fractal and fractional

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Hermite–Hadamard-Type Inequalities via Caputo–Fabrizio Fractional Integral for h-Godunova–Levin and (h_1, h_2)-Convex Functions

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Abstract: This note generalizes several existing results related to Hermite–Hadamard inequality using h-Godunova–Levin and (h_1, h_2)-convex functions using a fractional integral operator associated with the Caputo–Fabrizio fractional derivative. This study uses a non-singular kernel and constructs some new theorems associated with fractional order integrals. Furthermore, we demonstrate that the obtained results are a generalization of the existing ones. To demonstrate the correctness of these results, we developed a few interesting non-trivial examples. Finally, we discuss some applications of our findings associated with special means.

Keywords: h-Godunova–Levin; (h_1, h_2)-convexity; Hermite–Hadamard inequality; Caputo–Fabrizio operator

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Citation: Afzal, W.; Abbas, M.; Hamali, W.; Mahnashi, M.M.; De la Sen, M. Hermite–Hadamard-Type Inequalities via Caputo–Fabrizio Fractional Integral for h-Godunova–Levin and (h_1, h_2)-Convex Functions. *Fractal Fract.* **2023**, *7*, 687. https://doi.org/10.3390/fractalfract7090687

Academic Editor: Haci Mehmet Baskonus

Received: 23 August 2023
Revised: 9 September 2023
Accepted: 14 September 2023
Published: 15 September 2023

1. Introduction

Fractional calculus has grown exponentially in popularity, which enables the definition of fractional derivatives and fractional integrals in a variety of ways. It is worth noting that Leibniz and L'Hospital proposed the first fractional calculus idea in 1695. The origins and principles of fractional calculus have recently been the subject of intense research, particularly in light of the shortcomings of conventional calculus. The study of fractional order integrals and derivatives, as well as their applications in real and complex domains, is the focus of fractional calculus. Classical analysis arithmetic is required to generate more realistic results with fractional analysis. A variety of mathematical models can be solved using fractional differential equations and integral equations. Because they are special instances of fractional order mathematical models, mathematical models with fractional order have more broad and accurate conclusions than conventional mathematical models. In contrast to integer orders, fractional theory allows for the handling of any number of orders, real or integer, making it a more suitable method. We can calculate the precise stability and uniqueness of fractional differential equations using fractional integral inequalities. In today's world, almost no field of nonlinear disciplines or research is unaffected by fractional methods and instruments. Various fields of engineering have numerous applications, such as electrical engineering, control theory, mechanical engineering, viscoelasticity, rheology, optics, and physics; see Refs. [1–4].

A variety of methods and creative concepts are employed by researchers to generalize and extend the theory of convexity. There have been many developments, generalizations, and extensions of convexity in recent years, enabling us to solve problems arising both with concrete and applied sciences. In recent years, the study of convexity has been increasingly broadened by its relation to inequalities. There are numerous such inequalities reported in the literature as a result of applications of convexity in both pure and applied sciences. As a result of its many applications in mathematics, convexity is often used as the basis for estimating error bounds for a wide variety of problems; see Ref. [5]. An example of this is when the trapezoidal formula for numerical integration is used to estimate errors due to convexity; see Ref. [6]. Among others, nonlinear programming problems can be applied to special means; see Ref. [7]. As a result of Jensen's discovery of the first convex inequality, a long history of research has been conducted on convex inequalities. Convex inequalities have found applications in solving problems, optimizing, and theorizing probability. On the other hand, generalized convexity mapping can address a wide range of issues in both pure and nonlinear analysis. Consequently, it is possible to compare the Hermite–Hadamard inequality to a convex function that satisfies generalised convexity. Various inequalities are constructed by using related classes of convexity, such as Simpson, Ostrowski, Opial, Bullen, and the famous Hermite–Hadamard, which has been extended to various classes. There are a wide variety of convex classes and integral operators used in the construction of these inequality, including the standard Riemann integral, Caputo–Fabrizio, Riemann–Liouville, and k-fractional operators. Caputo fractional derivatives were first introduced by Michele Caputo in 1967; see Ref. [8]. As the kernel in the Caputo operator is not singular, it can be transformed into an integral using the Laplace transformation. The Caputo derivatives and integrals are generally used when physical models are presented because the physical interpretation is too clear and precise. The integral operator has recently been associated with integral inequalities, and several authors have utilized this notion and developed different inequalities using related classes of convexity. Butt et al. [9] have developed various inequalities using the Caputo–Fabrizio operater via exponetially convex mappings. As a result of the Caputo operator, Kemali et al. [10] proved some modified version of the famous double inequality for s-convex functions. A generalized form of these inequalities was provided by Abbasi and his colleagues for s-convex functions using a Caputo–Fabrizio integral operator, as well as bounds for the inequalities; see Ref. [11]. Gurbuz et al. [12] used convex mappings to create Hermite–Hadamard inequalities. Sahoo et al. [13,14] developed Hermite–Hadamard and midpoint inequalities via Caputo–Fabrizio operator. Nwaeze et al. [15] established these inequalities using strongly convex mappings. Utilizing h-convexity, Cortez et al. [16] created Hermite–Hadamard Mercer-type inequalities using the Caputo–Fabrizio operator. Tariq et al. [17] developed some new integral inequalities of the Hermite–Hadamard and Pachpatte types that incorporate the concept of preinvexity and Caputo–Fabrizio fractional integral operators. Nosheen et al. [18] developed several new integral inequalities involving the digamma function and special means for (s,m)-convex functions using Caputo fractional derivatives. Zhang et al. [19] presents a generalization of the Hermite–Hadamard-type inequalities for (p,h)-convex functions via the Caputo–Fabrizio fractional integral operator. Nasir et al. [20] developed bounds and novel connections for Hermite–Hadamard type inequalities for differentiable maps whose derivatives at certain powers are s-convex via the Caputo–Fabrizio operator. For some recent developments related to developed inequalities; see Refs. [21–37].

Using the Riemann integral operator, Afzal and his colleagues developed Jensen and Hermite–Hadamard type inequalities using h-Godunova–Levin and (h_1,h_2)-convex functions; see Refs. [38,39].

Theorem 1 (see [38])**.** *Let $h:(0,1)\to\mathfrak{R}^+$ and $h\left(\frac{1}{2}\right)\neq 0$. Let $\mathfrak{S}:\mathfrak{I}=[u,v]\to{\mathfrak{R}^+}_I$ be an $\mathcal{CR}$ interval-valued h-Godunova–Levin function defined on $[u,v]$ and $\mathfrak{S}\in L_1[u,v]$; then, one has*

$$\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right)\preceq_{\mathcal{CR}}\frac{1}{v-u}\int_u^v\mathfrak{S}(\mathfrak{i})d\mathfrak{i}\preceq_{\mathcal{CR}}(\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}.$$

Theorem 2 (see [39]). *Let $h_1, h_2 : (0,1) \to \mathfrak{R}^+$ and $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$. Let $\mathfrak{S} : \mathfrak{I} = [u, v] \to \mathfrak{R}^+{}_I$ be an $\mathcal{CR}$ interval-valued (h_1, h_2)-convex function defined on $[u, v]$ and $\mathfrak{S} \in L_1[u, v]$; then, one has*

$$\frac{1}{2H\left(\frac{1}{2}, \frac{1}{2}\right)}\mathfrak{S}\left(\frac{u+v}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{v-u}\int_u^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i} \preceq_{\mathcal{CR}} (\mathfrak{S}(u) + \mathfrak{S}(v)) \int_0^1 H(\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c}.$$

As a result of studying the Strong literature and specific articles [12,39,40], we reformulated the above two results based on Caputo–Fabrizio fractional integral operators.

