Positive operators and their applications on ordered vector spaces

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Magister Scientiae

by

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I love to dedicate the entirety of this work in memory of my younger brother Thabo Msíbí, who was brutally slain on Sep 23rd, 2023. Rest in Power, Little Bro, until we meet again.

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Abstract

A vector space X is called an ordered vector space if it is ordered through $\preceq$ for any elements $x, y, z \in X$ and $\alpha \in \mathbb{R}^+$, $x \preceq y \implies x + z \leq y + z$ and $0 \leq x$ implies $0 \leq \alpha x$. If in addition, X is a lattice, that is if for a pair $\{x, y\}$ the $\inf\{x, y\}$ and $\sup\{x, y\}$ exists, then X is a Riesz space (or a vector lattice). In this study, we discuss Banach lattices, ordered Banach spaces, operators on these spaces and their applications in economics, fixed-point theory, differential and integral equations.

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Declaration

I, ... declare that this Dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signed: . . . .

Date: . . . .

Introduction

A vector space X is called an ordered vector space if it is ordered through $\preccurlyeq$ for any $x, y, z \in X$ and $\alpha \in \mathbb{R}$, $\alpha > 0$, $x \leq y$ implies that $x + z \leq y + z$ and if $x \leq 0$ then $\alpha x \leq 0$. If X is also a lattice, then X is called a Riesz space (or vector lattice). The positive cone of ordered vector space X, denoted by X^+, is the set of positive elements of X. This set is closed under addition, closed under multiplication by positive scalars, and the only element of this set with an additive inverse is zero. In any non-trivial vector space X, we can define a cone K, with the properties mentioned above. In general, a cone K defines a partial ordering on the vector space X, as follows, $x \leq y \iff y - x \in K$ for all $x, y \in$ and the pair (X, K) is called an ordered vector space. A norm complete normed Riesz space or a complete ordered vector space is called a Banach lattice or ordered Banach space respectively.

In this study, we discuss the notion of operator theory in ordered Banach spaces (or lattices), with focus being on positive, regular, and order bounded operators. These operators have been studied since the early 1960s and they have applications in financial mathematics, economics and in applied mathematics. It is known that cones play an important role in order theory. We will discuss some cones in this study, which are generating. In particular, we discuss the Order Sequential Continuity (OSC) property, which is generated by some kind of a cone and it is more general than Lorentz cone. On the other hand, we discuss order continuity of a norm. Examples to show that OSC property and order continuity are not equivalent are provided.

The theory of operators has many applications, in physics, differential and integral equations, and those notions have implications in financial mathematics, mathematical biology and economics. We, therefore, discuss the relationship between positive, regular and order bounded operators. It is known that every positive or regular operator is order bounded but the converse is not true in general. We illustrate this, and other assertions, in the study.

Applications of Banach lattices operators in economics, in particular, the Leontief model is discussed. In these applications, it is shown that the notion of order continuous norm plays a role in determining whether the Leontief model has a solution or not. The second application is on differential equations and fixed-point theory. J. Nieto and R. Rodríguez-López [45] established some results on partially ordered spaces, to show existence and uniqueness of a fixed-point. Their results were more general than those of Tarski's as Tarski's theorem requires the space to be a lattice. Recently, M. Alfuraidan and M. Khamsi [7] studied some fixed-point of partial differential equations. In their study, the operator T is not continuous, and the continuity of the operator was replaced by OSC property on the space. We will discuss the above-mentioned results in this study.

In chapter one, we introduce partial ordering, ordered vector spaces and Riesz spaces, and some of their properties that are useful in our study. We present examples to show that ordered vector spaces are more general than Riesz spaces.

In chapter two, we discuss order continuity and OSC property. We illustrate by examples that OSC property and order continuity are not equivalent.

In chapter three, we discuss positive, regular, and order bounded operators. Examples to show that these operators are not equivalent are discussed and the conditions under which regular operators is equivalent to order bounded operators is discussed.

Lastly, in chapter four we discuss the applications of operators in Leontief model and fixed point theory cases.

1 Preliminaries

1.1 Partial ordering

Definition 1.1. Let X be a non-empty vector space. A relation " $\preccurlyeq$ " is called a partial order if for any $x, y, z \in X$,

1. $x \leq x$ (***reflexive***).
2. $x \leq y$ and $y \leq z$ then $x \leq z$ (***transitive***).
3. $x \leq y$ and $y \leq x$ then $x \leq y$ (***anti-symmetric***).

The pair $(X, \preccurlyeq)$ is a partially ordered set. In short, it is called a poset. Next, we follow with an example of a poset.

Example 1.2. Let $(X, \|\cdot\|)$, be a normed space and $f \in X^*$, where X^* is the norm dual of X. The relation $\preceq_f$ defined by

$$x \preceq_f y \iff \|x - y\| \leq f(y) - f(x)$$

is a partial ordering on X. First, we show *reflexivity*: that is for all $x \in X$ $x \preceq_f x$ if and only if

$$\|x - x\| = 0 \leq f(x) - f(x) = 0.$$

Next, we show *transitivity*, taking any elements $x, y, z \in X$, if

$$x \preceq_f y \iff \|x - y\| \leq f(y) - f(x)$$

and

$$y \preceq_f z \iff \|y - z\| \leq f(z) - f(y),$$

then, by the triangle-inequality, we have that

$$\|x - y\| + \|y - z\| \leq \|x - z\|.$$

Thus, implying that

$$x \preceq_f z \iff \|x - z\| \leq f(z) - f(x).$$

Lastly, we show *anti-symmetry*: that is, for any $x, y \in X$

$$y \preceq_f x \iff \|y - x\| \leq f(x) - f(y),$$

which is

$$0 \geq f(y) - f(x) \implies x = y.$$

Definition 1.3. Let A be a subset of a partially ordered set X. A point $x \in X$ is called

1. an ***upper-bound*** (or ***lower-bound***) of A if $x \geq y$ for every $y \in A$.

2. ***supremum*** in A if $x \leq y$ for every upperbound $z \in X$.

3. ***infimum*** in A if $x \geq y$ for every upperbound $w \in X$.

4. ***minimal element*** if for every $y \in A$ we will have that $x < y$.

5. ***maximal element*** if for every $y \in A$ we will have that $x > y$.

Definition 1.4. A partially ordered set X is called

1. ***Dedekind complete*** if every non-empty subset of X that is bounded above (bounded below) has a supremum (infimum).

2. σ-***Dedekind complete*** if every non-empty finite or countable subset of X that is bounded above (bounded below) has a supremum (infimum).

3. a ***lattice*** if every subset consisting of two points, x, y has a supremum denoted by $x \vee y$ and an infimum denoted by $x \wedge y$.

1.2 Ordered vector spaces

Definition 1.5. A vector space X is called an ordered vector space if it is endowed with an order $\geq$ and for $\lambda \geq 0$ then the following holds, for all $x, y, z \in X$

1. $x \leq y$ implies that $x + z \leq y + z$.

2. $x \geq 0$ implies that $\lambda x \geq 0$.

Definition 1.6. Let X be an ordered vector space and $x, y \in X$. Then the set $[x, y]= \{z \in X : x \leq z \leq y\}$ is called an ordered interval of X.

Definition 1.7. Let $A \subset X$ (X an ordered vector space). A is called

1. ***order convex*** if $[x, y] \subset X$ for all $x, y \in A$.

2. ***order bounded*** if $A \subset [x, y]$ for some $x, y \in A$.

Definition 1.8. If X is an ordered vector space, then the set $X^+ = \{x \in X : x \geq 0\}$ defines a cone in X . The cone X^+ is called the ***positive cone***.

Definition 1.9. An ordered vector space E is called a ***Riesz space*** (or a vector lattice) if it is also a lattice.

An ordered space that is not Riesz space will be shown, in Example 1.16 below. But first we will discuss the following.

Definition 1.10. Let E be a Riesz space and $x \in E$. Then,

1. $x^+ = x \vee 0$.
2. $x^- = (-x) \vee 0$.
3. $|x| = x \vee (-x)$.

Theorem 1.11. *Let E be a Riesz space, and $x, y, z \in E$. Then*

1. $x^- = (-x^+)$.
2. $x = x^+ - x^-, x^+ \wedge x^- = 0$ *and* $|x| = x^+ + x^-$.
3. $0 \leq x^+ \leq |x|$ *and* $0 \leq x^- \leq |x|$.
4. $x \leq y$ *if and only if* $x^+ \leq y^+$ *and* $x^- \leq y^-$.
5. $(x + y) \vee z = (x + z) \vee (y + z)$.
6. $x - (y \wedge z) = (x - y) \vee (x - z)$.

Proposition 1.12. *([13], Proposition 2.15.) If $x, y, z \in X^+$ and $z \leq x + y$, then there exist $u, v \in X^+$ such that $u \leq x, v \leq y$ and $z = u + v$.*

Proof. Taking $u = x \wedge z$ and $v = z - u$. Then $0 \leq u, v$ and $u \leq x$. Next, by using the second identity in [13] Proposition 2.8., that is, for any x, y, z in a Riesz space, if the identity

$$x + \sup\{y, z\} = \sup\{x + y, x + z\} \text{ and } x + \inf\{y, z\} = \inf\{x + y, x + z\}$$

is true, we will then get that

$$\begin{aligned} y - v &= y - z + (x \wedge z) \\ &= (y - z + x) \wedge y \geq 0. \end{aligned}$$

□

Remark 1.13. *The proof above is the Riesz decomposition property. Furthermore, this property will tend to be of great value when proving the distributive law [13].*

Lemma 1.14. ([6], Lemma 1.15.) An ordered vector space X is a Riesz space if and only if for every pair of vectors $x, y \in X$ their supremum $x \vee y$ exists in X. Furthermore, if x and y are elements in a Riesz space, then

$$x \vee y = -[(-x) \wedge (-y)] \text{ and } x \wedge y = -[(-x) \vee (-y)].$$

Proof. Assume that each ordered pair x, y of X have an infimum. Now, putting

$$z = (-x) \wedge (-y),$$

we will show that

$$x \vee y = -z.$$

Note that, we have

$$z \leq -x \text{ and } z \leq -y \text{ or } x \leq -z \text{ and } x \leq -z.$$

Thus, $-z$ is an upper bound of $\{x, y\}$. Suppose that there is a vector t satisfying

$$x \leq t \text{ and } y \leq t \implies -t \leq -x \text{ and } -t \leq -y$$

Therefore,

$$-t \leq (-x) \wedge (-y) = z, \text{ that is } -z \leq t.$$

Thus, we have that $x \vee y$ is true and so, clearly, the converse can easily be shown.

