A functional approach to distribution modelling: The spliced generalised normal distribution

by

Matthias Wagener

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I, Matthias Wagener, declare that the dissertation, which I hereby submit for the degree, Master of Commerce in Mathematical Statistics, at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Matthias Wagener
E-mail: matthias@dilectum.co.za

Abstract

A new body and tail generalisation of the normal distribution is introduced, the spliced generalised normal (SGN). A special case of the SGN, the tail-adjusted normal distribution, is further generalised with two-piece scaling to accommodate different combinations of skewness and tail weight in data. The two-piece scaled tail-adjusted normal (TPTAN) is thoroughly studied with the derivations of various statistical properties such as the probability density function, cumulative distribution function, quantile function, moments, and Fischer information. The applicability of the SGN distribution is demonstrated by the application of the TPTAN to light and heavy-tailed data sets. The small and large sample performance of the TPTAN is investigated with an extensive simulation study. The methods of estimation include maximum likelihood and Kolmogorov-Smirnov estimation. The goodness of fit is evaluated by likelihood criteria and hypothesis tests such as Akaike information criterion, Bayesian information criterion, consistent Akaike information criterion, Hannan-Quinn information criterion, and the KS and Bayes factor tests.

Keywords: generalized normal, normality, kurtosis, inferential statistics, Bitcoin

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Chapter 1

Introduction

In the field of flexible modelling the task of the researcher and practitioner is to model data with increasingly more complex structures. The dizzying number of existing proposals for flexible distributions is both a blessing and a curse. The blessing of course is that more types of data can be accommodated. The curse is that it is hard to determine which models should be used on practical and theoretical grounds [42].

1.1 Motivation

The plethora of available flexible distributions pose an essential problem for both the theorist and practitioner: theorists need to know what kinds of distributions to produce and practitioners need to know what distributions to use. The field of distribution theory is the foundation from which all statistical theory and application originates. Improvements in distribution theory therefore improves statistics as a whole. Here it will be endeavoured to derive a flexible model that balances and satisfies the theoretical and practical considerations of this field. In a broader sense the purpose of this would be, to give theorists information to consider when deriving a flexible model and to help practitioners to know what characteristics to look out for when choosing a flexible model.

1.2 Aims

- Develop and research a view of what makes a "good" model
- Follow a deliberate and conscious derivation ethic in producing a parsimonious flexible model
- Compare the resulting model to existing models through the lens of what makes a "good" model

1.3 Objectives

- Present a review of flexible modelling approaches and desirable model traits.
- Derive a four parameter model to accommodate skewness and tail weight separately. This is done by using a functional understanding of a probability density function (PDF).
- Derive various important statistical properties for this model such as the PDF, cumulative distribution function (CDF), quantile function, moments, and Fischer information.
- Compare this distribution on real and simulated data.

1.4 Contributions

- A review of four main flexible modelling approaches, desirable traits of a flexible model, and a paradigm for parameter interpretation.
- The very general spliced generalised normal (SGN) distribution and the derivation of various statistical properties such as the PDF, CDF, quantile function, and moments.
- The tail-adjusted normal (TAN) distribution with a specific parameter for accommodating light and heavy tails. Also, the same previously mentioned statistical properties are given.

- A skew version of the TAN, the tail-adjusted normal (TPTAN) distribution. This distribution is a four parameter distribution with specific parameters for location, scale, skewness, and kurtosis. Also, the same statistical properties are given as well as the Fischer information.

- A practical demonstration of inference and comparative preformance of the TPTAN on real data.

- A comparative simulation study of the small and large sample properties of the TPTAN.

1.5 Dissertation outline

- **Chapter 1** motivates and organises the dissertation with aim, objectives, contributions, and outline.

- **Chapter 2** explains the goal of flexible modelling, reviews four main approaches, gives a desirable traits of flexible model, and a paradigm for parameter interpretation.

- **Chapter 3** motivates the SGN using the concept of log-splines and the generalised normal (GN) distribution and provides important statistical properties.

- **Chapter 4** focusses on the TAN distribution a sub-model of the SGN which has an exclusive parameter for tail shape and provides various important statistical properties.

- **Chapter 5** gives a skew version of the TAN and the TPTAN, which is a distribution with specific parameters for skewness and tail shape.

- **Chapter 6** shows the practical significance of the TPTAN by studying its performance on real and simulated data.

- **Chapter 7** summarises how the objectives were met and provides suggestion for future work.

1.5.1 Appendixes

- **Appendix A** provides a list of the important acronyms used or newly-defined in the dissertation, as well as their definitions.
- **Appendix B** gives the simplified results of important integrals used in the derivation of TPTAN Fischer information. An important two-piece scaled distribution moments property is also derived.
- **Appendix C** provides the full derivations of the remaining TPTAN Fischer information elements not shown in in Section 5.6.
- **Appendix D** gives the program code used throughout the dissertation.

Chapter 2

Literature review and perspectives

The topic is introduced with an intuitive study of what a flexible model should accomplish in the real world. Thereafter, four current approaches of flexible modelling is discussed. The chapter concludes with the desirable traits of a flexible model and a paradigm for the interpretation of parameters in a flexible model.

2.1 A perspective on flexible modelling

The flexible modelling example is a sample of glass fibre data from the National Physical Laboratory in England. The sample consists of 63 observations of 1.5 cm glass fibre lengths previously studied by Smith and Naylor [57]. In Figure (2.1) the sample is modelled with a non-parametric kernel density estimate (KDE), a semi-parametric log-spline density estimate, and a parametric normal distribution estimate. The KDE is chosen as a baseline continuous representation of the sample data.

A clear picture of what a density function should achieve emerges from Figure (2.1). In the case of the normal model, a parabola is used to the determine the log-density and generates the data accordingly. Another approach is to use a natural cubic spline to determine the log-density. This is a more flexible approach that fits the sample KDE more closely and is discussed in Chapter 3. This principle essentially shows why a normal distribution is a relatively inflexible model, the log-density is restricted to different scaled and shifted parabolas. On the other hand, log-spline models are more flexible since the log-density takes on piecewise cubic polynomial functions with linear tails.