This work is significant and novel because it is the first time we have developed these inequalities using Caputo–Fabrizio fractional integral operators for h-Godunova–Levin and (h_1, h_2)-convex functions. These two classes of convexties are remarkable for the fact that they generalize several other related classes of convexities by setting some suitable parameters. In addition, we provide some remarks to show how our results generalize several existing findings. Moreover, we know that Godunova–Levin functions are extremely interesting; in that class, we have non-negative monotone and non-negative convex functions that are rarely used compared with classical convexity, so we hope these results will inspire readers to apply them to other approaches in the future. Furthermore, here are some other interesting properties that relate to the Godunova–Levin functions; see Refs. [41–44].

The paper structure consists of the following components: In Section 2, we begin with some known definitions and results that assist in proving the main findings of the paper. As discussed in Section 3, we developed some new variants of the Hermite–Hadamard type of inequalities involving h-Godunova–Levin functions. In Section 4, we introduced a more generalized type of convexity called (h_1, h_2)-convex functions. We used these functions to develop some variants of Hermite–Hadamard inequality involving Caputo–Fabrizio fractional operators. The purpose of Section 5 is to link our above results developed by the h-Godunova–Levin class of convexity to some applications to special means. In Section 6, we summarize our main findings and their applications. We also discuss future directions based on these new results. This structured organization presents an in-depth analysis of the introduced operators, establishes new inequalities, shows applications, and explores interesting results.

2. Preliminaries

The purpose of this section is to present some known definitions and results that can assist in proving the main findings of the article.

Definition 1 (see [38]). *(Convex function) . Let $\mathfrak{S} : v \to \mathfrak{R}$ defined on convex set $v \subset \mathfrak{R}^n$; then, $\mathfrak{S}$ is called to be convex if*

$$\mathfrak{S}(\mathfrak{c}u + (1-\mathfrak{c})v) \leq \mathfrak{c}\mathfrak{S}(u) + (1-\mathfrak{c})\mathfrak{S}(v), \tag{1}$$

holds for all $u, v \in v$ and $\mathfrak{c} \in (0,1)$.

Definition 2 (see [38]). *(h-convex function).Consider two non-negative functions $h, \mathfrak{S}$ such that $h : v \subset \mathfrak{R} \to \mathfrak{R}$ and $\mathfrak{S} : \mathfrak{I} \subset \mathfrak{R} \to \mathfrak{R}$; then, $\mathfrak{S}$ is called to be h-convex if*

$$\mathfrak{S}(\mathfrak{c}u + (1-\mathfrak{c})v) \leq h(\mathfrak{c})\mathfrak{S}(u) + h(1-\mathfrak{c})\mathfrak{S}(v), \tag{2}$$

holds for all $u, v \in v$ and $\mathfrak{c} \in [0,1]$.

Definition 3 (see [38]). *(h-Godunova–Levin function).Consider two non-negative functions $h, \mathfrak{S}$ such that $h : v \subset \mathfrak{R} \to \mathfrak{R}$ and $\mathfrak{S} : \mathfrak{I} \subset \mathfrak{R} \to \mathfrak{R}$; then, $\mathfrak{S}$ is known as h-Godunova–Levin if*

$$\mathfrak{S}(\mathfrak{c}u + (1-\mathfrak{c})v) \leq \frac{\mathfrak{S}(u)}{h(\mathfrak{c})} + \frac{\mathfrak{S}(v)}{h(1-\mathfrak{c})}, \tag{3}$$

holds for all $u, v \in v$ and $\mathfrak{c} \in (0,1)$.

If inequality (3) is altered, then mapping is considered to be in concave sense. The family of all convex (concave) h-Godunova–Levin functions are represented by $SGX(h,v)$, $SGV(h,v)$, respectively.

Definition 4 (see [39])**.** ***((h_1, h_2)-convex function)****. Consider non-negative functions $h_1, h_2, \mathfrak{S}$ such that $h_1, h_2 : v \subset \mathfrak{R} \to \mathfrak{R}$ and $\mathfrak{S} : \mathfrak{I} \subset \mathfrak{R} \to \mathfrak{R}$; then, $\mathfrak{S}$ is called to be (h_1, h_2)-convex if*

$$\mathfrak{S}(\mathfrak{c}u + (1-\mathfrak{c})v) \leq h_1(\mathfrak{c})h_2(1-\mathfrak{c})\mathfrak{S}(u) + h_1(1-\mathfrak{c})h_2(\mathfrak{c})\mathfrak{S}(v), \tag{4}$$

holds for all $u, v \in v$ and $\mathfrak{c} \in [0,1]$.

Definition 5 (see [12])**.** ***(Caputo–Fabrizio fractional time derivative)****. The Caputo derivative of order ε for any arbitrary function $\mathfrak{S}$ can be defined as*

$$D_{\mathfrak{t}}^{\varepsilon}\mathfrak{S}(\mathfrak{t}) = \frac{1}{\Gamma(1-\varepsilon)} \int_u^{\mathfrak{t}} \frac{\mathfrak{S}'(\mathfrak{i})}{(\mathfrak{t}-\mathfrak{i})^{\varepsilon}} d\mathfrak{i}, \tag{5}$$

and $u \in [-\infty, \mathfrak{t}), \mathfrak{S} \in H^1(u,v), u < v$. $H^1(u,v)$ is class of first order differentiable function with $\varepsilon \in (0,1)$. By changing this factor $\frac{1}{\Gamma(1-\varepsilon)}$ with $\frac{\mathcal{B}(\varepsilon)}{1-\varepsilon}$ and kernel $(\mathfrak{t}-\mathfrak{i})^{-\varepsilon}$ with the following exponential function $e^{\left(\frac{-\varepsilon(\mathfrak{t}-\mathfrak{i})^{\varepsilon}}{1-\varepsilon}\right)}$, where $\mathcal{B}(\varepsilon) > 0$ is a normalization function that is equally spaced with $\mathcal{B}(0) = \mathcal{B}(1) = 1$, we obtained modified version of fractional time derivative

$$(D_{\mathfrak{t}}^{\varepsilon}\mathfrak{S})(\mathfrak{t}) = \frac{\mathcal{B}(\varepsilon)}{1-\varepsilon} \int_u^{\mathfrak{t}} \mathfrak{S}'(\mathfrak{i}) e^{\frac{-\varepsilon(\mathfrak{t}-\mathfrak{i})^{\varepsilon}}{1-\varepsilon}} d\mathfrak{i}, \tag{6}$$