□

Theorem 1.15. *([6], Exercise 1.3.2.) An ordered vector space X is a Riesz space if and only if for $x \in X$ the supremum $x^+ = x \vee \mathbf{0} \in X$.*

Proof. First, suppose that X is a Riesz space. Then for x, $\mathbf{0} \in X$, we have, by definition of X,

$$x^+ = x \vee \mathbf{0} \in X.$$

Conversely, let $x^+ = x \vee \mathbf{0}$ be in an ordered vector space X. Then, for $x, y \in X$, we have $x - y \in X$ since X is a vector space and, by the assumption,

$$(x-y)^+ = (x-y) \vee \mathbf{0} \in X.$$

Therefore, applying property (4) of Theorem 1.11 , we get property (4) of Theorem 1.11

$$\begin{aligned} x \vee y &= [(x-y)+y] \vee (\mathbf{0} + y) \\ &= (x-y) \vee \mathbf{0} + y \\ &= (x-y)^+ + y \in X. \end{aligned}$$

By Lemma 1.14, ordered vector spaces are Riesz. □

Example 1.16. Let $X = C^1[0,1]$ be set of all continuously differentiable functions on $[0,1]$, with X pointwise ordering endowed. Then X is an ordered vector space which is not a Riesz space. Define $f \leq g$ if and only if $f(x) \leq g(x)$ for each $x \in [0,1]$. Now, since x is endowed with pointwise vector ordering, we have that it is partially ordered. To show that X is not a Riesz space, take functions

$$f(x) = x \text{ and } g(x) = 1 - x.$$

It is clear that both f and g are continuous and differentiable on the interval $[0,1]$ which implies that they are elements of X. Now,

$$f(x) \vee g(x) = \sup\{f, g\} = \left|x - \frac{1}{2}\right| + \frac{1}{2}.$$

Which equals x if $x \geq \frac{1}{2}$ or equals $1 - x$ if $x < \frac{1}{2}$. Thus, we have that $f(x) \vee g(x)$ is not differentiable at $x = \frac{1}{2}$ on the interval $[0, 1]$. And, the derivative

$$(f(x) \vee g(x))' = \begin{cases} 1, x > \frac{1}{2}, \\ -1, x < \frac{1}{2} \end{cases}$$

Therefore, $(X, \preceq)$ is an ordered vector space that is not a Riesz space, since $\sup\{f, g\}$ is not an element of X.

1.3 Cones and wedges

Definition 1.17. Let X be a vector space and $K \subseteq X$. We call K a cone if the following properties are satisfied

1. $K \neq \emptyset$ and $K \neq \{0\}$.
2. for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$ implies $\alpha x + \beta y \in K$.
3. $x \in K$ and $x \in -K$ implies that $x = 0$.

The set K is a wedge whenever it satisfies properties (1) *and* (2), *and not* (3).

We next show an example of a wedge which is not a cone.

Example 1.18. Let $X = \ell_\infty$ be the space of all bounded real sequences. Then X is a wedge but not a cone. Since ℓ_∞ is a vector space, it follows that ℓ_∞ is a wedge. But ℓ_∞ is not a cone, since the sequence $-(x_n) = (-1, -1, -1, ..., -1, ..) \in \ell_\infty$ and the same with $(x_n) = (1, 1, 1, ..., 1, ..) \in \ell_\infty$ but $(x_n) \neq 0$.

Proposition 1.19. *Let X be a vector space, and K a cone, then the relation $\preceq$ defined by*

$$x \preceq y \iff y - x \in K$$

for all $x, y \in K$, is a partial ordering on X.

Proof. Let x, y, z be arbitrary elements in X. Since $0 \in K$ then $x \preceq x$ for all $x \in X$ (*reflexivity*). Secondly, given $x \preceq y$ and $y \preceq x$ we have that $y - x \in K$ and $x - y \in K$. But $-(y - x) = x - y \in K$. By definition, $x - y = 0$, and therefore $x = y$ (*anti-symmetry*). Finally, suppose $x \preceq y$ and $y \leq z$ then $y - x \in K$ and also $z - y \in K$. Also, by definition, we have that $(y - x) + (z - y) = z - x \in K$ (*transitivity*). □

Corollary 1.20. *If X is an ordered vector space, with K a cone in X. Then $X^+ = K$, where the ordering in X^+ is induced by K.*

Proof. Suppose that X is an ordered vector space, and that K is a cone in X. Now, taking the positive cone of X, that is

$$\begin{aligned} X^+ &= \Big\{x \in X : x \geq 0\Big\} \\ &= \Big\{x \in X : x - 0 = x \in K\Big\} \\ &= K. \end{aligned}$$

Thus, the positive cone X^+ is K. □

Definition 1.21. Let $(X, \| \cdot \|)$ be a normed space, with a cone $K \subset X$. We say K is

1. ***generating*** if $X = K - K$.
2. ***Archimedean*** if $y \in X$, $x \in X^+$ and $ny \leq x$ for all $n \geq 1$ then $y \leq 0$.
3. ***normal*** if there exists $\lambda \geq 1$ such that $0 \leq x \leq y$ implying $\| x \| \leq \lambda \| y \|$ for all $x, y \in X$.
4. ***solid*** if there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset K$ for some x in K, where $B(x, \varepsilon)$ is an open ball.

Proposition 1.22. *Any solid cone K in a normed space $(X, \|\cdot\|)$ is generating.*

Proof. Take $x_0 \in \mathring{K}$, where $\mathring{K}$ is the interior of the cone K. Then there is $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset K$. Therefore,

$$x_0 - \frac{\varepsilon}{2\|x_1\|} \cdot x, x_0 + \frac{\varepsilon}{2\|x_1\|} \cdot x \in B_\varepsilon(x_0) \text{ for all } x \in X \backslash \{0\}$$

Now, let

$$\begin{aligned} u &= \frac{2\|x\|}{\varepsilon}\left(x_0 + \frac{\varepsilon}{2\|x_1\|} \cdot x\right) \\ &= \frac{2\|x\|}{\varepsilon} \cdot x_0 + \frac{x}{2} \end{aligned}$$

and

$$\begin{aligned} v &= \frac{2\|x\|}{\varepsilon}\left(x_0 - \frac{\varepsilon}{2\|x_1\|} \cdot x\right) \\ &= \frac{2\|x\|}{\varepsilon} \cdot x_0 - \frac{x}{2} \end{aligned}$$

Therefore, $u - v = x$ and $u, v \in K$. Thus, $x \in K - K$. So $X = K - K$.

□

Remark 1.23. *The converse of this result is not true in general as we see from the following example.*

Example 1.24. Consider the sequence space $(\ell_1, \|\cdot\|)$ and $K = {\ell_1}^+$. Then K is generating but has an empty interior. Since ℓ_1 is a Riesz space, K is generating. We only show that K has empty interior. To this end, suppose that K is solid. then there exists $a > 0$ such that $B_\varepsilon(y) \subset K$ for some $y \in \mathring{K}$. Since $y \in K$, there is $N \in \mathbb{N}$ such that,

$$0 \leq y_n < \frac{\varepsilon}{2} \text{ for all } n \geq N.$$

Now define $z \in \ell_1$ by,

$$z_n = \begin{cases} y_n, & \text{if } n \neq N \\ -\frac{\varepsilon}{2}, & \text{if } n = N \end{cases}$$

Then $z \in B_\varepsilon(a)$, since

$$\begin{aligned} \|y - z\| &= \sum_{n=1}^{\infty} |y_n - z_n| \\ &= |y_n + \frac{\varepsilon}{2}| \\ &\leq |y_n| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

But $z \notin K$, since $z_n = -\frac{\varepsilon}{2} < 0$. Now, since y and ε were arbitrary, K has empty interior.

Definition 1.25. A norm on a Riesz space E is a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.

Definition 1.26. A complete normed lattice $(E, \|\cdot\|)$ is called a ***Banach lattice***.

Proposition 1.27. *([13], Proposition 2.18.) Any normed lattice is Archimedean.*

Proof. Let $0 \leq y \leq n^{-1}x$, $n \in N$. By the lattice norm property, we have

$$\|y\| \leq \left\|n^{-1}x\right\| = n^{-1}\|x\|$$

for any n and hence $\|y\| = 0$, yielding $y = 0$. Therefore,

$$\inf\{n^{-1}x\} = 0.$$

□

Now, we also recall these two special norms on a Riesz space.

Definition 1.28. ([5], Definition 9.26.) A lattice norm on a Riesz space is

1. an M-norm if $x, y \geq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$.

2. an L-norm if $x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$.

Example 1.29. The space $(\ell_\infty, \|\cdot\|)$ of all bounded real sequences is a Banach lattice. We only show that the norn $\|\cdot\|_\infty$ is a lattice norm. To this end, let $|x| \leq |y|$, then

$$\begin{aligned}\|x\|_\infty &= \sup_{i\in\mathbb{N}}\{|x_i|\}\\ &\leq \sup_{i\in\mathbb{N}}\{|y_i|\}\\ &= \|y\|_\infty.\end{aligned}$$

An atom in an Archimedean vector lattice E is an element $a \in E^+$ such that $0 \leq b \leq a$ implies that b is a real multiple of a [60]. The definition below will be of useful reference in stating Theorem 3.18.

Definition 1.30. ([60], *page* 256.) Let E be an Archimedean vector lattice. E is called ***atomic*** if the only element of E that is disjoint from every atom is the zero element.

Definition 1.31. Let $(X, \|\cdot\|)$ be normed space, and that the cone $K \subseteq X$ is solid. An ***allowable sequence*** is any sequence $(x_n)_{n\in\mathbb{N}}^\infty \in \mathring{K}$ that approaches x as $n \to \infty$ and $x_n \geq x$ for all $n \in \mathbb{N}$.

The following example depicts an allowable squence. In particular, when taking the subset K of an ordered vector space X.

Example 1.32. ([17], Example 3.5.13.) Let $X = \mathbb{R}$, and $K = \mathbb{R}^+$ be a subset of the vector space X. Then the sequence $(x_n) = (1 - \frac{1}{n})$ is an allowable sequence in K. Note that, the sequence $x_n \in (0, \infty)$. Hence it is in $\mathring{K}$, the interior of cone K, and also that $x_n \to 1 \in \mathring{K}$ as $n \to \infty$. So (x_n) is allowable sequence since $\frac{1}{n} \in \mathring{K}$.

Proposition 1.33. *([17], Proposition 3.5.15.) Suppose X is an ordered Banach space, that is also Archimedian. Then X has an allowable sequence.*

Proof. To show this, we suppose that a solid cone exist in X. We then take a sequence

$$x_n = x + n^{-1}u,$$

where u is an order unit of X. Then, X is Archimedean with $n^{-1}u$ approaching 0 as $n \to \infty$. Thus,

$$\begin{aligned} x_n &= \left(x + n^{-1}u\right) \\ &= x + 0 \\ &= x \text{ as } n \to \infty. \end{aligned}$$

□

2 Some order properties on normed spaces

2.1 OSC property

We first discuss the following cone, and the order it generates, in relation to Order Sequential Continuity (OSC) property.

Proposition 2.1. *Let $(X, \|\cdot\|)$ be normed space, and X^* a norm dual of X. The set K defined by*

$$K = \{y \in X : \|y\| \leq f(y)\}, \text{ where } f \in X^* \text{ and } \|f\| = 1$$

is a solid cone in X.