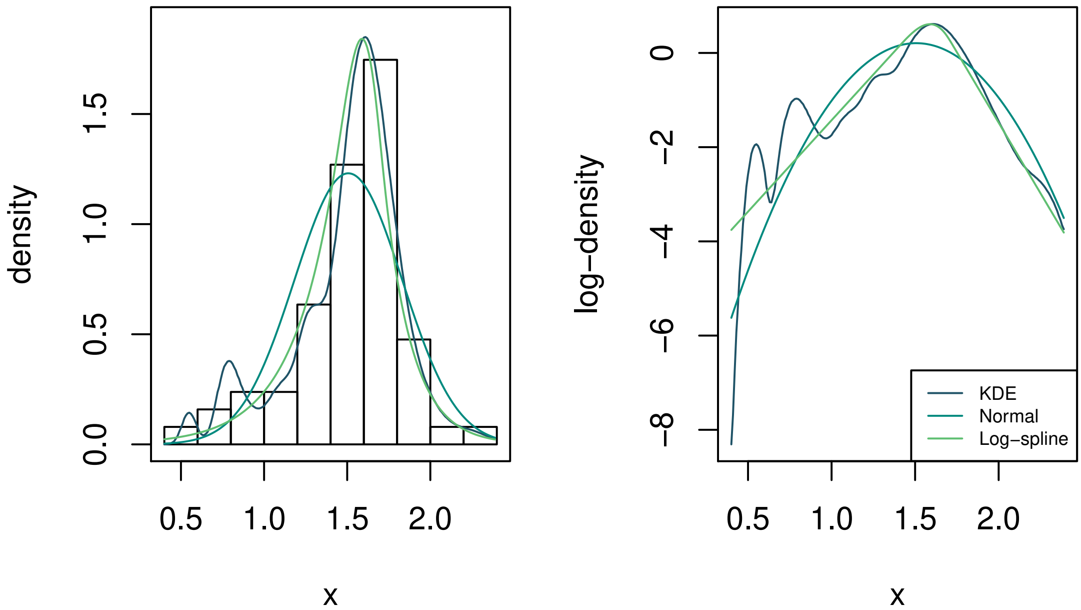

Figure 2.1: The KDE of the fibre sample data and a comparison of two data generation mechanisms. Note that for a good fit of the normal distribution, the log-density needs to be parabolic in nature.

Essentially speaking, in flexible modelling researcher is tasked with finding functions for log-densities which can mimic real data generating processes and have the desirable traits discussed in Section 2.3. It is interesting to note that the log-density paradigm has strong ties to the field of Information theory: the log-density is also known as the information content or surprisal of a random variable and the negative expectation of the log-density is know as the differential or Shannon entropy of a random variable [16].

2.2 Flexible model approaches

The importance of flexible distributions is underscored by the variety of applications in which they are necessary. In finance, returns data have market characteristics that lead to skew and heavy-tailed distributions [42], Ramirez-Cobo et al. modelled skew and heavy-tailed LAN and WAN traffic data [52], and Lin et al. analysed the bi-modal heavy right tail BMI data of men and women [45]. The impact of flexible modelling is further underscored by their successful integration into classical statistical approaches such as time series analysis [30], space-state models [48], random fields [2], regression models [7], linear mixed effects models [3], non-linear mixed-effects models [51], Bayesian statistics [54], and Bayesian linear mixed models [46].

At first glance, the problems faced in flexible modelling seem to have easy remedies. A mixture of normal distributions can be used for bi-modality and skewness and the t distribution for heavy tails. These basic approaches have their limitations since some data require extreme combinations of skewness and kurtosis. If elaborate models with many parameters are considered, the analyst encounters the risks of over-fitting, complex calculations, and problematic estimation [42]. Therefore, an ideal model balances flexibility with overall complexity. The predominant approach to generating flexible models is by the modification of a known distribution, see [37]. This is done with the intention to retain some of the properties of the original distribution which are well known. Approaches to generate flexible models include but are not limited to finite mixture models [47], variance-mean mixtures [10], copulas [49], the Box-Cox transformation [14], order-statistics-based distributions [35], probability integral transformations of [24], and the Pearson system of distributions. Here the focus will be on four famous flexible families of distributions: the Azzalini-type, transformation, two-piece, and log-spline distributions.

The Azzalini-type distributions have garnered a lot of success due to its good stochastic properties, elegant generating mechanisms, and good fitting properties, see [19] and [7]. Unfortunately certain Azzalini-type distributions suffer from inferential problems. Ley and Paindaveine [44] and Hallin and Ley [28] determined that, among others, the skew-normal distribution has singular Fischer information scores due to collinearity of the location and skewness scores. Additionally in some data sets the maximum likelihood estimate of the skewness parameter tends to infinity. That is to say that a half-normal distribution is fitted when values are observed on both halves of the distribution [6].

Another approach is to transform the data such that the transformed data follows a known distribution. A famous example of the transformation approach is the log-normal distribution derived by Galton in 1879 [20]. That of course is where the log of some data follows a normal distribution. This approach has expanded considerably since then, namely with the arcsinh transformed distributions [33], g-and-h distributions [61], sinh transformed distributions [53], and sinh-arcsinh distributions [39]. Some transformations such as the K-transformation [31] and the E-, J-transformations [25] only affect tail-weight. These approaches are highly flexible, since the flexibility is only restricted by the transformation used to transform the data. Transformed distributions do pose some problems. In the cases where the transformation does not admit an inverse, their density functions cannot be written out in closed form. The g-, h-, K-, E-, J-transformations all pose this problem and parameters are often estimated by quantile-based methods.

The two-piece and scale transformed distributions were pioneered by Fechner in 1897 [22]. These distributions work by re-scaling the density unequally on each side of the center of a symmetric distribution. The Bank of England and the Sveriges Riksbank used a two-piece normal model to show that their forecast of future inflation might be asymmetric. This application is what brought the two-piece normal distribution to public attention in the late 1990s [62]. Inferentially speaking the two-piece distributions have strong parameter orthogonality compared two other four parameter models [38]. A more recent approach to rescaled modelling is studied by [36], [34], and [27].

