Investigating the Role of Philosophy of Mathematics as the Bridge to Meaningful Mathematics Teaching in the Classroom

By

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ABSTRACT

This dissertation investigates the intricate relationship between the philosophy of mathematics and mathematics education, particularly within the South African context. The comprehensive exploration comprises five chapters, each with a distinct focus.

Chapter 1 serves as an introduction and contextualization of the study. In Chapter 2, there is an in-depth review of the philosophy of mathematics, exploring various schools of thought, their influence on mathematical knowledge, and their compatibility with teaching practices. This chapter also includes an examination of the landscape of mathematics education in South Africa. Chapter 3 delves into the research methodology, utilizing analytical autoethnography and a variant of grounded theory to collect insights from personal experiences and mathematics educators.

Chapter 4 unveils compelling findings, revealing how integrating philosophical perspectives into the classroom enhances curiosity, problem-solving skills, and interdisciplinary connections. Moreover, it investigates the infusion of philosophy into South African mathematics education, elucidating both challenges and opportunities.
In the concluding Chapter 5, the study draws from these insights to formulate actionable recommendations. These recommendations offer a roadmap for elevating mathematics education by merging philosophical concepts and pedagogical strategies, ultimately enriching the learning experience for South African learners.

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RESEARCHER'S FASCINATION WITH MATHEMATICS

FIGURE 1 What is infinity?

Source of FIGURE 1: Singh, Sunil (2021) Chasing Rabbits: A Curious Guide to a Lifetime of Mathematical Wellness Impress, LP

As a mere six-year-old, I am stirred from sleep by the ethereal glow of the full moon cascading over my face. An overpowering sense of awe envelops me as I lay there, my young mind grappling with an enigma that has echoed through the corridors of time – "What is infinity?" (see FIGURE 1)

In that profound moment, the heart of a child experiences an intangible chill, gripped by an unseen hand of icy trepidation. The absence of an answer, coupled with the absence of someone to consult, left an indelible mark, etching the question into the tapestry of my existence.

This solitary instance transforms into a guiding compass, guiding the trajectory of my lifelong pursuit. Over time, I am bestowed with the realization that the cosmos is an orchestra of countless other wondrous inquiries. The nature of infinitesimals, the fabric of space, the essence of matter and energy, the enigma of numbers – all these riddles invite contemplation.

Much akin to the grand birth of a universe, the remarkable instrument of mathematics unfurls its dimensions in every conceivable direction, evolving into the complex marvel that it is today. Eminent philosopher-mathematicians have ventured in search of an enduring scaffold upon which this exponential expansion can be securely erected.

Mathematics possesses the remarkable ability to traverse realms both infinite and infinitesimal, yielding answers with unwavering precision and unwavering accuracy. It assumes the role of an intrepid guide, illuminating pathways beyond our physical grasp.

I posit that a mathematics teacher, heart imbued with the spirit of a philosopher, wields a unique power. Within the haven of her classroom, she nurtures and kindles the boundless imaginations of her young charges. The amalgamation of mathematical pedagogy and philosophical contemplation forms a haven where learner’s minds can stretch, explore, and unfold in safety.

DECLARATION OF ORIGINALITY

I, Isabel Maria Schreiber, declare that this dissertation is my own original work. Where secondary material is used, this has been carefully acknowledged and referenced in accordance with university requirements.

I understand what plagiarism is and am aware of university policy and implication in this regards.

Student signature.

Date: 31/08/2023

ETHICS DECLARATION

I, Isabel Maria Schreiber, have obtained, for the research described in this work, the applicable research ethics approval (see ANNEXURE A).

Date of approval: 9 September 2022

Research ethics number:22958119 (HUM 007/0722)

I, Isabel Maria Schreiber, declare that I have observed the ethical standards required in terms of the University of Pretoria's Code of Ethics for Researchers and Policy Guidelines for Responsible Research.

Student signature.

Date: 31/08/2023

Chapter 1 - Introduction and Contextualization

1.1 Introduction

The state of mathematics education in South Africa sparked considerable apprehension due to the poor academic results of learners, highlighting concerns about the efficacy of professional development programs for mathematics teachers (Mouton et al., 2013:33). This study aimed to explore how integrating training in the philosophy of mathematics could enhance various aspects of lesson planning and teaching, such as fostering better comprehension, promoting in-depth understanding, and facilitating the recognition of interconnected concepts. The study explored the foundational aspects of mathematics, delving into various epistemological perspectives and ontological viewpoints that underlay the understanding of mathematics. The primary focus was on assessing the interconnections between these philosophical aspects and the field of mathematics teaching. To clarify, epistemology pertained to how we acquired and comprehended knowledge, while ontology dealt with our understanding of reality and existence.

Furthermore, this inquiry explored the specific ways in which integrating philosophy into the development of mathematical competencies could have impacted mathematics education. The investigation focused on the context of training mathematics educators in South Africa, aiming to address the pressing concerns that had been identified within the current system. By examining how philosophical perspectives could have informed and enriched teaching methodologies and pedagogical approaches, this study sought to contribute to the ongoing efforts to improve mathematics education in the region.

To ensure a coherent alignment between the study's objectives and the research methodology, this investigation adhered to qualitative research principles. More precisely, the study adopted an analytical autoethnographic approach in conjunction with a variant of grounded theory. The autoethnographic approach allowed the researcher to critically analyse their own experiences within the context, providing nuanced insights.

The grounded theory variant, with its reiterative analysis, offered a systematic means to construct a theoretical framework that expounded the potential impact of philosophical perspectives on mathematics education. This particular variant of grounded theory entailed the iterative analysis of diverse datasets, diverging from the conventional approach of analysing data obtained from repetitious interviews involving distinct groups. By taking this methodological trajectory, a more comprehensive exploration of the identified issue was facilitated, thereby enhancing the nuanced interpretation of the amassed data (Timans et al., 2019). Analytical autoethnography involved a critical analysis of the interplay between the philosophy of mathematics and personal experiences, presenting outcomes in an academically rigorous format (Pace, 2012). In contrast, this variant of grounded theory entailed a systematic dissection of relevant data, culminating in the construction of a theoretical framework of themes that explained a specific phenomenon (Charmaz 2006, Creswell 2009, Dey 1999).

This research specifically targeted teachers engaged in teaching mathematics across the Intermediate Phase, Senior Phase, and Further Education Training (FET) Phase in South Africa. The Senior Phase assumed particular significance, acting as a pivotal juncture in the mathematical development of learners by bridging the gap between the arithmetic-focused Intermediate Phase and the more advanced algebraic and calculus-focused FET Phase.

1.2 Research Question

The central research issue revolves around the observed deficiency in the training of mathematics educators in South Africa, which subsequently impairs their capacity to effectively teach mathematics and hinders the learning progress of their learners.

Research question:
Could philosophy of mathematics be the bridge to meaningful mathematics teaching in the classroom?

1.3 Research Problem

Various complex factors influence learning in the mathematics classroom, such as learners' attitudes, home conditions, teacher training, teaching methods, and school conditions (Ingersoll, et al., 2009). Much research exists connected to the abovementioned complex issues and many attempts to address these have been made. However, there is a concern that these activities are aimed at improving teaching methods at the cost of mathematical content and philosophy (Labuschagne, 2016; Chand, et al., 2021).

In the specific context of South Africa, mathematics educators face unique challenges. These challenges may include resource limitations, diverse learner populations with varying levels of preparedness, and a historical context that has shaped the landscape of mathematics education. Many teachers adhere to a transmission-based pedagogical approach, positioning themselves as the sole authoritative figures within the classroom (Moss, 2016). This disposition can be attributed to a variety of factors, including inadequate training, time constraints, or demanding workloads. However, this instructional paradigm may not be optimally aligned with the principles conducive to effective mathematics teaching and learning. This incongruity is notably accentuated by the distinctive nature of mathematical knowledge, which possesses an inherent uniqueness characterized by its precision, logical coherence, and abstract nature. Moreover, learners' imperative to uncover concealed patterns and intricate interconnections within mathematical concepts further exacerbates this incongruence. In light of these considerations, the integration of the philosophy of mathematics emerges as a potential remedy, furnishing teachers with a refined toolkit to steer learners in their quest for these insightful revelations.

The central objective of this study is to delve into the prospective role of the philosophy of mathematics in mitigating the prevalent challenges within mathematics education in the context of South Africa. As Paul Ernest aptly puts it, "… *the philosophy of mathematics is undoubtedly an important aspect of mathematics education*" (Ernest, 2016).

In a UNISA Study Guide aimed at mathematics teachers at the master's level, Ruttkamp et al. (2004) expound on this notion:
"*Given the historical and philosophical context of South African academia, science, and education, it is evident that a natural inclination towards a historical or philosophical perspective has not been pervasive... In our pursuit, we aspire for learners to impart mathematical knowledge within a firm and well-founded historical-philosophical framework. This aspiration is driven by the inherent uniqueness of mathematical knowledge... Philosophically, knowledge is invariably situated within specific contexts, although certain*

transcendent features persist. Honouring the knowledge embodied by mathematical processes — a perspective inherently integral to any mathematics teacher's perception of the subject — mandates that the subject be presented not in a vacuous or, worse yet, an unfounded void. This risk is mitigated by anchoring mathematical education within a historical-philosophical framework."

Within this comprehensive framework, this research aspires to reconcile an apparent dichotomy between theoretical constructs and practical implementation within the domain of mathematics education in South Africa. Furthermore, it seeks to invigorate renewed discourse regarding the intricate and consequential realm of the philosophy of mathematics, thereby redefining its significance and role within the pedagogical landscape of mathematics teaching.

In his essay about the philosophy of mathematics education, Stephen Brown (1995) asks a very pertinent question by posing a trichotomy (see FIGURE 2). Is the philosophical focus or dimension:

i. Philosophy applied to or of mathematics education.
ii. Philosophy of mathematics applied to mathematics education.
iii. Philosophy of education applied to mathematics education.

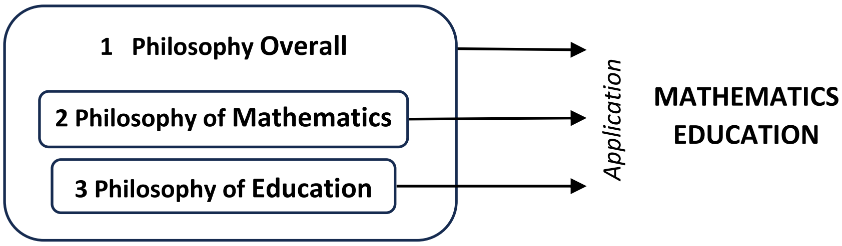

FIGURE 2 Application of Philosophy in Mathematics Education

Each of these three possible 'applications' of philosophy within mathematics education underscores distinct focal points that inherently illuminate diverse issues and challenges. These 'applications' not only manifest as substantial domains in their own right but also encompass intricate relationships and interactions spanning individuals, society, social constructs, knowledge representations, and communicative practices (Ernest, 2016).

In this context, the trichotomy suggests three perspectives through which philosophy interfaces with mathematics education, exploring how philosophy influences the teaching and learning of mathematics and vice versa. These dimensions help dissect the intricate connections between philosophical viewpoints and pedagogical practices within the realm of mathematics education.

This study centres its attention on the philosophy of mathematics applied to mathematics teaching, with the central premise being that the restoration or reintegration of the philosophy of mathematics at the heart of mathematical teaching is poised to address the deficit in the comprehension of mathematical concepts, structures, and their interrelations. Traditional views of professional development presume that we can train teachers to faithfully enact instructional methods and strategies that have proven effective elsewhere (Ernest, 2016). This is an inappropriate conception of professional development for teachers seeking to develop classroom practices that place learners' reasoning at the centre of instructional decision making. Instructional strategies that build on learner reasoning cannot simply be transferred to a new setting because, by their very nature, such adjusted instructional practices require refinement by teachers who have the necessary philosophical reflection, intellectual framework, and support to analyse, evaluate, and appropriately adjust practices based on learner understanding (Ernest, 2016).

This study aspires to address inquiries such as:

a) Will exposing teachers to the philosophy of mathematics potentially augment the effectiveness of teaching and learning mathematics?
b) What philosophies, skills, and knowledge can optimally equip teachers to meet their commitment to nurturing mathematical proficiency in every learner?

1.4 Proposed Research Methodology

Conducting research at the intersection of the philosophy of mathematics and education presents a vast intellectual landscape (Ernest, 2016). In this master's study, the choice of employing a mixed qualitative research approach serves as a deliberate means to illuminate the nuanced connections between philosophy and the mathematics classroom.

Philosophical aspects or considerations in the field of mathematics education are not receiving as much attention or emphasis as they should. (Labuschagne, 2016).

This research endeavours to bridge this gap by explicitly integrating experimental philosophy, a robust and valid form of philosophical inquiry, into its methodological framework. The deliberate inclusion of experimental philosophy aligns with the aim to investigate and elucidate the philosophical underpinnings shaping mathematics education. This methodological choice seeks to navigate the interplay between empirical investigation and philosophical inquiry within the domain of the philosophy of mathematics, as advocated by Knobe and Nichols (2018).

The significance of grasping the ontological presumptions (related to the nature of existence or reality) associated with mathematical knowledge and the consequent epistemological assumptions (related to the nature of knowledge and how it is acquired) is paramount in the context of shaping methodological choices for philosophical inquiry (Brown, 1995).

Hence, the conscious alignment of research methodologies with these philosophical underpinnings is not only essential but serves as a crucial justification for exploring the intricate connections between philosophy and mathematics education. This study recognizes the significance of philosophical methodology, particularly experimental philosophy, in elucidating the philosophical nature of the research while concurrently addressing pedagogical aspects within the realm of mathematics education.

1.4.1 Analytical Autoethnography

Analytical Autoethnography, as outlined by Denzin (2014), involves the rigorous analysis of data while intertwining the researcher's personal experiences and insights. This method enables a comprehensive exploration of individuals' experiences within a specific context, offering nuanced insights into the studied phenomenon (Ellis et al., 2011). The utilization of Analytical Autoethnography in this study aims to provide a deeper understanding of the interplay between the philosophy of mathematics and its impact on mathematics education.

1.4.2 Variant of Grounded Theory

Grounded theory, as expounded upon by Glaser and Strauss (2014), entails the systematic collection and analysis of data within a specific context, with the aim of comprehensively understanding a particular phenomenon. This study will utilize a variant of grounded theory, often referred to as abbreviated grounded theory (Willig, 2008). This variant does not focus on developing an emerging theory. Instead, it aims to identify emerging themes using some of the coding principles of grounded theory within the data sets at hand after a single data collection cycle. Unlike full grounded theory methodology, which involves multiple cyclical processes continuing until an emerging theory takes shape and saturation is reached, this adaptation is necessary due to time and resource constraints.

The qualitative data analysis methodology, anchored in this adapted grounded theory framework, will facilitate a thorough exploration of the potential role of philosophy of mathematics in enhancing the effectiveness of mathematics teaching. Through the amalgamation of diverse data sources, adherence to consistent coding procedures, and the triangulation of evidence, the analysis endeavours to uncover significant themes, relationships, and implications. The overarching goal is to shed illuminating insights on the potential impact of philosophy of mathematics in enhancing the quality and meaningfulness of mathematics education.

To gather data, the research will employ a multi-faceted approach, including literature reviews, informal interviews, discussions, and anonymous online questionnaires involving mathematics teachers across Grades 4 to 12. The collected data will be contextualized within the framework of relevant research reports, such as TIMSS (2019) and SACMEQ IV (2017).

1.5 Population and Sampling

"The population is the entire group that you want to draw conclusions about. The sample is the specific group of individuals that you will collect data from" (Mc Combes, 2021).

The selection of the Ekurhuleni North District in Gauteng was driven by practical considerations such as convenience, proximity, and financial constraints. However, this choice was accompanied by several challenges inherent to the sampling process. Some schools displayed reluctance to participate due to various factors including limited interest, concerns about disruptions, and potential distrust of the education department's involvement in research initiatives. These challenges affected the selection process, influencing the number and types of schools ultimately included in the study. Despite these obstacles, a purposive sampling approach was employed to carefully select a sample of 10 schools, comprising a mix of private and public institutions. These selected schools collectively represent around 6% of the total schools in the Gauteng region, and the inclusion of 27 mathematics educators from these schools contributes valuable insights to this research.

1.6 Data Management and Security

The collected data are subject to meticulous data management and security measures to ensure their integrity, security, and long-term availability. These measures include the utilization of Microsoft OneDrive for data storage and backup due to its robustness and accessibility. Additional security layers are implemented, such as antivirus software and multifactor authentication, to fortify protection against potential security breaches. Moreover, to safeguard participants' privacy, data will be anonymized during the analysis process. Access to the data is restricted exclusively to the researcher and supervisor, adding an extra layer of safeguarding to protect participants' privacy. Furthermore, the decision to retain the data for a duration of 10 years is driven by the necessity to uphold its preservation for potential future references, validations, and scholarly inquiry.

1.7 Possible Contribution of the Study

By delving into the integration of the philosophy of mathematics into South African mathematics education, this dissertation not only holds the promise of arousing a heightened interest in this intriguing subject of mathematics but also envisions practical implications. The exploration of philosophical underpinnings in education has the potential to bridge the existing training gap for educators, equipping them with a more refined pedagogical toolkit (Ernest, 1991; Pimm, 1987). By introducing philosophical perspectives, teachers can potentially engage students in a more meaningful manner, promoting deeper understanding and sparking curiosity (Lakatos, 1976; Cobb & Bauersfeld, 1995). Additionally, the investigation could pave the way for innovative pedagogical strategies that harness the essence of mathematics philosophy, enriching the overall learning experiences of students (Papert, 1980; Freudenthal, 1973). Through its potential to reshape teaching methodologies, this study aims to stimulate further research in the realm of mathematics education (Bishop, 1988).

1.8 Scope and Parameters of the Study: Acknowledging Constraints and Strategies for Rigor

In the pursuit of comprehending the philosophy of mathematics and its impact on learners, this study recognizes the inherent parameters guided by factors such as time and financial considerations. The sample selection strategically focuses on schools within the researcher's district of origin, Ekurhuleni North, particularly within Kempton Park, Gauteng. While this choice may impose a slight limitation on the generalizability of findings beyond this locale, the study adopts a comprehensive approach. It delves into a multifaceted examination of diverse dynamics contributing to learners' general challenges in mathematics achievement. While the primary emphasis is on exploring the philosophy of mathematics, the investigation extends its scope by integrating insights from teachers, heads of departments, and school principals. This broader approach aims to offer a comprehensive understanding of the intricate interplay between philosophical concepts and the practical challenges encountered within the educational landscape.

It is crucial to note that, although these constraints exist, the study is committed to employing strategies that address and mitigate potential limitations. Diligent efforts will be made to select a diverse range of schools within the chosen district, taking into account factors such as socioeconomic status and learner demographics. Additionally, proactive measures will be taken to ensure minimal disruptions to daily routines in participating schools, with a firm commitment to flexible research schedules and minimal intrusion into instructional activities. Leveraging the collaboration and support of the Gauteng Education Department (refer to ANNEXURES B1 & 2) will facilitate cooperation from schools and alleviate concerns related to research participation. These measures are designed to optimize the study's reliability and validity, acknowledging, and proactively managing potential limitations.

1.9 Summary

This chapter introduces the research topic, which focuses on the integration of the philosophy of mathematics into mathematics education in South Africa. The aim is to investigate whether incorporating training in the philosophy of mathematics can enhance lesson planning and teaching effectiveness. The research will explore the foundations of mathematics, different epistemological and ontological approaches, and their relevance to mathematics teaching. The study will use qualitative research methods, including analytical autoethnography and a variant of grounded theory, to analyse data collected from Grades 4 to 12 mathematics teachers.

The envisioned research journey holds the potential to address the existing void in the training of mathematics teachers, while concurrently refining pedagogical approaches and fostering enriched learning experiences for learners. Additionally, this study has the potential to make a significant contribution to the field of mathematics education and research by shedding light on the role of philosophy in enhancing teaching practices and learner learning outcomes.

However, it is acknowledged that limitations, including the restricted selection of sample schools within the researcher's district, complexities in isolating the effects of specific practices within the educational milieu, reluctance among schools to partake in research initiatives, and the need for direct coordination with individual schools, may impact the study's scope and implications. Despite these challenges, this research endeavours to advance our understanding of the relationship between philosophy and mathematics education, ultimately benefiting teachers and learners alike.

1. 10 Outline of Dissertation

Chapter 2: Literature Review

The literature review chapter offers a comprehensive overview of the philosophy of mathematics, discussing characteristics of mathematical knowledge, the nature of mathematical statements, and philosophical thought. It explores different schools of thought in the philosophy of mathematics, such as logicism, formalism, intuitionism, Platonism, and fallibilism. The chapter also examines the relationship between the philosophy of mathematics and mathematics teaching, considering the implications of different philosophies for classroom practices. Additionally, it explores the literature on mathematics education in South Africa, discussing the status of mathematics in public schools, curriculum policies, learner performance, and issues related to mathematical literacy and the language of instruction.

Chapter 3: Research Methodology and Design

This chapter introduces the research methodology used in the study. It discusses the research design, encompassing a qualitative research methodology in the data collection approach, with analytical autoethnography and a variant of grounded theory as research designs within the data approaches. The chapter covers data collection methods, such as questionnaires, field notes, and informal interviews. It also examines the categories of data, sampling and research sites, trustworthiness considerations, and ethical concerns. The study's limitations are discussed, and a conclusion is also presented.

Chapter 4: Research Results

This chapter presents a compelling exploration of the transformative power of the philosophy of mathematics in the classroom. Combining analytical autoethnography and a variant of grounded theory as data approaches, this chapter critically examines the researcher's own experiences and insights related to the integration of philosophical perspectives in teaching mathematics.

Through analytical autoethnography, the researcher delves into personal reflections, providing a unique perspective on the subject matter. Additionally, the variant of grounded theory analysis used offers empirical insights by exploring the perspectives of mathematics teachers in grades 4 to 9 in public and private schools. The chapter's findings shed light on how the philosophy of mathematics can enhance teaching and learning experiences, foster curiosity, promote problem-solving skills, and encourage interdisciplinary connections. This research contributes to a deeper understanding of the role of philosophy in nurturing teachers' and learners' appreciation for mathematics.

This chapter further focuses on the integration of the philosophy of mathematics into mathematics education in South Africa. Drawing on the data collected through the variant of grounded theory used, it systematically analyzes the dynamics contributing to learners' poor achievement in mathematics. The chapter explores the emerging theory and its implications for teaching and learning, with a specific focus on the philosophy of mathematics. The findings from the grounded theory study are presented and discussed."

Chapter 5: Conclusion and Recommendations

In the final chapter, the researcher provides recommendations for integrating the philosophy of mathematics into mathematics education in South Africa. Based on the findings and analysis from the analytical autoethnography and the variant of grounded theory studies, the chapter proposes strategies and resources to enhance the training of mathematics educators, improve pedagogical approaches, and create meaningful learning experiences for learners.

The concluding section of this chapter encapsulates a summary of the outlined recommendations, enriched by the perspectives of other South African education specialists. These insights collectively contribute to an informed assessment of their potential ramifications on the landscape of mathematics education within South Africa.

Chapter 2 - Literature review

2.1 Navigating the Mathematical Landscape: Exploring the Philosophy of Mathematics in Mathematics Education

2.1.1 Introduction

The journey into the philosophy of mathematics within the realm of mathematics education unfolds as a critical expedition to comprehend the intricate underpinnings of this discipline. As we traverse the mathematical landscape, it becomes evident that the philosophy of mathematics is not merely an esoteric pursuit but a cornerstone that shapes the very essence of how mathematics is understood, taught, and learned.

In this chapter, we embark on a quest to explore the profound interplay between the philosophy of mathematics and mathematics education. Avigad (2007) contends that unravelling the philosophical dimensions of mathematical knowledge is pivotal for advancing our understanding of mathematics as a discipline and, consequently, for refining our approaches to mathematics pedagogy.

The need for a comprehensive literature review is imperative at this juncture. A robust foundation in existing scholarship is paramount to navigating the philosophical landscape of mathematics effectively. By delving into the existing body of knowledge, we aim to discern the historical evolution of philosophical thought in mathematics (Kitcher, 1983), identify key concepts that have shaped its trajectory, and illuminate the various philosophical perspectives that have influenced mathematical education (Ernest, 1991).

This literature review serves as a compass, guiding our exploration through the philosophical intricacies of mathematics. By engaging with the scholarly discourse (Hersh, 1997), we aspire to not only uncover the theoretical underpinnings but also discern their practical implications for mathematics teaching. As we embark on this intellectual endeavor, it is crucial to acknowledge that a comprehensive understanding of the philosophy of mathematics lays the groundwork for fostering a more profound and nuanced approach to mathematics education.

In addition to establishing a robust philosophical foundation, the literature review endeavours to contextualize education within the challenging landscape of mathematics education in South Africa.

2.1.2 Characteristics of Mathematical Knowledge

Mathematical knowledge is a unique form of understanding that possesses distinctive characteristics, as outlined by Avigad (2007). This section explores the nature of mathematical knowledge and its essential attributes, shedding light on what sets it apart from other forms of knowledge.

i The Nature of Mathematical Knowledge

At its core, mathematical knowledge is marked by two fundamental qualities that differentiate it from other knowledge systems: certainty and consistency.

a) Certainty:

Mathematical knowledge stands as a beacon of certainty. Unlike many other knowledge systems, where doubt can linger, mathematical truths are unwavering. When something is proven true in mathematics, there is no room for scepticism or alternate interpretations. For instance, the statement 1 + 2 = 3 is known with absolute certainty.

b) Consistency:

Another defining characteristic of mathematical knowledge is its remarkable consistency. Once a mathematical truth is established, it remains unaltered and impervious to modification or correction through subsequent learning or discovery. The equation 1 + 2 = 3, for example, maintains its truth and structure throughout time.

Avigad (2007) further elaborates on the nature of mathematical knowledge, highlighting its status as not only necessarily true but also contingently true. The contingency aspect means that some mathematical truths may depend on specific conditions or assumptions. For instance, the statement "John is either 29 years old or 30 years old" is contingently true, contingent upon John's actual age.

Beyond these fundamental characteristics, mathematical knowledge serves a multifaceted role. It fosters mental discipline, sharpens logical reasoning, and instils rigor. It provides a framework of consistent and necessary truths that underpin the entire mathematical discipline, offering both a tool for solving practical problems and a source of intellectual exploration.

ii The Philosophical Exploration

The unique attributes of mathematical knowledge have captivated philosophers and mathematicians for millennia. These distinct characteristics raise profound questions about the nature of truth[1] and the relationship between mathematics and reality.

The roots of the philosophy of mathematics stretch back to antiquity, with philosophers like Plato (428/427 or 424/423 – 348/347 BC) offering their theories. In his seminal work *"The Republic,"* Plato posited numbers as mathematical entities housing inherent truths. According to Plato, these mathematical entities exist autonomously within a distinct realm known as the world of Forms (Monk, 2007). This perspective effectively detaches numbers from our empirical world.

Plato's profound recognition of the significance of mathematics in philosophy is exemplified by his famous inscription at the entrance to his academy: "*Let no one ignorant of geometry enter here.*" He underscored the pivotal role of mathematics as a foundational element of philosophical inquiry, emphasizing its ability to train the mind in precise reasoning and contemplation of abstract truths.

Platonism, which posits the existence of abstract mathematical entities, continues to

[1] In this context, "truth" pertains to the philosophical contemplation and investigation of the inherent nature of mathematical concepts, their resonance with reality, and the fundamental verities they represent. This inquiry involves questioning whether mathematical principles are products of human construction or whether they embody objective realities that exist autonomously from human cognition. The reference to the "disconnect between the study of mathematics and its relation to truth" underscores an exploration of how the teaching and comprehension of mathematics are influenced by these philosophical inquiries into the essence of mathematical truths (Shapiro, 1997).

albeit in a new form known as structuralism[2]. Plato's view that numbers exist objectively and abstractly aligns with his belief in a formal reality beyond our sensory perceptions. He considered our sensory experiences as inconsistent, while he regarded formal mathematical knowledge as consistent and superior, showcasing the power of reason over our senses (Monk, 2007).

In contrast to Plato's perspective, Aristotle, one of Plato's students, held different views. Aristotle argued that numbers were not independent objects but rather properties of objects. For instance, the number four is a property of a collection of four objects. Unlike Plato, Aristotle did not posit a separate world of Forms. He saw mathematics as a description of features of our physical world. This fundamental disagreement between Plato and Aristotle represents early tensions between philosophical theories concerning mathematics (Monk, 2007).

Advancing several centuries to the Enlightenment era[3] where in his notable work "*Meditations on First Philosophy*" the mathematician and philosopher René Descartes played a pivotal role in the development of rationalism[4]. His innovative approach to geometry, connecting it to algebra and introducing Cartesian coordinates, revolutionized both mathematics and philosophy. Descartes' contributions significantly shaped modern epistemology, emphasizing the importance of reason and mathematical thinking (Hatfield, 2018).

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[2] Structuralism in mathematics focuses on studying mathematical objects based on their relationships and structural properties rather than individual elements or interpretations. It emphasizes the importance of patterns, symmetries, and transformations within mathematical structures, aiming to uncover deep connections across different branches of mathematics (Shapiro, 1997).

[3] The Enlightenment era in mathematics roughly spans from the late 17th century to the late 18th century. It was a period characterized by significant advancements in mathematical thought and the application of reason and logic to mathematical problems. During this time, mathematicians such as Isaac Newton, Gottfried Wilhelm Leibniz, Leonhard Euler, and many others made groundbreaking contributions to fields such as calculus, number theory, probability theory, and algebra. The Enlightenment era marked a shift towards rigorous mathematical reasoning, mathematical formalism, and the establishment of mathematical foundations that laid the groundwork for modern mathematics (Posy, 1992).

[4] Rationalism in mathematics prioritizes reason and logical deduction in the acquisition of mathematical knowledge, emphasizing innate understanding and logical consistency.

Immanuel Kant (1724-1804), another influential philosopher, was a transcendental idealist who proposed that our knowledge is limited to phenomena—the way things appear to us. Kant believed that mathematics indirectly describes our world, with its source lying in sensory experience. In his seminal work *"Critique of Pure Reason"* Kant posited that mathematical knowledge is synthetic a priori, meaning it is both derived from experience and necessarily true. Kant's transcendental idealism offered a framework for understanding how mathematical concepts shape our perception of the phenomenal world (Avigad, 2007).

Moving into the 20th and 21st centuries, the philosophy of mathematics has continued to evolve and influence various fields. The emergence of mathematical logic, set theory, and formal systems through the works of mathematicians like Gödel, Russell, and Tarski expanded the horizons of mathematical philosophy. These developments led to profound insights into the nature of mathematical truth, the limits of formal systems, and the relationship between mathematics and logic (Avigad, 2007).

Additionally, contemporary philosophers of mathematics, such as Quine and Putnam, have engaged in debates about the indispensability of mathematics in scientific theories and the nature of mathematical objects. The ongoing discussions in the philosophy of mathematics continue to shape our understanding of this dynamic field, demonstrating its enduring relevance in the modern intellectual landscape (Avigad, 2007).

Avigad (2007) further categorizes knowledge claims according to Kant's framework, creating a technical terminology for the philosophy of mathematics:

a) **Analytically true sentences**: These are necessarily true propositions, such as "All bachelors are unmarried."
b) Analytical knowledge does not add new information to the subject; it is a tautology[5].

[5] In mathematics, a tautology refers to a statement or proposition that is always true, regardless of the truth values of its components. It is a logical expression that is true in every possible interpretation or combination of truth values for its variables. Tautologies play a fundamental role in mathematical reasoning and proof, serving as foundational principles or logical truths that are relied upon to derive further conclusions. They provide a basis for establishing the validity and consistency of mathematical arguments (Russell et al., 1910).

c) **Synthetically true sentences**: These propositions are not necessarily true and require empirical verification. For example, "All bachelors are unhappy" is a synthetic proposition that may or may not hold true based on empirical evidence.

d) **A priori truth**: Propositions that are knowable independently of experience, derived from reason and not contingent on empirical observations. For instance, "all squares are polygons."

e) **A posteriori truth**: Propositions that are knowable dependent on experience, derived from empirical observations. For instance, "the coffee is sweet."

Kant believed that mathematics, as synthetic a priori knowledge, falls into a fifth category: knowledge that is both necessarily true and derived from sensory perception. Geometry, as a form of spatial intuition, relates to our perception of three-dimensional Euclidean space. Arithmetic, dealing with numbers as properties of temporal sequences, corresponds to our experience of time as a one-dimensional sequence. Together, they provide the framework for our spatial-temporal understanding of the world (Avigad, 2007).

While this overview provides a glimpse into the landscape of the philosophy of mathematics, it is essential to engage in a more comprehensive analysis of the nature of mathematical knowledge. Subsequent sections will further explore the interconnectedness between objective mathematical truths and the empirical world, the role of axioms and proofs, and the various schools of thought within the philosophy of mathematics, offering a more comprehensive understanding of this intriguing field.

2.1.3 The Nature of Mathematical Knowledge

Knowledge, a fundamental concept in the realm of human understanding, is a term frequently invoked in discussions across various disciplines. In the field of epistemology, which delves into the nature and limits of human knowledge, knowledge is distinct from mere belief or opinion. It stands as an understanding founded on conclusive evidence, supported by justification, and aligned with truth. While beliefs and opinions can be volatile and subject to doubt, knowledge, in its truest sense, provides a robust foundation for uncovering reliable truths about the diverse array of concepts and subjects that comprise our understanding of the world (Baruwa et al., 2022). This section will explore the nature of knowledge and its characteristics, differentiating it from beliefs, and then delve into the specific attributes of mathematical knowledge.

Mathematics has long been regarded as a source of the most certain knowledge available to humankind. Before exploring the nature of mathematical knowledge, it is important to consider knowledge in general. While ontology studies the existence of things humans can know, epistemology delves into the nature of human knowledge—its source, landscape, limitations, justification, reliability, and certainty. Epistemology plays a crucial role in core philosophy, philosophy of mathematics, and philosophy of education (Baruwa et al., 2022).

Firstly, it is crucial to differentiate between knowledge and belief or opinion. Beliefs and opinions are subject to doubt and volatility. In contrast, knowledge is typically based on conclusive evidence. This type of knowledge allows us to discover reliable truths about various concepts and subjects, including:

a) laws of physics and natural phenomena,
b) biological processes and ecosystems,
c) historical events and timelines,
d) economic principles and market trends,
e) cultural practices and societal norms,
f) linguistic structures and communication patterns, and
g) geographical features and landscapes.

These examples illustrate the diverse range of concepts and subjects to which knowledge can be applied to uncover reliable truths. It deals with factual knowledge, which involves meaningful statements that assert something about the universe (Corcoran, 2007).

Beliefs alone cannot establish something as truth. You may believe that the Earth is flat, but that belief does not make it true. Therefore, knowledge must be a belief that is true, supported by justification. For example, the belief that "the Earth is spherical" can be justified by the evidence provided by scientist Galileo Galilei, who proved it beyond a reasonable doubt (Baruwa et al., 2022). Plato defined knowledge as a "justified true belief," although even such beliefs can be questioned (Akinpelu, 1981). Gettier (1963) emphasized the importance of understanding the nature of knowledge fully, as it relates to our understanding of ourselves. However, there is no consensus among epistemologists, and the questions surrounding knowledge remain puzzling. The conceptual possibilities regarding knowledge are still abundant.

Bamisaiye (1989) highlights four key criteria of knowledge: certainty, validity, veracity, and utility. Knowledge must be valid, proved, reliable, consistent, certain, and of direct benefit to the knower, with the potential for creating new knowledge. In the context of mathematical propositions, the belief condition must be supported by the truth condition, according to Sheffler (1965), and Woozley (1949).

The core of mathematical knowledge is built upon the foundation that allows us to establish the truth of mathematical statements, often achieved through logical reasoning. As explained earlier, truth in mathematics refers to how well mathematical statements match the logical reality they represent. A mathematical statement is considered true if it accurately describes the properties, relationships, and patterns of mathematical objects or ideas, following the accepted basic principles, rules, and logical thinking within the mathematical system. A mathematical proof is a finite sequence of statements that concludes with a proposition, satisfying the property that each statement is an axiom drawn from a previously stipulated set of axioms, or is derived from evidence and reasoning. The set of axioms includes all statements admitted into a proof without demonstration, such as axioms, postulates, and definitions (Baruwa et al., 2022).

For example, the statement "1 + 1 = 2" can be proven using the transitive property of equality. Any number times one equals itself, and any number multiplied by two equals itself plus itself. Therefore, 1 * 2 = 2 and 1 * 2 = 1 + 1 or 5 * 2 = 5 + 5. According to the transitive property of equality, if x = y and y = z, then x = z. Since both 1 + 1 and 2 are equal to 1 * 2, they must be equal to each other, resulting in 1 + 1 = 2. This proof establishes "1 + 1 = 2" as an item of mathematical knowledge or truth definitions (Baruwa et al., 2022).

In general, mathematical knowledge consists of mathematical statements justified by proofs, which rely on mathematical axioms and deductive reasoning. This reasoning has been accepted for nearly 2,500 years (Maddy, 1990). The subsequent section will further discuss the nature of mathematical statements.

2.1.4 The Nature of Mathematical Statements

At the core of mathematical knowledge lies the exploration of mathematical statements—propositions or assertions within the realm of mathematics that can be either true or false. These statements serve as the building blocks of mathematical reasoning and proof. Understanding the nature of mathematical statements is pivotal, as mathematicians rely on them to construct valid arguments and rigorously establish their validity (Lakatos, 1976). Moreover, mathematical statements often transcend the boundaries of mathematics, finding applications in science and philosophy. This subsection elucidates the significance of mathematical statements and their role in the broader landscape of knowledge and discovery.

The nature of mathematical knowledge is closely linked to the nature of mathematical statements. Mathematical knowledge is derived from the study of mathematical statements, which are propositions or assertions made within the realm of mathematics. These statements can be either true or false and form the foundation of mathematical reasoning and proof. As stated in the previous section, mathematical knowledge is based on the systematic exploration and analysis of mathematical statements using logical deduction and rigorous proof techniques. Mathematicians aim to establish the truth or falsity of these statements by constructing valid arguments that demonstrate their validity (Lakatos, 1976).

Often, abstract patterns that mathematicians study for their own sake later prove to be useful in science. For example, Riemann geometry served as a precursor to the General Theory of Relativity. Over the past two decades, there has been increased attention to mathematical explanation in both science and mathematics. Philosophers of science and philosophers of mathematics have collaborated to bridge the gaps between them. While progress has been made, more empirical evidence is needed to better understand the variety of explanatory uses of mathematics within the context of science (Mancosu, 2018).

In a letter to Robert Thornton in 1944, Albert Einstein stated, "*A knowledge of the historic and philosophical background gives that kind of independence from prejudices of his generation from which most scientists are suffering. This independence, created by philosophical insight, is—in my opinion—the mark of distinction between a mere artisan or specialist and a real seeker after truth*." This quote attributed to Albert Einstein, captures his belief in the significance of historical and philosophical understanding in fostering independent thinking among scientists.

Philosophical benefits stem from multiple directions (Mancosu, 2018):

1. **Mathematics Empowering Thought Formulation:** Mathematicians, scientists, and philosophers have often highlighted the remarkable power of mathematics in formulating new scientific theories. Wigner (1960) famously claimed that *"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve*." Kant asserted that "*In any special doctrine of nature, there can be only as much proper science as there is mathematics therein*" (Watkins et al., 2014)

2. **Mathematics Unveiling Phenomena's Secrets:** Mathematics is the study of quality, pattern, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth through rigorous deduction from carefully chosen axioms and definitions. Causal explanations in science are based on the assumption that finding and mathematically explaining the cause of a phenomenon leads to an understanding of the phenomenon (Burgess, 2015).

3. **Indispensability Argument:** The philosophy of mathematics grapples with a significant challenge regarding the *justification of mathematical beliefs* concerning abstract objects. This issue is highlighted by the renowned Indispensability Argument, which seeks to provide a rationale for our acceptance of mathematical truths without relying solely on intuitive or rational insight. The argument underscores the pivotal role of mathematics not only in making empirical predictions but also in providing elegant and concise representations of scientific theories, such as through equations. The language of mathematics serves as an indispensable tool in the realm of science, making it

exceedingly difficult to envision formulating complex theories like quantum mechanics and general relativity without a substantial reliance on mathematical concepts (Colyvan, 2023).

However, this apparent harmony in the role of mathematics in science is countered by a philosophical conundrum initially posed by Paul Benacerraf in 1973. This dilemma pertains to the potential incompatibility between plausible explanations of mathematical truth and knowledge. To illuminate this discord, we turn to two contrasting philosophical positions: Mathematical Realism and Mathematical Constructivism.

- **Mathematical Realism**, one of these positions, posits that mathematical truths exist independently of human thought or existence. In essence, it asserts that mathematical objects and facts possess an objective reality, and our mathematical knowledge comprises the discovery of these pre-existing truths. From a mathematical realist perspective, irrational numbers such as the square root of 2 or pi are considered inherent mathematical truths, not contingent on human cognition, and can be unearthed through rigorous mathematical proofs.
- **Mathematical Constructivism** advocates that mathematical knowledge is a product of human mental activity. In this viewpoint, mathematical objects and truths are not perceived as existing independently but are rather created by humans through cognitive processes. Within this framework, irrational numbers, for instance, are not pre-existing entities waiting to be discovered but are actively constructed through methods like Dedekind cuts or infinite decimal expansions. In essence, mathematical constructivists contend that we do not discover the existence of irrational numbers; we bring them into existence as needed.

This fundamental disparity between mathematical realism, which posits the independent existence of mathematical truths, and mathematical constructivism, which asserts that mathematical knowledge is a human creation, gives rise to a philosophical schism in our understanding of mathematical truth and knowledge.

It is worth acknowledging that this philosophical debate has persisted in the realm of mathematics for over a century. Various mathematicians and philosophers hold diverse views on this matter, with some attempting to reconcile these opposing perspectives, while others staunchly defend one position over the other (Marcus, 2007).

In light of this ongoing philosophical discourse, the Indispensability Argument in the philosophy of mathematics endeavours to resolve this paradox by demonstrating that our epistemology aligns with conventional forms of mathematical truth claims. This argument seeks to provide a rationale for our knowledge concerning abstract mathematical objects, such as numbers and structures, grounded in empirical evidence and experiential factors. It also acknowledges the possibility of error or revision in our understanding, adhering to the principle of fallibilism. The core of the indispensability argument comprises two premises: first, that mathematical objects exist if they are indispensable to our best scientific theories, and second, the recognition that, indeed, mathematical objects are essential components of our scientific theories. Consequently, the argument concludes that belief in the existence of abstract mathematical objects is a justifiable stance (Marcus, 2007).

2.1.5 The Characteristics of Philosophical Thought

The pursuit of mathematical knowledge and philosophical thought may at first glance appear distinct, each with its unique domain and methods. However, as we delve into the nature of mathematical knowledge, we find an intriguing connection between these seemingly disparate realms. Mathematical knowledge, rooted in rigorous logic and deduction, shares common ground with philosophical thought in its quest for truth, unity, and deeper understanding. In this section, we explore the distinctive characteristics of philosophical thought and its relevance to the pursuit of mathematical knowledge. By shedding light on the shared traits and the dynamic interplay between these disciplines, we aim to highlight the value of philosophical inquiry in fostering critical thinking and enriching mathematical education (Cahan, 2003; Frank, 1952; Pecorino, 2001; Solomon & Higgins, 2017).

As highlighted in earlier sections, knowledge is never a frail assertion but stands firm when supported by proof. Yet, what kind of thought can persistently reason and seek truth? In his work 'Introduction to Philosophy,' Professor Phillip Pecorino (2001) presents a comprehensive exposition of philosophical thought's characteristics. Pecorino (2001), alongside eminent philosophers like Russell (1912), Rescher (2000), and Scruton (1996), underscores that philosophical thought embodies distinct traits that set it apart as a singular mode of inquiry.

Philosophical thought is:

1. **Critical and Comprehensive**: It vigorously questions issues, revealing their interconnection and underlying unity (Scruton, 1996; Pecorino, 2001).
2. **Analytic and Synthetic**: It applies reason to relate and coordinate human-made thoughts, ideas, and observations. It employs both analytical and synthetic approaches to establish relationships and unity among concepts (Rescher, 2000; Pecorino, 2001).
3. **Practical and Theoretical**: It is useful when questioning assumptions, beliefs, current presuppositions, common sense, ideas, and it can also be abstract in that it is concerned with or involving the theory of a subject or area of study rather than its practical application (Pecorino, 2001).
4. **Logical and Empirical**: It is consistent, coherent, pragmatic, adequate and applicable (Russell, 1912; Pecorino, 2001).

Scholars further elaborate on philosophical thought and state that such thoughts:

1. **Probe Assumptions**: They diligently examine assumptions, meanings, word usages, beliefs, and theories.
2. **Develop Clear Definitions**: They strive for precise definitions of propositions and maintain expression precision.
3. **Embrace Logical Criteria**: They uphold consistency and coherency:
   a) Consistency avoids contradiction and ensures term meanings remain constant.
   b) Coherency requires meaningful terms and phrases that relate cohesively.
4. **Hold Empirical Criteria:** They follow adequacy and applicability:
   a) Adequacy ensures all evidence is considered in analysis and explanation.
   b) Applicability ensures relevance to available evidence.
5. **Are Comprehensive**: They unveil interconnections of issues, offering insights into diverse aspects of existence. They reveal unity across issues, providing insights into varied life facets (Solomon & Higgins, 2017; Pecorino, 2001).

6. **Are Synthetic**: They harmonize ideas, values, and distinctions to answer fundamental questions.
7. **Are Practical**: They employ inquiry to find solutions, analyse phenomena, and assist planning. They practically question assumptions, beliefs, and common sense (Pecorino, 2001).
8. **Are Speculative**: They explore beyond practical matters, tackling abstract problems and metaphysical issues. They advance human thought and better the human condition (Rescher, 2000; Pecorino, 2001).

Although the above provides a comprehensive characterization of philosophical thought, it is still incomplete. The question remains: What is the value of philosophy, and why should it be studied? In his book "The Problems of Philosophy," Bertrand Russell (1952) suggests that some individuals, including prominent figures in science, economics, and social studies, tend to doubt whether philosophy is anything more than meaningless small talk, hair-splitting distinctions, and debates about topics where knowledge seems unattainable. However, Russell argues that the value of philosophy lies primarily in its ability to provide knowledge that gives unity and structure to various scientific and mathematical disciplines. Knowledge resulting from critical examination of our convictions, prejudices, and beliefs is something worth pursuing (Russell, 1952).

There exists another frequently misunderstood point: the moment a subject becomes susceptible to definite understanding, it sheds its philosophical label and transforms into an autonomous scientific discipline. For instance, consider astronomy – once a facet of philosophy, yet over time, it progressed into an independent scientific domain. Similarly, the scrutiny of the human mind, once intertwined with philosophy, metamorphosed into the distinct realm of psychology. Furthermore, historical evidence reveals that up until the 19th century, the prevailing term for exploring the physical realm was "natural philosophy." This encompassing term contained not only what we presently denote as physics but also subsumed disciplines like botany, zoology, anthropology, and chemistry within its expansive framework (Cahan, 2003).

Consequently, the perceived uncertainty inherent in philosophy often masks a subtler reality. Questions that have clear and definite answers usually end up being seen as sciences. On the other hand, the questions that don't have those clear answers yet, they stay in the philosophy realm for now (Russell, 1952). This way of looking at this issue helps us tell when something has become a science and when it's still a philosophy.

As early as 1789, Immanuel Kant sought to determine the limits and scope of metaphysics (metaphysics explores the "big questions" about reality, existence, and the nature of the universe, often venturing into areas that go beyond what can be observed or measured directly). Einstein, in his book "The Origin of the Separation between Science and Philosophy," expressed that there has been a disconnect between scientists and philosophers since the 19th century. Scientists became sceptical of philosophical speculations that seemed imprecise and addressed insoluble problems. Likewise, philosophers lost interest in specific sciences due to their narrow focus. This separation, however, has been detrimental to both philosophers and scientists (Frank, 1952). Einstein also noted that highly capable students, those with independent judgment, showed great interest in the theory of knowledge. They engaged in discussions about the aims and methods of science, emphasizing the importance of such issues to them (Frank, 1952).

Regarding mathematics, it is worth considering which mathematics teacher would not want their learners' thinking patterns to resemble the characteristics mentioned above. We aim for our learners to develop metacognition, integrate knowledge, and constructively pursue problem-solving strategies. Therefore, introducing this mode of inquiry into the mathematics classroom would benefit both teachers and learners. As teachers, it is our responsibility to be role models and facilitators of such thinking.

2.1.6 Conclusion

Section 2.1 has provided a comprehensive exploration of mathematical knowledge and its intersection with philosophy. It has illuminated the distinct characteristics that differentiate mathematical knowledge, notably its unwavering certainty and enduring consistency. These qualities underscore the foundational role of mathematics in both abstract and empirical domains. The section has also ventured into the broader realm of epistemology, distinguishing knowledge from mere belief and elucidating the criteria that underpin its validity, including justification and truth.

This inquiry has revealed that mathematics, often celebrated for its unequivocal certainty, has historically been a focal point of philosophical inquiry. Philosophical luminaries such as Plato and Aristotle have contributed contrasting perspectives on the ontological status of mathematical entities, offering valuable insights into the relationship between mathematical truths and empirical reality. Immanuel Kant's transcendental idealism has further enriched this discourse by positing that mathematical knowledge is synthetic a priori, signifying a synthesis of experiential elements and necessary principles.

The section has underscored the pivotal role of mathematical statements in shaping our understanding of the world, transcending disciplinary boundaries, and finding applications in both scientific and philosophical realms. The symbiotic relationship between mathematics and science, as exemplified by Albert Einstein's perspective, underscores the profound influence of mathematical reasoning on the comprehension of the natural world. Furthermore, this exposition has unveiled the distinctive characteristics of philosophical thought, highlighting its inherent connection with mathematical inquiry. Philosophical thought, characterized by its critical, synthetic, and speculative nature, shares common ground with mathematical reasoning in their pursuit of truth, unity, and deeper comprehension. Both disciplines are instrumental in fostering critical thinking and enriching educational experiences.

In summary, Section 2.1 has laid a solid foundation for the exploration of the philosophy of mathematics. It has elucidated the unique attributes of mathematical knowledge, its pivotal role in shaping our understanding of reality, and its intricate relationship with philosophical investigation. As we delve further into this captivating field, we are poised to delve deeper into the nature of mathematical truths, the significance of axioms and proofs, and the diverse array of philosophical schools of thought that continue to enhance our grasp of mathematical knowledge.

2.2 Schools of Philosophical Thought

2.2.1 Introduction

To gain insight into the diverse schools of philosophy of mathematics, it is imperative to grasp the intellectual milieu from which they emerged. During the late 19th and early 20th centuries, mathematicians and philosophers undertook the task of establishing a solid foundation for mathematics. This endeavour led to the emergence of logicism, intuitionism, and formalism. These approaches sought to establish the reliability and irrefutability of mathematical truths. However, despite their noble goals, these classical perspectives encountered technical challenges and faced critical scrutiny (Rosenhouse et al., 2017).

2.2.2 The Crisis in the Foundation of Mathematics

The crisis in the foundation of mathematics came to the forefront with Bertrand Russell's discovery of an inconsistency in Gottlöb Frege's "naive set theory," known as Russell's paradox (Avigad, 2007). The paradox arises when we "consider the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself. Hence the paradox" (Irvine and Deutsch 2021). In concrete terms, the paradox considers a set of men that do not shave themselves but are only shaved by a barber. If this barber does not shave himself, it leads to a contradiction because if the barber shaves himself, he shouldn't be in the set, but if he doesn't, he should be in the set, but that would mean that he does indeed shave himself, so he should be both inside and outside of the set. This paradox challenged Frege's principle of comprehension, which aimed to define mathematical concepts using logical terms.

2.2.3 Reactions to the Crisis

The crisis in the foundation of mathematics prompted three main reactions: (1) logicism, (2) formalism, and (3) intuitionism, all of which rejected Platonism (Avigad, 2007). Bertrand Russell formulated logicism as a response, aiming to establish mathematics as an extension of logic. David Hilbert advocated for formalism, which emphasized the manipulation of symbols within formal systems. Luitzen Egbertus Jan Brouwer promoted intuitionism, focusing on the idea that mathematical truth is derived from constructive mental processes. Overall, these different schools of thought emerged as responses to the challenges posed by the crisis in the foundation of mathematics, each proposing distinct ways to address the foundational issues and reconcile mathematical knowledge.

2.2.4 Logicism and Its Challenges

Logicism, championed by Gottlöb Frege and Bertrand Russell, aimed to establish mathematics as a branch of logic. Frege dedicated a significant portion of his career to demonstrating how mathematical concepts could be derived from logical foundations. He aligned logicism with Platonism, emphasizing the existence of abstract mathematical entities. Russell expanded upon this work through his seminal work "*Principia Mathematica*" (1910), focusing on number theory. However, logicism has experienced a decline in popularity over time (Rosenhouse et al., 2017).

Factors Contributing to the Decline of Logicism

The decline of logicism as a foundational approach in the philosophy of mathematics can be attributed to several factors:

1. Gödel's Incompleteness Theorems:

One pivotal development was Kurt Gödel's Incompleteness Theorems in the early 20th century. These theorems demonstrated that any formal mathematical system with sufficient expressive power is either incomplete (containing true statements that cannot be proven within the system) or inconsistent (yielding both true and false statements). This undermined the central aim of logicism: reducing all of mathematics to a consistent and complete logical system.

To grasp the significance of Gödel's Incompleteness Theorems, consider the example of prime numbers. Prime numbers are integers greater than 1 that have no divisors other than 1 and themselves, such as 2, 3, 5, and 7. The conjecture "There are infinitely many prime numbers" is a mathematical truth but proving it within a specific set of rules can be challenging. Gödel's theorems revealed that there are limits to what any given mathematical system can prove.

2. Rise of Set Theory and Axiomatic Approaches:

Another significant shift in the field was the rise of set theory and axiomatic approaches as alternative frameworks for understanding the logical foundations of mathematics. Set theory, particularly exemplified by Zermelo-Fraenkel set theory (ZF), gained popularity for providing a more flexible and robust foundation.

3. Challenges in Various Mathematical Fields:

The strict reductionist approach of logicism encountered significant challenges as mathematicians and scholars recognized that mathematics extended beyond the confines of pure logic. Disciplines like geometry and arithmetic introduced principles and methodologies that defied complete reduction to logical formalisms. For instance, Euclidean geometry, which explores geometric shapes and spatial relationships, featured a fundamental axiom known as the "Parallel Postulate." This postulate stipulates that when presented with a straight line and a point not lying on it, one can draw exactly one parallel line to the given line, no matter how far both lines extend.

A compelling endeavour emerged within mathematical circles, aiming to establish the Parallel Postulate's truth through rigorous logical reasoning, analogous to proving basic arithmetic truths like "1 + 1 = 2." However, despite concerted efforts, mathematicians were unable to demonstrate its veracity solely through pure logic. Instead, they uncovered a surprising revelation - the Parallel Postulate stood apart from other axioms of Euclidean geometry. It assumed the character of a special rule that defied complete explication or proof using only logic and the existing geometric principles. This profound discovery disrupted the prevailing notion that all of mathematics could be reduced to the realm of pure logic. It underscored the existence of mathematical elements, such as the Parallel Postulate in geometry, that resisted full comprehension through logical means alone. This realization instigated a paradigm shift in the philosophy of mathematics, reshaping our understanding of the intricate relationship between mathematics and logic (Shapiro, 1997).

Furthermore, other illustrative examples underscore the challenges posed by geometry and arithmetic to strict logical reductionism:

a) **Geometry Challenge**: Consider the Pythagorean theorem, which asserts that in a right-angled triangle, the square of the hypotenuse's length equals the sum of the squares of the other two sides' lengths. While this theorem can be expressed and deduced symbolically using logical rules, its essence lies in the spatial relationships inherent to the triangle. It necessitates measurements, visualizations, and a profound grasp of geometric properties. The attempt to entirely reduce the Pythagorean theorem to pure logic overlooks the visual and spatial dimensions that define geometry's uniqueness. This example exemplifies how geometry introduces concepts that transcend formal logic, emphasizing the indispensable role of spatial understanding (Corry, 2007).

b) **Arithmetic Challenge**: In the realm of arithmetic, the concept of "equality" offers another illustration. In formal logic, equality is represented by a symbol (e.g., "="), accompanied by specific logical rules governing its usage. However, mathematics imbues equality with a more profound significance. When asserting "2 + 5 = 7," it extends beyond a mere logical assertion; it encapsulates a mathematical truth rooted in structures, operations, numerical entities, and their intricate relationships. This goes beyond the confines of formal logic, highlighting the multifaceted nature of equality in mathematics (Shapiro, 2000).

4. Emergence of Alternative Schools of Thought:

The emergence of alternative schools of thought, namely intuitionism and constructivism, further challenged the traditional logicist perspective. These approaches emphasized the role of intuition and constructive methods in mathematical reasoning, contrasting with logicism's focus on formal proof systems.

5. Practical Obstacles:

Practical obstacles arose in attempting to fully formalize all of mathematics within a single logical system. Certain areas of mathematics, such as higher-order arithmetic and analysis, proved difficult to fully capture within a purely logical framework.

6. Diversification of Mathematics:

As mathematics evolved and diversified into specialized areas, the relevance and practicality of reducing all of mathematics to a purely logical foundation diminished. Consequently, logicism experienced a decline in popularity as a comprehensive foundational approach in the philosophy of mathematics.

Nonetheless, despite its decline, logicism retains historical significance, and various philosophical perspectives continue to contribute to discussions about the nature and foundations of mathematics (Brown, 2008; Mendelson, 2010).

2.2.5 Formalism

Formalism, established by David Hilbert, is an anti-Kantian school that expresses mathematics as formal, logical systems without concerning itself with their meaning. According to formalists, mathematical claims are meaningless strings of symbols manipulated according to precise formal rules. Hilbert argued that meaningful mathematical claims could be expressed as finite strings of claims about finite numbers. However, claims that cannot be expressed in this manner are considered devoid of meaning (Rosenhouse, et al 2017).

Formalism can be approached in different ways, with metamathematical formalism developed by Haskell Curry being one of the simplest and most straightforward versions. Curry's stance emphasizes that mathematics deals with symbols and rules for manipulating them, namely formal systems. For formalists, "4 is even" does not imply the number 4 itself is even, but rather that the sentence "4 is even" follows from arithmetic axioms. Formalism is often associated with nominalism[6] (Avigad, 2007).

[6] Nominalism: the view according to which mathematical objects, relations, and structures do not exist at all e.g., the quantity of 3 apples does exist but the number 3 is a label that exists only in our heads. Retrieved from: Bueno, Otávio, "Nominalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2020 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/fall2020/entries/nominalism-mathematics/>.

While Hilbert and his contemporaries developed formal systems for logic, geometry, and set theory, Gödel's Incompleteness Theorems showed the limitations of formalism. These theorems demonstrated that one cannot prove the consistency of a mathematical theory within that theory, rendering the formalist program untenable.

In any reasonable mathematical system, there will always be true statements that cannot be proved (Ruttkamp, et al., 2004).

2.2.6 Intuitionism

Intuitionism, proposed by Luitzen Egbertus Jan Brouwer and further developed by Arend Heyting, is a school of thought influenced by Immanuel Kant's philosophy. It challenges the notion of a pre-existing universe of mathematical entities waiting to be discovered and instead sees mathematics as a constructive activity. Intuitionists argue that mathematical entities, including numbers, are products of mental acts performed by mathematicians. According to intuitionism, mathematical claims can be considered true if they are supported by constructive arguments, relying on explicit recipes for constructing the objects in question (Ruttkamp et al., 2004).

While Brouwer and Heyting were both associated with the constructivist school of thought, they held differing perspectives on the relationship between mathematics and language. Brouwer advocated for a radical view, asserting that prearranged logic fails to capture all the rules for correct mathematical thought. He believed that mathematical thinking and proof construction transcend the limitations of language and are conscious activities (Ruttkamp et al., 2004). In contrast, Heyting, influenced by Brouwer's ideas, developed intuitionistic logic, which aimed to provide a formal system aligned with constructivist principles (Ruttkamp et al., 2004). Although Brouwer and Heyting shared a constructivist approach[7], their specific views on the role of language and formalism within mathematics differed significantly.

[7] Constructivism in mathematics is a philosophical perspective that highlights the active role of mathematicians in constructing mathematical knowledge. It posits that mathematical objects and concepts are not discovered but rather created through mental processes and human activities. This viewpoint rejects the notion of an independent mathematical reality and emphasizes the constructive nature of mathematical truth.

A significant point of contention arose between Brouwer and David Hilbert regarding the position of logic. Hilbert considered logic as an independent and complete science that could be freely applied to other branches of mathematics, whereas Brouwer believed that logic should only come after the construction of mathematics. Intuitionism requires an underlying logic that rejects the Law of the Excluded Middle, which states that for any statement 'p,' either 'p' or not 'p' is true. This rejection of non-constructive existence proofs distinguishes intuitionism from other approaches (Ruttkamp et al., 2004).

While intuitionism gained some acceptance in the early 20th century, particularly in foundational debates, its divergence from classical mathematics, particularly in areas such as mathematical analysis, reduced its popularity among the mathematical community (Troelstra et al., 1988).

2.2.7 Perspectives from Philosophy: Impact on Mathematics Teaching

Within the realm of philosophy, differing viewpoints on the nature of mathematical knowledge construction have significant implications for mathematics education.

2.2.7.1 Intuitionism

Intuitionism posits that mathematical knowledge is primarily constructed through individual mathematical intuition and the mental processes of mathematicians (Ernest, 1998). This perspective asserts that mathematical objects and truths arise from intuitive processes and that mathematical knowledge is inherently subjective, constructed through personal cognitive activities.

2.2.7.2 Constructivism

In contrast, constructivism emphasizes that learners actively build their understanding of mathematical concepts and problem-solving strategies. It places importance on learner-centred approaches, hands-on experiences, and active engagement in knowledge construction. Constructivism views mathematical knowledge as a product of learners' interactions with their environment and social interactions (Ernest, 1998; Bishop, 1988;

Lerman, 1996). These insights have profound implications for pedagogical approaches in mathematics education.

2.2.7.3 Platonism

Historically, Platonism dominated mathematical philosophy until the 20th century. This perspective posits that mathematical entities exist in an abstract realm separate from the physical world. Platonists view mathematics as a process of uncovering pre-existing truths rather than inventing or constructing them (Ruttkamp et al., 2004). However, Platonism faced challenges over time.

One early challenge emerged with the discovery of irrational numbers, like √2, which cannot be expressed as the quotient of two integers. Complex numbers and alternative geometries, such as Riemannian geometry, further questioned the Platonist stance. These developments expanded the Platonist view to encompass coherent mathematical constructs. Albert Einstein's Theory of Relativity, based on non-Euclidean geometries, revealed that the curvature of physical space challenges the conventional Euclidean geometric perspective (Monk, 2007).

The introduction of complex numbers added to the setbacks faced by Platonism, as these numbers demonstrated independent existence. Non-Euclidean geometries, discovered by mathematicians like Nikolai Lobachevsky and Bernhard Riemann, challenged the assumptions of Euclidean geometry and had significant implications for theoretical physics, particularly in the realm of relativity theory. Einstein's theory, rooted in Riemannian geometry, linked light, matter, and spacetime, transforming abstract mathematical insights into practical theories (Monk, 2007).

These revelations underscored that space could exhibit curvature, and alternative geometries are valid, with practical implications for the physical world.

2.2.7.4 Fallibilism

Imre Lakatos introduced fallibilism, an epistemological theory asserting that no belief or theory can be conclusively justified or logically supported. This perspective acknowledges

the potential for future falsification and revision and challenges the dominance of foundationalism and absolutism in mathematics (Lakatos, 1976). Fallibilism recognizes that mathematics, like natural sciences, is subject to criticism, revision, error, and correction. It underscores the evolutionary nature of mathematics and rejects the notion of a final, unassailable body of absolute truths (Ruttkamp et al., 2004).
These philosophical perspectives play a crucial role in shaping approaches to teaching and learning mathematics and underscore the dynamic, evolving nature of mathematical knowledge.

2.2.8 Conclusion

In the exploration of philosophical schools of thought in mathematics, we have witnessed the profound impact of diverse perspectives on the foundational understanding of this discipline. The late 19th and early 20th centuries marked a pivotal period when mathematicians and philosophers grappled with the task of establishing a robust foundation for mathematics. This endeavour gave rise to logicism, intuitionism, and formalism, each seeking to solidify the reliability and certainty of mathematical truths.

However, the journey through these philosophical schools revealed intricate challenges and critical examinations. Russell's paradox exposed the limitations of naive set theory, sparking the crisis in the foundation of mathematics and prompting various reactions. Logicism, epitomized by Frege and Russell, aimed to position mathematics as an extension of logic but faced substantial challenges, notably through Gödel's Incompleteness Theorems and the emergence of alternative foundational frameworks.

Formalism, championed by Hilbert, viewed mathematics as a manipulation of formal symbols, yet it grappled with its own limitations, as exemplified by Gödel's theorems. Intuitionism, advanced by Brouwer and Heyting, emphasized mathematical knowledge as a construct of mental acts and constructive processes, challenging classical mathematical thought but struggling to gain widespread acceptance.

These philosophical perspectives, while experiencing shifts in popularity, continue to influence the teaching and learning of mathematics. Intuitionism and constructivism highlight the importance of actively constructing mathematical understanding, while Platonism's recognition of abstract mathematical entities persists in influencing mathematical thought. Fallibilism reminds us of the dynamic and evolving nature of mathematical knowledge.

In essence, the philosophical landscape of mathematics reflects the ongoing dialogue between foundational rigor and the practical needs of mathematics as a living, evolving discipline. These schools of thought, though often in contention, collectively contribute to the rich tapestry of mathematical inquiry, underscoring the multifaceted nature of this timeless and ever-evolving field.

2.3 Philosophy of Mathematics in Mathematics Education

2.3.1 Introduction

Understanding the nature of mathematics and its role in nurturing knowledge and creativity among learners is a pivotal aspect of mathematics education (Ernest, 2012). Furthermore, there exists a significant correlation between a teacher's personal philosophy of mathematics and the experiences of learners within the mathematics classroom (Ernest, 2012; 1996).

Renowned French philosopher and mathematician René Thom highlighted the importance of integrating a philosophy of mathematics into pedagogy, asserting that regardless of teachers' awareness, their teaching approaches are inherently shaped by their philosophy of mathematics (Thom, 1973; Ekeland, 2002). Thom's perspective underscores that even when teachers are not consciously aligning with a particular philosophy, their teaching practices are influenced by it (Ekeland, 2002).

By adopting a philosophy of mathematics in pedagogy, educators can deepen learners' understanding and engagement with mathematical concepts, promoting a profound appreciation for the subject and fostering independent thinking. Thom's approach encourages teachers to move beyond mere theorem proving, delivering meaningful mathematical experiences that stimulate critical thinking and creativity.

2.3.2 Philosophy of Mathematics and Teacher Beliefs

This section delves deeper into the nature and significance of philosophy of mathematics in education, drawing insights from various scholars.

Philosophers, mathematicians, scientists, and educators have constructed philosophical frameworks to interpret the world and apply them to their respective domains (Thom, 1973; Ekeland, 2002). These frameworks involve the intellectual process of clarifying confusion, challenging assumptions, evaluating opinions, and seeking reasons (Ernest, 2012).

Importantly, philosophy of mathematics is not solely a teaching method or personal belief; it constitutes a distinctive mode of thinking (Ernest, 2012). It involves scrutinizing worldviews, questioning conceptual frameworks, and synthesizing knowledge (Ernest, 2012), thereby integrating seamlessly with the objective of cultivating a holistic intellect and realizing human potential (Ernest, 2012).

As elucidated earlier, philosophy is a quest for wisdom and deeper comprehension of the essence of things (Ernest, 2012). It shapes cognition, behaviour, and interactions with the world. It isn't confined to subjective preferences, beliefs, or emotions; rather, it involves rational inquiry and broadening of experiences (Ernest, 2012).

The original inquiry revolved around whether philosophy of mathematics can serve as a bridge to meaningful mathematics teaching. This query doesn't advocate for the addition of philosophy as an optional component in teacher training. Instead, it underscores that philosophy should be inherently interwoven into the process of becoming and being a mathematics teacher. It's an exploration of why mathematics is intrinsic to science, the nature of mathematical phenomena, and the existence of mathematical structures. By embracing philosophy of mathematics, educators can enrich their comprehension and elevate their teaching practices (Cooney 1988).

Professor Thomas J. Cooney's research corroborates that teachers' views and beliefs significantly shape their instructional strategies (Cooney, 1988). Therefore, the philosophy of mathematics embedded in teachers' beliefs holds substantial sway over curriculum, lesson development, and assessment planning (Ernest, 1996). A study by A. G. Thompson also bolsters this assertion, revealing a congruence between teachers' conceptions of mathematics and their instructional approaches (Thompson, 1984).

Distinguishing between beliefs and knowledge poses a challenge, as the two often blur. Beliefs, deeply personal and immune to coercion, can arise from chance, experiences, or events. They encompass individuals' beliefs about themselves and others. For instance, a teacher might hold the belief that failing learners are lazy or that memorization is the sole path to learning mathematics. Socialization influences beliefs, shaping perceptions, behaviours, and information processing (Pajares, 1992).

While knowledge emanates from cognitive thought and beliefs from affective outcomes, beliefs also contain cognitive elements. While it's essential to differentiate them, research tends to focus on teachers' thought processes (Ernest, 1989). Cognitive knowledge inherently encompasses affective and evaluative dimensions. The notion that knowledge is pure and closer to objective truth often neglects the embedded judgment and evaluation. The intricate relationship between truth, knowledge, and judgment merits a more in-depth exploration.

The subsequent section delves into various epistemic knowledge systems within the realm of philosophy of mathematics.

2.3.3 Absolutist Philosophies of Mathematics

Absolutism encompasses several distinct views within the philosophy of mathematics, including logicism, formalism, intuitionism, and Platonism. Prominent proponents of logicism include Leibniz, Frege, Russell, Whitehead, and Carnap. Carnap, known as the founder of logical empiricism, defined logicism as the proposition that mathematics is reducible to logic, making it a mere part of logic (Leitgeb et al., 2023).

Absolutists perceive mathematical knowledge as certain and unchangeable, with the exception of logical truths and statements that are true by virtue of the meanings of terms. However, as alluded before, this perspective faced challenges when paradoxes and contradictions were discovered within mathematical thought processes (Kline, 1980).

Ernest (1991) notes that many philosophers, both modern and traditional, adhere to absolutist views, considering mathematical knowledge as neutral, pure, certain, and absolute. They argue that mathematical knowledge is firmly based on deductive logic. However, Ernest highlights that deductive logic merely channels mathematical truths through logically valid steps; it does not generate new knowledge. The certainty of a logical proof is no stronger than its weakest argument (Ernest, 1991).

A.J. Ayer, a philosopher and logical positivist, believed that the certainty of mathematics is beyond dispute. He argued that mathematical and logical truths are necessary and certain, constituting analytic propositions or tautologies. Ayer stated that a priori propositions derive their certainty from being tautologies, which are true solely based on the meanings of their constituent symbols and cannot be confirmed or refuted by empirical evidence (Ayer, 1946).

Research indicates that the notion of absolute truth in mathematics strongly influences the way mathematics is taught in classrooms (Ernest, 1988, 1989; Kantner, 2008). This absolutist perspective presents mathematics as rigid, fixed, logical, absolute, inhuman, cold, objective, pure, abstract, remote, and highly rational. Interestingly, this image aligns with the general public's perception of mathematics as difficult, cold, abstract, theoretical, and predominantly masculine. Mathematics is often viewed as isolated knowledge accessible only to a select few with "mathematical minds" (Ernest, 1996).

Dorothy Buerk (1982) states that an absolutist view of mathematics may manifest in classrooms through unrelated routine tasks that focus on applying learned procedures. The emphasis lies on each task having a unique, fixed, and objectively correct answer, with criticism and disapproval directed at any failure to achieve that answer. While this approach may not align with the philosophies of mathematicians or philosophers, it results in an absolutist conception of the subject.