Proof. We first show that K is a cone. To this end, take $x, y \in K$.

$$\begin{aligned} \|x + y\| &\leq \|x\| + \|y\| \text{ (triangle ineq.)} \\ &\leq f(x) + f(y) \text{ (by definition of } K) \\ &= f(x + y) \text{ (since } f \in X^*). \end{aligned}$$

That is, $x+y \in K$. Secondly, for $\alpha \geq 0$

$$\begin{aligned}\|\alpha x\| &\leq |\alpha| \cdot \|x\| \\ &= \alpha\|x\|, \alpha > 0 \\ &\leq \alpha f(x) \\ &= f(\alpha x),\ (f \in X^*)\ .\end{aligned}$$

Thus $\alpha x \in K$. Lastly, $x \in K$ and $-x \in K$, $\|x\| \leq f(x)$ and $\|-x\| \leq f(-x) \implies \|x\| \leq -f(x)$

$$\begin{aligned}\|x\| + \|x\| \leq f(x) - f(x) \\ 2\|x\| \leq 0.\end{aligned}$$

Thus $x = 0$. Now, we show that K is solid. Let $x_0 \in S_x$ be unit sphere in X, such that $f(x_0) > 1$. The function $\varphi(x) = \|x\| - f(x)$ is continuous, since both $\|\cdot\|$ and f are continuous and $\varphi(x_0) < 0$. This follows, since $\varphi(x_0) + f(x_0) = \|x_0\|$ implies that $\|x_0\| > \varphi(x_0) + 1$ and since $x_0 \in S_x$, we have that

$$1 > \varphi(x_0) + 1.$$

Thus, $\varphi(x_0) < 0$. This implies that there is $\varepsilon > 0$ such that $\|x - x_0\| < \varepsilon$ implies $\varphi(x_0) < 0$. Thus, $B_\varepsilon(x_0) \subset K$. Hence K is solid. □

Definition 2.2. ([3], Definition 1.) A partially ordered normed space $(X, \|\cdot\|, \preccurlyeq)$ is said to satisfy the Order Sequential Continuity (OSC) property if $x \preccurlyeq x_n$ for all $n \in N$, for every sequence (x_n) in X such that $x_n \overset{\|\cdot\|}{\to} x$ and $x_{n+1} \preccurlyeq x_n$ for all $n \in \mathbb{N}$.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a normed space. Then the partial ordering induced by K as follows,*

$$x \preceq_K y \iff y - x \in K, \|y - x\| \leq f(y) - f(x),$$

satisfies the OSC property, for $f \in X^*$ and $\|f\| = 1$.

Proof. Suppose (x_n) is a decreasing sequence in X such that $x_n \to x \in (X, \|\cdot\|)$. Then

$$x_{n+1} \preceq_K x_n \iff \|x_n - x_{n+1}\| \leq f(x_n) - f(x_{n+1}).$$

Now,

$$\begin{aligned}
\|x_n - x_{n+k}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{n+k-1} - x_{n+k}\| \\
&\leq f(x_n) - f(x_{n+1}) + f(x_{n+1}) - f(x_{n+2}) + ... + f(x_{n+k-1}) - f(x_{n+k}) \\
&\leq f(x_n) - f(x_{n+k}).
\end{aligned}$$

Thus,

$$\|x_n - x_{n+k}\| \leq f(x_n) - f(x_{n+k}).$$

□

We will discuss OSC property on L_p spaces. But first recall some useful definitions.

Definition 2.4. ([11], Definition 7.1.) Let (X, Σ, μ) be a measure space. A μ-measurable function $f : X \to Y$ is $p-$integrable $(1 \leq p < \infty)$ if $|f|^p$ is an integrable function.

1. $L_p(\mu), 1 \leq p < \infty$ will denote the space of all integrable functions. The norm of $L_p(\mu)$ is defined by

$$\|f\|_{L_p} = \left(\int_X |f(x)|^p dx \right)^{\frac{1}{p}},$$

   where $1 \leq p < \infty$ and $f \in L_p(\mu)$.

2. $L_\infty(\mu)$ will denote the space of all essentially bounded measurable functions. The norm on $L_\infty(\mu)$, *i.e.* the essential supremum of f or $ess \sup f$, is defined by

$$\|f\|_\infty = ess \sup f = \inf \Big\{ m > 0 : |f(x)| \leq m \text{ for } \mu \text{ almost all } x \Big\},$$

where f is a μ-integrable function $f : X \to \mathbb{R}$.

Remark 2.5. *The essential supremum captures the essential upper bound of the function f by considering the set of points where f is grater than any m and by removing points with negligible impact on its overall behavior.*

Theorem 2.6. *The positive cone $K = \{f \in X : x \geq 0 \ a.e.\}$ of $L_1[0,1]$ has the OSC property.*

Proof. Let $0 \leq f_{n+1} \leq f_n$ be a decreasing sequence in K such that

$$\lim_{n\to\infty} \int_0^1 |f_n(t) - h(t)|dt = 0 \text{ for some } h \in L_1[0,1].$$

Then (f_n) is $a.e.$ convergent to $f = \inf_n f_n$. This implies that the decreasing sequence $g_n = f_n - f$ is $a.e.$ convergent to zero. By the monotone convergence theorem, it follows that

$$\lim_{n\to\infty} \int_0^1 |f_n(t) - f(t)|dt = \lim_{n\to\infty} \int_0^1 (f_n(t) - f(t))dt = 0.$$

That is, (f_n) is $\|\cdot\|_1$ convergent to f. By uniqueness of the $\|\cdot\|_1$ limit we have that $h = f$. □

Remark 2.7. *This proof is for increasing sequences but is used for decreasing sequences, since monotone convergence theorem applies to both increasing and decreasing sequences.*

Lemma 2.8. The norm convergence in $L_\infty[0,1]$ can be characterized, with $f_n \xrightarrow{u} f$ denoting uniform convergence, as follows, where $\lambda(P)$ is the set with measure zero,

$$f_n \xrightarrow{\|\cdot\|_1} f \iff \text{there exists } P \subset [0,1], \lambda(P) = 0 \text{ such that } f_n \xrightarrow{u} f \text{ on } [0,1] \backslash P.$$

Theorem 2.9. *The positive cone $K = \{f \in L_\infty : f \geq 0 \ a.e.\}$ of $L_\infty[0,1]$ has the OSC property.*

Proof. Let $0 \leq f_{n+1} \leq f_n$ be a decreasing sequence in K such that

$$f_n \xrightarrow{\|\cdot\|_1} f \text{ for some } f \in L_\infty[0,1].$$

Let $P \subset [0,1]$ with $\lambda(P) = 0$ be such that $f_n \xrightarrow{u} f$ in $[0,1] \backslash P$. Then

$$0 \leq f(x) \leq f_n(x) \text{ for all } x \in [0,1] \backslash P \text{ and } n \in \mathbb{N}.$$

If $g \in L_\infty[0,1]$ is such that $g \leq f_n$ for all $n \in \mathbb{N}$. That is, for all $x \in [0,1] \backslash B$ for some subset B of $[0,1]$ with $\lambda(B) = 0$. Then

$$g(x) \leq f(x) \text{ for } x \in [0,1] \backslash (P \cup B).$$

That is, $g \leq f$ in $L_\infty[0,1]$. Thus $f = \inf\limits_n f_n$. □

2.2 Order continuity

Definition 2.10. A Banach lattice X is said to have an order continuous norm if any (sequence) net decreasing to 0 is norm convergent to 0.

Theorem 2.11. *([57], Theorem 5.19.) For a Banach lattice X*

1. *the norm on X is order continuous.*

2. *X is σ-Dedekind complete such that $\|x_n\| \to 0$, as $n \to \infty$, for any decreasing sequence (x_n).*

Theorem 2.12. *The Banach lattice $(L_p(\mu), \|\cdot\|_p)$, $1 \leq p < \infty$ has order continuous norm.*

Proof. Take f_α in $(L_p(\mu), \|\cdot\|_p), 1 \leq p < \infty$, such that $f_\alpha \downarrow 0$. Let

$$\int |f_\alpha|^p d\mu \downarrow s > 0.$$

We show that $s = 0$. Now, take an increasing sequence (α_n) such that

$$\int |f_\alpha|^p d\mu \downarrow s.$$

We show that$|f_\alpha|^p d\mu \downarrow 0$. To see this, let $|f_\alpha|^p d\mu \downarrow f \geq 0$ for some fixed α. For each $n \in \mathbb{N}$, there exists $B_n \geq \alpha$ and $B_n \geq \alpha_n$. Let us assume that $B_{n+1} \geq B_n$ for all $n \in \mathbb{N}$. If

$$|f_{B_n}|^p d\mu \downarrow g \geq 0,$$

then

$$f \geq g$$

and

$$\int |f|^p d\mu = \int |g|^p d\mu.$$

Hence $f = g$. Therefore, for each α,

$$f = g \leq f_{B_n} \leq f_\alpha.$$

Now, since $f_\alpha \downarrow 0$, we have that $f = 0$. Thus,

$$f_{\alpha_n} \downarrow 0.$$

So,

$$s = \lim_{n\to\infty} \int |f_\alpha|^p d\mu = 0.$$

Thus, $L_p(\mu)$, $1 \leq p < \infty$ has order continuous norm. □

Example 2.13. The space $(L_\infty[0,1], \|\cdot\|_\infty)$ does not have order continuous norm. To show this, consider a characteristic function $\chi_{(0,\frac{1}{n})}$, where $\chi_{(0,\frac{1}{n})} \in (L_\infty[0,1], \|\cdot\|_\infty)$. Then, we get that $\chi_{(0,\frac{1}{n})} \downarrow 0$ but $\left\|\chi_{(0,\frac{1}{n})}\right\|_\infty = 1$.

Remark 2.14. *In this section, we showed that OSC property is more general than order continuity. Later, we will show application of the OSC property.*

3 Operators in ordered vector spaces

Definition 3.1. Let X and Y be two vector spaces. A map T is called linear if, for all $x, y \in X$ and $\alpha \in \mathbb{R}$, then $T(\alpha x + y) = \alpha T(x) + T(y)$.

Definition 3.2. Let X and Y be two ordered vector spaces: An operator $T : X \to Y$ is called,

1. ***positive*** if $T(X^+) \subseteq Y^+$.

2. ***strictly positive*** if $T(X^+) \subset Y^+$ for all $x > 0$.

3. ***regular*** if $T = T_1 - T_2$ where T_1, T_2 are positive.

4. ***order bounded*** if T maps order bounded subsets of X to order bounded sets of Y.

The space of linear operators, between real ordered vector spaces, is denoted by $\mathcal{L}(X, Y)$ is an ordered vector space if for all T_1, T_2, we have that $T_1 \geq T_2$ whenever $T_1 - T_2 \geq 0$, that is $T_1 - T_2 \in Y^+$. Next, we give two more classes of operator spaces, We denote by, $\mathcal{L}_r(X, Y)$ the space of all regular operators, and $\mathcal{L}_b(X, Y)$ is the space of all bounded operators.

The sets of operators follows the following inclusion,

$$\mathcal{L}_r(X, Y) \subseteq \mathcal{L}_b(X, Y) \subseteq \mathcal{L}(X, Y).$$

There are several articles and results addressing the reverse inclusion. We discuss some of such results in the study. First, we will show that order bounded operators are bounded.