Definition 6 (see [12])**.** *Let $\mathfrak{S} \in H_1(u,v), u < v, \varepsilon \in (0,1)$; then, the left version of Caputo–Fabrizio derivative is defined as follows:*

$$\left({}_{u}^{CFC}D^{\varepsilon}\mathfrak{S}\right)(t) = \frac{\mathcal{B}(\varepsilon)}{1-\varepsilon} \int_u^k \mathfrak{S}'(\mathfrak{i}) e^{\frac{-\varepsilon(\mathfrak{t}-\mathfrak{i})^{\varepsilon}}{1-\varepsilon}} d\mathfrak{i}, \tag{7}$$

As a result, the integral associated with this fractional derivative is

$$\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(\mathfrak{t}) = \frac{1-\varepsilon}{\mathcal{B}(\varepsilon)}\mathfrak{S}(\mathfrak{t}) + \frac{\varepsilon}{\mathcal{B}(\varepsilon)} \int_u^{\mathfrak{t}} \mathfrak{S}(\mathfrak{i}) d\mathfrak{i}. \tag{8}$$

We have now defined the right Caputo–Fabrizio fractional derivative as follows:

$$\left({}^{CFC}D_v^{\varepsilon}\mathfrak{S}\right)(\mathfrak{t}) = \frac{-\mathcal{B}(\varepsilon)}{1-\varepsilon} \int_{\mathfrak{t}}^{v} \mathfrak{S}'(\mathfrak{i}) e^{\frac{-\varepsilon(\mathfrak{i}-\mathfrak{t})^{\varepsilon}}{1-\varepsilon}} d\mathfrak{i}, \tag{9}$$

and the integral associated with this fractional derivative is

$$\left({}^{CF}I_v^{\varepsilon}\mathfrak{S}\right)(t) = \frac{1-\varepsilon}{\mathcal{B}(\varepsilon)}\mathfrak{S}(t) + \frac{\varepsilon}{\mathcal{B}(\varepsilon)} \int_k^{v} \mathfrak{S}(\mathfrak{i}) d\mathfrak{i}. \tag{10}$$

There have been recent attempts to generalize existing kernels using fractional derivative operators and integral operators. By extending a Caputo–Fabrizio fractional integral operator, we will generalize the kernel that Dragomir and Agarwal proposed; see Ref. [45].

Lemma 1. *Let $\mathfrak{S} : \mathring{v} \subseteq \mathfrak{R} \to \mathfrak{R}$ be a differentiable mapping on $\mathring{v}$ and $u, v \in v$ with $u < v$. If $\mathfrak{S}' \in L_1[u, v]$ and $\mathfrak{c} \in (0, 1)$; then, we have*

$$\frac{\mathfrak{S}(u) + \mathfrak{S}(v)}{2} - \frac{1}{v-u}\int_u^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i} = \frac{v-u}{2}\int_0^1 (1-2\mathfrak{c})\mathfrak{S}'(\mathfrak{c}u - (1-\mathfrak{c})v)d\mathfrak{c}. \tag{11}$$

Lemma 2 (see [12])**.** *Let $\mathfrak{S} : \mathring{v} \subseteq \mathfrak{R} \to \mathfrak{R}$ be a differentiable mapping on $\mathring{v}$ and $u, v \in v$ with $u < v$. If $\mathfrak{S}' \in L_1[u, v]$ and $\varepsilon \in (0, 1)$; then, we have*

$$\begin{aligned} &\frac{v-u}{2}\int_0^1 (1-2\mathfrak{c})\mathfrak{S}'(\mathfrak{c}u - (1-\mathfrak{c})v)d\mathfrak{c} - \frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k) \\ &= \frac{\mathfrak{S}(u) + \mathfrak{S}(v)}{2} - \frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}\right)(k)\right]. \end{aligned} \tag{12}$$

where $k \in [u, v]$ and $\mathcal{B}(\varepsilon) > 0$ is a normalization function.

Theorem 3 (Hölder inequality see [46])**.** *Let $\mathfrak{p} > 1$ and $1/\mathfrak{p} + 1/\mathfrak{q} = 1$. If $\mathfrak{S}$ and $\mathfrak{G}$ are real-valued mappings defined on $[u, v]$ with $|\mathfrak{S}|^{\mathfrak{p}}, |\mathfrak{G}|^{\mathfrak{q}}$ are integrable functions on $[u, v]$, then*

$$\int_u^v |\mathfrak{S}(\mathfrak{x})\mathfrak{G}(\mathfrak{x})|d\mathfrak{x} \le \left(\int_u^v |\mathfrak{S}(\mathfrak{x})|^{\mathfrak{p}}d\mathfrak{x}\right)^{1/\mathfrak{p}}\left(\int_u^v |\mathfrak{G}(\mathfrak{x})|^{\mathfrak{q}}d\mathfrak{x}\right)^{1/\mathfrak{q}}. \tag{13}$$

3. Hermite–Hadamard Inequality via *H*-Godunova–Levin Functions Involving Caputo–Fabrizio Fractional Operator

As part of this section, we used a concept of Godunova–Levin mappings and developed some new variants of Hermite–Hadamard inequalities using Caputo–Fabrizio fractional operators.

Theorem 4. *Let $\mathfrak{S} : \mathfrak{I} = [u, v] \to \mathfrak{R}$ be an h-Godunova–Levin function defined on $[u, v]$ and $\mathfrak{S} \in L_1[u, v]$. If $\varepsilon \in (0, 1)$, then we have*

$$\begin{aligned} &\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right) \le \frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)\right] \\ &\le (\mathfrak{S}(u) + \mathfrak{S}(v))\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})}, \end{aligned} \tag{14}$$

where $k \in [u, v]$ and $\mathcal{B}(\varepsilon) > 0$ is a normalization function.

Proof. The Hermite–Hadamard inequality for *h*-Godunova–Levin function is as follows:

$$\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i} \le (\mathfrak{S}(u) + \mathfrak{S}(v))\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})}.$$

Since $\mathfrak{S}$ is *h*-Godunova–Levin function on $[u, v]$, we have

$$\frac{2h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right) \le \frac{2}{v-u}\int_u^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i} = \frac{2}{v-u}\left(\int_u^k \mathfrak{S}(\mathfrak{i})d\mathfrak{i} + \int_k^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\right). \tag{15}$$

Multiplying both sides of (15) by $\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)$, we get

$$
\begin{aligned}
&\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)+\frac{\varepsilon(v-u)}{\mathcal{B}(\varepsilon)}\left(\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right)\right)\\
&\leq \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)+\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\left[\int_u^k \mathfrak{S}(\mathfrak{i})d\mathfrak{i}+\int_k^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\right]\\
&=\left(\frac{(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)+\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\int_u^k \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\right)+\left(\frac{(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)+\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\int_k^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\right)\\
&=\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k).
\end{aligned} \tag{16}
$$

We obtain the first part after appropriately rearranging (16). Let us prove the right side of required result. The Hermite–Hadamard inequality for h-Godunova–Levin function is

$$
\begin{aligned}
&\frac{2}{v-u}\int_u^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\leq 2\left[(\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}\right],\\
&\frac{2}{v-u}\left[\int_u^k \mathfrak{S}(\mathfrak{i})d\mathfrak{i}+\int_k^v \mathfrak{S}(\mathfrak{i})d\mathfrak{i}\right]\leq 2\left[(\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}\right].
\end{aligned} \tag{17}
$$