2.3.4 Fallibilist Philosophies of Mathematics

Fallibilistic philosophy of mathematics, as highlighted by Ernest (1996), places emphasis on the practice of mathematics and its human aspect. Fallibilists acknowledge that mathematical knowledge, its proofs, and concepts are imperfect and open to revision. Imre Lakatos, a renowned Hungarian mathematician, and philosopher, rejects the absolutist image of mathematics and emphasizes the presence of controversy and human striving within the field. He likens mathematics to a gourmet restaurant, where perfect dishes are presented to patrons while chaos reigns in the kitchen. This analogy portrays an image of absolute mathematics to the outside world, but behind the scenes, mathematicians engage in constructing new knowledge amidst debate and the inevitable human challenges.

According to Ernest (1996), fallibilism recognizes both the processes and the products of mathematics as essential components of the discipline. Thomas Tymoczko, specializing in logic and philosophy of mathematics, argues that mathematics is not a body of pure and abstract knowledge existing in a superhuman, objective realm, but rather a set of social practices intertwined with history, individuals, institutions, symbolic forms, purposes, and power relations (Tymoczko, 1986). Fallibilistic views align with the values of connectedness, emphasizing relationships, empathy, caring, feelings, and intuition. This perspective tends to be holistic, human-centred, and appreciative of the wonder and beauty inherent in mathematics (Gilligan, 1982).

Conventionalism, another fallibilistic view, posits that the foundations of mathematics are based on linguistic conventions or agreements. Proponents of conventionalism include Quine, Hempel, and Wittgenstein. Fallibilists, such as C. S. Peirce, C. R. Popper, and Imre Lakatos, reject the absolutist image of mathematics and advocate for the idea that mathematical objects come into existence through social processes. However, these fallibilistic philosophies, along with logicism, formalism, and intuitionism, have been criticized for their fundamental weaknesses and their inability to provide a satisfactory account of mathematics (Brown, 2009).

Ernest proposes social constructivism as a philosophy of mathematics that draws upon conventionalism and Lakatos' quasi-empiricism. Social constructivism considers the social and linguistic foundations of mathematical knowledge and its connections to physics, social sciences, and computational mathematics. However, some argue that conventionalism and fallibilism are reductionistic, reducing mathematics to linguistic and social aspects of reality (Brown, 2009).

2.3.5 Reconciling Absolutism and Fallibilism in the Mathematics Classroom

The views of absolutism and fallibilism in mathematics present contrasting perspectives. Absolutism emphasizes the objective and value-free nature of mathematics, while fallibilism promotes a more connected, humanistic, and creative approach. These differing views raise the question of how to find a middle ground that can improve accessibility and transform the public and school perceptions of mathematics (Cockcroft, 1982).

According to research by Ernest (1996), there are mathematics specialists who combine absolutist ideas about the subject with positive and progressive attitudes towards mathematics and its teaching. On the other hand, non-mathematics-specialists, such as future primary school teachers, may not correlate fallibilist conceptions with positive attitudes towards mathematics and its teaching. This suggests that the relationship between beliefs and attitudes towards mathematics is complex and multifaceted.

While mathematics is often seen as consistent and neutral, many teachers and mathematicians recognize the value of embracing humanistic connected values in mathematics education. However, the association between an absolutist philosophy of mathematics and a transmission-style pedagogy is common due to the resonances and considerations between different aspects of an individual's philosophy, ideology, values, and belief systems (Ernest 1988, 1991).

Similarly, a fallibilist philosophy of mathematics combined with connected humanistic values can lead to a humanistic connected view of school mathematics and a classroom that communicates a connected image of mathematics. Interestingly, a deep commitment to progressive mathematics education can coexist with a belief in the objectivity and neutrality of mathematics, particularly among mathematics educators (Ernest, 1996).

Ernest argues that there is no conclusive evidence linking classroom results to the logical implications of a philosophy of mathematics. Additional values, aims, and assumptions are required to reach such conclusions. Although an absolutist philosophy of mathematics does not logically necessitate a transmission-style pedagogy, the two are often associated due to the alignment of different aspects of a person's philosophy, ideology, values, and belief systems.

2.3.6 Conclusion

In conclusion, the philosophy of mathematics plays a crucial role in shaping mathematics teaching and the experiences of learners in the classroom. It serves as a unique style of thinking, involving the examination of worldviews, questioning conceptual frameworks, and synthesizing knowledge. By embracing a philosophy of mathematics, teachers can enhance their understanding of the subject and promote meaningful teaching practices that foster critical thinking, creativity, and independent learning.

The philosophies of mathematics can be broadly categorized into absolutist and fallibilist perspectives. Absolutist views perceive mathematics as certain, unchangeable, and based on deductive logic. Fallibilistic philosophies, on the other hand, recognize the imperfections and openness to revision in mathematical knowledge, emphasizing the human aspect and the practice of mathematics. These contrasting perspectives raise important questions about finding a middle ground in classroom mathematics, one that combines the objective nature of mathematics with a more connected, humanistic, and creative approach.

It is important to note that a teacher's personal philosophy of mathematics significantly influences their instructional practices, curriculum content, lesson development, and assessment planning. The beliefs and perspectives held by teachers shape their teaching methods and approaches, impacting learners' engagement and understanding of mathematical concepts.

Finding a balance between absolutism and fallibilism can lead to an inclusive and transformative mathematics education that fosters a deeper appreciation for the subject and promotes positive attitudes towards mathematics. By incorporating both objective and humanistic values, educators can create a classroom environment that communicates a connected and accessible image of mathematics.

In moving forward, it is essential to continue exploring the philosophies of mathematics and their implications for teaching and learning. By critically examining our own beliefs and embracing a philosophy of mathematics that promotes inclusivity, creativity, and critical thinking, we can enhance mathematics education and empower learners to become independent thinkers and lifelong learners in the field.

The next section we will explore structuralism in the world of mathematics as a science of structures.

2.4 Exploring Structuralism: Mathematics as the Science of Structures

2.4.1 Introduction

Structuralism, a prominent philosophy of mathematics, fundamentally asserts that mathematics transcends the mere manipulation of numbers and quantities. Instead, it ventures into the realm of structures, offering profound insights into the nature of mathematical entities and their interconnectedness (Leitgeb, et al., 2023). As a philosophy of mathematics, structuralism explores the essence of mathematics itself, challenging conventional notions and emphasizing the study of structures as its core principle.

Rudolf Carnap, a distinguished philosopher, and mathematician, championed this perspective, contending that mathematics is, at its core, the science of structure. Under the banner of structuralism, the assertion stands firm: through the meticulous examination of structures, we can unveil the true nature of the objects they encompass. To illustrate this concept, consider the natural number structure—a paradigmatic example that resonates with any countably infinite system on the number line.

Within the natural number structure, one encounters a foundational initial object and a successor relationship that dutifully adheres to the induction principle. The profound essence of each natural number emerges from its intricate relationships with its numerical companions. In the structuralist worldview, structures themselves are conceived as ante rem universals, existing independently of other entities, and serving as the vessels that house mathematical objects within their intricate architectures.

Furthermore, structuralism invites us to view the language of structures as a concise representation of an ongoing dialogue—a profound discourse between systems of mathematical objects. Stewart Shapiro (2008), a luminary philosopher and mathematician, argues that while the realist ante rem structuralist[8] approach provides a clear and compelling account of mathematics, it also acknowledges the existence of alternative interpretations within the framework of structuralism.

Structuralism, as a philosophy of mathematics, diverges from other philosophical perspectives that regard mathematics as the study of numbers, quantities, basic set theory, or purely formal theories for calculation. Instead, structuralism highlights the significance of the relationships between mathematical objects within a structure and their connections to other structures (Reck et al, 2023).

Although structuralism emerged in the 1960s through the works of Paul Benacerraf and Hilary Putnam, it gained substantial attention in the 1980s and 1990s with contributions from scholars like Michael Resnik and Stewart Shapiro. Various challenges faced by structuralism have led to the development of different views within the field. Understanding these diverse perspectives and the challenges they address provides a contextual background and allows for a comparison with other philosophies of mathematics (Reck et al, 2023).

2.4.2 Different Perspectives of Structuralism

[8] Realism is a philosophical position that believes in the existence of an objective reality independent of our perception or language. In contrast idealism, which suggests that reality is fundamentally constructed by the mind or language. The ante rem structuralist views the relationship between mathematical structures and their places as inseparable and mutually dependent. The places only make sense within the context of the structure, and the structure gains substance and definition through its places. Therefore, the existence of both the structure and its places is regarded as necessary and independent of human thought or language. Retrieved from: Ahti-Veikko Pietarinen, "A Scholastic-Realist Modal-Structuralism", Philosophia Scientiæ 18-3 (2014) Logic and Philosophy of Science in Nancy (I)

In the realm of the philosophy of mathematics, a multitude of perspectives have been proposed by influential philosophers, each offering unique insights into the nature of mathematical entities and their relationships.

Paul Benacerraf advocates for a viewpoint known as structuralism, which posits that numbers are best understood as positions within specific structures rather than as independent entities. Benacerraf underscores the significance of relationships and progressions within structures such as the natural number sequence {1, 2, 3...} and the integer sequence {...-2, -1, 0, 1, 2...} (Benacerraf, 1965: 70).

In his article "*Mathematics Without Foundations*" Hilary Putnam, a distinguished philosopher, and mathematician, challenges the foundationalist stance of the widely accepted Zermelo-Fraenkel set theory (ZF) – an axiomatic system for mathematics (Putnam, 1983). Foundationalism posits that mathematical knowledge rests on a fixed set of axioms, serving as an immutable foundation. Putnam introduces an alternative perspective termed universalist structuralism, which rejects the notion that mathematical knowledge is solely predicated on axioms. Instead, it accentuates the significance of understanding the structural relationships between mathematical entities.

In the domain of arithmetic theorems, Putnam suggests scrutinizing them as "if-then" statements, transcending a narrow focus on foundational axioms to encompass how these axioms correlate with broader mathematical structures. They typically include statements asserting the existence of a starting number (often 0 or 1), rules for forming successive numbers (successor function), and principles such as mathematical induction that define the properties of natural numbers. The axioms serve as a basis for deriving the properties of arithmetic operations such as addition and multiplication over the set of natural numbers. Putnam's contention is that by probing the relationships between mathematical entities and their structures, a more comprehensive and pragmatic grasp of mathematics can be attained. This approach challenges the notion that mathematics is exclusively about symbolic manipulation and underscores the importance of apprehending the underlying structures and relationships inherent in mathematics (Reck et al, 2019).

Building upon Benacerraf's framework, Michael Resnik directs his focus towards pattern recognition in mathematics. Resnik perceives mathematical entities as positions within corresponding patterns, accentuating the importance of understanding mathematical statements for their literal meaning. Terms such as '0', '1', '2', and so forth, are construed as singular terms without necessitating specific structural associations (Reck et al, 2019). Stewart Shapiro presents a realist manifestation of structuralism, challenging nominalist, and constructivist viewpoints. He extends Resnik's concepts by considering positions within structures as "offices" that can be occupied by various objects. Shapiro proposes that these positions themselves can be treated as objects, alongside the abstract structure. He constructs a comprehensive theory of structures to support "ante rem structuralism," which posits the existence of structures independently of any particular embodiment (Reck et al, 2019).

Both Resnik (1997) and Shapiro (1997) acknowledge the presence of patterns and positions within mathematical structures, accentuating the concept of isomorphism – signifying essential similarity between distinct structures. Isomorphism is a formal mathematical concept denoting correspondence (Reck et al, 2019).
Geoffrey Hellman (2003) introduces eliminative structuralism, which concentrates on generalizing statements concerning positions within structures rather than centring on specific structures themselves. Hellman aims to formulate a form of "structuralism without structures" by substituting the abstract structures of Resnik and Shapiro with modal aspects. Hellman's modal structuralism offers a nominalist philosophy of mathematics asserting that mathematicians can systematically articulate truths without postulating the existence of mathematical entities (Reck et al., 2019).

These diverse philosophical perspectives within the realm of the philosophy of mathematics contribute significantly to ongoing debates and explorations, illuminating the foundations of mathematical knowledge and the intricate interconnections among mathematical concepts (Reck et al., 2019).

2.4.3 Structuralism in Mathematics Education

Warren (2005, 305) asserts, "*The power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, and the goal of mathematics learning*" (Mulligan, et al., 2009).

Research has shown that an understanding of pattern and structure is important in early mathematics learning. Low achievers consistently produced poorly organised pictorial (recognition) and iconic (associations) representations lacking in structure whereas high achievers used abstract notations with well-developed structures from the start (Mulligan, 2002).

Other research that focused on imagery and arithmetic have shown that learners who recognise the structure of mathematical processes and representations acquire deep conceptual understanding (Gray, et al., 2000; Pitta-Pantazzi, et. al., 2004; Thomas et al., 1995).

With regards to numbers, studies have examined the role of pattern and structure in young children's understanding of number concepts and processes, these include:

- counting – determining the total number of objects in a set or a group,
- subitising – look at a group of objects and realise how many there are without counting,
- partitioning – breaking large units into smaller units, and
- numeration – expressing numbers in words (Wright, 1994; Young-Loveridge, 2002).

In a study of partitioning, Hunting (2003) found that learners' ability to change concentration from counting individual items to identifying the structure of a group of numbers was fundamental to the development of their number knowledge. A study done by Cobb, et al., (1997) highlighted first graders' coordination of units of 10 and 1 in terms of the structure of collections. Similarly, Thomas, et al., (2002) recognised structural elements of the base ten-system found embedded in learners' images and recordings of the numbers 1 to 100, such as:

- grouping – for example associative law of addition (a + b) + c = a + (b + c)
- partitioning – learners aware that a two-digit number is made up of tens and ones, and

- patterning – building and recognising patterns.

Van Nes et al., (2014) also found a strong link between developing number sense and spatial structuring in Kindergartners' finger patterns and subitising structures. Studies of partitioning and part-whole reasoning (Young-Loveridge, 2002; Mulligan & Mitchelmore, 2009) indicate the importance of unitising and spatial structuring in developing fraction knowledge.

Extensive research on addition and subtraction concepts has highlighted young learners' strategies in recognising the structure of word problems (Mulligan & Vergnaud, 2006) as well as structural relationships such as:

- equivalence - 1 + 1 = 2 (left side is equivalent to right side),
- associativity - ((a + b) + c = a + (b + c)), and
- inversion - (inverse of a number 2 = $^{2}/_{1}$ and the inverse of that is ½) (Warren et al., 2003).

Also, studies conducted on multiplication and division have indicated that the composite structure is central to multiplicative reasoning and aids the learners in the transition from addition to multiplication (Confrey et al., 1995; Steffe, 1994).

With regards to measurement and space, some researchers have studied young learners' understanding of spatial structuring, defined by Battista (1999) as: "*The mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating, and combining these components, and establishing interrelationships between components and the new object*" (Battista, 1999: p. 418).

Carraher, et al., (2006) conducted research on early algebra and modelling which emphasised that young learners could grasp generalisations and develop abstract mathematical skills that reflect mathematical structure if they are exposed to it from an early age. Further studies have also supported this concept that young learners can develop functional thinking based on their understanding of structures (Blanton & Kaput, 2005). Other studies with young learners have highlighted the importance of structural relationships in the development of analogical reasoning (English, 2004).

2.4.4 Conclusion

In conclusion, structuralism offers a profound perspective on the nature of mathematics, focusing on the study of structures rather than merely numerical quantities. This philosophy, championed by philosophers like Benacerraf, Putnam, Resnik, and Shapiro, underscores the importance of relationships, patterns, and connections between mathematical entities within various structures. The diverse interpretations within structuralism, from ante rem universals to nominalist modal aspects, contribute to a rich philosophical landscape within the realm of mathematics.

In the context of education, the principles of structuralism find practical relevance in the mathematics classroom. Research has shown that understanding patterns and structures is fundamental to the development of deep mathematical understanding among learners. From early number concepts like counting and subitising to more complex operations like addition, multiplication, and spatial structuring, recognizing, and comprehending mathematical structures aids in building a strong foundation for mathematical knowledge. Moreover, exposure to structural thinking from an early age can pave the way for more advanced mathematical skills and abstract reasoning.

Moving forward, it's important to link these philosophical perspectives to effective mathematics teaching. By incorporating structuralist ideas, educators can foster learners' ability to recognize patterns, analyse relationships, and develop a deeper understanding of mathematical concepts. This connects seamlessly with the next section, 2.5, which delves into how different philosophies of mathematics, including structuralism, intersect with effective strategies for teaching mathematics.

2.5 Review of the Current State of Mathematics Education in South Africa

2.5.1 Introduction

This section aims to explore the South African education system and shed light on the challenges it faces, particularly in mathematics education. The system is currently trapped in a cycle characterized by weak Initial Teacher Education (ITE), ineffective Continuous

Professional Development (CPD), and below-average learning outcomes, resulting in a decline in teacher standings (Taylor, 2022). Understanding the problems and reasons behind the current state of mathematics education is crucial.

Despite the formal end of apartheid over three decades ago, its enduring legacy continues to cast a shadow over the South African education system. The historical repercussions of Bantu Education, marked by institutionalized racism and segregation, persist in the form of structural inequalities, resource disparities, and curriculum challenges (Gumede, 2018; Sehoole & Mokoko, 2019). While current generations of South African learners might not have directly experienced apartheid, the intergenerational impacts, compounded by ongoing socio-economic disparities, hinder equitable access to quality education (Taylor & Van Wyk, 2019; Van Wyk, 2014).

However, it is imperative to acknowledge that contemporary failures in the South African education system, as underscored in Spaull's research (2013), may also attributed to the authorities responsible for educational governance, particularly evident in the lamentable state of mathematics education (Feza, 2014; Jojo, 2019). Effectively addressing both the enduring historical legacies and the present challenges within mathematics education is pivotal to fostering a more inclusive and equitable educational landscape in South Africa.

2.5.2 Status of Mathematics in South African Public Schools - Cause for Concern?

After the 1994 democratic elections, efforts were made in South African public schools to ensure that all learners would be exposed to some form of mathematics by the time they complete matric. The reform process began with the development of legal and regulatory policy frameworks aimed at dismantling the apartheid structures and establishing a unified education system that benefits all learners (Jojo, 2019).

Initially, an Outcomes-Based Education (OBE) curriculum was introduced in 1997 to address past curricular divisions. However, after implementation challenges and poor results, the curriculum was revised, leading to the implementation of the Curriculum 2005 (C2005) in 1998. C2005, driven by OBE, emphasized cooperative learning, making it difficult for teachers to identify struggling learners in mathematics understanding (Jojo, 2019). However, the implementation of C2005 faced significant challenges, including overcrowding and inadequate teacher training (Bjorklund, 2015).

In 2004, the Revised National Curriculum Statement (RCNS) for Grades R-9 and the National Curriculum Statement (NCS) for Grades 10-12 were introduced to address the shortcomings of C2005. Despite these efforts, implementation challenges and insufficient understanding of the curriculum persisted, contributing to a cycle of inadequate mathematics education (Bjorklund, 2015).

2.5.3 Curriculum and Assessment Policy Statement (CAPS)

In 2012, the Curriculum and Assessment Policy Statement (CAPS) was introduced as a replacement for previous curricula. CAPS aimed to provide greater specification of content, detailed teaching schedules, and introduce new content in some cases (Jojo, 2019). However, OBE still underpins CAPS as its underlying philosophy (Carnoy et al., 2012).

CAPS outlines specific teaching and learning objectives, emphasizing critical and creative thinking, effective collaboration, responsible learning management, data analysis and evaluation, effective communication, and an understanding of the interconnectedness of problem-solving contexts. The curriculum seeks to develop learners' correct use of mathematics, numerical skills, critical thinking, problem-solving abilities, and awareness of mathematics in real-life situations (Jojo, 2019).

While CAPS provides a structured framework and specific content areas to be covered in each grade – with some constructivist ideals even – some argue that it restricts teachers' professional autonomy (Ramatlapana et al., 2012). The structured nature of CAPS has been seen as beneficial for teaching low-achieving learners from disadvantaged backgrounds, but it has also faced criticism for its dogmatic approach (Feza, 2014).

2.5.4 The Continuous Poor Results of South African Learners

During the implementation of CAPS, teachers found the curriculum to be too standardized and restrictive, which impacted learner performance (and prevented realization of constructivist ideals). This insistence on standardization was driven by the desire to address poor performance observed in previous curricula (Ramatlapana et al., 2012).

The Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) has consistently shown South Africa's below-average performance in numeracy compared to neighbouring countries like Zimbabwe, South West Africa (SWA)[9], and Botswana (SACMEQ IV 2017). The South African education system, particularly in mathematics education, faces significant challenges. The history of curriculum reforms highlights the need for a critical examination of the underlying philosophies and approaches to teaching mathematics. The current state of mathematics education calls for further research and innovative strategies to improve learning outcomes and empower learners in the field of mathematics (Ramatlapana et al., 2012).

2.5.4.1 The Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ IV 2017)

The Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) carries out large-scale cross-national research studies in member countries in the Southern and Eastern African region, consisting of 16 member ministries of education, of which South Africa is a member, and its objectives are to assess the conditions of schooling and performance levels of learners and teachers in the areas of literacy and numeracy. South Africa is currently not only below average but also lower in relation to its neighbours e.g., Zimbabwe, South West Africa (SWA) and Botswana (See FIGURE 3).

[9] It is important to note that since 1990, the country previously known as South West Africa is now recognized as the Republic of Namibia

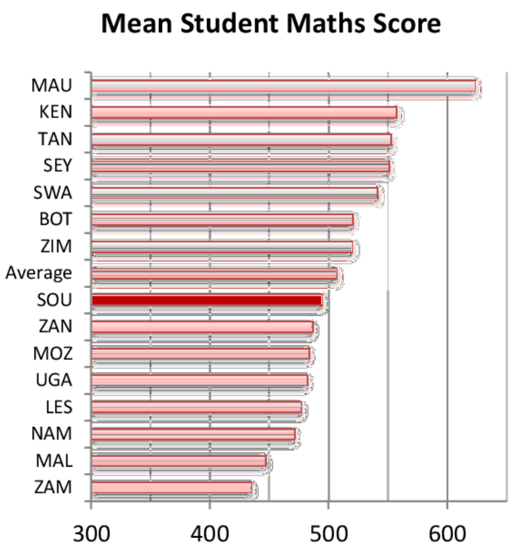

FIGURE 3 SACMEQ Student Mathematics Results in 2017 SACMEQ Student Mathematics Results in 2017

Retrieved from: https://www.acer.org/files/AssessGEMs_SACMEQ.pdf

2.5.4.2 The Trends in Mathematics and Science Study (TIMMS 2019)

The Trends in Mathematics and Science Study (TIMSS) is an international consortium sponsored by the International Association for the Evaluation of Educational Achievement (IEA) and conducted in the United States by the National Center for Education Statistics (NCES). TIMSS aims to gather detailed information on curriculum, instructional practices, and school resources in order to assess and improve teaching and learning in mathematics and science. It conducts large-scale assessments to measure the effectiveness of countries in teaching these subjects, providing valuable insights into learners' performance. The results of TIMSS have been widely used by policymakers, researchers, and the public to understand and address educational challenges.

TIMSS assesses the mathematics and science performance of 4th and 8th-grade learners in different countries every four years, comparing the achievement levels of learners in the United States with their counterparts in other countries. The assessments are designed to measure knowledge and skills aligned with participating countries' curricula (National Center for Education Statistics, 2008a). TIMSS is unique in that it also tracks changes in education quality over time.

South Africa has participated in TIMSS since 1995 (see FIGURE 4), and the results have consistently shown that South African public-school learners perform below the international average in mathematics and science. Even when South African grade 5 and grade 9 learners completed the TIMSS grade 4 and grade 8 tests, their performance remained below average, as reflected in the 2019 results (Juan et al., 2019). These findings highlight the ongoing challenges faced by South African learners in these subjects.

It is essential to take these results into consideration and use them as a basis for further analysis and improvement in mathematics and science education in South Africa. The TIMSS data provide valuable insights into the areas that require attention and can inform targeted interventions and policies to enhance teaching and learning outcomes in these critical subjects.

In FIGURE 5, it is noteworthy that, under the leadership of Singapore, the group of five East Asian countries demonstrated notably superior performance compared to other TIMSS participant nations in both fourth and eighth-grade assessments. Conversely, South Africa's performance remains below the TIMSS average score of 400.

A thought-provoking discovery stemming from the TIMSS 2019 survey pertains to the interplay between levels of confidence and job satisfaction among mathematics teachers and the subsequent academic achievement of their learners. Within this context, South African educators demonstrated a notable degree of confidence in their teaching capabilities; however, their learners' academic performance fell noticeably below the international average.

The **TIMSS** achievement scale for Science and Mathematics has a *centre point* of **500**.

Learners who achieve a score below 400 do not demonstrate the proficiency for the grade assessed.

A score *between* **400** and **475** indicates some knowledge of the subject, a score *between* **475** and **550** the ability to apply subject knowledge, and a score *above* **550** the ability to apply knowledge and to reason.

In the 1990s, the **HSRC** initiated and facilitated South Africa's participation in **TIMSS**. The 1995, 1999 and 2003 rounds were funded by the **HSRC's** parliamentary grant allocation.

Grade 8 114 schools | 4 491 learners

Mathematics score: 267

1995

Grade 8 194 schools | 8 147 learners

Mathematics score: 275

1999

Grade 8 *225 schools | 8 952 learners*

Mathematics score: 264

2003

Grade 9 238 schools | 4 261 learners

Mathematics score: 285

2006

2006 Introduction of Mathematics Literacy ML (Gr 10, 11 & 12) as an alternative to Pure Mathematics. ML does have a place in education, but the problem is that there are not enough qualified teachers, and its introduction has been poorly managed by the education department (Workman, 2021) See also section 2.4.7

The **HSRC** recommends that South Africa does *not participate in* **TIMSS** 2007 as the country's education system had undergone radical restructuring (*Review of SA Education system in 2007 by* ***OECD***).

2007

In 2011, the Department of Basic Education included **TIMSS** in its strategic plan, adopting its measures as key indicators of educational performance. The department would fund subsequent **TIMSS** rounds.

Testing was also changed from **Grade 8** to **Grade 9** The shift was judged as having enabled a better match between the content knowledge presented to learners in **TIMSS** and the curriculum coverage in South Africa. ***(THIS IS ALARMING: SA Grade 9s are tested by Grade 8 TIMSS standards?)***

2011

Grade 9 *285 schools | 10 085 learners*

Mathematics score: 352

The department decided to monitor progress in primary education, therefore, **Grade 5** mathematics learners participated in **TIMSS** 2015 for the first time.

Grade 5 *297 schools | 10 932 learners*

Mathematics score: 376

2015

Grade 9 *292 schools | 12 514 learners*

Mathematics score: 372

Grade 5 science learners participated in **TIMSS** 2019 for the first time.

Grade 5 *297 schools | 11 903 learners*

Mathematics score: 374

2019

Grade 9 *519 schools | 20 829 learners*

Mathematics score: 389

FIGURE 4 TIMSS A Concise Historical Perspective in South Africa

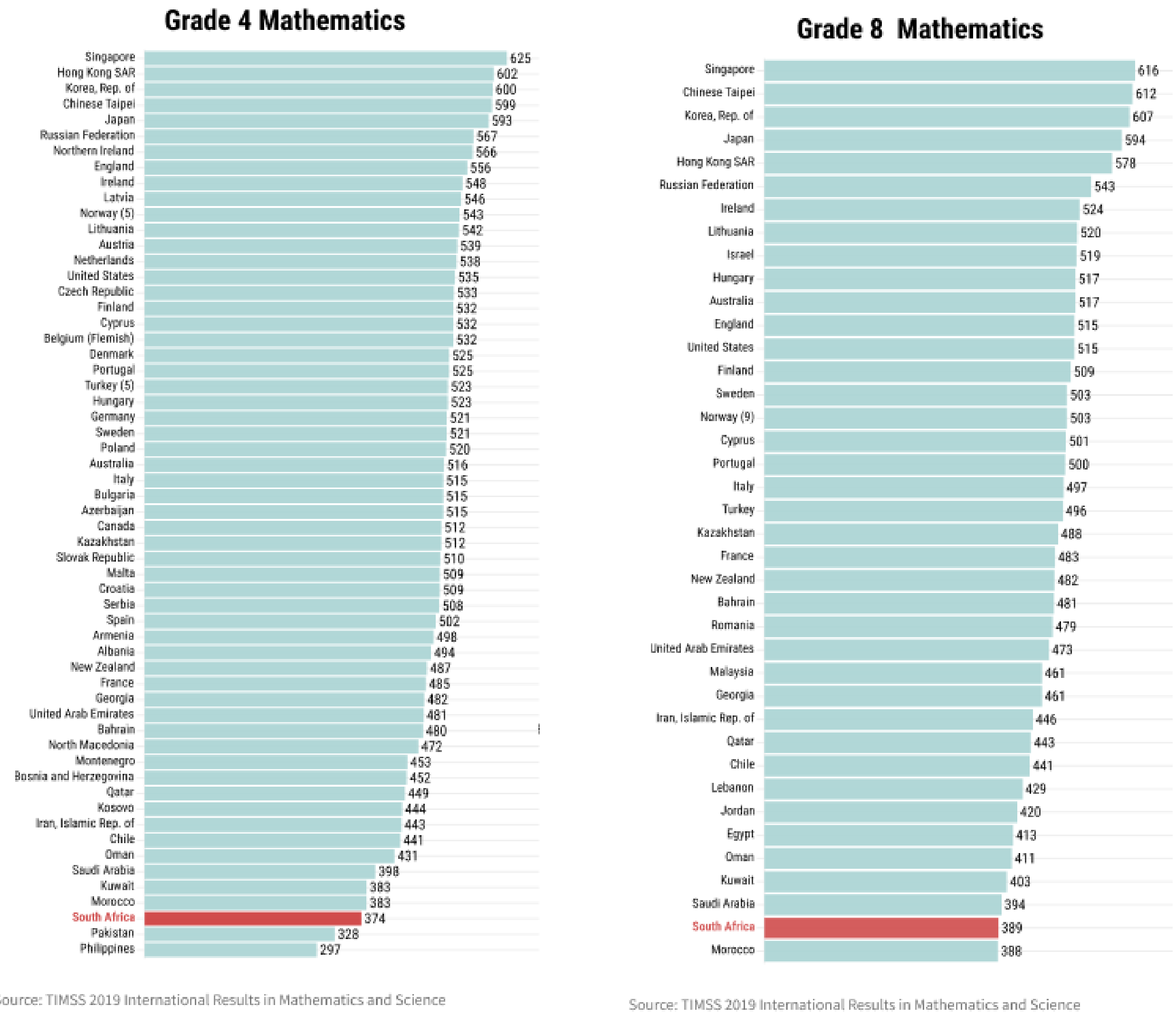

FIGURE 5 TIMSS Grade 4 and 8 achievements in Mathematics in 2019

Conversely, educators in various Asian nations reported comparatively lower levels of confidence and job satisfaction, yet consistently yielded above-average results in mathematics.

Within the TIMSS 2019 survey framework, a comprehensive evaluation was undertaken through the introduction of a Teachers' Job Satisfaction scale. This scale encompassed five essential inquiries directed at Grade 5 and Grade 9 mathematics teachers. The primary objectives of this survey were to assess teachers' professional contentment, ascertain the depth of their commitment to their vocation, and quantify the measure of motivation and pride derived from their educational role.

Significantly, the analysis of data concerning the proportion of teachers expressing elevated levels of job satisfaction yielded an intriguing outcome: South Africa secured the 9th position among the 58 participating countries.

Remarkably, countries globally recognized for the exceptional academic accomplishments of their learners—such as China and Japan—exhibited a distinct paradox in terms of their teachers' confidence in their instructional efficacy, as revealed by their responses within the TIMSS 2019 survey (TIMSS 2019).

This disparity between teacher confidence and learner performance raises important questions regarding the complex factors influencing educational outcomes. It suggests that factors beyond teachers' confidence and job satisfaction play a crucial role in learners' mathematical achievement.

Further research and analysis are warranted to gain a deeper understanding of the underlying dynamics and to identify effective strategies for improving mathematics education in this country.

2.5.4.3 The CDE Report

According to research conducted by the Centre for Development and Enterprise (CDE), an independent policy research and advocacy organization, the state of mathematics education in South African schools has been a cause for concern (CDE, n.d.). The Trends in International Mathematics and Science Study (TIMSS) results from 2011 showed that South African learners had the lowest performance among 21 middle-income countries surveyed (Spaull, 2013).

Independent studies conducted by Nicholas Spaull[10] and Professor Charles Simkins[11], commissioned by the CDE in 2013, confirmed these findings and highlighted the poor

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[10] Nicholas Spaull is a senior researcher in the Research on Socioeconomic Policy group (RESEP) at Stellenbosch University in South Africa. He has held resident fellowships at Stanford's Graduate School of Education and at the OECD in Paris. Retrieved from: https://nicspaull.com/about/

[11] The late (2022) Prof. Simkins completed his bachelor's degree at University of the Witwatersrand, and was then awarded a Rhodes scholarship, with which he obtained a MA (Philosophy, Politics and Economics) at Oxford University. He later completed a PhD (Economics) at the University of Kwazulu Natal in

teaching of mathematics in the majority of schools. The data collected indicated that despite some improvement, South Africa is still seriously underperforming in maths education. Furthermore, South Africa's extremely high youth unemployment, currently at 50%, is closely linked to the quality of schooling, numeracy, and mathematics competency (Spaull, 2013).

To address these challenges, Spaull (2013) suggests several considerations. These include basing the selection, appointment, and promotion of mathematics teachers on teaching qualities rather than personal relationships or union affiliations. The possibility of establishing a system of teacher rewards tied to learner performance in mathematics, independently overseen by an external body, is also proposed. Allocating more resources to teaching in lower school grades with the most significant deficiencies is recommended to positively impact matric level results (Spaull, 2013).

In a more recent report, it was found that the requirements for a Bachelor of Education (B.Ed) degree, which qualifies individuals to become teachers, are lower compared to other degrees. Many B.Ed students also have low performance rates in matric Mathematics. This shortage of qualified maths and science teachers further contributes to the country's low standards in these subjects (Resep[12], 2021).

Informed by the research conducted, Spaull (2013) highlighted issues that must be kept in mind when addressing South Africa's numeracy and mathematics schooling challenges. These include the urgent need to improve mathematics teaching and learning in public schools. The focus on primary school level interventions is emphasized, as the learning deficits children acquire early on impede their ability to learn in later years. The development of early childhood and special needs mathematics education programs is also recommended (Spaull, 2013).

Pietermaritzburg, South Africa. He was the first program director of the CPPA at University of the Witwatersrand in South Africa. Until his death, he was the head of research at Helen Suzman Foundation.

[12] RESEP 2021 - Research on Socio-Economic Policy (RESEP) University of Stellenbosch

It is evident that urgent action is needed to improve mathematics teaching and learning in South African schools. This includes sustained focus on teacher training and development, particularly in mathematics, as well as addressing systemic issues such as dropout rates and early childhood education programs (Spaull, 2013). By implementing these recommendations, South Africa can work towards improving numeracy and mathematics skills, ensuring that young people have the necessary knowledge for employment and socioeconomic growth.

2.5.4.4 The Outlier Report

The Outlier Report is an independent publication by The Media Hack Collective (MHC) that specializes in using data to create narratives and representations for public service. Their focus is on important issues that have a significant impact on people's lives, including education, health, climate, politics, economics, and state services. In response to the CDE report, Outlier commissioned journalist and researcher Gemma Gatticchi (2022) to conduct a survey of public schools in South Africa from 2013 to 2021, which forms the basis of the information presented in this section. Despite the government's promises to improve technical and science skills in the education system, the situation is concerning. By 2020, only 30% and 40% of learners who wrote the National Senior Certificate exams took the Physical Science and Mathematics papers, respectively, marking the lowest percentages since 2015 (see FIGURE 6).

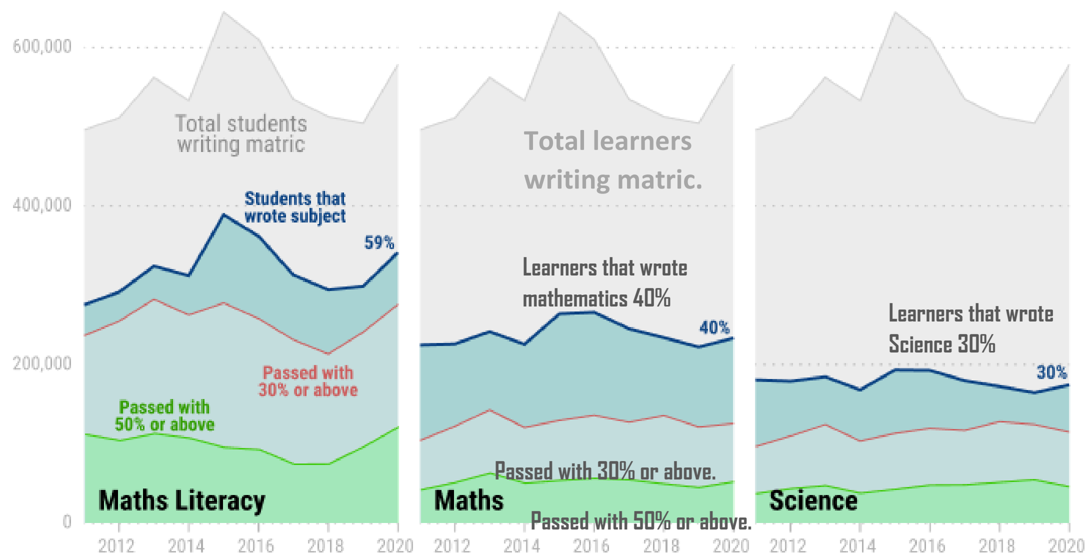

FIGURE 6 OUTLIER REPORT 2011 to 2020 results for Mathematics Literacy, Mathematics and Natural Science

Another alarming statistic is that 59% of learners opted for Mathematics Literacy, indicated by FIGURE 6, instead of Mathematics (Gatticchi, 2022).
Furthermore, there has been a decline in the number of schools offering Mathematics and Physical Science subjects. In 2020, an additional 64 high schools stopped offering these subjects, bringing the total to 446 schools (6.5% of schools) compared to 382 (5.7%) in 2019. Additionally, the proportion of learners who passed Mathematics with a score of 50% or more was relatively low, with only about one in four (26%) achieving this level in Physical Science and one in five (22%) in Mathematics in 2020 (Gatticchi, 2022).

These statistics highlight the challenges faced in mathematics and science education in South Africa and the need for urgent attention and improvement in these subjects. The declining participation rates and low pass rates in critical subjects like Mathematics and Physical Science indicate a pressing need for comprehensive interventions to enhance teaching and learning outcomes in these areas. In a disconcerting development, the number of high schools in South Africa that discontinued offering Mathematics and Physical Science subjects increased significantly in 2020 (see FIGURE 7).

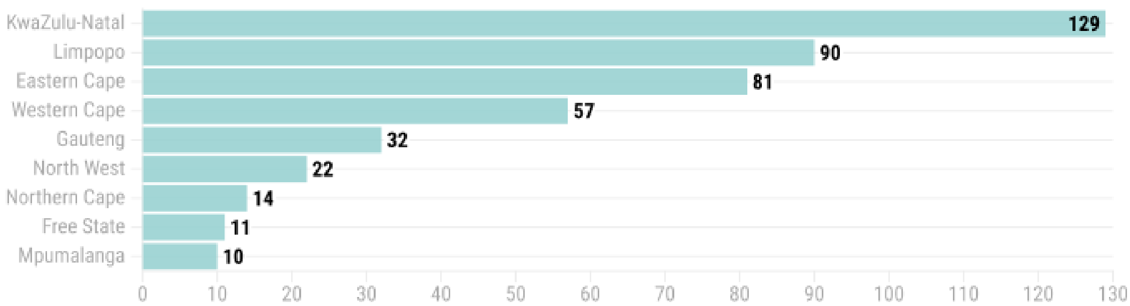

FIGURE 7 OUTLIER REPORT South African Public Schools that did not offer Mathematics or Natural Science subjects in 2020

An additional a total of 446 schools or 6.5% of all schools, made the decision to remove these subjects from their curriculum, compared to 382 schools or 5.7% in 2019 (Gatticchi, 2022). Disturbingly, the situation worsened in 2021, as indicated by FIGURE 7 in the study conducted by Gatticchi (2022).

A matter of concern pertains to the proportion of matric learners attaining a pass threshold of 30% or above in the field of Mathematics. The year 2020 witnessed an approximate 22% of examinees meeting or exceeding this stipulated benchmark in the mathematics examination, as detailed by Gatticchi's (2022) research study. Visual interpretation of these results is presented through FIGURE 8 within Gatticchi's study.

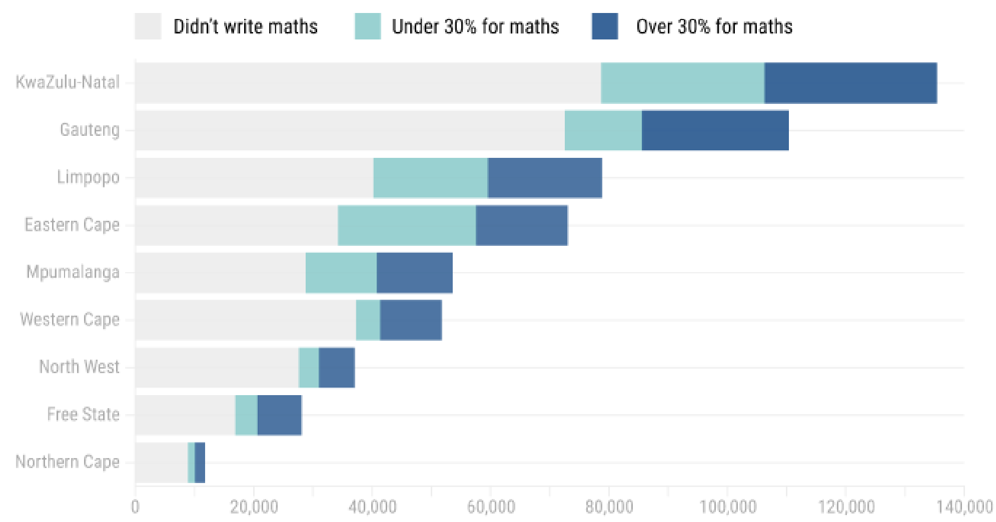

FIGURE 8 OUTLIER REPORT Matric learners' mathematics performance in 2020

2.5.5 No Fees and No Mathematics

In 2003, Mrs. Naledi Pandor, the former Minister of Education, introduced a policy to convert certain schools into no fee schools as part of the government's efforts to address poverty and promote equitable access to quality education. This initiative involved classifying public schools into quintiles, with quintile one, two, and three schools designated as no fee schools, while quintile four and five schools remained fee-paying (Nkosi, 2011).

While the government's commitment to addressing poverty and promoting equality is commendable, the implementation of the no fee school policy has faced various challenges, which continue to affect the education system in the country. The performance of schools is influenced by multiple factors, including the effective allocation and utilization of financial resources, as well as the equitable distribution of qualified educators and educational materials, such as textbooks (Nkosi, 2011).

It is important to note that simply providing additional resources to a school does not guarantee improved performance. The effective management and utilization of these resources by qualified staff are essential. For instance, during a research visit to a rural school, an educator who had recently completed a doctorate in foundation phase mathematics found that despite the presence of well-packaged school resources, such as a jungle gym, geometric shapes, and base ten blocks, they were not being utilized due to time constraints and other factors (Kakoma, 2014).

Furthermore, despite the government grants aimed at addressing disparities, there are still significant inequalities, particularly in poor rural and working-class communities. These communities often face challenges such as large class sizes, inadequate physical conditions, lack of learning resources, and limited availability of mathematics as a subject in no fee schools (Nkosi, 2011).

The implementation of free education has had unintended consequences, including the enrolment of individuals who had previously dropped out of the education system. Unfortunately, this influx of former dropouts into public schools has been associated with an increased likelihood of engaging in high-risk behaviours such as premature sexual activity, early pregnancy, delinquency, crime, violence, alcohol and drug abuse, and suicide.

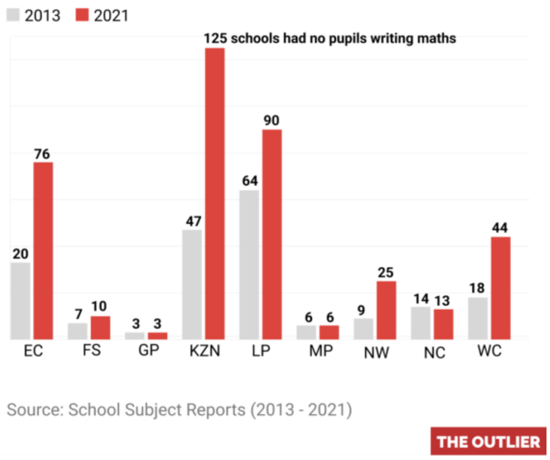

FIGURE 9 OUTLIER REPORT Notable rise in the quantity of no-fee schools

These behaviours are significantly more prevalent among individuals who have discontinued their education (Nkosi, 2011).

Illustrating the findings of The Outlier research in FIGURE 9, the data underscores a noteworthy trend wherein an increasing number of no-fee schools have opted to discontinue providing mathematics instruction to their learners.

The Outlier Report, reveals that within the context of South Africa, a total of 392 schools did not include mathematics as part of their curriculum for matriculation pupils during the year 2021 as seen in FIGURE 10.

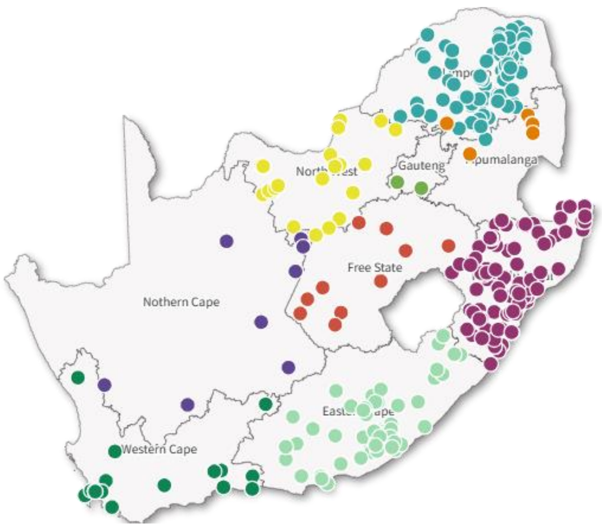

FIGURE 10 OUTLIER REPORT Spread of the 392 schools in South Africa that had no matric learners writing mathematics in 2021

Furthermore, the notion of free education has led to some parents becoming reluctant to contribute financially to the school. There has been a decline in parental involvement and accountability within no fee schools, as some parents no longer feel obligated to supervise their children's homework, ensure regular school attendance, and promote punctuality. This shift in parental attitude and involvement poses additional challenges to the effective functioning of these schools (Nkosi, 2011).

On June 20, 2017, the Portfolio Committee on Basic Education undertook a series of oversight visits to various no-fee schools situated in the Ekurhuleni North, Ekurhuleni South, and Tshwane North Education Districts within the Gauteng Province. The committee's subsequent report summarises a comprehensive synthesis of their observations drawn from their observations of five schools situated in the Ekurhuleni North district. For illustrative instances of the report's outcomes, please consult Table 1, titled "Summary of Report from Portfolio Committee on Basic Education – June 2017" (DoE, 2017). It is noteworthy that this table will highlight only two schools - one of primary level and the other of secondary level - as exemplars.

TABLE 1: Summary of Report from Portfolio Committee on Basic Education – June 2017.

(1) 700261123 Mvelaphanda Primary School 712, Seaparankwe, Tembisa, Kempton Park, 1628 Ekurhuleni North				
Learner Enrolment	**Learners Per Class**	**Performance**	**Learning and Teaching Support Material (LTSM)**	**Infrastructure**
The school experienced a learner enrolment of 1015 learners spanning across Grades 1 to 7. It is noteworthy that a significant number of these learners come from households where they assume the role of family heads, with their grandparents serving as their primary caregivers.	The classrooms are facing the challenge of overcrowding, with an average of approximately 46 learners per class.	There is a noticeable decline in academic performance in Mathematics and Economic and Management Sciences (EMS) during the Senior Phase (Grade 7) and Intermediate Phase (Grades 4 to 6).	Insufficient availability of textbooks and stationery was observed.	Infrastructure is generally well-maintained, with the exception of gutters requiring attention. The school yard may be insufficient to accommodate extra mobile classrooms. Toilets are in good working order.
(2) 700400138 Phomolong Secondary School Secondary 2847, Maduna Street, Tembisa, Kempton Park, Kempton Park, 1632 Ekurhuleni North				
Learner Enrolment	**Learners per class.**	**Performance**	**Learning and Teaching Support Material (LTSM)**	**Infrastructure**
The school has a learner enrolment of 1,624 learners in Grades 8 to 12, including 500 orphans from child-headed families. Unfortunately, there is a concerning prevalence of substance abuse and teenage pregnancies among the learners.	The classes at the school are facing issues of overcrowding, with approximately 35 learners per class. Additionally, the school has been experiencing a decline in registrations, which may be attributed to an increase in dropouts.	The school is facing challenges with academic performance, with learners performing below the expected standards. Furthermore, providing adequate support to progressed learners has been an area of concern.	The implementation of textbooks on tablets faced issues due to poor management, as learners misused the tablets and frequently damaged or lost them. Additionally, there is a lack of smartboards in 22 classrooms.	The school building requires maintenance, as certain class panels are open, ceilings in the toilets are falling, and some toilets are leaking. Additionally, there are broken windows that need repair. The perimeter fence of the school, particularly at the back, needs frequent fixing due to break-ins.

In his 2023 State of the Nation Address (SoNa 2023), President of South Africa Cyril Ramaphosa, stated the following with regards to education:

"We need to start with children who are very young, providing them with the foundation they need to write and read for meaning, to learn and develop. It is, therefore, significant that the number of children who receive the Early Childhood Development (ECD) subsidy has more than doubled between 2019 and 2022, reaching one-and-a-half million children. The Department of Basic Education is streamlining the

requirements for ECD centres to access, support and enable thousands more to receive subsidies from government. While at the other end of the basic education journey, we must applaud last year's matric pass rate of 80%, with all provinces showing improved results. This was up from 76% the year before. The share of passes in no-fee schools improved from 55% in 2019 to 64% in 2022. This means that the performance of learners from poorer schools is steadily improving, confirming the value of the support that government provides to them. What these results reveal is that there is a silent revolution taking place in our schools".

Considering the 2017 Portfolio Committee on Basic Education report and other research findings on South African schools, it is unclear as to what is meant by a "silent revolution taking place in our schools".

2.5.6 The call for decolonisation of Mathematics - Ethnomathematics

One distinguishing aspect of mathematics, setting it apart from other subjects in the arts and humanities, is its widely held perception as a universal and objective discipline. However, within the context of South African universities, students and academics are engaged in the process of "decolonization of mathematics," which challenges this notion. This movement aligns with the principles of "ethnomathematics," initiated by Ubiratàn D'Ambrosio, a Brazilian educator and philosopher, in the 1970s. Ethnomathematics aims to incorporate indigenous knowledge into mathematics education. Another influence on this perspective is Paul Ernest, a philosopher of mathematics at the University of Exeter, who advocates for a "fallibilist" understanding of mathematics. According to Ernest, mathematical truths are not absolute but should be seen as relative to a particular system or context. It is important to consider these perspectives in light of the ongoing debate surrounding absolutism versus fallibilism, as discussed in a previous section.

The exact implications of decolonizing mathematics are not fully defined, encompassing a range of potential changes. These include curriculum revisions to address educational inequalities, increased openness to non-mainstream ideas, and even questioning the fundamental philosophical principles of mathematics itself. Henri Laurie (2023), a senior lecturer in Mathematics and Applied Mathematics at the University of Cape Town, expresses reservations about the concept, particularly regarding its relevance in the scientific and mathematical domains. He emphasizes the importance of being connected to the international community in these fields. Similarly, Professor Bernhard Weiss (see ANNEXURE F), a philosopher at the University of Cape Town, concurs with Laurie, emphasizing the significance of the foundational truths inherent in pure mathematics. Understanding and appreciating the essential core of mathematics, beyond its applications, is vital to its comprehension (Adler et al., 2016; Lewton, 2019).

This means that those within the discipline must consider other aspects, such as:

1 Fostering Learner Identity:

Learner identity refers to an individual's awareness of self and self-worth, which forms the foundation for their growth as a learners. The development of a positive sense of self as a learner is influenced by the relationships learners form with their teachers and peers within the learning community. By acknowledging learners' experiences and encouraging their active participation in classroom discussions, teachers can empower learners to perceive themselves as capable contributors to the mathematics classroom. One's sense of self as a learner is developed through relationships e.g., teachers and fellow learners, and is identified as the individual relates their own experiences "as a participant in the conversations of the learning community" (Crick et al., 2005; Parkinson et al., 2021).

***The Importance of Fostering Learner Identity**:*

Recognizing learner identity in mathematics education is crucial for fostering a supportive and inclusive learning environment. When learners feel valued and confident in their abilities, they are more likely to engage actively with the subject matter and explore mathematical concepts with curiosity and enthusiasm. Moreover, promoting learner identity can address issues of epistemic justice, combating identity prejudice and ensuring that all learners are recognized as potential credible knowers (Fricker 2007).

2 Embracing Indigenous Knowledge Systems:

Another avenue for enriching mathematics education is the incorporation of Indigenous Knowledge Systems (IKS). IKS refers to knowledge developed within indigenous societies, predating the modern scientific knowledge system. This traditional knowledge offers unique insights into various aspects of life, including mathematics. For instance, practices like acupuncture from China exemplify the depth and significance of IKS in different domains (Tharakan, 2017).

However, it is essential to acknowledge the challenges that educators may face in integrating IKS into the mathematics curriculum. Research conducted by Professor Vimolan Mudaly in 2018 revealed that some teachers struggle to move beyond the conventional factual approach to teaching mathematics (Mudaly, 2018 67-84). They find it challenging to establish meaningful connections between mathematics and real-world contexts, hindering the effective integration of IKS.

The Importance of embracing Indigenous Knowledge Systems*:*

Embracing Indigenous Knowledge Systems in mathematics education promotes cultural diversity and encourages learners to recognize the value of various knowledge traditions. By integrating IKS, educators can provide learners with a broader understanding of mathematics and its applications, fostering a sense of appreciation for different ways of knowing and problem-solving. Overcoming the challenges related to IKS integration can lead to a more inclusive and culturally enriched mathematics curriculum, benefiting all learners.

3 Recognizing Non-Western Contributions to Mathematics

An often overlooked but significant aspect of mathematics education is the recognition of non-Western mathematicians and scientists throughout history. Too often, the historical narrative of mathematics tends to focus on Western contributors, neglecting the immense contributions made by mathematicians and scientists from diverse cultures and regions.

An illustrative example is a project conducted by a researcher with Grade 7 classes, where learners were asked to select famous mathematicians and scientists of their choice. The researcher noted with sadness that none of the learners included non-Western mathematicians or scientists.

The Importance of recognizing Non-Western Contributions to Mathematics*:*

Recognizing the contributions of non-Western mathematicians and scientists in mathematics education is vital for providing a comprehensive and accurate historical perspective. This acknowledgment fosters a sense of cultural appreciation and respect, enabling learners to perceive mathematics as a global and collaborative endeavour. By shedding light on diverse mathematical achievements, educators can inspire learners from different backgrounds and identities to pursue mathematics and contribute to its ongoing development.

2.5.7 The Mathematical Literacy (ML) issue.

The subject Mathematical Literacy (ML) was introduced in South African schools in 2006 as a compulsory alternative to mathematics. This was done to ensure that school learners in South African public schools were exposed to some form of mathematical skill which they can use in their personal and/or work-related life.[13]

Mathematical Literacy (ML) is a context driven subject that is taught and learnt from a contextual framework. Although the Department of Education had good intentions, not all the objectives of the subject have been accomplished, for two main reasons:

a) at the time of its introduction, there were *no* qualified teachers of Mathematical Literacy, and
b) it was aimed at grade 10 learners who did not perform well in pure mathematics and had a weak pass in their grade 9 exams, and what is interesting to note is that such learners also struggled to perform well in languages and other subjects (Ramatlapana et al., 2012).

The aim of Mathematics Literacy (ML) becoming a high-quality subject standing independently with its own set of objectives and not to be compared with mathematics, has also not been accomplished. What is also interesting is the announcement that ML had since 2014 also not shown any improvement in learners' poor performance up to 2016 (Jojo, 2019).

However, a group of concerned mathematics educators, known as Maths Excellence [14], reports that the debate regarding Mathematics Literacy (ML) as an alternative to

[13] DoE. (2011). Curriculum and assessment policy statement (CAPS): Mathematics Grades 10-12. Retrieved July 23, 2011, from http://www.thutong.doe.gov.za

[14] **Maths Excellence**: Founded by Professor **Aslam Mukadam** B.Sc. HDE FDE* (UCT). The group established a network whereas professionals, can debate issues that affect us and the learners we teach; exchange ideas

Mathematics continues and many teachers expressed their concern about the existence of two streams of mathematics and the way in which it is managed (Wiliam, et al., 2011).

2.5.8 The language of instruction issue

Classroom activities play an important role in developing learners' mathematical thinking and reasoning. Research has indicated that the language problem is one of the major factors impacting on the poor performance of many learners in mathematics (Barton, et al., 2003). The data collected by Nath (2009) point to the fact that linguistic factors have a significant effect on the learning of mathematics. Language makes it possible for the learner to actualise and hypothesize about the world around them, and language is a key mode of expression and exchange of thought in the classroom (Nath, 2009).

In South Africa, most learners are taught in English, which is often not their mother tongue. According to Venkat et al., (2009) contextualisation (understanding of the concept) and integration (method of adding or summing up the parts to find the whole) can become problematic due to the increased English language demands. Other researchers also reported on difficulties learners experience with regards to the contextualisation and integration of problems and the role language plays in conceptual understanding (Mbekwa, 2007; Setati, 2005).

In his research, Maree (1999) in his turn also suggests that insufficient language skills and language usage play an important role in under-achievement of learners in Mathematics. In his categorisation of learners' mistakes in mathematics, language challenges were the most significant causes identified. He expressed his concern about learners having to solve problems in mathematics that require sophisticated language skills.

What is disturbing is that most learners lacked the minimum language skills to even understand what is being asked (Krugel et al., 2014).

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and resources; and become an effective pressure group for change – in this we have also established a dedicated mathematics website to achieve this goal.
Retrieved from http://www.mathsexcellence.co.za/article.html.

2.5.9 Conclusion

This section delved into the realm of mathematics teaching and learning from a philosophy of mathematics perspective, emphasizing the importance for educators to comprehend the nature of mathematical knowledge, and thought processes. It highlighted the need for teachers to conduct meaningful mathematics lessons that foster deep understanding and engagement among learners.

Acknowledging the complexity of education systems, this section drew inspiration from Kemmis et al., (2012) analogy of education systems as complex ecosystems. It cautioned against simplistic causal attributions when evaluating educational outcomes, using the example of Shanghai's success in the Programme for International Student Assessment (PISA) results. It emphasized that education systems are not closed entities and that learning extends beyond the classroom, encompassing factors such as home learning environments, socio-economic status, and parental involvement.

Furthermore, the section underscored the importance of what transpires within the mathematics classroom itself. It cautioned against assuming that policies and teaching practices automatically manifest as intended outcomes. Inefficiencies such as teacher incompetence, inadequate school management, policy implementation challenges, and limited parental involvement have hindered the realization of desired educational objectives. The need for knowledgeable educators and a concerted effort to develop professional teachers was highlighted as crucial for improving the education system.

Despite the attention and efforts by the Department of Basic Education (DBE) and other stakeholders to address mathematics education in South Africa, minimal improvement in results has been observed. This necessitates a re-evaluation of the approach to teaching mathematics at every phase, calling for a unified and progressive mathematics trail that is rich in meaning, logically structured, and constructively learned, benefiting both learners and teachers.

This chapter further emphasized the fundamental significance of the mathematical way of thinking. Mathematics provides the means to organize and structure knowledge, enabling professionals in various fields such as science, engineering, computer science, and technology to generate systematic, replicable, and transmittable knowledge. From practical applications in architecture, engineering, and technology to the exploration of cosmic wonders and the inherent beauty and symmetry of nature, mathematics permeates diverse realms. The allure of mathematics, as an inspiring force for teachers and a desired outcome for parents, was highlighted.

In the next chapter, Chapter 3, the focus will shift to research methodology. This encompasses the systematic and organized processes employed to conduct research and acquire new knowledge or insights. Researchers rely on a set of techniques, tools, and procedures to collect, analyse, and interpret data, forming the foundation for advancing knowledge in the field.

Chapter 3 - Research Methodology and Design

3.1 Introduction

Conducting research in the expansive realm of the philosophy of mathematics can be formidable, particularly for novice researchers like myself. The vast expanse of available topics resembles a galaxy within the cosmological horizon. After thoughtful consideration, I have chosen the following research question as the cornerstone of my study: *Could philosophy of mathematics be the bridge to meaningful mathematics teaching in the classroom?*

To comprehensively investigate this research question and transcend initial impressions, I have instinctively opted for a mixed qualitative research approach, melding autoethnography with a variant of abbreviated grounded theory. This method will allow me to cultivate a holistic understanding of the phenomenon of philosophy of mathematics and its influence within the mathematics classroom. It emphasizes the importance of employing systematic methodologies to derive empirical insights and understanding.

Historically, quantitative research methodologies dominated the social sciences, including educational research. However, a "methodology crisis" emerged in the 1980s as advocates of quantitative and qualitative research collided (Guba, 1990; Tashakkori & Teddlie, 1998). Some researchers contended that these two approaches could not coexist due to their conflicting philosophies, particularly concerning generalization (Antwi et al., 2015). While qualitative studies aim to provide an intricate, context-rich understanding of human experiences without generalization, it remains pivotal to consider the significance of generalization in a scientific environment that prizes verified facts and practical solutions (Polit et al., 2010).

To elucidate the distinctions between qualitative and quantitative research from ontological, epistemological, and methodological perspectives, it is crucial to highlight their unique attributes. Bryman (2001:106) posits that the choice between these perspectives hinges on technical considerations aligned with addressing specific research questions.

Ontological presumptions, like the essence of mathematical knowledge, give rise to epistemological assumptions, including how this knowledge is acquired and its benefits for the learner. These assumptions, in turn, guide methodological choices and the selection of an appropriate research methodology. Moreover, researchers must determine the questions to pose and how data's connections, generality, and applicability can be established (Cohen, Manion, & Morrison, 2002).

Research methodology encompasses the entire research process, including research methods, focused activities, sampling techniques, data collection and analysis, and the ultimate drawing of conclusions and potential solutions (McMillan & Schumacher, 2001, p. 74). It is an organized process that justifies why the researcher has elected to delve into a particular topic, signifying its importance and addressing research gaps (Denzin & Lincoln, 2000, p. 157).

Therefore, the discussion of research methodology aims to grasp how the inquiry process unfolds and unveil its connections to the research question (Cohen & Manion, 1994, p. 39). It represents a remarkable journey of exploring and studying the intricacies of time, matter, space, and human existence, with the goal of defining and comprehending a field, structure, or activity (Brown & Dowling, 2001, p. 7).

3.2 Research Design

The research design aims to delineate the chosen qualitative research approach, specifically in my case, the fusion of autoethnography and a variant of abbreviated grounded theory built on aspects of multimethod and triangulation grounded theory, and its alignment with the research question. It commences by spotlighting the epistemological and ontological underpinnings of qualitative methodology, highlighting its interpretivist and constructionist essence. This standpoint asserts that meaning is constructed from participants' experiences and facilitated by the researcher's observations and reflections (Merriman, 1998).
Also, to fully encapsulate the philosophical underpinnings guiding this study, it's essential to introduce the **philosophical methodology** adopted in this research. Embracing the principles of **experimental philosophy** (Knobe & Nichols, 2008), this study navigates the interplay between empirical investigation and philosophical inquiry within the domain of the philosophy of mathematics.

The process of analytical autoethnography is a research approach that underscores the researcher's personal experiences and reflections within the research journey. It involves methodically and rigorously analysing personal narratives, situating them within broader social and cultural contexts. The stages of autoethnography typically encompass:

1. immersion in the research context,
2. personal reflections and journaling,
3. data collection through diverse methods like, observations, discussions, articles, books,
4. data analysis using thematic (identifying patterns) or narrative analysis techniques,
5. interpretation of findings, and
6. ongoing reflexivity to acknowledge and address the researcher's subjectivity (Ellis et al., 2011; Chang, 2016).

On the flip side, the approach of the variant of abbreviated grounded theory (Willig 2008), I will be employing is not directed towards the development of an emerging theory; instead, its focus lies in the identification of emerging themes through some of the coding principles of grounded theory but only within the data sets at hand after a single data collection cycle. Multiple cyclical processes that continue until an emerging theory takes shape and saturation is reached as per full grounded theory methodology were not possible due to time and resource constraints.

In this research methodology, using a variant of grounded theory, a systematic approach was employed to collect and analyse data, drawing on a diverse range of sources. The data collection process involved:

a) Collecting Comprehensive Data:

Data were sourced from various channels, including extensive literature reviews, autoethnography report, field notes derived from informal interviews, discussions, observations, documents, and also results obtained from an anonymous questionnaire. This comprehensive approach aimed to capture a rich and multifaceted dataset (Charmaz, 2006; Strauss & Corbin, 1998).

b) Initiating Open Coding:

The data analysis process commenced with initial open coding, focusing on discerning specific concepts and themes within the collected information. This initial coding phase facilitated the identification of key elements relevant to the research inquiry (Charmaz, 2006; Strauss & Corbin, 1998).

c) Progressing Through Coding Phases:

Subsequently, the analysis progressed through focused coding, axial coding, and selective coding. These phases involved a meticulous refinement and consolidation of the initially identified themes, contributing to a deeper understanding of the data (Charmaz, 2006; Strauss & Corbin, 1998).

d) Ongoing Comparison Across Data Sources:

A crucial aspect of the analysis was ongoing comparison, aiming to unearth relationships and patterns within the data. This process involved a comprehensive review of information obtained from various sources, including literature review (Chapter 2), questionnaire data (Annexures E1 & E2), field notes (Annexure F), and autoethnography (Chapter 4, Section 4.1) (Charmaz, 2006; Strauss & Corbin, 1998).

e) Utilizing Theoretical Memos:

The analysis incorporated the use of theoretical memos to capture emerging themes and ideas. These memos (see Annexures G1, G2 and G3) served as a tool for documenting and organizing evolving conceptualizations during the analysis process (Charmaz, 2006; Strauss & Corbin, 1998).

f) Undertaking Theoretical Integration:

The final phase involved theoretical integration, where a comprehensive understanding of the evolving themes was constructed as set out in Chapter 4 Section 4.2. This synthesis aimed to shed light on the phenomenon under examination, providing a holistic interpretation informed by the collected data (Charmaz, 2006; Strauss & Corbin, 1998).

By synergizing autoethnography and this specific variant of abbreviated grounded theory, the research design I am employing aspires to foster a holistic comprehension of the philosophy of mathematics phenomenon and its repercussions within the realm of the mathematics classroom. The autoethnographic component facilitates a profound exploration of my personal experiences and reflections, while the selected variant of grounded theory offers a structured framework tailored for theme identification, all grounded within the process of data and thematic analysis.

3.3 Addressing Limitations and Justification of Chosen Methodologies

The utilization of specific research methodologies often brings forth unique strengths and limitations, prompting researchers to navigate between various approaches to ensure a comprehensive understanding of the subject matter. In this section, the limitations inherent in analytical autoethnography and grounded theory methodologies are identified and subsequently addressed. Moreover, the rationale behind integrating these two distinct methodologies is explicated, illustrating how their combined application aims to offset individual constraints, fostering a more encompassing exploration within the domain of mathematics education. Below are some of these issues:

a) **Analytical Autoethnography Limitations:** While analytical autoethnography offers rich and introspective insights into personal experiences, it inherently relies on subjective reflections, potentially limiting objectivity (Ellis, 2004). Its reliance on self-generated data might introduce bias, and the interpretive nature of this approach could raise concerns about the generalizability of findings (Chang, 2008).

b) **Grounded Theory Limitations:** Conversely, grounded theory, although known for its systematic approach to theory development from empirical data, may face challenges in capturing deeply personal experiences as it primarily focuses on data interpretation (Grbich, 2007). Its reliance on categorization and theory generation might overlook individual intricacies embedded in lived experiences.

c) **Justification for Employing Both Methodologies:** The decision to combine analytical autoethnography and grounded theory methodologies stems from their complementary strengths. The amalgamation aims to mitigate the limitations inherent in each approach (Ellis, 2004; Grbich, 2007). Analytical autoethnography's emphasis on personal narratives, when juxtaposed with grounded theory's systematic analysis, cultivates a holistic perspective, offering a nuanced understanding of both individual experiences and broader patterns within mathematics education (Chang, 2008). By combining the deeply personal with the methodically structured, these methodologies intertwine to form a more comprehensive and balanced exploration, enriching the study's depth and breadth (Ellis, 2004; Grbich, 2007; Chang, 2008).

d) **Transparency in Methodology: Acknowledging and Clarifying the Limited Use of Quantitative Data:** It is crucial to explicitly acknowledge the limited quantity of quantitative data in this research endeavour. The primary focus of this study revolves around a mixed qualitative research approach, combining autoethnography with a variant of abbreviated grounded theory. While the methodological design is predominantly qualitative, the inclusion of limited quantitative data was a strategic decision made in consideration of the research question and the challenging landscape of obtaining large-scale quantitative responses from schools. This intentional choice stems from the inherent complexities of engaging educational institutions in extensive quantitative research efforts. As discussed in Chapter 1, only a subset of schools responded positively to the research request, underscoring the practical challenges faced. Consequently, the ensuing sections will explicitly highlight the constrained use of quantitative data, emphasizing its supplementary nature in enhancing the overall qualitative findings. This transparency aims to align the reader's expectations with the methodological choices made and underscore the significance of the mixed approach in capturing the nuanced interplay of philosophy of mathematics within the classroom context.

3.4 Emerging Themes: A Multimethod Exploration of Teaching Perspectives through Grounded Theory and Autoethnography

3.4.1 Analytical Autoethnography

As previously mentioned, one of the selected research methodologies for this study is analytical autoethnography, a qualitative research approach. Given my role as a mathematics teacher deeply concerned about the status and understanding of mathematics, this methodology holds significant relevance. Qualitative research approaches, including analytical autoethnography, draw from constructivism, pragmatism, and participatory perspectives, emphasizing interpretive and naturalistic studies of phenomena in authentic settings (Creswell, 2003). These approaches involve capturing the essence of phenomena in their genuine context (Denzin & Lincoln, 2000).

3.4.1.1 Analytical Autoethnography: An Overview

Analytical autoethnography, a distinctive qualitative methodology, draws from narrative research, autobiography, and ethnography. By utilizing narrative dialogue, self-study/autobiography, and memory work, researchers construct stories based on their own experiences, allowing for an exploration of their personal journey with mathematics (Butler-Kisber, 2010). While some scholars categorize analytical autoethnography under narrative methods, others place it within the ethnographic tradition (Marshall & Rossman, 2011; Robben & Sluka, 2012).

3.4.1.2 My Role as an Autoethnographer

As an autoethnographer, I engage in self-observation, compiling ethnographic field notes, and self-reflection concerning my experiences and perceptions related to the research topic. However, analysing my own data presents both strengths and challenges, as I must navigate the dual role of both "audience" and "dancer on the dance floor" (Heifetz, Grashow, & Linsky, 2009).

3.4.1.3 Data Collection and Analysis

Analytical autoethnography involves collecting data in the form of words, emphasizing the process over the final product. It centres on how I derive meaning from my life experiences, particularly as a teacher, using expressive language (Creswell, 2003). In this study, autoethnography serves as a revealing research method, shedding light on my self-identity as a mathematics teacher (Ellis, 2004). My relationship with mathematics, teaching, and connections with other teachers and learners are integral to the research (Reed-Danahay, 1997).

