pmatrix} -\mu_h & u\rho & \sigma & 0 & -t_h^* \\ 0 & -(\mu_h + \gamma_h + \delta + u) & 0 & 0 & t_h^* \\ 0 & \gamma_h + u(1-\rho) & -(\mu_h + \sigma) & 0 & 0 \\ 0 & -\pi t_v \dfrac{\Lambda_v}{\Lambda_h} \dfrac{\mu_h}{\mu_v(r)} & 0 & -\mu_v(r) & 0 \\ 0 & \pi t_v \dfrac{\Lambda_v}{\Lambda_h} \dfrac{\mu_h}{\mu_v(r)} & 0 & 0 & -\mu_v(r) \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are

$\lambda_1 = -\mu_h,\ \lambda_2 = -\mu_v(r), \lambda_3 = -(\mu_h + \sigma), \lambda_4 = 0, \quad \lambda_5 = -\big(\mu_h + \gamma_h + \delta + u + \mu_v(r)\big)$.

Now, we denote by $w = (w_1, w_2, w_3, w_4, w_5)^T$ a right eigenvector corresponding to the zero eigenvalue. Then,

$$w = \left(\left(-\frac{\gamma_h + u(1-\rho)}{\mu_h + \sigma} - \frac{\mu_h + \delta}{\mu_h}\right) w_2, w_2, \frac{\gamma_h + u(1-\rho)}{\mu_h + \sigma} w_2, - \pi t_v \frac{\Lambda_v}{\Lambda_h} \frac{\mu_h}{\mu_v^2(r)} w_2,\ \pi t_v \frac{\Lambda_v}{\Lambda_h} \frac{\mu_h}{\mu_v^2(r)} w_2\right)^T$$

Now, it is not difficult to prove that the left eigenvector $v = (v_1, v_2, v_3, v_4, v_5)$ corresponding to the zero eigenvalue such that $v.w = 1$ is given by

$$v = \left(0,\ \frac{\pi t_v \Lambda_v \mu_h}{\Lambda_h(\mu_v(r) + \mu_h + \gamma_h + \delta + u)},\ 0,\ 0,\ \frac{(\mu_h + \gamma_h + \delta + u)\mu_v(r)}{\mu_v(r) + \mu_h + \gamma_h + \delta + u}\right).$$

Let $f_i (i = 1, 2, 3, 4, 5)$ be the vector on the right hand side of system (4). The non-vanishing second-order partial derivatives of $f_i (i = 1, 2, 3, 4, 5)$ at the disease-free equilibrium E_0 are:

$$\begin{aligned}
\frac{\partial^2 f_2}{\partial I_h \partial I_v}(E_0, t_h^*) &= \frac{-\pi \mu_h t_h^*}{\Lambda_h}; \quad \frac{\partial^2 f_2}{\partial R_h \partial I_v}(E_0, t_h^*) \\
&= \frac{-\mu_h t_h^*}{\Lambda_h}; \quad \frac{\partial^2 f_4}{\partial S_h \partial I_h}(E_0, t_h^*) = \frac{-\pi t_v \Lambda_v \mu_h^2}{\mu_v(r, b)\Lambda_h^2}; \\
\frac{\partial^2 f_4}{\partial I_h \partial R_h}(E_0, t_h^*) &= \frac{-\pi t_v \Lambda_v \mu_h^2}{\mu_v(r)\Lambda_h^2}; \quad \frac{\partial^2 f_4}{\partial I_h \partial S_v}(E_0, t_h^*) \\
&= \frac{\pi t_v \mu_h}{\Lambda_h}; \quad \frac{\partial^2 f_4}{\partial I_h^2}(E_0, t_h^*) = \frac{-2\pi^2 t_v \Lambda_v \mu_h^2}{\mu_v(r)\Lambda_h^2}.
\end{aligned}$$

