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# Pricing two-asset rainbow options with the fast Fourier transform

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In this paper, we present a numerical method based on the fast Fourier transform (FFT) to price call options on the minimum of two assets, otherwise known as two-asset rainbow options. We consider two stochastic processes for the underlying assets: two-factor geometric Brownian motion and three-factor stochastic volatility. We show that the FFT can achieve a certain level of convergence by carefully choosing the number of terms and truncation width in the FFT algorithm. Furthermore, the FFT converges at an exponential rate and the pricing results are closely aligned with the results obtained from a Monte Carlo simulation for complex models that incorporate stochastic volatility.

*Keywords:* Characteristic function, Fast Fourier transform, Rainbow option, Three-factor stochastic volatility, Two-factor geometric Brownian motion.

## 1. Introduction

A rainbow option refers to an option that depends on more than one underlying risky asset, where each asset is seen as a colour of the rainbow (Ouwehand and West, 2006). These options come in various forms including the "best-of-$d$" and "worst-of-$d$" call options on $d$ underlying assets. Pricing rainbow options is often challenging due to the absence of a closed-form solution; hence, numerical methods must be employed.

The market for rainbow options is illiquid and these options are typically structured on demand. However, this could change in the future. Roberts (2018) lists various applications for rainbow options in the industry. Firstly, rainbow options can be used to gain exposure to the market at a lower cost whilst reducing risk (Klyueva, 2014). Furthermore, "best-of-$d$" call options can be used to hedge currency risk if a company has the option to settle their liabilities in various foreign currencies (Guillaume, 2008). Due to their potential, numerous methods have been proposed in the literature to price rainbow options.

The first contribution made in the literature on the pricing of two-asset rainbow options can be traced back to Stulz (1982). The author derived formulas for European call and put options on the minimum or maximum of two risky assets that involve the calculation of the bivariate normal density function. Ouwehand and West (2006) derived formulas for "best-of-3" and "worst-of-3" call options using a change of numeraire technique. Eberlein et al. (2010) presented a general framework for the

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Fourier transform method of Carr and Madan (1999), which is used when certain conditions hold like the existence of the dampened characteristic function. The framework allows for the pricing of "best-of-*d*" and "worst-of-*d*" call and put options.

Fourier transformation is a mathematical method that decomposes a function into the sum of simpler trigonometric functions. When uniformly spaced samples of a continuous function are input to a Fourier transform, the transformation is called the discrete-time Fourier transform (DTFT). The input data are discrete and the output DTFT is a continuous function. If samples of equal length are taken from the DTFT output, then the transformation is called the discrete Fourier transform (DFT). Computing the DFT directly requires a total of $N^2$ operations to be performed, where $N$ is the number of input data points. Therefore, the total computation time is $O(N^2)$.

The fast Fourier transform (FFT), pioneered by Cooley and Tukey (1965), is an efficient algorithm for computing the DFT. The Cooley-Tukey FFT algorithm requires $N$ to be a power of 2, *i.e.*, $N = 2^m$, where $m \geq 0$. This requirement leads to significant time saving compared to the direct computation approach, with the total running time for Cooley-Tukey FFT algorithm being $O(N \log(N))$.

Carr and Madan (1999) developed a Fourier transform for the price of a European call option in terms of the characteristic function of the log of the stock price at the option maturity. The authors applied the FFT algorithm to compute European call option prices given the Fourier transform and concluded that the FFT yields significant improvement in terms of computation speed.

Hurd and Zhou (2010) extended the Fourier transform of Carr and Madan (1999) to the two-dimensional case of spread options and concluded that the FFT produces accurate and efficient spread option prices.

Building on the work of Eberlein et al. (2010), Roberts (2018) applied the two-dimensional FFT method of Hurd and Zhou (2010) to price "worst-of-2" call options based on the two-factor geometric Brownian motion (gBm) model. Roberts (2018) claims to be the first author to have applied the two-dimensional FFT to price two-asset rainbow options.

In order to compute option prices using the two-dimensional FFT, the double integral that appears in the Fourier transform must be approximated by truncating the domain $\mathbb{R}^2$ to a suitable lower and upper bound. The lower and upper bounds are user defined and termed the "truncation width".

Roberts (2018) shows that the FFT price for a "worst-of-2" call option can converge to the price obtained from the Stulz (1982) formula using as few as 512 terms. However, Roberts (2018) mentions that there is no single truncation width that works consistently well for all terms in the FFT and one first needs a price estimate before an appropriate value for the truncation width can be chosen. Moreover, further investigation is needed to test the accuracy of the FFT method for further in-the-money rainbow options. Roberts (2018) showed that an in-the-money "worst-of-2" call option did not converge to three decimals whereas all other strike prices did. Lastly, Roberts (2018) suggests that more complex underlying models be considered to price rainbow options other than the two-factor gBm.