Lastly, for the sake of completeness, the lesser known log-splines modelling approach is considered. Originally defined by Neyman [50] as alternatives to goodness-of-fit tests. The approach works by estimating the log of the density function with a natural cubic spline. It is important to note that the log-spline model is therefore semi-parametric and does not have interpretable parameters. The maximum likelihood estimation of log-splines was independently studied by Stone [58] and Barron and Sheu [11]. A method for automatic knot selection and quantile confidence bounds is studied by [32]. Also, it has been shown that the standard errors of the log-spline CDF and quantile function are asymptotically equal to their respective empirical functions [58]. For further reading, see [17], [41] and [56] who study various similar ways of directly estimating the log-density functions.

2.3 Flexible model traits

In *Flexible modelling in statistics: past, present and future* [42] and *On Families of Distributions with Shape Parameters* [37], the respective authors formulate some of the desirable traits of a flexible model.

1. A finite number of well interpretable parameters: optimally these include a parameter for location, scale, skewness and kurtosis. This excludes parameters that cause a collateral trait as an accidental choice of parameters in the density function. An example of this would be "collateral bi-modality" which could likely force bi-modality on data without such a trait.

2. Parsimony and interpretation above ultra flexibility: this affords generalisability of results, particularly in prediction accuracy, by avoiding over-fitting. Also, this aids in parameter interpretability and ease of estimation, traits (1) and (3) respectively.

3. Favourable estimation properties: it is crucial to be able to estimate parameters correctly as to ensure correct predictions and inferences from the model.

4. Good inferential properties: a flexible model should inferentially be useful in tests for normality. An added benefit would be if such a distribution has general power among the different families of distributions.

5. Separated roles for model traits: appropriate measures of skewness should ideally not depend on the kurtosis parameter and vice versa. The measures of these traits can be based on moments [13] or [43], quantiles [8], the mode [4], and gradients [18].

6. Simple tractability: despite modern computational power, tractability is still to be preferred to non-tractability. Mathematical formulae should not only be convenient to work with but should describe specific distributional characteristics. A secondary benefit is, of course, improved computational speed.

7. Data generating mechanism: it is useful that a model can identify latent structures in the data, provided it gives an adequate fit.

2.4 A paradigm for flexible model parameters

The current paradigm for characterising data is usually done with location, scale/dispersion, skewness, and kurtosis. In non-normal data, the location parameter may be more naturally described as the node around which the structure of the data is formed. In some cases of uni-modality the node will be equal to the mode.

Therefore in non-normal structures the location parameter will not be equal to the overall mean of the data. Likewise the overall variance of the population is not equal to the squared scale parameter of the model. If the node paradigm is used it can aid in achieving the goal of the parameter interpretability. In this study the parameter interpretation will be as follows:

- μ: the location of the node.
- σ: the overall dispersion (not necessarily relative to the node).
- ψ: skewness as a relative measure of scale left of the node to the scale right of the node.
- ν: the kurtosis of the density relative to the baseline distribution kurtosis.

2.5 Summary

In this chapter the problem of flexible modelling is introduced in Section 2.1 and four approaches to this problem is discussed in Section 2.2. In Section 2.3 a standard for assessing the quality of a flexible modelling approach is discussed. In Section 2.4 a paradigm for achieving parameter interpretability is given which is an import desirable quality in flexible modelling. The next chapter introduces the SGN and its properties as a building block for flexible modelling.

Chapter 3

The spliced generalised normal distribution

In this chapter the SGN is introduced and its relation to log-splines and GN distribution exhibited. Various statistical properties of the SGN distribution is derived, such as the PDF, CDF, quantile function and moments.

3.1 An introduction to log-splines

A mathematical spline function is a piecewise polynomial function, $f(x)$, which is obtained by dividing the domain of x into continuous intervals, and letting f be equal to a separate polynomial in each interval. In the literature, splines are represented using basis functions and "knots" that denote the boundaries of different intervals [26]. As an example consider a linear and continuous spline with two knots ξ_1 and ξ_2 and four basis functions:

$$B_1(x) = 1, B_2(x) = x, B_3(x) = (x - \xi_1)_+, B_4(x) = (x - \xi_2)_+,$$

and where $(x)_+ = max(0, x)$.

The spline is then given by:

$$f(x) = \sum_{i=1}^{4} \theta_i B_i(x), \tag{3.1}$$

where the θ_i's are constants. In general, a mathematical spline with K basis functions is given by:

$$f(x) = \sum_{i=1}^{K} \theta_i B_i(x), \tag{3.2}$$

see [26]. It is clear from Figure (2.1) in Chapter 2 that density estimation can be seen as approximating an unknown log-density function. Naturally, this can be done with a spline function by estimating the unknown basis coefficients, θ_i for $i = 1, 2, \ldots, K$, by method of maximum likelihood. The resulting methodology, log-spline density estimation, has been studied by [32] and [58]. For the interested reader a formal definition of log-splines as given by [32] follows:

Given the numbers L and U, where $-\infty \leq L < U \leq \infty$, and the sequence of K knots $\xi_1, \ldots, \xi_K$ with $L < \xi_1 < \cdots < \xi_K < U$. Let S be the space of twice-continuously differentiable functions s on (L, U) such that the restriction of s to each of the intervals $(\xi_1, \xi_2], \ldots, (\xi_{K-1}, \xi_K]$ are a cubic polynomials. Further restrict the functions of S to be linear on $(L, \xi_1]$ and (ξ_K, U). The K-dimensional space of S, and the functions in this space are referred to as natural cubic splines. Set $p = K - 1$. Then S has a basis of the form $1, B_1, \ldots, B_p$. The slopes $B_1, \ldots, B_p$ is chosen such that B_1 is linear with a negative slope on $(L, \xi_1]$, $B_1, \ldots, B_p$ are constant on $(L, \xi_1]$, B_p is linear with positive slope on $[\xi_K, U)$, and $B_1, \ldots, B_{p-1}$ are constant on $[\xi_K, U)$. A column vector $\underline{\theta} = (\theta_1, \ldots, \theta_p)^t \in \Re^p$ is said to be feasible if

$$\int_L^U \exp\left(\theta_1 B_1(x) + \cdots + \theta_p B_p(x)\right) dx < \infty$$

or, equivalently, if (1) either $L > -\infty$ or $\theta_1 < 0$ and (2) either $U < \infty$ or $\theta_p < 0$. Let Θ denote the set of all such feasible column vectors. Given that $\underline{\theta} \in \Theta$, the log normalising constant is given as:

$$C(\underline{\theta}) = \log\left(\int_L^U \exp\left(\theta_1 B_1(x) + \cdots + \theta_p B_p(x)\right) dx\right). \tag{3.3}$$

Finally, the PDF of a log-spline is then given by:

$$f(x; \underline{\theta}) = \exp\left(\theta_1 B_1(x) + \cdots + \theta_p B_p(x) - C(\underline{\theta})\right), L < x < U. \tag{3.4}$$

The intention is to derive a density function that is constructed in the context of modelling different shapes of tail and body configurations with the spirit of log-spline density estimation.