3.4.1.4 Accessing Primary Data Source

Analytical autoethnography provides me direct access to the primary data source: myself. This accessibility offers a privileged perspective in data collection and analysis (Chang, 2008). Consequently, my autoethnography predominantly features first-person accounts, offering a detailed description of significant events and individuals that have shaped my identity as a mathematics teacher. It delves into my pursuit of meaning and the evolution of my teaching practices as my identity as a mathematics teacher evolves. By exploring the inner workings of my world, it invites readers to connect their own experiences with my narrative, underscoring the profound impact of critical reflection in this process (Jones, 2002; Patten, 2004).

3.4.1.5 A Lens on Personal Experiences

Denzin and Lincoln (2000) define autoethnography as a genre of research that intertwines personal thoughts with multiple layers of consciousness. This lens allows me, as the researcher, to focus outwardly on the social and cultural dimensions of my personal experiences, formulating interpretations while embracing vulnerability. Analytical autoethnography casts me, the researcher, as the main character, with others assuming supporting roles in my lived experiences, emphasizing the social element (Chang, 2008). My narrative represents a continuous construction of my identity as a mathematics teacher, acknowledging the contributions of mathematicians, learners, teachers, schools, and society.

3.4.1.6 Power of Analytical Autoethnography

Analytical autoethnography has emerged as a potent research source in humanistic disciplines like education, counselling, and social work (Chang, 2008). In this dissertation, autoethnography is chosen as the methodology due to its capacity to narrate change and growth, merging experiences and theories to offer explanations resonating with mathematics teachers, particularly concerning mathematics' significance, and understanding (Lewis, 2018).

3.4.2 Grounded Theory

3.4.2.1 Introduction

Grounded theory is a qualitative research methodology that emphasizes theory development from empirical data. It is particularly suited for exploring social processes and interactions (Grbich, 2007). The interpretive research paradigm and grounded theory design are well-matched for understanding teachers' perspectives on teaching and learning mathematics in schools. This approach is selected for its ability to delve into educators' experiences, thoughts, feelings, actions, and interactions within their classrooms and school environments. Grounded theory enables investigation into meanings and relationships, making it suitable for studying the philosophy of mathematics in teaching and learning contexts. It aligns with the exploratory nature of this study, focusing on inductively building theories from underexplored areas (Grbich, 2007).

3.4.2.2 Adapting Grounded Theory for Diverse Data Sources

In this study, a modified abbreviated grounded theory approach closely aligned with the Multimethod Grounded Theory or Triangulated Grounded Theory methodologies is employed, accommodating diverse data sources without strict categorization (Creamer, 2021; Howell Smith et al., 2020). This approach acknowledges the necessity of integrating varied data sources to achieve a comprehensive understanding of the research question.

3.4.2.3 Methodology Overview

This section provides an overview of the chosen methodologies for the research, highlighting the utilization of a Multimethod Grounded Theory approach and a Triangulated Grounded Theory Methodology. These approaches involve synthesizing diverse data sources and employing triangulation to ensure a comprehensive and credible analysis of the phenomenon under study – the philosophy of mathematics as a bridge to meaningful mathematics teaching.

a) Multimethod Grounded Theory Approach:

The Multimethod Grounded Theory approach involves synthesizing diverse data sources, each contributing distinct viewpoints that collectively enrich the understanding of the studied phenomenon. These sources include literature reviews (Chapter 2), educators' online questionnaires (ANNEXURE E1 & 2), field notes from discussions and informal interviews (ANNEXURE F), and analytical autoethnography (Chapter 4, Section 4.1). By combining these sources, the research aims to present a comprehensive analysis enriching the understanding of the research topic: philosophy of mathematics as a bridge to meaningful mathematics teaching.

b) Triangulated Grounded Theory Methodology:

Triangulation plays a pivotal role in enhancing analysis validity. Within the Triangulated Grounded Theory methodology (Bryant et al., 2019), various data sources are employed to validate emerging themes, patterns, and relationships, bolstering overall research credibility. This approach seeks to incorporate multiple perspectives and forms of evidence, thereby reinforcing the findings' reliability and dependability.

3.4.2.4 Qualitative Data Analysis Stages

The process of qualitative data analysis involves interconnected stages contributing to the exploration of philosophy of mathematics' potential role in effective mathematics teaching.

a) Open Coding and Focused Open Coding: Deconstructing and Refining Data

The initial phase, open coding, systematically breaks down gathered data. Each data segment receives a descriptive code capturing its essence. Open coding dissects data into smaller units, assigning codes to encapsulate their significance. This foundation sets the stage for understanding underlying concepts and patterns within the data.

Subsequently, the process advances to focused open coding, intensively analysing specific data segments to refine emerging categories and subcategories (see ANNEXURES G1 & 2). This iterative process develops and interconnects codes, facilitating deeper exploration of aspects like respondents' profiles, opinions on mathematics, teaching methodologies, and learning experiences. This refinement unveils data nuances and relationships, contributing to comprehensive understanding.

b) Axial Coding: Establishing Connections and Relationships

Building on focused open coding outcomes, axial coding identifies connections and relationships among generated codes. This phase identifies associations, interconnections, overarching categories, and subcategories. Axial coding examines the interplay between emerging themes and concepts (see ANNEXURE G3).

c) Selective Coding: Refining Core Concepts

Selective coding culminates the coding process. Here, the researcher identifies a central or core category unifying the data. This phase integrates and links categories and subcategories identified earlier. Selective coding establishes a pivotal category or concept encapsulating the primary theme or narrative within the data.

d) Analysis, Interpretation, and Comparisons

Following selective coding, a comprehensive analysis and interpretation of the data ensue. This meticulous review of relationships, patterns, and meanings within and across categories involves comparative analysis, revealing similarities, differences, and variations. This comparison offers insights into philosophy of mathematics' potential impact on meaningful teaching within mathematics education.

3.4.2.5 Concluding Insights

The qualitative data analysis methodology, rooted in an adapted grounded theory framework, facilitates comprehensive exploration of philosophy of mathematics' role in enhancing mathematics teaching effectiveness. By integrating diverse data, maintaining consistent coding procedures, and triangulating evidence, the analysis seeks to unveil significant themes, relationships, and implications in the data. The ultimate goal is to illuminate philosophy of mathematics' potential to elevate mathematics education's quality and meaning.

The description of the qualitative data analysis process effectively outlines the methodologies employed to investigate the relationship between philosophy of mathematics and meaningful mathematics teaching. The approach underscores the amalgamation of unique perspectives and triangulation of evidence, guided by the framework's adaptations (Creamer, 2021; Howell Smith et al., 2020).

It is important to further highlight that throughout the qualitative data analysis process, Microsoft Excel was employed as a supportive aid to manage and organize the gathered data segments (Miles et al., 2014). While specialized qualitative data analysis software was not utilized, Excel served as a valuable tool for sorting, organizing, and categorizing the collected qualitative data (Guest et al., 2012). This platform facilitated the systematic breakdown of data segments and assisted in structuring information derived from diverse sources (Saldana, 2015). The coding process, performed manually within Excel, allowed for a comprehensive understanding of emerging categories and relationships within the dataset (Charmaz, 2014; Creswell, 2007). Excel was instrumental in maintaining the integrity of the qualitative analysis by ensuring thorough categorization and systematic organization of the data (Saldana, 2015).

3.5 Data Collection

Grounded theory's data collection is a multifaceted process encompassing stages of collection, refinement, and coding (Strauss & Corbin, 2008). This systematic approach combines rigorous data collection, coding, analysis, and theoretical sampling to generate theory closely connected to the data and suitable for further testing (Conrad et al., 1993; Glaser & Strauss, 1967). Again, given limited time and resources, my focus as per abbreviated grounded theory was not on generating a theory, but only on identifying concepts, categories, and themes in the data resulting from a single data collection cycle.

Data collection commenced with an online anonymous questionnaire (see ANNEXURES E1 & 2) for teachers, supplemented by informal interviews, observations, discussion notes, email exchanges, and document analysis culminating in field notes (see ANNEXURE F). This multifaceted approach enabled a comprehensive and varied data collection (Creswell, 2007; Taylor & Bogdan, 1998).

As data collection progressed, the researcher reflected and compared various data sources: questionnaire results, field notes, autoethnography and literature review. This comparison was done to identify similarities, differences, and to develop concepts and categories emerging from the data (Strauss & Corbin, 2008). This iterative process refines and consolidates codes and categories, deepening the understanding of the phenomenon under study.

3.6 Sampling Focus and Research Sites

The process of participant selection adopted an emergent approach, which was guided by the challenges related to school access and some teachers' initial reluctance. Instead of predetermined selection, participants were chosen based on theoretical insights and emerging concepts that surfaced during the ongoing data collection and analysis. This approach is often referred to as "emergent sampling" or "purposive emergent sampling" (Creswell, 2007).

The focus of the sampling was on schools within the Ekurhuleni North District, encompassing both public and private institutions. A total of sixty schools were contacted via email, resulting in ten schools agreeing to participate and contribute data. The research involved the engagement of twenty-seven teachers, including Heads of Departments (HODs), who completed anonymous questionnaires (see ANNEXURES E1 & 2). In addition to questionnaires, participants took part in group discussions, informal interviews, and email exchanges (see ANNEXURE F). The study also included informal interviews with two Vice Principals. Notably, all participants granted permission to take part in the study (see ANNEXURES C1 & 2).

The sampling strategy of the study was designed to ensure the inclusion of diverse participants while remaining aligned with the research objectives. The selected participants and research sites provided valuable insights into the phenomenon under investigation.

3.7 Ensuring Trustworthiness: Validity and Reliability

Trustworthiness, underpinned by reliability and validity, enhances research credibility. Both concepts contribute to scientific integrity (Trochim, 2001; Silverman, 2004).

a) **Validity Enhancement:**

Validity ensures accurate and truthful research statements that precisely measure intended phenomena. Strategies like fieldwork, constant comparative method, triangulation, and inclusion of contradictory findings enhance validity (McMillan & Schumacher, 2001; Ritchie & Lewis, 2003).

b) **Reliability Consideration:**

Reliability, traditionally associated with repeatability, is complex in qualitative research due to human behaviour's influence (Merriam, 1998). It encompasses external and internal reliability. The first relates to replication in similar studies, while the latter focuses on correlation and support among analysis results (Ritchie et al., 2003).

To ensure reliability, a code-recode strategy maintains consistency in coding. The researcher's position and biases are transparent. Triangulation, employing multiple data collection methods, enhances understanding. An audit trail documents procedures and daily transcribed data verification (Ary et al., 2002; Merriam, 1998).

Research questions guide the analysis process, ensuring alignment with study purpose. Addressing trustworthiness, validity, and reliability aims to produce ethical, valid, and reliable knowledge.

3.8 Ethical Considerations

Ethical considerations safeguard participant rights, well-being, and research integrity. Upheld ethical principles include:

1. **Informed Consent**: Participants receive detailed information, including purpose, procedures, risks, benefits, confidentiality measures, and voluntary participation. Consent is obtained, ensuring autonomy (Ary et al., 2002; Denzin & Lincoln, 2002). (see ANNEXURES C1 & 2.)
2. **Beneficence and Non-Maleficence**: Balancing benefits and harm, minimizing risks to participants, and assessing potential risks and benefits (Denzin & Lincoln, 2000).
3. **Privacy and Confidentiality**: Strictly preserving participants' information and using anonymized data to ensure privacy (Denzin & Lincoln, 2000).
   Furthermore, special consideration is given to safeguarding students' identities and confidential information throughout the research process. To ensure anonymity and privacy, stringent measures are in place to protect the identities of minors, or any individuals mentioned in the study. Any references to students will be anonymized, using pseudonyms or codes, to prevent the disclosure of personal information, aligning with established ethical standards in educational research (Ary et al., 2002; Denzin & Lincoln, 2000).
4. **Research Integrity**: Upholding honesty, transparency, and accuracy, avoiding misconduct (McMillan & Schumacher, 2001).
5. **Respect for Participants**: Respecting autonomy, dignity, and rights, considering cultural and individual differences (Ary et al., 2002).
6. **Transparency and Openness:** Communicating research process, findings, and conflicts of interest transparently (Denzin & Lincoln, 2000).

The study adheres to ethical guidelines, ensuring participants' rights and research integrity.

3.9 Limitation of the Study

This research endeavour aimed to investigate the factors contributing to learners' subpar achievement in mathematics, with a primary focus on the philosophy of mathematics as perceived by teachers, heads of departments, school principals, and education authorities. However, the study faced several limitations that merit consideration for a comprehensive interpretation of the research outcomes.

As mentioned in Chapter 1, section 1.8, the selection of sample schools within the researcher's district of origin, Ekurhuleni North, specifically Kempton Park, Gauteng, was constrained by both time and financial limitations. Although this approach yielded valuable insights, the findings might have limited applicability to a broader population due to the localized nature of the sample.

Moreover, educational research inherently grapples with complexity, entailing the examination of the intricate interplay of factors impacting learner learning. These factors encompass classroom dynamics, socioeconomic status, learner motivation, parental involvement, and teacher quality. The necessity to account for these variables introduces intricacy and time-consuming aspects to the research process (McMillan and Schumacher, 2001).

Furthermore, the reluctance of schools and teachers to participate in research projects, as evidenced by Bannister et al., (2018), posed a challenge. Concerns regarding disruptions to daily routines and instructional time might have influenced participation rates, potentially affecting the comprehensiveness of the research. These factors also impacted on the extent to which grounded theory methodology could be followed.

In any research endeavour, the human element, as emphasized by Merriam (1998), carries inherent limitations such as errors, missed opportunities, and personal biases. These limitations, inherent to the researcher's human nature, could inadvertently impact various facets of the study, including data collection and interpretation.

Additionally, schools, as public institutions, are influenced by external environmental factors and undergo changes in various aspects, such as grade levels, programs, teaching methods, administrative roles, and policies, as outlined by McMillan and Schumacher (2001). The ever-evolving nature of educational institutions, coupled with variations in how respondents process ideas and the complexities of different school environments, might have introduced additional challenges to the research.

The study's qualitative research approach, while valuable in exploring perspectives on mathematics teaching, also presented its own set of limitations. Upholding rigor in qualitative research can be demanding due to the volume of data and the time-intensive nature of analysis and interpretation. Additionally, the presence of the researcher during data collection, often unavoidable in qualitative studies, might have influenced participants' responses (McMillan and Schumacher, 2001).

In the quest to comprehend learners' lacklustre achievement in mathematics, the study observed evident cultural diversity among the participating schools, encompassing both private and public institutions. This diversity introduces further complexity to the research, necessitating a careful consideration of context-specific influences on attitudes toward mathematics teaching.

Despite these limitations, the researcher strived to adopt an inclusive approach in exploring the dynamics contributing to learners' subpar achievement in mathematics. Acknowledging and addressing these limitations bolster the credibility of the research, enabling readers to interpret the findings within the appropriate context and make informed assessments.

3.10 Conclusion

In adopting a robust methodological approach, this study acknowledges the foundational role of philosophical methodology. Philosophical methodology, in the context of qualitative research, involves an awareness and consideration of the underlying philosophical assumptions, orientations, and paradigms shaping the research process (Crotty, 1998; Guba & Lincoln, 1994). Grounded in both autoethnography and an abbreviated grounded theory, this research methodology artfully merged personal experiences with theoretical foundations (Chang, 2016; Charmaz, 2014). The study embraced the interpretive nature inherent in social sciences research, recognizing the intricate interplay between theory and empirical data (Creswell, 2007). This approach allowed for a profound exploration of teachers' lived experiences within the realm of mathematics education, respecting the philosophical stance that shapes qualitative inquiry (Denzin & Lincoln, 2000). By intertwining empirical observations with philosophical sensitivity, the study ventured into the intricate philosophy of mathematics teaching and learning, aligning seamlessly with the objective of constructing theories from rich empirical insights (Glaser, 2017)."

The research methodology utilizing autoethnography and abbreviated grounded theory has demonstrated a rigorous approach to understanding and exploring the research topic. By integrating qualitative data collection methods such as questionnaires, field notes, informal interviews, and email correspondence, the researcher has gained profound insights into the experiences and viewpoints of mathematics teachers in the South African education system.

The utilization of questionnaires allowed for the collection of standardized data from a larger sample, providing statistical probabilities and broader applicability to the teacher population at large. The use of Likert scale items accurately captured participants' beliefs and opinions, enhancing the validity of the findings. The subsequent data analysis, conducted using Microsoft Excel, facilitated the computation of descriptive statistics and supported the research objectives.

Moreover, the inclusion of field notes and informal interviews added depth and context to the research findings. Engaging directly with participants enabled the capture of their perceptions and exploration of their experiences within the mathematics classroom. Through purposeful sampling and a focus on typicality, the research sample encompassed various characteristics relevant to the research problem, contributing to the depth and richness of the analysis.

The research's trustworthiness was upheld through considerations of validity, reliability, and ethical aspects. Validity was strengthened through fieldwork, sustained observation, the constant comparative method, and triangulation with literature resources. Reliability was established through replication logic, code-recode strategy, researcher reflexivity, triangulation of data collection methods, and the maintenance of an audit trail. Ethical considerations were adhered to, ensuring informed consent, maintaining confidentiality, safeguarding participants' privacy, upholding research integrity, and respecting participants' autonomy and dignity. The transparency and openness of the research process further contributed to its ethical execution.

Overall, this research methodology underscores a comprehensive and rigorous approach to grasping the experiences of mathematics teachers in the South African education system. The integration of autoethnography and grounded theory facilitated a profound exploration of the research topic, merging personal experiences with theoretical perspectives. By adhering to research rigor, encompassing validity, reliability, and ethical considerations, the credibility and trustworthiness of the research findings are reinforced, contributing to the broader realm of educational research.

Chapter 4 presents the research findings through an ethnographic report and an exhaustive grounded theory analysis of the collected data, encompassing information derived from questionnaires, informal interviews, and email correspondence. The central objective is to glean valuable insights into respondents' viewpoints concerning the essence of mathematics, the practice of mathematics teaching, and the process of mathematics learning.

Chapter 4 - Research Results

4.1 The Autoethnography Story: Unveiling the Power of Philosophy of Mathematics in The Classroom

4.1.1 Introduction

My research journey revolves around investigating the potential links between the philosophy of mathematics and effective teaching strategies in the mathematics classroom. To undertake this exploration, I have chosen analytical autoethnography as one of my research methodologies (the other being a variant of Grounded Theory) —a process that entails personal reflection and writing on a subject of profound personal significance. Throughout my career as an educator specializing in mathematics, science, and technology, I have encountered several noteworthy incidents that have greatly impacted my trajectory. I would like to share a few of these significant occurrences.

During a school assembly, the principal expressed the belief that all science would eventually be disproven by religion, I found this assertion preposterous. However, it sparked a determination in me to address this matter subtly by integrating it into my teaching approach.

I also observed a perplexing contradiction among some of my fellow teachers. While expressing concerns about certain abstract concepts being difficult for learners, they would then ask these same learners to engage in prayer. This incongruity left me astounded and motivated me to delve into the interconnections among science, mathematics, and technology.

Additionally, during a measurement lesson, I witnessed a teacher explaining that measuring starts at the "1" mark on a ruler, deviating from established conventions. This raised doubts in me and emphasized the need for a comprehensive understanding of mathematical principles.

These incidents, among others, have significantly influenced my educational journey, igniting a deep-rooted determination to explore the interconnections among science, mathematics, and technology. My aim was to cultivate a coherent understanding among my learners while unravelling the intricate relationships that exist within these fields.

I designed the Grade 7 curriculum by seamlessly integrating mathematics, natural science, and technology, as illustrated in FIGURE 11. At the commencement of the academic year, I presented this document to the students, offering a thorough insight into the curriculum's structure across four terms. The emphasis was placed on highlighting the interconnections between these three subjects, underscoring their intrinsic relationships. Furthermore, a detailed explanation was provided to the learners, emphasizing the significance and relevance of these interrelationships.

Interdisciplinary education, especially in the context of mathematics, natural science, and technology, plays a crucial role in fostering a holistic understanding of the subjects. By integrating these disciplines, students gain a more comprehensive view of how these areas interconnect and influence one another. This approach not only enhances their critical thinking skills but also promotes a broader perspective on problem-solving. Moreover, interdisciplinary education prepares students for real-world challenges by reflecting the interconnected nature of knowledge in various professional fields. It encourages a more well-rounded and adaptable approach to learning, ultimately equipping students with the skills needed to navigate an increasingly complex and interconnected world (Malcolm, 2017).

I have always taken a combination of a structuralist, constructivist, fallibilist and absolutist view of mathematics, understanding the interconnectedness of various concepts and the

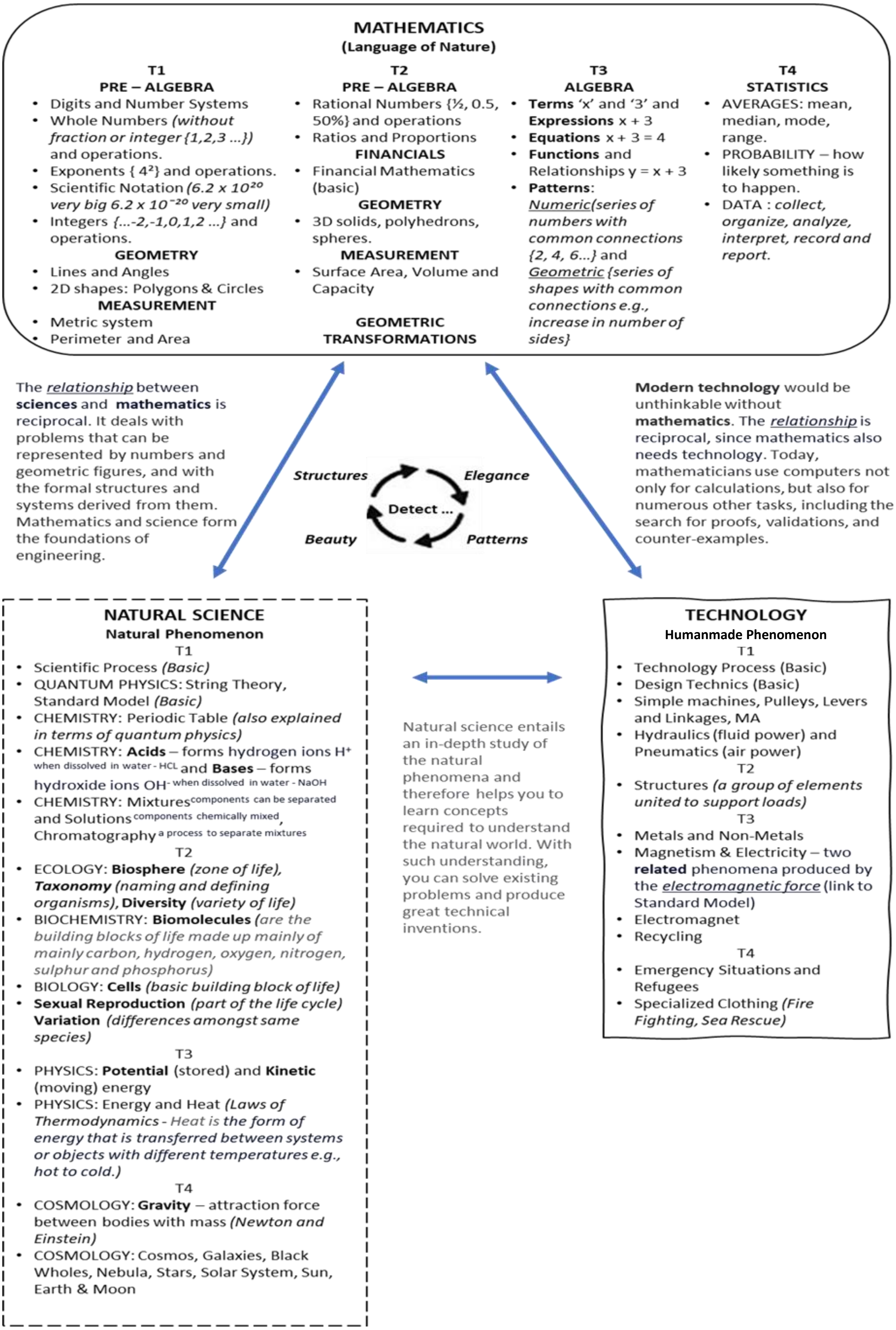


FIGURE 11 Integrated Grade 7 Curriculum for Mathematics, Natural Science, and Technology

importance of seeing the bigger picture. However, I have also encountered difficulties in teaching abstract concepts and noticed a lack of clarity and coherence in educational materials.

This has led me to believe that philosophy can play a significant role in understanding the nature of mathematical and scientific knowledge and finding constructive ways to impart it. Throughout my teaching career, I have been inspired by books and articles written by mathematicians, physicists, and philosophers. These individuals have provided valuable insights into the nature of mathematics and its broader connections. By incorporating their perspectives into my pedagogical approach, I have witnessed first-hand the profound impact it had on my learners' understanding, motivation, and enthusiasm for the subject.

In the following section of my autoethnography, I aim to shed light on the significance of incorporating the philosophy of mathematics into teacher training and advocating for a multidisciplinary approach that intertwines mathematics, science, and technology. This approach emphasizes the need for meaningful integration and application of mathematical principles to foster a deeper understanding and engagement with the subject.

These encounters have empowered me to design lessons that integrate mathematics, science, and technology in a meaningful and engaging manner. By incorporating their insights into my pedagogical approach, I have witnessed first-hand the profound impact it has on my learners' understanding, motivation, and enthusiasm for the subject.

Through the exploration of the philosophy of mathematics, I aim to inspire fellow educators to seek knowledge and inspiration beyond traditional educational resources. By embracing the insights and perspectives of experts in related fields, we can enrich our teaching practices and unlock new avenues for interdisciplinary learning.

Together, let us embark on a journey of discovery, drawing from the wisdom of mathematicians, physicists, and philosophers to create a transformative educational experience that fosters a deeper understanding and appreciation of mathematics, science, and technology.

4.1.2 Alexandre Grothendieck - The Dancing Star

How does one explain to a child "What is mathematics?" and "What is a mathematician?" To answer this question, I turned to the insights of mathematician Alexandre Grothendieck[15] (1986). I would, however, provide learners with a simplified explanation: Mathematics is like a special language that helps us understand how things work in the world. It is a way of thinking and solving problems using numbers, shapes, patterns, and logic. Mathematicians are like detectives who love to explore this language and solve puzzles. They ask questions, make discoveries, and find creative solutions. They use their imagination to explore new ideas and unlock the secrets of the universe. Just like magicians with numbers and shapes, mathematicians can amaze us by seeing hidden patterns and connections all around us. What follows is Grothendieck's explanation of both questions mentioned above.

4.1.2.1 What Is a Mathematician?

According to Grothendieck (1986), mathematicians can be divided into two groups: the Heirs and the Builders.

4.1.2.1.1 The Heirs of Mansions

"*The windows and shutters are carefully closed, probably on account of a fear that a foreign wind would blow in*." Extract from: Récoltes et Semailles: Réflexions et témoignages sur un passé de mathématicien (Grothendieck 1986).

The Heirs are mathematicians who confine themselves within an established conceptual framework or "universe" of mathematics. They work within the existing knowledge and tools passed down to them, occasionally making improvements or additions. They are hesitant to question the origins or structure of their mathematical universe and prefer to maintain the status quo.

[15] Alexandre Grothendieck (1928-2014) was a renowned mathematician known for his revolutionary contributions to algebraic geometry and the development of new mathematical theories. His profound insights and rigorous approach led to significant advancements in the field. Grothendieck's work fundamentally transformed algebraic geometry, paving the way for modern mathematical research and inspiring generations of mathematicians. Book: Grothendieck, Alexandre. (1986) Recoltes et Semailles

4.1.2.1.2 The Builders of Mansions

"*I tell you: one must have chaos inside, to give birth to a dancing star. I tell you: you have yet chaos in you*." Friedrich Nietzsche, Zarathustra (2006).

On the other hand, the Builders are mathematicians who have a natural inclination to continuously construct new mathematical ideas and frameworks. They not only build new mathematical structures but also invent the tools and concepts required for their construction. They thrive on the process of exploration and creation, often venturing into uncharted territory. The Builders find joy in discovering new questions, notions, and statements that were previously unseen or unexplored. Grothendieck (1986) identifies himself as belonging to the lineage of Builders, emphasizing their role in pushing the boundaries of mathematics. He sees himself as an explorer and visionary who discovers fertile viewpoints, unearths missing ideas, and brings unity to disparate mathematical concepts.

4.1.2.1.3 Fertile Viewpoints

The fertile viewpoint is a crucial tool for discovery, as it reveals hidden questions, notions, and statements, connecting them as parts of a larger whole. By combining multiple viewpoints, the mathematician can gain a deeper understanding of complex mathematical realities and sometimes even create new visions that transcend individual perspectives.

4.1.2.2 The Explorer and The Visionary

In this section, I expound Alexandre Grothendieck's view on the roles of the Explorer and the Visionary in mathematics.

4.1.2.2.1 The Explorer

The Explorer excels in discovering crucial questions, new notions and mathematical statements that were previously unseen or unexpressed. Grothendieck (1986) emphasizes his own inclination towards discovering fertile viewpoints rather than solely focusing on solving problems left by predecessors. Explorers believe that their ability to introduce and develop entirely novel themes is their most essential contribution to mathematics.

Grothendieck (1986) explains that these questions, notions, and statements only make sense to them when subjected to a fertile viewpoint or when they arise spontaneously from it. The fertile viewpoint serves as a unifying and organizing principle, revealing interconnectedness, and giving meaning to the discovered ideas. It is likened to a light in a pitch-black night, revealing contours and connecting various mental widgets into a cohesive whole.

4.1.2.2.2 The Visionary

On the other hand, the Visionary is described as someone who grasps the multiple viewpoints of a common reality as ONE. By combining complementary viewpoints, the Visionary's gaze can penetrate further into the understanding of things. Grothendieck (1986) suggests that a sheaf (bundle) of converging viewpoints can give rise to a "novel thing" that transcends each partial perspective, much like a living being transcends its individual parts. This new thing is referred to as a Vision, which unites known viewpoints while revealing previously ignored ones.

4.1.2.3 What Is Mathematics?

Grothendieck (1986) reflects on the three aspects of mathematics traditionally recognized: number, size, and shape. While these aspects often interact, there is often a prevalence of one over the others. It is important to recognize the wonder and interconnectedness of these concepts:

a) **Numbers:** Numbers are not just abstract symbols; they represent quantities and provide a way to count and measure things.
b) **Measurement**: Measurement helps us understand the size, length, and weight of objects, using numbers to describe and compare quantities.
c) **Shapes**: Shapes are formed by combining and arranging different numbers of points, lines, and angles.

By understanding the connections between numbers, measurement, and shape, we can explore fascinating relationships and patterns. For example, we can use numbers to calculate the area of a shape or understand the symmetry of geometric figures. Recognizing the wonder in numbers and their applications in measurement and shape allows us to appreciate the beauty and utility of mathematics in our everyday lives.

Grothendieck (1986) identifies himself as being more inclined towards shape or the structure hidden in mathematical things. He finds the structure of things fascinating and emphasizes the importance of unravelling and discovering it rather than inventing it. The process of expressing the discovered structure through language and constructing theories involves a continual back-and-forth motion between apprehension and expression.

He concludes by emphasizing the value of attentively listening to the voice of things in mathematics. The ingenuity and imagination of a mathematics researcher lie in their ability to attentively listen to and understand the requirements and possibilities of the mathematical terrain. The most beautiful house, metaphorically representing mathematical theories, is one that faithfully reflects the hidden structure and beauty of things.

4.1.2.4 The Novel Geometry - The Marriage of Number and Size – Algebraic Geometry

In this section, I briefly discuss Grothendieck's focus on the novel geometry of algebraic geometry, which combines number and size in a unified framework. He explains the traditional distinction between arithmetic, the study of discrete structures e.g., whole numbers which are positive integers that do not include fractions or decimals, and analysis, the study of continuous structures e.g., real numbers are continuous on the number line in both directions. Geometry has historically investigated both discrete and continuous properties of geometric figures.

However, a divorce between discrete geometries e.g., polygons and polyhedra, and continuous geometries e.g., classical Euclidean geometry, non-Euclidean geometries such as spherical geometry and hyperbolic geometry and focuses on the study of smooth, continuous curves, surfaces, and higher-dimensional spaces, emerged with the development of Abstract Algebraic Geometry in the early 20th century. This new branch of geometry, closely connected to arithmetic, deals with geometric objects possessing continuous structures. It marked a significant expansion and exploration of geometric concepts.

Grothendieck (1986) made significant contributions to the development of a novel geometry by introducing two fundamental concepts: schemes and topoi. Schemes are mathematical structures that serve as a unifying framework for algebraic varieties, enabling the integration and connection of different varieties. For instance, a scheme can encompass a geometric object defined by polynomial equations, such as $x^2+y^2+z^2=1$, which represents a sphere in three-dimensional space.

In the context of algebraic geometry, "topoi" refers to the category of sheaves of sets, which captures essential information about the local properties of a given space. This category allows for the systematic study of the geometric and topological characteristics of the space (properties of spaces that are constant under any continuous deformation), providing a powerful tool for examining properties and transformations of objects within that space.

4.1.2.5 Magical Fan or Innocence

The concept of a scheme acts as a "magical fan" that connects various branches of algebraic geometry, allowing for a common framework to study different types of varieties. Grothendieck (1986) reflects on the discovery of schemes and the initial scepticism he faced from fellow mathematicians. He describes the idea of schemes as being of childlike simplicity, requiring months of intense and solitary work to convince himself and others of its validity. Through this discovery, he obtained powerful tools that allowed him to recover old results in a more meaningful way, surpass previous limitations, and tackle new problems.

4.1.2.6 Conclusion

Grothendieck (1986) concludes by highlighting the importance of innocence in the exploration of the universe, whether in mathematics or beyond. Innocence, characterized by humility and audacity, allows us to penetrate into the heart of things, listening to their voices and absorbing their meaning. It is a crucial power that transcends intellectual gifts and ambitions, enabling us to cross boundaries and engage in profound exploration.

In the context of effective learning in the mathematics classroom, the insights from Grothendieck's philosophy of mathematics are highly relevant.

By embracing the mindset of the Builder mathematician, teachers can encourage learners to explore and construct new mathematical ideas and frameworks.

Fertile viewpoints can be introduced to reveal hidden connections and provide a deeper understanding of complex mathematical realities. The emphasis on understanding the interconnectedness of numbers, measurement, and shape fosters a holistic approach to mathematics education, allowing learners to appreciate the relevance and applicability of mathematical concepts in the real world.

Furthermore, the incorporation of schemes and the recognition of the marriage between number and size in algebraic geometry can inspire teachers to present mathematics as a unified and interconnected subject.

By highlighting the magical fan that connects different branches of mathematics, teachers can help learners see the broader landscape of mathematical knowledge and the relationships between various concepts and structures.

Overall, the philosophy of mathematics, as exemplified by Grothendieck's insights, can serve as a bridge to more effective learning in the mathematics classroom. It encourages a multidisciplinary approach, promotes curiosity and exploration, and fosters a deeper understanding and appreciation of the subject.

By embracing the philosophy of mathematics, educators can create transformative educational experiences that empower learners to engage with mathematics in meaningful ways.

4.1.3 Leonard Susskind - Illuminating the Cosmos, One Quantum Leap at a Time

Mathematics is the language of the universe, and its beauty lies in its ability to unlock the secrets of nature's inner workings. - Leonard Susskind (2006)

As I embarked on the intellectual exploration of the concept of "infinity," the scholarly contributions of distinguished physicist Leonard Susskind have proven to be a beacon of enlightenment in unravelling this intricate phenomenon. Susskind's body of work serves as a compelling nexus where mathematics, science, and technology intricately converge. While underscoring the pivotal role of scientific theories in shaping our understanding of the natural world, the task of effectively conveying this profound knowledge to others remains a formidable challenge. I am enthusiastic about sharing some of the profound insights derived from Leonard Susskind's works, particularly his notable contributions from 2006, which I have also imparted to my learners.

4.1.3.1 What Physicists Mean by "Beautiful"

Physicists often employ aesthetic criteria to evaluate theories, utilizing terms such as "elegant," "beautiful," "simple," "powerful," and "unique" in their discourse. Though the nuances of these terms might vary based on individual interpretations, there exists overarching definitions that physicists generally agree upon. While distinctions between elegance and simplicity may exist, they are often subtle and not easily distinguishable. Mathematicians and engineers also adopt these terms, using them interchangeably with physicists. In their contexts, an elegant engineering solution achieves the desired outcome using minimal technological resources, often involving a single component for multiple functions, reflecting a minimalistic approach. Ultimately, the pinnacle of elegance minimizes complexity while retaining efficacy.

4.1.3.2 The Beauty Queens from the World of Science

(1) Euler's Identity: Capturing Fundamental Mathematical Principles

One theory that I have often shared with my learners is Euler's formula, an epitome of simplicity, uniqueness, and elegance: $\mathbf{e^{ix} + 1 = 0.}$ Euler's formula encapsulates fundamental mathematical principles, incorporating five pivotal numbers: '- 1', '0', 'π', 'i', and 'e', along with three foundational operations: addition, multiplication, and exponentiation. The constituents are as follows:

- '**e**': A mathematical constant, approximately **2.71828**, comparable to '**pi'**, irrational, and of unknowable exact value.
- '**i**': The imaginary unit, satisfying $\mathbf{i^2 = -1}$.
- **π**: Pi, signifying the ratio of a circle's circumference to its diameter.

This formula's simplicity bears immense significance across diverse mathematical domains, with pi's relevance spanning from ancient Babylonian times to modern applications like planetary exploration, spacecraft launches, and even the DNA double helix.

(2) Einstein's Theory of Relativity: An Elegance Masterpiece

Einstein's General Theory of Relativity stands as an exemplar of beautiful theory. Originating from a rudimentary concept of gravity, Einstein's equations encapsulate a profound comprehension of the universe. Mathematics continues to offer profound insights into the mechanics of nature, a revelation I personally find beautiful. Equations like Newton's 2^{nd} Law of Motion (**F = ma**) and Newton's Universal Law of Gravity ($\mathbf{F = G\, m_1 m_2 / d^2}$) have paved the way for and are embedded into Einstein's General Theory of Relativity:

$$\mathbf{Rmn - ½\, gmnR = 8\pi G\, Tmn}$$

This formula, laden with symbols, encompasses the entire theory of gravitational phenomena, and its predictions have been empirically validated.

4.1.3.3 Embracing Complexity: Lessons from "Ugly" Theories

However, not all theories exhibit the elegance discussed earlier. Domains like nuclear physics and chemistry prove more intricate and resistant to simplification. This realization underscores that beauty in nature isn't always apparent but deserves investigation.

(1) Dirac's Equation

Let's consider Dirac's equation, a contemporary of Einstein. Dirac's equation marries Einstein's special theory of relativity with quantum mechanics, elucidating electron behaviour near light speed. This equation pioneered Quantum Field Theory, predicting antimatter's existence: $\mathbf{\{p_0 + e/c\, A_0 - \rho_1 (\sigma, p + e/c\, A) - \rho_3\, mc\}\, \psi = 0}$. While lacking simplicity, this equation wields substantial power.

(2) Schrödinger Equation

The Schrödinger Equation (see FIGURE 12), another formidable construct, depicts matter as waves rather than particles. It yields astonishing predictions, including instantaneous particle communication over distances, and identifying previously unseen particles such as the Higgs Boson and soon, the elusive Graviton. Quantum Mathematics offers astounding precision, predicting electron magnetic properties down to multiple decimal places.

Schrödinger's Equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r},t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

i is the imaginary number, $\sqrt{-1}$.
$\hbar$ is Planck's constant divided by 2π: 1.05459×10^{-34} joule-second.
$\psi(\mathbf{r},t)$ is the wave function, defined over space and time.
m is the mass of the particle.
∇^2 is the Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
$V(\mathbf{r},t)$ is the potential energy influencing the particle.

FIGURE 12 The Schrödinger Equation

4.1.3.4 Empowering Minds: Revealing Beauty in Complexity

Young learners do not have to understand the intricate and involved workings of complicated equations, but knowing about their existence is important. Unveiling the mesmerizing allure of intricate theories nurtures a deeper understanding and empowers inquisitive minds.

4.1.3.5 Constants and Beauty: Offerings from Mathematics and Nature

The existence of these numbers is not understood; they are bestowed upon us by pure mathematics, consistently proving their precision on numerous occasions. Below are a few examples which I often shared with my learners when we covered scientific notation:

(1) Atomic Mass Unit (AMU)

Nuclear physics introduces the empirical ratio of proton mass to neutron mass within an atom's nucleus, known to seven significant figures: 1.001378. While derived from mathematical physics, its accuracy relies on laboratory measurement. However, my learners were satisfied with 1: 1 ratio when we covered the Periodic Table.

(2) The Fine Structure Constant α

Denoted as α (Greek alpha), the Fine Structure Constant is crucial yet mysterious, approximated as 1/137. Its exact value, 0.007297351, remains inexplicable. This constant symbolizes coupling strengths in quantum field theory, underpinning fundamental quantum events e.g., pairing of two electrons resulting in the emission of a photon.

(3) The Cosmological Constant λ (Greek letter lambda)

A few curious facts of our understanding of the cosmological constant:

a) **Its Equivalent to Vacuum Energy:** The Cosmological Constant is equivalent to vacuum energy, which perplexingly exists in empty space due to the intricacies of quantum mechanics. How can empty space contain something? Yet, it does.
b) **The Right Magnitude:** The Cosmological Constant must maintain a delicate balance; too large, and it disrupts celestial bodies, too small, and its purpose remains cryptic.
c) **Divergent Perspectives**: Cosmologists and physicists diverge on the cosmological constant's interpretation; cosmologists consider it a measurable parameter, while physicists seek a deeper mathematical reason for its value.
d) **The Enigma:** The enigma lies in the delicate cancellation of 119 decimal places, such as 10^{-119}, followed by the unexpected emergence of a '2.' This conundrum has eluded physicists' attempts for decades.

(4) The World at the Planck Scales

The Planck scale represents the theoretical limits at which our current understanding of physics, including quantum mechanics, breaks down due to the extreme conditions. The values provided for the Planck length, Planck mass, Planck time, and Planck temperature correspond to the orders of magnitude associated with these fundamental scales.
The Planck scale is a realm where space, time, and particles undergo radical transformations, and quantum mechanics encounters limitations.
Examples:

- Planck length 1.616255(18) × 10^{-35} m (the infinitesimal – shortest measurable distance)
- Planck mass 2.176434(24) × 10^{-8} kg (this is lightweight at its extreme)
- Planck time 5.391247(60) × 10^{-44}s (this is the epitome of a fleeting moment)

- Planck temperature 1.416784(16) × 10^{32} Kelvin (this is indeed very hot)

These examples demonstrate the real-world application of mathematical concepts, specifically how scientific notation is used to represent extremely large or small values that are encountered in various fields of science, including physics. By using constants associated with the Planck scale, a teacher provides practical instances where scientific notation is essential for conveying the magnitudes of these quantities effectively. This approach helps learners connect mathematical principles with their applications in the physical world, enhancing their understanding of both mathematics and physics.

4.1.3.6 Leonard Susskind Dares to Dream – The Beautiful Story of the Possible Birth and Death of Our Universe

(1) The Mathematical Cosmic Landscape and Potential Energy

In the vast cosmic landscape of mathematical abstraction with its infinite possibilities e.g., 10^{500}, universes emerge as real entities, occupying specific locations within the intricate terrain. Just as a smooth ball rolls along the landscape's valleys, pocket universes navigate the highs and lows of potential energy (see FIGURE 13).

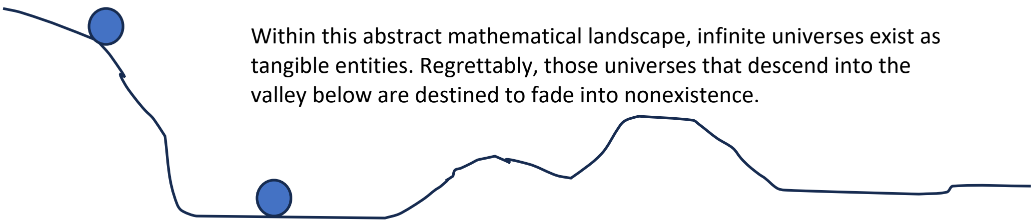


FIGURE 13: The "Landscape" is an abstract mathematical landscape with infinite possibilities.

(2) Geometry Is Fate

Einstein's gravity theory reshapes our understanding of geometry, linking it to mass and influencing the cosmic fate. The three possible geometries (Euclidean (flat), spherical, and hyperbolic (saddle)) intertwine with the universe's destiny.

(3) The Different Possible Fates of Our Universe Under Different Conditions

The destiny of our universe hinges on mass density and its effect on geometry. A large mass density results in a closed universe (hence collapsing into a singularity), a smaller density leads to an open universe (hence expanding forever), and a balance yields a flat universe (hence remaining a static universe). Telescopic observations offer insights into these potential fates.

(4) The Expanding Universe

Astronomical observations reveal the universe's expanding nature, driven by cosmic mass density. The Hubble constant defines this expansion, while peculiarities like Andromeda's (our neighbouring galaxy) movements provide intriguing clues.

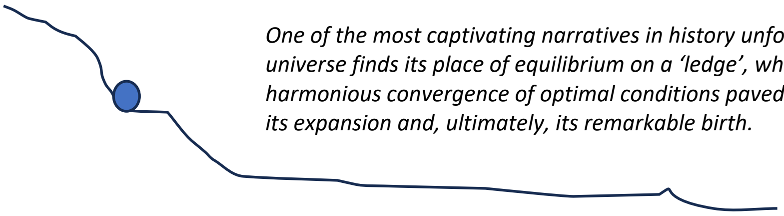
It is interesting to note that the renowned physicist Brian Greene (2020) predicts this as the end of our universe. Due to entropy[16] eventually our universe will become depleted of all energy and continue to expand into a dark infinity.

(5) The Birth of the Infant Universe

The evolution of the universe, transitioning from a high-energy state to a state of relative stability, is characterized by a dynamic process influenced by fluctuations in vacuum energy. Notably, some universes following this trajectory ultimately reach a state of cosmic insignificance.

However, our universe stands as an exceptional case, marked by a remarkable occurrence. It has settled into a unique equilibrium, a metaphorical ledge, where precisely orchestrated conditions have converged to facilitate its expansion and, consequently, its birth (see FIGURE 14).

Within this cosmic narrative, the principles of quantum mechanics play a pivotal role. Quantum fluctuations become agents of structure formation, imprinting the universe with its distinct characteristics, and laying the foundation for the emergence of galaxies and celestial structures.



One of the most captivating narratives in history unfolds as our universe finds its place of equilibrium on a 'ledge', where a harmonious convergence of optimal conditions paved the way for its expansion and, ultimately, its remarkable birth.

[16] Entropy is a measure of disorder or randomness in a system. In simple terms, it represents how spread out or disorganized the particles or components of a system are. In addition to representing the degree of disorder or randomness in a system, entropy is also associated with the spread of heat. It reflects the tendency of heat to spontaneously disperse and become more evenly distributed in a closed system over time. So, entropy can be thought of as a measure of both disorder and the spreading of energy or heat.

FIGURE 14: Our infant universe came to rest on a 'ledge' where its fate changed from those who role down to the valleys below.

(6) Flying the Flag for String Theory and Unravelling the Myth

Enrico Rinaldi, a physicist at the University of Michigan states the following: "*In Einstein's general relativity theory, there are no particles—there's just space-time and gravity in 3dimensions. and in the standard model of particle physics, there's no gravity — there's just particles in 2dimensions. connecting the two different theories is a longstanding issue in physics—something people have been trying to do since the last century*."

The pursuit of a unified theory of nature propels the exploration of String Theory—a complex concept that intricately weaves together quantum mechanics and gravity through the delicate dance of tiny vibrating strings. This theory's profound mathematical intricacy captivates both physicists and mathematicians, casting an intellectual spell on those who dare to engage with it. Within the scientific community, passionate debates among physicists and cosmologists about the validity of String Theory thrive, bearing witness to the captivating allure of delving into the realm of complex mathematics. These spirited discussions are an integral facet of the scientific endeavour, particularly when confronting the intricacies of mathematical concepts that defy simplicity. It's worth noting that these debates can occasionally be misconstrued by certain groups, leading to misinterpretations that science lacks credibility. Embracing a fallibilistic perspective, I've shared with my learners a distinctive insight: the truths inherent in mathematics and science are dynamic and in constant evolution, mirroring our ongoing quest to refine our understanding of the vast universe. Guided by this perspective, my teaching approach emphasizes illuminating the dynamic nature of mathematical and scientific truths. By doing so, I empower my learners with an appreciation for the perpetual growth and transformation of our knowledge—an essential reminder that humility is a cornerstone of genuine understanding. This ever-evolving comprehension resonates with the very essence of the universe we strive to untangle.

a) **The Marriage of the Small Infinitesimals to the Large Infinites**: In the realm where gravity's presence is conspicuously absent within the Standard Model, a deeper understanding beckons—one that bridges the chasm between quantum mechanics and gravity. Here, String Theory emerges as an alluring candidate, a theory poised to reconcile these two fundamental forces that shape our reality. Beyond being a mere theoretical construct, it presents insights that reverberate across the landscape of particle physics and cosmology, hinting at the intricate threads that unite the fabric of our universe.

b) **Practical Reasons for Digging Deeper:** The pursuit of understanding the universe's fundamental structure and its early growth impels the exploration of the interplay between gravity and quantum mechanics. In this pursuit, the significance transcends academic curiosity; it is a necessity for grasping the essence of our cosmos. Enter String Theory, a concept that not only strives to harmonize these distinct forces but carries within its mathematical intricacies the potential to revolutionize our understanding of particle physics and cosmology. Such implications promise to reshape our perception of the cosmos at its most fundamental level.

c) **String Theory is Different and Beautiful:** String Theory, distinct from its predecessors, boasts a unique feature: its intricate connection between the realms of quantum mechanics and gravity. This feature lends itself as a potential solution to the persistent conflict that has long eluded resolution. However, the journey to embrace String Theory's elegance and uniqueness is not without debate. Its unconventional path defies the conventional expectation of neat equations, yet within this seeming discord, it offers a rigorous framework and mathematical boundaries. The theory's evolution and its ever-expanding realm of possibilities serve to enrich its enigmatic allure, inviting continued exploration and contemplation.

d) **Is Nature Elegant?** Within the hallowed halls of academia, String Theory is not exempt from scepticism and criticism. Disparate viewpoints, including the advocacy for emergent theories and the practical perspectives of experimentalists, contribute to the complexity of the discourse. What lends depth to this contentious debate is the inherently personal and philosophical nature of the subject matter. It delves into the essence of being a physicist, intertwining different senses of virtue, identity, and versions of the legacy left by Einstein himself. Amid these differing positions, String

Theory perseveres as a complex mathematical framework with the potential to offer profound insights into the universe's inner workings (van Dongen, 2021).

e) **Could Inelegance and Lack of Uniqueness Be a Strength?** String Theory, with its apparent lack of elegance and uniqueness, paradoxically might hold a strength that mirrors the intricate complexity of the universe it seeks to explain. This very quality that some might consider a weakness could be its virtue. In a cosmos that defies simplistic explanations, String Theory's capacity to address multifaceted phenomena resonates with the richness of reality. By embracing this intricate nature, we might uncover that the very lack of conventional elegance is, in truth, a testament to its ability to navigate the intricate fabric of the universe itself.

4.1.3.7 Conclusion

Delving into the realm of mathematics and its interplay with the intricate fabric of the universe has illuminated not only the beauty of scientific exploration but also the potential of philosophy to serve as a bridge to meaningful mathematics teaching. The journey through the works of esteemed physicists like Leonard Susskind and the exploration of concepts such as beauty, elegance, and complexity have demonstrated the profound connection between mathematical principles and their real-world applications. As we navigate the depths of String Theory's complexity and grapple with the mysteries of the cosmos, we are reminded that the pursuit of understanding is not only about deciphering equations but also about unravelling the very nature of existence.

The pursuit of a unified theory of nature, the investigation of constants, and the exploration of the Planck scale all showcase the symbiotic relationship between mathematical concepts and their manifestations in the universe. The enigma of the cosmological constant's intricate cancellation of 119 decimal places and the emergence of a "2" underscores the limits of human comprehension and the potential significance of embracing complexity. Even as physicists debate the elegance and uniqueness of String Theory, its role as a potential unifier of forces speaks to the broader theme of bridging gaps, be they theoretical or pedagogical.

Addressing the research question, "*Could philosophy of mathematics be the bridge to meaningful mathematics teaching?*" is an endeavour that echoes the very essence of the scientific journey. Just as physicists grapple with theories that challenge conventional notions of elegance and simplicity, educators must navigate the diverse paths of individual understanding and learning. Embracing philosophy as a bridge offers an avenue to cultivate deep insights into the foundations of mathematics and its real-world implications. By infusing philosophical discussions into the teaching of mathematical concepts, educators have the potential to instil a profound sense of wonder and curiosity in their learners, fostering a deeper and more meaningful engagement with the subject.

In this narrative of exploration and discovery, the parallels between the pursuit of scientific truths and the art of effective teaching become apparent. As we seek to understand the universe's mysteries, we also strive to unlock the mysteries of effective mathematics education. In both endeavours, philosophy acts as a guiding light, illuminating the connections between the abstract and the tangible, and paving the way for a richer and more profound comprehension of the world around us. Just as Leonard Susskind aptly observed that mathematics is the language of the universe (echoing Galileo), we can envision that the philosophy of mathematics can be the bridge that helps learners comprehend, appreciate, and apply the beauty and meaning inherent in the realm of numbers and equations.

4.1.4 Brian Greene – Guiding Minds on a Journey through the Wonders of the Universe

"… *When we learn mathematics in the classroom, mathematics is represented as 'this body of knowledge' … there is no sense that this knowledge emerged from the struggles of human thinking* …" (Brian Greene[17])

Brian Greene[18] a renowned scientist, has made substantial contributions to mathematics, physics, and scientific education, leaving a profound impact on both scholars and learners.

[17] This introductory statement was given by Brian Greene during a panel discussion convened at the World Science Festival in 2021, wherein the discourse revolved around the fundamental query of whether mathematics is an outcome of discovery or invention.

[18] Brian Greene, born in 1963, is a renowned theoretical physicist known for his contributions to string theory and his ability to communicate complex scientific concepts to the general public. He currently works as a professor of physics and mathematics at Columbia University

His reflections emphasize the enduring and fundamental truths within mathematics, fostering curiosity and awe in the classroom. By highlighting the interconnectedness of mathematics with the natural world, educators can instil a deeper appreciation for the elegance and complexity of mathematical concepts.

In the context of thermodynamics, mathematics plays a pivotal role in unravelling the intricate dynamics of energy in the universe. Greene's exploration of entropy underscores the central importance of energy as the foundation of existence, with Kelvin 0^o symbolizing the lowest conceivable point. Although energy remains elusive and dissipates, mathematical models and equations developed by mathematicians, physicists, and engineers have advanced our understanding of thermodynamics, enabling practical applications and progress in energy conversion processes.

Moreover, mathematics serves as a vital tool in the study of biomolecules, including proteins and DNA. It aids in deciphering amino acid sequences, understanding protein folding, DNA sequencing, and developing computational models. This integration of mathematics and molecular biology enhances learners' comprehension of cellular processes and the molecular foundations of life.

Brian Greene's extensive body of work, spanning literature and enlightening discourse on science and mathematics, provides an inspiring journey. His profound insights, conveyed through captivating videos, delve into intriguing scientific subjects, further enriching the educational experience.

Furthermore, Greene's venture into the realm of children's literature with "Icarus at the Edge of Time" offers an imaginative narrative that introduces young readers to complex physics and astronomy concepts. The story follows a boy named Icarus[19], who defies his

[19] Instead of using wax wings to approach the Sun, as in the Greek fable, the boy in this story constructs his spaceship. Against his father's warning, he ventures close to a black hole. Black holes are often thought of as cosmic vacuums, but they don't work that way — if you remain outside the black hole's boundary, known as the event horizon, you can escape its grasp. The boy does just that. However, as Albert Einstein's theories have taught us, black holes have a profound impact on the passage of time. When the boy returns from his journey, a staggering 10,000 years have passed. The life he once knew has vanished, and his father is no longer present.

father's warnings to explore a black hole, experiencing time dilation (the relativistic effects of the black hole's immense gravitational field – time slows as gravity increases) and returning to a transformed world. While the story may appear sombre, it sparks curiosity in young minds and encourages questions about scientific principles, demonstrating how storytelling can make science more accessible and engaging for children.

Brian Greene's multifaceted contributions demonstrate the profound connection between mathematics, science, and storytelling in education. His work not only underscores the importance of the philosophy of mathematics but also illustrates how imaginative narratives can bridge the gap to effective mathematics teaching by inspiring curiosity and critical thinking in learners of all ages.

4.1.5 Reflection on Bertrand Russell's Influence on My Mathematics Teaching

Bertrand Russell likens mathematics to a temple, he writes: "*Dry pendants*[20] *possess themselves of the privilege of instilling this knowledge: they forget that it is to serve but as a key to open the doors of the temple; though they spend their lives on the steps leading up to those sacred doors, they turn their backs upon the temple so resolutely that its very existence is forgotten, and the eager youth, who would press forward to be initiated to its domes and arches, is bidden to turn back and count the steps*" (Russell 1959).

Bertrand Russell's profound insights into the fundamental purpose and ideals of education, particularly within the domain of mathematics, significantly influenced my approach to mathematics instruction during my teaching career. As I embarked on my journey as a mathematics teacher, Russell's wisdom served as a guiding beacon, urging me to reassess the core objectives of teaching mathematics and its profound impact on the lives of learners. In this reflective discourse, I delved into the ways in which Bertrand Russell's philosophy of mathematics indelibly shaped my pedagogical methodology, facilitating a bridge between abstract mathematical concepts and meaningful classroom experiences.

[20] Bertrand Russell uses the term "pendants" to refer to those who become overly pedantic and isolated in their pursuit of knowledge, neglecting the meaningful and transformative aspects of learning.

Russell's assertion that all forms of human endeavour necessitate periodic contemplation of their purpose and ideals resonated profoundly with my teaching philosophy. Aligned with this perspective, I embraced an ongoing regimen of pedagogical introspection. I acknowledged that mathematics education transcends the mere transmission of rules and formulas; it is an endeavour aimed at fostering a profound appreciation for the elegance and significance of mathematical concepts. Inspired by Russell's entreaty to reflect upon the contributions of mathematics to the broader tapestry of human existence, I diligently wove narratives into my instructional approach, spotlighting the historical and cultural contexts surrounding mathematical discoveries.

Furthermore, Russell's critique of sterile pedantry and the mechanistic diffusion of knowledge propelled me to adopt a synthesis of structuralism, constructivism, fallibilism, and absolutism (as expounded in Section 2.6) within my teaching paradigm. I recognized that the optimal apprehension of mathematical concepts materialized when learners actively constructed their understanding through tactile experiences and interactive engagements. Accordingly, I seamlessly integrated collaborative problem-solving exercises and dynamic simulations into my curriculum, affording learners the opportunity to explore mathematical intricacies within a meaningful framework. This shift engendered an evolution from passive recipients to active co-creators of knowledge, permitting them to unravel the interconnectedness of mathematical concepts while embracing their intrinsic absolutes.

Moreover, Russell's attention to rationality and logical rigor in mathematics motivated me to prioritize conceptual comprehension over rote memorization. Drawing inspiration from his insights, I undertook a reimagining of my approach to introducing intricate subjects such as algebra. Rather than presenting algebra as an enigmatic realm of symbols, I guided learners toward grasping the foundational principles of universal truths. By accentuating the potency of abstract reasoning and the capacity to navigate the abstract, I endeavoured to impart a sense of cognitive empowerment within my learners.

Russell's advocacy for a transition from poor reasoning to meticulous logical reflection aligned with my commitment to nurturing critical thinking acumen. I seamlessly incorporated exploratory exercises that galvanized learners to interact with mathematical constructs through inquisitive discovery. Through these activities, learners became first-hand witnesses to the origin of conclusions, fostering an authentic assimilation of mathematical underpinnings.

In conclusion, Bertrand Russell's philosophy of mathematics served as a guiding light throughout my pedagogical expedition. His conceptualization of mathematics as a sublime and meaningful pursuit propelled me to engender a teaching milieu wherein learners not only acquired mathematical proficiency but also cultivated an enduring reverence for the discipline. Russell's entreaty to bridge the chasm between theoretical constructs and practical applications metamorphosed into the bedrock of my pedagogical approach. As I embarked on the composition of my autoethnography, scrutinizing whether the philosophy of mathematics could indeed serve as a conduit to purposeful mathematics instruction, I remained indebted to Russell's insights, which persisted in illuminating my trajectory as a teacher of mathematics.

4.1.6 Contemplating Stewart Shapiro's Structural World

Perhaps one of the most important lessons of philosophy is to teach us how to live with the questions unanswered, rather than settle for unsatisfactory but popular answers."
— Stewart Shapiro (2000, Chapter 12)

Stewart Shapiro's profound commitment to structuralist philosophy has significantly shaped my teaching path as an enthusiast of structuralism. His insights have not only guided the direction of my dissertation but have also intricately woven into the very fabric of my teaching philosophy. Within the pages of this dissertation, which delves into the profound implications of the philosophy of mathematics in education, Shapiro's seminal contributions, notably his exposition of 'offices' as fundamental mathematical structures, occupy a pivotal position, as detailed in section 2.4 on structuralism.

4.1.6.1 A Conceptual Framework for Understanding Mathematical Structures

Shapiro's (2000, 2014) structuralist viewpoint, as expounded in "Thinking About Mathematics: Philosophy of Mathematics" and in "Philosophy of Mathematics: Structure and Ontology," has provided me with a robust conceptual framework to delve into the intricate interplay between mathematical structures and their pedagogical implications.

Through his analogy of 'offices,' I have metaphorically envisioned mathematical concepts as integral components within a larger organizational framework, each serving a distinct purpose and interconnected with others. This analogy has been instrumental in explaining the inherent relationships between mathematical entities and their functional roles, facilitating my analysis of how these structures can be harnessed to enrich the learning experience. Moreover, Shapiro's discussions of the absolutist nature of certain mathematical principles, such as the unequivocal identity between natural numbers, have been pivotal in shaping my perspectives on effective mathematics teaching.

4.1.6.2 Recognizing Structures and Patterns: Ante-Rem Structuralism

Shapiro's "places-are-offices" concept, where positions within a structure possess properties rather than standalone objects, has resonated with my understanding of mathematical objects and their relation to structures. As an ante-rem structuralist (structures exist independent of our perceptions akin to Plato's world of forms), Shapiro contends that existing structures are recognized through the observation of patterns, infusing my exploration with a deeper understanding of how mathematical constructs emerge and are identified.

4.1.6.3 Structural Understanding in Learning: A Pedagogical Emphasis

Shapiro's philosophical insights have deeply informed my exploration of effective learning strategies within the mathematics classroom. His delineation of the epistemological foundations of mathematical knowledge has motivated me to investigate how learners assimilate mathematical concepts and the crucial role of structural comprehension in their cognitive development. In alignment with Shapiro's emphasis on structural understanding, I have formulated pedagogical recommendations that prioritize not only procedural proficiency but also a profound grasp of the underlying structural relationships.

4.1.6.4 Harmonizing Metaphysics and Pedagogy: A Coherent Approach

Shapiro's work has facilitated a harmonious integration of metaphysical considerations and pedagogical imperatives within my dissertation. By seamlessly weaving together metaphysical notions of mathematical objects and structures with practical teaching approaches, I have constructed a compelling argument for the philosophy of mathematics as a bridge to effective learning. This synthesis mirrors Shapiro's own endeavour to establish a coherent and reciprocal relationship between mathematical ontology and its implications for education.

4.1.6.5 Conclusion: A Transformative Influence

Stewart Shapiro's profound commitment to structuralist philosophy, encapsulated by his analogy of 'offices' as foundational mathematical structures, has undeniably left an indelible mark on my journey as both a passionate adherent of structuralism and an educator. His insights have not only guided the trajectory of my dissertation but have also been intricately woven into the very fabric of my teaching philosophy. Through Shapiro's philosophical lens, I've come to appreciate the power of embracing unanswered questions and the value of navigating intellectual territories where satisfactory answers might elude us.

This pivotal connection between Shapiro's ideas and my pedagogical journey seamlessly propels us forward to the subsequent section, 4.1.7, where I delve into the practical embodiment of structuralism in my mathematics lessons. Here, I will illustrate how the profound concepts put forth by Shapiro were transformed into tangible and transformative learning experiences within the classroom setting. Just as Shapiro's insights have resonated deeply with my own understanding, they have also found a practical resonance in the way I engage and inspire learners through structuralist principles.

4.1.7 Structuralism in the Researcher's Mathematics Lessons

Stewart Shapiro's analogy, likening positions within mathematical structures to offices in a building, provides a helpful framework for organizing and presenting mathematical concepts. This section offers a concrete demonstration of implementing such a framework in mathematics teaching. By adopting this analogy, I, the researcher herself, incorporated it into my mathematics classroom to enhance the structure and coherence of the curriculum aligned with the Curriculum and Assessment Policy Statement (CAPS).

4.1.7.1 The Building and Office Analogy

The analogy allowed me to arrange the diverse mathematical concepts systematically, ensuring a logical and interconnected progression that was accessible and comprehensible to learners. In this pedagogical approach, I introduced the analogy of buildings with offices, representing different areas of mathematics that learners would explore throughout the academic year. This provided a visual representation and a metaphorical map for learners to navigate their mathematical learning journey.

4.1.7.2 Applying the Analogy in Mathematics Teaching

To begin, I provided learners with an overview of the various domains within mathematics, highlighting the relevance and applicability of each domain to their learning. This overview, represented as "FIGURE 3: Charting the Expanding Mathematics Universe" served as a guide, enabling learners to understand the interrelationships and interconnectedness of mathematical concepts across different areas.

4.1.7.3 Visual Representation: The City Metaphor

By employing this analogy, I aimed to create a structured and coherent learning environment, where learners could perceive the organization of mathematical knowledge and its interconnected nature. This approach facilitated a comprehensive understanding of mathematics as a unified discipline while ensuring a clear progression through the curriculum outlined in the CAPS.

As shown in FIGURE 15, the visual representation helps us understand how complex the world of mathematics is, which leads us to ask deep questions about what we know (epistemology) and what really exists (ontology). In this complicated space, even simple questions can lead to answers that are detailed and have many different aspects.

4.1.7.4 Building-Specific Focus

The city metaphor FIGURE 16 was employed to represent the various areas of mathematics as depicted in FIGURE 15. Each area is symbolized as a distinct building within the cityscape. In my Grade 7 curriculum lesson, I directed the learners’ attention towards specific buildings within the city, namely the Number System, Geometry, Algebra I, and Statistics and Probability buildings, as illustrated in FIGURE 17.

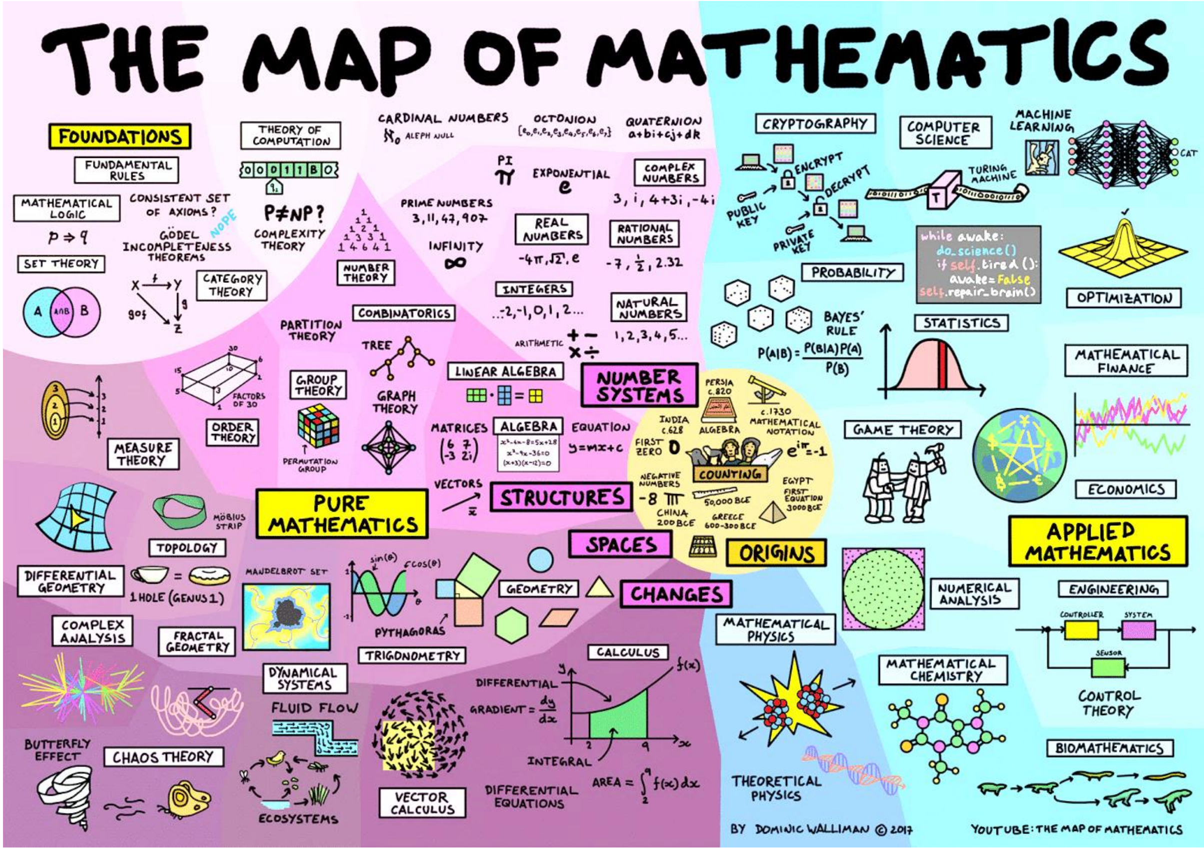


FIGURE 15 Charting the Expanding Mathematics This map illustrates the complexity of the mathematics world and how it lends itself to epistemological and ontological questions. Simple questions with complex answers.

Source of FIGURE 15: Science Infographics Breakdown STEM ...mymodernmet.com

FIGURE 16 The mathematics city metaphor.

Source of FIGURE 16: https://www.alamy.com/stock-photo/cartoon-buildings.html?blackwhite=1&sortBy=relevant

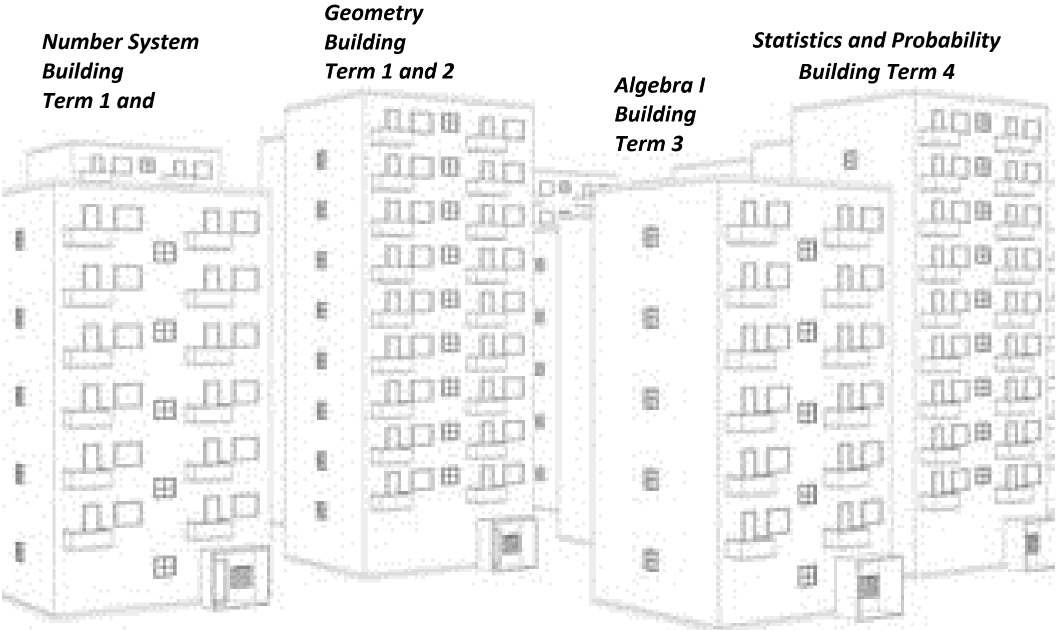


FIGURE 17 Foundation of Exploration: Journey into the Four Pillars of Mathematical Understanding

Source of FIGURE 17: https://www.alamy.com/stock-photo/cartoon-buildings.html?blackwhite=1&sortBy=relevant

4.1.7.5 Progression Within the Number System Building

In FIGURE 18 each floor offers an in-depth examination of the Number System Building, providing a detailed depiction of the various concepts that will be explored on each floor throughout the academic terms, aligning with the specified timelines from term one to term two.