In this paper, we first attempt to replicate the FFT results in Roberts (2018) to test whether we arrive at the same conclusions for "worst-of-2" call options. We then extend the work of Roberts (2018) per the author's suggestion by applying the three-factor stochastic volatility model in addition to the two-factor gBm model for the underlying assets. We then apply the two-dimensional FFT method of Hurd and Zhou (2010) to price "worst-of-2" call options based on these dynamics. Our contribution to the literature is, therefore, the pricing of two-asset rainbow options with stochastic volatility using the FFT. To our knowledge, we are the first authors to apply the FFT method of Hurd and Zhou (2010) to "worst-of-2" call options based on dynamics other than the two-factor gBm.

The field of quantitative finance draws on many concepts from mathematical statistics including moment generating functions, characteristic functions, stochastic processes, and the multivariate normal distribution, to name a few. This paper highlights the interplay between the fields of statistics and quantitative finance and that statistics plays a vital role in further development of quantitative finance.

The remainder of this paper is structured as follows. Section 2 introduces the underlying dynamics that will be considered to price two-asset rainbow options. Section 3 shows the characteristic functions for the two-asset gBm and three-factor stochastic volatility models. Section 4 introduces the Hurd and Zhou (2010) two-dimensional FFT algorithm. Section 5 shows the numerical results, and Section 6 concludes the paper.

## 2. Stock price dynamics for two-asset options

This section introduces two stochastic processes for options that depend on two underlying assets. The choice of model generally depends on the behaviour of the market. One generally aims to choose a model that describes the market dynamics of interest rates, stock prices, and volatility as closely as possible. The first, and most basic, model we consider is two-factor geometric Brownian motion (gBm).

### 2.1 Two-factor geometric Brownian motion

The two-factor gBm model was first introduced by Margrabe (1978) to price exchange options - the option to exchange one risky asset for another. The two-factor gBm model takes the following form:

$$\begin{cases} dS_1(t) &= (r-\delta_1)S_1(t)dt + \sigma_1 S_1(t)dW_{S_1}(t) \\ dS_2(t) &= (r-\delta_2)S_2(t)dt + \sigma_2 S_2(t)dW_{S_2}(t), \end{cases}$$

where $S_1(t), S_2(t)$ denote the stock prices at time $t$; $dS_1(t), dS_2(2)$ are the increments of the respective stock prices from time $t$ to time $t+dt$, with $dt$ an infinitesimal quantity; $\sigma_1$, $\sigma_2$ are the annualised volatility estimates for the two stocks; $\delta_1$, $\delta_2$ are the respective dividend yields; $r$ is the constant risk-free rate, and $dW_{S_1}(t)dW_{S_2}(t) = \rho_{S_1,S_2}dt$, with $\rho_{S_1,S_2}$ the correlation coefficient between the two stock prices.

The two-factor gBm model assumes that the two stock prices each follow a log-normal distribution with constant volatility. Clearly this assumption is very limiting since the model assumes symmetric returns for the log of the stock price based on a normal distribution. Empirical evidence has shown that returns for the log of the stock price tend to be negatively skewed (see Cont, 2001); hence, a model that can account for asymmetry is preferred to describe the behaviour of the equity market. Furthermore, the two-factor gBm model is unable to produce the fat-tailed returns often observed in equity markets (see Cont, 2001).

Next, we introduce the three-factor stochastic volatility model that addresses some of the short-comings in the two-factor gBm model.

### 2.2 Three-factor stochastic volatility

The three-factor stochastic volatility model was first introduced by Dempster and Hong (2002) to price spread options. The model assumes that both assets are driven by the same Cox et al. (1985) variance process and takes the following form:

$$\begin{cases} dS_1(t) &= (r-\delta_1)S_1(t)dt + \sigma_1\sqrt{v(t)}S_1(t)dW_{S_1}(t) \\ dS_2(t) &= (r-\delta_2)S_2(t)dt + \sigma_2\sqrt{v(t)}S_2(t)dW_{S_2}(t) \\ dv(t) &= \kappa\Big(\bar{v} - v(t)\Big)dt + \sigma_v\sqrt{v(t)}dW_v(t), \end{cases}$$

where $v(t)$ denotes the variance of the stock prices at time $t$; $dv(t)$ is the increment of the variance from time $t$ to time $t+dt$; $\kappa$ is the mean reversion speed of the variance; $\bar{v}$ is the long-run mean of the variance, and $\sigma_v$ is the volatility of the variance. The Brownian motions are correlated as follows:

$$\begin{aligned} dW_{S_1}(t)dW_{S_2}(t) &= \rho_{S_1,S_2}dt, \\ dW_{S_1}(t)dW_v(t) &= \rho_{S_1,v}dt, \\ dW_{S_2}(t)dW_v(t) &= \rho_{S_2,v}dt. \end{aligned}$$

The three-factor stochastic volatility model has the ability to produce asymmetric and fat-tailed returns and is, therefore, more flexible than the two-factor gBm model.

