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ORIGINAL PAPER

Recent developments in using digital technology in mathematics education

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Abstract
In this paper we review selected significant developments in the use of digital technology in the teaching and learning of mathematics over the last five years. We focus on a number of important topics in this field, including the evolvement of STEAM and critical making as well as the process of redefining learning spaces in the transformation of the mathematics classroom. We also address the increasing use of computer algebra systems and dynamic geometry packages; and the issue of student collaboration online, especially using learning environments and social media. We briefly touch on artificial intelligence systems, including hyper-personalisation of learning, multimodality and videos. We include a brief discussion on the impact of COVID-19 on mathematics education, and lastly on the more theoretical perspective of the epistemology of digital technology and the construct of humans-with-media. We conclude the discussion with some possible concerns and mentioning some possible new topics for research in the field.

Keywords Online learning and teaching · Blended learning · STEAM · Computer algebra system · Dynamic geometry · Moving classroom · Student collaboration · Social media · Artificial intelligence · Hyper-personalisation · Humans-with-media

1 Introduction

In previous papers, the authors reported on the status of blended and online learning in mathematics (Borba et al., 2016; Engelbrecht et al., 2020). Around the turn of the twenty-first century, the integration of technologies into education was rather slow. Currently, however, technologies are being implemented more rapidly—especially in developed countries (Lavicza et al., 2022). However, there is also a dedicated and growing interest of researchers in developing countries in the use of technology in mathematics and mathematics education. Because of substantial investments by both government and industry, combined with the widespread use of digital mobile technology and educational application development, fewer barriers to accessibility exist than before (Lavicza et al., 2022). With the recent COVID-19 pandemic, many schools and universities were forced to move to online courses and use a variety of software packages in their teaching. Before the COVID-19 pandemic, digital technology use in mathematics classrooms was reported to be inconsistent in quality and effectiveness and there were many questions regarding how and when it should be used, and whether its use transformed and improved student experiences of mathematics education (Attard & Holmes, 2022).

This transition happened alongside pedagogical shifts in both schools and universities to encourage more active student learning, foster greater engagement, and provide more flexible access to learning (Engelbrecht & Oates, 2022). Educational innovation has been suddenly moved from the margins to the core of the education system, providing opportunities to identify new strategies that can develop young people and prepare them for the changing times (Vegas & Winthrop, 2020). Countries, such as Austria, Denmark and Finland, are changing their assessment practices to allow the utilisation of advanced technologies in country wide assessment (Weinhandl & Lavicza, 2019).

In their study on the transformation of the mathematics classroom, Engelbrecht et al. (2020) highlighted different

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ways in which the use of digital technologies generates new ways of thinking, and about the settings in which mathematics is learnt. They suggested how mathematics teacher educators might frame new approaches to initial training and professional development, with one such approach may be through the vehicle of social media. Goos et al. (2020) conceptualised blended learning in terms of boundary crossing between face-to-face and computer-mediated modes of teaching and learning.

With the gradual implementation of technology into mathematics education, an ongoing change in the classroom configuration has been developing—from the traditional cubic model, to one with a different topology (Engelbrecht et al., 2020). Borba (2021) pointed out that the pandemic has created a new agenda for mathematics education, as new actors have been brought into the education scene. He also argues that the quality of internet access at home has become paramount for education.

Before the pandemic, teachers were either using technology to a limited or more sophisticated extent. Those with a low *pedagogical technology knowledge* (Thomas & Hong, 2013) used technology to a lower extent and focussed on operations, procedural and technical aspects of tech use. Others, with a higher pedagogical content knowledge, used technology with a more multi-representational approach (Brown, 2017).

Vegas and Winthrop (2020) suggested the idea of local learning ecosystems, in which new actors in the community—community sections that traditionally were not actively involved in education now become involved to support children's learning. These informal settings could be where school districts have engaged to offer general learning opportunities to families and children. Using the experience made during COVID-19 will hopefully harness new energies and relations between schools and communities to work together to support children's learning. Schools are establishing new relationships with social welfare organisations, media companies worked with education leaders, technology companies partner with governments, and local non-profit organisations and businesses contributed to supporting children's learning in innovative ways (Vegas & Winthrop, 2020).

2 Rationale of the review: methodological approach

In this paper we will analyse some of the important recent contributions to the use of digital technology in mathematics education. The approach that was used is a narrative review (Collins & Fauser, 2005) but some structure was used.