This focused view allows for a comprehensive understanding of the progression and sequencing of mathematical concepts within the Number System domain.

In the Real Numbers Building, which is often associated with arithmetic, the processing of different types of numbers takes place. The ground floor houses the Whole Number System, featuring various related offices such as place value, prime factorization, and exponents.

The foundations of this building are connected to learners’ prior knowledge, providing a link between their existing understanding and the classroom setting.

3rd Floor
Ratio, Rate and Proportion – Term 2

2nd Floor
Fractions, Decimals and Percentages – Term 2

1st Floor
Integers – Term 1

Ground Floor
Whole Numbers – Term 1

FIGURE 18 Ascending the Numerical Realm: Unveiling the Sequential Mastery of Concepts within the Number System Building

Source of FIGURE 18: https://www.istockphoto.com/illustrations/office-building-interior

4.1.7.6 Ground Floor: Whole Number System

In FIGURE 19 a detailed floor plan of the ground floor of the Number System Building, on this floor the focus is specifically on the definition and manipulation of whole numbers. The figure includes a floor-map illustrating the layout of different "offices" within the floor, each corresponding to a specific concept covered in the curriculum. During the learning process, each learner is provided with a copy of the floor plan. As they complete their work in each "office," they engage in a colouring activity, visually indicating their progress and completion of the corresponding concept. This approach fosters an organized and interactive learning experience, allowing learners to navigate and track their advancement through the various concepts within the whole numbers domain of mathematics.

After completing the exploration of whole numbers, learners move to the next stage represented by the first floor. Here, they enter the Integer Offices where they are introduced to new definitions and rules, further deepening their understanding in this area.

Moving up to the second floor, they encounter the Rational Numbers section, which includes offices dedicated to concepts related to fractions, decimals, and percentages.

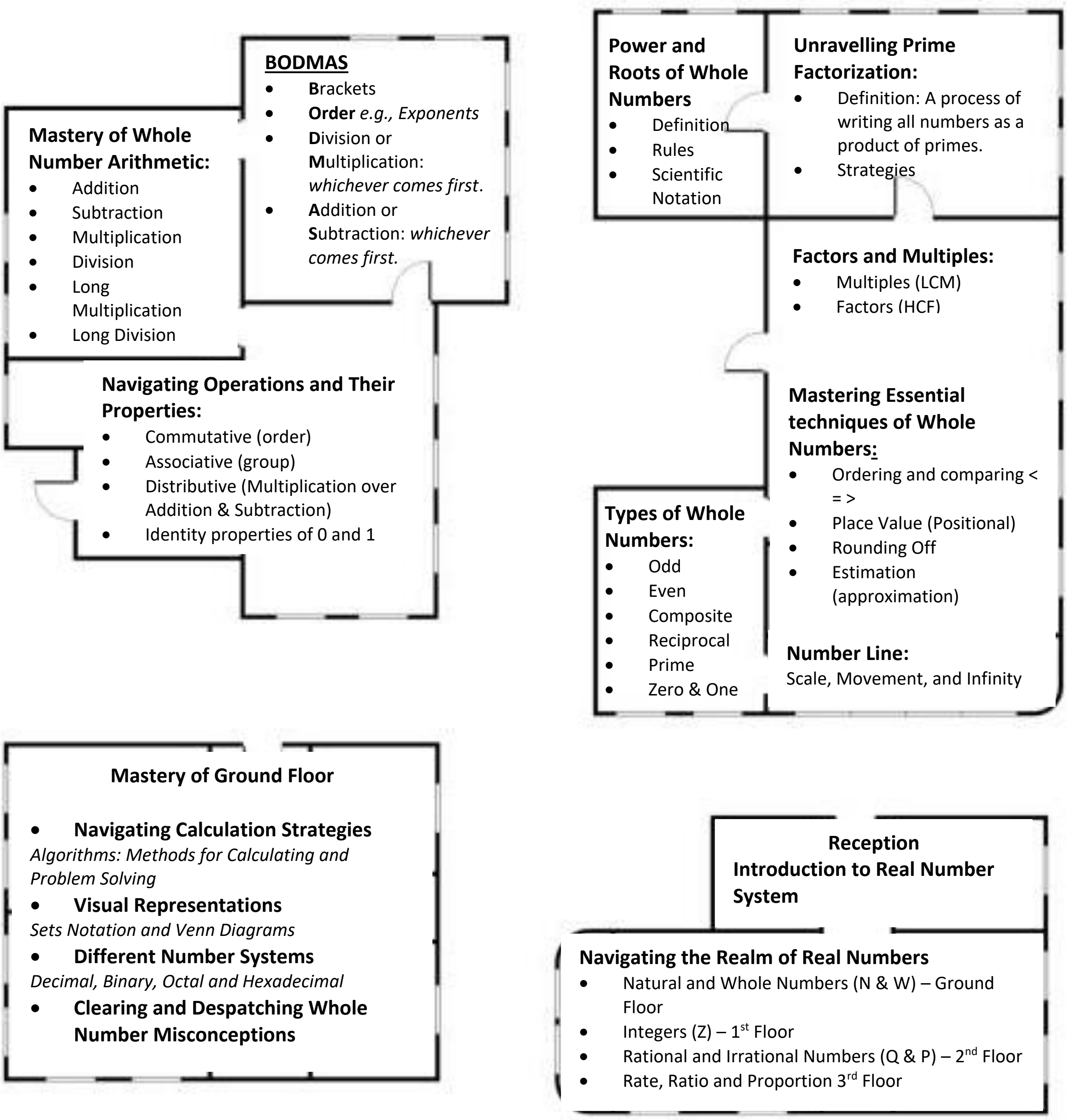


FIGURE 19 Number System Building's Ground Floor Plan: Whole Number Processing Offices

Source of FIGURE 19: https://www.istockphoto.com/illustrations/office-floor-plan

Progressing to the third floor, the focus shifts to Ratio, Rates, and Proportion, where learners engage with relevant offices that delve into these mathematical ideas. This structured progression allows learners to gradually expand their knowledge as they move through different floors and offices.

Each floor of the Real Numbers Building addresses a specific section of the Real Number system, maintaining interconnectedness to ensure the overall stability and coherence of the building. Periodically, there is a revisiting of previous floors to perform maintenance or revision work as necessary, reinforcing earlier concepts and connections.

4.1.7.7 Extension to Other Areas: Algebra and Geometry Buildings

Similarly, in the Algebra Building, distinct floors and offices are dedicated to terms, expressions, equations, polynomials, relations and functions, patterns, and other relevant topics. The Geometry Building features floors and offices covering points, lines (1D), angles, the Cartesian plane, 2D shapes, Pythagoras, 3D solids, measurement, transformations, and more. This organizational structure allows learners to track their progress within the building or specific office, while also facilitating connections to previously learned concepts and offering a glimpse of upcoming topics.

4.1.7.8 Conclusion

In conclusion, structuralism offers a unique perspective on mathematics as the study of structures, emphasizing the relationships and patterns within mathematical objects. The various perspectives within structuralism, including those presented by Benacerraf, Putnam, Resnik, Shapiro, and Hellman, shed light on the nature of mathematical objects and the interconnectedness of mathematical concepts.

In the mathematics classroom, understanding pattern and structure is essential for learners' learning and conceptual development. Research has shown that recognizing the structure of mathematical processes and representations leads to deep conceptual understanding and promotes critical thinking skills. Concepts such as counting, subitising, partitioning, and numeration are all influenced by the recognition and understanding of patterns and structures.

By incorporating the analogy of structures within mathematical buildings, teachers can provide a structured and coherent learning environment. This approach allows learners to navigate the interconnected nature of mathematics and perceive the organization of mathematical knowledge. It fosters a comprehensive understanding of mathematics while ensuring a clear progression through the curriculum. This approach not only aids learners in navigating the complex landscape of mathematical concepts but also fosters engagement and a deeper understanding of the subject. Through the incorporation of visual metaphors, interactive learning strategies, and a balanced teaching philosophy, this pedagogical approach demonstrates the potential to transform the way mathematics is taught and learned.

The incorporation of structuralism in the classroom aligns with both absolutist and fallibilist perspectives. It emphasizes precise mathematical terminology and organized workbooks, while also encouraging a growth mindset and the recognition of mathematics as an ongoing process. Group work and class discussions promote constructivist learning, where learners actively construct mathematical knowledge through collaboration and engagement.

In the next section, we will explore the constructivist philosophy of mathematics education and its implications for teaching and learning.

4.1.8 Absolutism and Fallibilism in the Researcher's Mathematics Lesson

4.1.8.1 Introduction

Central to my teaching philosophy is the inspiration drawn from the ideas of structuralist Stewart Shapiro, creating an environment that is both positive and engaging. Another distinguishing feature of my teaching approach is the deliberate integration of both absolutist and fallibilistic perspectives, a harmony that finds resonance in Ernest's work (1996).

This approach is a nuanced blend, highlighting the significance of precise mathematical language and meticulously structured resources, aligning cohesively with absolutist principles. However, concurrently, it encourages a fallibilistic mindset, urging learners to perceive mathematics as a continual journey of refinement, adaptation, and correction. This philosophy finds embodiment in our classroom motto, emphasizing the transparency and comprehension of processes used—whether mental calculations or calculator operations—ensuring accessibility for all.

Moreover, this teaching philosophy resonates with the practical application elucidated in the lesson plan concerning number systems (see ANNEXURE D). This supplementary lesson

demonstrates the practical convergence of structuralism with the philosophies of absolutism and fallibilism. It serves as a tangible illustration of how the fusion of precise mathematical terminology and a progressive learning trajectory yields a deeper and more comprehensive grasp of mathematical concepts.
Such alignment is pivotal, bridging the theoretical foundations of structuralism with the pragmatic sphere of teaching. It not only cultivates a comprehensive understanding but also nurtures a growth mindset among learners.

4.1.8.2 Symbolic Use of Pencils

Aligned with my fallibilist orientation, I have advocated for the exclusive use of pencils in my classroom, deliberately eschewing pens. This pedagogical stance is rooted in the recognition of pencils as symbolic tools representing an iterative and evolving process. They stand as an embodiment of the ethos of ongoing development and a readiness for adaptable adjustments. Engaging with pencil-based work signifies a pedagogical paradigm that embraces continuous advancement and openness to expansion. This instructional choice embodies an educational ethos valuing an uninterrupted trajectory of exploration and growth.

Within this context, the incorporation of pencils in the learning environment carries symbolic weight, reflecting an inherent capacity for perpetual evolution and fostering a receptive attitude towards enhancements. Thus, the endorsement of pencil-centric activities underlines an educational philosophy harmonizing with fallibilism's principles. This alignment resonates with the broader exploration of the potential of the philosophy of mathematics as a conduit to facilitate a more profound and meaningful approach to the pedagogy of mathematics instruction.

4.1.8.3 Conclusion

The symbiotic interplay of absolutism and fallibilism underscores the essence of my teaching philosophy, creating an environment where mathematical exploration and refinement coexist. As we transition into the upcoming Section 4.1.9, we delve into yet another compelling facet of my teaching methodology: constructivism. Just as these philosophical perspectives harmonize within my classroom, constructivism seamlessly integrates cognitive theories into a dynamic learning experience. Join me as we navigate the intriguing landscape of constructivism in my mathematics classroom, uncovering further layers of effective pedagogy.

4.1.9 Constructivism in the Researcher's Mathematics Lessons

4.1.9.1 Introduction

Constructivism, as a theory of learning, challenges the traditional absolutist view of mathematics as a static body of knowledge. Scholars like Polya, Lakatos, Hersh, and Davis have highlighted the exploratory and social nature of mathematics, presenting it as a human-friendly, socially constructed product (Ernest, 1991). The philosophy of constructivism recognizes that mathematical knowledge is actively constructed by learners and emphasizes the role of problem-solving, inquiry, and investigation in mathematics education.

Ernest (1991) argues for the centrality of problem posing and solving, inquiry, and fallibility in school mathematics. The transition from guided discovery to an investigatory approach involves a shift in power dynamics within the classroom, empowering learners to take ownership of their learning process. Learners gain autonomy and control over their solution methods and content, while teachers facilitate the learning environment. An inquiry-oriented approach encourages critical thinking, creative problem-solving, and collaborative exploration, aligning with the principles of constructivism.

Implementing a constructivist approach in the classroom requires considering the social context and power relations. Social constructivism emphasizes the importance of interaction between teachers and learners and among learners themselves. Learners support each other in creating richer meanings for new mathematical content, and the broader community of learners contributes to the knowledge-building process. Teachers play a facilitative role, encouraging learners to develop their own strategies for problem-solving (Lee, 2001).

Constructivism recognizes that mathematical understanding is built through integration, connection, and generalization of new knowledge with existing knowledge. Teachers create opportunities for learners to construct their own algorithms and make connections between procedures. The standard algorithms can be taught meaningfully, alongside alternative algorithms, using objects and physical manipulations to deepen learners' understanding of operations like addition and subtraction (Lee, 2001).

The transformation from deep structures of knowledge into behavioural objectives has often resulted in a distortion of knowledge in schools. Constructivism challenges this view and advocates for a more holistic presentation of mathematical concepts, allowing learners to make connections and gain new insights. Activities in constructivist classrooms are learner-focused, encouraging learners to ask questions, conduct experiments, make analogies, and arrive at their own conclusions. Negotiation of meaning plays a crucial role, as learners compare their existing knowledge with new experiences, engage in discussions, analyse different perspectives, and construct personal meanings aligned with the theories of others (Ernest, 1991; Lorsbach & Tobin, 1997).

In summary, implementing constructivism in the mathematics classroom involves creating a learner-centred environment, fostering inquiry, problem-solving, and negotiation of meaning. It shifts the role of teachers from information-givers to facilitators, empowering learners to construct their own mathematical knowledge. The approach recognizes mathematics as a human construction, emphasizes the tentative nature of knowledge, and encourages critical thinking, creativity, and collaboration among learners.

4.1.9.2 Designing a Constructivist Lesson for Number Systems

In accordance with the constructivist approach, a lesson on number systems can be designed to actively engage learners and encourage their participation in the learning process. The lesson begins by creating interest and curiosity among the learners regarding the topic of different number systems. This can be achieved through various means such as demonstrations, presenting data, or showing a short film that sparks their interest. Open-ended questions are then posed to probe the learners' preconceptions and elicit their initial understanding of the topic.

Next, the teacher presents information on different number systems, introducing new concepts that may challenge the learners' existing understanding. The aim is to create a cognitive dissonance that prompts learners to explore and reconcile their previous knowledge with the new information. The learners are given the opportunity to take charge of their learning, either individually or in small groups, fostering a sense of ownership and autonomy.

In small groups, learners collaborate to investigate the topic further and explore ways to reconcile their preconceptions with the new information. They are encouraged to make suggestions, conduct inquiries, and engage in problem-solving activities. Each group selects a spokesperson who will present their findings and discoveries to the rest of the class.
It is important to note that the suggested format for the lesson is not intended to be rigidly followed but rather serves as a guideline. The constructivist approach values flexibility and allows for adaptation to the specific needs and dynamics of the classroom. The lesson aims to promote active learning, critical thinking, and collaborative exploration, providing learners with opportunities to construct their own understanding of the topic.

By employing a constructivist framework, this lesson facilitates the development of learners' cognitive skills, problem-solving abilities, and ability to make connections between new and prior knowledge. It fosters a learner-centred environment where learners actively participate in the learning process, take ownership of their learning, and engage in meaningful interactions with their peers (see ANNEXURE D).

4.1.9.3 Constructivist Approach: Collaboration and Dialogue

Additionally, the cultivation of collaborative group activities and facilitated classroom dialogues exemplified an adherence to a constructivist educational philosophy. This approach acknowledges individuals as active architects of knowledge acquisition, encompassing the domain of mathematical knowledge, as will be elaborated upon in the ensuing section. This further underscores the inquiry into whether the philosophy of mathematics could serve as a conduit for fostering a more profound and enriched pedagogy in the realm of mathematics education.

4.1.9.4 Conclusion

In conclusion, the constructivist approach to teaching and learning mathematics challenges the traditional view of mathematics as a static body of knowledge. It recognizes that mathematical understanding is actively constructed by learners through problem-solving, inquiry, and investigation. By shifting the role of teachers from information-givers to facilitators, constructivism empowers learners to take ownership of their learning process, fostering autonomy, critical thinking, creativity, and collaboration.

Implementing constructivism in the mathematics classroom involves creating a learner-centred environment that promotes inquiry, problem-solving, and negotiation of meaning. It encourages learners to ask questions, conduct experiments, make connections, and arrive at their own conclusions.

Through interactions with their peers and the broader community of learners, learners construct personal meanings aligned with the theories of others, fostering a social constructivist perspective.

The example given of a constructivist lesson on number systems illustrates how it can actively engage learners and encourage their participation in the learning process. By creating cognitive dissonance and providing opportunities for exploration, collaboration, and presentation of findings, learners can construct their understanding of number systems. The lesson design values flexibility, adaptation to learners needs, and the integration of real-world applications, fostering active learning, critical thinking, and the development of problem-solving skills.

In embracing constructivism, teachers facilitate a transformative learning experience where mathematical knowledge is not merely transmitted but actively constructed by learners.

By recognizing mathematics as a human construction and emphasizing the tentative nature of knowledge, constructivism prepares learners to become lifelong learners who can apply their understanding of mathematical concepts to diverse contexts and challenges.

4.1.10 Analysis of The Autoethnography Report

4.1.10.1 The Transformative Power of Philosophy of Mathematics in Education

In the contemplative journey of this autoethnography, a symphony of insights emerges, resonating with the profound influence of the philosophy of mathematics on education. Drawing inspiration from the philosophies of Alexander Grothendieck, Leonard Susskind, Brian Greene, Bertrand Russell, and Stewart Shapiro, the narrative unfurls a tapestry that illustrates how mathematics can serve as a dynamic bridge, fostering a reimagined landscape of more effective mathematics classrooms. This transformative potential finds expression through four pivotal pillars:

a) Fostering Curiosity and Appreciation:

In line with Russell's sentiments, the narrative of my autoethnography underscores the pivotal role of mathematics in evoking curiosity and nurturing appreciation. By crafting a narrative that showcases the intricate interplay between mathematics and the natural world, educators possess the power to spark curiosity, lighting an inner flame that propels learners on a journey of wonder and exploration. This approach not only unveils the elegance and complexity of mathematical concepts but also fosters a deeper reverence for the subject. In my report, I demonstrate how I integrate philosophical perspectives, drawing from the insights of Grothendieck, Bertrand Russell, Leonard Susskind, and others, to ignite curiosity and foster appreciation among learners. Throughout the report, I emphasize concepts such as the beauty and elegance of mathematical principles, echoing Grothendieck's ideals. Furthermore, by incorporating fallibilism into my teaching approach, which acknowledges the ongoing nature of mathematical exploration, I encourage learners to wholeheartedly embrace the dynamic journey of refining and adapting their mathematical understanding.

b) Promoting Relevance and Applicability:

By delving into the synergy between mathematics and various scientific domains, my narrative highlights the transformative potential that arises from integrating mathematics with scientific concepts. This integration, as envisioned by Susskind and Greene, involves intertwining mathematical landscapes with notions of the expanding universe, entropy, thermodynamics, evolution, and even molecular biology. Through these intricate connections, the pragmatic applications of mathematical principles are vividly illuminated, transcending the confines of the classroom. As a result, the manifold ways in which mathematics contributes to our comprehension of the world become apparent. This emphasis on relevance not only empowers learners to view mathematics beyond a mere academic exercise but also as a potent tool endowed with real-world significance.

In line with Leonard Susskind's ideas, my report goes on to underscore the philosophy of mathematics as a bridge seamlessly linking abstract concepts to tangible real-world applications. Concepts like beauty, elegance, and complexity are explored to underpin my belief in the tangible significance of mathematical principles. This perspective is further complemented by my embrace of constructivism, which serves to actively involve learners through meticulously designed lessons such as those centred around number systems. This approach effectively engages learners and serves as a platform to showcase the practical applicability of various mathematical concepts.

c) Enhancing Problem-Solving and Analytical Skills:

Guided by Stewart Shapiro's structuralist philosophy, my narrative highlights the transformative potential embedded in immersing learners within the realm of mathematical models, algorithms, and equations. This engagement not only presents mathematics as a catalyst for innovative problem-solving and analytical thinking but also equips learners with tools to decipher intricate challenges—a notion influenced by Grothendieck's geometric perspective. Such a pedagogical shift empowers learners to confront complex problems with confidence, underpinned by the understanding that mathematical concepts transcend abstract notions, serving as potent instruments to unravel the intricacies of the world.
In parallel, the philosophy of mathematics, as illuminated through Bertrand Russell's insights, emerges as a catalyst for the enrichment of problem-solving and analytical skills.

The emphasis Russell places on critical thinking and logical rigor significantly shapes my approach to constructivism. This educational philosophy encourages active inquiry, problem-solving, and the negotiation of meaning. Creating a constructivist environment, I provide learners with the means to collaboratively construct a foundation of mathematical knowledge through interactive exploration and problem-solving activities.

d) Encouraging Interdisciplinary Connections:

Drawing inspiration from Leonard Susskind and Bertrand Russell, my report takes a focused approach to accentuate the philosophy of mathematics' inherent interdisciplinary nature. Through the connections I establish between mathematics and fields like physics and molecular biology, the potential for philosophy to nurture a more expansive perspective on interconnected knowledge is vividly showcased. This narrative is further enriched by the integration of absolutism and fallibilism, harmonizing the precision of mathematical terminology (absolutism) with the recognition of mathematics as an ongoing journey (fallibilism).

In alignment with this ethos, the contemplations offered by Susskind, Greene, and Russell serve as cornerstones, magnifying the profound implications of mathematics that reverberate across a spectrum of disciplines. By nurturing a holistic comprehension of the world, educators pave the way to illuminate the intricate threads that intricately connect mathematics, science, and technology.

This comprehensive approach, aligned with Susskind's vision, unravels a tapestry of knowledge transcending conventional boundaries. The fostering of interdisciplinary connections becomes a voyage that liberates learners from the confines of traditional silos, revealing a unified fabric of understanding, where mathematical fluency emerges as the shared language binding these diverse domains.

4.1.10.2 The Emerging Role of Philosophy of Mathematics in Effective Teaching and Learning in the Mathematics Classroom

The investigation at the core of this research process involves a systematic exploration employing open coding, axial coding, and selective coding. This rigorous analysis reveals an emerging role for the philosophy of mathematics in mathematics teaching, synthesizing insights from this autoethnography passage, the literature review section, the anonymous questionnaire completed by teachers (see ANNEXURE E1 & 2), and the field notes (see ANNEXURE F) as detailed in the next section 4.2.

This burgeoning role of the philosophy of mathematics in mathematics teaching signifies the potential for a substantial pedagogical transformation. The integration of this philosophy into the pedagogical landscape emerges as a potent catalyst, poised to elevate mathematics education to unprecedented levels. This theory encapsulates its transformative essence through a multifaceted lens:

DIMENSION 1: Philosophy of Mathematics as an Illuminating Beacon

In my report, I harness insights from Stewart Shapiro's structuralist philosophy, employing it as an illuminating beacon that breathes life into conventional teaching methods, transforming them into dynamic learning experiences. The analogy of 'offices' and my emphasis on mathematical structures serve as instruments through which philosophy sheds light on the path towards a more profound comprehension of mathematics. This illumination arises from the interplay between absolutism and fallibilism, a delicate equilibrium that simultaneously upholds precise terminology while welcoming ongoing exploration, thereby enriching the entire teaching process.

This dimension, rooted deeply in the philosophy of mathematics and influenced by the profound wisdom of Russell and Shapiro, emerges as a guiding light. Its essence draws from the intricate interplay connecting mathematics and the natural world. It not only enhances learners' understanding and appreciation of the subject but also inspires a holistic and interconnected perspective. This perspective urges learners to break free from the confines of the classroom and embark on explorations that unravel the profound applications of mathematical principles across various domains. This vision resonates with the profound insights of Grothendieck, echoing his concept of mathematics as a vehicle for boundless exploration.

DIMENSION 2: Interdisciplinary Threads: Unifying the Academic Tapestry

Interdisciplinary connections, mirroring the philosophy's plea for unity, serve as the foundational pillars of transformation. This dimension, guided by the philosophies of Susskind, Greene, and myself, intricately weaves together the threads that interconnect mathematics, science, and technology. In this fusion of disciplines, educators play the role of guides, leading learners toward an encompassing comprehension of the world.

This guidance highlights the pivotal role of mathematics as a binding agent, capable of unifying various academic realms. Such unification harmonizes seamlessly with Russell's and Shapiro's emphasis on interconnectedness, creating a harmonious symphony of knowledge. In alignment with the ideas of Susskind and Russell, I embark on weaving these interdisciplinary threads that seamlessly unite a diverse array of academic subjects beneath the expansive canopy of mathematics. My approach, driven by the principles of constructivism and fuelled by engagement with interdisciplinary concepts, showcases the profound potential of philosophy to transcend traditional subject boundaries. Within these connections, the tapestry of interconnectedness comes alive, mirroring Susskind's exploration of string theory and Russell's insistence on bridging the gap between theoretical constructs and practical applications.

DIMENSION 3: Cultivation of Critical Thought and Curiosity

Central to harnessing the transformative potential of the philosophy of mathematics in teaching is the cultivation of critical thinking, metacognition, curiosity, and a comprehensive grasp of the subject. Guided by Russell's eloquent articulation and Shapiro's structuralist viewpoints, the philosophy of mathematics stands as a guiding beacon. This beacon illuminates the path toward fostering a profound sense of wonder and inquisitiveness among learners. Through the fusion of theoretical constructs with real-world applicability, educators craft an environment that not only nurtures analytical prowess but also ignites an abiding passion for the subject. This nurturing aligns harmoniously with Shapiro's and Russell's perspectives on holistic understanding.

Inspired by Bertrand Russell's teachings and informed by my own insights, the philosophy of mathematics serves as a catalyst for nurturing critical thinking and curiosity within learners. Russell's emphasis on logical reflection significantly informs my application of constructivism—an educational approach that urges learners to engage in questioning, exploration, and discovery. Infused with fallibilism, my teaching approach acknowledges the notion that learning is an iterative journey of refinement. This combination empowers learners to actively construct their understanding, echoing Russell's call for the cultivation of rationality and logic in their educational journey.

4.1.11 Conclusion

In the culmination of this narrative, the convergence of personal experiences, Bertrand Russell's philosophical insights, and Stewart Shapiro's structuralist concepts unveils a profound and expansive perspective. Russell's likening of mathematics to a temple, intertwined with Shapiro's concept of 'offices,' forms a guiding motif that steers the transformative course of education. Russell's reminder that mathematics transcends dry facts resonates harmoniously with Shapiro's exploration of mathematical structures within a broader framework.

This harmony reinforces the central message of this narrative: the philosophy of mathematics is far from a detached abstraction; rather, it holds the potential to reshape the educational landscape.

Guided by the enlightening wisdom of Alexander Grothendieck, Leonard Susskind, and Brian Greene, this narrative reaches new heights, illuminating pathways where mathematics, science, and education intersect. Grothendieck's geometric visions inspire educators to delve deeper, uncovering the inherent beauty and interconnectedness within mathematics. Susskind's call for interdisciplinary thinking resonates, igniting a vision where mathematics acts as a bridge spanning diverse fields. Greene's emphasis on the significance of science beckons teachers to cultivate an educational environment where mathematics leads to the exploration of the universe's enigmas.

This narrative extends an open invitation to teachers, educators, scholars, and visionaries to embark on a journey guided by the philosophy of mathematics. This journey stretches beyond the confines of classrooms; it encompasses our perception, instruction, and comprehension of mathematics.

In culmination, a profound truth emerges: the philosophy of mathematics is not static; rather, it is a living force capable of shaping the trajectory of education. By embracing the transformative potential of the philosophy of mathematics, teachers evolve into architects of a novel educational paradigm. They lead a mathematical odyssey that transcends conventional norms, sparking curiosity and nurturing learners who actively explore the wonders of mathematics.

This conclusion marks the end of a chapter rather than the ultimate note of the symphony. The transformative capacity of the philosophy of mathematics remains a constant refrain, beckoning educators to harmonize and embrace their roles as conductors of an educational symphony where mathematics serves as a conduit not just for learning, but for transformative enlightenment, revelation, and limitless horizons. In the next section we will explore further the transformative power of philosophy of mathematics.

4.2 Exploring the Transformative Role of Philosophy of Mathematics in Enhancing Mathematics Teaching and Learning through Multimethod Grounded Theory Analysis

4.2.1 Introduction

In the realm of educational research, ongoing exploration focuses on innovative teaching approaches to address evolving challenges faced by educators and learners (Ernest, 1991; Lerman, 2007). This study draws inspiration from various sources, including the works of Weber (2012) and Cobb & Bauersfeld (1995), as well as insights from Lakatos (1976) and Kitcher (1984). It aims to explore the transformative potential of integrating the philosophy of mathematics into mathematics education, seeking to enhance teaching efficacy and learner experiences, as suggested by Pimm (1987) and Tymoczko (1986).

The study identifies a significant gap in the preparation of mathematics educators in South Africa, extending beyond the absence of training solely in the philosophy of mathematics. This broader issue relates to a limited holistic view and constructivist inclinations within mathematics education, calling for a deeper exploration of mathematical concepts, real-world applications, and critical thinking development among learners.

To address this multifaceted gap, the study delves into the integration of philosophical perspectives, drawing insights from prominent figures such as Susskind, Russell, Greene, Shapiro, and Grothendieck, as expounded in the Autoethnography framework. Grounded in the research methodologies outlined by Crotty (1998), Charmaz (2006), and Strauss & Corbin (1998), the primary objective is to illuminate how this integration can serve as a transformative bridge within mathematics education. Employing an abbreviated grounded theory methodology, following Glaser & Strauss (1967), accommodates diverse data sources and facilitates a comprehensive analysis, using the philosophy of mathematics as a potential bridge to enrich mathematics teaching.

The research methodology centres on synthesizing insights gathered from various sources, including literature reviews, educator questionnaires, field notes, and analytical autoethnography. This synthesis allows for a holistic analysis, providing a deeper understanding of the research topic.

The study's foundation lies in the multimethod grounded theory approach, combining insights from diverse data sources to offer a more comprehensive comprehension of the phenomenon. Triangulated grounded theory enhances the validity of the analysis by validating emerging themes, patterns, and relationships, reinforcing the credibility and reliability of findings. Subsequent sections will detail the adapted steps of open coding, focused open coding, axial coding, and selective coding, facilitating a systematic analysis. The focus remains on highlighting the transformative potential of philosophy in mathematics education, providing valuable insights to educators and policymakers.

4.2.2 Integration and Synthesizing of the Codes Across Multiple Sources Derived from Focused Coding

The process of integrating and synthesizing codes derived from focused open coding, as outlined in ANNEXURES G1 & 2, has been undertaken to comprehensively address three distinct themes within the broader context of mathematics education. Within each theme, the pertinent concepts and insights sourced from a diverse array of materials, including literature reviews, field notes, questionnaires, and autoethnography, have been systematically amalgamated and categorized under appropriate subthemes. This systematic organization facilitates both my understanding and that of the readers, allowing us to discern the interconnected nature of ideas across different sources. This, in turn, aids in recognizing recurring patterns, shared challenges, and the philosophical foundations underpinning the subject.

Through the integration and synthesis of these codes, my primary objective is to craft a coherent and cohesive narrative that effectively captures the intricacy and depth of the subject matter. This narrative serves to underscore both the diversity of viewpoints presented and the common threads that emerge from the dataset. The final integrated framework serves as a robust foundation upon which to formulate conclusions, generate actionable recommendations, and contribute to a more comprehensive comprehension of the topic's nuances. Ultimately, the meticulous integration and synthesis of codes from focused open coding in ANNEXURE G1 & 2 substantially elevate the depth and rigor of the analytical process, facilitating the extraction of meaningful insights from a wide spectrum of data sources.

THEME 1: Philosophy of Mathematics in Mathematics Teaching - Integration and Synthesizing of Codes Across Multiple Sources

This section provides a concise overview of the integrated and synthesized codes under Theme 1: Philosophy of Mathematics in Mathematics Teaching (see ANNEXURE G1 & 2). These codes and subcodes encapsulate a range of key concepts and insights derived from diverse sources such as literature reviews, field notes, questionnaires, and autoethnography. This thematic integration aims to unravel the multifaceted role of philosophy in mathematics education, spanning from philosophical perspectives on mathematical knowledge to its impact on teaching approaches, learner engagement, and the fostering of critical thinking.

a) **Role of Philosophy in Mathematics Education:**

- Exploration of philosophy as a bridge to meaningful mathematics teaching.
- Philosophical thought's role in enriching teaching.
- Integrating philosophy to enhance teaching.
- Teachers' perspectives on the CAPS curriculum reflect a particular philosophy.

b) **Teachers as Philosophical Role Models:**

- Teachers' personal philosophies influence teaching practices.
- Teachers' philosophies impact curriculum, lesson development, and assessment planning.
- Incorporating a philosophy of mathematics into pedagogy.

c) **Philosophical Perspectives on Mathematical Knowledge:**

- Mathematical Knowledge Characteristics: Certainty and consistency.
- Philosophical inquiry into the nature of mathematical knowledge and its relation to truth.
- Philosophy of mathematics can enhance learners' understanding and engagement.
- Moving beyond mere theorem proving to focus on meaningful mathematical experiences.

d) **Philosophers' Perspectives on Mathematical Entities:**

- Plato's perspective on mathematical entities existing in a separate realm of Forms.
- Aristotle's contrasting view of numbers as properties of objects.
- Kant's perspective on mathematical knowledge as synthetic a priori.
- Indispensability Argument: Justifying abstract mathematical objects' existence.

e) **Constructivist Approaches and Philosophies:**

- Constructivist perspectives emphasize learner-centred learning and active engagement.
- Intuitionism emphasizes individual mathematical intuition.
- Ethnomathematics and decolonization challenge universality and promote inclusivity.

- Non-Western Contributions to Mathematics: Acknowledging diverse historical perspectives.

f) Impact on Teaching and Learning:

- Philosophy of mathematics shapes the approach to mathematics education.
- Rational inquiry, exploration of assumptions, and synthesis of knowledge.
- Fallibilism recognizes imperfections and openness to revision.
- Structuralism provides insights into the organization and relationships within mathematical concepts.
- Encourages critical thinking, creativity, and active knowledge construction.
- In the contemplative journey of the autoethnography section, a symphony of insights emerged, resonating with the profound influence of the philosophy of mathematics on education, and it illustrated how philosophy of mathematics can serve as a dynamic bridge, fostering a reimagined landscape of more effective mathematics classrooms.

g) Historical Context and Conceptual Understanding:

- Teach history of mathematics to show evolution and context of knowledge.
- Break down concepts into manageable parts for understanding.
- Highlight connections and links between mathematical concepts.
- Understanding the origins and philosophies of mathematics is essential for teaching.

h) Multiple Representations and Interdisciplinary Impact:

- More than one representation should be used in teaching mathematics topics.
- Mathematics is intertwined with other subjects and contributes to various fields.

i) Cognitive Development and Characteristics of Mathematics:

- Mathematics teaches critical and logical thinking, fostering unique cognitive development.
- Mathematics is logical and follows a set of rules.
- Mathematics can be both theoretical and practical.
- Pure Mathematics vs. Applied Mathematics distinction is acknowledged.
- Philosophers who were mathematicians possess logical and critical thinking skills.
- Visualization is important in teaching mathematics.
- Mathematics requires both logical and creative thinking.

- Importance of understanding underlying concepts and principles.
- Mathematics is essential for understanding the physical world.
- The autoethnography report underscores the transformative potential of the philosophy of mathematics, manifesting through crucial pillars. Firstly, it nurtures curiosity and a sense of appreciation, akin to Russell's perspective, by unveiling the intricate connections between mathematics and the natural world, thus eliciting a sense of wonder. Secondly, it advances the concept of relevance by harmonizing mathematics with scientific principles, as exemplified by Susskind and Greene's portrayal of mathematical landscapes in the context of an expanding universe. Informed by Stewart Shapiro's structuralist philosophy, this pedagogical approach undergoes a metamorphosis, harnessing mathematics as a potent instrument for fostering problem-solving skills and analytical thinking.

j) Teaching Approach and Attitude/Mindset:

- Mathematics should be taught with understanding.
- Learner-centred teaching is important.
- Technology should be treated as a tool, not the process itself.
- Using technology to enhance mathematical understanding.
- Incorporating real-world applications of mathematical concepts.
- Fostering a love for mathematics through creative thinking and problem-solving.
- Motivating learners to engage with mathematics.
- Positive attitude towards mathematics is essential for success.
- Need to change learners' mindset about mathematics.
- Emphasizing the value of mathematics in daily life and other subjects.
- Mathematics as an important life skill.
- Guided by Stewart Shapiro's structuralist philosophy, the autoethnography report accentuates the transformative potential of immersing learners in the realm of mathematical models, algorithms, and equations. By portraying mathematics as a catalyst for innovative problem-solving and analytical thinking, educators can furnish learners with the means to decode intricate challenges, drawing inspiration from Grothendieck's geometric perspective and the concept of the builder mathematician. This paradigm shift empowers learners to approach complex

problems with confidence, fortified by the understanding that mathematical concepts transcend mere abstract notions, serving as potent tools for unravelling the intricacies of the world.

k) Philosophical Inspiration, Dynamic Bridge, and Fostering Curiosity:

- Mathematics as a dynamic bridge connecting philosophy to education. In the culmination of the autoethnography section, the amalgamation of personal experiences, Bertrand Russell's philosophical insights, and Stewart Shapiro's structuralist concepts coalesce to unveil a larger, more profound perspective. Russell's likening of mathematics to a temple, coupled with Shapiro's notion of 'offices,' serves as a guiding motif that steers the course of transformation in education.

  Russell's reminder that mathematics transcends mere dry facts resonates harmoniously with Shapiro's exploration of mathematical structures as integral components of a broader framework.

  This harmony reinforces the core message of this narrative: the philosophy of mathematics is not a detached abstraction; rather, it wields the power to reshape the educational landscape.

l) Relevance and Problem-Solving Enhancement:

- In the autoethnography report Greene's and Susskind's emphasis on connecting mathematics with scientific concepts for relevance and applicability is highlighted. The narrative underscores the transformative potential of integrating mathematics with scientific concepts such as mathematical landscapes, expanding universe, entropy, thermodynamics, evolution, and molecular biology, as envisioned by Susskind and Greene.
- Shapiro's structuralist philosophy emphasizing mathematical models and problem-solving skills.

m) Interdisciplinary Exploration and Cultivation of Critical Thought:

- Nurturing critical thinking and curiosity inspired by Shapiro's and Russell's insights.
- Susskind's, Greene's, and Russell's contemplations on the far-reaching implications of mathematics resonate as cornerstones, highlighting the interconnectedness between mathematics and other disciplines.

In fostering a holistic understanding of the world, educators can illuminate the threads that weave through mathematics, science, and technology, nurturing a comprehensive perspective envisioned by Susskind.

By fostering interdisciplinary connections, learners embark on a journey that transcends traditional silos, unveiling a cohesive tapestry of knowledge where mathematical fluency acts as the common language.

THEME 2: Challenges in Mathematics Education - Integration and Synthesizingof Codes Across Multiple Sources

This section provides a concise overview of the integration and synthesis of codes under Theme 2: Challenges in Mathematics Education (see ANNEXURE G1 & 2). This compilation delves into the multifaceted challenges faced in the realm of mathematics education. The codes and subcodes explore issues ranging from philosophical influences on knowledge definitions to curriculum complexities, teacher training challenges, diverse learning styles, debates, and innovative teaching approaches. It also encompasses considerations of support for learners, resistance to change, learner engagement, and the transformative potential of philosophy in reshaping education.

a) Defining Knowledge and Philosophical Influence:

- Differentiating between belief, opinion, and knowledge based on conclusive evidence.
- The role of epistemology in understanding knowledge and its sources.
- The significance of historical and philosophical understanding in fostering independent thinking among scientists.
- Russell's perspective on the value of philosophy in providing unity and structure to scientific disciplines.

b) Evolution of Mathematical Foundations:

- The emergence of logicism, formalism, and intuitionism in response to challenges in establishing reliable mathematical foundations.
- Gödel's Incompleteness Theorems demonstrating limitations in formal mathematical systems, undermining logicism's aim to reduce all of mathematics to a complete logical system.

- The rise of set theory and axiomatic approaches as alternative foundations for mathematics, challenging the traditional logicist perspective.
- Challenges from various mathematical fields highlighting that mathematics encompasses more than formal logic.
- Alternative schools of thought, including intuitionism and constructivism, further challenging the traditional logicist perspective.
- Diversification of mathematics into specialized areas reducing the practicality of reducing all mathematics to a single logical foundation.
- Fallibilism introducing the idea that mathematics, like the natural sciences, is subject to criticism, revision, error, and correction.

c) Curriculum Reforms, Teacher Training Challenges, and Literacy Issues:

- The challenges related to introducing and implementing various curricula, highlighting complexities in curriculum design, teacher training, and adapting to changing educational philosophies.
- The introduction of Mathematical Literacy as an alternative to Mathematics raises issues about objectives and impact on learner performance.
- The language barrier affecting learner engagement and understanding in mathematics.

d) Challenges in Mathematics Education:

- Curriculum Challenges, including curriculum overload and balancing content within limited instructional time.
- Teacher Challenges, encompassing excessive administrative workload, stress among teachers, and its impact on retention.
- Classroom Challenges, reflecting the impact of overcrowded classrooms on effective mathematics instruction.
- Course Selection Challenges, considering the declining number of learners selecting mathematics courses and issues related to mathematics literacy versus mathematics education.
- Concerns about content-heavy curriculum and emphasis on content delivery.

e) Diverse Learning Styles and Teaching Approaches:

- Acknowledgment of diverse learning styles and preferences for varied teaching approaches to cater to these styles.

f) Disagreements and Debates:

- Disagreements on mathematics abstraction, talent versus effort, and the effectiveness of practice for struggling learners.

g) Support and Guidance:

- Recognition of the importance of support and guidance for learners, as well as the need to consider individual differences.

h) Innovative Teaching and Resistance to Change:

- Discussion about incorporating innovative methods, overcoming resistance to change, and the constraints of traditional teaching methods.

i) Learner Engagement and Motivation:

- Recognition of issues related to learner engagement, motivation, aversion to math, and perception of difficulty.

j) Educational Transformation through Philosophy:

- Integration of philosophy as a catalyst for educational transformation, reshaping teaching practices, enhancing problem-solving skills, fostering interdisciplinary connections, and nurturing critical thinking.

THEME 3: The Nature of Mathematical Knowledge - Integration and Synthesizing of Codes Across Multiple Sources

This section provides a concise overview of the integration and synthesis of codes under Theme 3: The Nature of Mathematical Knowledge (see ANNEXURE G1 & 2). This compilation delves into the intricate exploration of mathematical knowledge's unique characteristics, delving into philosophical inquiries and perspectives. The codes and subcodes encompass diverse philosophical thought, challenges brought by decolonization and inclusivity, language's role, curriculum considerations, and varied perspectives on mathematics and learning approaches. Additionally, the relevance of history, learning strategies, skills, and the integration of disciplines are explored, shedding light on the multifaceted nature of mathematical knowledge and its interactions within broader educational contexts.

a) **Distinctive Characteristics and Philosophical Inquiry**:

- Exploration of the distinctive characteristics of mathematical knowledge, including its certainty and consistency.
- Philosophical inquiry into the nature of mathematical truths and their connection to the physical world.

b) **Philosophical Perspectives on Mathematical Entities:**

- Plato's perspective on mathematical entities existing in a realm of Forms, contrasting with Aristotle's view of numbers as properties of objects.
- Kant's transcendental idealism and the concept of mathematical knowledge as synthetic a priori.

c) **Diverse Philosophical Thought and Fallibilism**:

- Discussion of different schools of thought, including logicism, formalism, and intuitionism.
- Exploration of absolutist and fallibilist philosophies.
- The challenges of paradoxes and contradictions faced by absolutist philosophies.
- Fallibilism emphasizing the open nature of mathematical knowledge and its intertwining with history, individuals, and institutions.
- Constructivism emphasizing construction of knowledge.

d) **Decolonization, Ethnomathematics, and Inclusivity:**

- Challenges posed by decolonization and ethnomathematics to the idea of mathematics as an absolute and universal truth.
- Recognition of non-Western contributions to mathematics and their implications for a diverse understanding of mathematical knowledge.

e) **Language and Indigenous Knowledge:**

- Exploration of the impact of language on mathematical learning and the integration of indigenous knowledge systems into mathematics education.

f) **Curriculum Challenges and Pedagogical Concerns**:

- Concerns about shallow understanding and anxiety, as well as the tension between content-focused learning and deep conceptual understanding.
- Discussions on curriculum rigidity, the need for critical thinking and creativity, and the alignment with broader discussions on the nature of mathematical knowledge.

g) Perspectives on Mathematics and Learning Approaches:

- Variation in perspectives on mathematics as abstract or representing the real world.
- Recognition of mathematics' practical application and the importance of multidimensional learning, foundational concepts, and algorithmic learning.

h) Relevance of History, Learning Strategies, and Skills:

- Emphasis on the relevance of understanding the origins of mathematical concepts and the importance of mastering basic skills.
- The role of understanding reasoning behind mathematical ideas, connecting concepts, and promoting perseverance in problem-solving.

i) Integration of Disciplines and Teacher Competence:

- Highlighting interdisciplinary connections, the cultivation of critical thought, and the role of teacher competence in conveying mathematical knowledge.

4.2.2.1 Key Findings from Focused Coding

The compilation of these key findings stems from a comprehensive integration and synthesis of three distinct themes, namely "Philosophy of Mathematics in Mathematics Teaching," "Challenges in Mathematics Education," and "The Nature of Mathematical Knowledge". By systematically organizing and weaving together codes and subcodes derived from diverse sources such as literature reviews, field notes, questionnaires, and autoethnography, these key findings offer a succinct representation of the interconnected insights and concepts present across these themes.

These key findings depict a rich tapestry of ideas that collectively underscore the pivotal role of philosophy in shaping mathematics education. From the influence of philosophical perspectives on knowledge definitions to the challenges faced in curriculum design, teacher training, and learner engagement, the synthesis presents a holistic understanding of the subject. It highlights the transformative potential of philosophy in enhancing teaching practices, promoting critical thinking, and fostering interdisciplinary connections. Moreover, the findings reflect the intricate balance between deep conceptual understanding and practical application within the realm of mathematics education.

These findings provide a concise yet comprehensive overview of the multifaceted relationship between philosophy and mathematics education, underscoring both challenges and opportunities. The synthesis serves as a foundation for drawing insightful conclusions, formulating recommendations, and contributing to the broader understanding of the complex dynamics inherent in teaching and learning mathematics.

THEME 1: Philosophy of Mathematics in Mathematics Teaching

- Philosophy plays a significant role in enhancing mathematics education, acting as a bridge to meaningful teaching and enriching pedagogical approaches.
- Teachers' personal philosophies influence their teaching practices, impacting curriculum design, lesson development, and assessment planning.
- Philosophical perspectives on mathematical knowledge emphasize the characteristics of certainty and consistency, leading to a deeper understanding of mathematical truths.
- The integration of philosophy into mathematics education fosters critical thinking, creativity, and active knowledge construction.
- Understanding the historical context and philosophical foundations of mathematics is crucial for effective teaching and conceptual understanding.
- Embracing diverse teaching approaches, multiple representations, and interdisciplinary connections enhances the teaching and learning of mathematics.
- Mathematics promotes cognitive development, logical thinking, and creativity while having practical applications in various fields.
- Adopting a positive attitude towards mathematics and fostering a love for the subject is essential for learner engagement and success.

THEME 2: Challenges in Mathematics Education

- The definition of knowledge and its sources are influenced by philosophical perspectives, contributing to the understanding of beliefs, opinions, and conclusive evidence.
- The evolution of mathematical foundations has led to the emergence of diverse schools of thought, challenging traditional views and emphasizing the openness of mathematical knowledge.

- Challenges in mathematics education encompass curriculum complexities, teacher training issues, language barriers, and overcrowded classrooms.
- Curriculum challenges involve finding a balance between content and instructional time, while teacher challenges relate to workload and retention.
- Diverse learning styles and teaching approaches must be acknowledged to cater to the needs of all learners.
- Disagreements and debates exist regarding mathematics abstraction, talent versus effort, and effective teaching strategies.
- Support and guidance are essential for learner success and incorporating innovative teaching methods requires overcoming resistance to change.
- Learner engagement, motivation, and perception of mathematics difficulty are factors that influence the learning process.

THEME 3: The Nature of Mathematical Knowledge

- Mathematical knowledge possesses distinctive characteristics of certainty and consistency, leading to philosophical inquiries about its nature and connection to the physical world.
- Philosophical perspectives on mathematical entities vary, with Plato's and Aristotle's views offering contrasting notions of mathematical existence.
- Different schools of thought, including logicism, formalism, and intuitionism, present varying approaches to mathematical knowledge, emphasizing fallibilism and open inquiry.
- Challenges posed by decolonization and inclusivity challenge the concept of mathematics as an absolute and universal truth.
- The integration of indigenous knowledge systems and language into mathematics education enhances learning experiences.
- Curriculum considerations involve striking a balance between shallow understanding and deep conceptual learning, aligning with discussions on the nature of mathematical knowledge.
- Perspectives on mathematics span abstraction to practicality, with interdisciplinary connections and teacher competence playing crucial roles in conveying mathematical knowledge.

Overall, the integration and synthesis of these themes reveal that philosophy underpins mathematics education, influencing pedagogical approaches, knowledge definitions, and the very nature of mathematical understanding. These themes highlight challenges and opportunities in teaching and learning mathematics while underscoring the importance of fostering curiosity, critical thinking, and diverse perspectives to create a holistic and effective mathematical education experience.

4.2.2.2 Support Evidence - Key Findings Corroborated by the Data

The examination of the interplay between philosophy and mathematics education offers a nuanced perspective on the foundational elements that shape the teaching and learning of mathematics. This section delves into the corroborative evidence that supports the key findings presented earlier, reinforcing the intricate relationship between philosophical principles and the educational landscape. By drawing upon a range of sources including literature reviews (Chapter 2), educator insights (Questionnaire ANNEXURE E1 & 2 and Field Notes ANNEXURE F), and insight from the Autoethnography (Chapter 4 Section 4.1), this analysis aims to substantiate the role of philosophy in influencing pedagogical approaches, knowledge frameworks, and the nature of mathematical comprehension. Through a systematic exploration of the themes, the ensuing discussion underscores the potential impact of philosophical underpinnings on various facets of mathematics education.

THEME 1: Philosophy of Mathematics in Mathematics Teaching

1.1 Role of Philosophy in Enhancing Mathematics Education

Philosophy assumes a significant role in augmenting mathematics education, acting as a conduit for meaningful teaching and enriching pedagogical methodologies.

- Literature Review: Sections 2.2.7, 2.3, 2.4, 2.5.

1.2 Influence of Teachers' Personal Philosophies on Teaching Practices

The pedagogical practices of educators are shaped by their personal philosophical inclinations, which in turn affect curriculum design, lesson structuring, and assessment planning.

- Literature Review: Section 2.3.2.

- Questionnaire ANNEXURES E1 & 2: Insights from teacher inputs reveal the impact of philosophical perspectives on teaching.

1.3 Philosophical Perspectives on Mathematical Knowledge and Understanding

Philosophical viewpoints pertaining to mathematical knowledge emphasize traits of certainty and consistency, thereby fostering a profound comprehension of mathematical truths.

- Literature Review: Section 2.3.3.

1.4 Integration of Philosophy into Mathematics Education and Cognitive Development

The infusion of philosophical aspects into mathematics education cultivates critical thinking, creativity, and dynamic knowledge construction, stimulating cognitive advancement.

- Literature Review: Section 2.3.2.
- Autoethnography: Chapter 4, Section 4.1.

1.5 Historical and Philosophical Foundations of Mathematics in Effective Teaching

A grasp of the historical context and philosophical bedrock of mathematics is pivotal for competent teaching and holistic conceptual understanding.

- Literature Review: Sections 2.1.2, 2.5.

1.6 Enriching Teaching through Diverse Approaches and Interdisciplinary Connections

The adoption of diverse pedagogical techniques, manifold representations, and interdisciplinary interrelations elevates the quality of mathematics instruction and learning.

- Literature Review: Section 2.4.4.9
- Questionnaire ANNEXURES E1 & 2: Diverse teaching approaches as highlighted by teacher replies.

1.7 Positive Attitude, Learner Engagement, and Success in Mathematics

Nurturing a positive disposition towards mathematics and fostering a genuine affinity for the subject are fundamental for learner engagement and accomplishment. Mathematics promotes cognitive development, logical thinking, and creativity while having practical applications in various fields.

- Literature Review: Sections 2.2.7, 2.3, 2.4, 2.5.

THEME 2: Challenges in Mathematics Education

2.1 Influence of Philosophical Perspectives on Knowledge Definition

Philosophical underpinnings influence knowledge definition, thereby contributing to an enhanced comprehension of convictions, opinions, and conclusive evidence.

- Literature Review: Section 2.1.2, 2.1.3.

2.2 Evolution of Mathematical Foundations and Diverse Schools of Thought

The evolution of mathematical foundations has engendered a spectrum of viewpoints, challenged conventional perspectives, and accentuates the expansiveness of mathematical knowledge.

- Literature Review: Section 2.2

2.3 Challenges in Mathematics Education: Curriculum Complexity and Teacher Training

Challenges confronting mathematics education encompass intricacies in curriculum design, teacher training dilemmas, linguistic hurdles, and classroom overcrowding.

- Literature Review: Section 2.6
- Field Notes: ANNEXURE F

2.4 Curriculum and Teacher Challenges: Balancing Content and Instructional Time

Navigating curriculum complexities requires reconciling content with instructional time, while teacher challenges involve workload management and retention strategies. Real-world challenges faced by educators, including curriculum overload and inadequate resources.

- Literature Review: Section 2.6
- Field Notes: ANNEXURE F

2.5 Addressing Diverse Learning Styles and Teaching Approaches

Acknowledging varied learning styles and instructional approaches is imperative to cater to the diverse needs of learners.

- Literature Review: Sections 2.5
- Questionnaire: ANNEXURES E1 & 2

2.6 Disputes, Debates, and Strategies in Mathematics Education

Disputes and debates span domains such as mathematical abstraction, the dichotomy of talent versus effort, and the efficaciousness of pedagogical strategies.

- Literature Review: Sections 2.3, 2.4.
- Autoethnography: Section 4.1.2, 4.13, 4.14, 4.15, 4.1,6

2.7 Promoting Change and Innovative Methods in Mathematics Education

The promotion of transformative change and integration of innovative teaching methods necessitates overcoming resistance to change and embracing pedagogical innovation.

- Literature Review: Section 2.4, 2.5.
- Autoethnography: Chapter 4, Section 4.1.

2.8 Learner Engagement, Motivation, and Perceptions of Mathematics Difficulty

The learner experience is influenced by engagement levels, motivational factors, and the perceived complexity of mathematics.

- Literature Review: Sections 2.4, 2.5.
- Questionnaire: ANNEXURES E1 & 2

THEME 3: The Nature of Mathematical Knowledge

3.1 Characteristics of Mathematical Knowledge and Philosophical Inquiries

Mathematical knowledge possesses distinct attributes of certainty and consistency, prompting philosophical inquiries into its nature and correlation with the physical realm.

- Literature Review: Section 2.1.2, 213

3.2 Philosophical Perspectives on Mathematical Entities and Existence

Philosophical viewpoints on mathematical entities diverge, exemplified by the contrasting notions of mathematical existence presented by Plato and Aristotle.

- Literature Review: Section 2.1.2

3.3 Diversity of Schools of Thought and Approaches to Mathematical Knowledge

A range of schools of thought, encompassing logicism, formalism, and intuitionism, advocate diverse approaches to mathematical knowledge, emphasizing fallibilism and open inquiry.

- Literature Review: Section 2.1.3.

3.4 Challenges from Decolonization and Inclusivity in Mathematics Education

Challenges arising from decolonization and inclusivity probe the notion of mathematics as an absolute and universally applicable truth.

- Literature Review: Section 2.3, 2.6

3.5 Integration of Indigenous Knowledge Systems and Language in Mathematics Education

The incorporation of indigenous knowledge systems and language enriches mathematical learning experiences and fosters a holistic pedagogical environment.

- Literature Review: Section 2.6

3.6 Curriculum Considerations and the Nature of Mathematical Knowledge

Curriculum design requires striking a balance between superficial comprehension and profound conceptual learning, aligning with discussions on the essence of mathematical knowledge.

- Literature Review: Sections 2.3, 2.4, 2.5.
- Autoethnography: Chapter 4, Section 4.1.

3.7 Perspectives on Mathematics: From Abstraction to Practicality

Perspectives on mathematics span a spectrum from abstraction to practical application, with interdisciplinary connections and educator proficiency crucial for effective knowledge dissemination.

- Literature Review: Sections 2.3, 2.4, 2.5.
- Autoethnography: Chapter 4, Section 4.1.

4.2.2.3 Overall Integration and Synthesis

The integrated analysis and synthesis of these thematic threads unveil the overarching influence of philosophy on mathematics education, impacting pedagogical methodologies, knowledge delineation, and the fundamental nature of mathematical understanding.

4.2.2.4 Potential for Further Exploration and Vigilance in Analysis

As the analysis progresses, it is important to remain open to the possibility of emerging nuances, connections, and potential gaps that may not have been evident initially. While there are currently no identified gaps or inconsistencies based on the data analysed, as the analytical process continues it often reveals new insights as themes interact and intersect. Remaining aware of this and receptive to these developments ensures a comprehensive and thorough exploration of the research question.

4.2.2.5 Conclusion of Focused Open Coding

The process of focused coding[21] has yielded a rich tapestry of insights into the complex relationship between philosophy and mathematics education. Through the meticulous analysis of themes and codes derived from various data sources, including the literature review, field notes, autoethnography, and questionnaire responses, a multifaceted understanding of key issues, challenges, and opportunities within mathematics education has emerged.

The literature review's focused coding unearthed a profound interplay between distinct characteristics of mathematical knowledge, historical philosophical viewpoints, and the evolving nature of mathematics education. These themes underscore the enduring influence of philosophical thought on the foundations of mathematics, providing a holistic perspective that encompasses logical reasoning, historical evolution, and pedagogical practices.

Similarly, the focused coding of field notes illuminated the challenges faced by educators in mathematics education, the integration of philosophical insights into teaching approaches, and the imperative of fostering critical thinking. This analysis captured the multifarious

[21] The initial phase of open coding involved a thorough review of all collected materials, during which similar themes and concepts were identified and highlighted. These identified themes and concepts were documented using Microsoft Excel and subsequently organized to generate the focused open coding presented in ANNEXURE G1 & 2. Owing to the extensive nature of the initial open coding, a decision was made to include only the focused open coding in the final thesis document.

aspects of teaching, from administrative burdens to the integration of philosophy, underscoring the importance of adapting pedagogical methods to cater to diverse needs.

The autoethnography's focused coding revealed a personal journey deeply influenced by philosophical perspectives, leading to innovative pedagogical approaches that bridge disciplines and ignite intellectual curiosity. The integration of philosophy into teaching practices was a central theme, underscoring the potential of philosophical insights to enrich pedagogical strategies and deepen learners' engagement with mathematical concepts.

The focused coding of questionnaire responses provided a diverse array of perspectives from educators, highlighting the significance of exposing learners to the philosophy of mathematics and the role of interdisciplinary teaching approaches. These responses unveiled the varied challenges faced in mathematics education and the strategies employed to nurture learner motivation, engagement, and critical thinking skills.

When synthesizing these findings, it is evident that the interplay between philosophy and mathematics education is multifaceted and has far-reaching implications. From historical philosophical perspectives shaping mathematical knowledge to contemporary challenges in education, the synthesis of focused coding themes paints a comprehensive picture of a field in constant evolution.

These insights collectively underscore the need for an adaptable and holistic approach to mathematics education, where philosophical perspectives infuse pedagogy with depth and meaning. Moreover, the challenges and opportunities highlighted across different data sources emphasize the importance of ongoing research, curriculum reforms, and professional development to enhance the quality of mathematics education for learners across diverse contexts.

In the broader context, this section of focused coding serves as a foundational cornerstone for subsequent chapters, where the synthesis of themes and codes will inform in-depth discussions, implications, and recommendations. The multifaceted insights derived from focused coding reaffirm the intricate relationship between philosophy and mathematics education, offering a lens through which to navigate the dynamic landscape of educational practices and theories.

4.2.3 Axial Coding: Relationships between Themes

The process of axial coding plays a pivotal role within this grounded theory approach, enabling a comprehensive exploration of the intricate relationships and connections that exist among the identified themes. By meticulously examining the interplay between these themes, this section delves deeper into the data, revealing profound insights and discernible patterns. The axial coding exercise serves as a beacon, illuminating the intricate web of relationships that tie together the initially identified themes.

Through the axial coding process, the initial three themes have undergone a transformation, evolving into a more granular and insightful framework of eight distinct themes. This evolution is driven by the pursuit of a more thorough understanding of the research subject—philosophy of mathematics in teaching. By breaking down the overarching themes into more specific subcategories, we unlock a treasure trove of connections and nuances that might have remained hidden otherwise. This process of refinement sharpens our lens, enabling us to peer into the complexity that underlies the interactions within and between themes.

In essence, the axial coding exercise acts as a bridge, bridging the gap between broad conceptual themes and the intricate threads that weave them together. It unveils the layers of meaning, highlighting why each theme can be further subdivided to encapsulate the intricate relationships and cause-and-effect dynamics that contribute to a holistic and enriched understanding of the research landscape.

4.2.3.1 Relationships between Themes

In the preceding section, I presented the outcomes of focused open coding that led to the discernment of diverse themes. Within this section, the axial coding exercise unveils the intricate interplay and connections among these themes, offering a more structured and coherent perspective of their interactions. This organized portrayal, available in ANNEXURE G3, facilitates a more impactful presentation of findings, highlighting the central insights derived from the analysis.

THEME 1: Philosophy of Mathematics in Mathematics Teaching:

a) Connection to Enhancing Teaching and Learning:

- Integrating philosophical thinking into pedagogy.
- Bridging abstract concepts through philosophy.
- Shifting mindset and approach in education.
- Encouraging deeper learner engagement.

b) Application of Philosophical Ideas:

- Applying Grothendieck's, Susskind, Greene, Russell, Ernest, and Shapiro philosophies in teaching.
- Linking philosophy to engaging lessons and learning.
- Shaping teaching philosophies based on philosophy.

c) Role of Philosophy in Shaping Teaching Practices:

- Impact of philosophical understanding on teaching practices.
- Integration of philosophical insights enhances teaching.

d) Emphasis on Understanding and Application:

- Promoting deep understanding and real-world application.
- Encouraging comprehensive understanding of mathematical relationships.

THEME 2: Challenges in Mathematics Education:

a) Education System Challenges:

- Weak teacher education and outcomes.
- Curriculum challenges and reforms over time.
- Performance and assessment issues.
- Calls for improvement and decolonization.

b) Real-World Challenges Faced by Educators:

- Curriculum overload and content-heavy approaches.
- Teacher stress and challenges in managing diverse classrooms.
- Overcrowding and lack of support for struggling learners.

c) Need for Transformative Action:

- Nurturing critical thinking and creativity.
- Reflection on school engagement and limitations.
- Calls for change and creativity in teaching.

THEME 3: Nature of Mathematical Knowledge:

a) Characteristics of Mathematical Knowledge:

- Certainty, consistency, necessity, and contingency.
- Role of evidence, justification, and epistemological questions.

b) Differentiation from Belief and Opinion:

- Criteria of certainty, validity, veracity, and utility.
- Distinct nature of mathematical knowledge from belief.

c) Teacher Competence in Conveying Knowledge:

- Role of logical reasoning and clear explanations.
- Emphasis on accurate explanations, logical proofs, and solid grasp of concepts.

THEME 4: Teacher Reflection and Empowerment:

a) Role of Teachers in Education:

- Teachers as role models and educators.
- Influence of philosophy and experts on teaching approaches.

b) Incorporating Personal Experience:

- Linking personal journey with pedagogical strategies and insights.
- Acknowledgment of the impact of personal experiences on teaching philosophy.

THEME 5: Curriculum and Teaching Approaches:

a) **Curriculum Challenges and Approaches:**

- Curriculum overload, rigidity, and content-heavy methods.
- Encouragement of creative and transformative teaching approaches.
- Integration of philosophical insights into teaching strategies.

THEME 6: Educational System Improvement:

a) Calls for Educational Reforms:

- Inclusive and diverse mathematics education.
- Addressing issues of inequality, diversity, and decolonization.
- Need for educational improvements.

THEME 7: Connection Between Philosophy of Mathematics and Mathematics:

a) Relationship between Philosophy, Mathematics, and Teaching:

- Philosophy enhances teaching through discussions and open-ended approaches.
- Inspiring curiosity and deeper engagement in mathematics.

THEME 8: Interdisciplinary and Holistic Approach:

a) Promoting Holistic Learning:

- Interdisciplinary teaching approaches.
- Significance of critical thinking and holistic learning.
- Embracing complexity and exploring interconnectedness.

This axial coding exercise demonstrates the relationships between the themes identified in the focused open coding results in the previous section. It shows how these themes are interconnected and form broader categories that contribute to the understanding of philosophy of mathematics and its relationship to teaching of mathematics. This organized view can help present findings more effectively and highlight the main insights from the analysis.

4.2.3.2 Generate Subcategories

The identified themes are further refined into subcategories that provide a more detailed exploration of the connections between different aspects of the research subject. These subcategories, as outlined in ANNEXURE G3, capture the intricate interplay between philosophy of mathematics, teaching practices, challenges, and the nature of mathematical knowledge.

THEME 1: Philosophy of Mathematics in Mathematics Teaching:

This category examines the intricate interplay between philosophy and mathematics education, shedding light on how philosophical concepts contribute to teaching and learning. The following subcategories delve into the role of philosophy in enhancing pedagogy, the application of philosophical ideas to teaching methods, and the impact of philosophical perspectives on teaching practices. The discussions underscore the integration of philosophy and effective mathematics instruction.

Subcategories:

a) Philosophy's Role in Enhancing Teaching and Learning:

- *Integrating* philosophical thinking into pedagogy.
- *Shifting* mindset and approach in education.
- *Bridging* abstract concepts through philosophy.
- *Encouraging* deeper learner engagement.

b) Application of Philosophical Ideas in Teaching:

- *Applying* such philosophies as those of Grothendieck's, Susskind, Greene, Russell, and Shapiro in teaching (see autoethnography). Also, Ernest and others (see literature review).
- *Linking* philosophy to engaging lessons and learning.
- *Shaping* teaching philosophies based on philosophy.

c) Impact of Philosophy on Teaching Practices:

- *Influence* of philosophical understanding on teaching practices.
- *Integration* of philosophical insights enhances teaching.
- *Emphasis* on Understanding and Application:
- *Promoting* deep understanding and real-world application.
- *Encouraging* comprehensive understanding of mathematical relationships.

THEME 2: Challenges in Mathematics Education:

Within this category, we explore the diverse challenges inherent in mathematics education. Through subcategories, we analyse the complexities within the educational system. We discuss systemic challenges, real-world issues faced by educators, and the necessity for transformative action. This category emphasizes the importance of addressing challenges to foster an improved learning environment.

Subcategories:

a) Education System Challenges:

- *Weak teacher education* and *outcomes*.
- Curriculum challenges and reforms over time.
- *Performance* and *assessment* issues.
- Calls for *improvement* and *decolonization*.

b) Real-World Challenges Faced by Educators:

- *Curriculum overload* and *content-heavy approaches*.
- *Teacher stress* and *challenges* in managing diverse classrooms.
- *Overcrowding* and *lack of support* for struggling learners.

c) Need for Transformative Action:

- *Nurturing critical thinking* and *creativity*.
- *Reflection on school engagement and limitations*.
- Calls for *change and creativity in teaching*.

THEME 3: Nature of Mathematical Knowledge:

This category delves into the fundamental nature of mathematical understanding. Subcategories navigate the distinct characteristics of mathematical knowledge, its differentiation from belief, and the competence required to convey it effectively. The discussions explore philosophical inquiries arising from the nature of mathematical knowledge, highlighting its role in shaping pedagogy and the comprehension of mathematical truths.

Subcategories:

a) Characteristics of Mathematical Knowledge:

- *Certainty*, *consistency*, *necessity*, and *contingency (eventualities)*.
- Role of *evidence*, *justification*, and *epistemological questions (nature, scope, and limits of knowledge)*.

b) Differentiation from Belief and Opinion:

- Criteria of *certainty*, *validity*, *veracity*, and *utility*.
- *Distinct nature of mathematical knowledge* from belief.

c) Teacher Competence in Conveying Knowledge:

- *Role of logical reasoning* and *clear explanations*.
- Emphasis on accurate explanations, logical proofs, and solid grasp of concepts.

THEME 4: Teacher Reflection and Empowerment:

Focusing on educators' pivotal role in mathematics education, this category explores how teachers influence the learning experience. Subcategories examine teachers as role models and educators, the impact of philosophy on teaching approaches, and the incorporation of personal experiences into pedagogical strategies. This category underscores the significance of empowering teachers and their valuable contribution to the educational landscape.

Subcategories:

a) Role of Teachers in Education:

- Teachers as *role models* and educators.
- *Influence of philosophy* and experts on teaching approaches.

b) Incorporating Personal Experience:

- *Linking personal journey* with pedagogical strategies and insights.
- *Acknowledgment of the impact of personal experiences* on teaching philosophy.

THEME 5: Curriculum and Teaching Approaches:

This category delves into curriculum design and teaching methodologies. Subcategories dissect challenges posed by curriculum overload and rigidity, while also exploring innovative teaching approaches. The discussions delve into how integrating philosophical insights enriches teaching strategies, fostering a dynamic learning experience.

Subcategories:

a) Curriculum Challenges and Approaches:

- Curriculum *overload*, *rigidity*, and *content-heavy* methods.

- Encouragement of creative and *transformative* teaching approaches.
- *Integration* of philosophical insights into teaching strategies.

THEME 6: Educational System Improvement:

Examining the need for progressive change within mathematics education, this category explores calls for educational reforms. Subcategories discuss inclusivity, diversity, and addressing issues of inequality and decolonization. The category emphasizes the necessity for improvements to create a balanced and equitable educational environment.

Subcategories:

a) Calls for Educational Reforms:

- *Inclusive* and *diverse* mathematics education.
- *Addressing issues* of inequality, diversity, and decolonization.
- Need for educational *improvements*.

THEME 7: Connection Between Philosophy and Mathematics:

Bridging philosophy, mathematics, and teaching, this category explores how philosophy enhances teaching through open-ended approaches. Subcategories highlight the fostering of curiosity and engagement in mathematics through the connection between philosophy and the subject.

Subcategories:

a) Relationship between Philosophy, Mathematics, and Teaching:

- *Philosophy enhances* teaching through discussions and open-ended approaches.
- *Inspiring* curiosity and *deeper engagement* in mathematics.