Theorem 3.3. *([50], Theorem 1.0.) Let E be a Banach lattice. If $x_n \to x$ in E then there exists a subsequence (x_{n_k}) of (x_n) and some $\alpha > 0$ such that $|x_{n_k} - x| \leq \frac{1}{n}\alpha$ for all $n \in \mathbb{N}$.*

Theorem 3.4. *([2], Theorem 1.31.) Every order bounded operator T from a Banach lattice E to a normed Riesz space F is continuous.*

Proof. Let the operator $T : E \to F$ be order bounded, from a Banach lattice E to a normed Riesz space F. On the contrary, also assume that T is not continuous. Therefore, there is a sequence (x_n) in E such that $x_n \to 0$ as $n \to \infty$ and $Tx_n \nrightarrow 0$ as $n \to \infty$ in F. We can find a subsequence (x_{n_k}) of (x_n) and for some $\varepsilon > 0$ such that $\|Tx_{n_k}\| > \varepsilon$ for each n_k. By Theorem 3.3 there is a subsequence y_{n_k} of (x_n) and some $u \in E^+$ such that

$$|y_{n_k}| \leq \frac{1}{n_k} u \text{ for some } n_k \in \mathbb{N}.$$

But T is order bounded, this implies that there exists $w \in F$ such that

$$T[-u, u] \subseteq [-w, w], \text{ since } n_k |y_k| \leq u.$$

Thus $n_k T(y_{n_k}) \leq w$ where n_k is in $\mathbb{N}$. But

$$\begin{aligned} 0 < \varepsilon &\leq \|T(y_{n_k})\| \\ &\leq \frac{1}{n_k} \|w\| \to 0. \end{aligned}$$

This is a contradiction. Therefore, the operator T is continuous.

□

To show that not all bounded operators are regular, we first recall the following results and definitions.

Theorem 3.5. *([2], Theorem 1.16.) Let E, F be Riesz spaces, with F Dedekind complete. Then the vector space $\mathcal{L}_r(E, F)$ is a Dedekind complete Riesz spaces such that*

$$\mathcal{L}_r(E, F) = \mathcal{L}_b(E, F).$$

Furthermore, lattice operators for $\mathcal{L}_r(E,F)$ *are given by*

1. $T^+ = \sup\{Ty : 0 \leq y \leq x\}$.

2. $T^- = \sup\{-Ty : 0 \leq y \leq x\}$ *(or* $T^-x = \inf\{Ty : 0 \leq y \leq x\}$*).*

3. $|T| = \sup\{Ty : -x \leq y \leq x\}$ $\forall T \in \mathcal{L}_r(E,F)$ *and all* $x \in E^+$*, where* $|T| = T \bigvee -T$ *is the modulus of* T*.*

Proof. First, we note that the operator T^+ and the T_1 in Theorem 3.10. of [13], where it is shown that $T_1 = T \vee 0$, coincides. Therefore, positive part of T is given as T^+. Now, applying Proposition 2.10 and Proposition 2.8 (2. & 3.),

$$\begin{aligned} -T^-x &= Tx - T^+x \\ &= Tx - \sup\{Tv : 0 \leq y \leq x\} \\ &= Tx + \inf\{-Ty : 0 \leq y \leq x\} \\ &= \inf\{T(x-y) : 0 \leq y \leq x\}. \end{aligned}$$

Concluding the proof for (2). Now, in order to show (3), we have

$$\begin{aligned} |T|(x) &= T^+x + T^-x \\ &= \sup\{Ty : 0 \leq y \leq x\} + \sup\{-Tz : 0 \leq z \leq x\} \\ &= \sup\{T(y-z) : 0 \leq y \leq x, 0 \leq z \leq x\}. \end{aligned}$$

Now, since $0 \leq y \leq x, 0 \leq z \leq x$ implies that $|y-z| \leq x$, we have further

$$|T|(x) = \sup\{Tf : |f| \leq x\}.$$

On the other hand,

$$|f| \leq x \implies Tf \leq |Tf| \leq |T|(|f|) \leq |T|(x).$$

And so

$$\sup\{Tf : |f| \leq x\} \leq \sup\{|Tf| : |f| \leq x\} \leq |T|(x).$$

Thus proving (3). □

The following definitions will recap or touch on Rademacher functions, *that is* a family of orthonomal functions, of which are bounded but not order bounded linear operators. Before defining the Rademacher functions, we will first state the definition of the signum function of real numbers.

Definition 3.6. The signum function of $x \in \mathbb{R}$ is a piecewise function given by

$$sgn = \begin{cases} -1, \text{ if } x < 0, \\ 0, \text{ if } x = 0, \\ 1, \text{ if } x > 0 \end{cases}$$

Definition 3.7. ([55], *page* 733.) Rademacher functions are defined by $r_0 \equiv 1$ and $r_n(x) = sgn\ sin(2^n \pi t)$, for $t = [0, 1]$, $n = 1, 2, ...$ It is clear that,

$$r_n(t) = \begin{cases} -1, t \in \left(\frac{2k-1}{2^n}, \frac{2k}{2^n}\right), \\ 0, t \in \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, ..., \frac{2^n-1}{2^n}\right\}, \\ 1, t \in \left(\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right) \end{cases}$$

The Rademacher functions form an orthogonal system in this sense,

$$\int_0^1 r_m(t) r_n(t) dt = \begin{cases} 0, m \neq n, \\ 1, m = n \end{cases}$$

The functions r_n are bounded and integrable, so they belong to L_∞ since ${L_1}^+ = L_\infty$. Hence the functions can be viewed as continuous linear functionals on L_1, given that $r_n(x) = \int_0^1 x(t) r_n(t) dt$, for $x \in L_1$.

Example 3.8. ([20], Example 3.6.) Let $X = L_1([0,1])$, Y is the closed subspace of X generated by the Rademacher functions r_n, and let P be the positive cone of r_n. Recalling that $\{r_n\}$ is a basic sequence in $L_1([0,1])$, equivalent to the standard basis of the ℓ^2 space. Therefore, Y is is isomorphic to ℓ^2 and the cone P is reflexive. The Rademacher functions $\{r_n\}$, as elements of $L_\infty([0,1])$ are coefficient functionals of $\{r_n\}$ and, for each i, the dual cone of P is given by

$$P^* = \Big\{ f \in L_\infty[0,1] \colon f = \sum_{i=1}^{\infty} \lambda_i r_i, \lambda_i \geq 0 \Big\}.$$

The sum is taken in the weak* $\sigma(L_\infty, Y)$-topology of $L_\infty([0,1])$. Taking Theorem 3.5 [20], we have that P^* is not reflexive. Note that, Theorem 3.5. [20] implies that a reflexive cone P in a non reflexive space X. Nevertheless, the subspace $\overline{P-P}$, the closure of $P-P$, is a reflexive subspace. This does not hold in general as shown in Example 3.7. [20]. Furthermore, the subspace $\overline{P-P}$ generated by P is dense in X.

Definition 3.9. ([55], *page 727.*) Let X be a non-empty set, with the cone $K \subseteq X$. We define the characteristic function χ_K of K by

$$\chi_K = \begin{cases} 1, & if \quad x \in K \\ 0, & if \quad x \notin K \end{cases}$$

Proposition 3.10. *The Rademacher functions are defined by*

$$r_n(x) = \sum_{k=1}^{2^n} \Big(-1\Big)^k \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x),$$

where the set E have an indicator function given by

$$\chi_E(x) = \begin{cases} 0, x \notin E, \\ 1, x \in E. \end{cases}$$

Definition 3.11. Let (r_n) be Rademacher function on L_1, we say that $r_n \overset{w^*}{\to} 0$ if

$$\lim_{n\to\infty} \int_0^1 x(t) r_n(t) dt \text{ where } x \in L_1.$$

We state the following proposition whose results are of use in the subsequent proof of Theorem 3.14. First we will define weak* topology

Definition 3.12. Let X be a normed space. For each element $x \in X$, the functional of the form $\rho_x(\mu) = |\langle x, \mu \rangle|$, $\mu \in X^*$, is a seminorm on X^*. The topology induced by the family of seminorms $\{\rho_x\}_{x\in X}$ is the weak topology on X^*, known as the ***weak**-topology*** , and is denoted as $\sigma(X^*, X)$.

Proposition 3.13. *Consider the space $L_\infty[0,1]$ and $x_n, x \in L_\infty$. Then the following are equivalent,*

1. $x_n \overset{w^*}{\to} x$,

2. $\sup\limits_{n\in\mathbb{N}} \|x_n\| < \infty$ *and* $\lim\limits_{n\to\infty} \int_0^s x_n(t)dt = \int_0^s x(t)dt$ *for all* $s \in [0,1]$.

Theorem 3.14. *([40], page 105.) Consider the space $L_\infty[0,1]$ and the sequence of Rademacher functions $r_n \in L_\infty[0,1]$. Then*

$$r_n \overset{w^*}{\to} 0.$$

Proof. Let $y_n, y \in L_\infty[0,1]$. Then $y_n \overset{w^*}{\to} y$ is equivalent, for all $x \in L_1$, to

$$\lim_{n\to\infty} \int_0^1 x(t) y_n(t) dt = \int_0^1 x(t) y(t) dt.$$

Therefore, by Proposition 3.13, the implication of this is that $\sup\limits_{n\in\mathbb{N}} \|y_n\|_\alpha < \infty$ and, for all $s \in [0,1]$,

$$\lim_{n\to\infty} \int_0^s y_n(t) dt = \int_0^s y(t) dt.$$

And, thus, the sequence (y_n) is norm bounded in $L_\infty[0,1]$ where $z \in L_\infty[0,1]$, with the essential supremum (or the $ess\sup$) of f given as in the Definition 2.4,

$$\|z\|_\infty = ess\sup\{|z(t)| : t \in [0,1]\} = \inf_{\lambda(N)=0} \sup\{|z(t)| : t \in [0,1]\backslash N\}.$$

That is, the infimum is taken over all subsets $N \subset [0,1]$ of Lebesgue measure 0. In this case, the norm $\|r_n\| = 1$ for all $n \in \mathbb{N}$. So, we only show

$$\lim_{n\to\infty} \int_0^s r_n(t)dt = 0 \text{ for all } s \in [0,1].$$

Now, for $0 < s < 1$ and $n \in \mathbb{N}$, let K_n be the greatest number in $\{0, 1, ..., 2^{n-1} - 1\}$. Since $|r_n(t)| \leq 1$, we will have, as $n\to\infty$

$$\begin{aligned}\Big|\int_{\frac{2K_n}{2^n}}^{s} r_n(t)dt\Big| &\leq \int_{\frac{2K_n}{2^n}}^{s} |r_n(t)|dt \\ &\leq \int_{\frac{2K_n}{2^n}}^{\frac{2K_n+2}{2^n}} dt \\ &= \frac{2}{2^n}\to 0.\end{aligned}$$

Since the function r_n is alternating between -1 and 1, on successive open dyadic intervals, we will thus have that

$$\int_0^{\frac{2K_n}{2^n}} r_n(t)dt = 0.$$

□

Now we show an example of an operator T from $L_1([0,1])$ to c_0 that is bounded linear, but fails to be regular.