In the next section, we introduce the work of Eberlein et al. (2010) along with the characteristic functions for the two-factor gBm and three-factor stochastic volatility models. The next section forms an integral part of the two-dimensional FFT method of Hurd and Zhou (2010), which will be discussed in Section 4.

## 3. Rainbow options, Fourier transform and characteristic functions

Fourier transform methods have led to significant computational gains since their introduction to option pricing by Carr and Madan (1999). The Fourier transform framework was further generalised to multi-asset options by Eberlein et al. (2010) where the authors listed conditions under which Fourier transform formulas are valid. In this section, we first introduce the payoff functions for "best-of-2" and "worst-of-2" call options. Next, we discuss the general Fourier transform framework of Eberlein et al. (2010). Lastly, we show the characteristic functions for the two-factor gBm and three-factor stochastic volatility models.

### 3.1 Best-of-2 and worst-of-2 call options

Let $V_{max}\big(S_1(t), S_2(t)\big)$ and $V_{min}\big(S_1(t), S_2(t)\big)$ denote the values for a "best-of-2" and "worst-of-2" call option depending on two assets $\big\{S_1(t),\, S_2(t)\big\}$ at time $t$ with strike price $K$ and maturity $T$. At maturity $T$, the payoff formula for the call options is given by

$$V_{max}\big(S_1(T), S_2(T)\big) = \max\Big(\max\Big(S_1(T), S_2(T)\Big) - K, 0\Big), \tag{1}$$

$$V_{min}\big(S_1(T), S_2(T)\big) = \max\Big(\min\Big(S_1(T), S_2(T)\Big) - K, 0\Big). \tag{2}$$

From Eberlein et al. (2010), the Fourier transform corresponding to the payoff functions in equations (1) and (2) can be derived and will be discussed in the next subsection.

### 3.2 The result of Eberlein, Glau, and Papapantoleon

Eberlein et al. (2010) presented a general framework for which Fourier transform formulas are valid. For the two-dimensional case, the authors consider any payoff function $f : \mathbb{R}^2 \to \mathbb{R}_+$, for example equation (1) or (2), and the dampened payoff function

$$g(\boldsymbol{x}) := e^{-\boldsymbol{\alpha}^\top \boldsymbol{x}} f(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2,$$

where $\boldsymbol{\alpha} \in \mathbb{R}^2$ is the dampening coefficient.

Let $\hat{g}$ denote the Fourier transform of the function $g$; $L^1_{bc}(\mathbb{R}^2)$ be the space of bounded, continuous functions in $L^1(\mathbb{R}^2)$, where $L^1(\mathbb{R}^2)$ is the space of all integrable functions on $\mathbb{R}^2$. Moreover, let $\boldsymbol{X}(0) = \big(x_1(0), x_2(0)\big)^\top = \big(\log S_1(0), \log S_2(0)\big)^\top$ and let $M_{\boldsymbol{X}_T}$ denote the moment generating function for the random variable $\boldsymbol{X}(T) = \big(x_1(T), x_2(T)\big)^\top = \big(\log S_1(T), \log S_2(T)\big)^\top$ at the option maturity $T$.

Eberlein et al. (2010) make the following assumptions:

1. Assume that $g \in L^1_{bc}(\mathbb{R}^2)$ and $\hat{g} \in L^1(\mathbb{R}^2)$;
2. Assume that $M_{\boldsymbol{X}_T}(\boldsymbol{\alpha})$ exists;

Under these assumptions, the authors present the following Fourier transform formula for two-asset options:

**Theorem 1.** *If the asset price processes are modelled as two-factor gBm or three-factor stochastic volatility processes and assumptions 1 and 2 hold, then the payoff function for a two-asset option V at* $t = 0$ *can be written as*

$$V\big(\boldsymbol{X}(0)\big) = \frac{e^{\boldsymbol{\alpha}^\top \boldsymbol{X}(0)}}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\boldsymbol{u}^\top \boldsymbol{X}(0)} M_{\boldsymbol{X}_T}(\boldsymbol{\alpha} + i\boldsymbol{u}) \hat{f}(i\boldsymbol{\alpha} - \boldsymbol{u}) d\boldsymbol{u},$$

*where* $i = \sqrt{-1}$, $\boldsymbol{u} = [u_1, u_2] \in \mathbb{R}^2$, *and* $\hat{f}(\cdot)$ *denotes the Fourier transform of the payoff function.*

See Eberlein et al. (2010) for the proof.