Using t_h^* given by Eq. (20), we obtain:

$$\begin{aligned}
\frac{\partial^2 f_2}{\partial I_h \partial I_v}(E_0, t_h^*) &= -\frac{(\mu_h + \gamma_h + \delta + u)\mu_v^2(r)}{t_v \Lambda_v}; \qquad \frac{\partial^2 f_2}{\partial R_h \partial I_v}(E_0, t_h^*) \\
&= -\frac{(\mu_h + \gamma_h + \delta + u)\mu_v^2(r)}{\pi t_v \Lambda_v}.
\end{aligned}$$

According to definitions of the coefficients $\mathcal{A}$ and $\mathcal{B}$ in [6] (Theorem 4.1), it follows that:

$$\mathcal{A} = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_0, t_h^*), \quad x = (S_h, I_h, R_h, S_v, I_v),$$
$$= v_2 \left(2w_2 w_5 \frac{\partial^2 f_2}{\partial I_h \partial I_v}(E_0, t_h^*) + 2w_3 w_5 \frac{\partial^2 f_2}{\partial R_h \partial I_v}(E_0, t_h^*) \right) +$$
$$+ v_4 \left(2w_1 w_2 \frac{\partial^2 f_4}{\partial S_h \partial I_h}(E_0, t_h^*) + w_2^2 \frac{\partial^2 f_4}{\partial I_h^2}(E_0, t_h^*) + 2w_2 w_3 \frac{\partial^2 f_4}{\partial I_h \partial R_h}(E_0, t_h^*) \right.$$
$$\left. + 2w_2 w_4 \frac{\partial^2 f_4}{\partial I_h \partial S_v}(E_0, t_h^*) \right),$$

and

$$\mathcal{B} = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial t_h}(E_0, t_h^*) = v_2 w_5 \frac{\partial^2 f_2}{\partial I_v \partial t_h}(E_0, t_h^*) = v_2 w_5.$$

Substituting the eigenvectors and the above non-zero partial derivatives into $\mathcal{A}$ and $\mathcal{B}$ leads us to

$$\mathcal{A} = -\frac{2(\mu_h + \gamma_h + \delta + u)\mu_v(r)\,(m_v P(r) + 2\mu_v(r))}{\pi m_v P(r) \Lambda_v\,(\mu_h + \gamma_h + \delta + u + \mu_v(r))} \left[\pi - \pi^*\right],$$

and

$$\mathcal{B} = \pi m_v P(r) \frac{\Lambda_v}{\Lambda_h} \frac{\mu_h}{\mu_v(r)\,(\mu_h + \gamma_h + \delta + u + \mu_v(r))}. \tag{21}$$

Obviously, the coefficient $\mathcal{B}$ is positive. When $\pi < \pi^*$, $\mathcal{A}$ is positive. It follows that model (4) undergoes a backward bifurcation when $\pi < \pi^*$. If the reversed inequality holds, then the system exhibits a forward bifurcation.

Appendix B: Proof of Theorem 8

Let's recall that

$$\mathcal{A} = -\frac{2(\mu_h + \gamma_h + \delta + u)\mu_v(r)\,(m_v P(r) + 2\mu_v(r))}{\pi m_v P(r) \Lambda_v\,(\mu_h + \gamma_h + \delta + u + \mu_v(r))} \left[\pi - \pi^*\right]$$

When $r > r^*$, then $G(r) < G(r^*)$, thus $G < 1$. Therefore $G(\pi) < G(\pi^*)$.

$G(\pi) < G(\pi^*)$ is equivalent to $\pi < \pi^*$, thus we have $\mathcal{A} > 0$. therefore (4) undergoes a backward bifurcation when $r > r^*$. If the reversed inequality holds, then the system exhibits a forward bifurcation.