Example 3.15. ([59], Example 1.2.) Let $T : L_1([0,1]) \to c_0$ be an operator. Then T is a bounded linear operator but is not regular. Take a sequence (r_n) of Rademacher functions on $[0,1]$. Since, the sequence of Rademacher functions lie in $L_\infty[0,1]$ they may also, therefore, be regarded as elements of $L_1[0,1]^*$, since $L_1[0,1]^* = L_\infty[0,1]$. We have that (r_n) is in $L_1[0,1]^*$ and hence $r_n \to 0$ weak*. Now, define an operator $T : L_1([0,1]) \to c_0$ by

$$Tx = \sum_{n=1}^{k} r_n(x)e_n$$

where e_n is an element of c_0 such that $e_n = (0,0,...,1,0,0,...,0)$. We show that T is bounded. Now,

$$\begin{aligned} \|Tx\|_\infty &= \sup_{n\in\mathbb{N}} \left\{ \left| \sum_{n=1}^{k} r_n(x)e_n \right| \right\} \\ &\leq \sup_{n\in\mathbb{N}} |r_n(x)| \\ &\leq \|x\|_1 \cdot \|x\|_\infty. \end{aligned}$$

Thus T is bounded. And since $T(r_n) = e_n$ for any $n \in \mathbb{N}$ and $T(r_0) = 0$ we have, by the linearity of T, that

$$T(r_0 + r_n) = T(r_0) + T(r_n) = e_n.$$

Note that $r_0 + r_n \geq 0$, since r_0 is constantly one. Now, assuming $U \geq T, 0$, then

$$U(2r_0) \geq U(r_0 + r_n) \geq T(r_0 + r_n) = e_n.$$

So, $U(2r_0) \geq e_n$ for all $n \in \mathbb{N}$. But, this contradicts the fact that $U(2r_0) \subseteq c_0$. Hence, T is not order bounded, since c_0 is Dedekind.

Theorem 3.16. *([64], Proposition 4.0.35.) Let $T : E \to F$ be a regular linear operator from E to F, where E and F are Banach lattices. Then operator T is order bounded.*

Proof. Suppose that $[f,g]$ is an order interval, and T is regular, then $T = T_1 - T_2$ where

$T_1, T_2 \in F^+$. Therefore, T_1 and T_2 are order bounded meaning that

$$T_1[f, g] \subseteq [x_1, y_1]$$

and

$$T_2[f, g] \subseteq [x_2, y_2]$$

are order intervals in $W \subseteq F$. Therefore, we have that

$$T[f, g] \subseteq [x_1 - x_2, y_1 - y_2] = [x, y].$$

Thus, T is order bounded. □

Definition 3.17. Let X be a Banach lattice, we call X an

1. AL-space if $\|x + y\| = \|x\| + \|$ for all $x, y \in X^+$ with $x \wedge y = 0$, and
2. AM-space if $\|x_y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X^+$ with $x \wedge y = 0$.

Theorem 3.18. *([59], Theorem 2.10.) Let X be a Banach space, then the following conditions are equivalent,*

1. *X is atomic with order continuous norm.*
2. *If Y is isormophic to an AM-space then $\mathcal{L}(X, Y) = \mathcal{L}_r(X, Y)$.*
3. *If Y is isormophic to an AM-space then $\mathcal{L}(X, Y)$ is a lattice.*

Theorem 3.19. *([64], Definition 4.0.32.) Let X and Y be a ordered vector spaces with the linear operator T, defined as $T : X \mapsto Y$. The operator is said to be order bounded if and only if T maps $[0, \gamma]$ to an order interval in Y.*

Proof. First we show that T maps $[0, \gamma]$ to an order interval. By definition, T will be order bounded, and so

$$T[0,\gamma] \subseteq [x,y] \text{ with } [0,\gamma] \text{ an order interval in } Y.$$

Conversely, we show that T is order bounded. Suppose that,

$$T[0,\gamma] \subseteq [x,y] \in Y \text{ for all } \gamma \in \mathbb{R}^+.$$

Then for each $\rho \in X$ we have

$$T(\rho + [0,\gamma]) \subseteq T\rho + [a,b] = [a + T\rho, b + T\rho] = [\tilde{x}, \tilde{y}].$$

Both $\dot{x}$ and $\dot{y}$ are well defined since Y is a vector space. And so,

$$\rho + [0,\gamma] = [\rho, \gamma + \rho] \subseteq [\tilde{x}, \tilde{y}].$$

Now, letting $h := \gamma + \rho$, then $[\rho, h] \subseteq [\tilde{x}, \tilde{y}]h$ is also well defined with X being given as a vector space too. □

Definition 3.20. ([41], Definition 1.1.) Let E, F be Banach lattices. A mapping $T : E \to F$ is called positive linear if the following properties are valid

1. $T[0,x]$ is dense in $T[0,Tx]$ for each $x \in E$, $x > 0$.

2. T is a lattice homomorphism.

3. $T[0,x] = T[0,Tx]$ for each $x \in E$, $x > 0$.

Theorem 3.21. *([13], Theorem 1.71.) Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. A surjective linear operator $T\colon X \to Y$ is an isomorphism if and only if, whenever*

$x \in X$, there are positive constants ρ and γ such that

$$\rho\|x\|_X \leq \|Tx\|_Y \leq \gamma\|x\|_X.$$

Proof. Suppose that $K \subseteq X$, $P \subseteq Y$ are closed cones in the normed spaces respectively. Then the cone K is isomorphic to the cone P if there exits an one-to-one, onto map $T\colon K \to P$ such that, for each $x, y \in K$ and $\lambda, \mu \in \mathbb{R}_+$, we have

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

And the operators T, T^{-1} are continuous, in the norm-induced metric topologies of both cones K and P. By the continuity of T and T^{-1} at zero, there exist real constants $\rho, \gamma > 0$ such that, for any $x \in P$, we get

$$\rho\|x\| \leq \|T(x)\|_Y \leq \gamma\|x\|_X.$$

□

Definition 3.22. Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces. Then a linear operator $T\colon X \to Y$ is an isomorphism if T is a bijection and continuous and its inverse operator $T^{-1}\colon Y \to X$ is also continuous.

Theorem 3.23. *([5], Theorem 9.15.) For a linear operator $T : E \to F$ between Riesz spaces, the following statements are equivalent.*

1. *$T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in E$.*
2. *$T(x \wedge y) = T(x) \wedge T(y)$ for all $x, y \in E$.*
3. *$T(x^+) = (Tx)^+$ for all $x \in E$.*
4. *$T(x^-) = (Tx)^-$ for all $x \in E$.*
5. *$T(|x|) = |Tx|$ for all $x \in E$.*
6. *If $x \wedge y = 0 \in E$, then $Tx \wedge Ty = 0 \in F$.*

Proof. This proof is a direct application of the lattice identities in Riesz spaces. In particular, we will prove this result by establishing equivalence of statements (1) and (5). And so, assuming first that statement (1) is true. Then

$$\begin{aligned} T|x| &= T(x \vee (-x)) \\ &= T(x) \vee T(-x) \\ &= T(x) \vee (-T(x)) \\ &= |Tx|. \end{aligned}$$

Now assume that statement (5) holds. Then from $x \vee y = \frac{1}{2}(x + y + |x - y|)$, we will have that

$$\begin{aligned} T(x \vee y) &= \frac{1}{2}(Tx + Ty + T|x - y|) \\ &= \frac{1}{2}\Big(Tx + Ty + |Tx - Ty|\Big) \\ &= Tx \vee Ty. \end{aligned}$$

□

Definition 3.24. ([5], Definition 9.16.) A linear operator $T \colon E \to F$ between Riesz spaces is a *lattice homomorphism* (or a *Riesz homomorphism*) if T satisfies any of the statements of equivalence in Theorem 3.23.

A *lattice homomorphism* that is also one-to-one is a lattice isomorphism (or a *Riesz isomorphism*).

Remark 3.25. *Every lattice homomorphism $T \colon E \to F$ is a positive operator. Indeed, if $x \geq 0$, then*

$$Tx = T(x^+) = (Tx)^+ \geq 0.$$

Furthermore, note that, if $T \colon E \to F$ is a *lattice homomorphism*, then the range $T(E)$ is a Riesz subspace of F. In the case $T \colon E \to F$ is a lattice isomorphism, then $T(E)$ and E are considered as identical Riesz spaces. The Riesz spaces E and F are lattice isomorphic if there is a lattice isomorphism from E onto F. The basic relationships between these properties are described in the results of Proposition 3.26, pertaining the operator T.

Proposition 3.26. *([41] Proposition 1.2.) Let E, F be Banach lattices, and $T : E \to F$ be a positive linear operator. Then,*

1. *$T[0, x]$ is dense in $T[0, Tx]$ for each $x > 0 \in E$ if and only if T is a lattice homomorphism.*

2. *T is a lattice homomorphism if and only if $T[0, x] = T[0, Tx]$ for each $x > 0 \in E$.*

Proof. (1.) Assume that $T[0, x]$ is dense in $T[0, Tx]$ for each $x > 0 \in E$. Furthermore, suppose that $y^* \in F^*$, *that is* an element of the dual of F, then

$$
\begin{aligned}
(T^*y^*)^+(x) &= \sup_{z\in[0,x]} \langle z, T^*y^*\rangle = \sup_{z\in[0,x]} \langle Tz, y^*\rangle \\
&= \sup_{y\in[0,Tx]} \langle y, y^*\rangle \\
&= \langle Tx, (y^*)^+\rangle = \langle Tx, T^*y^{*+}\rangle.
\end{aligned}
$$

Consequently,

$$(T^*y^*)^+ = T^*(y^{*+}) \text{ for all } y^* \in F.$$

That is, T^* satisfies property (2) of Definition 3.23 Conversely, suppose that T is a lattice homomorphism, for for each $x > 0 \in E$, and also that $z \geq 0 \notin T[0, x]$. Thus, $T[0, x]$ is dense in $T[0, Tx]$.

(2.) We next show that $T[0, x] = T[0, Tx]$ given that T valid under (1), that is $T[0, x]$ is dense in $T[0, Tx]$ for each $x > 0 \in E$ if and only if T is a lattice homomorphism. Given $y^* \geq 0 \in F^*$ we will have that $T^*[0, y^*]$ and $[0, T^*y^*]$ are weak*-compact and thus $T^*[0, y^*]$ is $\sigma(E, E^*)$-dense in $[0, T^*y^*]$. Therefore, implying that T^* is is a lattice homomorphism. On the other hand, T^{**}

is also a lattice homomorphism since T^* is dense in $[0, T^*y^*]$ for all $y \geq 0$ in F^*. And since T^{**} is also a lattice homomorphism then so does T. ☐

Definition 3.27. Let E and F be Bananch lattices, with F Dedekind complete. An r-norm for a regular operator T is defined by $\|T\|_r = \|T\|$, where $\|T\|_r = \inf\{\|S\| : \pm T \leq S\}$.