Using Theorem 1, Eberlein et al. (2010) show that the Fourier valuation formula for a "worst-of-2" call option at $t = 0$ with strike $K$ and maturity $T$ is given by:

$$V_{min}\big(x_1(0), x_2(0)\big) = \frac{e^{-rT}}{4\pi^2} \int_{\mathbb{R}^2} e^{x_1(0)(\alpha_1 + iu_1)} e^{x_2(0)(\alpha_2 + iu_2)} M_{\boldsymbol{X}_T}(\alpha_1 + iu_1, \alpha_2 + iu_2)$$

$$\times \frac{K^{1-\alpha_1-\alpha_2-iu_1-iu_2}}{(\alpha_1 + iu_1)(\alpha_2 + iu_2)(\alpha_1 + \alpha_2 - 1 + iu_1 + iu_2)} du. \tag{3}$$

Assuming the moment generating function exists, the relationship between the moment generating function $M_{\boldsymbol{X}_T}$ and the characteristic function $\phi_{\boldsymbol{X}_T}$ is given by

$$M_{\boldsymbol{X}_T}(\boldsymbol{u}) = \phi_{\boldsymbol{X}_T}(-i\boldsymbol{u}),$$

hence, the moment generating function in equation (3) can be replaced by $\phi_{\boldsymbol{X}_T}$.

In the next section, we introduce the characteristic functions for the two-factor gBm and three-factor stochastic volatility models.

### 3.3 Two-factor gBm characteristic function

The characteristic function represents the joint distribution of $\boldsymbol{X}(T)$ at the option maturity $T$. From Dempster and Hong (2002), the characteristic function for the two-asset gBm model is given by

$$\phi_{gBm}(u_1,u_2) = \exp\Big(iu_1x_1(0) + iu_2x_2(0) + \zeta T + \sum_{j=1,2} u_j(r-\delta_j)T\Big),$$

where

$$\zeta := -\frac{1}{2}\Big[(\sigma_1^2u_1^2 + \sigma_2^2u_2^2 + 2\rho_{x_1,x_2}\sigma_1\sigma_2u_1u_2) + i(\sigma_1^2u_1 + \sigma_2^2u_2)\Big].$$

The characteristic function can be used as input to equation (3) to price "worst-of-2" call options under the two-factor gBm model.

As mentioned, the two-factor gBm model is not consistent with the behaviour of the equity market. Therefore, we consider the three-factor stochastic volatility model next.

### 3.4 Three-factor stochastic volatility characteristic function

Dempster and Hong (2002) derived an expression for the characteristic function of the three-factor stochastic volatility model where both assets are driven by a single Cox et al. (1985) variance process. The expression for the characteristic function is given by

$$\begin{aligned}\phi_{SV}(u_1,u_2) = \exp\Bigg(&iu_1x_1(0) + iu_2x_2(0) + \left(\frac{2\zeta(1-e^{-\beta T})}{2\beta-(\beta-\gamma)(1-e^{-\beta T})}\right)v(0)\\ &+ \sum_{j=1,2} u_j(r-\delta_j)T - \frac{\kappa\bar{v}}{\sigma_v^2}\Gamma\Bigg),\end{aligned}$$

where

$$\begin{aligned}\Gamma &:= \left[2\log\left(\frac{2\beta-(\beta-\gamma)(1-e^{-\beta T})}{2\beta}\right) + (\beta-\gamma)T\right],\\ \zeta &:= -\frac{1}{2}\Big[(\sigma_1^2u_1^2 + \sigma_2^2u_2^2 + 2\rho_{x_1,x_2}\sigma_1\sigma_2u_1u_2) + i(\sigma_1^2u_1 + \sigma_2^2u_2)\Big],\\ \gamma &:= \kappa - i(\rho_{x_1,v}\sigma_1u_1 + \rho_{x_2,v}\sigma_2u_2)\sigma_v,\\ \beta &:= \sqrt{\gamma^2 - 2\sigma_v^2\zeta}.\end{aligned}$$

In the next section, we introduce the two-dimensional FFT method of Hurd and Zhou (2010).

## 4. The two-fimensional FFT method

Hurd and Zhou (2010) initially developed the two-dimensional FFT method to price spread options. Using equation (3) and applying the same logic to "worst-of-2" call options with $K = 1$ and maturity $T$, the Fourier representation of the "worst-of-2" call option payoff function $V_{min}(x_1(T), x_2(T)) =$

$\max\big(\min(e^{x_1(T)}, e^{x_2(T)}) - 1, 0\big)$ is given by

$$V_{min}\big(x_1(0), x_2(0)\big) = \frac{e^{-rT}}{4\pi^2} \int_{\mathbb{R}^2+i\epsilon} e^{x_1(0)(\alpha_1+iu_1)} e^{x_2(0)(\alpha_2+iu_2)} \phi_{\{gBm,SV\}}(u_1 - i\alpha_1, u_2 - i\alpha_2)$$

$$\times \frac{1^{1-\alpha_1-\alpha_2-iu_1-iu_2}}{(\alpha_1+iu_1)(\alpha_2+iu_2)(\alpha_1+\alpha_2-1+iu_1+iu_2)} d\boldsymbol{u}, \tag{4}$$