Appendix C: Proof of Theorem 12

We use the geometric approach to establish the global asymptotic stability of $\widetilde{EE}$. First, note that Ω_0 is simply connected in $\mathbb{R}^4_+$ and System (19) has a unique endemic equilibrium in the interior of Ω_0 whenever $\pi_1 < \pi$. Moreover, the instability of the disease-free equilibrium implies the uniform persistence of System (19) see [13], i.e. there exists a constant $c > 0$ such that any solution $x(t, x_0) = (S_h(t), I_h(t), R_h(t), I_v(t))$ of (19) with the initial condition $x_0 = (S_h(0), I_h(0), R_h(0), I_v(0))$ in the interior of Ω_0 satisfies the inequality

$$\min\left\{\lim_{t\to+\infty} inf\, S_h(t),\quad lim_{t\to+\infty} inf\, I_h(t),\quad lim_{t\to+\infty} inf\, R_h(t),\right.$$
$$\left. lim_{t\to+\infty} inf\, I_v(t)\right\} > c.$$

The uniform persistence together with boundedness of Ω_0 is equivalent to the existence of a compact absorbing set K in the interior of Ω_0 [16]. Therefore, it remains to find conditions for which the Bendixson's criterion are verified. The Jacobian matrix J of System (19) is

$$J = \begin{pmatrix} -\dfrac{m_h P(r) I_v(\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} - \mu_h & \dfrac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} + u\rho \\ \dfrac{m_h P(r) I_v(\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} & -\dfrac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} - (\mu_h + \delta + \gamma_h + u) \\ 0 & \gamma_h + u(1-\rho) \\ -\dfrac{\pi m_v P(r)\left(\dfrac{\Lambda_v}{\mu_v} - I_v\right) I_h}{(\pi I_h + R_h + S_h)^2} & \dfrac{\pi m_v P(r)\left(\frac{\Lambda_v}{\mu_v} - I_v\right)(S_h + R_h)}{(\pi I_h + R_h + S_h)^2} \\ \dfrac{m_h P_h I_v S_h}{(\pi I_h + R_h + S_h)^2} + \sigma & -\dfrac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)} \\ -\dfrac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} & \dfrac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)} \\ -\mu_h - \sigma & 0 \\ -\dfrac{\pi m_v P(r)\left(\dfrac{\Lambda_v}{\mu_v} - I_v\right) I_h}{(\pi I_h + R_h + S_h)^2} & -\dfrac{\pi m_v P(r) I_h}{(\pi I_h + R_h + S_h)} - \mu_v \end{pmatrix}$$

The second additive compound matrix $J^{[2]}$ of J is

$$J^{[2]} = \begin{pmatrix} b_{11} & -\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} & \frac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)} \\ \gamma_h + u(1-\rho) & b_{22} & 0 \\ l_2 & -l_3 & b_{33} \\ 0 & \frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} & 0 \\ l_3 & 0 & \frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} \\ 0 & l_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -\sigma - \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} & \frac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)} & 0 \\ l_1 & 0 & k_1 \\ 0 & l_1 & k_2 \\ b_{44} & 0 & k_3 \\ -l_3 & b_{55} & k_4 \\ -l_2 & \gamma_h + u(1-\rho) & b_{66} \end{pmatrix},$$

where,

$$b_{11} = -\frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} - \mu_h - \frac{\pi m_h P_h(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} - (\mu_h + \delta + \gamma_h + u),$$

$$b_{22} = -\frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} - \mu_h - \mu_h - \sigma,$$

$$b_{33} = -\frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} - \mu_h - \frac{\pi m_v P(r) I_h}{(\pi I_h + R_h + S_h)} - \mu_v(r),$$

$$b_{44} = -\frac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} - (\mu_h + \delta + \gamma_h + u) - \mu_h - \sigma,$$

$$b_{55} = -\frac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} - (\mu_h + \delta + \gamma_h + u) - \frac{\pi m_v P(r) I_h}{(\pi I_h + R_h + S_h)} - \mu_v(r),$$

$$b_{66} = -\mu_h - \sigma - \frac{\pi m_v P(r) I_h}{(\pi I_h + R_h + S_h)} - \mu_v(r),$$

$$l_1 = \frac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} + u\rho,\ l_2 = \frac{\pi m_v P(r) \left(\frac{\Lambda_v}{\mu_v(r)} - I_v\right)(S_h + R_h)}{(\pi I_h + R_h + S_h)^2},$$

$$l_3 = \frac{\pi m_v P(r) \left(\frac{\Lambda_v}{\mu_v(r)} - I_v\right) I_h}{(\pi I_h + R_h + S_h)^2},$$

$$k_1 = \frac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)},\ k_2 = \sigma + \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2},\ k_3 = -\frac{m_h P(r) S_h}{(\pi I_h + R_h + S_h)},$$
$$k_4 = -\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2}.$$