Theorem 3.28. *([2], Theorem 1.32.) Let E and F be the Banach lattice with F Dedekind complete. Then the Dedekind complete Riesz space $(\mathcal{L}_r(E,F), \|\cdot\|_r)$ is a Banach lattice.*

Proof. First, we have to show that $\|\cdot\|_r$ is a norm on $\mathcal{L}_r(E,F)$. That is, we need to show that the following is true:

1. $\|T\|_r \geq 0$, $\|T\|_r = 0 \Rightarrow T = 0$.

2. $\|\alpha T\|_r = |\alpha|\|T\|_r$.

3. $\|T + S\|_r = \|T\|_r + \|S\|_r$.

That is,

1. Let $T \in \mathcal{L}(E,F)$. Then we have that $\|T\|_r \geq 0$, $\|T\|_r = 0 \Rightarrow T = 0$.

2. Let $\alpha \in \mathbb{R}$. Then we have that

$$\|\alpha T\|_r = \||\alpha T|\| = \|\alpha T\| = \alpha\|T\| = \alpha\||T|\| = \alpha\||T|\|_r.$$

3. Let $S, T \in$. Then we have that

$$\|T + S\|_r = \||T + S|\| = \|T + S\| \leq \|T\| + \|S\| = \||T|\| + \||S|\| = \|T\|_r + \|S\|_r$$

.

Now, for the norm $\|\cdot\|_r$, we can show that it is also a lattice norm. Suppose T is a positive operator, then

$$\||T|\|_r = \||T|\| = \|T\|.$$

Implying that $\|\cdot\|_r$ is absolute. In order to see that $\|\cdot\|_r$ is monotone, we let $S, T \in \mathcal{L}_r(E, F)$ such that $0 \leq S \leq T$ and $\|x\| = \||x|\| \leq 1$, then

$$|Sx| \leq S\|x\| \leq T\|x\|.$$

We see, from here then, that

$$\|Sx\| = \||Sx|\| \leq \|S|x|\| \leq \|T|x|\| \leq \|T\|.$$

Therefore,

$$\|s\| = \sup_{\|x\|\leq 1} \|S\| \leq \|T\|.$$

In particular, if $S, T \in \mathcal{L}_r(E, F)$ such that $|S| \leq |T|$ then

$$\|S\|_r = \||S|\| \leq \||T|\| = \|T\|_r.$$

Thus, concluding all the required steps in showing that $\|\cdot\|_r$ is a norm on $\mathcal{L}_r(E, F)$. Now, we show the norm completeness of $\mathcal{L}_r(E, F)$. This require that we show by [2], Theorem 4.8 that every increasing $\|\cdot\|_r$ sequence of positive operators is $\|\cdot\|_r$ convergent in $\mathcal{L}_r(E, F)$. To see this, let $(T_n), n \in \mathbb{N}$ be an increasing $\|\cdot\|_r$ Cauchy sequence of positive operators. Given that $\|T\|_r$ for all $T \in \mathcal{L}_r(E, F)$ we have that

$$\|T_n - T_m\| \leq \|T_n - T_m\|_r.$$

From which it follows (T_n) is Cauchy sequence with respect to the norm $\|\cdot\|$. The space $\mathcal{L}(E, F)$ is a Banach lattice and there exists $T \in \mathcal{L}(E, F)$ such that

$$\|T_n - T\| \to 0.$$

Now, for each $x \in E^+$ we have $0 \leq T_n x \uparrow$ and $\|T_n x - Tx\| \mapsto 0$, hence it must be that $T_n x \uparrow$ for all $x \in E^+$. Particularly, T is a positive operator and so $T \in \mathcal{L}_r(E, F)$ and $0 \leq T_n \leq T$ holds for all $n \in \mathbb{N}$. That is,

$$\|T_n - T|_r = \||T_n - T|\| = \|T - T_n\| \to 0.$$

And, therefore, $(\mathcal{L}_r(E, F), \|\cdot\|_r)$ is complete. □

Now we show an example of Banach lattices E, F, with F Dedekind complete, and a regular operator T from E into F such that $\|T\| \leq \|T\|_r$.

Example 3.29. ([2], Exercise 1.18.) Suppose E and F are Banach lattices, F Dedekind complete and $T : E \to F$ be a regular operator. Then $\|T\| < \|T\|_r$. Consider $E = F = \mathbb{R}^2$ with $\|\cdot\|_2$ and suppose that T is an order-bounded mapping defined by

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now, T is regular since F is Dedekind complete and the modulus of T is

$$|T| = T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The norms of these operators are

$$Tx = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$$

given some fixed

$$\begin{aligned}\|Tx\|_2 &= \left\| \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} \right\|_2 \\ &= \sqrt{(x_1 - x_2)^2 + (x_1 + x_2)^2} \\ &= \sqrt{x_1^2 + x_2^2 - 2x_1x_2 + x_1^2 + x_2^2 + 2x_1x_2} \\ &= \sqrt{2(x_1^2 + x_2^2)} = \sqrt{2}.\end{aligned}$$

We then compute the norm of $|T|$, as follows,

$$\begin{aligned}\|Tx\|_2 &= \left\| \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} \right\|_2 \\ &= \sqrt{(x_1 + x_2)^2 + (x_1 + x_2)^2} \\ &= \sqrt{x_1^2 + x_2^2 + 2x_1x_2 + x_1^2 + x_2^2 + 2x_1x_2} \\ &= \sqrt{2(x_1 + x_2)^2} \\ &= \sqrt{2} \cdot (x_1 + x_2).\end{aligned}$$

Thus

$$\begin{aligned}\|T\| = \sqrt{2} &< \||T|\| \\ &= \sqrt{2}(x_1 + x_2) \\ &= \|Tx\|_r.\end{aligned}$$

Therefore

$$\|T\| \leq \|T\|_r.$$

Now we furnish an example where $\|T\|$ and $\|T\mathbf{1}\|$ coincide.

Example 3.30. Let $T : C[0,1] \to C[0,1]$ be an opetator. Show that, if $f \geq 0$ implies $Tf \geq 0$ then T is continuous and $\|T\| = \|T\mathbf{1}\|$ where $\mathbf{1}$ is a constant function equal to 1 in $C[0,1]$. In order to show this, we suppose that $f \in C[0,1]$, with $\|f\|_\infty \leq 1$. Therefore, since f is positive and ≤ 1,

$$f + 1 \geq 0, \text{ and hence, } T(f+1) \geq 0.$$

Now, f being positive implies that

$$-T\mathbf{1} \leq Tf.$$

Therefore,

$$\|Tf\| \leq \|T\mathbf{1}\| \text{ for all } \|f\|_\infty \leq 1.$$

And, the boundedness of f implies that of T. Thus, T is continuous. Now, in order to complete the proof, we must show that

$$\|T\| = \|T\mathbf{1}\|.$$

Since the constant function is continuous the implication is that $\|T\| \leq \|T\mathbf{1}\|$. And, therefore, $\|T\| = \|T\mathbf{1}\|$ holds.

We show, below, that the operator T is almost order bounded while its modulus does not exist.

Corollary 3.31. *([42], Example 1.) Consider the continuous function $g : [0,1] \to [0,1]$ defined by*

$$g(x) = \begin{cases} x, 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}, \frac{1}{2} < x \leq 1 \end{cases}$$

Then, the operator $T : C[0,1] \to C[0,1]$ *given by*

$$Tf\left(x = f\Big(g(x)\Big)\right) - f\left(\frac{1}{2}\right).$$

is regular, and therefore, also an order bounded operator.

Proof. Given that, the space $C[0,1]$ is an AM-space with unit, then the operator T is almost order bounded. Noting that, the modulus of T does not exist ([4], Exercise 9). And, therefore, for cases where T is an almost order bounded operator, we will have that, indeed, the modulus of T exists, that is,

$$T = |T|.$$

Furthemore, this modulus is almost order bounded too. □

Proposition 3.32. *Let* E *and* F *be Banach lattices. Every positive operator* $T\colon E \to F$ *is regular.*

Proof. Suppose that $T\colon E \to F$ is linear operator. Since for any $x \in E$, we have that $Tx \in F$, which is a Riesz space. Then, by recalling Theorem 1.11, we will have that

$$Tx = (Tx)^+ - (Tx)^- = T^+(x) - T^-(x),$$

where

$$T^+(x) = (Tx) \vee 0 \text{ and } T^-(x) = (-Tx) \vee 0.$$

Thus,

$$T^+(x) \geq 0 \text{ and } T^-(x) \geq 0.$$

Therefore, subsequently T is a regular operator since it is a difference of two positive operators. □

Definition 3.33. ([18], Definition 1.) Let (X, d) be a complete metric space. A function $f : X \to X$ is called a ***contraction*** if there exists $k < 1$ such that for any $x, y \in X$,

$$k \cdot d(x, y) \geq d(f(x), f(y)).$$

Next, we show an example of an order bounded operator which is not positive.

Example 3.34. ([4], Example 1.16.) Let the operator $T : C[-1, 1] \to C[-1, 1]$ be defined by

$$Tf(t) = f\Big(sin\Big(\frac{1}{t}\Big)\Big) - f\Big(sin\Big(1 + \frac{1}{t}\Big)\Big), \ 0 \leq |t| \leq 1 \text{ and } Tf(0) = 0.$$

Then T is an order bounded operator but not positive. Firstly, we show that T is linear operator, consider any $f, g \in C[-1, 1]$ and $\alpha, \beta \in \mathbb{N}$. Then by definition of T we get

$$\begin{aligned} T\left(\alpha f + \beta g\right)\left(t\right) &= f\left(sin\left(\frac{1}{t}\right)\right) + \beta g\left(sin\left(1 + \frac{1}{t}\right)\right) - \left(\alpha f\left(sin\left(1 + \frac{1}{t}\right)\right) + \beta g\left(sin\left(1 + \frac{1}{t}\right)\right)\right) \\ &= \alpha\left(f\left(sin\left(\frac{1}{t}\right)\right) - f\left(sin\left(1 + \frac{1}{t}\right)\right)\right) + \beta\left(g\left(sin\left(\frac{1}{t}\right)\right) - g\left(sin\left(1 + \frac{1}{t}\right)\right)\right) \\ &= \alpha Tf + \beta Tg. \end{aligned}$$

f is uniformly continuous, that is for all $x, y \in [-1, 1]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Recall that $sin(x)$ is a contraction, thus

$$\left|sin\left(\frac{1}{t}\right) - sin\left(1 + \frac{1}{t}\right)\right| \leq \left|\frac{1}{t} - \left(t + \frac{1}{t}\right)\right| = |t|.$$

Now, if $|t| < \delta$ then

$$|Tf(t) - Tf(0)| = \left|f\left(sin\left(\frac{1}{t}\right)\right) - f\left(sin\left(1 + \frac{1}{t}\right)\right)\right| \leq \varepsilon \text{ as } \left|sin\left(\frac{1}{t}\right) - sin\left(1 + \frac{1}{t}\right)\right| < \delta.$$

And f is uniformly continuous. Secondly, we want to show that T is order bounded, consider the order interval $[-1, 1]$, that is $f \in C[-1, 1]$ such that

$\|f\| \leq 1$. Then, we have

$$\begin{aligned}\left|Tf\left(t\right)\right| &= \left|f\left(sin\left(\frac{1}{t}\right)\right) - f\left(sin\left(1+\frac{1}{t}\right)\right)\right| \\ &\leq f\left(sin\left(\frac{1}{t}\right)\right) + f\left(sin\left(1+\frac{1}{t}\right)\right) \\ &\leq \left|f\left(sin\left(\frac{1}{t}\right)\right)\right| + \left|f\left(sin\left(1+\frac{1}{t}\right)\right)\right| \\ &\leq 1+1 = 2.\end{aligned}$$

Since $[-1, 1]$ is a compact interval, all continuous functions will attain their maximum on the set, and thus there exist $M \in \mathbb{R}$ and $f(t) \leq \max_{t\in[-1,1]} |f(t)|$ for each $t \in [-1, 1]$. Thus

$$|f| \cdot \frac{1}{M} \leq 1.$$

So, for all $f \in C[-1, 1]$ we have

$$|f| \leq M \cdot 1 \text{ or } f \in M[-1, 1] \implies Tf \in 2M[-1, 1]$$

Thus, T is order bounded.