where $\alpha_1, \alpha_2 > 0$, $\phi_{\{gBm,SV\}}$ is the characteristic function under either the two-factor gBm or three-factor stochastic volatility model, and $\epsilon_1, \epsilon_2 < 0$ with $\epsilon_1 + \epsilon_2 < -1$. The parameters $\epsilon_1, \epsilon_2$ can be chosen freely within their given constraints. Roberts (2018) showed that "worst-of-2" call option prices are insensitive to the choice of $\epsilon_1$ and $\epsilon_2$. Furthermore, let

$$\hat{V}_{min}(u_1, u_2) := \frac{1^{1-\alpha_1-\alpha_2-iu_1-iu_2}}{(\alpha_1+iu_1)(\alpha_2+iu_2)(\alpha_1+\alpha_2-1+iu_1+iu_2)}.$$

The integral in equation (4) can be approximated as follows. Let

$$\Gamma = \{\boldsymbol{u}(\boldsymbol{k}) = \big(u_1(k_1), u_2(k_2)\big) \mid \boldsymbol{k} = (k_1, k_2) \in \{0, 1, ..., N-1\}^2\},\ u_i(k_i) = -\bar{u} + k_i\xi,$$

where $N = 2^m$ with $m \geq 0$, $\xi$ is the lattice spacing, and $\bar{u} = \frac{N\xi}{2}$.

Furthermore, let the reciprocal lattice be given by

$$\Gamma^* = \{\boldsymbol{x}(\boldsymbol{l}) = (x_1(l_1), x_2(l_2)) \mid \boldsymbol{l} = (l_1, l_2) \in \{0, 1, ..., N-1\}^2\},\ x_i(l_i) = -\bar{x} + l_i\xi^*,$$

where $\xi^* = \frac{\pi}{\bar{u}}$ is the reciprocal lattice spacing and $\bar{x} = \frac{N\xi^*}{2}$.

The integral in equation (4) is approximated by the following double sum for each pair $\big(x_1(l_1), x_2(l_2)\big)$ in $\Gamma^*$:

$$V_{min}\big(x_1(l_1), x_2(l_2)\big) \approx \frac{e^{-rT}}{4\pi^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{x_1(l_1)(\alpha_1+i(u_1(k_1)+i\epsilon_1))} e^{x_2(l_2)(\alpha_2+i(u_1(k_1)+i\epsilon_1))}$$

$$\times \phi_{\{gBm,SV\}}\big(u_1(k_1)+i\epsilon_1-i\alpha_1, u_2(k_2)+i\epsilon_2-i\alpha_2\big)\hat{V}_{min}\big(u_1(k_1)+i\epsilon_1, u_2(k_2)+i\epsilon_2\big)$$

$$= (-1)^{l_1+l_2} e^{-rT} \left(\frac{\xi N}{2\pi}\right)^2 e^{(\alpha_1-\epsilon_1)x_1(l_1)+(\alpha_2-\epsilon_2)x_1(l_2)} [\text{ifft2}(H)](l_1, l_2),$$

where

$$H(k_1, k_2) = (-1)^{k_1+k_2} \phi_{\{gBm,SV\}}\big(u_1(k_1)+i\epsilon_1-i\alpha_1, u_2(k_2)+i\epsilon_2-i\alpha_2\big)$$
$$\times \hat{V}_{min}\big(u_1(k_1)+i\epsilon_1, u_2(k_2)+i\epsilon_2\big),$$

and $\text{ifft2}(H)$ represents the two-dimensional FFT applied to matrix $H$.

Recall that equation (4) is for the specific case where $K = 1$. Roberts (2018) explains that equation (4) can be generalised for any $K > 0$ by scaling the initial share prices

$$\boldsymbol{x}(0) = \left(\log\left(\frac{S_1(0)}{K}\right), \log\left(\frac{S_2(0)}{K}\right)\right).$$

To determine the option price, a grid search is done to find $\boldsymbol{x}(0)$ in $\Gamma^*$. If $\boldsymbol{x}(0)$ does not perfectly fall on the lattice $\Gamma^*$, an interpolation scheme must be used to find $\boldsymbol{x}(0)$.

The implementation of the two-dimensional FFT method is outlined in Alfeus and Schlögl (2018). Algorithm 1 below follows directly from their paper.