Consulting recent articles in a number of highly cited journals in the field (e.g. *ZDM Mathematics Education* and *Educational Studies in Mathematics*) as well as the proceedings of international conferences on mathematics education (e.g. ICME and PME), the authors developed a short list of topics in the field that are currently particularly relevant. Using these criteria and our own and other colleagues' experience in the field, we decided to focus on

- the evolvement of STEAM and critical making;
- the process of redefining learning spaces in the transformation of the mathematics classroom;
- the increasing use of computer algebra systems and dynamic geometry packages;
- the issue of student collaboration online, especially using learning environments and social media;
- the use of artificial intelligence systems, including hyper-personalisation of learning;
- multimodality and videos, including augmented reality and immersive virtual reality
- the impact of COVID-19 on mathematics education;
- the epistemology of digital technology and the construct of humans-with-media.

In each of these topics, we then (using the *Harzing's*[1] *Publish or Perish* database) considered the frequency of citings and our own experience to decide on which papers to include into the study. We do not claim a total absence of subjectivity in our narrative review—we strongly relied on our and our colleagues' experience as editors of journals, organisers of conferences and keynote addresses to choose these themes as well as the papers that we included in the review. In any quantitative or qualitative research there is always an element of subjectivity.

We distinguished between three categories of references. Firstly there are papers published within the period of review that are of special interest and form the fundamental basis of the review. In the list of references these papers have been highlighted **. Secondly, papers that have been highlighted * in the list of references also form a (smaller) part of the basis of the review. Finally, papers not highlighted in the list of references consist of additional literature in order to clarify the theoretical basis or the method part.

In any literature review, as with research in general, there are both subjective and objective elements. Even using bibliometric indices, there is a subjective element in believing that numbers translate objectivity, not even mentioning the social economic aspects in building such a database.

[1] *Harzing's Publish or Perish* is a software programme that retrieves and analyses academic citations. It analyses raw citations from a variety of databases (e.g. *Crossref, Google Scholar, Scopus, WoS*, etc.) and presents a range of citation metrics, including the number of papers, total citations and the *h*-indices.

Most important is to reveal exactly what criteria were used. With the selection mentioned above, we had a fair amount of objectivity as we analysed top journals and conferences, but not everyone will agree with our selection. So, we do not claim that the topics and papers are necessarily fully representative of the field.

3 STEAM and critical making

Over recent decades, the importance of proficiency, including prerequisite knowledge, skills, attitudes, values and experiences, in STEM subjects (science, technology, engineering and mathematics) has been heralded by governments because of the link to a country's technological innovation and economic development (Namukasa et al., 2021). However, the current STEM movements are criticised for the neglect of environment and society and for continuing to focus more on science education, including its pedagogies (e.g., problem-based learning) than on the other disciplines such as health and design (Namukasa et al., 2021). *STEAM* was proposed as a way to enhance participation and interest in STEM-related fields by focusing on students' creative problem solving skills and artistic, creative and design skills (Namukasa et al., 2021). The 'A' in STEAM could stand for aesthetics, creativity or the incorporation of the arts, as integral part of the approach.

The adoption of ideas from informal maker spaces and communities is sometimes referred to as *maker education*, and it is associated with initiatives that emphasise the integration of engineering design and technology in teaching (Namukasa et al., 2021). So the 'T' in STEAM then relates to engineering, programming or computer graphics. Using maker education, students and teachers learn processes and approaches to solving problems and thinking about ideas using simulations and experiments (Namukasa et al., 2021).

Critical making in mathematics education can be seen as analysing connections and complements between mathematics hands-on learning and experiments, and critical thinking in teaching, emphasising the relationship between technologies and the social environment and shifting away from the culture of making for the sake of it and of quick demonstration projects about simplified designs. So, critical making refers to the hands-on productive activities that link digital technologies to society—the relationship between technology and social life (Namukasa et al., 2021).

Teams of researchers have been exploring ways to integrate the teaching and learning of mathematics, the arts, and technology to demonstrate to teachers and students that mathematics can be a deeply aesthetic experience. They developed a way of collaborating with teachers to design arts-informed, technology-enriched, mathematics experiences for students. Building on earlier research working in preservice teacher education (e.g., Gadanidis et al., 2021; Namukasa et al., 2021; Scucuglia et al., 2020), they conducted qualitative research focused on how an integrated STEAM approach can be implemented through maker pedagogies.

Namukasa et al. (2021) address the issues of how preservice teacher candidates engage with concepts related to mathematics, when a STEAM-based, maker approach is used in the learning process. The frameworks of constructionism and low-floor-high ceiling learning theories, humans-with-media in STEAM and critical making are presented to help conceptualise the maker education STEAM and critical making in teacher education.