Let $x = (S_h, I_h, R_h, I_v)$. Choose now the matrix

$$P = P(S_h, I_h, R_h, I_v) = \begin{pmatrix} k_1/I_h & 0 & 0 & 0 & 0 & 0 \\ 0 & k_1/I_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1/I_h & 0 & 0 \\ 0 & 0 & k_2/I_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2/I_v & 0 \\ 0 & 0 & 0 & 0 & 0 & k_2/I_v \end{pmatrix}, \quad (22)$$

where k_1 and k_2 are two undetermined positive constants. Thus we define the matrix P_f by

$$\left(P_{ij}(x)\right)_f = \left(\frac{\partial P_{ij}(x)}{\partial x}\right)^T . f(x) = \nabla_{P_{ij}(x)} . f(x) \quad where, \quad f(x) = \left(\dot{S}_h, \dot{I}_h, \dot{R}_h, \dot{I}_v\right)^T .$$

Then, $P_f P^{-1} = \text{diag}\left(-\frac{\dot{I}_h}{I_h}, -\frac{\dot{I}_h}{I_h}, -\frac{\dot{I}_h}{I_h}, -\frac{\dot{I}_v}{I_v}, -\frac{\dot{I}_v}{I_v}, -\frac{\dot{I}_v}{I_v}\right)$, and the matrix $N = P_f P^{-1} + P J^{[2]} P^{-1}$ can be rewritten in the block form

$$N(S_h, I_h, R_h, I_v) = \left(N_{ij}\right)_{1 \le i,j \le 4}, \quad (23)$$

where,

$$N_{11} = b_{11} - \frac{\dot{I}_h}{I_h}, \qquad N_{12} = \left(-\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2}, \quad -\sigma - \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2}\right),$$

$$N_{13} = \left(\frac{k_1}{k_2}\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h}, \quad \frac{k_1}{k_2}\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h}\right), \qquad N_{14} = 0,$$

$$N_{21} = (\gamma_h + u(1-\rho), \quad 0)^T, \quad N_{22} = \begin{pmatrix} b_{22} - \frac{\dot{I}_h}{I_h} & l_1 \\ \frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} & b_{44} - \frac{\dot{I}_h}{I_h} \end{pmatrix},$$

$$N_{23} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_{24} = \left(\frac{k_1}{k_2}\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h}, \quad \frac{k_1}{k_2}\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h}\right)^T,$$

$$N_{31} = \left(l_2 \frac{k_2}{k_1} \frac{I_h}{I_v}, \quad l_3 \frac{k_2}{k_1} \frac{I_h}{I_v} \right)^T, \quad N_{32} = \begin{pmatrix} -l_3 \frac{k_2}{k_1} \frac{I_h}{I_v} & 0 \\ 0 & -l_3 \frac{k_2}{k_1} \frac{I_h}{I_v} \end{pmatrix},$$

$$N_{33} = \begin{pmatrix} b_{33} - \frac{\dot{I}_v}{I_v} & l_1 \\ \frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} & b_{55} - \frac{\dot{I}_v}{I_v} \end{pmatrix},$$

$$N_{34} = \left(\sigma + \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2}, \quad -\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} \right)^T,$$

$$N_{41} = 0, \qquad N_{42} = \left(l_3 \frac{k_2}{k_1} \frac{I_h}{I_v}, \quad -l_2 \frac{k_2}{k_1} \frac{I_h}{I_v} \right),$$

$$N_{43} = (0, \quad \gamma_h + u(1-\rho)), \qquad N_{44} = b_{66} - \frac{\dot{I}_v}{I_v}.$$

Now, define a vector norm $|.|$ in $\mathbb{R}_+^6$ by

$$|(y_1, y_2, y_3, y_4, y_5, y_6)| = \max \{|y_1|, |y_2| + |y_3|, |y_4| + |y_5|, |y_6|\}.$$