THEME 8: Interdisciplinary and Holistic Approach:

This category embraces an interdisciplinary and holistic approach to mathematics education. Subcategories explore how interdisciplinary methods promote comprehensive learning and the significance of critical thinking. The discussions underscore the importance of cultivating a well-rounded educational experience.

Subcategories:

a) Promoting Holistic Learning:

- *Interdisciplinary* teaching approaches.
- Significance of *critical thinking* and *holistic learning*.
- *Embracing complexity* and *exploring interconnectedness*.

4.2.3.3 Exploring Cause and Effect Relationships

Delving deeper into the relationships between subcategories, ANNEXURE G3 dissects cause-and-effect relationships that offer a finely detailed understanding of how specific causes lead to tangible effects within the context of the research subject. This analysis illuminates the underlying mechanisms that shape teaching practices, curriculum challenges, and the nature of mathematical knowledge.

THEME 1: Philosophy of Mathematics in Mathematics Teaching:

In the realm of Philosophy of Mathematics in Mathematics Teaching, the integration of philosophical thinking into pedagogy emerges as a cause that leads to a shift in the mindset and approach to education, fostering deeper learner engagement. Similarly, the application of philosophical ideas in teaching, such as the philosophies of Grothendieck, Susskind, Greene, Russell, Ernest, and Shapiro, serves as a cause that links to engaging lessons and the development of teaching philosophies. These causes have significant effects on shaping effective teaching practices and enhancing critical thinking skills among learners.

Subcategories*:*

a) Philosophy's Role in Enhancing Teaching and Learning:

Cause	**Effect**
a) Integration of philosophical thinking into pedagogy.	Shifting mindset and approach in education, leading to deeper learner engagement.
b) Bridging abstract concepts through philosophy.	Making complex concepts more significant and understandable for learners.

b) Application of Philosophical Ideas in Teaching:

Cause	Effect
a) Applying such philosophies as those of Grothendieck's, Susskind, Greene, Russell, and Shapiro in teaching (see autoethnography). Also, Ernest and others (see literature review).	Linking philosophy to engaging lessons and shaping teaching philosophies.
b) Shaping teaching philosophies based on philosophy of mathematics.	Enhancing teaching practices through the integration of philosophical insights.

c) Impact of Philosophy on Teaching Practices:

Cause	Effect
Influence of philosophical understanding on teaching practices.	Integration of philosophical insights enhances teaching, fostering critical thinking and problem-solving skills.

d) Emphasis on Understanding and Application:

Cause	Effect
Promoting deep understanding and real-world application.	Encouraging comprehensive understanding of mathematical relationships, leading to improved learning outcomes.

THEME 2: Challenges in Mathematics Education:

Turning to the theme of Challenges in Mathematics Education, causes such as weak teacher education and outcomes are seen to result in curriculum challenges and performance issues. Calls for improvement and decolonization are catalysts for the movement towards a more inclusive and diverse mathematics education system. Real-world challenges faced by educators, including curriculum overload and overcrowded classrooms, have effects that include teacher stress and potential hindrance in providing effective instruction. These causes and effects underscore the need for transformative actions in education, fostering critical thinking and creativity to overcome limitations.

Subcategories*:*

a) Education System Challenges:

Cause	Effect
a) Weak teacher education and outcomes.	Curriculum challenges and performance issues in mathematics education.
b) Calls for improvement and decolonization.	Movement towards inclusive and diverse mathematics education, addressing historical disparities.

b) Real-World Challenges Faced by Educators:

Cause	Effect
a) Curriculum overload and content-heavy approaches.	Teacher stress and challenges in managing diverse classrooms, potentially leading to decreased teacher effectiveness.
b) Overcrowding and lack of support for struggling learners.	Hindrance in providing effective mathematics instruction and addressing individual learner needs.

c) Need for Transformative Action:

Cause	Effect
Nurturing critical thinking and creativity.	Reflection on school engagement and limitations, sparking calls for change and creativity in teaching.

THEME 3: Nature of Mathematical Knowledge:

In the context of the Nature of Mathematical Knowledge, causes like the certainty, consistency, and necessity of mathematical knowledge are linked to its role as a logical basis for understanding and conveying mathematical truths. Similarly, the differentiation of mathematical knowledge from belief and opinion, facilitated by criteria of certainty, validity, veracity, and utility, emphasizes the importance of justification. Teacher competence in conveying knowledge is influenced by the role of logical reasoning and clear explanations, resulting in effective communication and improved learner understanding.

Subcategories:

a) Characteristics of Mathematical Knowledge:

Cause	Effect
a) Certainty, consistency, necessity, and contingency (eventualities) of mathematical knowledge.	Logical basis for understanding and conveying mathematical truths.
b) Role of evidence, justification, and epistemological ((nature, scope, and limits of knowledge) questions.	Differentiation of mathematical knowledge from mere belief, emphasizing the importance of justification.

b) Differentiation from Belief and Opinion:

Cause	Effect
a) Criteria of certainty, validity, veracity, and utility.	Clear differentiation of mathematical knowledge from subjective belief and opinion.
b) Distinct nature of mathematical knowledge from belief.	Emphasis on logical reasoning and accurate explanations for conveying mathematical knowledge.

c) Teacher Competence in Conveying Knowledge:

Cause	Effect
Role of logical reasoning and clear explanations.	Effective communication of mathematical knowledge, fostering learner understanding.

THEME 4: Teacher Reflection and Empowerment:

This theme sheds light on the crucial role of teachers in mathematics education. It explores how teachers' actions and beliefs influence the learning experience. Causes related to teachers as role models and the influence of philosophy on teaching lead to effects that positively impact learner engagement and instructional quality. Additionally, the incorporation of personal experiences by educators serves as a cause that results in enriched pedagogical strategies and a deeper connection between teachers and learners.

Subcategories:

a) Role of Teachers in Education:

Cause	Effect
a) Teachers as role models and educators.	Positive influence on the learning experience, shaping learners' perceptions and attitudes toward mathematics.
b) Influence of philosophy and experts on teaching approaches.	Adoption of innovative teaching methodologies and strategies, promoting learner-centred learning.

b) Incorporating Personal Experience:

Cause	Effect
a) Linking personal journey with pedagogical strategies and insights.	Enriched teaching approaches, infusing personal experiences for relatability and connection.
b) Acknowledgment of the impact of personal experiences on teaching philosophy.	Enhanced pedagogical authenticity, contributing to the development of a teacher's unique philosophy.

THEME 5: Curriculum and Teaching Approaches:

This theme revolves around curriculum challenges and innovative teaching strategies. Curriculum overload and rigid methods serve as causes that hinder effective instruction, while encouragement of transformative teaching approaches is a cause that brings about positive change.

Integrating philosophical insights into teaching methods is another cause that enriches instructional strategies, resulting in more engaged and motivated learners.

Subcategories:

a) Curriculum Challenges and Approaches:

Cause	Effect
a) Curriculum overload, rigidity, and content-heavy methods.	Impaired teacher effectiveness and reduced learner enthusiasm for mathematics.
b) Encouragement of creative and transformative teaching approaches.	Enhanced learner engagement and comprehension, promoting critical thinking and problem-solving skills.

THEME 6: Educational System Improvement:

This theme focuses on the need for educational reforms and improvements. Causes related to inclusive and diverse mathematics education, addressing issues of inequality and decolonization, drive the need for change. These causes lead to effects that emphasize the creation of a more balanced and equitable educational environment that benefits all learners.

Subcategories:

a) Calls for Educational Reforms:

Cause	Effect
a) Inclusive and diverse mathematics education.	Mitigation of historical disparities and fostering a more equitable learning experience.
b) Addressing issues of inequality, diversity, and decolonization.	Creating a balanced and inclusive educational environment that reflects diverse perspectives.
c) Need for educational improvements.	Enhanced quality of mathematics education, aligning with evolving educational needs.

THEME 7: Connection Between Philosophy and Mathematics:

This theme explores the symbiotic relationship between philosophy, mathematics, and teaching. Causes related to integrating philosophical discussions and open-ended approaches lead to effects that inspire curiosity, deeper engagement, and a holistic understanding of mathematics.

The interplay between these causes and effects encourages learners to approach mathematics with a more inquisitive mindset.

Subcategories:

a) Relationship between Philosophy, Mathematics, and Teaching:

Cause	Effect
a) Philosophy enhances teaching through discussions and open-ended approaches.	Fostering an environment of inquiry and exploration, promoting critical thinking and creativity in learners.
b) Inspiring curiosity and deeper engagement in mathematics.	Enhanced interest and involvement in mathematical concepts, promoting a lifelong appreciation for the subject.

THEME 8: Interdisciplinary and Holistic Approach:

This theme embraces an interdisciplinary and holistic approach to mathematics education. Causes related to interdisciplinary teaching methods and the significance of critical thinking lead to effects that encompass comprehensive learning and a well-rounded educational experience. Embracing complexity and exploring interconnectedness serves as a cause that fosters a broader perspective on mathematics and its relevance.

Subcategories:

a) Promoting Holistic Learning:

Cause	Effect
a) Interdisciplinary teaching approaches.	Broadened understanding of mathematics in relation to other disciplines, fostering a holistic view of knowledge.
b) Significance of critical thinking and holistic learning.	Enhanced problem-solving skills and a deeper comprehension of mathematical concepts through a holistic lens.
c) Embracing complexity and exploring interconnectedness.	Encouraging learners to see mathematical concepts as interconnected and reflective of real-world complexity.

These cause-and-effect relationships offer deeper insights into how different aspects interact and influence each other within the identified subcategories. This type of analysis can help understand the dynamics at play and provide a more comprehensive interpretation of your research findings.

4.2.3.4 Conclusion of Axial Coding

The axial coding process has served as a pivotal tool in unravelling the intricate relationships that underlie the themes identified within this study. By delving into these connections, a more nuanced and organized perspective emerges, shedding light on the multifaceted interactions shaping the landscape of philosophy of mathematics in teaching.

The structured presentation of these relationships, as presented in ANNEXURE G3, offers a holistic view that enhances the impact of our findings. The subsequent exploration of cause-and-effect relationships within the subcategories further deepens our insights, uncovering the mechanisms through which influences translate into tangible effects within the realm of mathematics education. This comprehensive analysis fosters a more profound understanding of how philosophy, challenges, knowledge, and teaching practices intersect and contribute to the broader discourse surrounding mathematics education. As we move forward, the insights derived from this axial coding exercise will serve as a foundation for the subsequent stages of this research, enriching the understanding of the intricate connections that shape the philosophy of mathematics in teaching. The next stages in this variant of grounded theory is elective coding.

4.2.4 Selective Coding

In selective coding, one essentially selects specific themes or categories to be the central focus of the analysis, and work towards weaving these themes together into a coherent and meaningful storyline. This process helps to build a more refined and nuanced understanding of the phenomenon under scrutiny.

Selective coding is a crucial step in qualitative data analysis as it moves beyond identifying patterns and relationships to synthesizing and integrating these patterns into a cohesive and insightful interpretation of the data.

4.2.4.1 Interpretation and meaning-making of the themes that emerged during the coding process

Selective coding marks a crucial phase in the qualitative analysis process, where the nuanced connections and underlying insights gleaned from axial coding are synthesized into a coherent narrative. This integration highlights the broader implications and significance of the identified themes. By carefully examining the relationships between themes, one can unveil a richer understanding of how various aspects interact within the context of the research subject.

THEME 1: Philosophy of Mathematics in Mathematics Teaching

The integration of philosophy into mathematics education goes beyond a mere union of disciplines; it emerges as a transformative force that reshapes pedagogy and nurtures engagement. Philosophical concepts bridge the gap between abstract mathematical ideas and practical teaching methods, encouraging a profound shift in educational perspectives. As educators incorporate philosophical thinking, they inspire curiosity and promote deeper learner involvement. This theme reinforces that philosophy, when woven into teaching, not only enhances comprehension but also fosters a genuine connection between learners and mathematics.

THEME 2: Challenges in Mathematics Education

This theme unearths the complexities that beset mathematics education, going beyond surface-level obstacles. These challenges are deeply intertwined with philosophical considerations and systemic issues within education. It calls for a holistic approach to reform, rooted in both practical adjustments and philosophical insights. By tackling challenges head-on, educators can create a dynamic learning environment that nurtures critical thinking, creative solutions, and equitable education for all.

THEME 3: Nature of Mathematical Knowledge

The intrinsic characteristics of mathematical knowledge – certainty, logic, and distinction from belief – underpin effective teaching and learning. This theme underscores that mathematical truths are not simply facts to be conveyed but concepts to be justified. Philosophical explorations surrounding certainty and validity enhance educators' pedagogical competence, allowing them to communicate mathematical knowledge more effectively. Through this theme, it's evident that the marriage of philosophy and mathematics engenders a profound appreciation for the logical foundations of mathematical truths.

THEME 4: Teacher Reflection and Empowerment

Teachers emerge as the cornerstones of mathematics education, influencing learners not only through their teachings but also through their beliefs and experiences. Philosophy's impact on teaching approaches underscores the pivotal role educators play in shaping the learning experience. The incorporation of personal experiences enriches pedagogical strategies, creating a classroom environment where authenticity and connection thrive. This theme highlights that teacher empowerment is a catalyst for meaningful learner engagement and the development of a vibrant educational ecosystem.

THEME 5: Curriculum and Teaching Approaches

Curriculum challenges and innovative teaching strategies intersect to create a dynamic interplay within this theme. The tension between rigid methods and the encouragement of transformative approaches exemplifies the necessity for a balanced educational landscape. Integrating philosophical insights into teaching methods brings forth an enriched learning experience that fosters curiosity and critical thinking. This theme underscores that harmonizing curriculum design with pedagogical flexibility creates an environment conducive to comprehensive learner growth.

THEME 6: Educational System Improvement

Educational reform surfaces as a fundamental need within mathematics education. The intersection of inclusivity, diversity, and addressing historical disparities reinforces the call for change. This theme emphasizes the imperative to transform the educational landscape into one that is inclusive, equitable, and supportive of all learners. By understanding the philosophical underpinnings of education, stakeholders can collaborate to build a well-rounded system that nurtures excellence and equality.

THEME 7: Connection Between Philosophy and Mathematics

Philosophy, mathematics, and teaching form an intricate tapestry that fosters a holistic learning experience. This theme unravels the ways in which philosophical discussions infuse mathematics with curiosity and open-ended exploration. By embracing philosophy, educators kindle a lifelong appreciation for mathematical concepts, encouraging learners to view mathematics as a vibrant and ever-evolving subject. This theme underscores that the fusion of philosophy and mathematics nurtures an inquiry-driven approach to learning.

THEME 8: Interdisciplinary and Holistic

Approach Embracing an interdisciplinary lens transforms mathematics education into a multifaceted journey. This theme illustrates how critical thinking, coupled with an appreciation for interconnectedness, enhances the educational experience. Integrating philosophy bolsters this approach, infusing it with depth and relevance. The interconnectedness of various disciplines underscores that a holistic education equips learners with the tools to decipher complexities and make informed contributions to a dynamic world.

4.2.4.2 Link of Emerging Themes to Established Educational Theories

In the process of qualitative data analysis, the "Link to Established Themes" step serves as a critical bridge between the emergent findings from the coding process and the existing theoretical and conceptual frameworks in the field. This step is essential for grounding the newly identified themes in established educational theories, philosophies, and pedagogical perspectives. By connecting the emergent themes to well-established frameworks, one gains a deeper understanding of the theoretical underpinnings and broader implications of the research findings. This linkage allows us to contextualize the research within the existing scholarly discourse and enriches the overall significance of the study.

Theme	Link to Educational Theory
1. Philosophy of Mathematics in Mathematics Teaching	Constructivist theories, fallibilist principles, and structuralist view of logical organization.
2. Challenges in Mathematics Education	Critical pedagogy, sociocultural theories, and transformative reform.
3. Nature of Mathematical Knowledge	Epistemological inquiries and philosophical investigations into truth and certainty.
4. Teacher Reflection and Empowerment	Humanistic theories, narrative approaches, and established pedagogical philosophies.
5. Curriculum and Teaching Approaches	Constructivist approaches, personalized learning, and balanced pedagogical strategies.
6. Educational System Improvement	Social justice pedagogies and the broader philosophical goal of creating just educational systems.
7. Connection Between Philosophy and Mathematics	Constructivist principles and the active role of learners in their educational journey.
8. Interdisciplinary and Holistic Approach	Educational philosophies that recognize the interconnected nature of knowledge and emphasize critical thinking.

4.2.4.3 Comparison of Emerging Themes with Established Educational Theories

This section provides a comprehensive comparative analysis that situates the research findings within the broader context of education and philosophy. By systematically exploring both the points of convergence and divergence between the emergent themes and established educational theories, this analysis offers a nuanced comprehension of the thematic alignment and novel contributions inherent in the research.
Through this evaluative process, the study seeks to illuminate the originality and significance of the insights presented. Convergence serves to signify the resonance and consistency between the research outcomes and existing prevailing theories. Conversely, divergence accentuates areas in which the research findings introduce distinctive perspectives or make unique contributions, thereby enhancing our understanding of the subject matter (Muller, et al., 2016).

THEME 1: Philosophy of Mathematics in Mathematics Teaching

- **Convergence**: Aligns with constructivist theories emphasizing active learning and fallibilist principles acknowledging uncertainty.
- **Convergence**: Resonates with the structuralist view of logical organization, connecting abstract mathematical ideas with practical teaching methods.
- **Divergence**: Goes beyond traditional disciplinary boundaries, transforming pedagogy through the integration of philosophy.

THEME 2: Challenges in Mathematics Education

- **Convergence**: Resonates with critical pedagogy, addressing systemic issues and advocating transformative reform.
- **Convergence**: Aligned with sociocultural theories, recognizing the impact of societal and cultural contexts on education.
- **Divergence**: Explores challenges deeply intertwined with philosophical considerations, going beyond surface-level obstacles.

THEME 3: Nature of Mathematical Knowledge

- **Convergence**: Resonates with epistemological inquiries, exploring the characteristics and justification of mathematical knowledge.
- **Convergence**: Aligns with philosophical investigations into truth, certainty, and validity in mathematical truths.
- **Divergence**: Emphasizes the distinction between mathematical knowledge and mere belief, highlighting the unique nature of mathematical truths.

THEME 4: Teacher Reflection and Empowerment

- **Convergence**: Reflects humanistic theories valuing individual growth and personal experiences in education.
- **Convergence**: Resonates with narrative approaches that recognize the power of personal stories in pedagogy.
- **Divergence**: Explores the impact of philosophical thinking on teaching approaches, emphasizing teacher empowerment and authentic connection.

THEME 5: Curriculum and Teaching Approaches

- **Convergence**: Reflects constructivist approaches, encouraging learner-centred and inquiry-based learning.
- **Convergence**: Aligns with personalized learning, tailoring education to individual learner needs.
- **Divergence**: Integrates philosophical insights into teaching methods, enriching the learning experience, and fostering critical thinking.

THEME 6: Educational System Improvement

- **Convergence**: Aligns with social justice pedagogies, advocating for inclusivity and addressing historical disparities.
- **Convergence**: Resonates with the broader philosophical goal of creating equitable and just educational systems.
- **Divergence**: Emphasizes the need for transformation in the educational landscape, acknowledging the imperative for change.

THEME 7: Connection Between Philosophy and Mathematics

- **Convergence**: Aligns with constructivist principles, emphasizing the active role of learners in inquiry-driven learning.
- **Convergence**: Resonates with constructivist goal of fostering curiosity and deeper engagement.
- **Divergence**: Explores the infusion of philosophical discussions into mathematics, nurturing a lifelong appreciation for the subject.

THEME 8: Interdisciplinary and Holistic Approach

- **Convergence**: Reflects educational philosophies recognizing the interconnected nature of knowledge.
- **Convergence**: Emphasizes critical thinking, aligning with educational approaches that prioritize higher-order thinking skills.
- **Divergence**: Integrates philosophy to infuse depth and relevance into the interdisciplinary lens, equipping learners to navigate complexities.

4.2.4.4 New Insights and Gaps Identified

NEW INSIGHTS:

1. **Philosophy as a Transformative Element:** The analysis reveals a novel insight into the transformative power of philosophy in mathematics education. It suggests that the integration of philosophical concepts doesn't just enrich pedagogy but fundamentally reshapes it, fostering deeper learner engagement and connection with mathematics.

2. **Philosophy's Role in Teacher Empowerment:** The exploration of the impact of philosophical thinking on teaching approaches offers a new perspective on teacher empowerment. It highlights how philosophical reflection empowers teachers to create authentic connections with learners and enhances their role in shaping the learning experience.

3. **Incorporation of Personal Experience:** The analysis underscores the importance of incorporating personal experiences in pedagogy, which is often associated with humanistic approaches. However, it introduces the idea that philosophy can provide a

structured framework for effectively integrating personal experiences, fostering an environment of authenticity and connection.

GAPS:

1. **The Specifics of Philosophical Integration:** While the analysis discusses the transformative role of philosophy in education, it doesn't delve deeply into specific strategies or methods for integrating philosophical concepts into mathematics teaching. Addressing this gap could provide educators with actionable insights on how to practically infuse philosophy into their teaching.

2. **Empirical Evidence for Impact:** While the comparative analysis presents theoretical alignment, it doesn't extensively address empirical evidence that demonstrates the actual impact of integrating philosophy into mathematics education. Research studies or examples showcasing tangible improvements in learner engagement, critical thinking, or other outcomes could strengthen the argument.

3. **Ethical Considerations in Educational System Improvement:** The analysis touches upon the need for educational system improvement and inclusivity but doesn't explicitly address the ethical considerations that come into play when implementing changes. Exploring the ethical implications and potential challenges in creating an equitable educational system could add depth to this theme.

4.2.4.5 Conclusion of Selective Coding

The research findings converge with established ideas in education and philosophy, while also offering novel insights by integrating philosophy into mathematics education. This integration bridges theoretical discussions with practical applications, contributing to a more comprehensive understanding of the intersection between philosophy of mathematics and mathematics teaching. In the next section conclusions and recommendations for integration of philosophy of mathematics into mathematics education will be discussed.

Chapter 5 Conclusion and Recommendations for Integrating Philosophy of Mathematics into Mathematics Education in South Africa

5.1 Introduction

This study embarked on a journey to explore the integration of philosophy of mathematics into mathematics education within the context of South Africa. Through a rigorous process of data collection, analysis, and interpretation, it uncovered insights that underscore the transformative potential of incorporating philosophical perspectives in mathematics instruction. The interconnectedness of philosophy, teaching practices, challenges, and the essence of mathematical knowledge emerged as critical areas that warrant attention and action.

Chapter 5 serves as the culmination of this exploration, presenting a set of recommendations grounded in the findings and analysis presented in earlier chapters. These recommendations are strategically designed to address the existing gaps within mathematics education, enhance pedagogical methodologies, and create meaningful learning experiences for learners. By proactively implementing these recommendations, South Africa has the opportunity to propel its mathematics education system toward advancement, equipping learners with the essential skills and perspectives required for success in an ever-evolving world.

5.2 Philosophical Integration in Mathematics Education: Bridging Theory and Practice for Meaningful Teaching

The research findings derived from focused, axial, and selective coding significantly address the central inquiry: "*Could philosophy of mathematics be the bridge to meaningful mathematics teaching in the classroom*?"

5.2.1 Philosophy's Role in Teaching Enhancement

The focused coding insights reveal a convergence in affirming that integrating philosophical ideas into pedagogy enhances teaching practices. For instance, applying philosophical concepts elucidated by Grothendieck, Susskind, Greene, Russell, and Shapiro into teaching methodologies (as noted in the autoethnography) reflects a deeper convergence. These applications bridge abstract mathematical concepts, encouraging a comprehensive understanding among learners. This integration directly answers the research question by showcasing how philosophical elements act as a bridge to foster deeper engagement and understanding in mathematics teaching.

5.2.2 Transformative Impact of Philosophy

The findings also converge on the transformative impact of philosophical integration. They reflect alignment in the recognition of philosophy as a transformative element. The integration of personal experiences, a reflection on various philosophies such as those by Ernest and others, directly empowers educators. This convergence illustrates how the incorporation of personal experiences alongside philosophical insights empowers teachers to restructure their teaching philosophies. This directly correlates with the research question by demonstrating how philosophy serves as a catalyst for transformation in mathematics education.

5.2.3 Identified Gaps and Future Research

Despite convergence, identified gaps—like the need for empirical evidence and specifics in philosophical integration—stand out. For instance, the lack of empirical evidence for the impact of philosophical integration in actual classroom settings emphasizes divergence. This divergence signals opportunities for further exploration. Addressing these gaps could involve conducting empirical studies that measure the direct impact of incorporating philosophical concepts on student engagement and learning outcomes.

These insights serve as directions for future research aimed at bolstering the relationship between philosophy of mathematics and effective teaching in classrooms, further aligning with the initial research question's intent to bridge meaningful mathematics teaching through philosophical integration.

5.3 Implications and Recommendations

5.3.1 Implications for Theory

The research findings offer significant implications for the theoretical landscape of both mathematics education and philosophy of education. By highlighting the role of philosophical perspectives in enhancing teaching practices and addressing educational challenges, this research contributes to a deeper understanding of how philosophy can be integrated into educational theories. It opens avenues for interdisciplinary exploration and prompts a re-examination of epistemological assumptions, enriching the theoretical foundations of these fields.

5.3.1.1 Enriching Theoretical Framework

This research offers an opportunity to enrich existing theoretical frameworks in both mathematics education and philosophy of education. By highlighting the role of philosophy of mathematics in enhancing teaching practices and addressing challenges, these findings contribute to a more comprehensive understanding of how philosophical insights can be integrated into educational theories.

5.3.1.2 Integration of Disciplines

This research demonstrates the potential for fruitful integration between philosophy of mathematics and mathematics teaching. This interdisciplinary approach challenges traditional boundaries and suggests that philosophy can serve as a bridge to enhance the effectiveness of teaching and learning in STEM[22] disciplines.

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[22] STEM (Science, Technology, Engineering, and Mathematics) is an interdisciplinary educational approach that integrates knowledge and skills from the fields of science, technology, engineering, and mathematics. The primary goal of STEM education is to foster critical thinking, problem-solving abilities, creativity, and innovation among students. By engaging students in hands-on activities, real-world applications, and collaborative projects, STEM programs aim to prepare primary and secondary students for higher education, advanced studies, and careers in STEM-related fields. This holistic approach emphasizes not only subject-specific knowledge but also the development of transferable skills essential for success in an increasingly technology-driven and interconnected world.

5.3.1.3 Revisiting Epistemological Assumptions

These findings encourage a re-examination of epistemological assumptions within mathematics education. The emphasis on logical reasoning, certainty, and justification, coupled with the nuanced interplay of structured and constructed knowledge along with elements of fallibilism, prompts educators and researchers to reflect on how these concepts impact the conveyance and understanding of mathematical knowledge.

5.3.2 Implications for Practice

In the realm of practice, the research findings offer actionable insights for educators and educational institutions. These implications guide the development of teacher professional development programs that integrate philosophical perspectives, thereby enhancing pedagogical approaches. Furthermore, the findings advocate for curriculum designs that prioritize structured and constructed critical thinking, as well as philosophical discussions, aiming to foster a more profound understanding of mathematical concepts among learners. By harnessing philosophy's potential, educators can create engaging, transformative, and interdisciplinary learning experiences that address challenges and promote holistic learning approaches, embracing the nuanced elements of fallibilism.

5.3.2.1 Teacher Professional Development

This research suggests that incorporating philosophical perspectives into teacher professional development programs can enhance educators' pedagogical practices, fostering a profound understanding of the fundamental mathematical concepts of relationships, structure, and patterns. By providing educators with the tools to integrate philosophy into their teaching, it can lead to more engaging and effective classroom experiences.

5.3.2.2 Curriculum Design

These findings underscore the importance of curriculum design that goes beyond content delivery. Designing curricula that incorporate philosophical discussions, such as those related to structure, connections, and patterns, as well as critical thinking exercises that explore possibilities and verification, can help learners develop a deeper understanding of mathematical concepts.

5.3.2.3 Learner Engagement and Critical Thinking in the Classroom

Educators can use the insights from this research to foster learner engagement and critical thinking in mathematics. By infusing philosophical questions and discussions, educators can encourage learners to explore mathematical concepts from different angles. It also highlights the role of the teacher as not only role models but also facilitators of learning, underlining their responsibility in creating an environment that promotes exploration and deeper understanding through philosophical discussions.

5.3.2.4 Addressing Challenges

This research offers a unique perspective on addressing challenges in mathematics education. Teachers can draw on philosophical insights to approach challenges such as curriculum overload, negative learner attitudes, and teacher stress in innovative and transformative ways.

5.3.2.5 Holistic and Interdisciplinary Approach

This research promotes holistic and interdisciplinary approaches to education. Teachers can consider integrating philosophical discussions and perspectives not only in mathematics classes but also across other disciplines, fostering a broader understanding of the interconnectedness of knowledge.

5.3.2.6 Personalized Learning

The research's emphasis on incorporating personal experiences into teaching practices suggests the value of personalized learning experiences. For example, teachers can encourage learners to bring their own perspectives and experiences into the learning process, enhancing engagement and relevance. This might include exploring concepts such as infinity, the infinitesimal, and everything in between, all within the safe environment of a well-trained mathematics teacher.

5.3.2.7 Ethical and Social Implications

1 Ethical Decision-Making

Aligned with the tenets of philosophical tradition, the inclusion of this segment addressing ethical considerations and the transformative potential of mathematics holds significant importance. Ethical dimensions within mathematics encompass the societal and moral implications stemming from the application of mathematical concepts. While mathematics is often perceived as an abstract field, its practical applications possess the potential for profound societal impacts. Here are some thought-provoking examples tailored for Grade 7 to 12 learners:

- **Data Ethics and Privacy**: In the contemporary landscape of extensive data usage, mathematicians and analysts handle large volumes of personal information. Ethical considerations revolve around the judicious and secure handling of this data to safeguard individuals' privacy.
    - *Social Media and Data Privacy*: Social media platforms collecting user data for targeted advertising, leading to discussions about consent and privacy.
    - *Healthcare Data*: Ensuring that patient medical data is kept secure and confidential while still allowing for advancements in medical research.
    - *Social Media and Bullying:* The ethical dimension extends to addressing cyberbullying on social media platforms, necessitating a responsible approach to protect users from harm and foster a safe digital environment.
- **Algorithmic Accountability**: Mathematical algorithms are pivotal in various domains, including artificial intelligence and machine learning. Yet, unchecked algorithmic development can perpetuate biases inherent in training data, leading to unjust or prejudiced outcomes.
    - *Criminal Justice Algorithms*: Algorithms used in predicting crime rates may perpetuate biases against certain communities, sparking discussions about fairness in legal decisions.
    - *Facial Recognition Technology*: Biased algorithms in facial recognition technology can lead to misidentification and violations of personal privacy.

- **Cryptocurrency Complexities**: The emergence of cryptocurrencies like Bitcoin sparks discussions about ethical implications, such as concerns about money laundering, tax evasion, and the environmental ramifications of cryptocurrency mining.
- **Equitable Resource Allocation**: Mathematical models play a decisive role in shaping resource allocation, such as healthcare distribution or public service funding. Ethical considerations encompass ensuring fairness and preventing undue harm.
    - *COVID-19 Resource Allocation*: During the pandemic, decisions on allocating medical supplies highlighted the ethical challenges of ensuring fair access to limited resources.
    - *School Funding*: Distributing educational funds to schools in a way that doesn't disadvantage certain communities can pose ethical dilemmas.
- **Inclusivity in Education**: Mathematics can inadvertently create barriers for certain learner groups due to inequitable access to resources. Ethical deliberations focus on cultivating inclusive and just learning environments.
    - *Online Learning Disparities*: The digital divide, where some learners lack access to online learning resources, raises concerns about equitable access to education.
    - *Culturally Relevant Math Education*: Ensuring math curricula acknowledge diverse cultural backgrounds to create inclusive learning environments.
- **Surveillance and Ethics**: The application of mathematical principles in surveillance technologies raises ethical concerns about striking a balance between security and individual privacy rights.
    - *Government Surveillance*: Debates surrounding the balance between national security and citizens' right to privacy through surveillance programs like mass data collection.
    - *Smart Home Devices*: The implications of using Internet of Things (IoT) devices that collect personal data in homes, raising questions about privacy.
- **Environmental Modelling Challenges**: Mathematical models help interpret intricate environmental phenomena like climate change. Ethical concerns entail accurately representing uncertainty and potential consequences within these models.
    - *Climate Change Modelling*: Creating mathematical models to predict the impacts of climate change on sea levels, temperature, and extreme weather events.

   - *Biodiversity Monitoring*: Using mathematical tools to assess the health of ecosystems and biodiversity, crucial for making informed environmental policies.
- **Ethics of Military Applications**: Mathematics holds significant importance in military contexts, including cryptography and weapon guidance systems. Ethical discussions centre around the responsible deployment of these technologies. Exploring documentaries like the one on Oppenheimer and the Manhattan Project can provide profound insights into the ethical dimensions and societal impacts of such applications.

These instances underscore the intricate interplay between mathematics and societal dynamics. Encouraging reflective discussions on the ethical implications of mathematical knowledge empowers senior learners to navigate the broader implications of their mathematical studies thoughtfully and responsibly.

Here are some recent documentaries that teachers can share with their learners:

- **"The Social Dilemma"** (2020): While not solely focused on artificial intelligence, this documentary explores the negative impact of social media and technology on society, including issues related to AI-driven algorithms and their influence on human behaviour.
- **"Do You Trust This Computer?"** (2018): This documentary examines the rise of artificial intelligence and its potential consequences, including discussions on privacy, bias, and the implications of AI on society.
- **"Machine"** (2017): While not the most recent, this documentary delves into the world of artificial intelligence and its impact on human life, exploring both the potential benefits and ethical challenges.

2 Equity and Inclusion

Integrating philosophical perspectives can help address issues of equity and inclusion in mathematics education. By encouraging diverse viewpoints and engaging with questions of social justice, educators can create a more inclusive learning environment. For instance, by embracing and unifying Western and non-Western histories of mathematics and their applications can contribute to this inclusivity. Some examples of non-western histories of mathematics:

- **Indian Mathematics and the Concept of Zero**: The concept of zero as a placeholder and its integration into the decimal numeral system originated in India. This innovation revolutionized mathematics and numerical representation.
- **Islamic Mathematics and Algebra**: Scholars from the Islamic world made significant contributions to algebra, helping to develop and advance algebraic concepts and solutions.
- **Mayan Mathematics and Numerical Systems**: The ancient Mayans developed a complex numerical system that included the use of a base-20 system and a symbol for zero.
- **African Mathematical Traditions**: Various African cultures have unique mathematical traditions, including techniques for counting, measuring, and spatial reasoning e.g., The Morabaraba Board Game whose configurations of the tokens on the board use mathematics concepts in the game: geometric shapes, ratio and proportion, symmetry, logical reasoning and counting.
- **Arabic Numerals and Number System**: The Arabic numeral system, which includes the digits 0 to 9, was a significant advancement in mathematics and made arithmetic operations more efficient.
- **Chinese Mathematics and Trigonometry**: Ancient Chinese mathematicians developed trigonometric methods for solving practical problems related to surveying, architecture, and astronomy.
- **Indigenous Knowledge and Ethnomathematics**: Indigenous cultures around the world have their own mathematical knowledge, often deeply intertwined with their cultural practices, such as navigation techniques or patterns in art and design.
- **Babylonian Mathematics and Base-60 System:** The Babylonians used a base-60 numeral system, which is the basis for our division of hours, minutes, and seconds in modern timekeeping.

3 Cultivating Critical Citizens

By fostering critical thinking and philosophical inquiry, this research contributes to the development of critical citizens who can engage with complex societal and ethical issues beyond the classroom.

This research has far-reaching implications for both theory and practice in mathematics education and philosophy of education. By integrating philosophy of mathematics into mathematics teaching, new avenues are opened up for enhancing teaching practices, addressing challenges, and fostering deeper engagement and critical thinking among learners. Additionally, this work contributes to the ongoing dialogue about the intersections between philosophy, education, and the broader societal context.

5.4 Limitations of this Study

5.4.1 Generalizability

The study's findings are context-specific to the South African education system and may not be directly applicable to other educational contexts. The unique challenges, curriculum structures, and cultural factors in South Africa could limit the generalizability of the results to different regions.

5.4.2 Sampling Bias

The participants in this study, such as teachers and educators, may have personal beliefs and experiences that influence their perspectives. This potential sampling bias could impact the overall representation of views related to the integration of philosophy in mathematics education.

5.4.3 Self-Report Bias

Although anonymity and informal interactions add to minimizing bias, data collected through anonymous questionnaires, informal discussions and informal interviews rely on participants' self-reported experiences and opinions. This may introduce bias due to participants' subjectivity, social desirability, or limited recall accuracy.

5.4.4 Limited Quantitative Data

The research heavily relies on qualitative data sources, such as field notes and informal interviews and discussions, and an autoethnography report which provide rich insights but might lack the quantitative rigor of larger-scale surveys or experiments. This limitation could impact the statistical generalizability of the findings.

5.4.5 Lack of Control Variables

Due to the inherent nature of qualitative research, maintaining complete control over external variables might be challenging. Consequently, isolating the precise impact of integrating philosophy into teaching outcomes from potential confounding factors becomes a complex task. These factors encompass various aspects, including the broader social milieu and specific cultural disparities within the realm of mathematics education. For example, differing perspectives on mathematics literacy versus mathematics proficiency could introduce additional layers of complexity. Additionally, the existing educational constraints, such as issues related to overcrowding, discipline, and heavy workloads, further contribute to the intricate web of influences on teaching outcomes.

5.4.6 Researcher Bias

My personal involvement as the researcher in the autoethnography component could introduce bias in data interpretation and analysis. My own experiences and beliefs might inadvertently influence the findings. Autoethnography involves the researcher's reflection on their own experiences and perspectives, which could lead to bias in interpretation, selective recall, or personal subjectivity. As such it is important to acknowledge and critically assess these potential sources of bias in the analysis and interpretation of the data.

5.4.7 Depth vs. Breadth

The research focuses on a specific set of themes and subcategories derived from the data, which might not encompass the entire spectrum of potential insights related to the integration of philosophy in mathematics education. For instance, the study could expand its scope to include perspectives from rural schools and township schools across different provinces.

5.4.8 Single-Method Approach

While this approach to data collection and analysis is comprehensive, relying solely on qualitative methods limits the ability to triangulate findings with quantitative data, potentially hindering the attainment of a more comprehensive perspective. However, conducting quantitative research demands a significantly larger scale of interdisciplinary involvement and carries the necessity of higher authority to overcome the prevalent reluctance of schools to engage with research. For instance, out of 60 schools approached, only 10 responded positively to the research request, underlining the challenges faced in obtaining participation.

5.4.9 Temporal Limitation

The study's data collection was conducted within a specific time frame, and educational and philosophical contexts could evolve. This temporal limitation might affect the currency and relevance of the findings over time.

5.4.10 Ethical Considerations

Ethical considerations in qualitative research, such as obtaining informed consent and maintaining confidentiality, are crucial. However, addressing all ethical dimensions might pose challenges in capturing the full complexity of participants' experiences.

5.5. Addressing Study Limitations: Examples and Mitigation Strategies for a Comprehensive Understanding

This section acknowledges and addresses limitations in the study to ensure research validity. By providing specific examples and corresponding mitigation strategies, we aim to present a nuanced understanding of potential constraints. These measures contribute to methodological robustness, fostering transparency and trust in the study's outcomes.

5.5.1 Generalizability

Acknowledged the South African education system's unique challenges and emphasized caution in applying findings to other contexts in the study's discussion and conclusion.

5.5.2 Sampling Bias

Employed a purposeful sampling technique that included educators from various socio-economic backgrounds, different schools with diverse school settings to capture a wider array of perspectives in the study e.g., public schools, private schools, Montessori school.

5.5.3 Self-Report Bias

Triangulated self-reported data from questionnaires and interviews with field notes and autoethnography to validate and cross-reference participant responses, ensuring a more comprehensive and less biased analysis.

5.5.4 Limited Quantitative Data

Attempted to complement qualitative data with available statistical information from the questionnaires, educational authorities e.g., headmasters, department of education and experts, or relevant studies to offer a more well-rounded perspective in the research findings.

5.5.5 Lack of Control Variables

Discussed the potential confounding factors in the study, such as varying educational backgrounds, socio-economic status, and cultural influences, in the study's limitations section to highlight potential impacts on outcomes.

5.5.6 Researcher Bias

Involved peer researchers in the analysis process, conducting peer reviews and discussions to ensure diverse perspectives and minimize personal biases in data interpretation and findings.

5.5.7 Depth vs. Breadth

Acknowledged the study's focused approach and highlighted the need for future research to include a broader range of perspectives from rural and township schools to capture a more comprehensive understanding of the topic.

5.5.8 Single-Method Approach

Clearly stated the study's reliance on qualitative methods and provided a rationale for this choice while acknowledging potential limitations and challenges faced in obtaining quantitative data.

5.5.9 Temporal Limitation

I mentioned the specific period of data collection and discussed how evolving educational and philosophical contexts could impact the relevance and applicability of findings over time in the study's conclusion.

5.5.10 Ethical Considerations

Described the steps taken to ensure ethical practices in the research, including obtaining informed consent, maintaining confidentiality, and adhering to institutional ethical guidelines, ensuring ethical integrity throughout the study.

5.6 Recommendations for Integrating Philosophy of Mathematics into Mathematics Education

5.6.1 Curriculum Enhancement

Policymakers should initiate comprehensive curriculum reforms that transcend content mastery alone. The curriculum design should strike a balance between foundational knowledge and the cultivation of critical thinking, problem-solving skills, and philosophical perspectives. By integrating philosophical dimensions into mathematics education, curriculum reform can foster intellectual curiosity and creative exploration.

5.6.2 Professional Growth for Educators

Educational institutions should commit to ongoing professional development for mathematics educators, with a particular focus on the integration of philosophy into teaching practices. Offering workshops, training sessions, and collaborative discussions can empower educators to seamlessly incorporate philosophical insights into their pedagogical strategies. This approach aligns with the evolving needs of educators and ensures accessibility, echoing the insights of Ernest (1991).

5.6.3 Teaching resources

Create high-quality teaching resources that align with the principles of philosophy of mathematics. These resources should support educators in implementing effective teaching practices and engaging learners in meaningful mathematics learning experiences. The resources should emphasize the practical application of philosophical perspectives, promote critical thinking, problem-solving skills, and highlight the interconnectedness of mathematical concepts. Accessibility and availability of these resources should be ensured to reach a wide range of educators e.g., Video Lectures or Games with Philosophical Insights, Philosophical Puzzle Kits – Tangrams With Philosophical Twists, Multiple Solution Puzzles (Pimm, 1987).

5.6.4 Learner-Centred Learning Environment

Embracing learner-centred learning approaches can create an environment conducive to open-ended philosophical discussions. Providing learners with a safe space to explore questions related to the nature of mathematics fosters a deeper understanding and appreciation for the subject.

This approach resonates with the pedagogical principles that prioritize meaningful engagement and holistic learning experiences.

5.6.5 Assessment Beyond Recall

Shifting assessment strategies beyond traditional content recall is crucial. Developing assessment methods that evaluate learners' critical thinking, logical reasoning, and their ability to apply philosophical insights to mathematical problem-solving can provide a more comprehensive measure of learning outcomes. This aligns with contemporary educational paradigms that emphasize holistic skill development. Here are some examples of assessment strategies that align with these principles:

1. **Socratic Problem-Solving**: Present learners with open-ended mathematical problems that require them to analyse, question assumptions, and consider multiple solution strategies. Encourage them to engage in Socratic dialogues, discussing their thought processes and rationale behind their chosen approach (Peterson & Curtis, 2019).
2. **Philosophical Reflection Papers**: Assign learners to write reflective essays that connect philosophical concepts discussed in class (e.g., fallibilism, certainty, paradoxes) with mathematical topics they have learned. They should explore the implications of these concepts on their understanding of mathematics and their problem-solving methods.
3. **Case Studies:** Provide real-world case studies where learners must apply mathematical concepts to analyse complex situations e.g., Urban Traffic Flow Optimization – solving congestion. Ask them to critically evaluate the ethical, social, and practical implications of different mathematical approaches and solutions.
4. **Mathematical Debates:** Organize classroom debates on controversial mathematical topics or philosophical questions related to mathematics e.g., The Role of Intuition in Mathematical Proof (logic). Learners must use logical reasoning and evidence to support their arguments and engage in thoughtful discussions.
5. **Portfolio Assessment**: Have learners create portfolios showcasing their problem-solving journey throughout the course. Include reflections on their understanding of mathematical concepts, their application of philosophical insights, and how their problem-solving strategies have evolved e.g., Mathematics Is The Study Of Structures, or How I Solved My Own Misconceptions.

6. **Collaborative Projects:** Assign group projects that require learners to solve multifaceted mathematical problems. They should integrate diverse viewpoints, draw on philosophical concepts, and present their solutions in a coherent and logical manner e.g., Exploring Math and Ethics in Everyday Situations.
7. **Scenario-Based Assessments**: Present learners with scenarios that require mathematical analysis in context. Ask them to provide solutions while considering the broader implications and potential limitations of their approaches e.g., The Water Conservation Challenge.
8. **Critical Analysis of Mathematical Texts:** Assign readings from historical or contemporary mathematical texts that explore philosophical perspectives. Have learners analyse and critique the arguments presented, assessing the validity of the mathematical claims and the philosophical assumptions e.g., Exploring Mathematical Paradoxes such as Zeno's paradox of Achilles and the Tortoise, Plato's discovery of irrational numbers and the concept that space is not flat.
9. **Mathematical Inquiry Projects:** Allow learners to choose a mathematical topic of interest and guide them through an inquiry-based project. They should investigate the topic from multiple angles, applying philosophical insights to their exploration and drawing conclusions e.g., explore the Morabaraba Board Game whose configurations of the tokens on the board use mathematics concepts in the game: geometric shapes, ratio and proportion, symmetry, logical reasoning and counting.
10. **Collaborative Problem-Solving Challenges:** Organize problem-solving challenges that require teams of learners to collaborate on intricate mathematical problems. Include elements that prompt them to consider alternative approaches and justify their choices e.g., The Mathematical Escape Room - Fibonacci's Enigma (Brookfield & Preskill, 1999; Davis, 1993; Frederick, 1981 for points 2 to 9).

5.6.6 Continuation of Research Endeavours

This study establishes a foundational understanding of the integration of philosophy into mathematics education. Future researchers can build upon this foundation by conducting larger-scale studies, longitudinal investigations, and comparative analyses. By further validating and refining the identified themes, research can contribute to an evidence-based approach that informs educational practices.

5.6.7 Collaborative Synergy

Establishing collaborative initiatives involving educators, philosophers, and researchers can lead to the design of innovative teaching methodologies. These collaborations address the gap between theoretical philosophical insights and their real-world application in the classroom. By fostering synergy among diverse stakeholders, this recommendation nurtures an enriched educational ecosystem (Roth, 2007).

5.6.8 Interdisciplinary Innovation

Encouraging educators to explore interdisciplinary approaches that amalgamate philosophy with mathematics can lead to innovative teaching methodologies, for example, a mixture of structuralism, constructivism, absolutism, and fallibilism, similar to Grothendieck's (1986), Russell's (1959) and other's visionary perspectives. Collaborations between philosophy and mathematics departments can result in strategies that enhance learners' comprehension and engagement. These collaborative initiatives reflect a bridge between theoretical philosophical insights and practical classroom application.

5.6.9 Monitoring and Feedback Mechanisms

This step is centered on monitoring and feedback mechanisms within the context of philosophy-integrated teaching methods. It talks about regular evaluation, considering feedback from various stakeholders, and the importance of an adaptive response based on evaluation findings. The purpose here is to highlight the significance of ongoing assessment and the use of feedback to make adaptive changes, ensuring continuous improvement in the implementation of philosophy-integrated teaching methods.

5.6.10 Resourceful Pedagogical Tools

The development and dissemination of high-quality pedagogical resources, including lesson plans, teaching guides, and multimedia materials, can support educators in integrating philosophical perspectives into mathematics education. These practical resources equip educators with tools to implement philosophy-based teaching practices that foster critical thinking and problem-solving skills.

5.6.11 Holistic Policy Implementation

Education policymakers should take a holistic approach to policy implementation, integrating philosophical dimensions into teacher training programs, curriculum development, and educational policy frameworks. This transformative integration facilitates a shift toward more engaging and holistic mathematics education. Such policy considerations align with the insights of scholars advocating for comprehensive educational reforms.

TABLE 2: Roadmap for Successful Integration of the Philosophy of Mathematics into the Classroom

Steps	Sub Steps	Action To Be Taken	Rationale
1 Curriculum Reform and Development:	**a. Initial Assessment:**	Evaluate the current curriculum to identify areas that lack philosophical perspectives.	This step involves a comprehensive review of the existing curriculum to pinpoint any gaps or deficiencies in philosophical content.
	b. Design Philosophy-Integrated Curriculum:	Develop a curriculum framework that integrates philosophical dimensions into mathematical topics and problem-solving tasks.	This crucial phase focuses on the creation of a curriculum that seamlessly weaves philosophical perspectives into mathematical concepts, enhancing students' understanding and critical thinking skills.
	c. Balanced Integration:	Ensure a balanced blend of traditional mathematical content and philosophical inquiry, focusing on critical thinking and problem-solving.	Striking a balance is vital to avoid overshadowing mathematical fundamentals. This sub step ensures that philosophical elements complement rather than overshadow traditional content, promoting a comprehensive learning experience.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
2 Educator Training and Professional Development:	**a. Workshops and Training Sessions:**	Conduct regular professional development programs to familiarize educators with philosophical concepts and their integration into teaching practices.	Continuous training ensures educators are equipped with the knowledge and skills to effectively integrate philosophy into their teaching, fostering a supportive and informed teaching community.
	b. Resource Provision:	Provide educators with access to materials, lesson plans, and guides that support philosophy-based teaching strategies.	Accessible resources are essential for successful implementation. This action ensures educators have the necessary tools to confidently

			incorporate philosophy into their teaching methodologies.
	c. **Collaborative Learning:**	Encourage educators to collaborate and share experiences, methodologies, and best practices related to integrating philosophy into mathematics teaching. This teacher collaboration should be professionally facilitated.	Facilitated collaboration promotes a community of practice, enhancing the effectiveness of integration by allowing educators to learn from one another and share successful strategies.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
3 Pedagogical Tools and Classroom Resources:	a. **Development of Resources:**	Create and distribute pedagogical tools, such as multimedia materials, games, and videos, that effectively incorporate philosophical elements into mathematics lessons.	Providing diverse tools accommodates different learning styles and ensures that educators have engaging resources to reinforce philosophical integration in their teaching.
	b. **Resource Accessibility:**	Ensure easy access and availability of these resources to educators across various educational settings.	Accessibility is key to successful implementation. This ensures that educators, regardless of their location or teaching environment, can readily access the necessary materials to support philosophy-integrated teaching.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
4 Student Engagement and Learning Strategies:	a. **Promotion of Critical Thinking:**	Implement teaching methodologies that encourage critical thinking, problem-solving, and open-ended philosophical discussions within the classroom.	This action focuses on actively engaging students, promoting deeper understanding and application of philosophical concepts to enhance their critical thinking skills.
	b. **Learner-Centred Approaches:**	Foster an environment where students feel safe to explore philosophical questions related to mathematics, promoting deeper understanding and engagement.	A learner-centered approach ensures that students are active participants in their learning, fostering a positive environment for exploring philosophical dimensions.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
5 Assessment Strategies Beyond Recall:	a. **Diverse Assessment Methods:**	Develop diverse assessment tools that evaluate not only content recall but also critical thinking, logical reasoning, and the application of philosophical insights to mathematical problem-solving.	Diverse assessments align with the integrated curriculum, ensuring that students' understanding goes beyond memorization, emphasizing the practical application of philosophical insights to problem-solving.
	b. **Authentic Assessments:**	Include methods like philosophical reflection papers, case studies, debates, and collaborative projects to assess learners' deeper understanding and application of philosophical concepts in mathematics.	Authentic assessments mirror real-world applications, providing a comprehensive evaluation of students' ability to apply philosophical insights in practical scenarios.

Steps	Sub Steps	Action To Be Taken	Rationale
6 Continuous Research and Evaluation:	**a. Longitudinal Studies:**	Conduct ongoing research to evaluate the effectiveness of philosophy-integrated teaching methods on student learning outcomes over time.	Longitudinal studies provide insights into the sustained impact of the integrated approach, informing continuous improvement and adaptation based on evolving student needs.
	b. Adaptive Modifications:	Continuously assess and modify the integration strategies based on research findings and evolving educational needs.	Adaptability ensures that the curriculum remains responsive to emerging educational trends and continuously evolves to meet the changing needs of students and the educational landscape.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
7 Policy Integration and Advocacy:	**a. Policy Formulation:**	Advocate for policy reforms that recognize the importance of philosophical integration in mathematics education, influencing curriculum standards and teacher training programs.	Policy advocacy ensures that systemic changes support and sustain the integration of philosophy into mathematics education, fostering a conducive environment for implementation.
	b. Collaborative Synergy:	Encourage collaboration among policymakers, educators, philosophers, and researchers to implement effective policies and practices.	Collaborative synergy ensures that policies are informed by diverse perspectives, promoting effective implementation, and addressing the multifaceted aspects of curriculum reform.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
8 Holistic Approach and Collaborative Initiatives:	**a. Interdisciplinary Collaboration:**	Foster collaborations between mathematics and philosophy departments to create innovative teaching methodologies that bridge theoretical insights with practical applications.	Interdisciplinary collaboration enriches the curriculum by combining the strengths of mathematics and philosophy, offering students a holistic educational experience that transcends disciplinary boundaries.
	b. Stakeholder Engagement:	Engage diverse stakeholders (educators, researchers, policymakers) to contribute to a rich and inclusive educational ecosystem.	Stakeholder engagement ensures that a variety of perspectives are considered, enriching the curriculum reform process and fostering a sense of collective responsibility for its success.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
9 Monitoring and Feedback Mechanisms	**a. Regular Evaluation:**	Establish mechanisms to monitor and evaluate the implementation of philosophy-integrated teaching methods, considering feedback from educators, students, and other stakeholders.	Regular evaluation provides insights into the effectiveness of the integrated approach and allows for timely adjustments based on feedback, ensuring continuous improvement.
	b. Adaptive Response:	Use evaluation results to make adaptive changes and improvements in the integration process.	An adaptive response based on evaluation findings ensures that the curriculum remains responsive to the dynamic needs of educators, students, and the educational environment.

Steps	Sub Steps	Action To Be Taken	Rationale
10 Sustainability and Resource Development:	**a. Continuous Development:**	Continuously develop and update teaching resources and strategies to align with evolving educational trends and philosophies.	Continuous development ensures that teaching resources remain relevant and effective in the face of evolving educational
	b. Sustainability Measures:	Ensure the sustainability of philosophy-integrated teaching practices through institutional support, funding, and ongoing collaboration.	Sustainability measures, including institutional support and ongoing collaboration, are essential to ensure the longevity and continued success of philosophy-integrated teaching practices.
Steps	**Sub Steps**	**Action To Be Taken**	**Rationale**
11 Holistic Policy Implementation		This step emphasizes a holistic approach to policy implementation in education. It suggests integrating philosophical dimensions not only into teacher training programs but also into curriculum development and overall educational policy frameworks.	The purpose is to advocate for a comprehensive transformation in education policy that incorporates philosophical elements, leading to a more engaging and holistic mathematics education.

By following this roadmap, educational institutions and policymakers can effectively integrate the philosophy of mathematics into the mathematics classroom, fostering a more engaging, meaningful, and holistic learning experience for students.

5.7 Conclusion

In essence, these implications and recommendations converge to create a comprehensive roadmap for integrating philosophy of mathematics into mathematics education. By embracing these recommendations, South Africa's mathematics education system can evolve into a dynamic and inclusive landscape that nurtures critical thinkers, problem solvers, and learners who appreciate the intricate interplay between mathematics and philosophy. These recommendations, collectively, paint a vivid picture of a future where education resonates with holistic growth, intellectual curiosity, and innovation.

As we reflect on the journey of this study, it is imperative to underscore the potential that these recommendations hold for South Africa's mathematics education landscape. By embracing philosophical perspectives, educators can bridge the gap between theoretical understanding and practical application, resulting in pedagogical practices that resonate deeply with learners. The vision presented in this conclusion aligns with the insights of Prof. Loyiso Nongxa, Chairperson of the National Research Foundation NRF, envisions a mathematical community thriving with inclusivity and gender equity.

The challenges within South Africa's mathematics education system demand a comprehensive approach, and the recommendations presented here respond to this call. The collaborative initiatives championed by Prof Francesco Petruccione[23] reflect an avenue to invigorate learner interest and engagement through interdisciplinary approaches.

Integrating diverse graduates into educational institutions, as proposed by Prof Nongxa, addresses the escalating demand for qualified educators and underscores the urgency for proactive action.

As we envision the future, it becomes evident that fostering a vibrant mathematical community is essential. Initiatives such as the 'Wisaarkhu' magazine[24] championed by Dr. Sophie Marques exemplify creative efforts to reshape perceptions of mathematics while fostering a global network of mathematical thinkers. Collaborative initiatives with esteemed institutions like the African Institute of Mathematical Sciences (AIMS)[25] and the Centre of Excellence – Mathematical and Statistical Sciences (MASS)[26] attest to a dedicated commitment to nurturing mathematical talent and promoting skills development.
In a landscape characterized by urgency and complexity, the steadfast dedication of Stellenbosch University[27] and its partners is commendable.

[23] Department of Physics, Stellenbosch University, South Africa

[24] https://wisaarkhu.co.za/

[25] https://aims.ac.za/

[26] https://www.wits.ac.za/coe-mass/

[27] For further information, please refer to the press release issued by Stellenbosch University on 18 July 2023. The release highlights the critical issues and initiatives discussed within this conclusion and reflects the commitment of academia and stakeholders to shape the future landscape of mathematics education in South Africa. (Stellenbosch University Press Release, 18 July 2023)

This commitment encompasses enhancing mathematical education, embracing diversity, and nurturing the mathematical talent that will shape the trajectory of the discipline in the coming decades.

Finally, this study has not only shed light on the integration of philosophy in mathematics education but has also provided actionable recommendations that resonate with the voices of scholars and practitioners alike. By heeding these recommendations and infusing philosophical perspectives into mathematics education, South Africa paves the way for a future where mathematics becomes a beacon of enlightenment, innovation, and societal progress. Through collaborative efforts and unwavering dedication, the mathematics education landscape in South Africa possesses the potential to transcend challenges and catalyse a brighter future for generations to come.

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ANNEXURES

ANNEXURE A: UP Research Ethics Committee Approval

Faculty of Humanities
Fakulteit Geesteswetenskappe
Lefapha la Bomotho

Humanities 100.
Since 1919

09 September 2022

Dear Mrs IM Schreiber

Project Title:	Investigating the role of philosophy of mathematics as the bridge to meaningful mathematics teaching in the classroom
Researcher:	Mrs IM Schreiber
Supervisor(s):	Prof EB Ruttkamp-Bloem
Department:	Philosophy
Reference number:	22958119 (HUM007/0722)
Degree:	Masters

I have pleasure in informing you that the above application was **approved** by the Research Ethics Committee on 09 September 2022. Please note that before research can commence all other approvals must have been received.

Please note that this approval is based on the assumption that the research will be carried out along the lines laid out in the proposal. Should the actual research depart significantly from the proposed research, it will be necessary to apply for a new research approval and ethical clearance.

We wish you success with the project.

Sincerely,

Prof Karen Harris
Chair: Research Ethics Committee
Faculty of Humanities
UNIVERSITY OF PRETORIA
e-mail: tracey.andrew@up.ac.za

Room 7-27, Humanities Building, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa
Tel +27 (0)12 420 4853 | Fax +27 (0)12 420 4501 | Email pghumanities@up.ac.za | www.up.ac.za/faculty-of-humanities

ANNEXURE B1: Gauteng Province Department of Education – Research Approval

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	**02 June 2022**
Validity of Research Approval:	**08 February 2022– 30 September 2022** **2022/214**
Name of Researcher:	**Schreiber IM**
Address of Researcher:	**129 Quintus van der Walth Drive**
	Norkem Park
	Kempton park
Telephone Number:	**084 605 8224**
Email address:	**Oortjies.piexie@gmail.com**
Research Topic:	**Investigating the role of Philosophy of Mathematics as the bridge to meaningful mathematics teaching in the classroom**
Type of qualification	**MA Philosophy**
Number and type of schools:	**Primary Schools and Secondary Schools**
District/s/HO	**Ekurhuleni North**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

02/06/2022

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below are met. Approval may be withdrawn should any of the conditions listed below be flouted:

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

ANNEXURE B2: Gauteng Province Department of Education – Letter of Endorsement

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

EKURHULENI NORTH DISTRCIT

Enquiries: Kagiso Mogaswane (T) 011 746 – 8295 (E) Ramabowa.Mogaswane @gauteng.gov.za
Sub-Directorate: Information Systems and Strategic Planning

TO : **SCHREIBER IM**

FROM : **MRS N.P. NTUTA**
DISTRICT DIRECTOR

DATE : **25TH AUGUST 2022**

SUBJECT : **LETTER OF ENDORSEMENT TO CONDUCT MASTERS DEGREE RESEARCH ON PRIMARY AND SECONDARY SCHOOLS UNDER EKURHULENI NORTH DISTRICT OFFICE**

The Ekurhuleni North District is situated at 78 Howard Avenue, Munpen Building under the Gauteng Department of Education. The district office is servicing a total 284 both public ordinary and independent (some state subsidised) schools.

This letter serves as an endorsement for the approval by Gauteng Department of Education to Schreiber IM to conduct research. The Ekurhuleni North District office hereby endorses permission granted by GDE for the said research to be conducted as per the pre-scripts of Gauteng Department of Education: Education Research and Knowledge Management.

This research study is for a Masters degree in Philosophy valid from 08th February 2022 to 30th September 2022**(2022/214)** to be conducted in 18 primary and 08 secondary schools within the Ekurhuleni North district. **See attached list of schools**.

The research topic is **"Investigating the role of Philosophy of Mathematics as the bridge to meaningful mathematics reaching in the classroom"**.

Kind regards.

MRS NP NTUTA
DISTRICT DIRECTOR: EKURHULENI NORTH

Making education a societal priority
Office of the District Director: Ekurhuleni North
Munpen Building, 78 Howard Avenue, Benoni, 1500
Private Bag X 059, Benoni, 1500
Tel: (011) 746-8000 Fax: (011) 746-8027/70
Website: www.education.gpg.gov.za

ANNEXURE C1: Letter to Headmaster – Introduction and Permission to Conduct Research.

Consent to Participate in Research:
Investigating the role of philosophy of mathematics as the bridge to meaningful mathematics teaching in the classroom.
Page 1 of 2

Letter of Introduction and Permission to conduct Research

Research Question:

Could Philosophy of Mathematics be the bridge to meaningful teaching in the mathematics classroom?

Researchers:

I M Schreiber

Supervisor:

Prof E. Ruttkamp-Bloem, Department of Philosophy, University of Pretoria

1 Introduction and Explanation of the Study:

The researcher is conducting this study to learn more about the research question posed above as it has not been studied much in the past in a South African context.

A letter of consent to participate will be given to each educator. This study will be submitted for approval to the Research Ethics Committee of the Faculty of Humanities, University of Pretoria.

The Gauteng Department of Education has given consent for the research. Please see attached letter.

If you agree to this research taking place at your school, you will be asked to invite Mathematics Educators for Grades 7, 8 and 9 and their HOD's to a presentation to introduce and contextualize the research. During the presentation they may ask any relevant questions to clarify issues of concern. (30min) Then they will be invited to complete an online questionnaire to establish their own views on the issues involved. Participation in the questionnaire will be voluntary and anonymous. (30min) *(A link to the questionnaire will be provided at the presentation.)*

- Only Mathematics HOD and Mathematics Educators for Grades 7, 8 and 9 will take part. However, should you wish to involve other educators you are welcome to invite them.
- The entire session should not be more than 1 hour.
- The research is to take place during the 3rd school term (from 1 July to 30 September as stipulated by the Department of Education).
- Date and times will be negotiated.

2 Confidentiality:

All information collected will be confidential and will only be used for research purposes. Only the researcher and supervisor will have access to the study data and information.

No names or identifying information will be collected throughout the study.

The researcher will safely keep all information and data collected on UP's server in a password-protected manner.

Researcher will keep the abovementioned information for five years after the publication of this research, after which, all files will be destroyed.

3 Risks and Benefits:

There are no risks for participants involved in this research. The presentation before-hand will familiarize participants with the topic of research which will help break the ice and minimize any anxiety.

The anticipated benefit of participation is the opportunity to discuss feelings, perceptions, and concerns related to the philosophy of mathematics and mathematics education, and to contribute to greater understanding of the subject matter of the study.

4 Voluntary Participation:

Educator's participation is entirely voluntary.

5 Further Questions and Follow-Up

You are welcome to ask the researchers any questions that occur to you before and during the research.

Isabel Schreiber
oortjies.piexie@gmail.com
+27 84 605 8224

6 Permission Granted

Signature: ……………………………………………

Name: ……………………………………………

Date: ……………………………………………

ANNEXURE C2: Letter to Teachers – Introduction and Permission to Conduct Research

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Humanities
Fakulteit Geesteswetenskappe
Lefapha la Bomotho

Department of Philosophy

Humanities 100
Since 1919

Letter of Introduction and Permission to conduct Research

Study:
Investigating the role of philosophy of mathematics as the bridge to meaningful mathematics teaching in the classroom.

Research Question:
Could the philosophy of mathematics provide a space within which well-designed tasks can promote class discussions, and where hidden structures, connections, and characteristics, can be discovered by the learners in the process of learning.

Researchers:
I M Schreiber

Supervisor:
Prof E. Ruttkamp-Bloem, Department of Philosophy, University of Pretoria

1 Introduction and Explanation of the Study:

The researcher is conducting this study to learn more about the research question posed above as it has not been studied much in the past in a South African context.

This study will be submitted for approval to the Research Ethics Committee of the Faculty of Humanities, University of Pretoria.

The Gauteng Department of Education has given consent for the research. Please see attached letter.

Department of Philosophy
Faculty of Humanities, University of Pretoria
Room 20-02, Level 20, Humanities Building
Tel: +27 (0)12 420 3039
Email: emma.ruttkamp-bloem@up.ac.za
www.up.ac.za
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa

If the headmaster agrees to this research taking place at your school, they will be asked to invite Mathematics Educators for Grades 7, 8 and 9 and their HOD's to a presentation to introduce and contextualize the research. During the presentation you may ask any relevant questions to clarify issues of concern. (30min)

A letter of consent to participate will be given to each educator. Then you will be invited to complete an online questionnaire to establish their own views on the issues involved. Participation in the questionnaire will be voluntary and anonymous. (30min)

A link to the questionnaire will be provided at the presentation to everyone who has signed the letters of consent.

- The entire session should not be more than 1 hour.
- The research is to take place during the 3rd school term (from 1 July to 30 September as stipulated by the Department of Education).
- Times will be negotiated.

2 Confidentiality:

All information collected will be confidential and will only be used for research purposes. Only the researcher and supervisor will have access to the study data and information.

No names or identifying information will be collected throughout the study. No data will be re-used without seeking further consent, except in the case of a possible publication of an article in a peer-reviewed journal.

The researcher will safely keep all information and data collected on UP's server in a password-protected manner.

Researcher will keep the abovementioned information for 15 years after the publication of this research, after which, all files will be destroyed.

3 Risks and Benefits:

There are no risks for participants involved in this research. The presentation before-hand will familiarize participants with the topic of research which will help break the ice and minimize any anxiety.

The anticipated benefit of participation is the opportunity to discuss feelings, perceptions, and concerns related to the philosophy of mathematics and mathematics education, and to contribute to greater understanding of the subject matter of the study.

4 Voluntary Participation:

Educator's participation is entirely voluntary.

5 Further Questions and Follow-Up

You are welcome to ask the researchers any questions that occur to you before and during the research.

Signed:

Signed:

Isabel Schreiber
portiles.piexie@gmail.com
+27 84 605 8224

Emma Ruttkamp-Bloem
emma.ruttkamp-bloem@up.ac.za
012 420 3039

6 Permission Granted PLEASE ADD DATE STAMP from SCHOOL to this letter or write a permission note on a school letterhead and attach to this letter.

Signature: ....

Name: ....

Date: ....

ANNEXURE D: Lesson Example

Ground Floor: Mastering Whole Numbers - Lesson on Number Systems

1 Introduction

Welcome to the captivating world of number systems!

Today, we embark on an exhilarating journey to explore the captivating realm of numbers and their diverse representations. Prepare yourself to dive deep into the intricacies of decimal, binary, octal, and hexadecimal number systems, as we unveil their mysteries and grasp their significance in both our mathematical and digital worlds.

2 Classification of Number Systems

Number systems, like the pieces of a puzzle, come in various shapes and sizes. They are crafted based on their unique base or radix. Think of the base as the cornerstone of a number system, shaping its structure and characteristics.

Let's closely examine the four major types of number systems:

- **Binary (base 2):** In the binary realm, only two digits are at play: 0 and 1. This simplicity grants binary an advantage when dealing with computers. It serves as the language of machines, with each 1 or 0 representing an ON or OFF state. Can you deduce the base of binary? Correct, it's 2!

- **Decimal (base 10):** Ah, the familiar decimal number system. It's the base 10 system that we encounter daily. With ten digits from 0 to 9, it enables us to express any number, no matter its size. But have you ever pondered why it's termed "decimal"? It's because the base, or radix, of this system is 10.

- **Octal (base 8):** Octal, as its name implies, revolves around groups of eight. Comprising digits 0 through 7, the octal number system offers a concise and organized representation. Its base is 8, making it ideal for handling arrays of numbers. Can you visualize its utility when working with computers?

- **Hexadecimal (base 16):** Brace yourself for the hexadecimal number system, where numbers collaborate with alphabets. Hexadecimal includes 10 digits (0-9) and six additional alphabets (A, B, C, D, E, F). This system finds extensive use in computer systems, especially microprocessors and microcontrollers. Any guesses regarding the base of hexadecimal? You're right, it's 16!

3 Key Components for Determining Digit Value

Now that we comprehend the various number systems, let's demystify the process of determining the value of a digit within them. Three essential components come into play:

1. **The digit itself:** Each number system possesses its own set of digits. In binary, it's 0 and 1; in decimal, it's 0 to 9; in octal, it's 0 to 7; and in hexadecimal, it's 0 to 9 and A to F.

2. **Position of the digit within the number:** The position (or place) of a digit within a number holds immense significance. It defines its place value, signifying its magnitude.

3. **Base of the number system:** The base of a number system reveals the number of available digits for expressing any digit in a specific place value. Understanding the base is crucial for deciphering and converting numbers across different systems.

Let's now delve deeper into each number system and explore their distinct attributes.

4 Decimal Number System

The dependable decimal number system comprises 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Its base is 10, reflecting the quantity of available digits. However, there's more beneath the surface. Every digit in a decimal number carries two critical attributes: the face value and the place value.

The face value is the digit itself. For instance, in the number 7896, the face value of 8 is 8. Yet, the true magic lies within the place value, which represents the digit's magnitude. In the same number, the place value of 8 is 100. Thus, the face value of 8 remains 8, but its place value within the number 7896 is 800!

5 Expansion of Decimal Number

Let's take our number, 7896, and deconstruct it using the power of 10:
7896 = 1000 x 7 + 100 x 8 + 10 x 9 + 6

To denote a number in terms of base 10, we can utilize the positional value as a superscript of 10:
7896 = (10^3 x 7) + (10^2 x 8) + (10^1 x 9) + (10^0 x 6)

6 Binary Number System

Now, let's step into the domain of binary, where everything revolves around 0 and 1. The binary system's base is 2, as it solely employs these two digits. Though it might appear straightforward, binary serves as the language of computers, forming the basis of their operations.