4 Redefining learning spaces

There is documented history of reports describing how technological developments are transforming educational processes and the educational environment (e.g. Attard & Holmes, 2022; Borba et al., 2016; Engelbrecht et al., 2020). Whereas traditional classroom engagement is limited to the time in which the students are actually physically in the classroom, with discussion outside the classroom (e.g. on social online platforms) engagement is prolonged. Noroozi et al. (2020) believe that traditional education can no longer meet the needs of the world, because digitalisation is the order of the day and higher order thinking and argumentation are standard requirements. The terms *classroom in movement* or a *distributed classroom* are used—it moves from the traditional cubic space to a combination of a classroom with a bedroom for one student, an office for another, and some kind of computer centre for others (Borba et al., 2016; Engelbrecht et al., 2020). Bini et al. (2020) claim that the walls of the classroom confine not only the body of the student, but also the mind, but the internet has produced a digital environment in which students are immersed and where knowledge is easily accessible.

In this same line of thinking, the idea of a *flipped classroom* has now become a popular topic. The flipped approach is intended to optimise classroom time. Rather than expose students to new materials within mathematics lessons, students are expected to access materials before their lessons (Attard & Holmes, 2022). The approach strives to put pupils and learning processes at the centre of teaching and learning (Weinhandl et al., 2020). Flipped learning is sometimes considered as a specific model of blended learning that has a clear delineation between online and face-to-face instruction (Polly & Casto, 2019).

In a systematic literature review on the use of flipped classrooms Cevikbas and Kaiser (2023) demonstrated that flipped classrooms can be a promising pedagogy that has numerous benefits for mathematics teaching and learning, but also pointed out some significant pitfalls. In their experiment, they found that using a flipped approach, although the assessment strategies did not really change, the central elements of teaching—teaching and learning environments, interaction mode, feedback and scaffolding—underwent a radical change in accordance with the perspectives of social constructivism (Vygotsky, 1978). Attard and Holmes (2022) found that a flipped approach promoted self-paced learning among students.

Weinhandl et al. (2020) experimented with changing the technological orientation of flipped education away from exclusively using videos to utilising the computer algebra system *GeoGebra* for exploring mathematics. In their study it became apparent that when combining flipped approaches and *GeoGebra*, feedback (technological and personal) is an essential element of instructional designs for students. They also found that students do not want a diverse mix of technologies when learning mathematics, but that the number of technologies used should be kept to a small number to facilitate pupils ‘ orientation when learning.

5 Computer algebra systems and dynamic geometry

Devlin (2019) considers the appearance of computer algebra systems (*Mathematica, Maple,* and others) running on personal computers, in the late 1980s as a major revolution in mathematics. These packages were mainly used in university mathematics, physics or engineering departments since they were expensive, challenging to use, and needed a good computer to run on, keeping them away from many schools (Devlin, 2021). This situation changed in 2009 with the release of *Wolfram Alpha*, which made the *Mathematica* manipulations available in a free application with a simple interface that could easily be accessed from any PC, tablet, or smart phone. Now, mastering procedures is no longer needed to do mathematics, although certain technological advancements may necessitate new skills and expertise in certain instances. In 2011, the mobile-based system *Desmos* arrived—a very powerful graphing calculator designed for school mathematics education. The teaching of school mathematics has been drastically impacted (or should have been) by these packages in that the focus is no longer on the technical issues of an algorithm or technique, but more on conceptual understanding of it (Devlin, 2021).

The mathematics we teach at school is radically changing from what it was. Mathematicians do not need to calculate or perform symbolic manipulations—these calculations and manipulations are done by computers; much faster and more accurate than any human (Devlin, 2021). It is important, however, to keep in mind the potential risks and issues associated with using this technology in mathematics education—we do not want our students to start thinking of technology as merely a black-box—and other possible hazards.

Following up on earlier work on using computer algebra systems in mathematics assessment by Sangwin (2013), which he is quite well known for, Sangwin (2022) and Bickerton and Sangwin (2021) used a computer algebra assessment tool, *STACK*, to assess students' ability to do mathematical proofs.

Weinhandl et al. (2020) introduced *GeoGebra* in a flipped classroom approach and used design-based research and grounded theory to develop core basic principles for using the approach. Lavicza et al. (2022) used the same research design and found that cutting edge technologies change so rapidly that they need to precipitate alterations in the teaching materials and pedagogies, independent of the feedback data acquired during the implementation.

6 Student collaboration: learning environments and social media

Much has been writen about the social (collaborative) aspect of the learning process and researchers have suggested that learning mathematics using a collaborative environment such as a personal learning environment, a learning management system or social media, can be modelled on Vygotsky's social contructivist theory (Gerstein, 2013). As students leave their zone of current development, they benefit from social interactions and move towards reaching their learning potential by engaging with peers (Kurt, 2020). The virtual environment of learning through social media seems conceptually well-aligned with social constructivist views of learning. Many studies support this theory to improve student learning and engagement (e.g. Engelbrecht & Oates, 2022).