3.2 Spliced generalised normal kernel

The SGN has its origins from the generalised normal (GN) distribution as originally proposed by [59]. The GN distribution includes the normal and Laplace distributions for shape parameter value of $s = 2$ and $s = 1$ respectively. The GN PDF and CDF is given below:

$$\phi(x;\mu,\sigma,s) = \frac{s}{2\sigma\Gamma(\frac{1}{s})}\exp\left(-\left|\frac{x-\mu}{\sigma}\right|^{s}\right), \tag{3.5}$$

and

$$\Phi(x;\mu,\sigma,s) = \begin{cases} \frac{\Gamma_l\left(\frac{\mu-x}{\sigma}\right)^{s}}{\Gamma(1/s)} & x \le \mu \\ 1 - \frac{\Gamma_l\left(\frac{x-\mu}{\sigma}\right)^{s}}{\Gamma(1/s)} & \mu > x \end{cases}, \tag{3.6}$$

where the location, dispersion, and shape is given by parameters $\mu \in \Re$, $0 < \sigma$, and $0 < s$. In Figure (3.1) it can be seen that for smaller values of s, the GN is more heavy-tailed, and more light-tailed for larger values of s. This provides an opportunity to model the tail and body of a distribution with two different GN distributions. Importantly, this approach allows for heavier and lighter tails relative to the body of the distribution.

For the construction of such a distribution, a general spliced kernel is studied given in the equation below:

$$k(z;\alpha,\beta,\gamma) = \begin{cases} \mathrm{e}^{-|z|^{\alpha}} & |z| < z_c \\ \mathrm{e}^{-\gamma|z|^{\beta}+c} & z_c \le |z| \end{cases}, \tag{3.7}$$

where $\alpha, \beta, \gamma, z_c > 0$ and c is a constant.

The log-density of Equation (3.7) is constrainded to have a valid first derivative which ensures the density to be smooth and act similarly to a mathematical spline. The parameters in Equation (3.7) then have the following significance; α determines the shape of the body, β the shape of the tails, c ensures point continuity, γ the relative

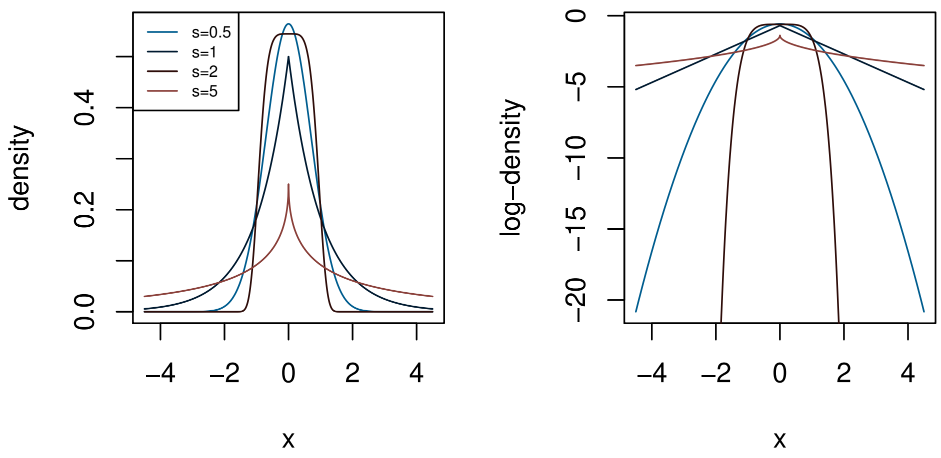

Figure 3.1: The GN PDF for different values of s. The GN PDF is heavy-tailed for values of $s < 2$ and light-tailed for values $2 < s$. This property can be exploited to add heavier or lighter tails to the body of another GN distribution.

scale of the tails, and z_c is some threshold of transition. Another key advantage of such an approach is that this kernel is a piecewise function of GN kernels. This permits convenient derivations of statistical properties in analytic form. Next, the kernel in Equation (3.7) is transformed into a valid PDF by deriving the critical point z_c, the point continuity constant c, and a normalising constant δ. The steps of derivation for z_c and c are graphically shown in Figure (3.2). The resulting general PDF is then SGN PDF.

The log of Equation (3.7) is contrained to have a valid first derivative. Therefore the critical point, z_c, is calculated such that the first derivative exists:

$$\begin{aligned}
\frac{d}{dz_c}\left(-z_c^{\alpha}\right) &= \frac{d}{dz_c}\left(-\gamma z_c^{\beta}\right) \\
\alpha z_c^{\alpha-1} &= \gamma\beta z_c^{\beta-1} \\
\ln(\alpha) + (\alpha-1)\ln(z_c) &= \ln(\gamma) + \ln(\beta) + (\beta-1)\ln(z_c) \\
\therefore z_c &= e^{\frac{-\ln(\alpha)+\ln(\beta)+\ln(\gamma)}{\alpha-\beta}}. \qquad (3.8)
\end{aligned}$$

Next, the constant c is calculated to ensure point continuity at the critical point, z_c, of Equation (3.7):