In the binary system, each 1 or 0 is referred to as a bit. Bits function as the building blocks of information processing. A group of 4 bits constitutes a nibble, and 8 bits form a byte. Bytes

hold significance in binary, as processors are tailored to operate with specific bit lengths: 8, 16, 32, 64, and so forth.
To convert binary to decimal, we multiply the binary digits by the power of 2 corresponding to their positional value. Let's consider an example: 10011 = (2^4 x 1) + (2^3 x 0) + (2^2 x 0) + (2^1 x 1) + (2^0 x 1) = 16 + 2 + 1 = 19. Hence, 10011_2 in binary equals 19_{10} in decimal.

Practice Activity 1

Pair up with a friend and convert the following binary numbers to decimal:
• 111_2
• 10101_2

7 Octal Number System

Now, let's delve into the realm of octal numbers. Octal employs a set of 8 digits: 0, 1, 2, 3, 4, 5, 6, and 7. With a base of 8, it offers a more compact representation compared to binary.

To convert octal to decimal, we multiply each digit by 8 raised to the power of its positional value. Let's take an example: 431 = (8^2 x 4) + (8^1 x 3) + (8^0 x 1) = (64 x 4) + (8 x 3) + (1 x 1) = 256 + 24 + 1 = 281. Therefore, 431_8 in octal is equivalent to 281_{10} in decimal.

Practice Activity 2

Try converting the following octal numbers to decimal:
• 63_8
• 352_8

8 Hexadecimal Number System

Prepare to unlock the potential of hexadecimal! It amalgamates 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) with 6 alphabets (A (10), B (11), C (12), D (13), E (14), F (15)). The alphabets signify values 10 to 15. With a base of 16, hexadecimal finds extensive application in microprocessors and microcontrollers.

To convert hexadecimal to decimal, we multiply each digit by 16 raised to the power of its positional value. Consider this example: 5B52 = (16^3 x 5) + (16^2 x B) + (16^1 x 5) + (16^0 x 2) = (4096 x 5) + (256 x 11) + (16 x 5) + (1 x 2) = 20480 + 2816 + 80 + 2 = 23378. Thus, $5B52_{16}$ in hexadecimal is equal to 23378_{10} in decimal.

Practice Activity 3

Try converting the following hexadecimal numbers to decimal:
• $10D_{16}$
• 64_{16}
• $2A_{16}$

9 Significance of Number Systems

Number systems serve as the foundation of both mathematics and digital systems. In the realm of digital systems, binary, octal, and hexadecimal numbers function as inputs to generate powerful outputs. Grasping number systems is pivotal for comprehending the operations of information technology.

10 Discussion Questions

Within your respective groups, explore the following:

1. In the base ten number system, place values take the form of powers of ten: 10^0 (units), 10^1 (tens), 10^2 (hundreds), 10^3 (thousands), and so forth. How would you describe the place values in binary, octal, and hexadecimal number systems?
2. Apart from the base ten number system, which of the other number systems do you find the easiest to understand and use? Explain your reasoning.
3. Why does any number to the power of 0 equal 1? For example: $10^0 = 1$; $2^0 = 1$; $8^0 = 1$; $16^0 = 1$.

11 Creating Your Own Number System

Crafting your own base number system can be an engaging and creative exercise. Here's an example of a base-6 number system:
• Base: 6
• Digits: 0, 1, 2, 3, 4, 5 (total of six digits)
In this system, digits 0 to 5 represent the same numerical values as in the decimal system.

However, with a base of 6, each digit's positional value increases by a factor of 6 as you move left.

Examples within this number system:
• 34 in base-6: Represents 3 groups of 6 and 4 units. In decimal notation, it's equivalent to 22 (6^1 x 3) + (6^0 x 4).
• 105 in base-6: Represents 1 group of 36, 0 groups of 6, and 5 units. In decimal notation, it's equivalent to 41 (6^2 x 1) + (6^1 x 0) + (6^0 x 5).
• 520 in base-6: Represents 5 groups of 36, 2 groups of 6, and 0 units. In decimal notation, it's equivalent to 1,500 (6^2 x 5) + (6^1 x 2) + (6^0 x 0) - (36 x 5) + (6 x 2) + (1 x 0).

Creating your base number system allows you to explore different positional values and arithmetic operations within that system. It's an enjoyable way to comprehend the workings of various bases and expand your mathematical thinking.

12 Challenge

Having learned to convert from binary, octal, and hexadecimal to decimal, it's time to reverse the process. Convert the decimal number 124_{10} to binary, octal, and hexadecimal number systems.

HINT: To convert from decimal to binary, octal, or hexadecimal, you'll need to divide the decimal number by the base and keep track of remainders. Can you crack the code?

Now it's your turn to infuse life into these vibrant number systems through your enthusiasm and exploration. Delight in unravelling the secrets of numbers and their incredible representations!

13 Discussion Points: Connections to Other Concepts in Mathematics

1. **BODMAS and Number System Conversions**: BODMAS can be applied when performing operations during number system conversions. For instance, when converting numbers from one base to another, you might need to perform addition, subtraction, multiplication, or division. Applying the BODMAS rules ensures the correct order of operations is followed, allowing for accurate conversions.
2. **Exponents and Number System Bases**: Exponents play a role in understanding the positional value of digits in different number systems. For example, in the decimal system, each digit's value is determined by multiplying it with a power of 10, where the exponent represents the position of the digit. Similarly, in other number systems like binary or hexadecimal, the base of the system determines the exponent used for positional values.
3. **Place Value and Number System Bases**: Place value is closely tied to the base of a number system. In the decimal system, the base is 10, and each digit's value depends on its position (ones, tens, hundreds, etc.). Similarly, in binary, the base is 2, and each digit's value depends on its position (ones, twos, fours, etc.). Understanding place value is crucial when converting between number systems as it helps determine the correct positional values in the target system.

By leveraging these connections, you can effectively perform operations, conversions, and understand the positional values of digits within different number systems. These concepts provide a foundation for working with numbers across various bases and performing calculations within those systems.

14 Challenge Memorandum:

Upon completion, learners will receive the challenge memorandum to review and discuss their own work:

To convert 124 from decimal to binary, we divide the decimal number by 2 repeatedly and record the remainders. Here's the step-by-step process:

124 ÷ 2 = 62, remainder 0 → 62 ÷ 2 = 31, remainder 0 → 31 ÷ 2 = 15, remainder 1 → 15 ÷ 2 = 7, remainder 1 → 7 ÷ 2 = 3, remainder 1 → 3 ÷ 2 = 1, remainder 1 → 1 ÷ 2 = 0, remainder 1

By reading the remainders from the last division to the first, we obtain the binary representation of 124, which is 1111100.

To convert 124 from decimal to octal, we divide the decimal number by 8 repeatedly and record the remainders. Here's the process:

124 ÷ 8 = 15, remainder 4 → 15 ÷ 8 = 1, remainder 7 → 1 ÷ 8 = 0, remainder 1
Reading the remainders from the last division to the first, we get the octal representation of 124, which is 174.

To convert 124 from decimal to hexadecimal, we divide the decimal number by 16 repeatedly and record the remainders. Here's the process:

124 ÷ 16 = 7, remainder 12 (C in hexadecimal) → 7 ÷ 16 = 0, remainder 7

Reading the remainders from the last division to the first, we arrive at the hexadecimal representation of 124, which is 7C.

15 End of Lesson

ANNEXURE E1: Mathematics Teacher Questionnaire[28]

NOTE: *There are no "**right**" or "**wrong**" answers to any of these items. The questionnaire is designed to **provide information** about teachers' professional experiences, opinions, and classroom activities.*

REASONS FOR TYPES QUESTIONS: *The online anonymous questionnaire was designed to collect responses anonymously and included a range of questions, each serving a specific purpose aligned with the research objectives. The questionnaire encompassed various types of variables, including descriptive, comparative, and attitudinal variables.*

- *Descriptive variables, such as age, gender, educational qualifications, and teaching experience, were included to provide an overview of the demographic composition of the participants. These variables help to contextualize the responses and understand how different backgrounds might influence perspectives.*
- *Comparative variables were employed through close-ended questions that asked participants to make choices between predefined options. For instance, inquiries about teaching methodologies used, technology integration, and preferred communication tools yielded comparative data, enabling the identification of prevalent practices and trends.*
- *Attitudinal variables were captured through Likert-scale questions, where participants were asked to indicate their level of agreement with statements related to teaching challenges, job satisfaction, or opinions on specific educational approaches. These items provided a measure of participants' attitudes and allowed for quantitative analysis of sentiments.*
- *Open-ended questions were also incorporated to collect qualitative data, encouraging participants to provide detailed insights, suggestions, or personal experiences related to the research topic. These responses enriched the understanding of the underlying factors influencing various teaching practices and attitudes.*

In summary, the questionnaire design encompassed a well-rounded mix of descriptive, comparative, and attitudinal variables, as well as both quantitative and qualitative data collection methods. This comprehensive approach ensured that the study gathered a diverse range of information to address the research objectives thoroughly.

SECTION A: BACKGROUND INFORMATION

*(**Demographic Questions**: The purpose of Section A in this questionnaire is to gather essential demographic information about the teachers participating in the study. This section aims to provide context to the survey responses by capturing key characteristics such as age, gender, educational qualifications, teaching experience, and possibly other relevant factors like school location or subject specialization. Understanding the diverse backgrounds of the teachers involved can help in analysing how different demographics might influence their perspectives, experiences, and responses in the subsequent sections of the questionnaire. This information ensures that the findings of the study are interpreted with a nuanced understanding of the teachers' backgrounds, contributing to a more comprehensive and insightful analysis.)*

1 How old are you? *Tick the correct box.*
(Multiple choice question, descriptive variable: age.)

Under 25	
25-29	
30-39	
40-49	
50-59	
60 or more.	

[28] *Adapted from a TIMSS 2019 (International Results in Mathematics and Science) questionnaire.*

2 **By the end of this school year, how many years will you have been teaching altogether?**
Please round to the nearest Whole Number. ……………………………………
(Descriptive variable: number of years in teaching.)

3 **What is *highest level* of formal education you have completed?**
(Comparative variable: highest level of formal education.)

i. Certificate/Diploma in education ……………………………………
ii. Bachelor of Education ……………………………………
iii. Honours in Education ……………………………………
iv. Master's in education ……………………………………
v. Other ……………………………………

4 **While studying to obtain your bachelor's degree or equivalent, what was your *major* or *main area of study*?** *Tick one box in each row.*
(Comparative variable major/main area of study.)

	Yes	*No*
Mathematics		
Maths Literacy		
Biology		
Natural Science		
Chemistry		
Technology		
Social Sciences (Geography)		
Social Sciences (History)		
Information Technology		
Languages e.g., English, Afrikaans etc		
Other		

5 **If there are any other subjects other than those mentioned in QUESTION 4, please state below what the subjects are.**
……………………………………

6 **Did you have any exposure to the Philosophy of Mathematics during your training** *(NOTE: Not the same as Philosophy of Teaching)***?**

(It's important to clarify the distinction between these two concepts, as they are distinct areas of study. The note provides clarity and context for the respondents, ensuring that they understand the specific focus of the question and why it is different from a related but separate concept. This manner of questioning will likely lead to more accurate and meaningful responses.)

Yes	*No*

SECTION B: TEACHER'S VIEWS OF MATHEMATICS AS A DISCIPLINE

This section of the questionnaire aims to delve into teachers' perspectives on mathematics as a subject, their teaching methods, and their personal beliefs about the nature of mathematics and its role in education. The questions are designed to gather insights into the participants' understanding of mathematics, their teaching approaches, and their attitudes towards various aspects of the subject.

7 Should Mathematics be taught alongside Natural Science as twin disciplines?

This question explores whether participants believe that Mathematics should be taught in conjunction with Natural Science. It takes note of the fact that teaching assignments change as learners' progress through grade levels e.g., Grade 4 teacher will teach all subjects versus Grade 7 where different teachers will teach mathematics and science, is a and aims to understand the teachers' perspectives on interdisciplinary teaching approaches.

……………………………………………………………………………………

8 Do you think mathematics is different to any other subject in terms of what it has to offer? Why or why not?

An open-ended qualitative question that investigates whether participants consider Mathematics to be distinct from other subjects and why. This question seeks to uncover the teachers' perspectives on the unique qualities of Mathematics as a discipline.

……………………………………………………………………………………

9 How do you think mathematics should be taught?

i)	teach some background information on History of Mathematics	*Yes / No*
ii)	brake concepts down into bite sizes *(reductionism)*	*Yes / No*
iii)	highlight connections and links between concepts	*Yes / No*
iv)	scaffolding	*Yes / No*

10 With regards to QUESTION 9: Please list any other methods you may have in mind or practice e.g., Realistic Mathematics Education (RME), Maths in Context (MiC), Critical Thinking etc.

……………………………………………………………………………………

This question presents a set of teaching methods and asks participants to indicate whether they employ any of these methods. It provides a range of options for instructional approaches and encourages teachers to reflect on their practices, including their use of historical context, reductionism, concept connections, and scaffolding.

11 To what extent do you agree or disagree with each of the following statements?
Tick one box in each row.

	Strongly Disagree	*Disagree*	*Agree*	*Strongly Agree*
a) Mathematics is ***primarily*** an abstract subject.				
b) Mathematics is *primarily* a formal way of representing the real world.				
c) Mathematics is *primarily* a practical and structured guide for addressing real situations.				
d) If learners are having difficulty, an effective approach is to give them *more practice* by themselves during the class.				
e) Some learners have a *natural talent* for mathematics and others do not.				
f) More than one representation *(picture, concrete material, symbol set, etc.)* should be used in teaching a mathematics topic.				
g) Mathematics should be learned as *sets of algorithms* or *rules* that cover all possibilities.				
h) Basic computational skills *(add, subtract, multiply, divide)* the part of the teacher *is sufficient* for teaching elementary school mathematics.				
i) An *understanding* of the origins of concepts and Philosophies of Mathematics is essential to teach mathematics.				
j) An *understanding* of the fundamental concepts of mathematics and their connections.				

Utilizing a Likert-scale format, this question prompts participants to express their level of agreement or disagreement with statements concerning the nature of Mathematics, its representation, and effective teaching methods.

12 Is mathematics logical?

………………………………………………………………………………………

13 Was mathematics *discovered* or *invented* by humans?

………………………………………………………………………………………

14 How do you, as a teacher, experience mathematics e.g., is it practical, is it procedural, is it interesting etc?

………………………………………………………………………………………

15 What do you understand of the difference between pure mathematics and applied mathematics?

………………………………………………………………………………………

16 Could we know about the physical world without mathematics? Why or why not?

……………………………………………………………………………………

17 Why do you think some of the greatest philosophers have been mathematicians?

……………………………………………………………………………………

18 Do you think understanding of mathematical concepts and their fundamental connections is essential to successful mathematics teaching?

……………………………………………………………………………………

19 When creating a <u>'realistic' mathematics lesson</u>, teachers should be careful that it's not done at the cost of <u>delivering authentic mathematics</u>. *(Many <u>mathematical ideas</u> are unlikely to evolve <u>outside</u> of mathematically rich environments i.e., use correct and relevant vocabulary, rules, formulae etc.).*

What do you understand by the above statement: Do you agree or disagree?

……………………………………………………………………………………

20 How important is visualization in teaching mathematics?

……………………………………………………………………………………

21 What is the difference between a misconception and a mistake in mathematics? Give some examples from your own experience.

……………………………………………………………………………………

22 What is the rationale for picking out certain elements of mathematics for schooling?

……………………………………………………………………………………

23 What are YOUR Beliefs and Perceptions Regarding Math?

Are the statements about mathematics ability in the boxes below True or False? Read each statement and circle your selection.

a) Some learners are born with the ability to do mathematics whereas others are not. **T / F**
b) High mathematics ability means being able to calculate with accuracy. **T / F**
c) Mathematics requires logical rather than creative thinking. **T / F**
d) Reading skills are important for success in mathematics. **T / F**
e) The main objective in mathematics is to obtain the correct answer. **T / F**
f) In addition to knowing basic mathematic facts, learners need to understand the underlying concepts of the skill they are learning. **T / F**
g) Mathematics teachers should know how to use effective teaching practices and have knowledge of the mathematics they are teaching. **T / F**
h) Males are better at mathematics. **T / F**
i) Transformation to less conventional methods of teaching results in fear and reluctance from teachers, who find the change hard and risky. **T / F**
j) The interactive classroom must be aware of the mathematical concept being taught and not the ability of using technology as the novelty to be learned. **T / F**
k) Technology should be treated as part of the learning process but not the process itself. **T / F**
l) Learner-centred teaching is not about putting the learner in the centre but encourages teachers to put *learners' learning in the centre*. **T / F**
m) Many teachers are still using conventional teaching and have noted that in conventional teaching classrooms, while they are explaining and writing on the board, some learners will be copying the same thing onto their notes, some daydreaming, and some sleeping. **T / F**
n) In some cases, conventional teaching constrains creative thinking. **T / F**
o) Conventional thinking seldom considers individual differences amongst learners. **T / F**
p) Do you believe that the teaching and learning process can be reinvented *e.g., introduction of learner cantered lessons, Information Technology etc.,* because learners are well prepared in this current information age? **T / F**
q) It is greatly important to understand the *roles* which can be played by *teachers* and *learners* in the learner cantered classroom. **T / F**
r) An ineffective internet lessons is as hopeless as an ineffective worksheet lesson. **T / F**

Questions 10-24: These questions delve into various aspects of participants' beliefs and perceptions about mathematics, pedagogy, and teaching practices. They encompass True/False statements, open-ended queries, and Likert-scale items. These questions explore participants' thoughts on mathematics ability, teaching practices, the role of technology, learner-centred teaching, and the relationship between conventional and innovative methods.

Each question in this section serves a unique purpose, ranging from gauging teachers' beliefs about mathematical concepts and teaching methods to understanding their views on interdisciplinary education and the role of technology. The mix of question types - qualitative, close-ended, Likert-scale, and True/False - ensures a comprehensive exploration of teachers' perspectives, experiences, and philosophies related to the discipline of mathematics.

SECTION C: CLASSROOM PRACTICES

This section of the questionnaire focuses on understanding teachers' classroom practices related to teaching mathematics. It seeks to gather information about the number of teaching periods, the subjects taught, teachers' preparedness for various topics, their influence over curriculum decisions, the use of teaching materials, and the integration of technology.

24 In one typical calendar week from Monday to Sunday, for how many periods are you formally scheduled to teach each of the following subjects? *Write zero if none.*

This question inquiries about the number of periods teachers are scheduled to teach various subjects in a typical week. It provides insight into teachers' workload distribution across mathematics, natural science, and technology.

	Number of Single Periods	*Number of Double Periods*	*Grade*
a) Mathematics	_______	_______	_______
b) Natural Science	_______	_______	_______
c) Technology	_______	_______	_______

25 To be good at mathematics at school, how important do you think it is for learners to: *Tick one box in each row.*

A Likert-scale question that assesses teachers' perceptions of the importance of different attributes for learners' success in mathematics. It covers aspects such as memory, creative thinking, understanding concepts, and real-world application.

	Not Important	*Somewhat Important*	*Very Important*
a) remember formulas and procedures			
b) think in a sequential and procedural manner			
c) understand mathematical concepts, principles, and strategies			
d) be able to think creatively			
e) understand how mathematics is used in the 'real world'.			
f) be able to provide reasons to support their solutions.			
g) have some knowledge of the origins of mathematic concepts			
h) have knowledge of fundamental mathematical concepts and their links			

26 How well prepared do you feel you are to teach? *Tick one box in each row.*

Teachers rate their level of preparedness to teach various mathematical topics, offering insights into their self-assessment of their content knowledge and teaching readiness.

	I do not teach these topics	*Not well prepared*	*Somewhat Prepared*	*Very well Prepared*
a) Fractions, Decimals, and Percentages				
b) Ratios and Proportions				
c) Whole Numbers				
d) Exponents				
e) Measurement – *units, instruments, and accuracy*.				
f) Perimeter, Area, Capacity, and Volume				
g) Geometric Figures – definitions and properties				
h) Geometric Figures – symmetry, motions and transformations, congruence, and similarity.				
i) Coordinate Geometry				
j) Algebraic Representation				
k) Evaluate and perform operations on Algebraic Expressions				
l) Solving Linear Equations and Inequalities				
m) Solving Quadratic Equations				
n) Solving Functions				
o) Solving Numeric and Geometric Patterns				
p) Representation and Interpretation of Data in Graphs, Charts, and Tables				
q) Simple Probabilities – understanding and calculations.				
r) Financial Mathematics				

27 How much influence do you have on each of the following:
Tick one box in each row.

Teachers indicate the extent of their influence over subject matter, textbook choices, resource budgeting, and purchasing decisions, revealing their involvement in shaping the classroom environment.

	None	*Little*	*Some*	*A Lot*
a) subject matter to be taught				
b) specific textbooks to be used				
c) the amount of money to be spent on resources				
d) what resources are purchased				

28 How many learners are in your mathematics class?

This question gathers information on class size, providing insights into classroom dynamics and demographics.

Write in a number for each. Write 0 (zero) if there are none.

Boys	*Girls*

29 What concepts do you emphasize most in your mathematics class?
Tick one box only.

Teachers choose the mathematical concepts they emphasize the most in their classrooms, giving a glimpse into their teaching priorities, whether it's general mathematics, geometry, algebra, or other topics.

i.	general mathematics *(e.g., Whole Numbers, Fractions, Decimals, Percentages, BODMAS, Exponents, Operations etc.)*	
ii.	geometry	
iii.	algebra	
iv.	combined algebra and geometry	
v.	combined algebra, geometry, numbers, etc.	
vi.	other, please specify:	

30 If there are other concepts than the ones mentioned in QUESTION 29, please specify.

....

31 Do you use a textbook in teaching mathematics to your class?

Yes	*No*

If yes, what textbook(s) do you use:

....

Questions 30 and 31: *Teachers can specify additional concepts they emphasize and whether they use textbooks in their teaching.*

32 Do you create your own teaching material?

Yes	*No*

33 Do you create your own assessment material?

Yes	*No*

Questions 32 and 33: *These questions inquire whether teachers create their own teaching and assessment materials, reflecting their autonomy in instructional design.*

34 To what extent are the learners in your mathematics class permitted to use calculators during mathematics lessons?

Tick one box only.

		Yes	*No*
i.	unrestricted use		
ii.	restricted use		
iii.	calculators are not permitted		

Teachers express the extent to which they permit calculator usage during mathematics lessons, providing insights into their stance on technology integration.

35 Do the learners in your mathematics class have computers available to use during mathematics lessons?
Tick one box in each row.

		Never or almost never	*Some lessons*	*Most lessons*	*Every lesson*
i)	in the classroom				
ii)	in other instructional rooms (computer labs, science lab, reading lab, library, etc.)				

36 If computers are available …

		Yes	*No*
iii)	do any of the computers have access to the Internet?		
iv)	do you use the Internet for instructional/educational purposes?		

Questions 35 and 36: *These questions explore the availability of computers in classrooms, their internet access, and teachers' utilization of online resources.*

37 In your mathematics lessons, how often do you usually ask learners to do the following?
Tick one box in each row.

		Never or almost never	*Some lessons*	*Most lessons*	*Every lesson*
i.	*explain* the reasoning behind an idea				
ii.	*represent* and *analyse* relationships using tables, charts, or graphs				
iii.	*work on problems* for which there is no immediately obvious method of solution *(perseverance, productive struggles)*				
iv.	*use* computers to solve exercises or problems				
v.	*write* equations to *represent* relationships				
vi.	*practice* computational skills				

vii.	*use* graphing calculators to solve exercises or problems				
viii.	*groupwork e.g., solving quizzes etc*				
ix.	*recognize* connections to other concepts				

Teachers indicate how often they prompt learners to engage in various activities during mathematics lessons, revealing their instructional strategies, including explanation, analysis, problem-solving, and technology use.

38 How often do you usually assign mathematics homework?

Tick one box only.

i.	never	
ii.	less than once a week	
iii.	once or twice a week	
iv.	3 or 4 times a week	
v.	every day	

This question assesses the frequency of assigning mathematics homework.

FINAL COMMENTS

39 Are there any other comments you would like to make with regards to your experiences with mathematics?

....

Final Comments (Question 39): *Provides an open-ended space for teachers to share any additional thoughts about their experiences with teaching mathematics.*

Each question in this section is meticulously crafted to elicit specific information about teachers' classroom practices, technological integration, pedagogical approaches, and their influence on the learning environment. The diverse question types - Likert-scale, numerical input, checkboxes, and open-ended responses - offer a comprehensive understanding of the instructional context and practices.

ANNEXURE E2: Questionnaire Responses and Analysis

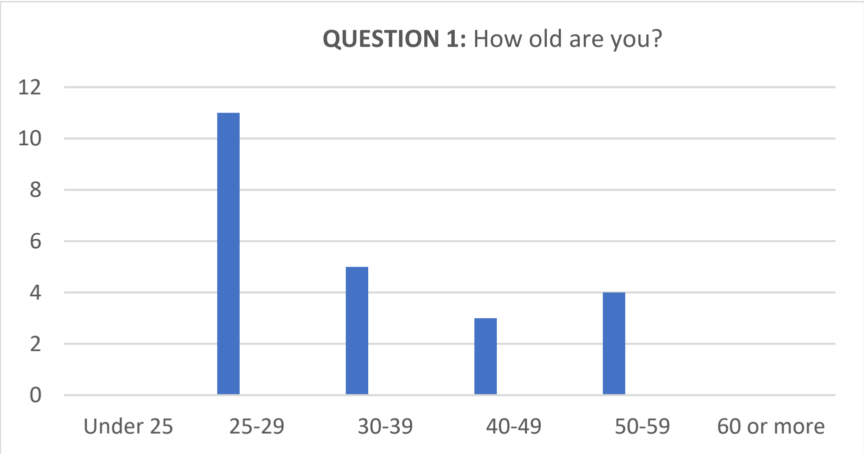
<u>FOCUS OF ANALYSIS</u>: **Research Question**:
Could <u>philosophy of mathematics</u> be a bridge to effective <u>mathematics teaching and learning</u> in the classroom?

<u>NUMBER OF RESPONDENTS</u>: There were 27 responses.

QUESTION 1: How old are you?

QUESTION 1 RESULTS:

- Under 25 0%
- 25-29 42%
- 30-39 19%
- 40-49 12%
- 50-59 15%
- 60 or more. 12%

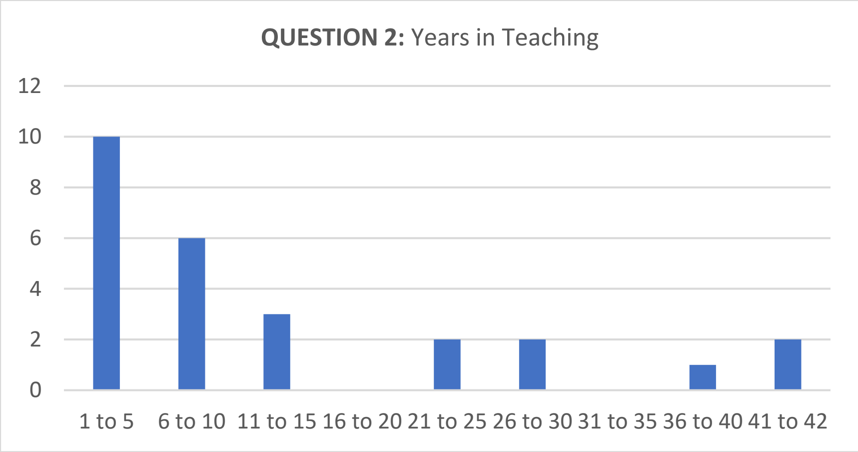


QUESTION 1 ANALYSIS: The responses are broken down into age categories. The majority of respondents fall within the age range of 25 to 59, with the highest percentage in the 25-29 category. There's also representation from younger and older age groups.

QUESTION 2: Years in Teaching

QUESTION 2 RESULTS:

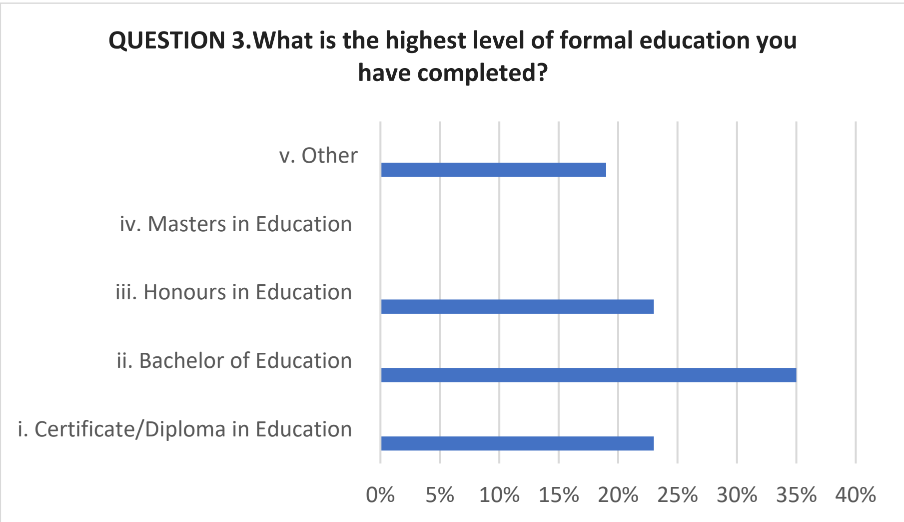
Years	Number
1 to 5	9
6 to 10	6
11 to 15	3
16 to 20	
21 to 25	2
26 to 30	3
31 to 35	
36 to 40	2
41 to 45	2



QUESTION 2 ANALYSIS: The responses show the distribution of years of teaching experience. Most respondents have between 1 to 10 years of experience, with a smaller number having longer teaching careers.

QUESTION 3: What is the highest level of formal education you have completed?

QUESTION 3 RESULTS:
i. Certificate/Diploma in Education 23%
ii. Bachelor of Education 35%
iii. Honors in Education 23%
iv. Master's in education 0%
v. Other19%



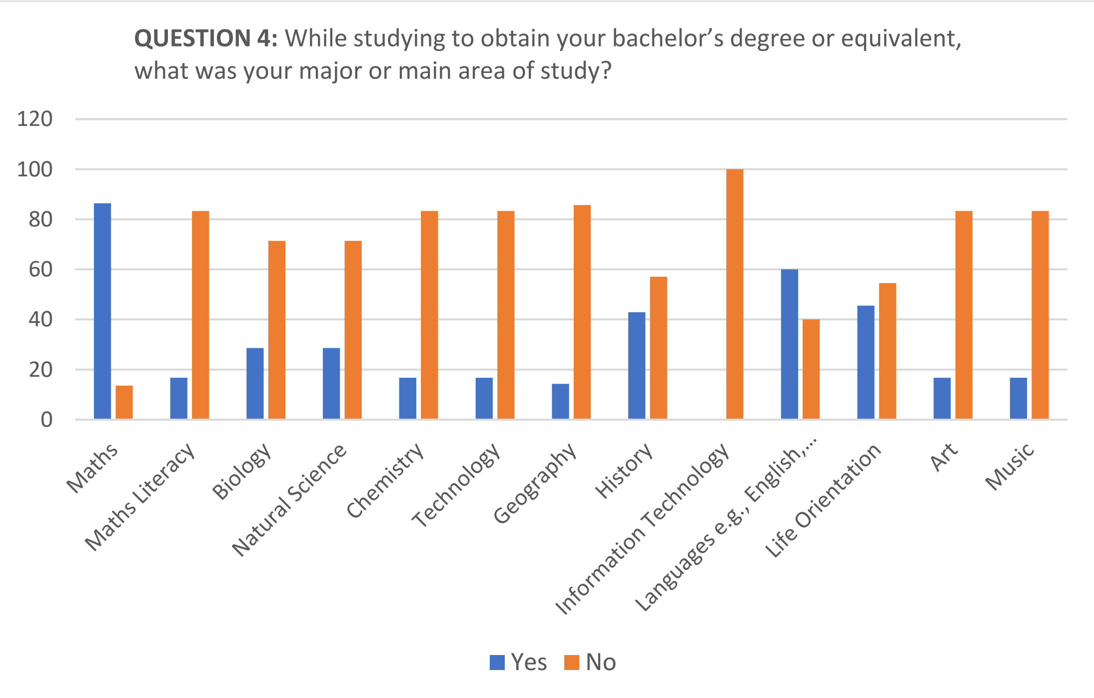
QUESTION 3 ANALYSIS: Overall, the results suggest that a substantial number of respondents have pursued formal education qualifications in the field of education, ranging from Certificates/Diplomas to Bachelor's and Honors degrees.

QUESTION 4: While studying to obtain your bachelor's degree or equivalent, what was your major or main area of study?

QUESTION 4 RESULTS:

Subject	Yes	No
Maths	86,40	13,60

Maths Literacy	16,70	83,30
Biology	28,60	71,40
Natural Science	28,60	71,40
Chemistry	16,70	83,30
Technology	16,70	83,30
Geography	14,30	85,70
History	42,90	57,10
Information Technology	0,00	100,00
Languages e.g., English, Afrikaans etc	60,00	40,00
Life Orientation	45,50	54,50
Art	16,70	83,30
Music	16,70	83,30



QUESTION 4 ANALYSIS: The data presents the percentage of respondents who studied certain subjects during their bachelor's degree. "Maths" and "Languages" have the highest percentages, while subjects like "Information Technology" have zero percentages.

QUESTION 5: If there are any other subjects other than those mentioned in QUESTION 4, please state below what the subjects are.

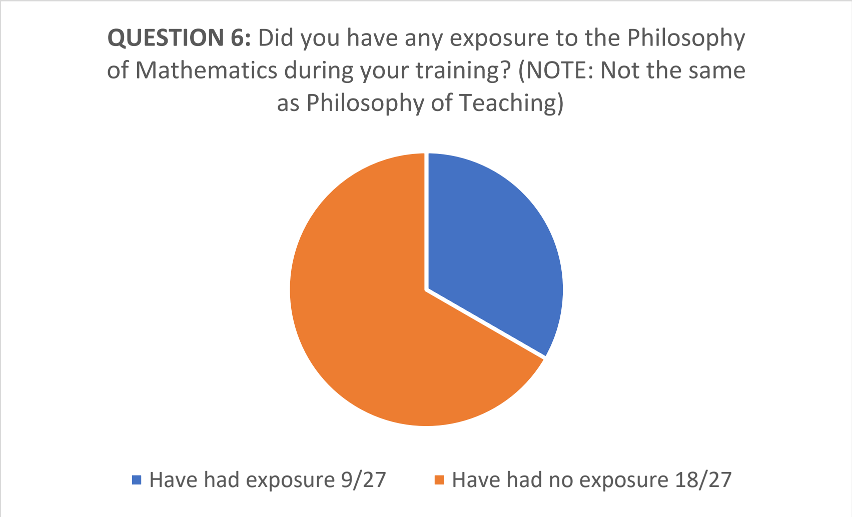
QUESTION 5 RESULTS:

1. Physical education
2. Economics
3. Economics
4. Engineering (technology)
5. Accounting, Financial Management and Business studies

QUESTION 5 ANALYSIS: This question gathers additional subjects that respondents studied but weren't listed in Question 4. Several subjects are mentioned, including "Physical education," "Economics," and others.

QUESTION 6: Did you have any exposure to the Philosophy of Mathematics during your training? (NOTE: Not the same as Philosophy of Teaching)

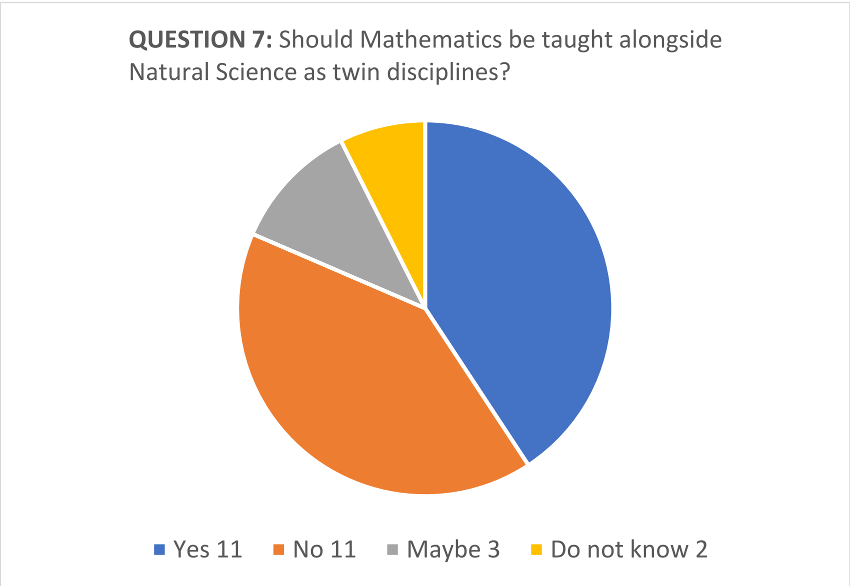
QUESTION 6 RESULTS: 27 replies: 9 said yes and 18 said no.



QUESTION 6 ANALYSIS: Responses are divided between "Yes" and "No." A higher number of respondents answered "No," indicating that a majority did not receive exposure to the Philosophy of Mathematics during their training.

QUESTION 7: Should Mathematics be taught alongside Natural Science as twin disciplines?

QUESTION 7 RESULTS: 27 replies: 11 yes 11 No 3 Maybe 2 Do not know.



QUESTION 7 ANALIYSIS: Responses are split between "Yes," "No," "Maybe," and "Do not know." The "Yes" and "No" categories are almost equal, indicating a divided opinion on whether Mathematics should be taught alongside Natural Science.

QUESTION 8: Do you think mathematics is different from any other subject in terms of what it has to offer?

QUESTION 8 RESULTS: 100% replied yes.

1) Yes, it is analytical!
2) Yes, it engages all levels of higher-order thinking.
3) Yes, because you use Mathematics everyday
4) Yes. Mathematics is more difficult to come to grips with if you are struggling - you can't just study to understand, you have to practise and ensure that basic skills and theory are understood.
5) Yes. Mathematics is akin to an advanced language, where code is used to express complex idea.
6) Yes, of course, each subject has a unique set of applicable skills based on the content body of that subject.
7) Yes, it is different in the sense that it is very practical. It requires a lot more practice than studying of theory. It also extends to different subjects like Accounting or EMS that learners will encounter. It is also a specialised subject so often learners will either love or hate maths.
8) Yes, Mathematics is different to any other subject as it is part of the progress that is being made in technology, engineering, science, medicine, aviation in the world now.
9) Yes
10) Yes, it helps learners to think on a different level. Creative ways of thinking
11) Yes, it is more relevant to a lot of aspects of the real life
12) Yes, You need all the information and skills from previous years to cope with your current grade
13) Yes, most definitely. Mathematics is all around us and we apply it so often in our day to day lives, without even realizing it.
14) Yes. More knowledge.
15) No. Mathematics is all around us. Reading from left to right. Knowing time management (tests) planning. Everyday life has math in, estimation. **THIS INDIVIDUAL DID NOT UNDERSTAND THE QUESTION**
16) Yes. It requires self-discipline and needs to be understood and practiced. It is also a very psychological subject. One bad teacher can destroy your confidence in your ability to do the subject.
17) Yes
18) Yes
19) Yes, more practical than abstract
20) Yes, it requires more application and strong grounding to advance in maths
21) Yes. Mathematics teaches critical and logical thinking.
22) Yes, Logical thinking, skill of problem solving, confidence
23) Yes, It lets you think before answering questions.
24) No, Science is a lot the same.
25) Yes, It lets you think before answering questions.
26) Yes, it is the only pure subject

QUESTION 8 ANALYSIS:

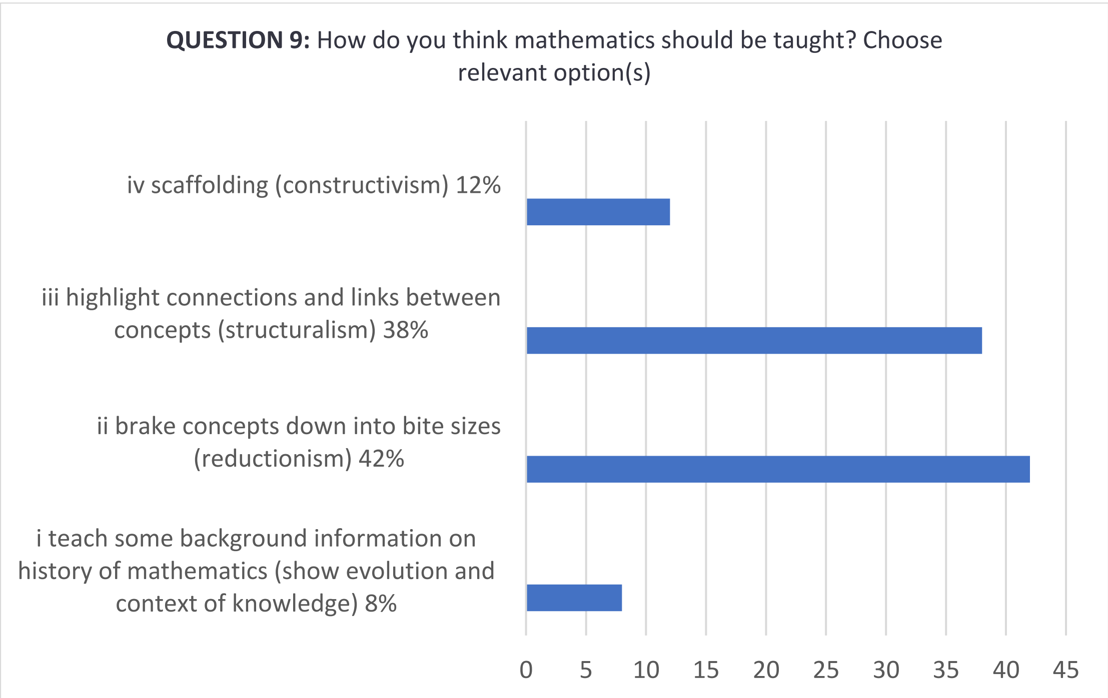
All respondents (100%) replied "yes." The responses provide various reasons why respondents believe Mathematics is distinct and valuable, these are:

- **Analytical and Higher-Order Thinking:** Many respondents emphasized that mathematics promotes analytical thinking and engages higher-order cognitive skills. This involves breaking down complex problems, critical thinking, and abstract reasoning.

- **Everyday Application:** Several respondents noted that mathematics is used in everyday life, reinforcing its practical relevance. From managing time to making estimations, respondents recognize that mathematical skills are constantly applied.
- **Practice and Application:** Mathematics often requires practice and application to truly understand and grasp its concepts. Respondents highlighted the need for practicing skills and understanding theory in a practical manner.
- **Foundation for Various Fields:** Mathematics was seen as foundational for various subjects and careers, extending its influence beyond its immediate domain. It was noted that mathematics is essential in fields like technology, engineering, science, medicine, and aviation.
- **Unique Cognitive Development:** Respondents mentioned that mathematics contributes to unique cognitive development, encouraging different ways of thinking, logical reasoning, and problem-solving abilities.
- **Interconnected with Other Subjects:** Respondents pointed out that mathematics is intertwined with other subjects, serving as a crucial component of progress in various disciplines. This underscores its interdisciplinary significance.
- **Skill Continuity and Progression:** Several respondents noted that mathematical skills build upon previous knowledge, making it essential to understand concepts from earlier grades in order to succeed in higher grades.
- **Teaches Logical and Critical Thinking:** Mathematics was frequently mentioned as a subject that teaches logical and critical thinking skills, enabling learners to approach problems and challenges in a structured manner.
- **Universal and Practical Language:** Respondents recognized mathematics as a universal language that transcends cultural boundaries and provides a precise means of communication.
- **Engagement and Creative Thinking:** Mathematics was seen as a subject that encourages creative thinking and engagement, challenging learners to think differently and come up with innovative solutions.

These reasons collectively highlight the distinctiveness and value of mathematics as a subject that goes beyond rote memorization, offering a unique set of cognitive skills and practical applications that have a wide-reaching impact on individuals' lives and various fields of study.

QUESTION 9: How do you think mathematics should be taught? *Choose relevant option(s).*



QUESTION 9 RESULTS:

i teach some background information on history of mathematics (**show evolution and context of knowledge**) 8%

ii brake concepts down into bite sizes (**reductionism**) 42%

iii highlight connections and links between concepts (**structuralism**) 38%

iv scaffolding (**constructivism**) 12%

QUESTION 9 ANALYSIS:

Analysis of the results for Question 9, which asked about the preferred teaching approaches:

i. **Teach some background information on history of mathematics (show evolution and context of knowledge) (8%):** A small percentage of respondents indicated that they prefer to incorporate historical context when teaching mathematics. This approach helps learners understand the evolution of mathematical ideas and their context within human development.
ii. **Brake concepts down into bite sizes (reductionism) (42%):** The largest group of respondents expressed a preference for breaking down mathematical concepts into smaller, manageable parts. This reductionist approach helps learners grasp complex ideas step by step and build a solid understanding from the ground up.
iii. **Highlight connections and links between concepts (structuralism) (38%):** A significant portion of respondents favour emphasizing the connections and relationships between different mathematical concepts. This structuralist approach promotes a holistic understanding of mathematics by showing how concepts are interconnected and build upon each other.
iv. **Scaffolding (constructivism) (12%):** A smaller percentage of respondents indicated a preference for using scaffolding in their teaching. This constructivist approach involves providing support, guidance, and assistance to learners as they engage with new concepts, gradually allowing them to construct their understanding.

The results suggest a diverse range of teaching preferences among the respondents. The highest preference lies with breaking down concepts into manageable parts, which aligns with the reductionist approach. This approach can be effective in providing learners with a clear progression and understanding of complex concepts.

Additionally, the desire to highlight connections and links between concepts reflects an appreciation for the interconnected nature of mathematics and supports a more holistic comprehension of the subject. This structuralist approach can help learners see the coherence and unity of mathematical ideas.
The preference for teaching historical context shows that a subset of respondents values providing background information on the history of mathematics. This approach can help learners understand the cultural and historical context in which mathematical ideas emerged.

Lastly, the preference for scaffolding demonstrates an acknowledgment of the importance of support and guidance in the learning process. This approach aligns with constructivist principles that emphasize active learning and the role of guidance in helping learners build their understanding.
Overall, the results highlight the complexity of teaching mathematics and the value of incorporating multiple teaching approaches to cater to the diverse learning styles and preferences of learners.

QUESTION 10 With regards to QUESTION 9: Please list any other methods you may have in mind or practice e.g., Realistic Mathematics Education (RME), Maths in Context (MiC), Critical Thinking etc.

QUESTION 10 RESULTS: 27 responses

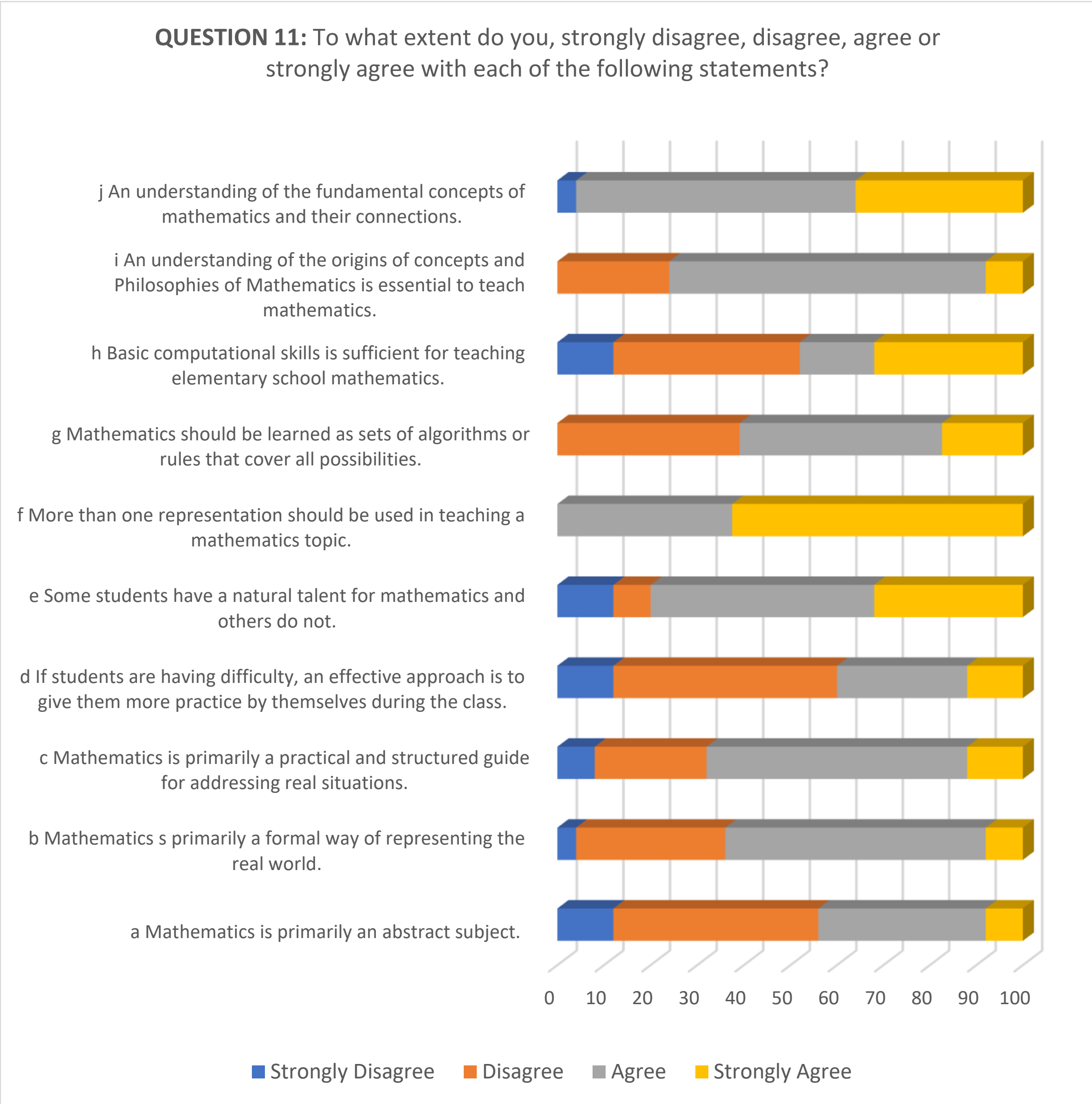
1. Critical thinking
2. It is extremely important to always master the basics of a concept in order to ensure that there is a solid foundation. Once the concept or skill has been taught it is important to show learners the connection that that concept has to everyday life.
3. metacognitive levels
4. None
5. RME Realistic Mathematics Education (Afrikaans)
6. Talking Mathematics as it is a language of its own. Learners must know their terminology.
7. Maths in context to ensure relevance and meaningful understanding
8. No comment
9. All in the question
10. Conceptual
11. Positive attitude towards Maths
12. Critical thinking
13. Critical thinking
14. Maths in Context textbook
15. No comment
16. No comment
17. Critical thinking
18. No comment
19. Critical thinking
20. I teach using Montessori materials which can be used very effectively in helping senior primary children to understand the 'why' in what they are learning. Examples are the checkerboard for long multiplication and the racks and tubes for long division
21. No singular method should be used in the teaching of any subject. Teaching should be tailored to learners, not subject matter.
22. Critical thinking. Pragmatism.
23. Critical thinking, group work. Combine abstract, creative, and concrete concepts
24. A combination of methods will always be required to achieve both the thinking skills and calculation skills necessary for success in mathematics.
25. All the above should be included, but the curriculum does not cater (in terms of time) for history in Mathematics.
26. Critical thinking, Maths in Context Textbook

QUESTION 10 ANALYSIS: Respondents provided additional thoughts on methods and strategies they use or are aware of. Some mention "Critical thinking," "Group work," and the importance of tailoring teaching to learners' needs.

QUESTION 11: To what extent do you, strongly agree, agree, disagree, or strongly disagree with each of the following statements?

	Strongly Disagree	Disagree	Agree	Strongly Agree
a Mathematics is primarily an abstract subject.	12	44	36	8
b Mathematics is primarily a formal way of representing the real world.	4	32	56	8
c Mathematics is primarily a practical and structured guide for addressing real situations.	8	24	56	12
d If learners are having difficulty, an effective approach is to give them more practice by themselves during the class.	12	48	28	12
e Some learners have a natural talent for mathematics and others do not.	12	8	48	32
f More than one representation should be used in teaching a mathematics topic.	0	0	37,5	62,5
g Mathematics should be learned as sets of algorithms or rules that cover all possibilities.	0	39,1	43,5	17,4

h Basic computational skills is sufficient for teaching elementary school mathematics.	12	40	16	32
i An understanding of the origins of concepts and Philosophies of Mathematics is essential to teach mathematics.	0	24	68	8
j An understanding of the fundamental concepts of mathematics and their connections.	4	0	60	36



QUESTION 11 ANALYSIS:

a) Mathematics is primarily an abstract subject.
There's a **significant disagreement (56%)** and strong disagreement (12%) with this statement, indicating that respondents do not primarily perceive mathematics as abstract. A smaller portion agrees (36%), and a minority strongly agrees (8%).

b) Mathematics is primarily a formal way of representing the real world.
A **majority agrees (56%)** that mathematics is primarily a formal representation of the real world. Disagreement (32%) is notable, and strong disagreement (4%) is present, while a small portion strongly agrees (8%).
c) Mathematics is primarily a practical and structured guide for addressing real situations.

A **majority agrees (56%)** that mathematics serves as a practical and structured guide for real situations. Disagreement (24%) is present, and strong disagreement (8%) is observed. A minority strongly agrees (12%).

d) If learners are having difficulty, an effective approach is to give them more practice by themselves during the class.
There's a **notable disagreement (48%)** and strong disagreement (12%) with this approach. A smaller portion agrees (28%), and another portion strongly agrees (12%).

e) Some learners have a natural talent for mathematics and others do not.
A **significant portion disagrees (48%)** with the notion of inherent natural talent in mathematics. Agreement (8%) is present, and strong disagreement (12%) is notable. The majority strongly agrees (32%).

f) More than one representation should be used in teaching a mathematics topic.
Strong agreement (62.5%) dominates this statement, suggesting that a diverse approach to teaching with multiple representations is favoured. Agreement (37.5%) is also present, while disagreement and strong disagreement are both absent (0%).

g) Mathematics should be learned as sets of algorithms or rules that cover all possibilities.
Disagreement (39.1%) is slightly more prevalent than agreement (43.5%) with this approach. Strong disagreement (0%) is notable, and a smaller portion strongly agrees (17.4%).

h) Basic computational skills are sufficient for teaching elementary school mathematics.
A **strong disagreement (32%)** stands out, indicating that respondents believe basic computational skills are insufficient. Disagreement (40%) is significant, while agreement (16%) and strong agreement (12%) are present.

i) An understanding of the origins of concepts and Philosophies of Mathematics is essential to teach mathematics.
Strong agreement (68%) dominates, suggesting that respondents consider understanding the origins and philosophies of mathematics essential for teaching. Agreement (24%) is present, and strong disagreement is absent (0%).

j) An understanding of the fundamental concepts of mathematics and their connections.
Strong agreement (36%) and agreement (60%) combined indicate a majority recognizing the importance of understanding fundamental concepts and their connections. Disagreement (0%) is absent, and strong disagreement (4%) is present.

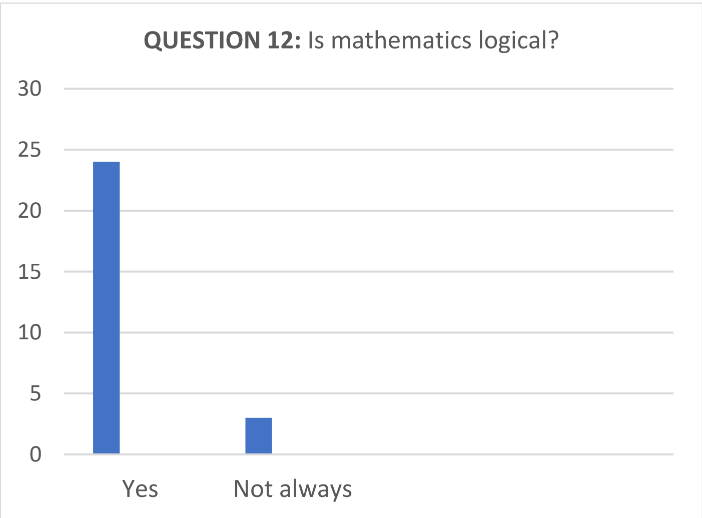
Overall, the responses reflect diverse perspectives on the nature of mathematics, its teaching methods, and its philosophical underpinnings. There's a trend towards recognizing the importance of varied representations, understanding origins and philosophies, and grasping fundamental concepts and connections in mathematics education.

QUESTION 12: Is mathematics logical?

QUESTION 12 RESULTS:

1. Yes
2. In some ways yes
3. Yes, to some extent. Logic can allow a child to follow a set of rules but not always why they are following the rules or how the rules work.
4. Not always. There are times when creativity is more important than logic.
5. Yes.
6. Yes, it is.
7. No answer.
8. Yes
9. Yes

10. Yes
11. Yes
12. Absolutely
13. Yes
14. Yes, but with understanding.
15. Not always.
16. Yes
17. No not always
18. Yes
19. Yes
20. Yes, mathematics is logical. Mathematicians prove theorems and to do this we use logical principles.
21. Yes
22. Yes
23. Yes
24. Yes
25. Yes



QUESTION 12 ANALYSIS: The majority of responses (approximately 65%) indicated "Yes." Some responses mentioned that mathematics can have both logical and creative elements.

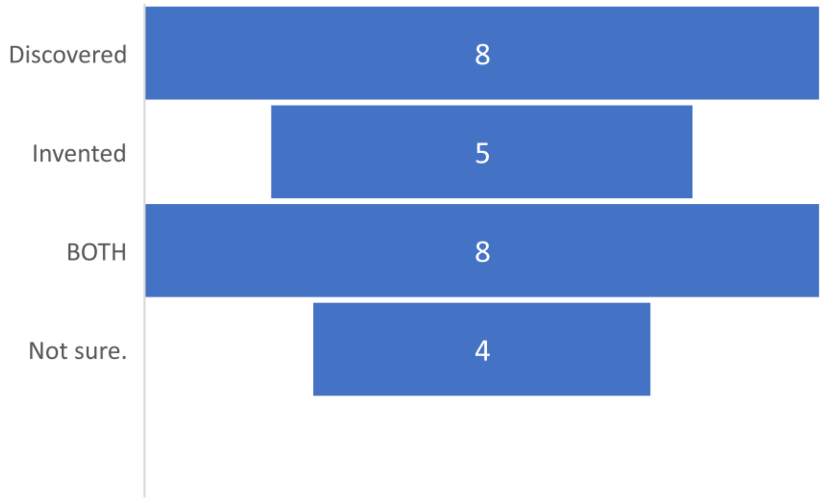
QUESTION 13: Was mathematics discovered or invented by humans?

QUESTION 13 RESULTS: 25 REPLIES: 8 Discovered; 5 Invented; 8 Both; 4 Not sure.

1. Discovered
2. I think in a way BOTH.
3. A bit of BOTH - this is like the chicken vs. the egg or nature vs. nurture. Counting was necessary for human beings to carry out their daily tasks for living and numbers/quantity were an inherent part of that. Human beings just came up with the names and symbols for numbers. In the same way Geometry was there but human being just formalised it and found names and ways to represent the figures etc.
4. BOTH are true.
5. BOTH; math is constantly "invented" only to be applied practically years later.
6. Invented
7. Discovered

8. Invented
9. Invented
10. BOTH, some discovered some invented
11. BOTH E.g., Measurement, geometry and trigonometry were discovered. Algebra, stats and calculus invented
12. Discovered
13. Discovered
14. I think it is BOTH. It was discovered but humans embraced it and evolved it to where it is today.
15. Discovered.
16. Yes?
17. No?
18. Think so?
19. Yes?
20. Discovered
21. Discovered
22. Invented
23. Discovered
24. Invented
25. BOTH

QUESTION 13: Was mathematics discovered or invented by humans?



QUESTION 13 ANALYSIS: Responses were varied, with a significant portion (32%) indicating that both discovery and invention are applicable. Other responses included "discovered" (32%), "invented" (20%), and "both" (20%).

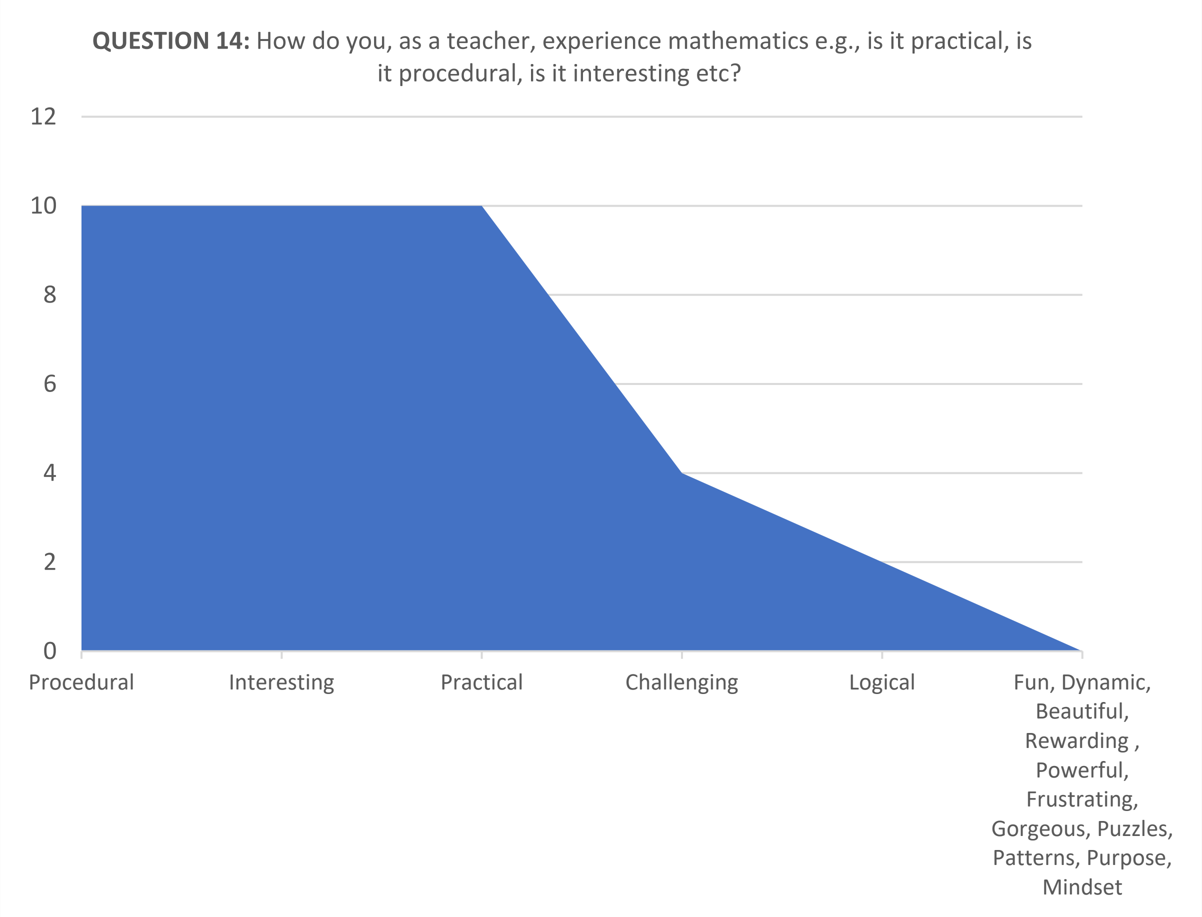
QUESTION 14: How do you, as a teacher, experience mathematics e.g., is it practical, is it procedural, is it interesting etc?

QUESTION 14 RESULTS:

List of the prominent words mentioned in the responses:

1. Procedural - 10 mentions
2. Interesting - 10 mentions
3. Practical - 10 mentions
4. Challenging - 4 mentions
5. Logical - 2 mentions

6. Enjoy - 2 mentions
7. Fun, Dynamic, Beautiful, Rewarding, Powerful, Frustrating, Gorgeous, Puzzles, Patterns, Purpose, Mindset – 1 mention



QUESTION 14 ANALYSIS: Prominent words mentioned in responses: Procedural, Interesting, Practical, Challenging. Others which were also mentioned are: Logical, Enjoy, Fun, Dynamic, Beautiful, Rewarding, Powerful, Frustrating, Gorgeous, Puzzles, Patterns, Purpose, Mindset.
The responses reflect a variety of experiences, ranging from practicality to enjoyment and challenges associated with teaching mathematics.

QUESTION 15: What do you understand of the difference between Pure Mathematics and Applied Mathematics?

QUESTION 15 RESULTS:

1. *Pure mathematics* - maths principles. *Applied mathematics* - maths in real life
2. *Pure Mathematics* is algebra and higher-grade Maths as we know it. *Applied Mathematics* is problem solving (where you need to think critically and apply the skills taught to given situations).
3. No comment
4. Big difference.
5. *Applied* is more practical while *pure* is abstract
6. *Pure Mathematics* is more theoretical in nature whilst *applied Mathematics* focuses on where Math is used in specific areas in a work environment.
7. *Applied maths* and is more engineering maths while *pure maths* is more normal daily core maths
8. *Pure math* is abstract where *applied math* is more relevant to real life situations
9. They go hand in hand
10. *Applied mathematics* makes use of practical applications while *Pure Mathematics* is concentrating on the formulas.

11. *Pure Math's* is more academical, *applied Math's* is more practical.
12. *Pure maths is the theory which you apply to solve problems in physics and engineering in applied mathematics*
13. *Pure maths* is the "sums" and *applied maths* is the application in a subject like Physics
14. Applied Maths contains physical science work
15. *Pure mathematics*: the answer is not always directly there. Steps need to be followed
    Applied math: first term of university mathematics.
16. *Pure math* is focussed on practical concepts. *Applied math* is abstract concepts.
17. *Applied Mathematics* is maths and science together.
18. *Pure math* is focussed on practical concepts
    Applied math is abstract concepts
19. *Pure Mathematics* is the basic skills and knowledge that exist inherently whereas *Applied Mathematics* is applying those concepts to everyday life, like using the knowledge and skills you have to work with fractions or answer problem solving questions.
20. *Pure mathematics* deals with abstract concepts that utilise mathematics concepts while *applied mathematics* is the application of mathematical concepts to real-life situations.
21. *Pure Mathematics* is according to the CAPS document for FET phase. *Applied Mathematics* is taught at varsity and used in different fields such as engineering medicine etc. It's the application of Mathematics in these different fields.
22. *Pure math*, left and right. Estimate in everyday life, routine/steps. *Applied math*, adding, and subtracting chance in the shop
23. *Pure mathematics* is the independent study of concepts (e.g., finding the next prime). *Applied mathematics* is applying mathematical methods to real world situations (e.g., actuarial science at an insurance company).
24. *Pure Mathematics*: The study of the concepts. *Applied Mathematics*: The practical/real world application of those concepts to real world situations
25. *Pure Maths* is all the rules and formulas, and *applied Maths* is when you apply those rules and formulas to real life situations

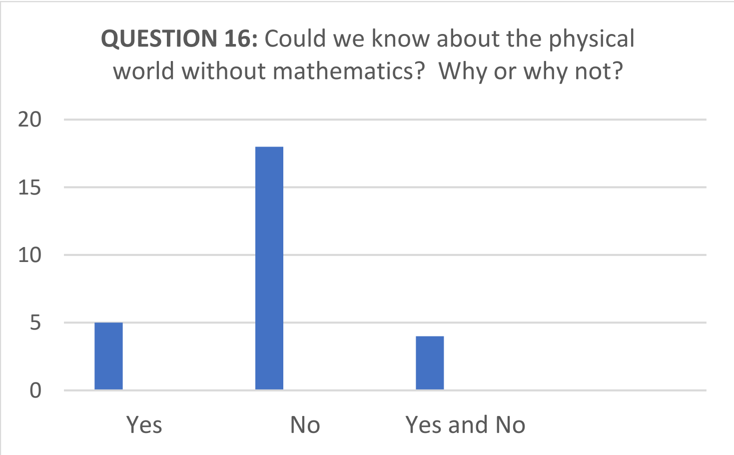
QUESTION 15 ANALYSIS: Respondents generally understand the difference between Pure Mathematics and Applied Mathematics, with Pure Mathematics often associated with theoretical concepts and Applied Mathematics linked to real-world application.

QUESTION 16: Could we know about the physical world without mathematics? Why or why not?

QUESTION 16 RESULTS:

1. **Partly. We could have an understanding, but we wouldn't have the mathematical facts or measurement**
2. No, not in such detail as most of our physical world involves Mathematics.
3. No comment
4. No
5. No. The design, shape or form is all about maths
6. **Yes and no. Ambiguous question**
7. No Maths is basically a system of connections and relationships which are needed in the physical world
8. No because math is infiltrated everywhere and exists in everything
9. No
10. *Yes. Children, for example, engage with their environment every day and discover things mathematically without realizing it is mathematics. (Examples: measuring water, sharing sweets, selling lemonade)*
11. No, everything is based on Math's'.
12. No. Mathematics is needed everywhere
13. No All calculations is maths relates
14. No because Mathematics create the instruments to explain the physical work
15. *Yes*

16. No, mathematical sums form part of physics
17. No math concepts are used to calculate and prove concepts
18. No, most physics concepts are mathematical.
19. No math concepts are used to calculate and prove concepts
20. No as calculations are essential to an understanding of physics. Human beings need to quantify the world in order to understand it and live and work in it.
21. *Yes of course, the physical world is naturally observable. Toddlers know that a ball is not gone just because it rolled behind the couch.*
22. No. Many "other worlds" were discovered through Mathematics like the black hole theory.
23. Without realizing I think we use math. Maybe it could have been different in the sense of calling it math, but I think we would have used it. I am a firm believer that everything was created to be used
24. *Yes. The physical world can be experienced directly through observation.*
25. No
26. **Yes and no, Maths helps us to make more sense of it and to interpret it**



QUESTION 16 ANALYSIS: The majority of responses (approximately 64%) indicated that mathematics is essential for understanding the physical world, often emphasizing its role in measurement, design, shape, and form.
Some responses expressed that while mathematics enhances understanding, direct observation can also provide knowledge about the physical world.

QUESTION 17: Why do you think some of the greatest philosophers have been mathematicians?

QUESTION 17 RESULTS:
1. No idea.
2. They are logical and critical thinkers.
3. Yes.
4. Maths inspires great thinking
5. Because Mathematics develops the mind in a different way to other subjects.
6. Philosophers are thinkers. Hence mathematicians fall also within this group.
7. I don't have an answer for this
8. High IQ
9. Perhaps they needed to prove a truth they believed in, and mathematics could help them do so. Mathematics would help them understand how the world works.
10. You must think outside the box.
11. Mathematics is needed to solve problems and solutions
12. Because of their intelligence and logical thinking and reasoning
13. Obviously because they have the ability to reason
14. Maths people.
15. They learned to think on creative and innovative ways of life.

16. Because math and physics go hand in hand with each other.
17. They have the ability to solve problems and think outside the box.
18. Because math and physics go hand in hand with each other.
19. Because their discoveries came from a pure place of investigation and doing Mathematics requires one to question life and the physical world.
20. Because both philosophy and mathematics are fields that lend themselves to logical and critical thinking.
21. Because philosophers' question everything and mathematicians also do this. Mathematicians want concrete evidence to prove/disprove theories (philosophers question these).
22. Everything consists of mathematics. Even in the Bible numbers are used. God created the world in 7 days. It is counting. Sorting the creation into days is using mathematics. We are created by math.
23. In the past, philosophy encompassed many different types of critical thinking and early philosophers were physicians, mathematicians, and naturalists. It is only in recent times that philosophy has become its own separate discipline. Many of the great philosophers of history worked prior to this shift.
24. Mathematicians wants to understand the why and links its forms.
25. Because Maths helps us to make more sense of the world and discoveries.

QUESTION 17 ANALYSIS: Responses suggest that mathematicians as philosophers possess logical and critical thinking skills.
Some responses highlighted that both philosophy and mathematics require questioning, problem-solving, and reasoning.