We have achieved this by employing the same operator as with (15) in (17); we have

$$
\begin{aligned}
&\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\\
&\leq \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)+\frac{\varepsilon(v-u)}{\mathcal{B}(\varepsilon)}\left((\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}\right).
\end{aligned} \tag{18}
$$

As a result of rearranging (18), we obtain the required output that is

$$
\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right)\leq \frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)\right]\\
&\leq (\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}.
\end{aligned} \tag{19}
$$

□

Example 1. *Consider* $[u,v]=[0,1]$, $h(\mathfrak{c})=\frac{1}{\mathfrak{c}}$ *with* $\mathcal{B}(\varepsilon)=1$ *and for all* $\mathfrak{c}\in(0,1)$. *If* $\mathfrak{S}:[u,v]\to\mathfrak{R}^{+}$ *is defined as*

$$
\mathfrak{S}(\mathfrak{c})=\mathfrak{c}^2
$$

then

$$
\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{S}\left(\frac{u+v}{2}\right)=\frac{1}{4},\\
&\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)\right]=\frac{1}{3},\\
&(\mathfrak{S}(u)+\mathfrak{S}(v))\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})}=\frac{1}{2}.
\end{aligned}
$$

Consequently,

$$
\frac{1}{4}\leq\frac{1}{3}\leq\frac{1}{2}.
$$

This verifies above Theorem.

The following remark proves that our result is a generalization of an existing result.

Remark 1. *(i)* *Taking* $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ *in above result, we obtain [12] [Theorem 2].*
(ii) *Taking* $h(\mathfrak{c}) = \mathfrak{c}^s$ *in above result with* $\varepsilon = 1$*, we obtain [47] [Theorem 2.1].*
(iii) *Taking* $h(\mathfrak{c}) = 1$ *in above result with* $\varepsilon = 1$*, we obtain [48] [Theorem 1].*

Theorem 5. *Let* $\mathfrak{S}_1, \mathfrak{S}_2 : \mathfrak{I} \subseteq \mathfrak{R} \to \mathfrak{R}$ *be two h-Godunova–Levin functions on* $\mathfrak{I}$*. If* $\mathfrak{S}_1\mathfrak{S}_2 \in L^1([u,v])$*; then, we have the following inequality:*

$$\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\right]$$
$$\leq \left(2\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + \left(2\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v), \tag{20}$$

where

$$M(u,v) = \mathfrak{S}_1(u)\mathfrak{S}_2(u) + \mathfrak{S}_1(v)\mathfrak{S}_2(v),$$

and

$$N(u,v) = \mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u).$$

Proof. By definition of h-Godunova–Levin, we have

$$\mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v) \leq \frac{\mathfrak{S}_1(u)}{h(\mathfrak{c})} + \frac{\mathfrak{S}_1(v)}{h(1-\mathfrak{c})}, \forall \mathfrak{c} \in (0,1), u, v \in \mathfrak{I}, \tag{21}$$

and

$$\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v) \leq \frac{\mathfrak{S}_2(u)}{h(\mathfrak{c})} + \frac{\mathfrak{S}_2(v)}{h(1-\mathfrak{c})}, \forall \mathfrak{c} \in (0,1), u, v \in \mathfrak{I}. \tag{22}$$

Multiplying both sides of (21) and (22), we have

$$\mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v)\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v)$$
$$\leq \frac{\mathfrak{S}_1(u)\mathfrak{S}_2(u)}{(h(\mathfrak{c}))^2} + \frac{\mathfrak{S}_1(v)\mathfrak{S}_2(v)}{(h(1-\mathfrak{c}))^2} + \frac{[\mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u)]}{h(\mathfrak{c})h(1-\mathfrak{c})}. \tag{23}$$

Integrating (23) and changing variables, we obtain

$$\frac{1}{v-u}\int_u^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} \leq \mathfrak{S}_1(u)\mathfrak{S}_2(u)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2} + \mathfrak{S}_1(v)\mathfrak{S}_2(v)\int_0^1 \frac{d\mathfrak{c}}{(h(1-\mathfrak{c}))^2}$$
$$+[\mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u)]\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})},$$

which implies

$$\frac{2}{v-u}\left[\int_u^k \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} + \int_k^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}\right]$$
$$\leq 2\left[\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}[\mathfrak{S}_1(u)\mathfrak{S}_2(u) + \mathfrak{S}_1(v)\mathfrak{S}_2(v)]\right]$$
$$+\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}[\mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u)]$$
$$\leq 2\left[\left(\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + \left(\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v)\right]. \tag{24}$$

Multiplying both sides of (24) by $\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$, we obtain

$$\begin{aligned}
&\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\left[\int_u^k \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} + \int_k^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}\right] + \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k) \\
&\leq \frac{\varepsilon(v-u)}{\mathcal{B}(\varepsilon)}\left[2\left(\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + 2\left(\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v)\right] \\
&+ \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k).
\end{aligned} \tag{25}$$

Thus,

$$\begin{aligned}
&\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k) \\
&\leq \frac{\varepsilon(v-u)}{\mathcal{B}(\varepsilon)}\left[2\left(\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + 2\left(\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v)\right] \\
&+ \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k).
\end{aligned} \tag{26}$$

As a result of rearranging (26), we obtain the required output that is

$$\begin{aligned}
&\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\right] \\
&\leq \left(2\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + \left(2\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v).
\end{aligned} \tag{27}$$

□

Example 2. *Consider* $[u,v] = [0,1]$, $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ *with* $\mathcal{B}(\varepsilon) = 1$ *and for all* $\mathfrak{c} \in (0,1)$. *If* $\mathfrak{S}_1, \mathfrak{S}_2 : [u,v] \to \mathfrak{R}^+$ *are defined as*

$$\mathfrak{S}_1(\mathfrak{c}) = \mathfrak{c}^2 \ \ and \ \ \mathfrak{S}_2(\mathfrak{c}) = 2\mathfrak{c}^2 + 1,$$

then

$$\begin{aligned}
&\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\right] = \frac{22}{15}, \\
&\left(2\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2}\right)M(u,v) + \left(2\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right)N(u,v) = \frac{7}{3}.
\end{aligned}$$

Consequently,

$$\frac{22}{15} \leq \frac{7}{3}.$$

This verifies above Theorem.