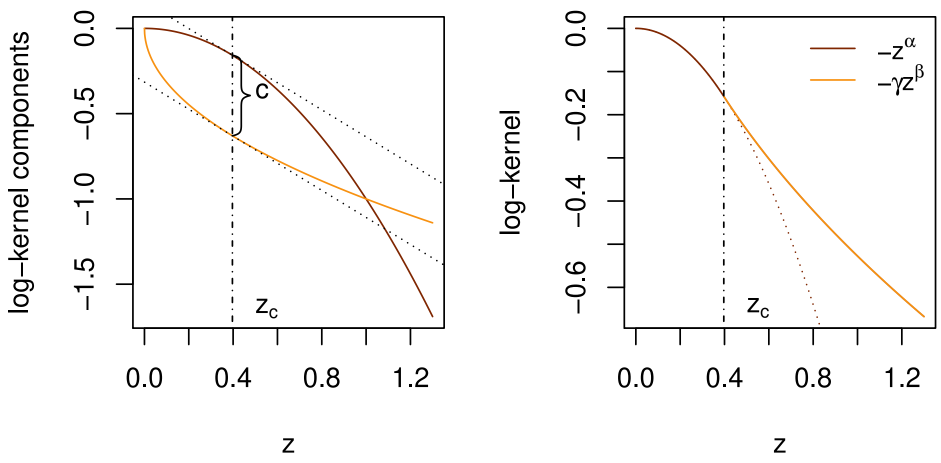

Figure 3.2: The construction of the SGN kernel. For a given shape parameter α, β, and γ, the critical point z_c is where the slopes of $-z^{\alpha}$ and $-\gamma z^{\beta}$ are parallel to each other, and c ensures that the kernel is point continuous at z_c.

$$\begin{aligned} -\gamma z_c^{\beta} + c &= -z_c^{\alpha} \\ \therefore c &= \gamma z_c^{\beta} - z_c^{\alpha}. \end{aligned} \tag{3.9}$$

Finally, the normalising constant, δ, for the kernel in Equation (3.7) is given by:

$$\begin{aligned} \int_{\Re} k(z; \alpha, \beta, \gamma) dz &= 2 \int_0^{\infty} k(z; \alpha, \beta, \gamma) dz \\ &= 2 \left(\int_0^{z_c} \mathrm{e}^{-s^{\alpha}} ds + \int_{z_c}^{\infty} \mathrm{e}^{-\gamma t^{\beta} + (\gamma z_c^{\beta} - z_c^{\alpha})} dt \right). \end{aligned} \tag{3.10}$$

Let $l = s^{\alpha}$ and $m = \gamma t^{\beta}$, which implies $s = l^{\frac{1}{\alpha}}$ and $t = \gamma^{-\frac{1}{\beta}} m^{\frac{1}{\beta}}$. Therefore:

$$\begin{aligned}
\int_{\Re} k(z;\alpha,\beta,\gamma)dz &= 2\left(\int_0^{z_c^\alpha} \mathrm{e}^{-l}\frac{1}{\alpha}l^{\frac{1}{\alpha}-1}dl + \mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}\int_{\gamma z_c^\beta}^{\infty} \mathrm{e}^{-m}\frac{\gamma^{-\frac{1}{\beta}}}{\beta}m^{\frac{1}{\beta}-1}dm\right) \\
&= 2\left(\frac{1}{\alpha}\int_0^{z_c^\alpha} l^{\frac{1}{\alpha}-1}\mathrm{e}^{-l}dl + \frac{\mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}}{\gamma^{\frac{1}{\beta}}\beta}\int_{\gamma z_c^\beta}^{\infty} m^{\frac{1}{\beta}-1}\mathrm{e}^{-m}dm\right) \\
&= 2\left(\frac{1}{\alpha}\Gamma_l\left(\frac{1}{\alpha}, z_c^\alpha\right) + \frac{\mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}}{\gamma^{\frac{1}{\beta}}\beta}\Gamma_u\left(\frac{1}{\beta}, \gamma z_c^\beta\right)\right) \\
&= \delta^{-1}, \qquad (3.11)
\end{aligned}$$

where $\Gamma_l(\cdot,\cdot)$ and $\Gamma_u(\cdot,\cdot)$ are the lower and upper incomplete gamma functions, see Equation (B.2).

3.3 Spliced generalised normal probability density function

The spliced generalised normal PDF is finalised using Equations (3.7, 3.8, 3.9, 3.10). The SGN PDF is therefore given by:

$$f(z;\alpha,\beta,\gamma) = \begin{cases} \delta \mathrm{e}^{-|z|^\alpha} & |z| < z_c \\ \delta \mathrm{e}^{-\gamma|z|^\beta + c} & z_c \leq |z| \end{cases}, \qquad (3.12)$$

where $\alpha,\beta,\gamma > 0$, $c = \gamma z_c^\beta - z_c^\alpha$, $z_c = \mathrm{e}^{\frac{-\ln(\alpha)+\ln(\beta)+\ln(\gamma)}{\alpha-\beta}}$, and

$$\delta^{-1} = 2\left(\frac{1}{\alpha}\Gamma_l\left(\frac{1}{\alpha}, z_c^\alpha\right) + \frac{\mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}}{\gamma^{\frac{1}{\beta}}\beta}\Gamma_u\left(\frac{1}{\beta}, \gamma z_c^\beta\right)\right).$$

From here on forward a SGN distribution and its parameters will be denoted by $SGN(\alpha,\beta,\gamma)$. A graphical representation of the SGN PDF for different values of α,β, and γ is given by Figures (3.3), (3.4), and (3.5).

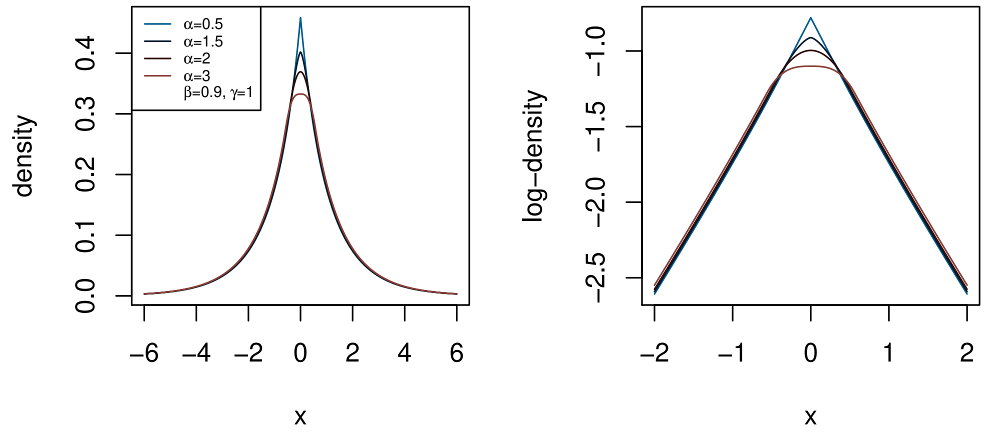

Figure 3.3: The SGN PDF for varying body parameters α. Note that for increasing values of α the body shape becomes flatter. For $\alpha = 1$ and $\alpha = 2$ the body shape resembles the Laplace and normal distribution respectively.