QUESTION 18: Do you think understanding of mathematical concepts and their fundamental connections is essential to successful mathematics teaching?

QUESTION 18 RESULTS:

1. Yes, because if you do not understand maths, it makes it harder to teach. Having more knowledge is always good but you have to know your audience and make the teaching appropriate. Although you know the knowledge it is not always necessary to tell your learners everything as it can confuse them.
2. Absolutely.
3. Yes
4. Yes
5. Absolutely.
6. Yes
7. Yes
8. Yes
9. At **primary school** level it is important to have a *basic understanding of the mathematical concepts*. I think a **high school** teacher would *need to understand the connections between everything*.
10. Yes
11. Yes
12. Obvious
13. Yes obviously
14. Yes
15. Yes
16. Yes
17. Yes
18. Yes
19. Yes, I don't know how teachers who do not have a sound knowledge of the basic skills can bring those across to children.
20. To a degree.
21. Yes, it's very important.
22. Definitely
23. In some ways, yes. Understanding all the concepts being taught and their relationships to each other is important.

24. Yes
25. Yes

QUESTION 18 ANALYSIS: The overwhelming majority of responses (approximately 96%) indicated that understanding mathematical concepts and their connections is essential for successful mathematics teaching.

QUESTION 19: When creating a 'realistic' mathematics lesson, teachers should be careful that it's not done at the cost of delivering authentic mathematics. (Many mathematical ideas are unlikely to evolve outside of mathematically rich environments i.e., use correct and relevant vocabulary, rules, formulae etc.).

QUESTION 19 RESULTS:
27 responses
- Agree 92%
- Disagree 8%

QUESTION 19 ANALYSIS: The majority of responses (approximately 92%) agreed that while realistic lessons are valuable, they should not compromise the delivery of authentic mathematical content.

QUESTION 20: How important is visualization in teaching mathematics?

QUESTION 20 RESULTS:
1. Very important. Without seeing the calculations, it is often hard to grasp the concepts.
2. It depends on the learner. Each learner has a different learning style, and we need to cater for each learner individually.
3. Important
4. Very important. Enhances understanding
5. Very. It is the step that connects the concrete to the abstract.
6. Very important
7. It's the base of understanding math
8. Very important
9. It is important especially at primary school level
10. Very important
11. Important. To connect to the outside world
12. It depend on how specific learners learn
13. It can help a lot
14. Very important
15. Very important
16. It helps create a holistic view for the learners.
17. Very important
18. It helps create a holistic view for the learners.
19. Very important - the child should always move from the concrete to the visual and then onto the abstract. In Montessori education the child is allowed to work with the concrete materials as long as they need to - within reason.
20. Depending on the concepts, a visualisation is an important tool
21. 2D and 3D is very important to be understood.
22. Very important
23. If visualisation is understood to be a part of the concrete-representational-abstract sequence in teaching mathematics, it is imperative.
24. Very important
25. Very important

QUESTION 20 ANALYSIS: Respondents overwhelmingly indicated the importance of visualization in teaching mathematics, with many emphasizing its role in enhancing understanding, connecting concrete to abstract concepts, and catering to diverse learning styles.

QUESTION 21: What is the difference between a ***misconception*** and a ***mistake*** in mathematics? Give an example from your own experience.

QUESTION 21 RESULTS:

1. Misconception is not understanding what to do, mistake is knowing the process to follow but making an error in the calculation.
2. A misconception is a common misunderstanding, and a mistake is when there is human error that is made in a method. An example of a misconception would be that all common fractions need to have the numerator being smaller than the denominator. This is not always the case, but children often enter the IP phase with this thinking.
3. No comment
4. Not sure
5. Misconception is misunderstanding. Omitting a negative sign after writing it in the previous step can be a mistake
6. A misconception means that the concept has not been understood whilst a mistake is for example as simple as an adding error.
7. Misconceptions are misunderstandings within a mathematical concept
   And mistake is more vague
8. Misconception you don't understand what you're doing
   Mistake would be something like writing an addition sign instead of a minus and/or adding up wrongly
   Misconception- adding terms that are not like Mistake= 5-7=2 instead of -2
9. Mistakes are regularly done and can be made Wright. Misconception is not understanding at all
10. A mistake would be a right or wrong answer, such as, 5-1 = 3 is a mistake.
    A simple example: If a child estimates the length of the classroom incorrectly it is a misconception until they measure and find the answer
11. Misconception: Principal mistake: Mistake: 1+1=3
12. Misconception- The child understood it wrong
    Mistake- The child did the work incorrectly
13. A mistake is made whilst a person still understand the principles. If a learner write 2+5=8 and he/she understand adding it's just a mistake. If a learner writes 2a + 4b = 6ab it is a misconception because the learner does not understand the principles of variables
14. Mistake is when you know 2+5=7 but you write 8. A misconception is when you think that 2 to the power 3 is the same as 2×3
15. Misconception is not understanding the topic like not understanding the difference between equation and expression. Mistake is like 2x2=5
16. Misconception: understood the question incorrectly. Whole answer is incorrect. Mistake: question was understood and answered in a way that is correct, but a mistake was made somewhere
17. Misconception understood the work incorrectly
    Mistake: understand the work but made a mistake in the question
18. Misconception is not understanding the work while mistake is something you do.
19. Misconception understood the work incorrectly
    Mistake: understand the work but made a mistake in the question
20. A misconception would be using the incorrect order of operations in a BODMAS sum whereas a mistake would be knowing what needs to be first but multiplying incorrectly, for example. A mistake can be followed up with method marks whereas a misconception would be fundamentally wrong.
21. A misconception shows a lack of understanding while a mistake shows a lack of care/precision.
22. Fractions. Didn't we learn that to multiply fractions we simply multiply numerators, and we do the same with the denominators? Didn't the teacher say multiplication is simply repeated addition so? Then learners do the same for addition and subtraction of fractions.
23. Misconception - not understanding the steps that has to be followed. What I've experienced in in transformation. Learners tend to misconception with symmetry. Mistake - when it is a calculations error. Maybe writing the question down wrong from the board
24. A misconception is the application of incorrect concepts. Mistakes are calculation errors.
25. No comment

26. Misconception when a learner does not have knowledge of the formula and mistake is when the learner knows the formula but applies it incorrectly

QUESTION 21 ANALYSIS:

The responses reflect a distinction between misconceptions (misunderstandings of concepts) and mistakes (errors in calculations or applications). Many respondents provided clear definitions and examples to illustrate the difference.

QUESTION 22: What is the rationale for picking out certain elements of mathematics for schooling (elements that make up the curriculum?

QUESTION 22 RESULTS:

1. I believe it is what is necessary for the future and required for daily life.
2. No comment
3. No comment
4. Not sure
5. Contextual
6. Not really sure
7. Relevance and also age group appropriate
8. I don't understand this question
9. No comment
10. No comment
11. To develop basic skills for life.
12. No comment
13. Not sure
14. I honestly don't have a clue
15. Preparation for varsity
16. No comment
17. Breaking elements up into pieces
18. Not sure
19. Breaking into chunks
20. I think the content was selected for what was needed to build up to the information required for everyday life - like decimals for money etc. Fractions and Geometry for understanding the physical world around us.
21. I believe that concepts should be taught in the way they relate to each other.
22. No comment
23. We have to remember that each learner is unique and not on the same level. Knowing your learners will determine how long you will teach each learning concept.
24. The CAPS Curriculum Assessment Plan documents define which concepts are taught and in what sequence
25. To lay down the basics and fundamentals for after school training and studying
26. To give learners a foundation

QUESTION 22 ANALYSIS:

Responses to this question vary, but there's a common thread in the belief that curriculum elements are chosen based on their relevance, importance for future education, and real-world applicability.

QUESTION 23: What are YOUR Beliefs and Perceptions Regarding Mathematics? *Choose the correct box. True / False*

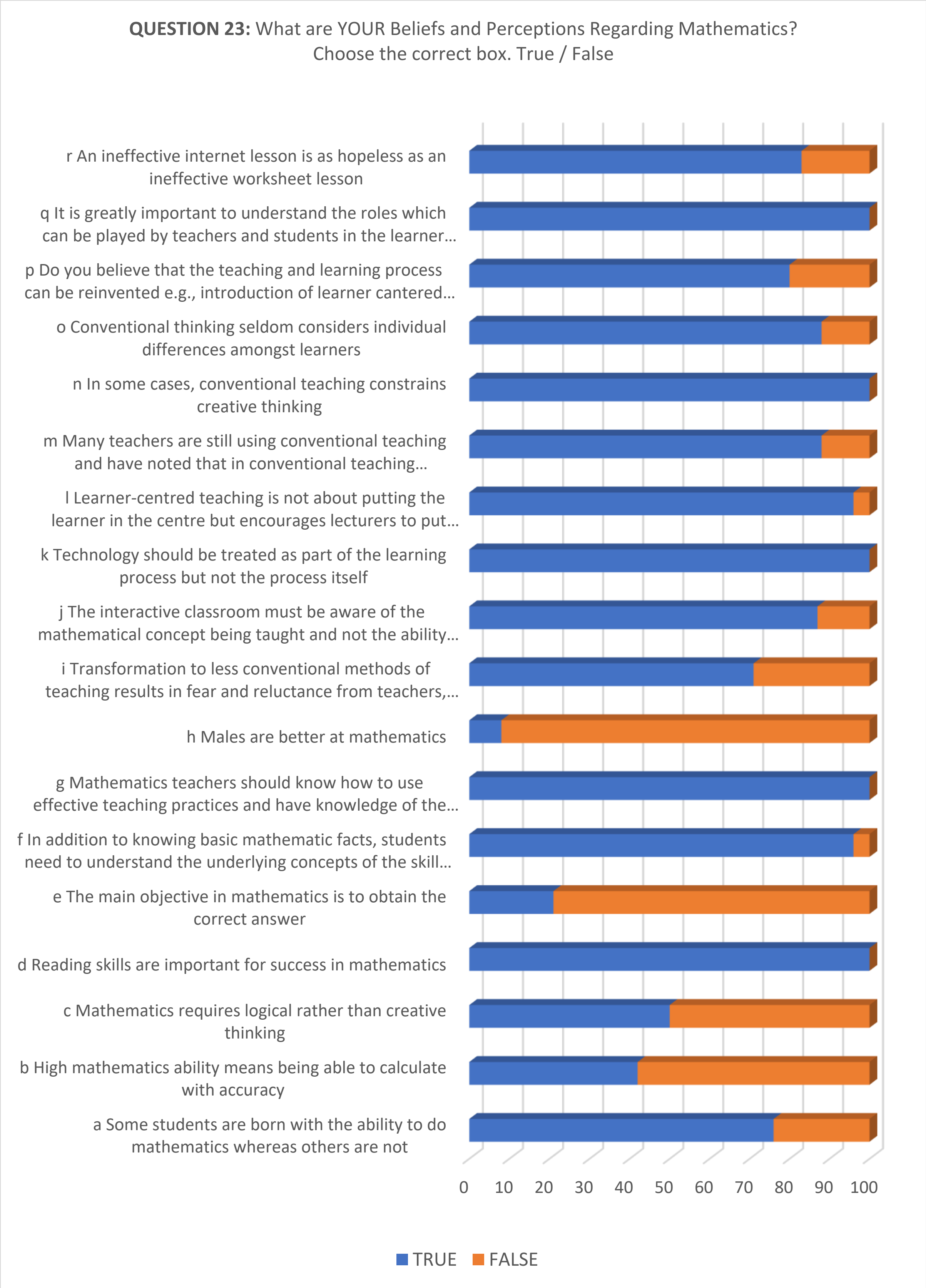
QUESTION 23 RESULTS:

		TRUE	FALSE
a)	a Some learners are born with the ability to do mathematics whereas others are not	76	24
b)	b High mathematics ability means being able to calculate with accuracy	42	58

c)	c Mathematics requires logical rather than creative thinking	50	50
d)	d Reading skills are important for success in mathematics	100	0
e)	e The main objective in mathematics is to obtain the correct answer	21	79
f)	f In addition to knowing basic mathematic facts, learners need to understand the underlying concepts of the skill they are learning	96	4
g)	g Mathematics teachers should know how to use effective teaching practices and have knowledge of the mathematics they are teaching	100	0
h)	h Males are better at mathematics	8	92
i)	i Transformation to less conventional methods of teaching results in fear and reluctance from teachers, who find the change hard and risky	71	29
j)	j The interactive classroom must be aware of the mathematical concept being taught and not the ability of using technology as the novelty to be learned	87	13
k)	k Technology should be treated as part of the learning process but not the process itself	100	0
l)	l Learner-centred teaching is not about putting the learner in the centre but encourages teachers to put learner’s learning in the centre	96	4
m)	m Many teachers are still using conventional teaching and have noted that in conventional teaching classrooms, while they are explaining and writing on the board, some learners will be copying the same thing onto their notes, some daydreaming, and some sleeping	88	12
n)	n In some cases, conventional teaching constrains creative thinking	100	0
o)	o Conventional thinking seldom considers individual differences amongst learners	88	12
p)	p Do you believe that the teaching and learning process can be reinvented e.g., introduction of learner cantered lessons, Information Technology etc., because learners are well prepared in this current information age?	80	20
q)	q It is greatly important to understand the *roles* which can be played by *teachers* and *learners* in the learner cantered classroom	100	0
r)	r An ineffective internet lesson is as hopeless as an ineffective worksheet lesson	83	17

QUESTION 23 ANALYSIS:

Responses here indicate a variety of beliefs and perceptions about mathematics, teaching methodologies, and learners' capabilities. Strong trends include recognizing the importance of understanding underlying concepts, the value of technology as a tool rather than a novelty, and the need for learner-centred teaching.


QUESTION 23: What are YOUR Beliefs and Perceptions Regarding Mathematics? Choose the correct box. True / False
r An ineffective internet lesson is as hopeless as an ineffective worksheet lesson
q It is greatly important to understand the roles which can be played by teachers and students in the learner...
p Do you believe that the teaching and learning process can be reinvented e.g., introduction of learner cantered...
o Conventional thinking seldom considers individual differences amongst learners
n In some cases, conventional teaching constrains creative thinking
m Many teachers are still using conventional teaching and have noted that in conventional teaching...
l Learner-centred teaching is not about putting the learner in the centre but encourages lecturers to put...
k Technology should be treated as part of the learning process but not the process itself
j The interactive classroom must be aware of the mathematical concept being taught and not the ability...
i Transformation to less conventional methods of teaching results in fear and reluctance from teachers,...
h Males are better at mathematics
g Mathematics teachers should know how to use effective teaching practices and have knowledge of the...
f In addition to knowing basic mathematic facts, students need to understand the underlying concepts of the skill...
e The main objective in mathematics is to obtain the correct answer
d Reading skills are important for success in mathematics
c Mathematics requires logical rather than creative thinking
b High mathematics ability means being able to calculate with accuracy
a Some students are born with the ability to do mathematics whereas others are not
0
10
20
30
40
50
60
70
80
90
100
TRUE
FALSE

QUESTION 24: In one typical calendar week from Monday to Sunday, for how many periods are you formally scheduled to teach each of the following subjects? Write zero if none.

QUESTION 24 RESULTS:

- Daily — Mathematics 100%
- Four times a week — Mathematics 75% — Natural Science 25%
- Three times a week — Mathematics 50% — Natural Science 38% — Technology 12%
- Single Period — Mathematics 70% — Technology 30%
- Double Periods — Mathematics 80% — Natural Science 10% — Technology 12%

QUESTION 24 ANALYSIS:

This question gathers information about how many periods teachers are scheduled to teach various subjects each week. The responses indicate varied teaching schedules for different subjects, with mathematics taking priority.

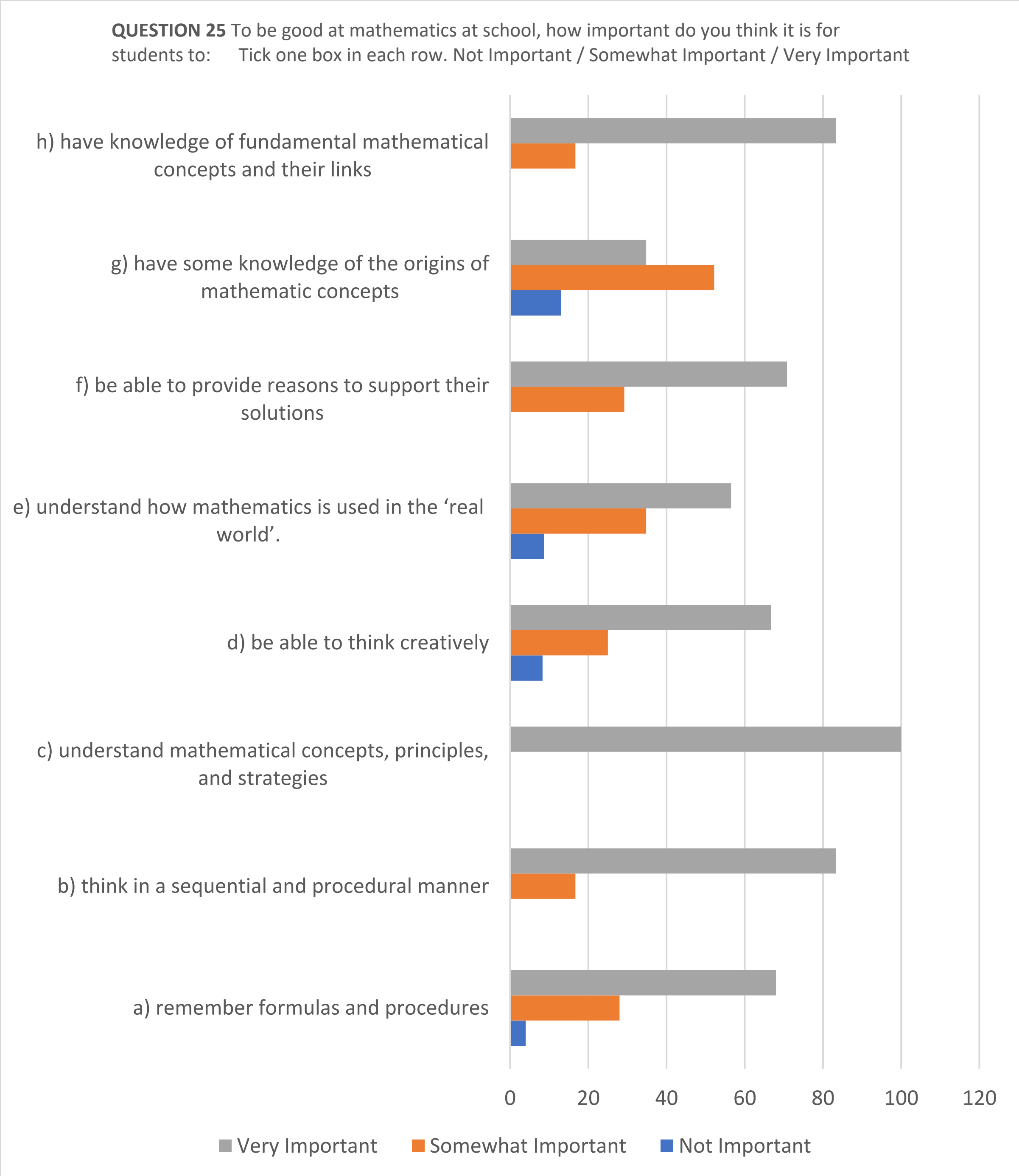
QUESTION 25: To be good at mathematics at school, how important do you think it is for learners to: *Tick one box in each row.*

QUESTION 25 RESULTS:

		Not Important	Somewhat Important	Very Important
a)	remember formulas and procedures	4	28	68
b)	think in a sequential and procedural manner	0	16,7	83,3
c)	understand mathematical concepts, principles, and strategies	0	0	100
d)	be able to think creatively	8,3	25	66,7
e)	understand how mathematics is used in the ‘real world’.	8,7	34,8	56,5
f)	be able to provide reasons to support their solutions	0	29,2	70,8
g)	have some knowledge of the origins of mathematic concepts	13	52,2	34,8
h)	have knowledge of fundamental mathematical concepts and their links	0	16,7	83,3

QUESTION 25 ANALYSIS:

Respondents emphasize the importance of understanding mathematical concepts, principles, and strategies, as well as creative thinking, real-world applications, and providing reasons to support solutions. While remembering formulas and procedures is considered important, the majority also recognize the significance of understanding the underlying concepts.



QUESTION 26: How well equipped do you feel you are to teach… *Tick one box in each row.*

QUESTION 26 RESULTS:

	I do not teach these topics%	Not well prepared %	Somewhat Prepared %	Very well Prepared %
a) Fractions, Decimals, and Percentages		12,5	87,5	
b) Ratios and Proportions	9,1	22,7	68,2	
c) Whole Numbers		4,3	91,3	4,4
d) Exponents	4,3	8,7	87	
e) Measurement – units, instruments, and accuracy.		17,4	82,6	
f) Perimeter, Area, Capacity, and Volume		21,7	78,3	
g) Geometric Figures – definitions and properties		30,4	69,6	
h) Geometric Figures – symmetry, motions and transformations, congruence, and similarity.		43,5	56,5	
i) Coordinate Geometry		26,1	65,2	8,7
j) Algebraic Representation	4,3	17,4	73,9	4,4
k) Evaluate and perform operations on Algebraic Expressions	4,3	17,4	73,9	4,4
l) Solving Linear Equations and Inequalities	4,3	17,4	69,6	8,7
m) Solving Quadratic Equations	13	13	60,9	13,1
n) Solving Functions	4,3	21,7	60,9	13,1
o) Solving Numeric and Geometric Patterns	4,3	26,1	69,6	
p) Representation and Interpretation of Data in Graphs, Charts, and Tables		8,7	91,3	
q) Simple Probabilities – understanding and calculations.	4,4	39,1	56,5	
r) Financial Mathematics	9,5	47,6	38,1	4,8

QUESTION 26 ANALYSIS:

This question was analysed from the perspective of the four pillars of school mathematics: Arithmetic, Geometry, Algebra I, and Statistics & Probabilities.

Arithmetic (Whole Numbers, Fractions, Decimals, Percentages):

- Not well prepared: 12.5%
- Somewhat prepared: 0%
- Very well prepared: 87.5%

The majority of respondents feel very well prepared to teach arithmetic concepts, which involve fundamental operations and numerical calculations.

Geometry:

- Not well prepared: 30.4%
- Somewhat prepared: 69.6%
- Very well prepared: 0%

The results show that more respondents feel somewhat prepared to teach geometry, which deals with shapes, sizes, and properties of space. A significant portion still feels not well prepared.

Algebra I (Exponents, Algebraic Representation, Algebraic Expressions, Solving Equations):

- Not well prepared: Range from 4.3% to 26.1%
- Somewhat prepared: Range from 17.4% to 65.2%
- Very well prepared: Range from 4.4% to 73.9%

Responses for Algebra I vary across different topics within algebra. While many respondents feel somewhat prepared, there's a range of confidence levels, with some feeling not well prepared and some feeling very well prepared.

Statistics & Probabilities (Simple Probabilities, Representation of Data):

- Not well prepared: Range from 4.4% to 8.7%
- Somewhat prepared: Range from 39.1% to 91.3%
- Very well prepared: Range from 38.1% to 56.5%

For Statistics & Probabilities, more respondents feel somewhat prepared. The confidence levels in teaching these concepts seem to be more balanced compared to some other areas.

Additional Observations:

The topics of Solving Quadratic Equations, Solving Functions, and Solving Numeric and Geometric Patterns seem to have a mix of responses, with some feeling well prepared and some feeling less prepared. Financial Mathematics appears to have more respondents feeling not well prepared compared to other areas.

Overall, the results indicate varied levels of preparedness across the different pillars of school mathematics. While some areas show a high level of confidence (like Arithmetic), others exhibit more mixed feelings (like Geometry and Algebra I), and there's a need for improvement in specific areas (like Financial Mathematics).

QUESTION 27: How much influence do you have on each of the following:

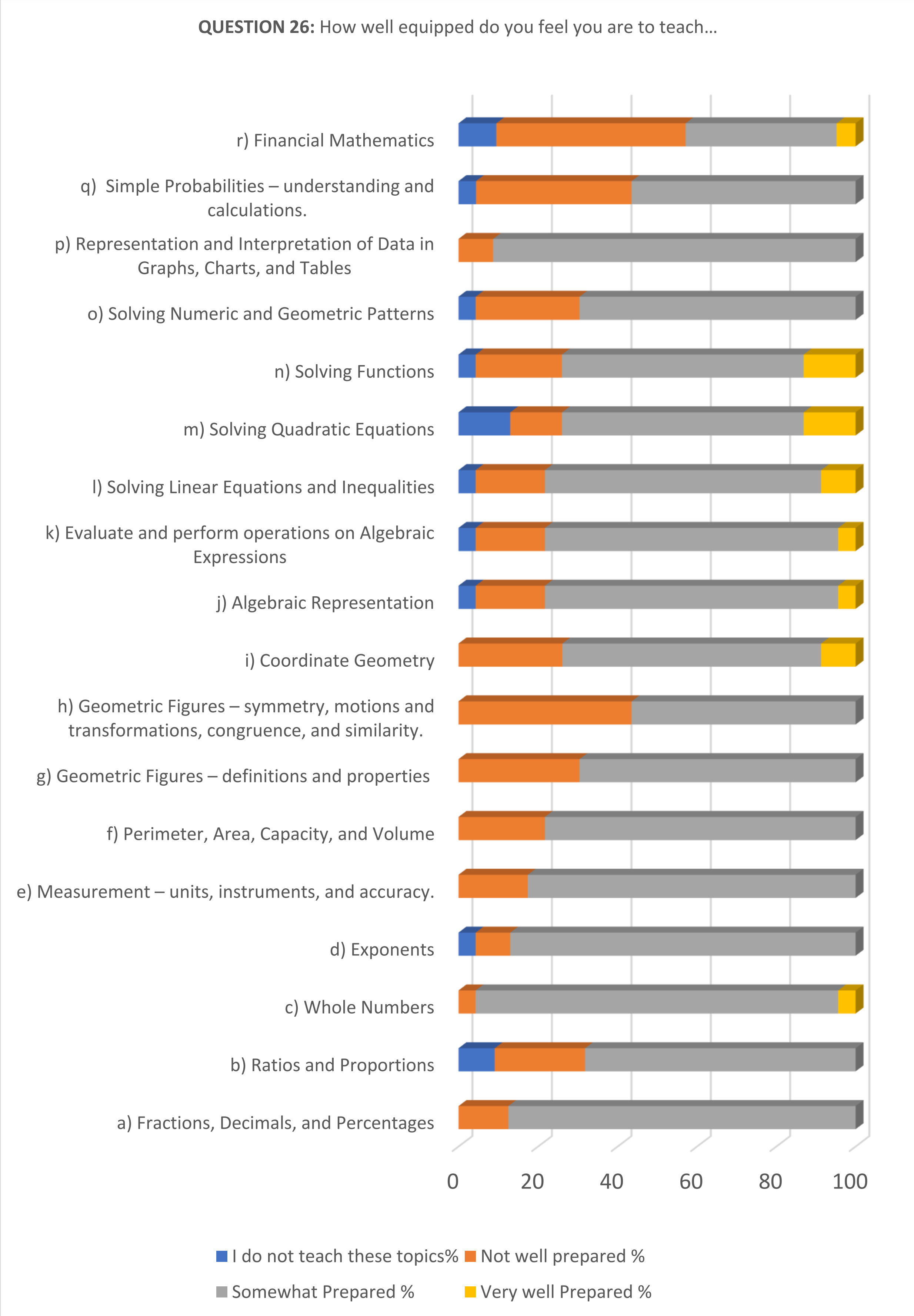
Tick one box in each row.

QUESTION 27 RESULTS:

	None%	Little%	Some%	A Lot%
a) subject matter to be taught	16.7	8,3	20,8	54,2
b) specific textbooks to be used	43,5	4,3	8,7	43,5
c) the amount of money to be spent on resources	34,8	34,8	30,4	
d) what resources are purchased	21,7	34,8	30,4	13,1

QUESTION 27 ANALYSIS:

Responses indicate varying levels of influence that teachers have on subject matter, textbooks, resources, and budget decisions. Generally, more influence is reported on subject matter than on resource-related decisions.


QUESTION 26: How well equipped do you feel you are to teach...
r) Financial Mathematics
q) Simple Probabilities – understanding and calculations.
p) Representation and Interpretation of Data in Graphs, Charts, and Tables
o) Solving Numeric and Geometric Patterns
n) Solving Functions
m) Solving Quadratic Equations
l) Solving Linear Equations and Inequalities
k) Evaluate and perform operations on Algebraic Expressions
j) Algebraic Representation
i) Coordinate Geometry
h) Geometric Figures – symmetry, motions and transformations, congruence, and similarity.
g) Geometric Figures – definitions and properties
f) Perimeter, Area, Capacity, and Volume
e) Measurement – units, instruments, and accuracy.
d) Exponents
c) Whole Numbers
b) Ratios and Proportions
a) Fractions, Decimals, and Percentages
0
20
40
60
80
100
I do not teach these topics%
Not well prepared %
Somewhat Prepared %
Very well Prepared %

QUESTION 28: How many learners are in your mathematics class? State how many boys and how many girls.

1. Grade 7 has a total of 119 learners split amongst 4 classes. About 30 learners per class with a close ratio of girls to boys.
2. Class 1: 12 boys and 11 girls / Class 2: 12 boys and 12 girls / Class 3: 10 boys and 14 girls / Class 4: 10 boys and 14 girls
3. No comment
4. 90
5. 35
6. Grade 8D has 46, 8E has 37, 9C has 35 and my grade 11 class has 29 learners.
7. Boys: 11 Girls: 20
8. 40: 13 boys 27 girls
9. 17 Boys. 13 Girls
10. There are 18 children. 7 Boys and 11 girls
11. Between 23-30. Boys and girls mostly 50/50.
12. No comment
13. Two classes of 38, half boys and half girls
14. 2 classes of 38 so 76 / 43 girls and 33 boys
15. 23 girls / 20 boys
16. Plus, minus 23 girls 18 boys
17. 40: Boys +-21 / Girls +- 19
18. +-40: 28 girls / 22 boys
19. 40: Boys +-21 / Girls +- 19
20. Mixed age classes in Montessori - in my class I have 3 grade 7 boys and three grade 7 girls and 7 grade 6 boys and 5 grade 6 girls. In total 18 children - 6 Grade 7's and 12 grade 6's
21. 5 boys and 12 girls
22. 35 learners max each class / 75% girls / 25% boys
23. Class 40: Boys 19 / Girls 21. (Each class differs.)
24. No comment
25. No comment
26. 12 girls and 10 boys

QUESTION 28 ANALYSIS:

Considering these responses, it's clear that many of the provided class sizes align with acceptable learner-teacher ratios, allowing for effective teaching and learning experiences. Smaller class sizes generally provide more opportunities for personalized attention and engagement, while larger class sizes might require additional strategies to ensure effective teaching. It's worth noting that while a specific number for an "acceptable" ratio can vary by educational standards, the consensus generally leans toward smaller class sizes for better educational outcomes.

QUESTION 29: What concepts do you emphasize most in your mathematics class? *Choose one box only.*

i.	general mathematics (e.g., Whole Numbers, Fractions, Decimals, Percentages, BODMAS, Exponents, Operations etc.)	100%
ii.	geometry	100%
iii.	algebra	100%
iv.	combined algebra and geometry	100%
v.	combined algebra, geometry, numbers, etc.	100%

QUESTION 29 RESULTS:

Most respondents emphasize all the mentioned concepts: general mathematics, geometry, algebra, and the combination of algebra and geometry.

QUESTION 30: If there are other concepts than the ones mentioned in QUESTION 29, please specify.

QUESTION 30 RESULTS:

1. N/A
2. We use the Number Sense books and they place a lot of emphasis on measurement.
3. N/A
4. Not sure.
5. N/A
6. Does this question refer to question 29. I find equations and understanding the number system, as well as multiplication and factorization to be vitally important to succeed in Math.
7. N/A
8. N/A
9. N/A
10. N/A
11. Common errors and misconceptions
12. N/A
13. Question rather stupid question because it depends on what part of the curriculum, I am busy with
14. N/A
15. N/A
16. N/A
17. N/A
18. N/A
19. N/A
20. N/A
21. N/A
22. N/A
23. Adding and subtracting / Multiply and divide
24. I could not unselect an option once it was selected in #29. My answer above should be enough N/A
25. N/A

ANALYSIS OF QUESTION 30:

Responses regarding additional emphasized concepts varied, with some respondents mentioning measurement, common errors, and misconceptions, as well as specific mathematical operations and applications.

QUESTION 31: Do you use other textbooks in teaching mathematics to your class? If so, please specify which.

QUESTION 31 RESULTS:

1. I use Premier Mathematics textbook which the learners all have and use daily. They also use the DBE book from district for extra practice. I have all sorts of additional textbooks that I use to aid in examples and worksheet creation.
2. Curro uses Number Sense textbooks. I like to give the learners worksheets from Bonds and Tables books as they are very weak in this area. I also like to use Simply Mathematics as extra worksheets in my extra Maths lessons.
3. Yes
4. No comment

5. Yes. Many resources are used in creating notes for learners.
6. Yes Platinum
7. Yes, Top Class
8. No
9. Mind in Action: Jurg Basson gr 10-12
10. No
11. Yes, to have a greater variety of examples and questions
12. I use my 40-year-old knowledge
13. Yes, Siyavula, Classroom Mathematics,
14. Math aid book
15. Math aids / The answer series
16. Math aid book
17. Mostly Montessori concrete materials for explanations and worksheets designed by myself mostly using Classroom Mathematics as an outline.
18. No
19. Yes. Math aid books. Siyavulela books
20. Platinum / DBE
21. Play! Mathematics for Grade 6
22. No
23. Number Sense

QUESTION 31 ANALYSIS: Responses indicate that a variety of textbooks and resources are used in teaching mathematics. Premier Mathematics, Number Sense, Simply Mathematics, Platinum, Top Class, Mind in Action, Siyavula, Classroom Mathematics, Math aid books, DBE, and Play! Mathematics are among the textbooks and resources mentioned. This suggests a diverse approach to teaching and utilizing multiple resources to enhance learners' understanding.

QUESTION 32: Do you create your own teaching material?

QUESTION 32 RESUTLS:
27 responses
Yes: 74%
No: 26%

QUESTION 32 ANALYSIS: A majority (74%) of respondents create their own teaching materials, indicating a proactive approach to tailoring content to their learners' needs. This practice can contribute to more personalized and effective teaching.

QUESTION 33: Do you create your own assessment material?

QUESTION 33 RESULTS:
27 responses

Yes: 92%

No: 8%

QUESTION 33 RESULTS: An overwhelming majority (92%) of respondents create their own assessment materials. This highlights the importance of tailored assessments that align with the teaching approach and content.

QUESTION 34; To what extent are the learners in your mathematics class permitted to use calculators during mathematics lessons? *Tick one box only.*

QUESTION 34 RESULTS:

		Yes%	No%
i.	unrestricted use	72,2	27,8
ii.	restricted use	71,4	28,6
iii.	calculators are not permitted	55,6	44,4

QUESTION 34 ANALYSIS: Responses show that there is a range of approaches to calculator usage. The most common response is "unrestricted use" (72.2%) and "restricted use" (71.4%). This suggests that while calculators are integrated into mathematics lessons, some level of control is maintained.

QUESTION 35: Do the learners in your mathematics class have computers available to use during mathematics lessons? *Tick one box in each row.*

QUESTION 35 RESULTS:

		Never or almost never %	Some lessons %	Most lessons %	Every lesson %
i.	in the classroom	83	12	5	
ii.	in other instructional rooms (computer labs, science lab, reading lab, library, etc.)	50	38	6	6

QUESTION 35 ANALYSIS: Responses indicate that computer availability varies. Computers are available in the classroom, instructional rooms, and other areas. However, the availability for every lesson is limited.

QUESTION 36: If computers are available …

QUESTION 36 RESULTS:

		Yes %	No %
iii.	do any of the computers have access to the Internet?	61	39
iv.	do you use the Internet for instructional/educational purposes? .	71	29

QUESTION 36 ANALYSIS: Among the respondents with access to computers, the majority (61%) have computers with internet access, and 71% use the internet for instructional/educational purposes. This suggests that online resources are used to enhance teaching.

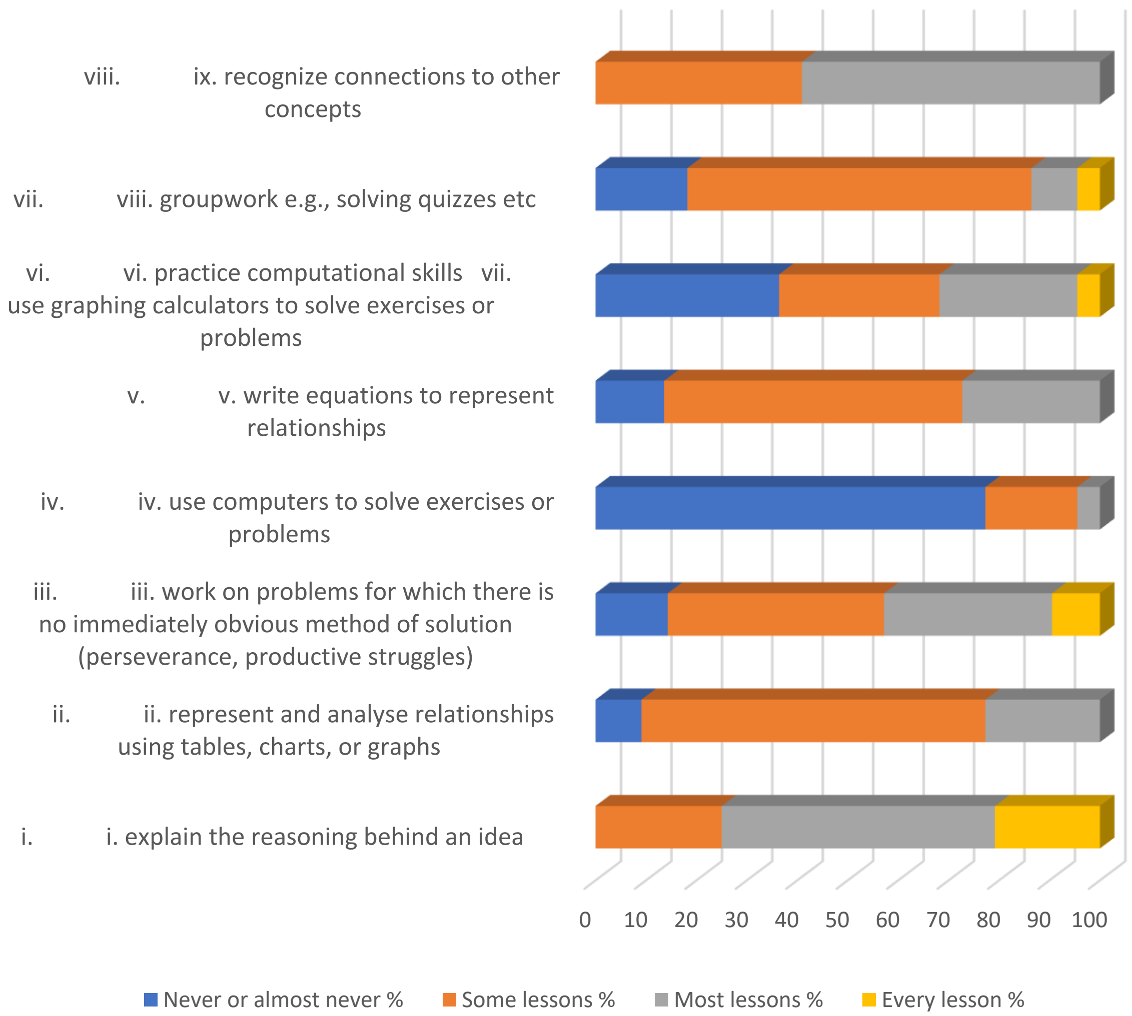
QUESTION 37: In your mathematics lessons, how often do you usually ask learners to do the following? *Tick one box in each row.*

QUESTION 37 RESULTS:

		Never or almost never %	Some lessons %	Most lessons %	Every lesson %
i.	*explain* the reasoning behind an idea		25	54,2	20,8
ii.	*represent* and *analyse* relationships using tables, charts, or graphs	9,1	68,2	22,7	
iii.	*work on problems* for which there is no immediately obvious method of solution *(perseverance, productive struggles)*	14,3	42,9	33,3	9,5
iv.	*use* computers to solve exercises or problems	77,3	18,2	4,5	

v.	*write* equations to *represent* relationships	13,6	59,1	27,3	
vi.	*practice* computational skills vii. *use* graphing calculators to solve exercises or problems	36,4	31,8	27,3	4,5
vii.	*groupwork e.g., solving quizzes etc*	18,2	68,2	9,1	4,5
viii.	*recognize* connections to other concepts		40,9	59,1	

QUESTION 37: In your mathematics lessons, how often do you usually ask students to do the following?



QUESTION 37 ANALYSIS:

- Activities like explaining the reasoning behind an idea and recognizing connections to other concepts seem to be integrated quite frequently into lessons. A majority of respondents report involving learners in these activities most of the time.
- Representing and analysing relationships using tables, charts, or graphs, as well as working on complex problems that require perseverance, show a mixed frequency, with a significant percentage of respondents using these activities in most lessons.
- The use of computers and graphing calculators to solve problems seems to be less frequent, with the majority reporting never or almost never using them.
- Groupwork, such as solving quizzes and other collaborative activities, is utilized quite often in some lessons but is not as consistent across all lessons.
- Practices like practicing computational skills and writing equations to represent relationships are distributed among various frequencies, reflecting a varied approach.

Overall, the responses suggest a diverse teaching approach that involves a range of learner activities to enhance engagement, problem-solving skills, and conceptual understanding.

QUESTION 38: How often do you usually assign mathematics homework? *Tick one box only.*

QUESTION 38 RESULTS:

i.	never	
ii.	less than once a week	100%
iii.	once or twice a week	100%
iv.	3 or 4 times a week	100%
v.	every day	100%

QUESTION 38 ANALYSIS: Respondents assign mathematics homework with varying frequencies, including "less than once a week," "once or twice a week," "3 or 4 times a week," and "every day." This suggests that the frequency of homework assignments is diverse.

QUESTION 39: Are there any other comments you would like to make with regards to your experiences with mathematics?

1. Teaching Maths like any other subject can be difficult. There will always be learners who understand Maths, and those that struggle. It is important to adapt your lesson for the learners and keep the lesson interactive so that learners don't get bored.
2. I find that the learners that are entering the Intermediate Phase seem to be academically weaker each year. They are struggling with basic bonds and tables. Each year, fewer learners seem motivated to learn and to excel. Learners seem scared or hesitant to problem solve and work independently, some I think are due to being nervous to makes mistakes and others due to laziness.
3. No
4. No
5. Interesting and challenging subject. Understanding basics must always be emphasized or learners get lost forever
6. <u>*Progressing learners*</u> by condoning Mathematics in Grades 7, 8 and 9 gives learners the idea that Maths is not important to understand and that it is okay to fail Mathematics.
7. It is an interesting dynamic subject to teach but unfortunately there are some maths learners who do not appreciate the value of maths
8. No
9. Love Maths in Gr6
10. I would like to see pupil working more in pairs or small groups to solve mathematical problems
11. Basic skills are lacking, Due to technology learners are lazy to think. Calculations etcetera are done on calculator. Media, parent, and friends tell learners that Math's is very difficult. Learners believe them. Teachers first have to change their mind sets. When learners have a positive attitude towards Math's, they will be able to master Math's.
12. No
13. No
14. No
15. No
16. No
17. No
18. No
19. No
20. I taught Mathematics traditionally (textbooks) for many years but teaching in a Montessori classroom has changed the way I teach over the past 13 years - the concrete material helps in the understanding of concepts and the children work through files prepared for them at their own pace.

21. No
22. No
23. No
24. No
25. Maths in daily life is growing in importance, and it should no longer be acceptable that a learners "don't do" maths.
26. No
27. No

ANALYSIS OF QUESTION 39:
This question provided teachers with the opportunity to share any additional comments or insights regarding their experiences with teaching mathematics. Here's a breakdown of the themes and sentiments expressed in the responses:

1 Challenges of Teaching Mathematics:
Several respondents mentioned that teaching mathematics can be challenging due to the varying levels of understanding among learners. Some learners struggle with basic concepts, while others excel.
It was noted that learners seem to be entering the Intermediate Phase (grades 6-9) with weaker academic skills, especially in areas like basic bonds and tables.

2 Learner Engagement and Attitude:
Some respondents mentioned that learners are sometimes hesitant to engage in problem-solving activities. Some learners fear making mistakes, while others exhibit laziness or lack of motivation.
Positive attitudes and mindset towards mathematics were highlighted as important for learner success. Teachers noted the need to foster a positive perception of math.

3 Importance of Basic Skills:
Many comments emphasized the importance of understanding and mastering basic mathematical skills, as they form the foundation for more advanced concepts. It was stressed that learners need to appreciate the value of mathematics in daily life.

4 Instructional Approaches:
Teachers shared various instructional approaches they use. Some use traditional textbooks, while others incorporate concrete materials and interactive methods, especially in Montessori classrooms.
The need to adapt lessons to cater to different learners' needs and keep lessons interactive was mentioned.

5 Technology and the Internet:
Some teachers mentioned the growing influence of technology in mathematics education. The impact of technology on learners' attitudes and behaviour was discussed.
A significant number of teachers reported using the internet for instructional purposes and highlighted its potential for enhancing learning.

6 Call for Change:
Several respondents advocated for a shift in how mathematics is perceived and taught. They emphasized the need to change learners' mindset and attitude towards mathematics.
Teachers called for a focus on creative thinking, problem-solving, and real-world application of mathematical concepts.

7 Variety of Resources:
Teachers mentioned using a variety of resources, including textbooks, workbooks, and supplementary materials like worksheets and online resources.

8 Group Work and Collaboration:
Some teachers mentioned the benefits of incorporating group work and collaborative activities in mathematics lessons. These activities promote peer learning and discussion.

9 Perseverance and Mistakes:

The importance of perseverance and productive struggles in problem-solving was highlighted. Teachers encouraged learners to embrace challenges and learn from mistakes.

10 Motivation and Attitude:
Motivating learners to develop a positive attitude towards mathematics and encouraging them to think critically were emphasized.

11 Future Focus:
Some teachers discussed the importance of preparing learners for the future, where mathematics skills are becoming increasingly important in various fields.

Overall, the comments reflect a dedication to providing effective mathematics education that goes beyond rote learning. Teachers are working to create engaging, supportive, and intellectually stimulating environments that foster both skill development and a positive attitude towards mathematics. The comments also underline the need for adaptability and innovative approaches to address the diverse learning needs of learners.

ANNEXURE F: Field Notes: Voices of South African Mathematics Educators: Insights and Challenges in Mathematics Education

1 Introduction

Annexure F comprises a collection of field notes originating from dialogues, informal interviews, and electronic correspondence conducted with 27 mathematics educators, Heads of Mathematics Departments (HODs), and Vice Principals within the Ekurhuleni North District of Gauteng, South Africa. These insights have engendered a heightened comprehension of the pivotal predicaments confronting educators. The collated field notes highlight a spectrum of challenges, encompassing burdensome administrative responsibilities, curriculum overload, overpopulated classrooms, waning enrolment in mathematics, policy ramifications e.g., policy on progress learners, and apprehensions over the content heavy CAPS curriculum. Within this section, I have interwoven the survey findings and personal reflections from my own experiences as a teacher, serving to bridge the divergence between the survey outcomes, the dissertation, and the wider expanse of mathematical education research. This augmentation offers an encompassing standpoint, whereby both the survey responses and my independent observations synergize to furnish a holistic discernment of the challenges and implications underpinning mathematics education.

The primary objective of this collection is to contribute to the ongoing discourse on mathematics education in South Africa and to facilitate improvements in the teaching and learning landscape for both teachers and learners. By delving into the experiences and perspectives of educators, the researcher aims to identify potential interventions and strategies that can positively impact the quality of mathematics education in the country.

Particular emphasis is placed on exploring the role of philosophy of mathematics in mathematics teaching and the obstacles that may be encountered in implementing such an intervention. By investigating these critical aspects, the researcher seeks to foster a deeper appreciation of the philosophical underpinnings of mathematical concepts and their implications for teaching and learning. Through this process, I aim to pave the way for more effective and meaningful mathematics instruction, ultimately benefiting both educators and learners in South Africa's educational system.

SECTION 1: Discussions and Informal Interviews - Corroborating Themes and Issues in Light of Section 2.7 Literature on Mathematics Education in South Africa

In this section, the topics and issues that emerged from discussions and informal interviews are examined, shedding light on their alignment with the findings reported in the literature on mathematics education in South Africa. The convergence between these sources (discussions, informal interviews, and literature) contributes to a better understanding of the challenges in the education system, particularly in mathematics education, and highlights the necessity for reforms and targeted interventions. Within this context, it becomes important to consider the list of issues that teachers raised through discussions and interviews.

ISSUE 1: Excessive Administrative Work and Stress as Deterrents to Teacher Retention and Quality Education Delivery in South African Schools

Sub issue 1.1: Contextualizing the Challenge - The literature review highlights a significant concern in South African education related to the excessive administrative workload and stress experienced by teachers. The Teaching and Learning International Survey (Talis) report, conducted by the Organisation for Economic Co-operation and Development (OECD, 2020), revealed that the teaching environment in South Africa has led to acute stress among a quarter of its teachers, prompting many to consider leaving the profession.

Sub issue 1.2: Impact on Teacher Retention and Educational Quality - The findings in literature review section 2.7 and informal discussions with teachers underscore that the stress levels reported by South African teachers exceed those of their OECD counterparts. This elevated stress is attributed to multiple factors, including the burden of learner achievement expectations, extensive marking demands, and overwhelming administrative tasks. Urban schools, dealing with overcrowding, are particularly affected by stress related to marking, while curriculum reforms introduced in 2011 have amplified administrative responsibilities.

Sub issue 1.3: Implications for Educational Policy - Teachers experiencing high stress are 40% more likely to express a desire to leave the profession within the next five years (Nkosi, 2020). The literature also demonstrates that over a quarter of South African teachers are contemplating leaving their roles in education within the same timeframe. Inadequate compensation further exacerbates teacher dissatisfaction, with a significant majority expressing discontent with their salary levels. These findings underline the pressing need for educational policies that address the root causes of stress and work-related strain among teachers. The Minister of Basic Education, Angie Motshekga, has emphasized the necessity of comprehensive policies to address teacher recruitment, retention, and professional development. The concern regarding excessive administrative workload and stress potentially leading to a loss of skilled educators resonates with the broader narrative of South African education, where the profession's quality and sustainability are at stake (Nkosi, 2020).

The survey's findings appear to converge with the views that such stressors could contribute to professional attrition, indicating a level of agreement on this matter.

Issue 2: Addressing Curriculum Overload: Teachers' Perspectives, Challenges, and Strategies for Effective Classroom Learning

Sub issue 2.1: Defining and Assessing Curriculum Overload - Building upon insights from the literature review, the notion of curriculum overload emerges as a pertinent concern in South African education. The cyclical challenges faced by the education system, as outlined in the literature, including issues related to curriculum revisions, implementation difficulties, and performance outcomes, contribute to the conversation surrounding curriculum overload. The observed struggles in implementing various curricular frameworks, such as OBE, C2005, and CAPS, highlight the complexity of managing curriculum content within limited instructional time. This resonates with the concept of curriculum overload, where the sheer volume of content to be covered becomes a potential impediment to meaningful and effective learning experiences (Jojo, 2019; Ramatlapana et al., 2012).

Sub issue 2.2: Impacts on Teachers and Learners - Aligning with the findings from various reports, curriculum overload exerts a significant impact on both educators and learners. The intensified demands of fitting additional material into the curriculum often lead to a rushed approach, potentially undermining the depth of comprehension and critical thinking among learners (Jojo, 2019; Bjorklund, 2015). The concerns expressed by educators about coping with emerging areas of study, such as digital skills and computational thinking, underscore the challenges associated with addressing the evolving needs of education within the confines of an overloaded curriculum (Ramatlapana et al., 2012; Spaull, 2013). The parallel drawn between curriculum overload and the challenges faced in South African education reinforces the urgency of considering strategies to alleviate this issue and promote effective learning.

Issue 3: Navigating Overcrowded South African Classrooms: Challenges, Effects, and Collaborative Strategies

Sub issue 3.1: Balancing the Learner-Educator Ratio (LER) - South Africa's education landscape contends with a persistent challenge – overcrowded classrooms. This phenomenon emerges as a consequence of a shortage of qualified educators and inadequate school infrastructure. This sub issue underscores the current national learner-educator ratio (LER), which stands at 35:1, a value notably higher than the Organisation for Economic Co-operation and Development's (OECD, 2020) international average of 16:1. A more unsettling facet is revealed through classes reporting LER values surpassing 50:1, raising concerns about the educational experience and outcomes of learners.

Sub issue 3.2: Impact on Learning and Academic Performance - Delving into this issue, the study by Joyce West et. al., (2020) concentrates on the ripple effects of overcrowded classrooms, particularly in South Africa's Foundation Phase. The study's qualitative research approach engages with 27 participants, encompassing heads of departments, school principals, higher education lecturers, and a representative from the Department of Basic Education. The research findings illuminate the adverse repercussions of overcrowding, manifesting in the form of suboptimal academic performance among South African learners. This phenomenon is evident through diminished scores in both national and international assessments. The research by West et al., (2020) attributes these challenges to an insufficiency of educators and subpar school infrastructure, intricately linked to the elevated learner-educator ratio witnessed in the South African context.

Sub issue 3.3: Embracing Collaboration for Mitigating Overcrowding - West et al., (2020) study highlights the multifaceted impact of overcrowded classrooms on teaching and learning dynamics. Among the consequences identified are educational neglect, behavioural issues, and a decline in teacher morale. Significantly, the study underscores the significance of collaborative measures, advocating for the collective engagement of key stakeholders such as School Management Teams and educators to tackle the complexities posed by overcrowding. Throughout the literature review chapter, the study delves into the contributing factors fuelling overcrowded classrooms, spanning from educator shortages to infrastructural inadequacies. The narrative underscores the connection between class size and academic achievement, highlighting the potential gains of smaller class configurations in enhancing learning outcomes. Furthermore, the study acknowledges the broader ramifications of overcrowding, ranging from psychological distress to cognitive impairment, impacting both learners and teachers. Rooted in the theoretical construct of the "fight or flight response" to stress, the study illuminates the intricate interplay between stressors, teachers' and learners' responses, and the eventual educational consequences arising from chronic stress (West, et al., 2020).

Issue 4: Combating the Dwindling State of Mathematics Education: Educators' Concerns

As highlight in an article written in Business Tech website[29] and the literature review section a grave concern in the South African education landscape is the marked decline in both the number of learners selecting mathematics courses and the dismal pass rates among those who do. These pressing issues prompted teachers to vocalize their apprehensions.

1. **Scarce Availability of Qualified Mathematics Instructors:** An acute shortage of proficient mathematics teachers stands as a central driver behind the erosion of mathematics education quality. The inadequacy of well-trained mathematics teachers adversely affects the teaching methods and ultimately the standard of instruction delivered (Nkosi, 2011).

[29] Business Tech is South Africa's largest business news website, which was started in 2008 by the media company, Broad Media. The publication's Editor is Quinton Bronkhorst. It features articles on finance, technology, industry, education, investing, and marketing topics.

2. **Implications on Vocational Trajectories:** The choice to eschew mathematics in favour of Mathematics Literacy has profound ramifications for learners' future professional pursuits. Beyond accountancy, numerous fields require mathematics as a prerequisite for tertiary education, potentially closing doors for those opting out of mathematics education.

3. **Transition to Mathematics Literacy:** The academic system urges learners grappling with mathematics to transition to Mathematics Literacy upon entering Grade 10. The motive behind this move often stems from the fear of attaining subpar mathematics pass rates and incurring unfavourable ratings from the Department of Basic Education.

   FIGURES OBTAINED ON LEARNERS TAKING MATHEMATICS LITERACY VERSUS MATHEMATICS

   I have sought specific data regarding the enrolment rates of learners in mathematics compared to mathematical literacy. While I am unable to disclose the school's identity due to anonymity concerns, I can provide the following figures:
   In 2023, approximately 165 out of 370 Grade 10 learners have chosen to take Mathematics. Meanwhile, the remaining 205 learners, constituting 55% of the total, have opted for Mathematical Literacy.

   NOTE: The above observation related to mathematics literacy is supported by the findings emphasized in the literature review section 2.7.

4. **Teacher Competence and Conceptual Explanation:** Even among the limited cohort of mathematics teachers, the prevalence of challenges is palpable. Many teachers themselves harbour discomfort and unease with the subject matter, resulting in instances where they furnish solutions without a comprehensive grasp of the underlying principles.

5. **Holistic Significance of Mathematical Learning:** Mathematics extends beyond numerical manipulation, playing a pivotal role in nurturing cognitive competencies

vital for contemporary learning, such as analytical thinking, effective communication, collaboration, and problem-solving. Unfortunately, the existing pedagogical strategies often neglect the foundational logical tenets of mathematics.

6. **Imperative for Transformative Action:** To counteract the downward trajectory of both learner engagement and performance in mathematics, decisive steps must be taken. The entrenched notion that mathematics is best confined within classroom boundaries, with minimal homework, necessitates a paradigm shift. Proactive measures encompass early intervention strategies, learner support systems, and empowerment of specialized educators.

Issue 5: Progressed Learners in Focus: Teachers' Perspectives on Policy Implementation and Academic Implications in South African Classrooms

1) **Definition and Purpose of Progressed Learners:** The discourse surrounding "progressed learners" and the subsequent implications of their integration into the education system highlights critical points, as elucidated by educators. This sub-issue delves into the core aspects of progressed learners, their definition, and the purpose behind their inclusion within the South African education framework.

2) **Definition of Progressed Learners:** Progressed learners are defined as learners who have been promoted to the subsequent grade, despite not meeting the entirety of the promotion prerequisites. This policy was initially introduced to encompass the General Education and Training (GET) phase, spanning Grades R to 9, and subsequently extended to the Further Education and Training (FET) phase, encompassing Grades 10 to 12. The primary intention behind this policy is to deter unnecessary dropout rates and extend learners the opportunity to attain an exit qualification, such as the National Senior Certificate (Juan et al., 2023).

3) **Retention vs Progression:** Engaging in a perpetual debate, educators grapple with the contrasting impacts of social promotion (progression) versus grade retention. Advocates of retention claim that it might lead to elevated dropout rates and limited long-term benefits. Conversely, proponents of progression believe that moving

learners through the educational trajectory is essential. Research findings hint that retention may offer transient advantages but fail to ensure enduring achievements (Juan et al., 2023).

4) **Challenges and Pressures on Teachers:** The incorporation of progressed learners necessitates teachers to allocate supplementary time and resources for administering additional support. Teachers are tasked with helping these learners bridge gaps in their foundational skills and knowledge. This added responsibility places considerable strain on educators, straining their time and resource allocation (Juan et al., 2023).

5) **Implications and the Need for Revision:** The revelations from the discussions cast doubt on the comprehension of grade repetition and retention, while also questioning schools' ability to assess learners' achievements accurately. The merit of progressing learners within the FET phase remains under scrutiny, especially as the education system embraces diversified pathways. This prompts the exploration of potential policy revisions that could appropriately place learners in academic or technical and vocational education streams, contingent on their individual abilities. The broader implications of including progressed learners in the education system underscore concerns regarding policy effectiveness, the precision of assessment methodologies, and the urgency for educational trajectories tailored to the distinct needs of diverse learners (Juan et al., 2023).

FIGURES ON PROGRESS LEARNERS AND RETAINED LEARNERS

I requested specific figures regarding progress learners and retained learners below are the average figures from one high school on retained and progress learners.

Term 1 2023 District: Ekurhuleni North School Name:
….. High School Report date: 28/3/2023

Number of learners in Grade 8 2023	501
Number of learners in Grade 8 2023 achieved	123
Number of learners not achieved Gr 8	378
Number of *Progress Learners* in Term 4 2022 Gr 8 (promoted from Gr 7)	312
Number of *Progress learners* Term 4 2022 achieved in Term 1 Gr 8	1
Number of Retained Learners in T4 2022 Gr 8	38
Number of Retained Learners in Term 4 2022 achieved in Term 1 Gr 8	0

Number of learners in Grade 9 2023	479
Number of learners in Grade 9 2023 achieved	153
Number of *Progress Learners* in Term 4 2022 Grade 9	188
Number of Term 4 2022 in Grade 9 *Progress Learners* achieved in Term 1 2023	55
Number of *Retained Learners* in Term 4 2022 in Grade 9	24
Number of Term 4 2022 *Retained Learners* Achieved in Term 1 Grade 9	1

NOTE:

- PROGRESSION LEARNERS: Means that the learner has not met the minimum requirements to pass the grade but is being moved on to the next grade with support.
- RETENTION LEARNERS: Means the learner will remain in the same grade for another year.

NOTE: The above observation related to progress and retained learners is supported by the findings emphasized in the literature section 2.7.

Issue 6: The Challenge of a Content-Heavy CAPS Curriculum: Teachers' Perspectives on South African Education

Sub issue 6.1: Content-Heavy Curriculum and its Impact on Learners

The implications of the content-heavy nature of the CAPS curriculum emerged as a significant concern among teachers. The curriculum's extensive coverage of topics and concepts has led to what educators refer to as a "mile wide but inch deep" approach. This approach, highlighted during the PowerPoint presentation and informal interviews, has fostered math anxiety among learners. The overwhelming amount of content impedes learners from establishing a strong foundational understanding of the subjects. This anxiety not only hampers learning but also diminishes learners’ confidence in their mathematical abilities (Ramatlapana et al., 2012; Spaull, 2013; Gatticchi, 2022).

Moreover, the voluminous curriculum has contributed to a surge in homework assignments, encroaching on children's play and relaxation time. This extensive workload has stirred concerns about learners' emotional well-being and social development. Teachers raised alarms about how this excessive homework load may lead to burnout and hinder holistic growth (Ramatlapana et al., 2012; Gatticchi, 2022).

Sub issue 6.2: Inadequate Time for Concept Consolidation

Teachers highlighted the struggle with limited time for consolidating concepts within the curriculum. The rapid progression from one topic to another prevents learners from thoroughly comprehending and internalizing the taught material. This fast-paced approach poses challenges for learners, particularly those with average to below-average processing capabilities. The resulting confusion and feeling of being overwhelmed contribute to heightened anxiety and stress levels among learners (Ramatlapana et al., 2012; Gatticchi, 2022).

Sub issue 6.3: Rigidity, Over-Assessment, and Teaching Creativity

Educators expressed their frustration with the inflexibility of the CAPS curriculum, which restricts their autonomy in teaching. The predetermined schedule leaves minimal room for adjustments based on learners' learning pace or comprehension levels. Teachers stressed the need for a more flexible approach to address individual learner needs effectively. The curriculum's focus on content delivery also obstructs teachers' creativity in the classroom. They find it challenging to engage learners while simultaneously covering the extensive syllabus. The issue of over-assessment further compounds the challenges, as teachers emphasized that a sole reliance on formal tests might not accurately reflect learners' true understanding. Additionally, the introduction of formal assessments at the Grade 1-3 levels was considered excessive and potentially detrimental to young learners' emotional well-being (Ramatlapana et al., 2012; Gatticchi, 2022).

Sub issue 6.4: Nurturing Critical Thinking and Creativity

The pervasive content-driven approach of the curriculum has given rise to concerns regarding the development of essential skills such as critical thinking, creativity, and problem-solving. Teachers lamented that the rigorous focus on content delivery leaves little space for teaching learners how to think critically and creatively. This deficiency in skills development is particularly troubling considering the growing demand for individuals with these competencies in the workforce. Educators stressed that cultivating a curriculum that places greater emphasis on skills development would better prepare learners for their future roles as informed and adaptable citizens (Ramatlapana et al., 2012; Gatticchi, 2022).

Reflection on Raised Issues in Relation to the Research Question: Bridging Philosophy of Mathematics and Meaningful Mathematics Teaching in South African Education

i *Alignment of Issues with Literature Review*

The issues surfaced through discussions and informal interviews with teachers serve to reinforce the findings documented in the literature review, specifically within the context of South Africa's education system. The parallelism between these sources underscores the consistent challenges that persist within the education system, particularly within the realm of mathematics education. Noteworthy concerns, such as the burdensome administrative workload, stress levels, curriculum saturation, overcrowded classrooms, diminishing quality of mathematics education, challenges in policy implementation, and the encumbrance of content-heavy curricula, resonate with the existing literature. This confluence of teacher insights and established literature collectively provides an extensive and holistic perspective on the various challenges encountered.

ii *Relevance to Research Question*

The research question, "Can philosophy of mathematics be a bridge to meaningful mathematics teaching?" should not be considered as a sporadic addition to instruction but rather as an integral element woven into the educational fabric. This perspective gains even greater significance in the present context where identified challenges highlight the pressing need for comprehensive reforms and targeted interventions within the education system.

Embedding the philosophy of mathematics within the educational framework offers a promising pathway to address these multifaceted challenges. The philosophy of mathematics can serve as a constant, much like an ever-present "white noise[30]," fostering critical thinking, refining problem-solving skills, and nurturing a deep comprehension of mathematical concepts. This infusion of philosophical principles acts as a counterbalance to the prevalent content-centric approach, thus effectively enriching the learning experience.

This integration not only holds the potential to elevate learner engagement but also to cultivate a holistic and enriching learning environment. Moreover, the integration of philosophical inquiry into mathematics education could provide valuable guidance to educators navigating policy

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[30] White noise refers to a sound that contains a mixture of all audible frequencies in equal amounts, creating a consistent, uniform sound that lacks any distinct patterns or variations. In the context you mentioned earlier, the phrase "white noise" was used metaphorically to signify a constant presence or background element that maintains continuity. Just as white noise is a consistent sound, the integration of philosophy of mathematics is described as being continuously present, providing a consistent and constant influence on the educational process. It's a way of emphasizing the idea that philosophy should be an ongoing and ever-present factor rather than something sporadic or intermittent.

implementation, managing curriculum saturation, and advancing learner-centred pedagogies. In essence, this seamless integration of philosophy has the power to instigate a transformative shift in the landscape of mathematics education within South Africa.

2 Candid Reflection on School Engagement and Limitations of this Study

i *School Engagement*

In the progression of my research process concerning mathematics education, I initiated correspondence with a total of 60 schools, soliciting their involvement in my study. Among this group, 10 schools encountered constraints due to pre-existing staff commitments and workloads, leading to their regrettably expressed inability to accommodate my study's requirements. Conversely, a promising response emerged from 10 schools, granting me the privilege to conduct my research within their premises.

i *Acknowledging Non-Responses*

It is imperative to acknowledge that a notable portion of the contacted schools—40 schools or 66%, did not furnish any response to my initial solicitation. The motivations underlying their non-response are varied, possibly encompassing administrative delays and inadvertent email oversights.

ii *Persisting Commitment*

Undeterred by the limitations introduced by non-responsiveness, I remained unwavering in my dedication to executing a meticulous and comprehensive study on mathematics education. To mitigate this limitation, I meticulously scrutinized existing research, as elaborated upon in the literature review, aiming to glean insights that could supplement the viewpoints that may have been inadvertently omitted due to non-engagement.

iii *Future Prospects and Appreciation*

I extend my gratitude to the schools that warmly embraced and endorsed my research pursuits. I eagerly anticipate the collaborative prospects extended by these accommodating schools, recognizing the potential for gathering invaluable insights to enrich my study. Concurrently, I remain receptive to the potential for forging partnerships with other educational institutions in the future, should circumstances align favourably. This, naturally, would entail securing fresh permissions and adhering to relevant departmental protocols.

Section 2: Field Notes from email correspondence with various entities.

These e-mails are included to provide evidence of my communication and attempts to engage with various stakeholders for this research on mathematics education.

EMAIL 1: Sent to a Professor at UCT - 1/6/2021

This email outlines regarding the potential benefits of incorporating Philosophy of Mathematics into teacher training.

Dear xxx,

I hope this email finds you well. My name is Isabel, and I am a mathematics teacher currently facing some frustrations in my profession. After much contemplation, I have identified three main areas of concern: the changing nature of learners, the state of the curriculum, and the inadequacy of teacher training, particularly in mathematics education.

Among these concerns, the issue of teacher training stands out to me as a critical area that requires further exploration and improvement. I firmly believe that the training of mathematics teachers should encompass a deeper understanding of the Philosophy of Mathematics. My thoughts on this were strengthened after reading works by Bertrand Russell on the subject.

The inclusion of Philosophy of Mathematics in teacher training could help educators gain valuable insights into the nature of mathematics itself and the complexities involved in answering fundamental questions about the subject. However, I am aware that introducing philosophical aspects into teacher training might require careful consideration and planning, as it may not align with the immediate goals and expectations of learners, who often focus on mastering methods and techniques.

Given your expertise and experience, I am reaching out to seek your valuable advice and insights on this matter. I would greatly appreciate your thoughts on the potential benefits of incorporating Philosophy of Mathematics into teacher training and any possible challenges or considerations that may arise in the process.

Your perspective on this topic would be invaluable as I plan to pursue further study and research in the field of teacher training, with a particular focus on mathematics education. Your guidance will undoubtedly contribute to the development of effective strategies for enhancing the quality of mathematics education and teacher preparation.

Thank you for taking the time to consider my inquiry. I look forward to your response and hope for an opportunity to discuss this topic in greater detail.

Best regards,
Isabel

EMAIL 2: Received from Professor at UCT - 1/6/2021

Dear Isabel,

I agree that it would be good for teachers of mathematics to have reflected a little on what mathematics is, and how hard that question is to answer. I must say that I was working in a maths clinic for matric learners and wanted to get them to puzzle a little about maths and its methods; but I don't think they want a perplexity introduced into the equation (!); rather they just want to master a method. So, I think it'd take a good of deal of thinking to introduce the more philosophical aspects productively.

Best wishes,
xxx.

Reflection on UCT professor's response:

The response highlights a crucial aspect of incorporating Philosophy of Mathematics into teacher training – the need for thoughtful and careful consideration. While acknowledging the value of having teachers reflect on the nature of mathematics and its complexities, he emphasizes the practical challenges that may arise. His experience in working with matric learners and their focus on mastering methods underscores the existing educational landscape's priorities.
This correspondence resonates with the idea that introducing philosophical aspects into education requires a shift in both mindset and approach. It confirms the notion that philosophy cannot be a sporadic add-on but must be embedded into the very fabric of teaching.

This observation aligns with the view that teachers play a pivotal role in shaping learners' perspectives and attitudes. Their consistency as role models influences learners' perceptions and willingness to engage with philosophical inquiries.
The response reinforces the idea that transforming education requires a holistic approach that involves teacher preparation, curriculum development, and a shift in the overall educational culture. It's evident that integrating philosophy into education necessitates a well-considered strategy that addresses both teachers' readiness to embrace philosophical elements and learners' receptivity to such an approach. This reflective exchange highlights the importance of long-term, systemic changes that value philosophical thinking as an integral part of education, rather than a mere supplement.

EMAIL 3: CORRESPONDENCE WITH GAUTENG DEPARTMENT OF EDUCATION - SECOND EMAIL FROM GAUTENG DEPARTMENT OF EDUCATION (GPEDU) 26/10/22

Please see some thoughts on your questions below.

QUESTION 1 Mathematics Literacy does have a place, but it has not achieved its goals? It has become the 'dumping ground' for those who do not do well in Pure Mathematics.

ANSWER 1 All Grade 9 learners are given an opportunity to CONTINUE with ***mathematics*** in grade 10.

Learners who have obtained less than 30% in Gr 9 are more inclined to take *Mathematical Literacy* in Gr 10. Many learners who fail mathematics in the **FET (Gr 10 to 12)** phase do then opt for *Mathematics Literacy* in the next grade or in the current grade provided they comply with policy requirements on change of subjects. ***The policy is clear that the parent has the prerogative on which subjects a learner offers in Further Education and Training FET phase.***

QUESTION 2 Why does the department of Education even allow some schools to drop Mathematics and Natural Science. This is inconceivable.