Theorem 6. *Let* $\mathfrak{S}_1, \mathfrak{S}_2 : J \subseteq \mathfrak{R} \to \mathfrak{R}$ *be two h-Godunova–Levin functions on* J. *If* $\mathfrak{S}_1\mathfrak{S}_2 \in L^1([u,v])$, *then we have the following inequality:*

$$\begin{aligned}
&\frac{\left[h(\frac{1}{2})\right]^2}{2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right) - \frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right] \\
&+ \frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k) \leq M(u,v)\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})} \\
&+ \frac{1}{2}N(u,v)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2 + (h(1-\mathfrak{c}))^2},
\end{aligned} \tag{28}$$

Proof. By definition of h-Godunova–Levin, we have

$$\mathfrak{S}_1\left(\frac{u+v}{2}\right) \leq \frac{\mathfrak{S}_1((1-\mathfrak{c})u+\mathfrak{c}v)}{h\left(\frac{1}{2}\right)} + \frac{\mathfrak{S}_1(\mathfrak{c}u+(1-\mathfrak{c})v)}{h\left(\frac{1}{2}\right)}, \tag{29}$$

and

$$\mathfrak{S}_2\left(\frac{u+v}{2}\right) \leq \frac{\mathfrak{S}_2((1-\mathfrak{c})u+\mathfrak{c}v)}{h\left(\frac{1}{2}\right)} + \frac{\mathfrak{S}_2(\mathfrak{c}u+(1-\mathfrak{c})v)}{h\left(\frac{1}{2}\right)}. \tag{30}$$

Multiplying both sides of (29) and (30), we have

$$\begin{aligned}
&\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)\\
&\leq \frac{1}{\left[h\left(\frac{1}{2}\right)\right]^2}[\mathfrak{S}_1((1-\mathfrak{c})u+\mathfrak{c}v)\mathfrak{S}_2((1-\mathfrak{c})u+\mathfrak{c}v)+\mathfrak{S}_1(\mathfrak{c}u+(1-\mathfrak{c})v)\\
&\mathfrak{S}_2(\mathfrak{c}u+(1-\mathfrak{c})v)+\mathfrak{S}_1((1-\mathfrak{c})u+\mathfrak{c}v)\mathfrak{S}_2(\mathfrak{c}u+(1-\mathfrak{c})v)\\
&+\mathfrak{S}_1(\mathfrak{c}u+(1-\mathfrak{c})v)\mathfrak{S}_2((1-\mathfrak{c})u+\mathfrak{c}v)]\\
&\leq \frac{1}{\left[h\left(\frac{1}{2}\right)\right]^2}[\mathfrak{S}_1((1-\mathfrak{c})u+\mathfrak{c}v)\mathfrak{S}_2((1-\mathfrak{c})u+\mathfrak{c}v)+\mathfrak{S}_1(\mathfrak{c}u+(1-\mathfrak{c})v)\\
&\mathfrak{S}_2(\mathfrak{c}u+(1-\mathfrak{c})v)+\frac{2}{h(\mathfrak{c})h(1-\mathfrak{c})}\{\mathfrak{S}_1(u)\mathfrak{S}_2(u)+\mathfrak{S}_1(v)\mathfrak{S}_2(v)\}\\
&+\left.\frac{\{\mathfrak{S}_1(u)\mathfrak{S}_2(v)+\mathfrak{S}_1(v)\mathfrak{S}_2(u)\}}{\{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2\}}\right].
\end{aligned} \tag{31}$$

Integrating (31) and changing variables, we have

$$\begin{aligned}
&\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)\\
&\leq \frac{1}{\left[h\left(\frac{1}{2}\right)\right]^2}\left[\frac{2}{v-u}\int_u^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}+2M(u,v)\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right.\\
&\left.+N(u,v)\int_0^1\frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}\right].
\end{aligned} \tag{32}$$

Multiplying both sides of (32) by $\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}$ and subtracting $\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$, we obtain

$$\begin{aligned}
&\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\\
&\leq \frac{\varepsilon}{\mathcal{B}(\varepsilon)}\int_u^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}+\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}\left[2M(u,v)\int_0^1\frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}\right.\\
&\left.+N(u,v)\int_0^1\frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}\right]-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,.
\end{aligned} \tag{33}$$

Thus,

$$
\begin{aligned}
&\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\\
&-\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\left[\int_u^k \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}+\int_k^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}\right]\\
&\leq \frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}\left[2M(u,v)\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}+N(u,v)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}\right]\\
&-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,.
\end{aligned}
\tag{34}
$$

This implies that

$$
\begin{aligned}
&\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)-\left({}_u^{CF}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)+\left({}^{CF}I_v^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\\
&\leq \frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}\left[2M(u,v)\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}+N(u,v)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}\right]\\
&-\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,.
\end{aligned}
\tag{35}
$$

Multiplying (35) by $\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}$, we obtained the required output that is

$$
\begin{aligned}
&\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)\\
&-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_u^{CF}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)+\left({}^{CF}I_v^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right]+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\\
&\leq M(u,v)\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}+\frac{1}{2}N(u,v)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}.
\end{aligned}
\tag{36}
$$

□

Example 3. *Consider* $[u,v]=[0,1]$, $h(\mathfrak{c})=\mathfrak{c}$ *with* $\mathcal{B}(\varepsilon)=1$ *and for all* $\mathfrak{c}\in(0,1)$. *If* $\mathfrak{S}_1,\mathfrak{S}_2:[u,v]\to\mathfrak{R}^+$ *are defined as*

$$\mathfrak{S}_1(\mathfrak{c})=\mathfrak{c}^2 \ \ \textit{and} \ \ \mathfrak{S}_2(\mathfrak{c})=2\mathfrak{c}^2+1,$$

then

$$
\begin{aligned}
&\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)\\
&-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_u^{CF}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)+\left({}^{CF}I_v^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right]+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)=\frac{23}{30},\\
&M(u,v)\int_0^1 \frac{d\mathfrak{c}}{h(\mathfrak{c})h(1-\mathfrak{c})}+\frac{1}{2}N(u,v)\int_0^1 \frac{d\mathfrak{c}}{(h(\mathfrak{c}))^2+(h(1-\mathfrak{c}))^2}=\frac{13}{4}.
\end{aligned}
$$

Consequently,

$$\frac{23}{30}\leq\frac{13}{4}.$$

This verifies above Theorem.

4. Hermite–Hadamard Inequality via (H_1, H_2)-Convex Functions Involving Caputo–Fabrizio Fractional Operator

In this section, we used a concept of (h_1, h_2)-convex mappings and developed some new variants of Hermite–Hadamard inequalities involving Caputo–Fabrizio fractional operators.

Theorem 7. *Let $\mathfrak{S} : \mathfrak{I} = [u, v] \to \mathfrak{R}$ be an (h_1, h_2)-convex function defined on $[u, v]$ and $\mathfrak{S} \in L_1[u, v]$. If $\varepsilon \in (0, 1)$, then we have*

$$\frac{1}{2\Big[H\Big(\frac{1}{2},\frac{1}{2}\Big)\Big]}\mathfrak{S}\left(\frac{u+v}{2}\right) \leq \frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k) + \left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)\right]$$
$$\leq (\mathfrak{S}(u) + \mathfrak{S}(v))\int_0^1 H(\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c}, \tag{37}$$

where $k \in [u, v]$ and $\mathcal{B}(\varepsilon) > 0$ is a normalization function.