**Algorithm 1** Two-dimensional FFT

1. **Input :** $N$, a power of two; $\bar{u}$, truncation width; $\epsilon$, damping factor.
2. Set $\boldsymbol{x}(0) = \left(\log\left(\frac{S_1(0)}{K}\right), \log\left(\frac{S_2(0)}{K}\right)\right) \in \left(x_1(l_1), x_2(l_2)\right)$.
3. **for all** $\boldsymbol{k}, \boldsymbol{l} \in \{1, 2, ..., N-1\}^2$ **do**

$$H(k_1, k_2) = (-1)^{k_1+k_2}\phi_{\{gBm,SV\}}(u_1(k_1) + i\epsilon_1 - \alpha_1, u_2(k_2) + i\epsilon_2 - \alpha_2) \times \hat{V}_{min}(u_1(k_1) + i\epsilon_1, u_2(k_1) + i\epsilon_1);$$

$$C(l_1, l_2) = (-1)^{l_1+l_2}\left(\frac{\xi N}{2\pi}\right)^2 e^{(\alpha_1-\epsilon_1)x_1(l_1)+(\alpha_2-\epsilon_2)x_2(l_2)};$$

4. **end**
5. $V_{min}\left(x_1(l_1), x_2(l_2)\right) = \Re(C \times \text{ifft2}(H))$ where $\Re(\cdot)$ denotes the real part of the complex number.
6. $P \leftarrow K \times V_{min}\left(\boldsymbol{x}(0)\right)$ using an interpolation scheme to find $\boldsymbol{x}(0)$ in $\Gamma^*$.

**Output :** $P$.

In the next section, we present our numerical results for "worst-of-2" call options based on the FFT method of Hurd and Zhou (2010) and various dynamics for the underlying assets.

## 5. Numerical results

In this section, we first compare our pricing results of the FFT and two-factor gBm model with the results in Roberts (2018). We also test various values for the truncation width in the FFT algorithm to see whether an optimal value exists. Thereafter, we test the accuracy of the FFT under the three-factor stochastic volatility by comparing the FFT prices to the prices obtained from a Monte Carlo simulation.

The code was implemented in *Python* on an HP laptop *Intel(R) Core(TM) i5 – 1.60GHz with 16 GB memory*.

### 5.1 FFT and two-factor gBm

Table 1 shows our results for "worst-of-2" call options using the FFT and two-factor gBm model. The prices are compared to the prices obtained from the Stulz (1982) formula, as shown in Roberts (2018). Furthermore, we use the exact same values for $N$ and $\bar{u}$ as Roberts (2018) to compare the accuracy of our implementation.

Table 2 shows the absolute difference between our FFT prices and the Stulz (1982) prices across strike.

Our results do not support the findings in Roberts (2018). Firstly, our prices are noticeably

**Table 1**. FFT prices under two-factor gBm with $S_1(0) = 100$, $S_2(0) = 96$, $\delta_1 = 0.05$, $\delta_2 = 0.05$, $r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\rho_{x_1,x_2} = 0.5$, $\epsilon_1 = -3$, $\epsilon_2 = -1$, $\alpha_1 = 0.75$, $\alpha_2 = 0.75$, $T = 1$.

| $K$ | Stulz | $N = 64$ $\bar{u} = 30$ | $N = 128$ $\bar{u} = 50$ | $N = 256$ $\bar{u} = 90$ | $N = 512$ $\bar{u} = 120$ | $N = 1024$ $\bar{u} = 130$ | $N = 2048$ $\bar{u} = 130$ |
|---|---|---|---|---|---|---|---|
| 90 | 8.274176 | 8.276002 | 8.273253 | 8.274178 | 8.274173 | 8.274158 | 8.274158 |
| 92 | 7.118883 | 7.122826 | 7.177980 | 7.118794 | 7.118871 | 7.118862 | 7.118862 |
| 94 | 6.055238 | 6.041106 | 6.055158 | 6.055197 | 6.055214 | 6.055220 | 6.055220 |
| 96 | 5.087925 | 5.072448 | 5.087190 | 5.087892 | 5.087903 | 5.087914 | 5.087914 |
| 98 | 4.220092 | 4.210246 | 4.219182 | 4.220043 | 4.220087 | 4.220090 | 4.220090 |
| 100 | 3.452949 | 3.451947 | 3.452913 | 3.452951 | 3.452948 | 3.452948 | 3.452948 |
| 102 | 2.785485 | 2.791147 | 2.785838 | 2.785476 | 2.785481 | 2.785482 | 2.785482 |
| 104 | 2.214392 | 2.222128 | 2.214398 | 2.214413 | 2.214401 | 2.214396 | 2.214396 |

**Table 2**. Absolute difference between FFT and Stulz prices.

| $K$ | $N = 64$ $\bar{u} = 30$ | $N = 128$ $\bar{u} = 50$ | $N = 256$ $\bar{u} = 90$ | $N = 512$ $\bar{u} = 120$ | $N = 1024$ $\bar{u} = 130$ | $N = 2048$ $\bar{u} = 130$ |
|---|---|---|---|---|---|---|
| 90 | 0.001826 | 0.000923 | 0.000002 | 0.000003 | 0.000018 | 0.000018 |
| 92 | 0.006057 | 0.000903 | 0.000089 | 0.000012 | 0.000021 | 0.000021 |
| 94 | 0.014132 | 0.000080 | 0.000041 | 0.000024 | 0.000018 | 0.000018 |
| 96 | 0.015477 | 0.000735 | 0.000033 | 0.000022 | 0.000011 | 0.000011 |
| 98 | 0.009846 | 0.000910 | 0.000049 | 0.000005 | 0.000002 | 0.000002 |
| 100 | 0.001002 | 0.000036 | 0.000002 | 0.000001 | 0.000001 | 0.000001 |
| 102 | 0.005662 | 0.000353 | 0.000009 | 0.000004 | 0.000003 | 0.000003 |
| 104 | 0.007736 | 0.000006 | 0.000021 | 0.000009 | 0.000004 | 0.000004 |