Social constructivism enables us to understand the mental processes that are involved when students conceive ideas to deal with complex problems, forming unique problem-solving strategies through a shared way of thinking (Ofori-Kusi & Tachie, 2022). This model of social constructivism assists us in understanding how students can acquire and share mathematical knowledge while learning mathematics through using a social setting on a social media platform (Ofori-Kusi & Tachie, 2022). Interactive environments, such as social media platforms, create opportunities for students to reorganise their knowledge in the course of the social interaction (Engelbrecht & Oates, 2022).

The role of the internet in student collaboration as part of the process of developing students into independent learners, is addressed by Engelbrecht and Oates (2022).

A variety of social learning networks, such as learning management systems, personal learning environments and social media are employed in this collaborative learning process. Today students use social media platforms to communicate with friends within or outside their school or university environment to help feel they belong to a community and almost all students are now connected to social networking sites.

Greenhow and Chapman (2020) define social media as

> Web 2.0 internet-based applications that feature user-generated content, profiles created for the site or app by users, and the development of online social networks to connect a user's profile with those of other individuals or groups within the system (p. 342)

Engelbrecht et al. (2023b) discuss how social media is consistent with the process of developing students into independent learners, as explained by the views of humans and media interaction in the learning process. They elaborate on the challenges and affordances of using social media tools in teaching, considering research that demonstrates how these might align with best teaching practices and contribute effectively to student engagement. They consider how social media can be used to build communities of practice for professional learning, collaborative research, and to support student learning.

Apart from the communication features, students use mathematics-focused YouTube channels, such as the Khan Academy, on YouTube (more than 2 billion views), and Numberphile (one video has enjoyed 10 million views with 2171 comments).

In mathematics education, the use of social media provides students with access to information, to connect with other student groups and to other educational systems. Modern social media platforms have expanded from online communication platforms to platforms with a variety of functions, for example education, entertainment, and social interest (YouTube, Pinterest, Reddit, TikTok), careers (LinkedIn), business, e-commerce and travel (EBay, TripAdvisor), and communication (WhatsApp, Instagram, Snapchat), and Facebook, which encompasses many of these elements.

Studies have shown that active participation on social media platforms may increase learners' motivation and engagement with content that can help foster active learning (Greenhow & Chapham, 2020). Studies have also found that the use of social platforms and social media groups tends to increase student involvement in discussions and out-of-class communication with their teachers and peers (Biton & Segal, 2021), and promotes collaboration with others during (e.g. mathematical) problem-solving (Koichu & Keller, 2018).

Some academics claim that students prefer social media platforms to *learning management systems* (Alfalah et al., 2017). According to Chatti et al. (2010), the LMS-centric model of learning has failed to improve performance. In many instances the initially paper-based learning resource is simply converted into digital format, and classroom training is changed into an online course. Consequently, some LMS-driven models tend to suffer from an inability to satisfy the heterogeneous needs of many students (Borba et al., 2016). This discussion has suggested that social media may be the better pathway for connecting with our current and future generations of students.

7 Artificial intelligence and hyper-personalisation of learning

Over recent decades, artificial intelligence (AI) has gradually infiltrated all facets of society, including mathematics and mathematics education. Although AI raises numerous ethical questions, augmenting intelligence in a kind of human–machine partnership, it goes to the heart of knowledge development. A recent book, edited by Richard et al. (2023) highlights the contribution of AI to mathematics education, providing concrete ideas obtained through dynamic international collaboration, and addressing the interaction between humans and the machine.

Van Vaerenbergh and Pérez-Suay (2022) provide an overview of the different AI systems that can be used in mathematics education, clarifying what the possibilities of current AI technologies are, what is still out of reach, and what can be expected in the near future.

Despite our awareness of a disparity between students, we still tend to group students together by age. Personal characteristics such as creativity, a sense of humour, and special competencies are recognised as important in education (Paludan, 2006). Learning can be enhanced when the instructional process allows students a personalised approach to learning where they control their own pacing and even their own learning pathways (Chaney, 2016). The internet holds the potential of individualising the learning process to provide for the individual needs of students (Vasileiou, 2009). The idea of *hyper-personalisation* has become quite popular in internet marketing. We all leave behind evidence of information about ourselves nearly every time we browse the internet or shop online. Adaptive hypermedia platforms gather this personalised information and use it to customise their pitches to us—so they identify your preferences and provide in your specific needs. Adaptive learning platforms provide an alternative to the traditional approach in that they build a model of the goals, preferences and knowledge of each individual user and this model is used throughout the interaction with the user in order to adapt to the needs of that particular user (Kurilovas, 2016). We can just imagine how well this tool can be utilised in mathematics education (Frey, 2022).