ANSWER 2 Mathematics and Natural Sciences offered in the Gr 4 to Gr 9 Phase, are compulsory subjects.

Mathematics is an *elective subject* in FET Phase. At **Ekurhuleni North District (EN),** the only schools that do not offer mathematics in The FET phase are the LSEN schools – *schools for learners with special needs*.

The department advocates for learners to take Mathematics. To this end in EN has two technical schools that do not offer Mathematics Literacy. **(This is not the complete picture: See "No Fees School" in the Literature Review Chapter 6 and ISSUE 4 above.)**

QUESTION 3 Are the qualifications and experience of mathematics teachers checked by the department?

ANSWER 3 For teachers to be employed by the department they must be *professionally qualified* and *registered with SACE*. These are the minimum requirements that teachers must meet. Further qualifications and minimum teaching experience / teaching experience in a subject are stipulated in the advertisements (Government gazette) when posts are available. Schools then verify these qualifications and experience.

School Governing Bodies also employ teachers on the schools' payroll. Schools then allocate teaching subjects according to the school's needs.

NOTE: South African Council for Educators (SACE) is the professional council for educators, that aims to enhance the status of the teaching profession through appropriate Registration, management of Professional Development and inculcation of a Code of Ethics for all educators. The SACE Council.

QUESTION 4 For me mathematics is a non-negotiable subject shouldn't the department be stricter in this regard.

No answer / No comment.

QUESTION 5 The policy directive is that mathematics be an elective subject.

ANSWER 5 There is a percentage of learners who may not require mathematics in their chosen career paths on completion of school. Here Mathematical Literacy is deemed to equip the citizen with a functional level of mathematical literacy to conduct everyday transactions. The focus here is also for citizens not to be duped in everyday transacting.

QUESTION 6 What is your department called. GPEDU stands for Gauteng Department of Education - but I want to know specifically the department name.

ANSWER 6 The name is Gauteng Department of Education.

E-MAIL 4: CORRESPONDENCE WITH GAUTENG DEPARTMENT OF EDUCATION - FIRST EMAIL FROM GAUTENG DEPARTMENT OF EDUCATION 11/10/22

Subject: RE: REQUESTING A MEETING

Hi Isabel,

I can only answer the first few of your questions that pertain to grade 7,8 and 9, unfortunately, the rest will have to be answered by the FET officials.

QUESTION 1 Is your department the department that issues **ATPs (Annual Teaching Plan)** for the teachers? If so, would you be able send me copies from grade 7 to grade 12?

ANSWER 1 The **Annual Teaching Plan ATPs** are compiled by **Department of Basic Education DBE** and made available on the website, I have attached the CAPs document as well as the **RATPs** (**Recovery ATPs**, these are the trimmed ATPs to assist educators with time, so that revision of the topic from the previous year can take place before this year's work is started.)

QUESTION 2 Who is involved for curriculum content – is it expert mathematicians? Is it previous teachers?

ANSWER 2 Curriculum content is developed by DBE. *Expert mathematicians*, *University professors*, *studies, and input from various experts from different countries are consulted*. Provincial and District officials normally provide input. Educators are required to submit their inputs to District officials who also provide input which are submitted to Provincial officials.

QUESTION 3 ANA has been scraped – is there any other departmental tests available for grades lower than 12?

ANSWER 3 Yes, **Annual National Assessment ANA** has been discontinued for now. In **Ekurhuleni North District EN**, grade 7,8 and 9 we provide the *common baseline assessments* that are based on the previous years' work and is administered at the beginning of each term. The aim of these assessments is for schools to analyse the results and determine any errors, misconceptions, mistakes and to check if the topics were taught in the previous year or not. The purpose of the baseline is to ensure that any challenges are addressed during the teaching of the topics or to do a bridging module before the topic is to be taught. In EN we also provide exemplar **School Based Assessment SBA** tasks and there are common District set papers for the June and November test. The aim is to ensure that the appropriate standard and the correct topics are assessed in each grade. Across the Province, this year, selected secondary schools are piloting the **General Education Certificate GEC** project, these schools are implementing the integrated term 3 **School Based Assessment SBA** task (project) in **Mathematics, Science, and Technology MST** and other subjects, at the end of the year, these schools will write the *GEC November test*. (See attached presentation).

PRESENTATION SENT as mentioned in her Answer 3 above.
CURRICULUM GEC SUBJECT ADVISOR TRAINING:

NOTE: The General Education Certificate (GEC) is a flagship programme of the Department of Education (DBE) designed to expand the learning pathways and success opportunities of South African Grade 9 learners. The GEC is intended to formally recognise learners' achievements at the end of the compulsory phase of schooling.

SLIDE 1: Gauteng Department of Education (GDE) VISION AND MISSION

VISION

Every learner feels valued and inspired in our innovative education system.

MISSION

We are committed to providing functional and modern schools that enable quality teaching and learning to protect and promote the right of every learner to quality, equitable and relevant education.

SLIDE 2: BACKGROUND

Forms part of the **National Assessment Framework (NAF)**

The research model is applied

Implication of the model latches onto the fact that:

- Not perfect
- There will be significant learnings
- Teachers and implementers bound to experience challenges

Preparation for full scale roll out

That in itself creates a crucial need to document challenges experienced during the pilot phase to ensure that the same mistakes are not made when moving to scale and eventually system-wide

SLIDE 3: GBROADER GOALS

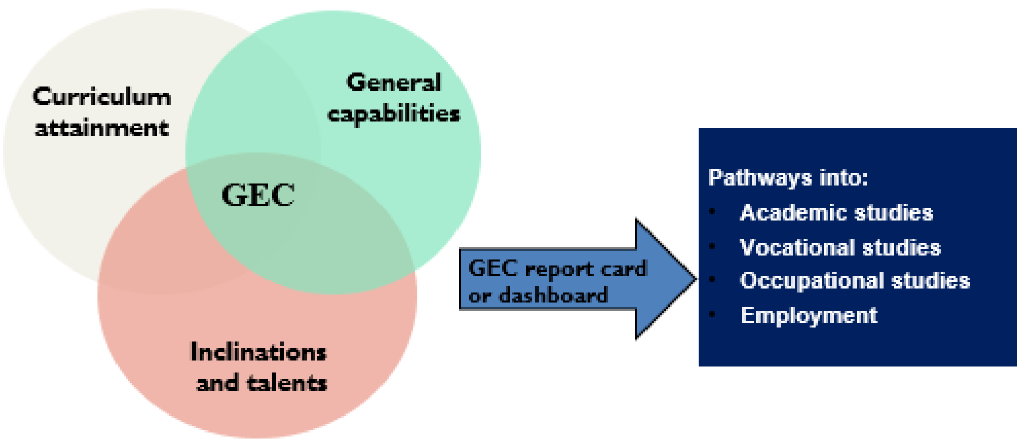
1) Recognize the **holistic skill set** for learners over schooling in GET.
2) Set **appropriate curriculum standards** to be achieved at the end of Grade 9.
3) Enable learners to **access further education streams** linked to the 3-stream model.
4) **Integrate and facilitate** the assessment of **21st-century skills** (critical thinking; creative thinking; collaboration; and communication).
5) To **award learners a report card/certificate** that reflects the skills, talents, and competencies that will be inclusive to all learners and assist them transition from school to work or further education.

SLIDE 4: GEC BREAKDOWN

- Inclinations
- Integrated Project
- Summative Test

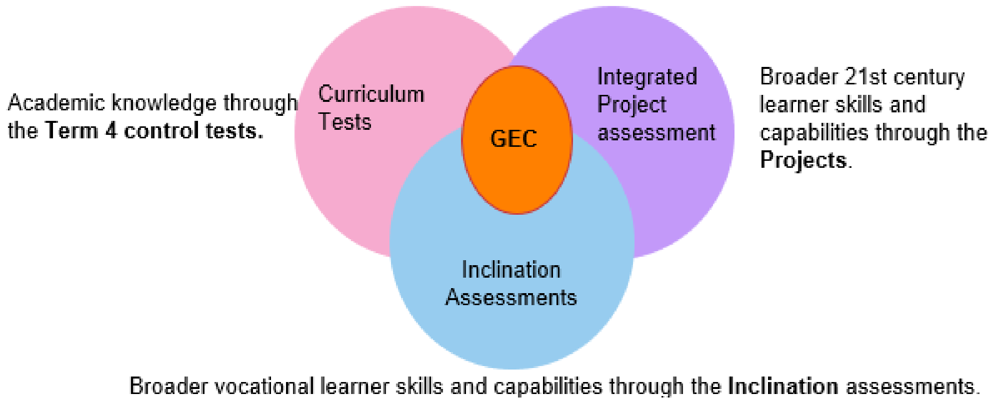
SLIDE 5: RE-SKILLING FOR THE 21st CENTURY

Holistically Assessing Learners

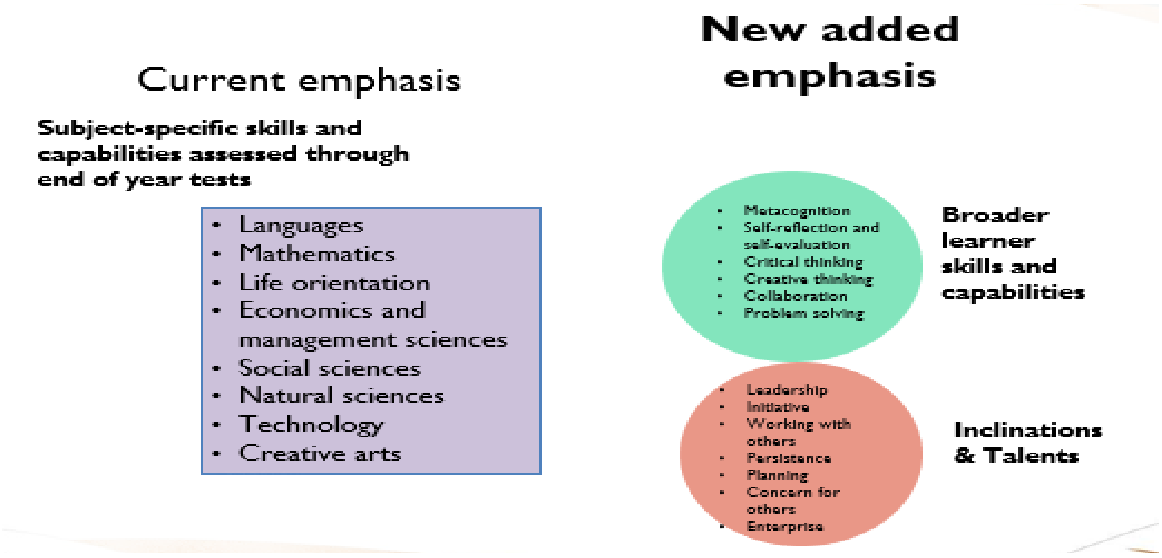


SLIDE 6: GEC MODEL

The model below was tailored to support the localised needs of our schools and learners.

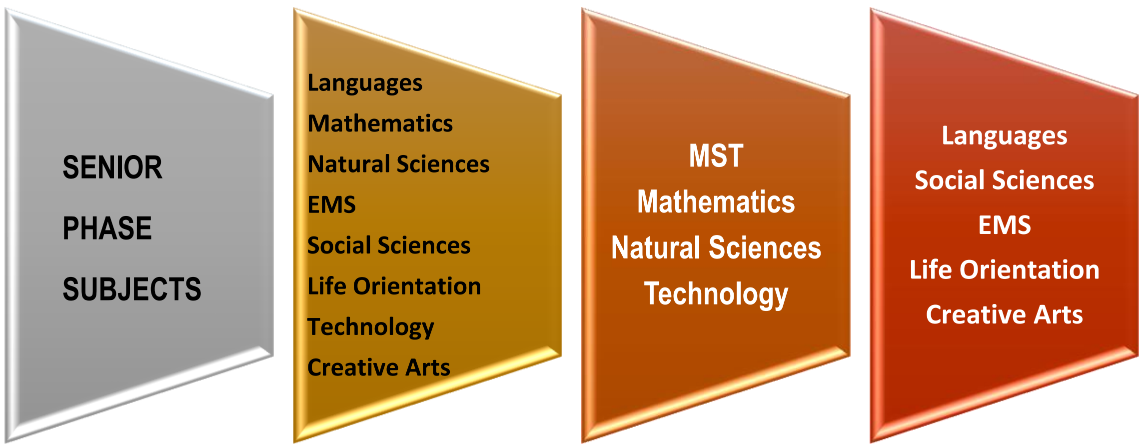


SLIDE 7: GEC COMPONENTS



SLIDE 8: EFFECT ON CURRENT STATUS

Need to integrate the content of the term: 9 Projects STREAMLINED to 2 Projects



SLIDE 9 PROJECT RESOURCES

Meant to simplify the implementation process

Resources provided by Department of Basic Education DBE:

- Videos,
- Learner Book with worksheets,
- Teacher Resource Book with notes and content explanation,
- Teacher's Project Notes (Assessment rubrics and mark allocation)

SLIDE 10 GUIDELINE: 13 STEPS FOR PROJECT BASED MODEL

PART 1: Inquiry-based learning	***PART 2: Problem-based learning***	***PART 3: Design-based learning***
Step 1: Prior knowledge	Step 5: Define	Step 9: Evaluate
Step 2: New knowledge	Step 6: Explore	Step 10: Prototype
Step 3: Order	Step 7: Brainstorm	Step 11: Feedback
Step 4: Apply	Step 8: Present	Step 12: Integration
		Step 13: Present
Assessment rubrics		

SLIDE 11 ASSESSMENT

This Integrated Project forms part of the school-based assessment.

- modeled on a Project based learning methodology,
- creating activities that more closely replicate real-life experiences, and thus develop real-life competencies.
- Includes School Based Assessment SBA project marks for all subjects (as per the amended Annual Teaching Plan ATP) and is aligned to the Curriculum Assessment Policy Statement CAPS amended ATP in each subject.

SLIDE 12 PROJECT TEAMS – SCHOOL-BASED – WHAT NEEDS TO HAPPEN AT SCHOOL?

- Need for teachers to work as team – collaboration
  - Be professional
  - Respect and support each other
  - Communicate effectively
- Planning crucial – Dedicated sessions for planning, reflection, and progress reports
- GRID – model of implementation (See slide 10)
- Change of mindset
- Create a safe environment for learners – non-judgemental

SLIDE 13 BASIC PRINCIPLES OF THE S.P.E.C.I.A.L CLASSROOM

1) S.P.E.C.I.A.L = *Social Interaction, Purpose, Enjoyment, Curiosity, Iteration, Active Engagement and Learner Autonomy*
2) A Safe and Caring Environment
3) Recognizing language barriers
4) Group work

5) The inclusive classroom
6) "Safe" assessment

NOTE: Slide 13 Is Last Slide as Part of The Field Notes: This presentation continuous, but I stopped at Slide 13 as it gives the jest of the presentation.

Reflections on The Above Departmental Presentation:

The subject at hand pertains to the inclusion of the philosophy of mathematics in teacher training and development within the context of South Africa's education system. Bearing in mind the extensive literature that discusses the failures of previous education department interventions in the country it is an issue that cannot be ignored. While intervention failures merits its own comprehensive study, it is essential to acknowledge its relevance when considering the potential impact of incorporating the philosophy of mathematics into teacher training initiatives. Such consideration is vital in evaluating whether this approach can address the challenges faced in the South African education landscape effectively. Rigorous examination of supporting evidence, feasibility, and potential drawbacks is imperative to determine the viability and importance of integrating the philosophy of mathematics in efforts to enhance mathematics education in South Africa.

QUESTION 4 Do we rely on TIMSS results for guidance?

ANSWER 4 We have used the TIMMS results to guide and reinforce the intervention and support resources provided to schools. Our learners struggled with the application and reasoning aspects of the TIMMs assessments. Therefore, we ensure that complex questions and problem-solving procedures are included in the resource material sent to schools and these cognitive levels are also addressed in our formal assessments, as per cognitive weightings in CAPS page 157. Our baseline assessments are compiled based on the findings from the DIBA assessment that was piloted by WITS (University of the Witwatersrand) in 2020.

NOTE: Trends in International Mathematics and Science Study (TIMSS)
The Trends in International Mathematics and Science Study (TIMSS) provides reliable and timely trend data on the mathematics and science achievement of U.S. learners compared to that of learners in other countries. TIMSS data have been collected from learners at grades 4 and 8 every 4 years since 1995, with the United States participating in every administration of TIMSS. TIMSS Advanced studies the achievement in advanced mathematics and physics of learners in their final year of secondary school. It has been conducted in 1995, 2008, and 2015, with the United States participating in 1995 and 2015. TIMSS and TIMSS Advanced are sponsored by the International Association for the Evaluation of Educational Achievement (IEA) and conducted in the United States by the National Center for Education Statistics (NCES).

QUESTION 5 How do you keep trace of progression learners?

ANSWER 5 For the progressed learners, there is a tracking tool that is compiled in SA SAMs, all SP maths educators are required to have an *intervention file*. In the file they need to have a list of the progressed learners, according to category ((1) progressed due to condonation, (2) progressed due to mark adjustment as well as (3) the retained learners are indicated). The learners' previous years formal assessments are analysed and the topics or aspects that learners are struggling with are recorded so that these aspects can be targeted during intervention and support. A timetable based on support and intervention for these learners are compiled at schools and evidence of the support must be available in the files and the learners' intervention and support books. The results of these learners for each SBA are tracked per term. Parents are also communicated with regarding an improvement or not per term. Officials monitor this intervention and support during school visits.

QUESTION: In the past we had Higher Grade Mathematics and Standard Grade Mathematics – this was scrapped because it was deemed discriminatory – However, my question is how is Mathematics Literacy not discriminatory?
NO RESPONSE

QUESTION: For me mathematics is a non-negotiable subject shouldn't the department be stricter in this regard.
NO RESPONSE

QUESTION 6: Are the qualifications and experience of mathematics teachers checked by the department

ANSWER 6: Yes, the qualifications of educators are checked at the time of employment, however, remember in primary schools, teachers manage and teach more than one subject and therefore do not necessarily have a maths qualification but would have teaching experience. In the secondary schools, the performing grade 8 and 9 maths teachers normally move up to FET and the grade 8 and 9 maths then becomes a filler subject for teachers who are short of periods and have maths teaching experience.

That concludes my responses, I hope that this is helpful. Don't hesitate to ask if anything else is needed. The FET officials should be able to assist you with the FET related questions.

EMAIL 5: TO GAUTENG DEPARTMENT OF EDUCATION REQUESTING A MEETING – after the replies from e-mails above.

This email is essential as it documents my attempt to set up a meeting with the Gauteng Department of Education to discuss various aspects related to mathematics education.

Subject: Request for Meeting - Research Inquiry

Dear xxxx,
I hope this email finds you well. My name is Isabel Schreiber, and I am currently conducting research on mathematics education in schools. Specifically, I am focusing on understanding the challenges faced by learners and educators in the field of mathematics.
I have read your informative responses to previous inquiries, and I believe that meeting with you would provide valuable insights for my research. I understand that this term is particularly busy for you, given the grade 12 exam monitoring, school visits, and Provincial meetings. Nevertheless, I am hoping to request a meeting with you to discuss some important aspects related to mathematics education.

I am particularly interested in gaining more information about the curriculum content development and the methods used to track progression learners. Additionally, I would like to explore the department's approach to addressing the challenges of learners who face difficulties in mathematics.

Given your expertise as a Senior Education Specialist and Subject Advisor for Senior Phase Mathematics, I believe your insights will be invaluable to my research.

If you are available for a meeting, I am flexible with the timing and can adjust to your schedule. Alternatively, if meeting in person is not feasible at the moment, I would greatly appreciate the opportunity to correspond via email to seek answers to specific questions related to my research. Your support and assistance in this matter would be highly appreciated, as I aim to shed light on the important issues surrounding mathematics education in our schools. I assure you that all information shared will be treated with the utmost confidentiality and used solely for academic purposes.
Thank you for your consideration, and I look forward to your response.
Best regards,

From: Department of Education

Subject: RE: REQUESTING A MEETING

Good day Ms Schreiber

This term is a bit of a challenging term for me, especially with the grade 12 exam monitoring, school visits and Provincial meetings taking place on most afternoons.
I will try and source some free time to meet, however it may not be in the immediate weeks ahead. In the interim if there is any information you require or questions that need to be answered, maybe send it via email and I can try and assist on that platform.

Reflection on the above communication with Gauteng Department of Education

The communication with the Gauteng Department of Education bears noteworthy significance within the scope of researching the philosophy of mathematics as a conduit to enhancing mathematics teaching and learning effectiveness. The following reflective points highlight the potential connections and contributions of this communication to the research:

1. **Insights into Policy Implementation**: The responses provide valuable insights into how educational policies are implemented on the ground. Understanding how policies like offering Mathematics Literacy as an alternative to Pure Mathematics are put into practice can shed light on the real-world implications of policy decisions.
2. **Alignment with Philosophical Approach**: By examining how the department's actions align with their educational philosophy, I can assess whether their practices reflect a focus on fostering deeper mathematical understanding and practical application, which is a central theme of this research. **Curriculum Development and Expertise**: The engagement of expert mathematicians, professors, and educational specialists in curriculum development underscores the significance attributed to subject expertise. Nonetheless, a disconnection between the department's initiatives and on-ground realities, as indicated in the literature review section 2.7, and my own observations, becomes evident. Further, this facet can enrich the research by illustrating how a philosophy of mathematics that underscores conceptual clarity harmonizes with the process of curriculum creation.
3. **Holistic Assessment and Intervention**: The information about baseline assessments, tracking of progressed learners, and intervention strategies provides evidence of the department's commitment to addressing individual learning needs and ensuring effective mathematics learning experiences. This aligns with this research's emphasis on a philosophy that supports varied learning styles and levels.

4. **Evidence-Based Practices**: The department's use of TIMSS results to guide teaching practices underscores the importance of evidence-based decision-making in education. This aligns with this research's focus on using philosophical principles rooted in evidence to enhance mathematics teaching.
5. **Collaboration and Engagement**: The initiation of a meeting and the department's receptiveness to providing a personalized appointment date and time, while also being open to dialogue through email, exemplify the prospects for collaboration between researchers and educational institutions. This interaction has fostered a reciprocal exchange of ideas and the validation of specific information discussed with educators, thus allowing this research to draw from real-world viewpoints while simultaneously offering valuable insights to educational stakeholders.
6. **Challenges and Considerations**: By exploring the challenges faced by learners, educators, and the department, I can gain a deeper understanding of how the philosophy of mathematics can address these challenges and foster a more effective teaching and learning environment.

In essence, the communication provides a rich source of data and insights that can be woven into this research narrative. It enables me to ground the emerging themes in the practical realities of education, showcasing how a philosophical approach to mathematics can bridge the gap between policy, curriculum, and effective teaching and learning strategies.

ANNEXURE G1: Initial Focused Open Coding

Research Question:

Could philosophy of mathematics be the bridge to meaningful mathematics teaching in the classroom?

The research question explores the potential role of the philosophy of mathematics as a bridge to meaningful mathematics teaching in the classroom. I intend to use open coding in a variant of grounded theory to identify emerging themes from various data sources such as

- literature review (Chapter 2)
- autoethnography (Chapter 4 Section 4.1),
- questionnaire responses (ANNEXURE E 1 & 2), and
- field notes: informal discussions and interview, and emails (ANNEXURE F).

Below is a breakdown of the approach and clarification of some aspects:

- **Open Coding and Grounded Theory**: Open coding is a qualitative data analysis method used in grounded theory research. It involves systematically examining data sources to identify concepts, categories, and themes without preconceived notions. Grounded theory aims to develop theories grounded in the data itself rather than imposing existing theories onto the data.
- **Emerging Themes:** When conducting open coding, one looks for concepts, ideas, or patterns that emerge from the data. These could be recurring words, phrases, or concepts that help make sense of the data and eventually highlight the emerging themes related to the research question.
- **Data Sources**: The data sources listed above provide diverse perspectives and insights into the research question.
- **Relating to Every Word of the Research Question**: While one does not necessarily need to find direct matches for every single word in your research question, one should aim to identify concepts, phrases, or themes that relate to the various components of the research question.

Literature Review Focused Coding Themes and Codes:

Philosophy of Mathematics

THEME 1: Distinct Characteristics of Mathematical Knowledge:

- Certainty and consistency in mathematical truths
- Necessity and contingency of mathematical truths
- Mathematics as a framework of consistent and necessary truths
- Role of certainty in mathematical education

THEME 2: *Historical Philosophical Views* on Mathematics:

- Plato's view on mathematical entities and Forms
- Aristotle's perspective on numbers as properties of objects
- Kant's transcendental idealism and mathematics

THEME 3: Nature of Mathematical Knowledge:

- Distinction between knowledge, belief, and opinion
- Role of evidence, justification, and epistemological questions
- Criteria of certainty, validity, veracity, and utility
- Knowledge built upon proofs, axioms, and reasoning

THEME 4: Logical Basis of Mathematical Knowledge:
- Logical reasoning and truth in mathematics
- Mathematical proof and deductive reasoning
- Justification through mathematical axioms and deductive reasoning

THEME 5: Development of Mathematical Knowledge:
- Evolution of mathematical reasoning over centuries
- Role of logical thinking and axioms in mathematical knowledge
- Relationship between mathematics and philosophical inquiry

THEME 6: *Philosophy of Mathematics* and *Mathematics Education*:
- Exploration of philosophy as a bridge to meaningful teaching
- Role of philosophical thought in mathematics education

THEME 7: Characteristics of Mathematical Knowledge:
- Certainty and consistency in mathematical knowledge
- Mathematics as mental discipline, logical reasoning, and rigor

THEME 8: *Historical Perspectives* on *Philosophy of Mathematics*:
- Ancient perspectives from philosophers like Plato and Aristotle
- Influence of philosophical thought on mathematics education

THEME 9: Characteristics of Philosophical Thought:
- Critical, comprehensive, analytical, synthetic, practical, theoretical, logical, empirical.

THEME 10: Philosophies: *Platonism*, *Logicism*, *Formalism*, and *Intuitionism*:
- Different responses to challenges in mathematical foundations

THEME 11: *Challenges and Shifts in Philosophical Approaches*:
- Decline of logicism, rise of set theory, challenges from various fields

THEME 12: Importance of Historical and Philosophical Understanding:
- Value of historical and philosophical insight for mathematics, science, and education

THEME 13: Philosophical Background: **Absolutism**:
- Various absolutist views on mathematical knowledge

THEME 14: Challenges to Absolutism:
- Paradoxes, critique of deductive logic, Ayer's perspective (see Section 2.3.3 Absolutist Philosophies of Mathematics)

THEME 15: Impact on Mathematics Education:
- Absolutist perspective's influence on teaching practices

THEME 16: Teacher Beliefs and Instructional Practices:
- Influence of philosophical views on teaching strategies

THEME 17: Discrepancies Between Philosophical Views and Practice:
- Gap between theoretical perspectives and classroom practice: Absolutism, Constructivism, Fallibilism and Structuralism.

THEME 18: Complex Relationship Between Truth, Knowledge, and Judgment:
- Interaction of cognitive and affective elements in beliefs

THEME 19: Influence of Socialization and Personal Beliefs:
- Impact of personal beliefs on teaching strategies

THEME 20: Teaching Mathematics with Absolutist Views:
- Focus on certainty and unchangeability

THEME 21: Perceived Nature of Mathematics:
- Public perception vs. diverse nature of mathematics

THEME 22: Encouraging Philosophical *Integration* in Teaching:
- Integrating philosophy to enrich teaching

THEME 23: Philosophical Identity of Mathematics Teachers:
- Correlation between teacher philosophy and practices

THEME 24: Shift from **Absolutism** to **Holistic Approaches**:

- Moving from fixed views to open-ended approaches

THEME 25: Evolution of Mathematics Teaching:

- Transformation from rigid to exploratory methods

THEME 26: Structuralist Perspectives:

- Various structuralist views on mathematics

THEME 27: Constructivist Perspectives:

- Constructivist approaches to mathematics education

THEME 28: Fallibilism Perspectives:

- Embracing fallibility and adaptation in learning

THEME 29: Absolutism Perspectives:

- Belief in absolute truths and unchanging principles

Bridge to Meaningful Mathematics Teaching

THEME 30: Importance of Mathematics:

- Universality and foundational nature of mathematics
- Principals of philosophical thought, meaning and characteristics

THEME 31: Mathematics as a Tool and Way of Thinking:

- Mathematics as a tool and cognitive approach
- Inquiry-based learning
- Socratic questioning
- Conceptual understanding
- Contextual learning
- Learner-centred teaching
- Interdisciplinary approaches
- Pedagogical approaches

THEME 32: Classroom Instruction

- Active learning
- Cooperative learning
- Differentiated instruction
- Assessment strategies
- Technology integration
- Classroom dynamics (teacher-learner ratio, phase)
- Teacher-learner interactions

Education System

THEME33: Education System Challenges:

- Weak teacher education, poor outcomes, curriculum challenges

THEME 34: Curriculum Reforms:

- Evolution of curriculum approaches over time

THEME 35: Performance and Assessment:

- Challenges and outcomes in numeracy performance

THEME 36: Calls for Improvement and Decolonization:

- Movement towards inclusive and diverse mathematics education

THEME 37: Mathematical Literacy and Language Issues:

- Role of language and literacy in mathematics education

THEME 38: Educational Ecosystem and Complexity:

- Education systems as complex environments

THEME 39: Teacher Competence and Professional Development:

- Importance of knowledgeable educators and professional growth

THEME 40: Parental Involvement and Learner Identity:

- Impact of parental involvement and learner identity

THEME 41: Indigenous Knowledge and Cultural Diversity:

- Incorporating indigenous knowledge and diversity in education

THEME 42: Inequalities and Policy Challenges:

- Addressing inequalities and policy implementation

THEME 43: Policy Implications and Recommendations:

- Suggestions for improving mathematics education

THEME 44: Mathematics Literacy Debate and Streams:

- Debate over mathematics streams and their management

THEME 45: Lack of Improvement and Ongoing Challenges:

- Challenges persisting in mathematics education

Field Notes Focused Coding:

THEME 1: Challenges in Mathematics Education:

- Excessive administrative workload and stress
- Teacher retention and educational quality
- Curriculum overload and challenges
- Overcrowded classrooms and balancing LER
- Impact on learning and academic performance
- Scarce availability of qualified instructors
- Implications on vocational trajectories
- Challenges of content-heavy curriculum
- Rigidity, over-assessment, and teaching creativity
- Nurturing critical thinking and creativity
- Progressed learners, definition, and purpose
- Retention vs. progression debate

THEME 2: Teacher-Related Challenges and Reflection:

- Challenges and pressures on teachers
- Inadequate time for concept consolidation
- Homework load and impact on learners
- Reflection on school engagement and limitations
- Frustrations in the teaching profession
- Concerns about teacher training
- Teacher competence and conceptual explanation
- Need for transformative action

THEME 3: Philosophy of Mathematics and Teaching (Prof Bernhard Weiss):

- Incorporating philosophy of mathematics
- Benefits of philosophical understanding
- Shifting mindset and approach in education
- Embedding philosophy into teaching
- Teachers as role models
- Systemic changes in education
- Philosophical thinking as integral

THEME 4: Engagement with Experts:

- Reflection on Professor Weiss's response

THEME 5: Department of Education Perspectives:

- Mathematics Literacy as a 'dumping ground'
- Learner choices, subjects, and qualifications

- Importance of mathematics as a subject
- Holistic assessment and learner autonomy
- Curriculum development, assessments, and skills

THEME 6: Teaching Approaches and Interventions:
- Annual Teaching Plans (ATPs)
- Project-based learning and cognitive levels
- GEC (General Education Certificate) goals
- Baseline assessments and interventions
- Tracking progression learners

Autoethnography Focused Coding:

THEME 1: Research Journey and Influence:
- Investigating philosophy of mathematics and teaching strategies.
- Personal experiences shaping the research journey.
- Impact of incidents and observations on teaching approach.
- Influence of experts' insights on pedagogy.

THEME 2: Curriculum Integration and Multidisciplinarity:
- Developing an integrated Grade 7 curriculum.
- Emphasizing interconnectedness of subjects.
- Connecting math, science, and technology.
- Addressing perception of math as abstract.

THEME 3: Philosophical Perspective on Mathematics:
- Various philosophical views of mathematics.
- Challenges in teaching abstract concepts.
- Belief in philosophy's role in understanding mathematical knowledge.

THEME 4: Influence of Experts and Philosophers (Grothendieck, Susskind, Greene, Russell, and Shapiro):
- Influence of mathematicians, physicists, and philosophers.
- Incorporating expert perspectives into pedagogy.
- Impact on learners' understanding, motivation, and enthusiasm.

THEME 5: Pedagogical Approaches and Philosophical Insights:
- Linking philosophy to engaging lessons.
- Applying Grothendieck's, Susskind, Greene, Russell, and Shapiro's philosophy in teaching.
- Encouraging exploration and construction.
- Bridging philosophy with effective teaching and learning.

THEME 6: Mathematics as a Universal Language:
- Mathematics as a tool to understand the universe.
- Role of mathematics in understanding nature's structure and behaviour.
- Aesthetic criteria in physics and mathematics.
- Examples of beautiful theories.

THEME 7: Complexity and Exploration:
- Exploration of complex and "ugly" theories.
- Role of complexity in deepening understanding.
- String Theory and unified theories.
- Exploring the fate of the universe.

THEME 8: Philosophy-Mathematics Teaching Nexus (links to science and technology):
- Connection between philosophy, mathematics, and teaching.
- Embracing complexity and philosophical discussions.
- Inspiring curiosity and deeper engagement.
- Shaping teaching philosophies based on philosophy.

THEME 9: *Emphasis* on Understanding and Curiosity:
- Prioritizing deep understanding and application.
- Cultivating intellectual curiosity.
- Promoting comprehensive understanding of relationships.

THEME 10: Interdisciplinary and Critical Thinking:
- Bridging gaps between disciplines.
- Nurturing critical thinking through exploration.
- Empowering learners to tackle complex challenges.

THEME 11: *Synthesis* of Metaphysics (first principals of knowledge, time, space, truth, infinite and infinitesimal etc) and Pedagogy:
- Harmonizing metaphysical considerations and pedagogy.
- Bridging abstract concepts and practical teaching.

THEME 12: Educator Empowerment and Continuity:
- Educators as architects of new education.
- Teachers leading learners on a mathematical journey.
- Philosophical ideas shaping education's trajectory.

Questionnaire Focused Coding:

THEME 1: Demographic Information:
- Age categories.
- Teaching experience.
- Highest level of education completed.
- Major/area of study during bachelor's.

THEME 2: Exposure to Mathematics and Philosophy:
- Exposure to philosophy of mathematics.
- Opinions on teaching mathematics with natural science.
- Views on the uniqueness of mathematics.
- Perception of mathematics' origin (discovered or invented).

THEME 3: Teaching Approaches and Methods:
- Preferred teaching approaches.
- Additional methods/approaches used or practiced.
- Beliefs about teaching methods and authenticity.

THEME 4: Views on Mathematics:
- Agreement with various statements about mathematics.
- Views on mathematics being logical.

THEME 5: *Teacher's* Experience and Perceptions:
- Teacher's experience with mathematics.
- Views on the difference between pure and applied mathematics.
- Emphasis on various aspects of mathematics.

THEME 6: Challenges and Strategies in *Teaching Mathematics*:
- Challenges of teaching mathematics.
- Learner attitude and engagement.
- Importance of basic skills.
- Instructional approaches and technology.
- Calls for change and creativity in teaching.
- Variety of resources used.
- Group work and collaboration.
- Emphasis on perseverance and learning from mistakes.
- Motivation and attitude in teaching.
- Preparing learners for the future.

THEME 7: General Comments:

- Miscellaneous comments, love for teaching math, interest in the subject / learner attitude, workload, progress learners, mathematics enrolment in favour of mathematics literacy.

ANNEXURE G2: Focused Open Coding: Dissecting and Refining Data

- **Data Dissection**: Throughout all stages, I dissected the data, breaking it down, and examining it closely. As I delved deeper into the data, I attempted to identify patterns, exceptions, and outliers. This process of breaking down the data is part of the analysis process.
- **Refinement**: Refinement happens throughout the entire analysis process. As I progressed from open coding to axial coding to selective coding, I continuously revisited and refined the themes and codes. This involved going back to the data, comparing codes, adjusting categories, and ensuring that the analysis accurately captures the nuances of the data.
- **Themes (From Open Coding)**: Themes are broader, overarching categories that emerge during the open coding process. They represent common threads or ideas that are present across the data. In this case, these themes are:
    1. Philosophy of Mathematics in Mathematics Teaching
    2. Challenges in Mathematics Education
    3. Nature of Mathematical Knowledge

THEME 1: Philosophy of Mathematics in Mathematics Teaching

a) Codes and Subcodes from Literature Review (Chapter 2):

Code 1: **Role of Philosophy in Mathematics Education:**
- **Exploration** of philosophy as a bridge to **meaningful** mathematics teaching.
- Philosophical thought's role in **enriching** teaching.
- **Integrating** philosophy to **enhance** teaching.

Code 2: **Teachers as Philosophical Role Models:**
- **Teachers' personal philosophies** influence teaching practices.
- Teachers' philosophies **impact** curriculum, lesson development, and assessment planning.
- **Incorporating** a philosophy of mathematics into **pedagogy**.

Code 3: **Philosophical Perspectives on Mathematical Knowledge:**
- **Mathematical Knowledge Characteristics: Certainty** and **consistency**.
- **Philosophical inquiry** into the **nature of mathematical knowledge** and its **relation to truth**.
- Philosophy of mathematics can **enhance learners' understanding** and **engagement**.
- **Moving** beyond mere theorem proving to **focus** on **meaningful mathematical experiences**.

Code 4. **Philosophers' Perspectives on Mathematical Entities:**
- **Plato's perspective** on mathematical **entities existing** in a **separate realm of Forms**.
- **Aristotle's contrasting** view of numbers as **properties of objects**.

- **Kant's perspective** on mathematical knowledge as **synthetic a priori**.
- **Indispensability Argument**: Justifying abstract mathematical objects' existence.

Code 5: **Constructivist Approaches and Philosophies:**
- **Constructivist perspectives** emphasize **learner-centred learning** and **active engagement**.
- **Intuitionism** emphasizes **individual mathematical intuition**.
- **Ethnomathematics** and **decolonization** challenge **universality** and promote **inclusivity**.
- **Non-Western Contributions to Mathematics**: Acknowledging **diverse historical perspectives**.

Code 6: **Impact on Teaching and Learning:**
- Philosophy of mathematics **shapes the approach** to mathematics education.
- **Rational inquiry**, **exploration of assumptions**, and **synthesis of knowledge**.
- **Fallibilism** recognizes **imperfections** and **openness** to revision.
- **Structuralism** provides insights into the **organization and relationships within mathematical concepts**.
- Encourages **critical thinking**, **creativity**, and **active knowledge construction**.

b) Codes and Subcodes from Field Notes (ANNEXURE F):

Code 1: **Role of Mathematics Education Philosophy**
- **Teachers' perspectives** on the CAPS curriculum reflect a particular philosophy.
- **Concerns** about **content-heavy** curriculum and **emphasis on content delivery**.

Code 2: **Impact on Holistic Learning**
- Curriculum's **workload** and **lack of time** raise questions about **holistic development**.
- **Balancing mathematical knowledge** with **emotional well-being** and **social growth**.

Code 3: **Challenges when Incorporating Philosophy of Mathematics and TEACHER-TRAINING**
- Highlighting **mindset shift** and **transformative education**.

c) Codes and Subcodes from Questionnaire (ANNEXURE E1 & 2)

Code 1: **Philosophy of Mathematics in Mathematics Teaching: Historical Context**
- Teach history of mathematics to **show evolution and context of knowledge**.

Code 2: **Conceptual Understanding**:
- **Break down** concepts into **manageable parts for understanding**
- **Link parts to whole**.
- **Highlight connections** and **links** between mathematical concepts.
- Understanding the **origins and philosophies** of mathematics is essential for teaching.

Code 3: **Multiple Representations**:
- **More than one representation** should be used in teaching mathematics topics.

Code 4: **Interdisciplinary Impact:**
- Mathematics is **intertwined** with **other subjects** and **contributes** to various **fields**.

Code 5: **Cognitive Development:**
- Mathematics teaches **critical** and **logical thinking**, **fostering unique cognitive development**.

Code 6: **Characteristics of Mathematics:**
- Mathematics is **logical and follows a set of rules**.
- Mathematics can be **both theoretical and practical**.
- Pure Mathematics vs. Applied Mathematics **distinction is acknowledged**.
- **Philosophers who were mathematicians possess logical and critical thinking skills**.

- **Visualization** is important in teaching mathematics.
- **Mathematics requires both** logical and creative thinking.
- **Importance of** understanding underlying concepts and principles.
- Mathematics is **essential for understanding the physical world**.

Code 7: **Teaching Approach:**

- Mathematics should be **taught with understanding**. (teacher)
- **Learner-centred** teaching is important.
- **Technology should be treated as a tool**, not the process itself.
- Using **technology to** enhance mathematical understanding.
- Incorporating **real-world applications of mathematical concepts**. (Other subjects: geography, natural science, technology, sport)
- Fostering a **love for mathematics through creative thinking and problem-solving**.
- **Motivating** learners to **engage** with mathematics.

Code 8: **Attitude and Mindset:**

- **Positive attitude** towards mathematics is essential for success.
- Need to **change learners' mindset** about mathematics.
- **Emphasizing the** value of mathematics in daily life and other subjects.
- **Mathematics as an important life skill**.

d) Codes and Subcodes from Autoethnography (Chapter 4, Section 4.1):

Code 1: **Philosophical Inspiration**

- Drawing from the philosophies of Grothendieck, Susskind, Greene, Russell, and Shapiro.

Code 2: **Dynamic Bridge**

- Mathematics as a dynamic bridge connecting philosophy to education.

Code 3: **Fostering Curiosity**

- Russell's perspective on igniting curiosity through mathematics.

Code 4: **Relevance and Appreciation**

- Highlighting Greene's and Susskind's emphasis on connecting mathematics with scientific concepts for relevance and applicability.

Code 5: **Problem-Solving Enhancement**

- Shapiro's structuralist philosophy emphasizing mathematical models and problem-solving skills.

Code 6: **Interdisciplinary Exploration**

- Susskind's, Greene's, and Russell's contemplations on interdisciplinary connections in mathematics.

Code 7: **Cultivation of Critical Thought**

- Nurturing critical thinking and curiosity inspired by Shapiro's and Russell's insights.

THEME 2: Challenges in Mathematics Education

a) Codes and Subcodes from Literature Review (Chapter 2):

Code 1: **Defining Knowledge**

- Differentiating between belief, opinion, and knowledge based on conclusive evidence.

Code 2: Epistemology in Mathematics

- The role of epistemology in understanding knowledge and its sources.

Code 3: Philosophy and Mathematics Connection

- The significance of historical and philosophical understanding in fostering independent thinking among scientists.

Code 4: Philosophical Inquiry

- Characteristics of philosophical thought, including critical analysis, logic, empiricism, and synthesis.

Code 5: Value of Philosophy

- Russell's perspective on the value of philosophy in providing unity and structure to scientific disciplines.

Code 6: Transition to Science

- The concept that clear answers often lead to the transition of a subject from philosophy to science.

Code 7: Disconnect between Science and Philosophy

- Einstein's observations on the disconnect between scientists and philosophers, leading to a separation between science and philosophy.

Code 8: Evolution of Mathematical Foundations: Challenges and Diverse Perspectives

- The crisis in the foundation of mathematics led to the emergence of logicism, formalism, and intuitionism in response to challenges in establishing reliable mathematical foundations.
- Gödel's Incompleteness Theorems demonstrated the limitations of formal mathematical systems, undermining logicism's aim to reduce all of mathematics to a complete logical system.
- The rise of set theory and axiomatic approaches offered alternative foundations for mathematics, challenging the traditional logicist perspective.
- Challenges from various mathematical fields, such as geometry and arithmetic, highlighted that mathematics encompasses more than just formal logic.
- Alternative schools of thought, including intuitionism and constructivism, further challenged the traditional logicist perspective.
- Diversification of mathematics into specialized areas reduced the practicality of reducing all mathematics to a single logical foundation.
- Fallibilism introduced the idea that mathematics, like the natural sciences, is subject to criticism, revision, error, and correction.

Code 9: Curriculum Reforms and Implementation Challenges

- The challenges related to the introduction and implementation of various curricula (OBE, C2005, RCNS, CAPS) highlight the complexities of curriculum design, teacher training, and adapting to changing educational philosophies.

Code 10: Teacher Training and Professional Development

- Weak Initial Teacher Education (ITE) and ineffective Continuous Professional Development (CPD) contribute to below-average learning outcomes. This emphasizes the need for ongoing training and support for teachers to improve mathematics education.

Code 11: Mathematics Literacy and Language Issues

- The introduction of Mathematical Literacy as an alternative to Mathematics raises issues about its objectives, impact on learners' performance.
- The language barrier affects learner engagement and understanding in mathematics.

b) Codes and Subcodes from Field Notes (ANNEXURE F):

Code 1: Challenges in Mathematics Education:

- **Curriculum Challenges**:
- Curriculum Overload
  - Discussion on curriculum overload and its impact on mathematics education.
  - Struggle to balance curriculum content within limited instructional time.
- **Teacher Challenges**:
- Teacher Stress and Retention
  - Excessive administrative workload, stress among teachers, and its impact on retention.
  - Relevance to mathematics educators due to the demanding nature of the subject.
- **Classroom Challenges**:
- Overcrowded Classrooms
  - Challenge of overcrowded classrooms affecting effective mathematics instruction.
  - Elevated learner-educator ratio and its impact on academic performance.
- **Course Selection Challenges**:
- Mathematics Literacy vs. Mathematics
  - Declining number of learners selecting mathematics courses.
  - Issues related to mathematics literacy versus mathematics education.
  - Implications of course selection on learners' academic and vocational trajectories.

c) Codes and Subcodes from Questionnaire (ANNEXURE E1 & 2)

Code 1: Diverse Learning Styles
- Preferences for varied teaching approaches to cater to different learning styles.

Code 2: Struggles with Abstraction
- Disagreement on mathematics being primarily an abstract subject.

Code 3: Talent vs. Effort
- Disagreement with the notion that some learners have a natural talent for mathematics.

Code 4: Practice for Struggling Learners
- Disagreement with the effectiveness of giving struggling learners more practice alone.

Code 5: Need for Support
- Recognition of the importance of support and guidance for learners.

Code 6: Curriculum Constraints
- Challenges in fitting diverse methods within the limited curriculum time.

Code 7: Transformation
- Transformation to less conventional teaching methods can be met with **reluctance**.

Code 8: Conventional teaching methods may constrain creative thinking.
- Ineffective teaching methods whether internet-based or worksheet-based, are not effective.

Code 9: Individual differences among learners need to be considered.
- Math teachers should know effective teaching practices, know underlying concepts and connections of the mathematics they teach.
- Some teachers may be resistant to change.
- Varying levels of understanding among learners.
- Learners entering Intermediate Phase with weaker academic skills.

- Learners' hesitance to problem solve and work independently.
- Fear of making mistakes and its impact on learning.
- The need for personalized and adaptable teaching methods.
- Incorporating interactive and engaging lessons.
- Addressing learner laziness and lack of motivation.
- Handling learners' aversion to math and perception of difficulty.
- Overcoming traditional approaches to teaching.

d) Codes and Subcodes from Autoethnography (Chapter 4, Section 4.1):

Code 1: Educational Transformation
- The potential for a transformative pedagogical shift.

Code 2: Philosophy as a Catalyst
- The infusion of philosophy of mathematics into education as a catalyst for improvement.

Code 3: Integration of Philosophy
- Integrating philosophy to elevate mathematics education.

Code 4: Philosophical Transformation
- Philosophy's role in reshaping teaching practices.

Code 5: Relevance and Real-World Connection
- Emphasis on mathematics' relevance and applicability beyond academia.

Code 6: Problem-Solving and Analytical Skills
- Enhancing problem-solving skills through mathematical concepts.

Code 7: Interdisciplinary Unity
- The role of interdisciplinary connections in education.

Code 8: Cultivation of Critical Thought
- Nurturing critical thinking and metacognition through philosophy-infused education.

THEME 3: Nature of Mathematical Knowledge

a) Codes and Subcodes from Literature Review (Chapter 2):

Code 1: Mathematical Knowledge Characteristics
- The distinctive characteristics of mathematical knowledge, including certainty and consistency.

Code 2: Philosophical Inquiry into Mathematics
- The exploration of mathematical truths, their nature, and their connection to the physical world.

Code 3: Plato's Perspective
- Plato's view of mathematical entities as existing in a realm of Forms.

Code 4: Aristotle's Perspective
- Aristotle's perspective on numbers as properties of objects in the physical world.

Code 5: Kant's Transcendental Idealism
- Kant's perspective on mathematical knowledge as synthetic a priori and derived from reason and experience.

Code 6: Nature of Mathematical Statements
- The examination of mathematical statements, their truth or falsity, and their role in mathematical reasoning.

Code 7: Philosophical Thought Characteristics

- Characteristics of philosophical thought, such as critical analysis, logical reasoning, synthesis, and practicality.
- Different schools of thought, including logicism, formalism, and intuitionism, offer distinct perspectives on the nature of mathematical knowledge.
- Absolutist philosophies view mathematical knowledge as certain, unchangeable, and objective, but they faced challenges when paradoxes and contradictions were discovered.
- Fallibilist philosophies recognize the imperfections and open nature of mathematical knowledge, emphasizing human engagement, controversy, and revision.
- Fallibilism aligns with the view that mathematics is a set of social practices intertwined with history, individuals, institutions, symbolic forms, and power relations.
- The relationship between beliefs and attitudes towards mathematics is complex, and individuals may hold combinations of absolutist and fallibilist views.
- A fallibilist philosophy combined with humanistic values can lead to a more connected and creative approach to mathematics education.

Code 8: Decolonization of Mathematics and Ethnomathematics

- These concepts challenge the notion of mathematics as an absolute and universal truth, suggesting that mathematical knowledge is influenced by cultural perspectives and contexts. This aligns with a fallibilist understanding of mathematics, emphasizing its relativity.

Code 9: Recognizing Non-Western Contributions to Mathematics

- Highlighting non-Western contributions emphasizes the diverse origins of mathematical knowledge and challenges the Eurocentric view of mathematics as solely a Western endeavour.

Code 10: Language of Instruction Issue

- The impact of language on mathematical learning underscores the importance of language as a medium for conceptual understanding and mathematical reasoning. It suggests that mathematical knowledge is not independent of the linguistic framework in which it is conveyed.

Code 11: Teacher Confidence and Learner Performance

- The disparity between teacher confidence and learner performance points to the complexity of factors influencing mathematical achievement. This highlights the multifaceted nature of mathematical knowledge acquisition and the role of various factors beyond pedagogical confidence.

Code 12: Integrating Indigenous Knowledge Systems

- The integration of indigenous knowledge systems implies that mathematical knowledge is not limited to formal academic contexts but can be derived from diverse cultural sources. This challenges the traditional view of mathematical knowledge as solely arising from formal mathematical disciplines.

b) Codes and Subcodes from Field Notes (ANNEXURE F):

Code 1: Shallow Understanding and Anxiety

- Discussion on the "mile wide but inch deep" curriculum approach.
- Concerns about learners' shallow understanding and associated anxiety.

Code 2: Content vs. Conceptual Learning

- Discussion on inadequate time for concept consolidation.
- Fast-paced curriculum delivery and tension between content-focused learning and deep conceptual understanding.

Code 3: **Creativity and Critical Thinking**

- Discussion on curriculum rigidity and lack of emphasis on critical thinking and creativity.
- Alignment with broader discussions about the nature of mathematical knowledge.

Correspondence and Insights:

Email 1: Sent to Professor Bernhard Weiss University of Cape Town - 1/6/2021

- Discussion of gaining insights into the nature of mathematics through Philosophy of Mathematics in teacher training.
- Professor Weiss's response emphasizing the philosophical complexity of mathematics and contrasting practical goals.

Email 5: To Gauteng Department of Education Requesting a Meeting

- Seeking deeper insights into curriculum content development, learner progression tracking, and the department's approach to addressing mathematical challenges.
- Gauteng Department of education responses explaining curriculum content development and consultation with experts for mathematics education.

c) Codes and Subcodes from Questionnaire (ANNEXURE E1 & 2)

Code 1: **Perspectives on Mathematics**

- **Representation and Realism:** Different perspectives on whether mathematics is primarily abstract or a formal representation of the real world.
- **Practical Application:** Recognition that mathematics serves as a practical and structured guide for addressing real situations.

Code 2: **Approaches to Learning**

- **Multidimensional Learning:** Importance of understanding fundamental concepts and their connections in mathematics.

Code 3: **Algorithmic Learning**

- Diverse opinions on whether mathematics should be learned as sets of algorithms or rules.

Code 4: **Computational Skills**

- Disagreement on whether basic computational skills are sufficient for teaching elementary school mathematics.

Code 5: **Relevance of History**

- Recognition that understanding the origins of mathematical concepts is important for teaching.

Code 6: **Inclusiveness of Representations**

- Agreement on using multiple representations in teaching mathematics topics.

Code 7: **Key Concepts in Teaching Mathematics**

- Mathematics is discovered and invented; both perspectives are present.
- Mathematics requires understanding beyond memorization.
- Real-world applications are emphasized.
- Mathematical concepts and connections are essential for successful teaching.
- Misconceptions vs. mistakes are differentiated.
- Mathematics involves both procedural and conceptual understanding.
- Understanding mathematics is more important than just obtaining the correct answer.
- Teachers should be knowledgeable and skilled in teaching practices.

Code 8: **Learning Strategies and Skills**

- Importance of mastering basic mathematical skills.
- Basic skills as a foundation for advanced concepts.

- Focus on understanding reasoning behind mathematical ideas.
- Connecting mathematical concepts and recognizing relationships.
- Perseverance and productive struggles in problem-solving.
- The importance of problem-solving skills in mathematics.
- Learning from mistakes and valuing trial and error.
- Collaboration and group work for peer learning.
- Internet as a valuable resource for learning and instruction.

d) Codes and Subcodes from Autoethnography (Chapter 4, Section 4.1):

Code 1: Philosophy's Influence

- Influence of philosophy on understanding mathematics' nature.

Code 2: Curiosity and Appreciation

- Emphasis on cultivating curiosity and appreciation for mathematics.

Code 3: Relevance and Applicability

- Connecting mathematical concepts to real-world applications.

Code 4: Problem-Solving Skills

- The role of mathematical models and problem-solving skills.

Code 5: Interdisciplinary Connections

- Highlighting connections between mathematics and other disciplines.

Code 6: Cultivation of Critical Thought

- Nurturing critical thinking and holistic comprehension.

Code 7: Philosophical Understanding

- Philosophical perspective deepening learners’ understanding of mathematics.

Code 8: Interdisciplinary Threads

- Unification of academic fields through interdisciplinary connections.

Code 9: Teacher Competence and Conceptual Explanation

- Emphasis on accurate explanations, logical proofs, and solid grasp of concepts. The role of teacher competence in conveying mathematical knowledge.

ANNEXURE G3: Axial Coding Matrix

The transition from open coding to axial coding marks a significant stage in the research analysis, where one delves deeper into the *interconnected relationships* and *cause-effect dynamics* that have emerged from our investigation into the philosophy of mathematics and its implications for mathematics education. While open coding allowed us to identify and categorize key concepts, axial coding takes us a step further by systematically organizing these concepts into patterns that elucidate the complex interplay among various categories and subcategories.

This annexure presents the results of axial coding, which have led to uncovering eight distinct patterns that capture the multifaceted interactions within the research. These patterns encapsulate the intricate connections between *philosophical insights*, *teaching practices*, *challenges in education*, and *the nature of mathematical knowledge*. They offer a comprehensive view of how these aspects intersect and contribute to the landscape of mathematics education, shedding light on the nuanced dynamics inherent in our study. These patterns not only help make sense of the rich data we have gathered but also provide valuable insights into the broader implications of our research findings. Each pattern highlights a particular facet of the relationship between philosophy and mathematics education, offering a deeper understanding of how these elements influence and shape each other. By documenting these patterns, one aims to provide a structured analysis that allows one to explore the underlying themes and connections, ultimately contributing to a more comprehensive and holistic interpretation of the research.

The patterns presented here are not isolated concepts but rather interwoven threads that collectively contribute to the intricate fabric of the philosophy of mathematics and its impact on teaching practices. As one delves into these patterns, one is taken through the interconnected world of philosophical exploration, educational challenges, and the transformative potential of mathematics education.

Axial Coding: Graphic Representation of Relationships and Connections

Patterns	**Subcategories**	**Cause-Effect Relationships**
1. Philosophy of Mathematics in Mathematics Teaching	Philosophy's Role in *Enhancing* Teaching and Learning	• Integration of philosophical thinking → Shifting mindset and approach in education • Bridging abstract concepts through philosophy → Making complex concepts relevant
	Application of Philosophical Ideas in teaching	• Applying philosophical ideas → Linking philosophy to engaging lessons • Shaping teaching philosophies based on philosophy of mathematics → Enhancing teaching practices
	Impact of philosophy on teaching practices – emphasis on understanding and application	• Influence of philosophical understanding → Integration, enhances teaching, fosters critical thinking and problem-solving skills • Promoting deep understanding and application → Encourages comprehensive understanding of mathematical relationships
2. Challenges in Mathematics Education	Education System *Challenges*	• Weak teacher education and outcomes → Curriculum challenges and performance issues in mathematics education • Calls for improvement and decolonization → Inclusive and diverse mathematics education
	Real-World *Challenges* Faced by educators	• Curriculum overload and content-heavy approaches → Teacher stress and challenges in managing diverse classrooms • Overcrowding and lack of support for struggling learners (e.g., progress learners) → Hindrance in providing effective mathematics instruction and addressing individual learner needs.
	Need for *Transformative Action*	• Nurturing critical thinking and creativity → Reflection on school engagement and limitations → Calls for change and creativity in teaching
3. Nature of Mathematical Knowledge	*Characteristics* of Mathematical Knowledge	• Certainty, consistency, necessity, and contingency of mathematical knowledge → Logical basis for understanding and conveying mathematical truths • Role of evidence, justification, and epistemological questions → Differentiation of mathematical knowledge from mere belief, importance of justification

	Differentiation from Belief and Opinion	• Criteria of certainty, validity, veracity, and utility → Clear differentiation of mathematical knowledge from subjective belief and opinion • Distinct nature of mathematical knowledge → Emphasis on logical reasoning and accurate explanations for conveying mathematical knowledge.
	Teacher *Competence* in Conveying knowledge	• Role of logical reasoning and clear explanations → Effective communication of mathematical knowledge, fostering learner understanding
4. Teacher Reflection and Empowerment	*Role of teachers* in Education	• Teachers as role models and educators → Positive influence on the learning experience, shaping learners' perceptions and attitudes toward mathematics. • Influence of philosophy and experts on teaching approaches → Adoption of innovative teaching methodologies and strategies, promoting learner-centred learning.
	Incorporating Personal Experiences	• Linking personal journey with pedagogical strategies and insights → Enriched teaching approaches, infusing personal experiences for relatability and connection. • Acknowledgment of the impact of personal experiences on teaching philosophy → Enhanced pedagogical authenticity, contributing to the development of a teacher's unique philosophy.
5. Curriculum and Teaching Approaches	Curriculum *Challenges* and Approaches	• Curriculum overload, rigidity, and content-heavy methods → Impaired teacher effectiveness and reduced learner enthusiasm for mathematics. • Encouragement of creative and transformative teaching approaches → Enhanced learner engagement and comprehension, promoting critical thinking and problem-solving skills.
6. Educational System Improvement	Calls for Educational *Reforms*	• Inclusive and diverse mathematics education → Mitigation of historical disparities and fostering a more equitable learning experience. • Addressing issues of inequality, diversity, and decolonization → Creating a balanced and inclusive educational environment that reflects diverse perspectives. • Need for educational improvements → Enhanced quality of mathematics education, aligning with evolving educational needs.
7. Connection Between Philosophy and Mathematics	*Relationship between Philosophy, Mathematics and Teaching*	• Philosophy enhances teaching through discussions and open-ended approaches → Fostering an environment of inquiry and

		exploration, promoting critical thinking and creativity in learners. • Inspiring curiosity and deeper engagement in mathematics → Enhanced interest and involvement in mathematical concepts, promoting a lifelong appreciation for the subject
8. Interdisciplinary and Holistic Approach	*Promoting Holistic Learning*	• Interdisciplinary teaching approaches → Broadened understanding of mathematics in relation to other disciplines, fostering a holistic view of knowledge. • Significance of critical thinking and holistic learning → Enhanced problem-solving skills and a deeper comprehension of mathematical concepts through a holistic lens. • Embracing complexity and exploring interconnectedness → Encouraging learners to see mathematical concepts as interconnected and reflective of real-world complexity.

Documenting Patterns Based on Axial Coding Matrix

Patterns (from Axial Coding): Within the context of this research on the profound interplay between philosophy of mathematics and mathematics education, the axial coding matrix facilitates a structured analysis of the intricate relationships among various categories and subcategories. This section delves deeper into these relationships by documenting key patterns that emerge from the cause-and-effect connections identified within the matrix. These patterns provide us with a nuanced understanding of how different aspects interact and influence one another, shedding light on the multifaceted dynamics that underpin the philosophy of mathematics and its implications for teaching practices. By systematically examining these patterns, one gains valuable insights into the complex nature of this research findings and their broader significance in shaping the landscape of mathematics education. In this case the pattern are:

1. Philosophy of Mathematics in Mathematics Teaching → Integration of Philosophical Thinking in Teaching
2. Challenges in Mathematics Education → Application of Philosophical Ideas
3. Nature of Mathematical Knowledge → Influence of Philosophy on Teaching Practices
4. Teacher Reflection and Empowerment → Emphasis on Understanding and Application

5. Curriculum and Teaching Approaches → Challenges Within Education System
6. Educational System Improvement → Real-World Challenges Faced by Educators
7. Connection Between Philosophy and Mathematics → Nature of Mathematical Knowledge
8. Interdisciplinary and Holistic Approaches → Differentiation from Belief and Opinion

1) *Integration of Philosophical Thinking in Teaching:*

- The subcategories under "Philosophy of Mathematics in Mathematics Teaching" highlight the integration of philosophical perspectives into teaching practices.
- *Cause-Effect Relationship*: *Integration* of philosophical thinking leads → to a shift in educators' mindsets and approaches in the classroom, making abstract concepts more relevant and understandable for learners.

2) *Application of Philosophical Ideas:*

- Educators apply philosophical ideas like Grothendieck's, Susskind's, Greene's, Russell's, Ernest's, and Shapiro's philosophies in their teaching.
- *Cause-Effect Relationship*: *Application* of philosophical ideas → enriches lessons and teaching philosophies, fostering engaging and learner-centred learning experiences.

3) *Influence of Philosophy on Teaching Practices:*

- Philosophical understanding influences teaching practices, encouraging critical thinking and problem-solving.
- *Cause-Effect Relationship*: *Incorporating* philosophical insights → enhances teaching practices, leading to more effective instruction and deeper learner engagement.

4) *Emphasis on Understanding and Application:*

- There is a strong emphasis on promoting deep understanding and real-world application of mathematical concepts.
- *Cause-Effect Relationship*: *Focusing* on understanding and application → cultivates intellectual curiosity, encouraging comprehensive comprehension of mathematical relationships.

5) Challenges Within Education System:

- Challenges such as weak teacher education, poor outcomes, curriculum overload, and content-heavy approaches are consistently highlighted.
- *Cause-Effect Relationship*: *Challenges within the education system* → drive the need for transformative action, including curriculum reforms and embracing diversity for inclusive mathematics education.

6) Real-World Challenges Faced by Educators:

- Overcrowded classrooms, curriculum overload, and lack of support for struggling learners are common challenges.
- *Cause-Effect Relationship*: *Addressing real-world challenges* → requires transformative action, fostering critical thinking, creativity, and reflection on teaching practices.

7) Nature of Mathematical Knowledge:

- Philosophical exploration delves into the certainty, logic, and characteristics of mathematical knowledge.
- *Cause-Effect Relationship*: *Emphasizing* the logical basis of mathematical truths → enhances effective communication of mathematical knowledge and understanding among learners.

8) Differentiation from Belief and Opinion:

- Clear differentiation between mathematical knowledge and subjective belief/opinion is stressed.
- *Cause-Effect Relationship*: *Acknowledging* the distinction between mathematical knowledge and belief/opinion → supports accurate communication of concepts to learners.

Narrative Descriptions of Documented Patterns

The narrative descriptions of documented patterns, encapsulate the essence of the interconnected themes and subcategories. These narratives provide a comprehensive understanding of how philosophical insights, teaching practices, challenges, and the nature of mathematical knowledge intersect and collectively contribute to the landscape of mathematics education.

1) *Integration of Philosophical Thinking in Teaching:*

   The integration of philosophical perspectives into teaching practices is evident across the autoethnography section, literature review, questionnaire responses (ANNEXURE E1 & 2), and field notes (ANNEXURE F). This integration serves as a catalyst, prompting a transformation in educators' mindsets and approaches within the classroom. The incorporation of philosophical insights bridges the gap between abstract mathematical concepts and learners’ lived experiences, fostering a more relatable and comprehensible learning environment. As educators infuse philosophical thinking into pedagogy, they serve as role models, encouraging learners to engage more deeply with the subject matter.

2) *Application of Philosophical Ideas:*

   The practical application of philosophical ideas, such as those advocated by Grothendieck, Susskind, Greene, Russell, Ernest, and Shapiro, consistently emerges from the autoethnography section, literature review, and questionnaire responses (ANNEXURE E1 & 2). These philosophical concepts are actively woven into lessons, shaping teaching philosophies based on these perspectives. This application not only enriches the content of the curriculum but also connects abstract mathematical theories to engaging learning experiences. By bridging philosophy with effective teaching strategies, educators create an environment that encourages learners' curiosity and critical thinking.

3) *Influence of Philosophy on Teaching Practices:*

   The influence of philosophical understanding on teaching practices is evident across various sources, including the autoethnography section, and literature review. The incorporation of philosophical insights leads to a shift in pedagogical approaches. This shift is marked by a focus on encouraging critical thinking, problem-solving, and reflective learning. As educators embrace philosophical perspectives, they adopt learner-centred methods that go beyond mere content delivery. This transformation enhances teaching practices, encouraging deeper engagement with mathematics and the cultivation of intellectual curiosity among learners.

4) *Emphasis on Understanding and Application:*

The autoethnography section, literature review, and questionnaire responses (ANNEXURE E1 & 2) consistently highlight a strong emphasis on nurturing deep understanding and real-world application of mathematical concepts. The importance of moving beyond rote memorization to foster comprehensive comprehension is recognized. This emphasis aligns with the overarching goal of cultivating learners' curiosity and critical thinking. By prioritizing understanding and application, educators create an environment where learners can explore the intricate relationships within mathematics and see its relevance in their everyday lives.

5) *Challenges Within Education System:*

Acknowledgment of challenges within the education system, ranging from weak teacher education and poor outcomes to curriculum overload and content-heavy approaches, is consistent across the autoethnography section, literature review, questionnaire responses (ANNEXURE E1 & 2), and field notes (ANNEXURE F). However, these challenges also serve as catalysts for transformative action. Educators and stakeholders recognize the need for curriculum reforms, teacher empowerment, and the embrace of diversity to create an inclusive mathematics education environment that addresses these systemic issues.

6) *Real-World Challenges Faced by Educators:*

The autoethnography section, literature review, questionnaire responses (ANNEXURE E1 & 2), and field notes (ANNEXURE F) vividly illuminate educators' real-world challenges, such as overcrowded classrooms, curriculum overload, and a lack of support for struggling learners. This realization underscores the importance of transformative action. To effectively address these challenges, educators reflect on their teaching practices, embracing creative and learner-centred approaches that foster critical thinking, creativity, and engagement.

7) *Nature of Mathematical Knowledge:*

The exploration of the nature of mathematical knowledge consistently emerges across various sources, including the autoethnography section, literature review, and questionnaire responses (ANNEXURE E1 & 2). Philosophical perspectives delve into the certainty, logic, and characteristics that define mathematical truths. This exploration highlights the intrinsic nature of mathematical knowledge as distinct from belief and opinion. By emphasizing the logical basis of mathematical concepts, educators communicate these complex ideas more effectively, fostering a deeper understanding among learners.

8) *Differentiation from Belief and Opinion:*

The need for clear differentiation between mathematical knowledge and subjective belief or opinion is a recurring theme, spanning the autoethnography section, and literature review. This differentiation ensures that learneres grasp the foundation of mathematical concepts based on logical reasoning and evidence, promoting a deeper comprehension of the subject matter. By clarifying this delineation, educators empower learners to engage critically with mathematical content.

Axial Coding: Dissecting and Refining Data

The above was the process of documenting patterns and relationships in the axial coding matrix. Now, this data will be further dissected, and patterns will be refined to extract deeper insights and implications from the qualitative data analysis.

Below is a breakdown of each documented pattern identified above, providing a refined analysis, and extracting implications:

PATTERN 1: Integration of Philosophical Thinking in Teaching

Refined Analysis: The integration of philosophical perspectives enriches teaching practices by shifting educators' mindsets. It connects abstract concepts to learners' real-world experiences, enhancing comprehension and engagement.

Implications: Educational institutions could encourage educators to *explore* and *incorporate* philosophical perspectives. This could result in more *relevant teaching methods* and *foster deeper connections between learners and mathematical concepts*.

PATTERN 2: Application of Philosophical Ideas

Refined Analysis: Philosophical ideas like Grothendieck's, Susskind's, Greene's, etc., can be actively applied in teaching. This application enhances lesson content and teaching philosophies, leading to learner-centred learning experiences.

Implications: Curriculum development could *integrate* philosophical concepts, encouraging educators to *apply* these ideas creatively in their teaching. This could lead to more *engaging* and *thought-provoking lessons*.

PATTERN 3: Influence of Philosophy on Teaching Practices

Refined Analysis: Philosophical insights *influence* teaching practices, *promoting* critical thinking and problem-solving approaches. This incorporation *enhances* teaching methods and deepens learner engagement.

Implications: Professional development programs could *focus on integrating philosophical understanding into teaching strategies*. This might lead to more *effective teaching practices that prioritize holistic learner development*.

PATTERN 4: Emphasis on Understanding and Application

Refined Analysis: The *emphasis on deep understanding* and *real-world (science and technology)* application of mathematical concepts fosters *curiosity* and *comprehensive comprehension*.

Implications: Educational policies might prioritize learning outcomes that emphasize both understanding and application. This could lead to learners *developing a stronger conceptual grasp of mathematics*.

PATTERN 5: Challenges Within Education System

Refined Analysis: Challenges within the education system, including weak teacher education and curriculum overload, drive the need for transformative action and inclusive mathematics education.

Implications: *Policymakers* could address teacher training programs and curriculum design to alleviate challenges. *Embracing diversity and reforming the curriculum could lead to more equitable education*.

PATTERN 6: Real-World Challenges Faced by Educators

Refined Analysis: Overcrowded classrooms, curriculum overload, and lack of support require transformative action, fostering critical thinking and creativity in educators.

Implications: Schools and institutions might *prioritize class size reduction* and *offer resources for struggling learners*. *Encouraging creative teaching methods can help educators navigate challenges more effectively*.

PATTERN 7: Nature of Mathematical Knowledge

Refined Analysis: Philosophical exploration of mathematical knowledge's certainty and logical basis enhances effective communication and understanding.

Implications: Teacher training programs could *include elements that enhance* educators' ability to communicate the logical foundations of mathematical concepts to learners.

PATTERN 8: Differentiation from Belief and Opinion

Refined Analysis: Clear differentiation between mathematical knowledge and subjective belief supports accurate communication of concepts.

Implications: Pedagogical approaches could *focus on teaching learners how to critically differentiate between factual mathematical knowledge and subjective opinions*.

By refining the documented patterns, one gains a deeper understanding of the interplay between philosophy of mathematics and mathematics education. These refined insights can guide the development of educational strategies, curriculum design, and professional development programs that align with the nuanced dynamics that are identified in this research.