Proof. The proof is based on the same technique as the Theorem 4 and the result by Saeed et al. [39] [Theorem 4]. □

Example 4. *Consider $[u, v] = [1, 2]$, $h_1(\mathfrak{c}) = \mathfrak{c}, h_2(\mathfrak{c}) = 1$ with $\mathcal{B}(\varepsilon) = 1$ and for all $\mathfrak{c} \in (0, 1)$. If $\mathfrak{S} : [u, v] \to \mathfrak{R}^+$ is defined as*

$$\mathfrak{S}(\mathfrak{c}) = \mathfrak{c}^2 + 2$$

, then

$$\frac{1}{2\Big[H\Big(\frac{1}{2},\frac{1}{2}\Big)\Big]}\mathfrak{S}\left(\frac{u+v}{2}\right) = \frac{17}{4},$$
$$\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k) + \left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}(k)\right] = \frac{13}{3},$$
$$(\mathfrak{S}(u) + \mathfrak{S}(v))\int_0^1 H(\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c} = \frac{9}{2}.$$

Consequently,

$$\frac{17}{4} \leq \frac{13}{3} \leq \frac{9}{2}.$$

This verifies above Theorem.

Theorem 8. *Let $\mathfrak{S}_1, \mathfrak{S}_2 : \mathfrak{I} \subseteq \mathfrak{R} \to \mathfrak{R}$ be (h_1, h_2)-convex functions on $\mathfrak{I}$. If $\mathfrak{S}_1\mathfrak{S}_2 \in L^1([u, v])$; then, the following inequality holds:*

$$\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\right]$$
$$\leq \left(2\int_0^1 H^2(\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c}\right)M(u, v)$$
$$+\left(2\int_0^1 H(\mathfrak{c}, \mathfrak{c})H(1-\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c}\right)N(u, v), \tag{38}$$

where

$$M(u, v) = \mathfrak{S}_1(u)\mathfrak{S}_2(u) + \mathfrak{S}_1(v)\mathfrak{S}_2(v),$$

and

$$N(u, v) = \mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u).$$

Proof. The proof is based on the same technique as the Theorem 5 and the result by Saeed et al. [39] [Theorem 5]. □

Example 5. *Consider* $[u,v] = [0,1]$, $h_1(\mathfrak{c}) = \mathfrak{c}, h_2(\mathfrak{c}) = 1$ *with* $\mathcal{B}(\varepsilon) = 1$ *and for all* $\mathfrak{c} \in (0,1)$. *If* $\mathfrak{S}_1, \mathfrak{S}_2 : [u,v] \to \mathfrak{R}^+$ *are defined as*

$$\mathfrak{S}_1(\mathfrak{c}) = \mathfrak{c}^2 + 2 \ \ and \ \ \mathfrak{S}_2(\mathfrak{c}) = 2\mathfrak{c}^2 + 3,$$

then

$$\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\right] = \frac{262}{15},$$
$$\left(2\int_0^1 H^2(\mathfrak{c}, 1-\mathfrak{c})d\mathfrak{c}\right)M(u,v) + \left(2\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}\right)N(u,v) = \frac{109}{3}.$$

Consequently,

$$\frac{262}{15} \leq \frac{109}{3}.$$

This verifies above Theorem.

Remark 2. *Taking* $h_1(\mathfrak{c}) = \mathfrak{c}, h_2(\mathfrak{c}) = 1$ *in above Theorem, we obtain [12] [Theorem 3].*

Theorem 9. *Let* $\mathfrak{S}_1, \mathfrak{S}_2 : J \subseteq \mathfrak{R} \to \mathfrak{R}$ *are two* (h_1,h_2)*-convex functions on* J. *If* $\mathfrak{S}_1\mathfrak{S}_2 \in L^1([u,v])$, *then we have the following inequality:*

$$\begin{aligned}
&\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right) \\
&-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right] \\
&+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k) \leq M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c} \\
&+\frac{1}{2}N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}, \qquad (39)
\end{aligned}$$

Proof. By definition of (h_1,h_2)-convex function, we have

$$\begin{aligned}
&\mathfrak{S}_1\left(\frac{u+v}{2}\right) \\
&\leq H\left(\frac{1}{2},\frac{1}{2}\right)\mathfrak{S}_1((1-\mathfrak{c})u + \mathfrak{c}v) + H\left(\frac{1}{2},\frac{1}{2}\right)\mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v), \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{S}_2\left(\frac{u+v}{2}\right) \\
&\leq H\left(\frac{1}{2},\frac{1}{2}\right)\mathfrak{S}_2((1-\mathfrak{c})u + \mathfrak{c}v) + H\left(\frac{1}{2},\frac{1}{2}\right)\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v). \qquad (41)
\end{aligned}$$

Multiplying (40) and (41), we obtain

$$\begin{aligned}
&\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right) \\
&\leq \left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[\mathfrak{S}_1((1-\mathfrak{c})u + \mathfrak{c}v)\mathfrak{S}_2((1-\mathfrak{c})u + \mathfrak{c}v) + \mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v) \\
&\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v) + \mathfrak{S}_1((1-\mathfrak{c})u + \mathfrak{c}v)\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v) \\
&+\mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v)\mathfrak{S}_2((1-\mathfrak{c})u + \mathfrak{c}v)]
\end{aligned}$$

$$\leq \left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2 [\mathfrak{S}_1((1-\mathfrak{c})u + \mathfrak{c}v)\mathfrak{S}_2((1-\mathfrak{c})u + \mathfrak{c}v) + \mathfrak{S}_1(\mathfrak{c}u + (1-\mathfrak{c})v)$$
$$\mathfrak{S}_2(\mathfrak{c}u + (1-\mathfrak{c})v) + 2H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})\{\mathfrak{S}_1(u)\mathfrak{S}_2(u) + \mathfrak{S}_1(v)\mathfrak{S}_2(v)\}$$
$$+\left(H^2(\mathfrak{c},1-\mathfrak{c}) + H^2(1-\mathfrak{c},\mathfrak{c})\right)\{\mathfrak{S}_1(u)\mathfrak{S}_2(v) + \mathfrak{S}_1(v)\mathfrak{S}_2(u)\}]\,. \tag{42}$$

Integrating (42) and changing variables, we have

$$\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)$$
$$\leq \left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2 \left[\frac{2}{v-u}\int_u^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} + 2M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}\right.$$
$$\left.+N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}\right]. \tag{43}$$

Multiplying both sides of (43) with $\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}$ and subtracting $\frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$, we obtain

$$\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$$
$$\leq \frac{\varepsilon}{\mathcal{B}(\varepsilon)}\int_u^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} + \frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}\left[2M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}\right.$$
$$\left.+N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}\right] - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,. \tag{44}$$

Thus,

$$\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right) - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$$
$$-\frac{\varepsilon}{\mathcal{B}(\varepsilon)}\left[\int_u^k \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i} + \int_k^v \mathfrak{S}_1(\mathfrak{i})\mathfrak{S}_2(\mathfrak{i})d\mathfrak{i}\right]$$
$$\leq \frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}[2M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}$$
$$+N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}] - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,. \tag{45}$$

This implies that

$$\frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)$$
$$-\left({}_u^{CF}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k) + \left({}^{CF}I_v^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)$$
$$\leq \frac{\varepsilon(v-u)}{2\mathcal{B}(\varepsilon)}[2M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}$$
$$+N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}] - \frac{2(1-\varepsilon)}{\mathcal{B}(\varepsilon)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)\,. \tag{46}$$

Multiplying (46) by $\frac{2\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}$, we obtained the required output that is