Let $\zeta(.)$ denote the Lozinskii measure with respect to the above defined norm given by

$$\zeta(N) = \lim_{h \to 0^+} \frac{|I + hN| - 1}{h}.$$

Using a similar argument as in [29], we have the following estimate

$$\zeta(N) \leq \sup \{g_1, g_2, g_3, g_4\},$$

where,

$$g_i = \zeta_1(N_{ii}) + \sum_{j=1, j \neq i}^{4} |N_{ij}|,$$

$|N_{ij}| (i \neq j, i, j = 1, 2, 3, 4)$ are matrix norms with respect to the L^1 vector norm defined for a generic matrix $A = (a_{ij})$, $|A| = \max_{1 \leq k \leq n} \sum_{j=1}^{n} |a_{jk}|$ and ζ_1 is the Lozinskii measure of A with respect to that L^1 norm. To calculate the values of g_i, we firstly obtain that,

$$\zeta_1(N_{11}) = -\frac{m_h P(r) I_v (\pi I_h + R_h)}{(\pi I_h + R_h + S_h)^2} - \mu_h - \frac{\pi m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2} - (\mu_h + \delta + \gamma_h + u) - \frac{\dot{I}_h}{I_h},$$

$$|N_{12}| = \sigma + \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2},$$

$$|N_{13}| = \frac{k_1}{k_2} \frac{m_h P_h(r, b) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h}, \quad |N_{14}| = 0, \quad |N_{21}| = \gamma_h + u(1 - \rho),$$

$$\zeta_1(N_{22}) = -2\mu_h - \sigma - \frac{\dot{I}_h}{I_h},$$

$$|N_{23}| = 0, |N_{24}| = 2\frac{k_1}{k_2} \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h},$$
$$|N_{31}| < \pi m_v P(r) \frac{k_2}{k_1} \left(\frac{\Lambda_v}{\mu_v(r)} - I_v \right) \frac{I_h}{(\pi I_h + R_h + S_h)\, I_v},$$

$$|N_{32}| < 2\pi m_v P(r) \frac{k_2}{k_1} \left(\frac{\Lambda_v}{\mu_v(r)} - I_v \right) \frac{I_h}{(\pi I_h + R_h + S_h)\, I_v},$$

$$\zeta_1(N_{33}) = -\mu_h - \mu_v(r, b) - m_h P(r) \frac{I_h}{(\pi I_h + R_h + S_h)} - \frac{\dot{I}_v}{I_v},$$

$$|N_{34}| = \sigma + 2\frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)^2}, \quad |N_{41}| = 0,$$

$$|N_{42}| < \pi m_v P(r) \frac{k_2}{k_1} \left(\frac{\Lambda_v}{\mu_v(r)} - I_v \right) \frac{I_h}{(\pi I_h + R_h + S_h)\, I_v},$$

$$|N_{43}| = \gamma_h + u(1 - \rho), \quad \zeta_1(N_{44}) = -\mu_h - \sigma - \frac{\pi m_v P(r) I_h}{(\pi I_h + R_h + S_h)} - \mu_v(r) - \frac{\dot{I}_v}{I_v}.$$

Furthermore, from System (19), we have,

$$\frac{\dot{I}_h}{I_h} = \frac{m_h P(r) I_v S_h}{(\pi I_h + R_h + S_h)\, I_h} - (\mu_h + \delta + \gamma_h + u),$$

and

$$\frac{\dot{I}_v}{I_v} = \pi m_v P(r) \left(\frac{\Lambda_v}{\mu_v(r)} - I_v \right) \frac{I_h}{(\pi I_h + R_h + S_h)\, I_v} - \mu_v(r).$$