Theorem 3.35. *The Rademacher functions $(r_n) \subseteq L_1$ are pointwise convergent to 0, that is, for any $x \in L_1$, we have that*

$$\lim_{n\to\infty} \int_0^1 x(t)r_n(t)dt \to 0.$$

Below is an example of a norm-bounded operator that is not order bounded. First, we will state the definition of operator domination.

Definition 3.36. Let $T : E \to F$ between two Riesz spaces. We say an operator S is dominated by T whenever $|Sx| \leq |Tx|$ for all x in E.

Example 3.37. Suppose $X = L_1([0,1])$ and $Y = c_0$. Define the operator $T : L_1([0,1]) \to c_0$ by

$$Tx = \sum_{n=1}^{\infty} r_n(x)e_n,$$

where $e_n \in c_0$ is the n^{th} standard basis vector. T is a well-defined bounded linear operator since the sequence $(r_n(x))$ is in c_0.

$$\begin{aligned}
\|Tx\|_\infty &= \sup_{n\in\mathbb{N}} |r_n(x)| \\
&\leq \sup_{n\in\mathbb{N}} \|r_n(x)\|_\infty \|x\|_1 \\
&= \|x\|_1.
\end{aligned}$$

Therefore, T is bounded. Next, we show that T is not regular. Suppose $T(r_0) = 0$ and $T(r_n) = 1$, since $r_0 = 1$ then

$$T(r_0) = \sum_{n=0}^{\infty} sgn\Big(sin(2^n \pi r_0)\Big)e_n.$$

Now, for $n, m \in \mathbb{N}$ we have

$$\begin{aligned}
r_n(r_m) &= \sum_{k=1}^{2^n}\sum_{j=1}^{2^m} \Big(-1\Big)^k \Big(-1\Big)^j \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \cdot \chi_{\left(\frac{j-1}{2^n}, \frac{j}{2^n}\right)}(t) \\
&= r_n^2 \\
&= 1.
\end{aligned}$$

Only when $m = n$. Thus,

$$T(r_n) = \sum_{n=0}^{\infty} r_n^2 e_n = e_n.$$

Since T is a linear operator

$$T(r_n + r_0) = T(r_n) + T(r_0) = e_n.$$

If $r_0 = 1$ then we have that

$$r_n + r_0 \geq 0 \text{ for all } n \in \mathbb{N}.$$

Since T is regular, there exists U a positive operator such that

$$U \geq T.$$

So, it follows that

$$U(r_n + r_0) = U(2r_0) \geq U(r_0 + r_n) \geq T(r_0 + r_n) = e_n.$$

Thus

$$U(2r_0) \geq e_n, \text{ for all } n \in \mathbb{N}.$$

As per the assumption $U(2r_0) \in c_0$ which implies that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $\varepsilon > 0$ we will have that

$$\|U(2r_0)\|_\infty < \varepsilon.$$

For all $n \in \mathbb{N}$, this is clearly not the case since,

$$\|U(2r_0)\|_\infty \geq \|e_n\|_\infty = 1.$$

Which contradicts that U is in c_0. Therefore, T is not regular since there exists no dominant operator to T in c_0. Also, since c_0 is Dedekind complete, we conclude that T is not order complete.

Example 3.38. ([59], Example 2.1.) The space $\mathcal{L}_r(L_1([0,1]),c)$ is not a lattice. Define the operators $T, U : L_1([0,1]) \to c$ by

$$U_x = \|x\|$$

and

$$Tx = \sum_{n=1}^{k} \Big(r_n(x)\Big) e_{2n}.$$

Now, we have that

$$Tx \in c_0 \text{ as } r_n(x) \to 0.$$

And, clearly, $U \geq T, 0$ such that T is regular. Let S be a supremum in $\mathcal{L}_r(L_1([0,1]),c)$ of T and 0. Let P_k be a projection of c onto one dimensional band generated by e_k. Now, for k odd,

$$\Big(I - P_k\Big) \circ U_{\chi_{[0,1]}} \geq S_{\chi_{[0,1]}} \geq 0.$$

Therefore, the k^{th} entry in $S_{\chi_{[0,1]}}$ is zero. On the contrary, all event entries in $S_{\chi_{[0,1]}}$ are greater than or equal to 1. We thus conclude that, since $S_{\chi_{[0,1]}} \notin c$, we will have $\mathcal{L}_r(L_1([0,1]),c)$ not being a lattice.

4 Applications of operators in Banach lattices

4.1 Leontief model

In this section, we discuss the Leontief's model in Banach lattices. This is the most useful and known model in Mathematical Economics. Let E be a Riesz space and $T : E^+ \to E^+$ be a mapping such that $T(0) = 0$. We consider the equation

$$x = Tx + e, \tag{4.1}$$

where $x, e \in E^+$. This equation corresponds with the equation $x = Ax+e$, where A is an $n \times n$ consumption matrix, and e is the final demand n vector in $\mathbb{R}^n$, the linear Leontief model. For $e \in E^+$, Pe will denote the problem of determining a solution $x \in E^+$ of the Leontief equation.

Definition 4.1. Let E be a Banach lattice. The element $x \in E^+$ is called a subsolution of Pe if $x \geq Tx + e$, where T is a positive operator on E.

We write $P(e, x_0)$ if x_0 is a subsolution of Pe and we say that a Banach lattice E has the subsolution property if for every $e \in E^+, P(e, x_0)$ has a solution whenever Pe has a subsolution. We say that a Banach lattice E has a subsolution property if it has a subsolution for every positive operator $T : E \to E$.

Definition 4.2. Let X,Y be ordered vector spaces. An operator $T : X \to Y$ is called order preserving if $x \leq y$ then $Tx \leq Ty$.

We state the following *existence* theorem.

Theorem 4.3. *([49], Theorem 14.) If $T : E \to E$ is a positive linear operator and x_0 is a subsolution of Pe, $e \in E^*$, then there exists $x^{**} \in E^{**}$ such that $0 \leq x^{**} \leq x_0$ and*

$$x^{**} = T^{**}x^{**} + e.$$

Theorem 4.4. *([31], Theorem 1.) Let E be a Riesz space with the following conditions,*

1. *E is Dedekind complete.*

2. *T is order preserving.*

3. *There exists an $\tilde{x} \in E^+$ such that $\tilde{x} = T\tilde{x} + e$.*

Then there exists a solution $x_0 \in E^+$ such that $x_0 = Tx_0 + e$.

Proof. Taking the order interval $[0, \tilde{x}] = \{x : x \in E^+, 0 \leq x \leq \tilde{x}\}$, where $\tilde{x}$ is given as per the third assumption. Define an isotone map $\acute{T}$, that maps the order interval into itself, by

$$\acute{T} : x \in E^+ \to Tx + e.$$

Furthermore, define a set

$$D = \left\{x : x \in [0, \tilde{x}], x \geq \acute{T}x\right\}.$$

The third assumption assures us that D will be non-empty. Left to show that $x_0 = \inf(D)$ is a solution to Equation (4.1). By definition, we have that

$$x_0 \leq x \text{ for all } x \in D.$$

Also, with $\acute{T}$ being isotone,

$$\acute{T}x_0 \leq \acute{T}x \leq x \text{ for any } x \in D.$$

This implies that $\acute{T}x_0 \leq x_0$. Now, using $\acute{T}x_0 \leq x_0$, we will have that

$$\acute{T}(\acute{T}x_0) \leq \acute{T}x_0$$

and thus

$$\acute{T}x_0 \in D \implies x_0 = \inf(D) \leq \acute{T}x_0.$$

Therefore implying that $x_0 = \acute{T}x_0$. □

Theorem 4.5. *Let E be a Banach lattice with order continuous norm. Then E has the subsolution.*

Proof. If E has order continuous norm then it is Dedekind complete, by Theorem 2.11. Thus E has a subsolution, as shown in Theorem 4.5 (or [31] Theorem 1). □

Corollary 4.6. *Every space $L_p(\Omega, \Sigma, \mu)$, with $1 \leq p < \infty$, has the subsolution property.*

Theorem 4.7. *([49], Theorem 19.) The space $L_\infty(\Omega, \Sigma, \mu)$ has no subsolution property.*

Corollary 4.8. *([49], Corollary 2.) If the Banach lattice X is order continuous then X has the subsolution property.*

Proof. It follows, from the results of Proposition 1.a.8 [40], that all order continuous Banach lattices are order complete. Now, since

$$x_n \to \tilde{x}_w = \sum_{n \in \mathbb{N}} x_n \text{ have that } \tilde{x}_w = T\tilde{x}_w + e.$$

Analogously, if

$$\tilde{x}_w = \bigvee_{n \in \mathbb{N}} x_n = \sum_{n+1 \in \mathbb{N}} T_n e \implies \tilde{x}_w = T\tilde{x}_w + e.$$

□

Proposition 4.9. *([49], Corollary 3.) Let E be a Banach lattice. If E has one of the following properties,*

1. *E does not contain any latticed copy of c_0.*
2. *E is reflexive.*
3. *E is σ-order complete and it does not contain any lattice copy of ℓ_∞.*
4. *E is Dedekind and separable.*

Then it has a subsolution.

Remark 4.10. *The proof for Proposition 4.9 is a combination of [40] Theorem 1.a.5 and Proposition 1.a.7, respectively.*

Lemma 4.11. ([49], Lemma 10.) Let $X = (C(K), \|\cdot\|_\infty)$, with the positive cone $K = X^+$. If X has a subsolution property, then the following holds on X. If $x_1 \in X^+$ and $x_2 \in X$ such that $x_2 \leq 1$, then there exists $x_0 \in X^+$ such that $x_2 = x_1 \cdot x_2$.