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)$$
$$-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)+\left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right]+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)$$
$$\leq M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}+\frac{1}{2}N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}. \tag{47}$$

□

Example 6. *Consider* $[u,v]=[0,1]$, $h_1(\mathfrak{c})=\mathfrak{c}, h_2(\mathfrak{c})=1$ *with* $\mathcal{B}(\varepsilon)=1$ *and for all* $\mathfrak{c}\in(0,1)$. *If* $\mathfrak{S}_1,\mathfrak{S}_2:[u,v]\to\mathfrak{R}^+$ *are defined as*

$$\mathfrak{S}_1(\mathfrak{c})=\mathfrak{c}^2+2 \ \ \textit{and} \ \ \mathfrak{S}_2(\mathfrak{c})=2\mathfrak{c}^2+3,$$

then

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\mathfrak{S}_1\left(\frac{u+v}{2}\right)\mathfrak{S}_2\left(\frac{u+v}{2}\right)$$
$$-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}_1\mathfrak{S}_2\right)(k)+\left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}_1\mathfrak{S}_2\right)(k)\right]+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}_1(k)\mathfrak{S}_2(k)=\frac{63}{64},$$
$$M(u,v)\int_0^1 H(\mathfrak{c},\mathfrak{c})H(1-\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}+\frac{1}{2}N(u,v)\int_0^1 H^2(\mathfrak{c},1-\mathfrak{c})d\mathfrak{c}=\frac{70}{6}.$$

Consequently,

$$\frac{63}{64}\leq\frac{70}{6}.$$

This verifies above Theorem.

Remark 3. *Taking* $h_1(\mathfrak{c})=\mathfrak{c}, h_2(\mathfrak{c})=1$ *in above Theorem, we obtain [12] [Theorem 4].*

5. Results Concerning Caputo–Fabrizio Fractional Operator

In the following theorem, we present an inequality concerning Caputo–Fabrizio fractional operator in the setting of h-Godunova–Levin function.

Theorem 10. *Let* $\mathfrak{S}:\mathfrak{I}\subseteq\mathfrak{R}\to\mathfrak{R}$ *be a differentiable positive mapping on* $\mathfrak{I}^o$ *and* $|\mathfrak{S}'|$ *be* h*-Godunova–Levin on* $[u,v]$ *where* $u,v\in\mathfrak{I}$ *with* $u<v$. *If* $\mathfrak{S}'\in L_1[u,v]$ *and* $\varepsilon\in(0,1)$, *then we have*

$$\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}^{CF}_{u}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I^{\varepsilon}_{v}\mathfrak{S}\right)(k)\right]\right|$$
$$\leq\frac{v-u}{2}\left[E_1|\mathfrak{S}'(u)|+E_2|\mathfrak{S}'(v)|\right], \tag{48}$$

where

$$E_1=\left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|d\mathfrak{c}}{h(\mathfrak{c})}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(\mathfrak{c})}\right), \tag{49}$$

$$E_2=\left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|}{h(1-\mathfrak{c})}d\mathfrak{c}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(1-\mathfrak{c})}\right), \tag{50}$$

where $k \in [u, v]$ and $\mathcal{B}(\varepsilon) > 0$ is a normalization function.

Proof. As a result of Lemma 2, and since $|\psi'|$ is h-Godunova–Levin, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\right]\right| \\
&\leq \frac{v-u}{2}\int_0^1|1-2\mathfrak{c}|\left|\mathfrak{S}'(\mathfrak{c}u+(1-\mathfrak{c})v)\right|d\mathfrak{c} \\
&\leq \frac{v-u}{2}\int_0^1|1-2\mathfrak{c}|\left[\frac{|\mathfrak{S}'(u)|}{h(\mathfrak{c})}+\frac{|\mathfrak{S}'(v)|}{h(1-\mathfrak{c})}\right]d\mathfrak{c} \\
&= \frac{v-u}{2}\left(\int_0^{\frac{1}{2}}|1-2\mathfrak{c}|\left[\frac{|\mathfrak{S}'(u)|}{h(\mathfrak{c})}+\frac{|\mathfrak{S}'(v)|}{h(1-\mathfrak{c})}\right]d\mathfrak{c}+\int_{\frac{1}{2}}^1|2\mathfrak{c}-1|\left[\frac{|\mathfrak{S}'(u)|}{h(\mathfrak{c})}+\frac{|\mathfrak{S}'(v)|}{h(1-\mathfrak{c})}\right]d\mathfrak{c}\right) \\
&= \frac{v-u}{2}\left[\left|\mathfrak{S}'(u)\right|\left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|d\mathfrak{c}}{h(\mathfrak{c})}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(\mathfrak{c})}\right)\right. \\
&\left.+\left|\mathfrak{S}'(v)\right|\left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|d\mathfrak{c}}{h(1-\mathfrak{c})}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(1-\mathfrak{c})}\right)\right] \\
&= \frac{v-u}{2}\left[E_1\left|\mathfrak{S}'(u)\right|+E_2\left|\mathfrak{S}'(v)\right|\right].
\end{aligned} \tag{51}$$

This concludes the proof. □

Example 7. *Consider $[u, v] = [0, 1]$, $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ with $\mathcal{B}(\varepsilon) = 1$ and for all $\mathfrak{c} \in (0, 1)$. If $\mathfrak{S} : [u, v] \to \mathfrak{R}^+$ is defined as*

$$\mathfrak{S}(\mathfrak{c}) = \mathfrak{c}^2$$

, then

$$\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\right]\right| = \frac{1}{6},$$

$$E_1 = \left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|d\mathfrak{c}}{h(\mathfrak{c})}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(\mathfrak{c})}\right) = \frac{1}{4},$$

$$E_2 = \left(\int_0^{\frac{1}{2}}\frac{|1-2\mathfrak{c}|}{h(1-\mathfrak{c})}d\mathfrak{c}+\int_{\frac{1}{2}}^1\frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(1-\mathfrak{c})}\right) = \frac{1}{4},$$

$$\frac{v-u}{2}\left[E_1\left|\mathfrak{S}'(u)\right|+E_2\left|\mathfrak{S}'(v)\right|\right] = \frac{1}{4}.$$

Consequently,

$$\frac{1}{6} \leq \frac{1}{4}.$$

This verifies above Theorem.