Choosing $k_2 = 2k_1$ yields:

$$g_1 < -\mu_h + \sigma,$$

$$g_2 = -(\mu_h + \sigma) + \delta + 2\gamma_h + u(2-\rho),$$

$$g_3 < 5\frac{\dot{I}_v}{I_v} + 5\mu_v(r) + \sigma - \mu_h + 2m_h P(r)\frac{\Lambda_v}{\mu_v(r)}\frac{\mu_h + \delta}{\Lambda_h},$$

$$g_4 < \frac{\dot{I}_v}{I_v} + \mu_v(r) + \gamma_h + u(1-\rho) - (\mu_h + \sigma).$$

Set

$$d_1 = \mu_h - \sigma, \quad d_2 = \sigma + \mu_h - 2\gamma_h - u(2-\rho) - \delta,$$
$$d_3 = \mu_h + \sigma - \gamma_h - u(1-\rho) - \mu_v(r), \quad d_4 = \mu_h - \sigma - 5\mu_v(r) - 2\frac{m_h P(r)\Lambda_v(\mu_h + \delta)}{\mu_v(r)\Lambda_h},$$

and

$$d = \min\{d_1, d_2, d_3, d_4\}.$$

From condition (10), we have $d > 0$ and

$$g_1 \le -d, \quad g_2 \le -d, \quad g_3 < 5\frac{\dot{I}_v}{I_v} - d, \quad g_4 < \frac{\dot{I}_v}{I_v} - d,$$

The above relations hold alongside each solution $(S_h(t), I_h(t), R_h(t), I_v(t))$ to system (19) corresponding to the initial condition $(S_h(0), I_h(0), R_h(0), I_v(0)) \in K$, with K being a compact and absorbing in Ω_0 shown earlier, when $t > T$, we have

$$\begin{aligned}
&\frac{1}{t}\int_0^t g_1\,\mathrm{d}s \le -d,\\
&\frac{1}{t}\int_0^t g_2\,\mathrm{d}s \le -d,\\
&\frac{1}{t}\int_0^t g_3\,\mathrm{d}s < \frac{1}{t}\int_0^T g_3\,\mathrm{d}s + \frac{5}{t}\ln\frac{I_v(t)}{I_v(T)} - d\frac{t-T}{t},\\
&\frac{1}{t}\int_0^t g_4\,\mathrm{d}s < \frac{1}{t}\int_0^T g_4\,\mathrm{d}s + \frac{1}{t}\ln\frac{I_v(t)}{I_v(T)} - d\frac{t-T}{t}.
\end{aligned}$$

Moreover, we have,

$$\begin{aligned}
\frac{1}{t}\int_0^t \zeta(N)\,\mathrm{d}s \le \sup\Bigg\{&-d, \quad \frac{1}{t}\int_0^T g_3\,\mathrm{d}s + \frac{5}{t}\ln\frac{I_v(t)}{I_v(T)} - d\frac{t-T}{t},\\
&\frac{1}{t}\int_0^T g_4\,\mathrm{d}s + \frac{1}{t}\ln\frac{I_v(t)}{I_v(T)} - d\frac{t-T}{t}\Bigg\}.
\end{aligned}$$

Therefore,

$$\overline{q} = \lim_{t \to +\infty} \sup \sup_{x_0 \in K} \frac{1}{t} \int_0^t \zeta \left(N(x(s, x_0)) \right) \mathrm{d}s \leq -d < 0.$$

This proves the GAS of $\widetilde{EE}$.