Mohan (2013) predicted that the future of education—on all levels—will be hyper-personalised. In such an environment, the teacher, using adaptive hypermedia, will become a person who understands the unique needs of each student. Some students might surge ahead in mathematics while others in literature or art (Engelbrecht et al., 2020). These adaptive AI-driven learning platforms assist in identifying learning gaps and areas that need additional attention and they nurture important skills, such as critical thinking and problem-solving. By recognising a student's problem areas in mathematics, the AI platform can dynamically adjust the lesson, providing additional explanations, visual aids, or practice problems tailored to the student's unique learning style (Graham, 2023). These systems can analyse students' steps, identify misconceptions, and offer hints or explanations, allowing for personalised assistance and contributing to deeper understanding.

Graham (2023) discuss a number of other novel uses of AI in mathematics education, including AI-powered visualisations and interactive simulations to unravel complex concepts; enhancing student engagement through gamification and personalised mathematics challenges and providing AI-assisted remediation and support for struggling students. Other AI uses include automated grading and assessment; data analysis and learning analytics; virtual simulations and visualisation; and adaptive testing—in which AI algorithms design adaptive tests that dynamically adjust the difficulty level and content based on students' performance.

More recently large language models, such as *ChatGPT,* have entered our lives and our students refer to *bots* (robots). *ChatGPT* is probably the best known example of a large language model. Devlin (2023) does not consider these GPT (Generative Pre-trained Transformer) systems a threat in mathematics education—in fact, computer algebra systems can be seen as useful special kinds of AI systems. He emphasises that *artificial intelligence* is different from *human intelligence* and *machine learning* from *human learning*; as *driving a car* is diferent from *walking* or *riding a horse*. AI technology is different and should not be compared to existing technology—we need to learn how to employ this technology in a safe and appropriate way.

Examples of recently developed AI packages include the *ChatGPT*-powered *Tutor Eva*, (https://www.tutoreva.com/) and *Brainquake* (https://brainquake.com/) but this is a new field and we may expect many new developments in future.

While AI holds great potential for mathematics education, it is important to ensure ethical considerations, data privacy, and the human role in the learning process. Teachers will continue to play a vital role in guiding students' mathematical development, leveraging AI as a tool to enhance their teaching and support student learning. Bliss (2023) found that while AI can assist in getting information to a learner, it cannot do the thinking for them—it cannot help them truly learn. She is concerned that while educators attempt to spur interest in subjects and acquisition of skills, student uses of AI often involves an individual student working alone with a bot that will "get you instant answers" and does not compel students to think through or retain knowledge.

8 Multimodality: videos, virtual reality (VR) and augmented reality (AR)

Multimodality refers to the interplay between different representational modes, for instance, between images, technology, verbal and written text. Cendros Araujo and Gadanidis (2020) described a flexible and creative layout in developing a theory that describes the abundance of multimodal information contained in online collaborative mind mapping and that interprets the elements of meaning that have significance for knowledge construction in mathematics education. Their results describe how students interact and construct knowledge while they engage in online collaborative mind mapping, providing insights into the ways that collaborative and multimodal technologies affect mathematics education. They filled a gap in our understanding of how students construct knowledge when they interact through online collaborative mind mapping.

Dynamic media, such as watching or producing mathematical videos, have been prominent as technologies that stimulate the senses in the production of mathematical knowledge, leading to new ways of knowing (Domingues & Borba, 2021). Videos emerged as an element of the use of technology that has a significant impact on students' learning experiences in mathematics. The authors construct an argument that the multimodal nature of videos offers ways to combine various semiotic resources. The written forms of mathematical discourse combine with audio-visual resources to create dynamic representations of mathematical content.

Researchers are still trying to explain why videos are so popular with students. Traditionally, mathematics has been conducted and produced with different artefacts, such as compass, ruler, tablets, papyrus, paper and pencil and digital technology that were (or are) present in the collectives of humans-with-media that produce mathematical knowledge. Producing digital videos may be the next artefact of the twenty-first century (Domingues & Borba, 2021). When students produce videos, they participate in the "exhibition culture of the social media"—students are motivated to perform well because it will be displayed. The pedagogical use and value that watching and producing digital mathematical videos have, should be further investigated.

In a multimodal approach to mathematics education, *augmented* and *virtual reality* should be mentioned. *Augmented reality (AR)* is an interactive experience in which a real world environment is enhanced with computer-generated visual elements, sounds, and other stimuli. *Immersive virtual reality (VR)* is a simulated environment that engulfs a user, providing the illusion of being physically present in a different world or setting, involving using a head-mounted display that covers the user's eyes and sometimes ears, along with motion-tracking sensors to provide a realistic and interactive experience. AR uses a real-world setting while VR is completely virtual—so AR users can control their presence in the real world where VR users are controlled by the system.