Remark 4. *Taking $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ in above Theorem, we obtain [12] [Theorem 5].*

Theorem 11. *Let $\mathfrak{S} : \mathfrak{I} \subseteq \mathfrak{R} \to \mathfrak{R}$ be a differentiable positive mapping on $\mathfrak{I}^o$ and $|\mathfrak{S}'|$ be h-Godunova–Levin on $[u, v]$ where $u, v \in \mathfrak{I}$ with $u < v$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathfrak{S}' \in \mathcal{L}_1[u, v]$, and $\varepsilon \in (0, 1)$, we have*

$$\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\right]\right|$$
$$\leq\frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|\mathfrak{S}'(u)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(\mathfrak{c})}+|\mathfrak{S}'(v)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(1-\mathfrak{c})}\right]^{\frac{1}{q}}, \tag{52}$$

where $k\in[u,v]$ *and* $\mathcal{B}(\varepsilon)>0$ *is a normalization function.*

Proof. As a result of Hölder's inequality, Lemma 2 and the fact that $|\mathfrak{S}'|^{q}$ is h-Godunova–Levin function, we have

$$\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\right]\right|$$
$$\leq\frac{v-u}{2}\int_{0}^{1}|1-2\mathfrak{c}||\mathfrak{S}'(\mathfrak{c}u+(1-\mathfrak{c})v)|d\mathfrak{c} \tag{53}$$

$$\leq\frac{v-u}{2}\left[\left(\int_{0}^{1}|1-2\mathfrak{c}|^{p}d\mathfrak{c}\right)^{\frac{1}{p}}\left(\int_{0}^{1}|\mathfrak{S}'(\mathfrak{c}u+(1-\mathfrak{c})v)|^{q}d\mathfrak{c}\right)^{\frac{1}{q}}\right]$$
$$=\frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(|\mathfrak{S}'(u)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(\mathfrak{c})}+|\mathfrak{S}'(v)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(1-\mathfrak{c})}\right)^{\frac{1}{q}}. \tag{54}$$

which completes the proof. □

Example 8. *Consider* $[u,v]=[0,1]$, $h(\mathfrak{c})=\frac{1}{\mathfrak{c}}$, $p=q=2$ *with* $\mathcal{B}(\varepsilon)=1$ *and for all* $\mathfrak{c}\in\ (0,1)$. *If* $\mathfrak{S}:[u,v]\to\mathfrak{R}^{+}$ *is defined as*

$$\mathfrak{S}(\mathfrak{c})=\mathfrak{c}^{2}$$

, then

$$\left|\frac{\mathfrak{S}(u)+\mathfrak{S}(v)}{2}+\frac{2(1-\varepsilon)}{\varepsilon(v-u)}\mathfrak{S}(k)-\frac{\mathcal{B}(\varepsilon)}{\varepsilon(v-u)}\left[\left({}_{u}^{CF}I^{\varepsilon}\mathfrak{S}\right)(k)+\left({}^{CF}I_{v}^{\varepsilon}\mathfrak{S}\right)(k)\right]\right|=\frac{1}{6},$$

$$\frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(|\mathfrak{S}'(u)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(\mathfrak{c})}+|\mathfrak{S}'(v)|^{q}\int_{0}^{1}\frac{d\mathfrak{c}}{h(1-\mathfrak{c})}\right)^{\frac{1}{q}}=\frac{\sqrt{2}}{2\sqrt{3}}$$

Consequently,

$$\frac{1}{6}\leq\frac{\sqrt{2}}{2\sqrt{3}}.$$

This verifies above Theorem.

Remark 5. *Taking* $h(\mathfrak{c})=\frac{1}{\mathfrak{c}}$ *in above Theorem, we obtain [12] [Theorem 6].*

Application to Means

Means play an important role in both pure and applied mathematics, especially when it comes to verifying the accuracy of results using special means for real numbers u,v such that $u\neq v$. They are in the following order:

$$H\leq G\leq\mathcal{L}\leq I\leq A \tag{55}$$

The arithmetic mean of any two arbitrary positive numbers, $u>0,v>0$ is defined as

$$A(u,v)=\frac{u+v}{2},u,v\in\mathfrak{R} \tag{56}$$

The generalized form of logarithmic mean, is defined as follows:

$$\mathcal{L} = \mathcal{L}_p^p(u,v) = \frac{v^{p+1} - u^{p+1}}{(p+1)(v-u)},$$

where

$$p \in \Re \setminus [-1,0], u,v \in \Re, u \neq v.$$

Proposition 1. *Let $u, v \in [0, \infty)$, $(u < v)$; one has*

$$|A(e^u, e^v) - L(e^u, e^v)| \leq \frac{(v-u)}{2}[E_1 e^u + E_2 e^v]. \tag{57}$$

where

$$E_1 = \left(\int_0^{\frac{1}{2}} \frac{|1-2\mathfrak{c}|d\mathfrak{c}}{h(\mathfrak{c})} + \int_{\frac{1}{2}}^1 \frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(\mathfrak{c})} \right), \tag{58}$$

$$E_2 = \left(\int_0^{\frac{1}{2}} \frac{|1-2\mathfrak{c}|}{h(1-\mathfrak{c})} d\mathfrak{c} + \int_{\frac{1}{2}}^1 \frac{|2\mathfrak{c}-1|d\mathfrak{c}}{h(1-\mathfrak{c})} \right), \tag{59}$$

Proof. In Theorem 10, if we consider $\mathfrak{S}(\mathfrak{i}) = e^x$ with $\varepsilon = 1$ and $\mathcal{B}(\varepsilon) = \mathcal{B}(1) = 1$, we obtain the required output. □

Remark 6. *Taking $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ in above result, we obtain [12] [Proposition 2].*

Proposition 2. *Let $u, v \in [0, \infty)$ with $u < v$; one has*

$$\left| A(u^2, v^2) - L_2^2(u,v) \right| \leq (v-u)[E_1|u| + E_2|v|]. \tag{60}$$

Proof. In Theorem 10, if we consider $\mathfrak{S}(\mathfrak{i}) = x^2$ with $\varepsilon = 1$ and $\mathcal{B}(\varepsilon) = \mathcal{B}(1) = 1$, we obtain the required output. □

Remark 7. *Taking $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ in above result, we obtain [12] [Proposition 1].*

Proposition 3. *Let $u, v \in \Re^+$, $u < v$; then, one has*

$$|A(u^n, v^n) - L_n^n(u,v)| \leq \frac{n(v-u)}{2}\left[E_1 \left| u^{n-1} \right| + E_2 \left| v^{n-1} \right| \right]. \tag{61}$$

Proof. In Theorem 10, if we consider $\mathfrak{S}(\mathfrak{i}) = x^n$, where n is an even number, $\varepsilon = 1$ and $\mathcal{B}(\varepsilon) = \mathcal{B}(1) = 1$, we obtain the required output. □

Remark 8. *Taking $h(\mathfrak{c}) = \frac{1}{\mathfrak{c}}$ in above result, we get [12] [Proposition 3].*

6. Conclusions

This paper provides some novel inequalities of the Hermite–Hadamard types based on h-Godunova–Levin and (h_1, h_2)-convex functions using Caputo–Fabrizio fractional integral operators. As mentioned in the remarks, many existing results in the literature become particular cases for these results. To demonstrate the reliability of our findings, we use examples to demonstrate their accuracy. In addition, the results associated with h-Godunova–Levin functions have some applications to special means. It would be interesting if people constructed these results using the Mittag–Leffler function as a kernel and generalized them as well as other similar results in the future.

Author Contributions: Conceptualization, W.A. and M.A.; investigation, W.A., W.H., M.D.l.S., and W.A.; methodology, W.A. and M.A.; validation, A.M.M., M.A., W.A., and W.H.; visualization, M.D.l.S., W.A., and A.M.M.; writing—original draft, W.A., M.D.l.S., and M.A.; writing—review and editing, W.A. and M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research work was funded by the Basque Government , Grant IT 1555-22.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

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