Proof. Assume $x_1 \neq 0$ and define an operator T on X by

$$Tx = \left(1 - \frac{x_1 x}{\|x_1\|_\infty}\right) x \geq 0,$$

with $\left(x - \frac{x_1 x}{\|x_1\|_\infty}\right)$ and 1 is a sub-solution of Pe, since

$$1 - Tx = 1 - \left(1 - \frac{x_1}{\|x_1\|_\infty}\right) x = \frac{x_1}{\|x_1\|_\infty} \geq \frac{x_1 x_2}{\|x_1\|} \geq 0.$$

Therefore,

$$1 - Tx = \frac{x_1 x_2}{\|x_1\|} \geq 0.$$

Thus $x \geq Tx$. So x is a subsolution of Pe, with $e = \frac{x_1 x_2}{\|x_1\|}$. Therefore, there exist a $x_0 \in X^+$

such that

$$\frac{x_1 x_2}{\|x_1\|} = x_0 - Tx = \frac{x_1 x_0}{\|x_1\|}.$$

And thus,

$$x_1 \cdot x_2 = x_1 \cdot x_0.$$

□

4.2 Applications to differential equations

Example 4.12. ([45], Theorem 3.1.) Consider the periodic boundary problem. Suppose that

$$\begin{aligned} &u'(t) = f(t, u(t)),\ t \in C[0, T],\ \text{and} \\ &u(0) = u(T), \end{aligned}$$

where $T > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. It is known that any solution to this problem must be continuously differeriantble on $[0, T]$. Therefore, suitable space for this problem is $(C^1[0, T], \mathbb{R})$. This problem is equivalent to the integral problem (see. [48] and [45]).

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds \tag{4.2}$$

where $\lambda > 0$ and the Green function is given by

$$G(t, s) = \begin{cases} \frac{e^{(T+s-t)}}{e^{(T-1)}}, 0 \le s < t \le T \\ \frac{e^{(s-t)}}{e^{(T-1)}}, 0 \le t < s \le T \end{cases}$$

Define the mapping $T : (C[0, T], \mathbb{R}) \to (C[0, T], \mathbb{R})$ by

$$T(u)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds$$

Now, as stated in [7] if $u(t) \in (C[0,T],\mathbb{R})$ is a fixed point of T, then $u(t) \in (C^1[0,T],\mathbb{R})$ is a solution to the periodic boundary value problem. It is noted in [9] that under some condition, the mapping T satisfies the following conditions,

$$\begin{aligned} &(1.)\quad u(t) \leq v(t), \text{ then } T(u) \leq T(v) \\ &(2.)\quad u(t) \leq v(t), \text{ then } \|T(u) - T(v)\|_\infty \leq k\|u - v\|_\infty, \end{aligned}$$

for some constant $k \in (0,1)$, independent of u and v.

Remark 4.13. *This condition, however, is only valid for comparable functions in* $(C[0,T],\mathbb{R})$ *not to the entire space.*

In [45] the authors introduced a weaker version of the Banach Contraction Principle, for monotone non-decreasing functions.

Definition 4.14. Let $(X,\preceq)$ be be a partially ordered set and $f : X \to X$ a function, f is said to be monotone non-decreasing if $x \preceq y$ implies that $f(x) \preceq f(y)$ for any $x, y \in X$.

Theorem 4.15. *([45], Theorem 2.1.) Let* $(X,\preceq)$ *be be a partially ordered set and suppose that there is a metric* d *in* X *such that* (X,d) *is complete. Let* $f : X \to X$ *be a continuous and non-decreasing mapping such that there is* $k \in [0,1)$ *with* $d(f(x), f(y)) \leq k\cdot d(x,y)$ *for all* $x \geq y$. *If there exists* $x_0 \in X$ *with* $x_0 \leq f(x_0)$, *then* f *has a fixed point.*

Proof. If $f(x_0) = x_0$, then there is nothing to prove. Now, assume that $x_0 < f(x_0)$. Since

$x_0 \leq f(x_0)$ by assumption, and f is non-decreasing, we have that

$$\begin{aligned} x_0 &\leq f(x_0) \\ &\leq f(f(x_0)) \\ &= f^2(x_0) \\ &\leq f^2(f(x_0)) \\ &. \\ &. \\ &. \\ &\leq f^n(f(x_0)) \\ &\leq f^{n+1}(f(x_0)) \\ &. \\ &. \\ &. \end{aligned}$$

We now apply the Contraction Condition on f^n, f^{n+1} to obtain

$$d(f^{n+1}(x_0), f^n(x_0)) \leq k^n \cdot d(f(x_0), x_0), \text{ where } n \in \mathbb{N}. \tag{4.3}$$

If $n = 1$, and $f(x_0) \geq x_0$, we obtain,

$$d(f^2(x_0), f(x_0)) \leq k \cdot d(f(x_0), x_0)$$

Now, suppose that is true, for all $n \in \mathbb{N}$. Also, the fact that $f^{n+1}(x_0) \geq f^n(x_0)$ will give,

$$\begin{aligned} d(f^{n+2}(x_0), f^{n+1}(x_0)) &= d(f^{n+1}(x_0), f^n(x_0)) \\ &\leq k \cdot d(f^{n+1}(x_0), f^n(x_0)) \\ &\leq k \cdot k^n \cdot d(f(x_0), (x_0)) \\ &= k^{n+1} \cdot d(f(x_0), (x_0)). \end{aligned}$$

Therefore, $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Indeed, letting an arbirtrary $m > n$ in $\mathbb{N}$ we have that

$$\begin{aligned} d(f^m(x_0), f^n(x_0)) &\leq d(f^m(x_0), f^{m-1}(x_0)) + ... + d(f^{n+1}(x_0), f^n(x_0)) \\ &\leq (k^{m-1} + k^{m-2} + ... + k^n) d(f(x_0), (x_0)) \\ &= \frac{k^n - k^m}{1-k} d(f(x_0), (x_0)) \leq \frac{k^n}{1-k} d(f(x_0), (x_0)). \end{aligned}$$

Now, with (X, d) being a complete, we have that there exists $y \in X$ such that

$$\lim_{n\to+\infty} f^n(x_0) = y.$$

Finally, we show that y is the fixed point of f, that is $f(y) = y$. Suppose that $\epsilon > 0$, and using the continuity of f at y there exists $\delta > 0$ such that $d(z, y) < \delta$, given $\frac{\epsilon}{2} > 0$, implying that

$$d(f(y), (z)) < \frac{\epsilon}{2}$$

And since $(f^n(x_0)) \to y$, therefore given $\eta = \min\{\frac{\epsilon}{2}, \delta\} > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq n_0$, we get

$$d(f^n(x_0), (y)) < \eta$$

Therefore, for any $n \geq n_0 \in \mathbb{N}$,

$$d(f(y),(y)) \leq d(f(y), f(f^n(x_0))) + d(f^{n+1}(x_0), y) < \frac{\epsilon}{2} + \eta \leq \epsilon$$

This proves that $d(f(y),(y)) = 0$ and y is a fixed point of f. Thus, concluding the proof. □

The next theorem shows that the results still holds true even if f is not continuous, under additional assumptions.

Theorem 4.16. *([45], Theorem 2.2.) Let $(X, \preceq)$ be a partially ordered set and assume that there exists a metric d such that (X, d) is complete. In addition, assume that X satisfies that, if a non-decreasing sequence (x_n) converges to x then $x_n \leq x$ for all $n \in \mathbb{N}$. Let $f : X \to X$ be a monotone non-decreasing mapping such that there exists $k \in [0, 1)$ with*

$$d(f(x), f(y)) \leq k{\cdot}\, d(x, y) \text{ for all } x \geq y.$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Theorem 4.17. *([45], Theorem 2.3.) Let X be be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, if (X, d) is a complete metric space, for some metric d on X. If T is a continuous, monotone map from X to X such that there is $c \in (0, 1)$ $d(T(x), T(y)) \leq c \cdot d(x, y)$ for all $x \geq y$ there exists $x_0 \in X$ such that $x_0 \leq T(x_0)$ or $x_0 \geq T(x_0)$, then T has a unique fixed point $\bar{x}$. Moreover, for any $x \in X$,*

$$\lim_{n\to\infty} T^n(x) = \bar{x}.$$

Proof. This proof will be done in two distinct cases. That is, we need to show that that $d(x, y) = 0$ for any fixed point $x \in X$ other than $\bar{x}$. (1.) Suppose that x is comparable to $\bar{x}$ then,

$$\bar{x} \text{ then } T^n(x) = x \text{ is comparable to } T^n(\bar{x}) = \bar{x}$$

for all $n = 0, 1, 2, ...$, and

$$\begin{aligned} d(f(x), f(\bar{x})) = d(f^n(x), f^n(\bar{x})) &\leq d(f^n(x), f^n(\bar{x})) \\ &\leq k^n \cdot d(f(x), \bar{x}) \implies d(x, y) = 0. \end{aligned}$$

(2.) Now, suppose that x is not comparable to $\bar{x}$ then, there exists $z \in X$ comparable, as either a lower or upper bound, to x and $\bar{x}$. By monoticity, this implies that

$$f^n(z) \text{ is comparable to } f^n(x) = x \text{ and } f^n(\bar{x}) = \bar{x},$$

for all $n = 0, 1, 2, ...$ and

$$\begin{aligned} d(f(x), f(\bar{x})) &\leq d(f^n(x), f^n(z)) + d(f^n(z), f^n(\bar{x})) \\ &\leq d(f^n(x), f^n(\bar{x})) \\ &\leq k^n \cdot d(f(x), z) + k^n \cdot d(f(z), \bar{x}) \to 0 \text{ as } n \to \infty \end{aligned}$$

Therefore, implying also that $d(x, y) = 0$. □

Next, in Example 4.18 we show an operator T that is not continuous even though it consists of a fixed point.

Example 4.18. Consider the Banach lattice $(L_1[0,1], \|\cdot\|)$, and then define a positive cone $K = \{f \in X : f(x) \geq 0 \text{ almost everywhere } \}$ of X. Let $T : K \to K$ be defined by

$$Tf(x) = \begin{cases} f(x), & \text{if } \ f(x) > \frac{1}{2} \\ 0, & \text{if } \ f(x) < \frac{1}{2} \end{cases}$$

The mapping T has fixed points but it is not continuous. To show this, let $f \in K$ then, by Definition 3.2, $0 \leq Tf \leq f$. Also, taking the operator $T(T)$ on T is given as

$$T^2(f) = T(Tf) = Tf.$$

That is, Tf is a fixed point of T for any $f \in K$. For any $f \in c$, we obtain

$$\begin{aligned} d(f, Tf) &= \int_0^1 |f(x) - Tf(x)| dx \\ &= \|f\|_1 - \|Tf\|_1. \end{aligned}$$

But T is not continuous. To see this, let $f(x) = \frac{1}{2}$ and $f_n(x) = \frac{1}{2} + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\Big\| f_n - f \Big\|_1 = \int_0^1 \left| \frac{1}{2} + \frac{1}{n} - \frac{1}{2} \right| dx = \frac{x}{n}\Big|_0^1.$$

Thus, implying that $\frac{1}{n} \to 0$ as $n \to \infty$. But

$$\Big\| f_n - f \Big\|_1 = \int_0^1 \left| \frac{1}{2} + \frac{1}{n} \right| dx = \frac{x}{2} + \frac{x}{n}\Big|_0^1.$$

Implying that $\frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$ as $n \to \infty$. That is, Tf is a fixed point of T for any $f \in K$. Thus Tf is not continuous on K.

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