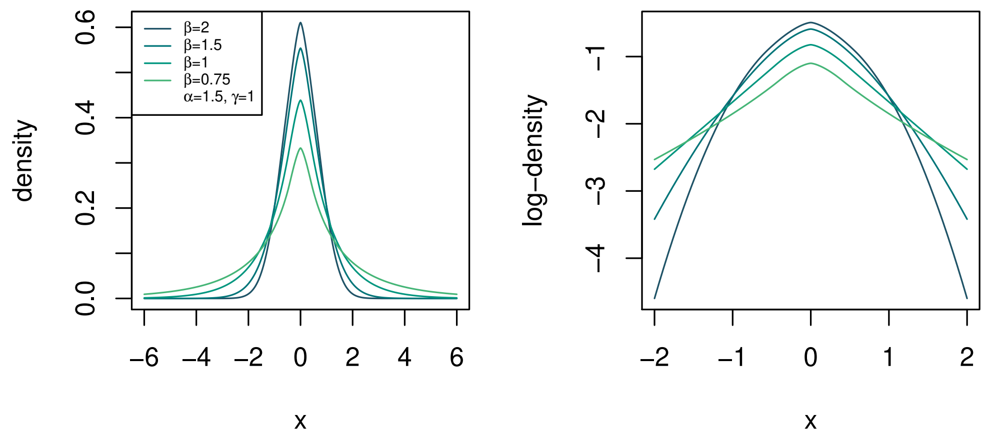

Figure 3.4: The SGN PDF for varying tail parameter β. Note that for decreasing values of β the tail becomes heavier. For $\beta = 1$ and $\beta = 2$ the tail shape resembles the Laplace and normal distribution respectively.

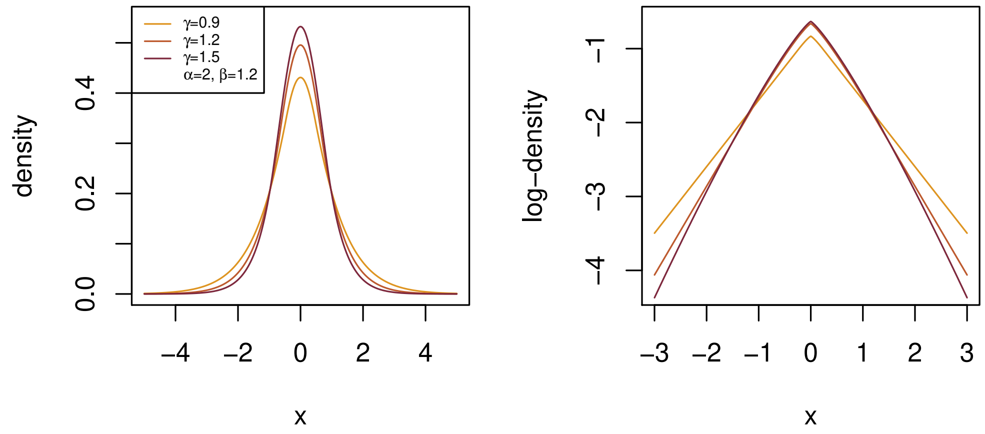

Figure 3.5: The SGN PDF for varying parameters of tail scale γ. Note that for increasing values of γ the effect of the tail shape parameter becomes less pronounced.

The non-standard SGN is simply derived by using the transformation $X = \sigma Z + \mu$, which implies $Z = \frac{X-\mu}{\sigma}$. The non-standard SGN density is therefore given by:

$$f(x;\mu,\sigma,\alpha,\beta,\gamma) = \begin{cases} \frac{\delta}{\sigma}\mathrm{e}^{-|\frac{x-\mu}{\sigma}|^{\alpha}} & \left|\frac{x-\mu}{\sigma}\right| < x_c \\ \frac{\delta}{\sigma}\mathrm{e}^{-\gamma|\frac{x-\mu}{\sigma}|^{\beta}+c} & x_c \leq \left|\frac{x-\mu}{\sigma}\right| \end{cases}, \tag{3.13}$$

where $\alpha,\beta,\gamma > 0$, $c = \gamma x_c^{\beta} - x_c^{\alpha}$, and $x_c = \mathrm{e}^{\frac{-\ln(\alpha)+\ln(\beta)+\ln(\gamma)}{\alpha-\beta}}$, and

$$\delta^{-1} = 2\left(\frac{1}{\alpha}\Gamma_l\left(\frac{1}{\alpha}, z_c^{\alpha}\right) + \frac{\mathrm{e}^{\gamma z_c^{\beta} - z_c^{\alpha}}}{\gamma^{\frac{1}{\beta}}\beta}\Gamma_u\left(\frac{1}{\beta}, \gamma z_c^{\beta}\right)\right).$$

3.4 Tail-adjusted normal cumulative distribution function

From the definition of a CDF and Equation (3.12), it is clear that:

$$\Psi(z;\alpha,\beta,\gamma) = \int_{-\infty}^{z} f(t;\alpha,\beta,\gamma)dt. \tag{3.14}$$

For $z \leq -z_c$:

$$
\begin{aligned}
\Psi(z;\alpha,\beta,\gamma) &= \int_{-\infty}^{z} \delta \mathrm{e}^{-\gamma|t|^{\beta}+c} dt \\
&= \delta \mathrm{e}^{\gamma z_c^{\beta} - z_c^{\alpha}} \frac{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})}{\beta} \int_{-\infty}^{z} \frac{\beta}{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})} \mathrm{e}^{-\left|\frac{t}{\gamma^{-\frac{1}{\beta}}}\right|^{\beta}} dt \\
&= \delta \mathrm{e}^{\gamma z_c^{\beta} - z_c^{\alpha}} \frac{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})}{\beta} \Phi\left(z; 0, \gamma^{-\frac{1}{\beta}}, \beta\right)
\end{aligned}
\tag{3.15}
$$