Both AR and VR can contribute to enhancing mathematics education by providing students with engaging and interactive learning experiences, through e.g. visualisation of concepts, interactive problem-solving and addressing real-world applications. Little research has been done on the impact that these technologies can have on mathematics education.

Cevikbas et al. (2023) did a systematic review of literature on AR and VR, identifying research trends, characteristics and methodologies, and exploring the potential benefits and drawbacks of AR/VR technologies in mathematics learning.

Their study points out that more research is needed to fully understand the potential benefits and limitations of AR/VR technology in mathematics education.

9 Impact of COVID-19 on mathematics education

After a number of years of emergency remote teaching (ERT) during the COVID-19 pandemic, we are now moving into a position where we can evaluate the impact that ERT has had on mathematics education. In a survey article, Engelbrecht et al. (2023a) focused on the actual mathematics curriculum, learning design and assessment, the role of collaborative activities and social media, educational videos, and the role of family and parents in future. Borba (2021) discussed the connections of the humans to the virus, how it has laid bare social inequality. Studies, such as Chan et al. (2021) and Bakker et al. (2021) also warned about the dangers of possible inequities such as unequal access to internet and to computers and other hardware, as well as the unequal availability of space at home for uninterrupted time for learning.

The pandemic prompted educators to explore innovative teaching methods and integrate technology more extensively. Teachers utilised various online platforms, video tutorials, interactive apps, and virtual manipulatives to enhance mathematics instruction. These tools offered opportunities for personalised learning experiences.

Bakker et al.'s (2021) experience showed that mobile phone applications such as WhatsApp and WeChat have become key tools in teaching and learning mathematics in many rural areas worldwide. During the pandemic smartphones have become central devices in mathematics education.

The COVID-19 pandemic highlighted the importance of mathematical problem-solving skills in addressing real-world challenges. Educators now increasingly seek to emphasise the practical applications of mathematics, such as modelling real-life scenarios and understanding statistical data.

Traditional methods of assessing mathematics, such as in-person examinations, had to be adapted to the remote learning environment. Educators explored alternative assessment methods, including online quizzes and projects (Sangwin, 2022).

The traditional classroom dynamics were disrupted: in the shift to remote learning, students were no longer able to work together in person. New avenues for collaboration had to be found and the role of social media in the teaching process is growing. Furthermore, because of the disruption caused by the pandemic, some students (in particular those from disadvantaged backgrounds or those without necessary support at home) missed out on essential mathematical content and skills.

10 Epistemology of digital technology and humans-with-media

With the increase in the use of technology, Borba et al., (2016, 2023) summarised five different phases in the utilisation of digital technology in mathematics education, documenting different ways that digital media influence mathematics education. Theoretical perspectives were developed that emphasise the role of artefact and the general role of technology in knowledge production. There is a growing trend to consider knowledge construction as not only developed by the collective that involves only humans, but *humans and things*. There are researchers inside and outside mathematics education who advocate such a position and propose other approaches to understand the way that digital technology is increasingly changing the way we know, and also what we know. Theories about how we know and what we know point to the *agency of things* (Borba et al. 2023).

The notion of humans-with-media was systematised by Borba et al. (2023), where they synthesised how the notion of humans-with-media could be understood based on the work of Borba and Villarreal (2005) and Lévy (1993) contributing to the notion that learning is social, not only in the sense that it involves more than one person, but that it also involves "things" e.g. pieces of software, hardware, and the internet. Borba (2021) showed how different media shape humans, and also provided examples of how humans shape technology. He reported on the interaction between software and the interaction of students with the software and with the teacher. If the correct problems are proposed, a collective of *students/teacher-with-software* could generate conjectures, test them and then collectives of *humans-with-software-with-paper-and-pencil* could develop proofs in the classroom. There are examples in which students and teachers used software in ways not planned by designers (e.g. spreadsheets). So the notion of humans-with-media as unity, produces knowledge, with technology shaping humans and humans shaping technology. Jacinto and Carreira (2017) used this notion to analyse how *students-with-Geogebra* solved problems in basic education. Others, such as Villarreal et al. (2023) used this construction to analyse how digital technology transformed teaching and learning during the pandemic.