more accurate than the prices in Roberts (2018). Based on the author's implementation, the FFT did not converge to three decimal places for $K = 90$. However, we find that the FFT converges for $K = 90$ up to five decimals using 256 or 512 terms. Roberts (2018) recommends that further investigation be done to conclude the accuracy of the FFT applied to further in-the-money options. Based on our implementation, there is nothing that suggests the FFT method is less accurate for further in-the-money options.

Another interesting observation is that our FFT results converge faster than the results in Roberts (2018). The author mentioned that all FFT prices apart from $K = 90$ converged to three decimal places from 512 terms. Based on our implementation, we find that the FFT converges to at least three decimals using 256 terms for all strikes. From 512 terms, the FFT converges to at least four decimals for all strikes.

Lastly, Roberts (2018) stated that it is preferable to use the Fourier-cosine series expansion (COS) method of Ruijter and Oosterlee (2012) to price rainbow options since it is faster and more robust than the FFT method. In terms of accuracy, our FFT implementation is very much aligned with the COS pricing results in Roberts (2018) (and in many cases even more accurate). This leads us to believe that

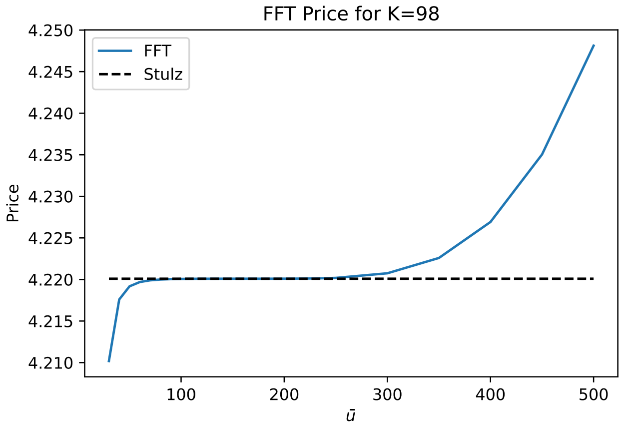


**Figure 1**. Two-factor gBm FFT Price as a function of truncation width $\bar{u}$ with $N = 512$.

there is an error in the FFT implementation in Roberts (2018). Upon further investigation, Roberts (2018) mentioned that the dampening factors in the FFT method should be restricted to $\alpha_2 < 0$ and $\alpha_1 + \alpha_2 > 1$, where, in fact, the dampening factors should be restricted to $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 > 1$ as per Eberlein et al. (2010). The question remains whether it is truly better to use the COS method rather than the FFT method for "worst-of-2" call options. We will attempt to clarify this.

### 5.2 Testing the truncation width

The FFT algorithm of Hurd and Zhou (2010) requires an appropriate choice for the truncation width $\bar{u}$. Figure 1 shows the FFT price for a "worst-of-2" call option with $K = 98$ and $T = 1$ as a function of $\bar{u}$.

From Figure 1, it is clear that the choice of $\bar{u}$ can have a significant impact on the FFT price. If $\bar{u}$ is chosen too small, the FFT price will be understated. Alternatively, choosing a value that is too large will overestimate the price.

Based on our implementation, the FFT converges to at least four decimal places for $\bar{u} \in [90, 200]$ for $K = 98$ and $N = 512$ compared to the Stulz (1982) price. Roberts (2018) mentioned that it is not possible to choose an optimal value for $\bar{u}$. However, we have shown that a value of $N = 512$ and $\bar{u} \in [90, 200]$ will be sufficient to achieve convergence up to four decimals under the two-factor gBm model. In Table 1, for $N = 512$, we used a value of $\bar{u} = 120$, which falls in the interval $[90, 200]$. All FFT prices converged to at least four decimal places, which further supports our finding.

An area where the COS method does seem to outperform the FFT is the rate of convergence. Based on the results in Roberts (2018), the COS method converged to three decimal places using as few as 64 terms. Based on our implementation of the FFT, convergence to three decimals was only achieved from 256 terms. Although the COS method does seem to be faster, there is not much that differentiates the pricing results between the COS and FFT methods. Therefore, if speed is an important factor, the COS method is preferable to the FFT method which aligns with the results in Roberts (2018). However, it takes only 3 seconds to price a single option when using the FFT method with 256 terms.