For $-z_c < z \leq 0$:

$$
\begin{aligned}
\Psi(z;\alpha,\beta,\gamma) &= \int_{-\infty}^{-z_c} \delta \mathrm{e}^{-\gamma|t|^{\beta}+c} dt + \int_{-z_c}^{z} \delta \mathrm{e}^{-|t|^{\alpha}} dt \\
&= \Psi(-z_c;\alpha,\beta,\gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \int_{-z_c}^{z} \frac{\alpha}{2\Gamma(\frac{1}{\alpha})} \mathrm{e}^{-|t|^{\alpha}} dt \\
&= \Psi(-z_c;\alpha,\beta,\gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \left(\Phi(z;0,1,\alpha) - \Phi(-z_c;0,1,\alpha)\right)
\end{aligned}
\tag{3.16}
$$

Since the SGN PDF is symmetric it follows that,

For $0 < z$:

$$\Psi(z;\alpha,\beta,\gamma) = 1 - \Psi(-z;\alpha,\beta,\gamma), \tag{3.17}$$

where $\alpha, \beta, \gamma > 0$, $z_c = \mathrm{e}^{-\frac{-\ln(\alpha)+\ln(\beta)+\ln(\gamma)}{\alpha-\beta}}$, δ is the normalising constant, and $\Phi(\cdot)$ is the CDF of a GN distribution.

The CDF of a non-standard SGN distribution, $X = \mu + \sigma Z$, $Z \sim SGN(\alpha, \beta, \gamma)$, is derived with the standard SGN CDF below:

$$
\begin{aligned}
P(X \leq x) &= P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \\
\therefore \Psi_X(x;\alpha,\beta,\gamma) &= \Psi\left(\frac{x-\mu}{\sigma};\alpha,\beta,\gamma\right).
\end{aligned}
\tag{3.18}
$$

3.5 Spliced generalised normal quantile function

The quantile function is derived using the CDF of the SGN, the inverse function, $\Phi^{-1}(\cdot)$, of Equation 3.6, and a dummy variable y.

For the CDF domain of $z \leq -z_c$, the quantile function is derived using Equation (3.14) below:

$0 < y \leq \Psi(-z_c; \alpha, \beta, \gamma)$:

$$\begin{aligned}
y &= \delta e^{\gamma z_c^{\beta} - z_c^{\alpha}} \frac{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})}{\beta} \Phi\left(z; 0, \gamma^{-\frac{1}{\beta}}\right) \\
\frac{y}{\delta e^{\gamma z_c^{\beta} - z_c^{\alpha}}} \frac{\beta}{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})} &= \Phi\left(z; 0, \gamma^{-\frac{1}{\beta}}\right) \\
\Phi^{-1}\left(\frac{y}{\delta e^{\gamma z_c^{\beta} - z_c^{\alpha}}; 0, \gamma^{-\frac{1}{\beta}}} \frac{\beta}{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})}; 0, \gamma^{-\frac{1}{\beta}}\right) &= z \\
\therefore \Psi^{-1}(y; \alpha, \beta, \gamma) &= \Phi^{-1}\left(\frac{y}{\delta e^{\gamma z_c^{\beta} - z_c^{\alpha}}} \frac{\beta}{2\gamma^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta})}; 0, \gamma^{-\frac{1}{\beta}}\right). \qquad (3.19)
\end{aligned}$$

For the CDF domain of $-z_c < z \leq 0$, the quantile function is derived using Equation (3.15) below:

$\Psi(-z_c; \alpha, \beta, \gamma) < y \leq \frac{1}{2}$:

$$\begin{aligned}
y &= \Psi(-z_c; \alpha, \beta, \gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \left(\Phi(z; 0, 1, \alpha) - \Phi(-z_c; 0, 1, \alpha)\right) \\
\frac{\alpha}{\delta 2\Gamma(\frac{1}{\alpha})} \left(y - \Psi(-z_c; \alpha, \beta, \gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \Phi_{0,1,\alpha}(-z_c)\right) &= \Phi(z; 0, 1, \alpha) \\
\Phi^{-1}\left(\frac{\alpha}{\delta 2\Gamma(\frac{1}{\alpha})} \left(y - \Psi(-z_c; \alpha, \beta, \gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \Phi_{0,1,\alpha}(-z_c)\right); 0, \gamma^{-\frac{1}{\beta}}, \beta\right) &= z
\end{aligned}$$

$$\therefore \Psi^{-1}(y; \alpha, \beta, \gamma) = \Phi^{-1}\left(\frac{\alpha}{\delta 2\Gamma(\frac{1}{\alpha})} \left(y - \Psi(-z_c; \alpha, \beta, \gamma) + \delta \frac{2\Gamma(\frac{1}{\alpha})}{\alpha} \Phi_{0,1,\alpha}(-z_c)\right); 0, \gamma^{-\frac{1}{\beta}}, \beta\right). \qquad (3.20)$$

For the CDF domain of $0 < z$, the quantile function is derived using Equation (3.16) below:

$\frac{1}{2} < y < 1:$

$$
\begin{aligned}
y &= 1 - \Psi(-z;\alpha,\beta,\gamma)\\
1 - y &= \Psi(-z;\alpha,\beta,\gamma)\\
\therefore \Psi^{-1}(y;\alpha,\beta,\gamma) &= -\Psi^{-1}(1-y;\alpha,\beta,\gamma). \qquad (3.21)
\end{aligned}
$$

The quantile function of a non-standard SGN distribution, $X = \mu + \sigma Z$, $Z \sim SGN(\alpha,\beta,\gamma)$, is derived with the standard SGN quantile function below:

$$
\begin{aligned}
y &= \Psi\left(\frac{x-\mu}{\sigma};\alpha,\beta,\gamma\right)\\
\Psi^{-1}(y;\alpha,\beta,\gamma) &= \frac{x-\mu}{\sigma}\\
\sigma\Psi^{-1}(y;\alpha,\beta,\gamma) + \mu &= x\\
\therefore \Psi_X^{-1}(y;\mu,\sigma,\alpha,\beta,\gamma) &= \sigma\Psi^{-1}(y;\alpha,\beta,\gamma) + \mu. \qquad (3.22)
\end{aligned}
$$