The question whether digital technologies would be able to provide alternative ways to conduct mathematics education has been asked often. Borba (2021) used the theoretical construct of *humans-with-media* to connect the pandemic to different trends in mathematics education: the use of digital technology, philosophy of mathematics education, and critical mathematics education. He extended the agency of digital media to also include *homes*. During the pandemic the role of homes became a fundamental actor in the teaching of mathematics more evident, showing that adequate conditions at home and available internet were paramount. It does not seem to loose its importance now after the pandemic. In fact, the pandemic helped to uncover the role that different conditions at home, the agency of home, had on learning mathematics (Borba et al. 2023). With the prominence of AI tools, the idea that technology has agency, embedded in the notion of humans-with-media, became even more powerful.

11 Concerns and possible pitfalls in using technology

In spite of technology offering numerous benefits, some of which were mentioned in this review, there are some potential concerns and disadvantages that need to be addressed to ensure its effective integration.

A major concern is regarding equity and access. Not all students have equal access to technology and the internet, creating a digital divide where some students have a significant advantage over others. We need to find ways to ensure equitable access to technology resources and opportunities.

Graphing software or equation solvers, may assist with expediting problem-solving but, in some instances, may not necessarily deepen conceptual understanding. Effective pedagogical strategies must be employed that promote conceptual understanding using the technology.

Virtual manipulatives and simulations cannot always replicate a physical object such as geometric shapes, blocks or measuring tools and a balance should be found between virtual and physical experiences. We still know too little about the effectiveness of multimodal resources such as videos, interactive simulations, AR and VR on students' understanding of mathematical concepts, spatial reasoning, mathematical reasoning and problem-solving skills.

There is also a danger that overreliance on technology can lead to a depersonalisation of the learning experience. Students may miss out on the personal interaction and mentorship that traditional classrooms and the face-to-face environment provide. Increased exposure to technology can even lead to social isolation.

There are some concerns that the ease of finding information online can discourage critical thinking and problem-solving skills. Students may be more inclined to search for quick answers rather than engage in deep, reflective thinking. On the other hand, the availability of independent learning, coming with an online environment, may promote critical thinking skills.

There are also concerns that as education becomes more technology-dependent, students may become increasingly reliant on technology, even for basic tasks, potentially impacting their ability to perform even trivial tasks without technological assistance.

The same platforms, such as social media, gaming, and other non-educational applications and technologies used for educational purposes, can also become sources of distraction and can divert students' attention from their studies.

Regarding teacher training, many educators require proper initial training and ongoing support to effectively integrate technology into their teaching. If not, technology can easily become a barrier rather than contributing to education.

As mentioned before, AI may cause overdependence on digital technology and may raise ethical issues regarding fake news. Trained machines may start bringing ridiculous and false theories to our classrooms.

12 Gaps in the research and new research topics

Apart from addressing the concerns mentioned in the previous section, there are some more obvious areas that warrant the attention of researchers in the field.

Current studies focus mainly on short-term outcomes of using technology in mathematics classrooms. More research is needed to investigate the long-term impact of technology integration on students' mathematical knowledge, problem-solving skills, and conceptual understanding.

Researchers have claimed that the regular classroom is changing—the question is how this discussion will evolve after the pandemic. Attard and Holmes (2022) suggest further and deeper investigation into how what appear to be small changes to technology use, can lead to improvements in the student experience of mathematics, potentially influencing students' choices to continue to study mathematics beyond the compulsory years.

While many educational technology tools are available, there is a need for more research into their effectiveness in mathematics education. More studies should assess how different technologies impact student learning outcomes and how effective they are for specific student populations.

The role of external resources, such as *social media*, is surely going to become an increasingly relevant research topic in mathematics education. The importance of the actual curriculum seems to be becoming blurred, as students increasingly turn to informal educational platforms for their learning interactions and students increasingly want to be involved not only in in how they are taught but also in what they are taught (Dekker, 2021; Mkandawire et al., 2018). They want to decide on what mathematics they learn and how, in a pull approach, rather than a curriculum that is pushed onto them by the educational system (Engelbrecht et al., 2023a). On the other hand, however, informal educational platforms can pose a risk to learners since the content shared on these platforms is typically not created or verified by subject matter experts. Therefore, this situation highlights the need for increased attention to the curriculum and its significance.

The integration of *generative artificial intelligence* (e.g. the application *ChatGPT* and adaptive platforms) may have a serious impact on mathematics education in general and in particular to theoretical discussions such as the nature of knowledge and who the "agent" is in collectives of humans-with-media-things. AI may play a central role in reshaping the relationship between humans and media. *ChatGPT* certainly has agency and can be active in a collective of *humans-with-ChatGPT*. New mathematical problems will have to be developed for collectives of student-teachers-with-*ChatGPT*. Similar to with the introduction of calculators, computers or the internet, this new technology will participate in education. This technology will be shaped by us humans, and AI will shape us. We need to examine how AI algorithms can personalise mathematics instruction, based on individual students' needs, preferences and learning styles. In fact, the issue of *personalised learning*, powered by technology, is a very promising approach, but more research is needed to understand how to tailor the needs of individual students effectively while addressing privacy and ethical concerns. We also need better understanding on how AI can provide personalised feedback, identify learning gaps, and improve educational outcomes.