References

1. Abboubakar, H., Buonomo, B., Chitnis, N.: Modelling the effects of malaria infection on mosquito biting behaviour and attractiveness of humans. Ricerche Mat. **65**, 329–34 (2016)
2. Agusto, F.B., Del Valle, S.Y., Blayneh, K.W., Ngonghala, C.N., Goncalves, M.J., Li, N., Zhao, R., Gong, H.: The impact of bed-net use on malaria prevalence. J. Theor. Biol. **320**, 58–65 (2013)
3. Aldila, D., Seno, H.: A population dynamics model of mosquito-borne disease transmission, focusing on mosquitoes biased distribution and mosquito repellent use. Bull. Math. Biol. **81**, 4977–5008 (2019)
4. Birget, P.L.G., Koella, J.C.: An epidemiological model of the effects of insecticide-treated bed nets on malaria transmission. PLoS ONE **10**, e0144173 (2015)
5. Buonomo, B.: Analysis of a malaria model with mosquito host choice and bed-net control. Int. J. Biomath. **6**, 1550077 (2015)
6. Castillo-Chavez, C., Song, B.: Dynamical models of tuberculosis and their applications. Math. Biosci. Eng. **1**, 361–404 (2004)
7. Chitnis, N., Hyman, J.M., Cushing, J.M.: Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bull. Math. Biol. **70**, 1272–1296 (2008)
8. Chitnis, N., Schapira, A., Smith, T., Steketee, R.: Comparing the effectiveness of malaria vector-control interventions through a mathematical model. Am. J. Trop. Med. Hyg. **83**, 230–240 (2010)
9. Curtis, C.F., Lines, J.D.: Impregnated fabrics against malaria mosquitoes. Parasitol. Today 1–147 (1985)
10. Chamchod, F., Britton, N.F.: Analysis of a vector-bias model on malaria transmission. Bull. Math. Biol. **73**, 639–657 (2011)
11. Darriet, F., Robert,V., Vien, T., Carnevale, P.: Evaluation of the efficacy of permethrin-impregnated intact and perforated mosquito nets against vectors of malaria. Who/VBC/84.899, World health organization, Geneva (1984). Accessed 30 Nov 2019
12. Davidson, G., Draper, C.: Field studies of some of the basic factors concerned in the transmission of malaria. Trans. R. Soc. Trop. Med. Hyg. **47**, 522–535 (1953)
13. Freedman, H.I., Ruan, S., Tang, M.: Uniform persistence and flows near a closed positively invariant set. J. Dyn. Differ. Equ. **6**, 583–600 (1994)
14. Goodman, C.A., Mills, A.J.: The evidence base on the cost-effectiveness of malaria control measures in Africa. Health. Policy. Plann. **14**, 301–312 (1999)
15. Hethcote, H.W.: The mathematics of infectious diseases. SIAM Rev. **42**, 599–653 (2000)
16. Hutson, V., Schmitt, K.: Permanence and the dynamics of biological systems. Math. Biosci. **111**, 1–71 (1992)
17. Killeen, G.F., Smith, T.A.: Exploring the contributions of bed nets, cattle, insecticides and excito-repellency to malaria control: a deterministic model of mosquito host-seeking behaviour and mortality. Trans. R. Soc. Trop. Med. Hyg. **101**, 867–880 (2007)
18. Killeen, G.F., Seyoum, A., Gimnig, J.E., Stevenson, J.C., Drakeley, C.J., Chitnis, N.: Made-to-measure malaria vector control strategies: rational design based on insecticide properties and coverage of blood resources for mosquitoes. Malar. J., 13–146 (2014). https://doi.org/10.1186/1475-2875-13-146
19. Kim, S., Masud, M.A., Cho, G., Jung, I.H.: Analysis of a vector-bias effect in the spread of malaria between two different incidence areas. J. Theor. Biol. **419**, 66–76 (2017)
20. Lacroix, R., Mukabana, W.R., Gouagna, L.C., Koella, J.C.: Malaria infection increases attractiveness of humans to mosquitoes. PLoS Biol. **3**, 298 (2005)
21. Lines, J., Myamba, J., Curtis, C.J.: Experimental hut trials of permethrin-impregnated mosquito nets and eave curtains against malaria vectors in Tanzania. Med. Vet. Entomol. **1**, 37–51 (1987)