**Table 3**. MC and FFT prices under three-factor stochastic volatility with $S_1(0) = 100,\ S_2(0) = 96,\ \delta_1 = 0.05,\ \delta_2 = 0.05,\ r = 0.1,\ \sigma_1 = 1,\ \sigma_2 = 0.5,\ v(0) = 0.04,\ \kappa = 1,\ \bar{v} = 0.04,\ \sigma_v = 0.05,\ \rho_{x_1,x_2} = 0.5,\ \rho_{x_1,v} = -0.5,\ \rho_{x_2,v} = 0.25,\ \epsilon_1 = -3,\ \epsilon_2 = -1,\ \alpha_1 = 0.75,\ \alpha_2 = 0.75,\ T = 1,\ N = 512,\ \bar{u} = 100$.

| $K$ | MC Price | FFT Price | Absolute Difference |
|---|---|---|---|
| 90 | 7.642450 | 7.642304 | 0.000145 |
| 92 | 6.436579 | 6.436327 | 0.000252 |
| 94 | 5.340944 | 5.340803 | 0.000141 |
| 96 | 4.363270 | 4.363219 | 0.000052 |
| 98 | 3.507569 | 3.507650 | 0.000081 |
| 100 | 2.773671 | 2.773815 | 0.000141 |
| 102 | 2.157198 | 2.157236 | 0.000038 |
| 104 | 1.650093 | 1.650149 | 0.000056 |

In summary, it is possible to choose an optimal value for $\bar{u}$ to achieve a certain level of convergence to the Stulz (1982) price. Next, we implement the three-factor stochastic volatility model of Dempster and Hong (2002).

### 5.3 FFT and three-factor stochastic volatility

In this subsection, we compare the FFT prices for "worst-of-2" call options under the three-factor stochastic volatility model of Dempster and Hong (2002) with a Monte Carlo simulation and show the convergence speed of the FFT by varying the number of terms $N$.

To test the accuracy of the FFT, Table 3 compares the FFT pricing results for "worst-of-2" call options based on the three-factor stochastic volatility model with a Monte Carlo simulation of 20,000,000 samples for various $K$, where the Monte Carlo simulation is seen as the benchmark price. We show the results up to 6 decimal places:

The results in Table 3 indicate that the FFT and Monte Carlo prices are aligned to at least three decimal places for all strikes considered which confirms the accuracy of the FFT.

To illustrate the FFT convergence, Figure 2 plots the logarithm of the absolute difference for a "worst-of-2" call option with $K = 98$ and $T = 1$ as a function of $N$, where $N = 8196$ is the benchmark price. The results are based on the same parameters shown in Table 3.

Figure 2 shows that the FFT converges at an exponential rate. Monte Carlo simulation, on the other hand, is known to converge at a rate of $O\left(\sqrt{N}\right)$ – to achieve a tenfold increase in accuracy, a hundredfold increase in the number of simulations is required (see Glasserman, 2003). FFT is far superior to Monte Carlo simulation in terms of efficiency.

## 6. Conclusion

In this paper, we implemented the two-factor gBm model and three-factor stochastic volatility model of Dempster and Hong (2002) to price "worst-of-2" call options. Using the two-dimensional FFT method of Hurd and Zhou (2010), we showed that it is possible to achieve a certain level of convergence to the Stulz (1982) and Monte Carlo prices under the two-factor gBm and three-factor

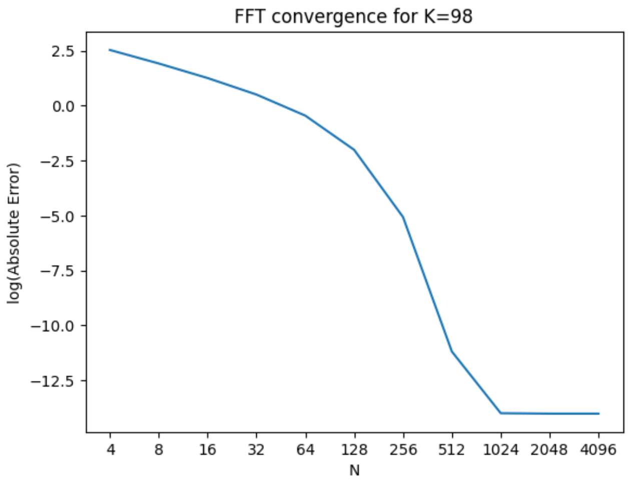


**Figure 2**. FFT convergence under the three-factor stochastic volatility model.

stochastic volatility models respectively. We also showed that an optimal value for the truncation width can be chosen which contradicts a previous finding in the literature.

The FFT gives practitioners a powerful way of calculating rainbow option prices without the need to perform a costly Monte Carlo simulation. As rainbow option trading becomes more popular, the FFT method is likely to shine.

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