Blockchains are a recent addition to the educational scene that enhances hyper-personalisation. A blockchain is jointly distributed and decentralised ledger, aiming to record transaction history with different networks. Crypto currency (e.g. Bitcoin) is probably the best-known application of blockchain technology. This technology is quite practical if used in the field of education to carry out digital certification and record keeping (Guustaaf et al., 2021) Blockchain's tokenisation and security features provide a key to future hyper-individualised learning environments (Frey, 2022).

Assessment will definitely be addressed. How should we design assessment in mathematics education to encourage online collaboration and provide students with formative feedback? We should explore the use of data-driven approaches to analyse students' performance, learning patterns, and misconceptions in mathematics, aiming to provide personalised feedback and support.

We did not address the issue of *gamification* in mathematics learning but we need to explore the integration of game elements, such as badges, leader boards, and interactive challenges, to engage students and promote motivation and persistence in learning mathematics.

We need to explore effective approaches to train new teachers and support existing teachers in integrating technology tools and resources effectively into their mathematics instruction. Educators must be equipped with the pedagogical knowledge to utilising technology tools to harness their full potential.

Addressing these gaps in research will help improve our understanding of how technology can be effectively used in mathematics education and inform the development of evidence-based practices and policies in this field.

13 Conclusions

With the concerns about student disengagement in advanced mathematics courses, Attard and Holmes (2022) consider a potential disruption by technology use in mathematics education as imperative. Other studies on student engagement (e.g. Murphy, 2016; Ní Shé et al., 2023) support this view. This can happen through the affordances that online environments offer for teachers to introduce meaningful interactions through blended and flipped learning approaches.

Although there have been changes in mathematics that resulted from the growth of computer technology, it is not really mathematics itself that has changed in the digital age. The most significant change is the way people use mathematics. With the availability of new technology, the focus in mathematics has moved from the calculation and the execution of procedures to actual mathematical thinking. This transition gives us the opportunity and the challenge to produce effective users of mathematics—mathematical thinkers (Devlin, 2021).

> Being able to calculate quickly, efficiently, and accurately used to be essential. Now, it is not required. In

place of that skillset (which took most people considerable time and effort to master, with many dropping by the wayside in the process) is a new set of skills. Those new skills—mathematical thinking—are, in fact, much closer to those in the humanities or the creative arts than most people yet realize or, in some cases, are willing to contemplate (Devlin, 2021, p. 43)

Apart from the new developments in the field of online and blended mathematics education mentioned in this paper, there have been several other developments in the field that we did not discuss here. Here are a few examples:

- Interactive online platforms: Some online platforms, such as Khan Academy and IXL, offer interactive exercises and tutorials that allow students to practise and learn math at their own pace. Some of these platforms use adaptive technology to personalise learning and provide instant feedback to students.
- Gamification: Game-based learning has become popular in online and blended mathematics classrooms. Games and simulations can make learning more engaging and provide students with opportunities to apply mathematical concepts in a fun and interactive way.
- Virtual manipulation: Online tools and simulations allow students to manipulate virtual objects to explore and understand mathematical concepts, proving quite helpful for visual learners who sometimes struggle with abstract concepts.

Applications, such as *ChatGPT,* have raised discussions on the impact of artificial intelligence on mathematics education. The status quo has been disrupted globally, as students use it to write their essay submissions. Therefore, it is essential to carefully evaluate the benefits and limitations of these technologies, including potential ethical concerns, and to adapt teaching and assessment strategies to align with the changing educational technology landscape.

Through history we have had to address similar questions: how will be adapt to successfully use the calculator in mathematics education?, how will we use the internet in mathematics education? Now, as a community of mathematics educators, we will have to develop a new pedagogy for collectives of humans-with-AI.

To summarise, online and blended environmnets in mathematics education are particularly well-suited for self-directed learners who can manage their own time and prefer to learn independently and are well-suited well-suited for students who benefit from the structure and accountability of traditional classroom instruction but also want the flexibility and convenience of online learning.

New technological developments in this field may enable us to increasingly individualise learning. Our students may soon be able to chart their own courses, study things that interest them and switch directions when they feel like it (Frey, 2022).

On the other hand, there is a concern that social inequality can make *education for all* increasingly difficult. Decades ago some authors dreamed that digital technology would help to have knowledge for all. This does not seem to be the case so far, as we have learnt from our experience with mathematics education during the pandemic (Borba, 2021; Engelbrecht et al., 2023a).

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