22. Lindsay, S., Adiamah, J., Armstrong, J.: The effect of permethrin-impregnated bed nets on house entry by mosquitoes (Diptera: Culicidae) in The Gambia. Bull. Entomol. R. **82**, 49–55 (1992)
23. Mohammed-Awel, J., Numfor, E.: Optimal insecticide-treated bed-net coverage and malaria treatment in a malaria-HIV co-infection model. J. Biol. Dyn. **11**, 160–191 (2017)
24. Moore, S., Shrestha, S., Tomlinson, K.W., Vuong, H.: Predicting the effect of climate change on African trypanosomiasis: integrating epidemiology with parasite and vector biology. J. R. Soc. Interface **9**, 817–830 (2012)
25. Ngonghala, C.N., Del Valle, S.Y., Zhao, R., Mohammed-Awel, J.: Quantifying the impact of decay in bed-net efficacy on malaria transmission. J. Theor. Biol. **363**, 247–261 (2014)
26. Ngonghala, C.N., Ngwa, G.A., Teboh-Ewungkem, M.I.: Persistent oscillations and backward bifurcation in a malaria model with varying human and mosquito populations: implications for control. J. Math. Biol. **70**, 1581–622 (2015)
27. Ogoma, S.B., Lorenz, L.M., Ngonyani, H., Sangusangu, R., Kitumbukile, M., Kilalangongono, E.M., Simfukwe, E.T., Mseka, A., Mbeyela, E., Roman, D., Moore, J., Kreppel, K., Maia, M.F., Moore, S.J.: An experimental hut study to quantify the effect of DDT and airborne pyrethroids on entomological parameters of malaria transmission. Malar. J. **13**, 131 (2014)
28. Rivera, R.C., Barradas, I.: Vector preference annihilates backward bifurcation and reduces endemicity. Bull. Math. Biol. **81**, 4447–4469 (2019)
29. Robert, H.: Martin, Jr: logarithmic norms and projections applied to linear differential systems. J. Math. Anal. Appl. **45**, 432–454 (1974)
30. Smith, L., Maire, N., Ross, A., Penny, M., Chitnis, N., Schapira, A., Studer, A., Genton, B., Lengeler, C., Tediosi, F., De Savigny, D., Tanner, M.: Towards a comprehensive simulation model of malaria epidemiology and control. Parasitology **135**, 1507–1516 (2008)
31. Tsanou, B., Kamgang, J.C., Lubuma, J.M.C., Houpa Danga, E.D.: Modeling pyrethroid repellency and its role on the bifurcation analysis for a bed net malaria model. Chaos Solitons Fractals **136**, 109809 (2020)
32. Teboh-Ewungkem, MI.: Malaria control: the role of local communities as seen through a mathematical model in a changing population-Cameroon. In: Advances in Disease Epidemiology (J.M.T& Z.M., eds), Nova Science Publishers, pp. 103–140 (2009)
33. Van den Driessche, P., Watmough, J.: Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. **180**, 29–48 (2002)
34. Vidyasagar, M.: Decomposition techniques for large-scale systems with nonadditive interactions: stability and stabilizability. IEEE Trans. Autom. Control **25**, 773–779 (1980)
35. Wang, X., Zhao, X.-Q.: A periodic vector-bias malaria model with incubation period. SIAM J. Appl. Math. **77**, 181–201 (2017)
36. Wang, X., Zhao, X.-Q.: A climate-based malaria model with the use of bed nets. J. Math. Biol. **1**, 1–25 (2018)
37. WHO.: Pesticides and their application for the control of vectors and pests of public health importance, 6th edn. WHO/CDS/NTD/WHOPES/GCDPP/2006.1, Geneva. https://apps.who.int/iris/handle/10665/69223 (2006). Accessed 26 Nov 2020
38. WHO: Guidances for malaria control vector. https://apps.who.int/iris/bitstream/handle/10665/310862/9789241550499-eng.pdf (2019). Accessed 28 Apr 2022
39. WHO: World malaria report. https://www.who.int/publications-detail/world-malaria-report (2019). Accessed 17 Mar 2020
40. Xu, Z., Zhao, X.-Q.: A vector-bias malaria model with incubation period and diffusion. Discrete Contin. Dyn. Syst. Ser. B. **17**, 2615–2634 (2012)
41. Xiulei, J., Shuwan, J., Daozhou, G.: Mathematical analysis of the Ross–Macdonald model with quarantine. Bull. Math. Biol. **82**, 47 (2020)
42. Zaim, M., Aitio, A., Nakashima, N.: Safety of pyrethroid-treated mosquito nets. Med. Vet. Entomol. **14**, 1–5 (2000)