3.6 Spliced generalised normal moments

For even or absolute moments r, from the definition of moments:

$$
\begin{aligned}
E(|Z|^r) &= \int_{\Re} z^r f(z;\alpha,\beta,\gamma)dz\\
&= 2\int_0^{\infty} z^r f(z;\alpha,\beta,\gamma)dz \qquad (3.23)\\
&= 2\left(\int_0^{z_c} \delta s^r \mathrm{e}^{-|s|^{\alpha}}ds + \int_{z_c}^{\infty} \delta t^r \mathrm{e}^{-\gamma|t|^{\beta}+(\gamma z_c^{\beta}-z_c^{\alpha})}dt\right)\\
&= 2\left(\delta\int_0^{z_c} s^r \mathrm{e}^{-s^{\alpha}}ds + \delta \mathrm{e}^{\gamma z_c^{\beta}-z_c^{\alpha}}\int_{z_c}^{\infty} t^r \mathrm{e}^{-\left(\frac{t}{\gamma}\right)^{\beta}}dt\right). \qquad (3.24)
\end{aligned}
$$

Let $l = s^{\alpha}$ and $m = \gamma t^{\beta}$, which implies $s = l^{\frac{1}{\alpha}}$ and $t = \gamma^{-\frac{1}{\beta}} m^{\frac{1}{\beta}}$.

$$
\begin{aligned}
E(|Z|^r) &= 2\left(\delta\int_0^{z_c^\alpha} l^{\frac{r}{\alpha}} \mathrm{e}^{-l}\frac{1}{\alpha}l^{\frac{1}{\alpha}-1}dl + \delta \mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}\int_{\gamma z_c^\beta}^{\infty}\left(\gamma^{-\frac{1}{\beta}}m^{\frac{1}{\beta}}\right)^r \mathrm{e}^{-m}\frac{\gamma^{-\frac{1}{\beta}}}{\beta}m^{\frac{1}{\beta}-1}dm\right)\\
&= 2\left(\frac{\delta}{\alpha}\int_0^{z_c^\alpha} l^{\frac{r+1}{\alpha}-1}\mathrm{e}^{-l}dl + \frac{\delta \mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}}{\gamma^{r+1}\beta}\int_{\gamma z_c^\beta}^{\infty} m^{\frac{r+1}{\beta}-1}\mathrm{e}^{-m}dm\right)\\
&= 2\delta\left(\frac{1}{\alpha}\Gamma_l\left(\frac{r+1}{\alpha}, z_c^\alpha\right) + \frac{\mathrm{e}^{\gamma z_c^\beta - z_c^\alpha}}{\gamma^{r+1}\beta}\Gamma_u\left(\frac{r+1}{\beta}, \gamma z_c^\beta\right)\right),
\end{aligned}
\tag{3.25}
$$

where $\Gamma_l(\cdot,\cdot)$ and $\Gamma_u(\cdot,\cdot)$ are the lower and upper incomplete gamma functions.

For odd moments of r, from the definition of moments:

$$
\begin{aligned}
E(Z^r) &= \int_{\Re} z^r f(z;\alpha,\beta,\gamma)dz\\
&= \int_0^\infty z^r f(z;\alpha,\beta,\gamma)dz + \int_{-\infty}^0 z^r f(z;\alpha,\beta,\gamma)dz\\
&= \int_0^\infty z^r f(z;\alpha,\beta,\gamma)dz - \int_0^\infty z^r f(z;\alpha,\beta,\gamma)dz\\
&= \left(\int_0^{z_c}\delta s^r \mathrm{e}^{-|s|^\alpha}ds + \int_{z_c}^\infty \delta t^r \mathrm{e}^{-\gamma|t|^\beta + (\gamma z_c^\beta - z_c^\alpha)}dt\right)\\
&\quad - \left(\int_0^{z_c}\delta s^r \mathrm{e}^{-|s|^\alpha}ds + \int_{z_c}^\infty \delta t^r \mathrm{e}^{-\gamma|t|^\beta + (\gamma z_c^\beta - z_c^\alpha)}dt\right)\\
&= 0.
\end{aligned}
\tag{3.26}
$$

3.7 Summary

In this chapter the origins of the SGN is given in Sections 3.1 and 3.2. The following SGN statistical properties were derived in each Section: 3.3 the PDF, 3.4 the CDF, 3.5 the quantile function, and in 3.6 the moments. The SGN model is a general model for which there are a plethora of sub-models. In the next chapter the TAN distribution is motivated as a normal-like distribution which can achieve light and heavy tails in data modelling.

Chapter 4

Tail-adjusted normal distribution

In this chapter the TAN is motivated as a special case of the SGN. The TAN is chosen is such a way that an additional parameter specifically controls the tail shape. Various statistical properties of the SGN distribution is derived, such as the PDF, CDF, quantile function and moments.

4.1 A special case of spliced generalised normal behaviour

It is clear that the non-standard SGN PDF in Equation (3.13) tends to the standard normal PDF if $\mu = 0, \sigma = \sqrt{2}, \alpha = 2$ and $\beta \to 2$ irrespective of γ. A normal-like distribution can thus be obtained by choosing the body parameter $\alpha = 2$. To study the effect of the tail parameters a special case of the non-standard SGN is studied where $\mu = 0, \sigma = \sqrt{2}, \alpha = 2$. The PDF for this special case is plotted in Figure (4.1). Notice that for both β and γ the kurtosis of this special case increases for decreasing values relative to the normal distribution. Importantly, it is possible to have lighter than normal tails when $\beta > 2$ which cannot be achieved by varying γ. To reduce the parameters of estimation either β or γ can be fixed. Since it is more desirable to have the possibilities of light and heavy tails, the value of γ is fixed instead of fixing the value β. A natural intuition and matter of convenience would be to fix $\gamma = 1$. The standard tail-adjusted normal distribution then follows a standard $SGN(2, \nu, 1)$ distribution. Empirically a related distribution, the two-piece tail-adjusted normal, has shown that this condition is useful in applications to real data, see Chapter 6. It remains for future work outside the scope of this study to verify whether this condition poses any problems.

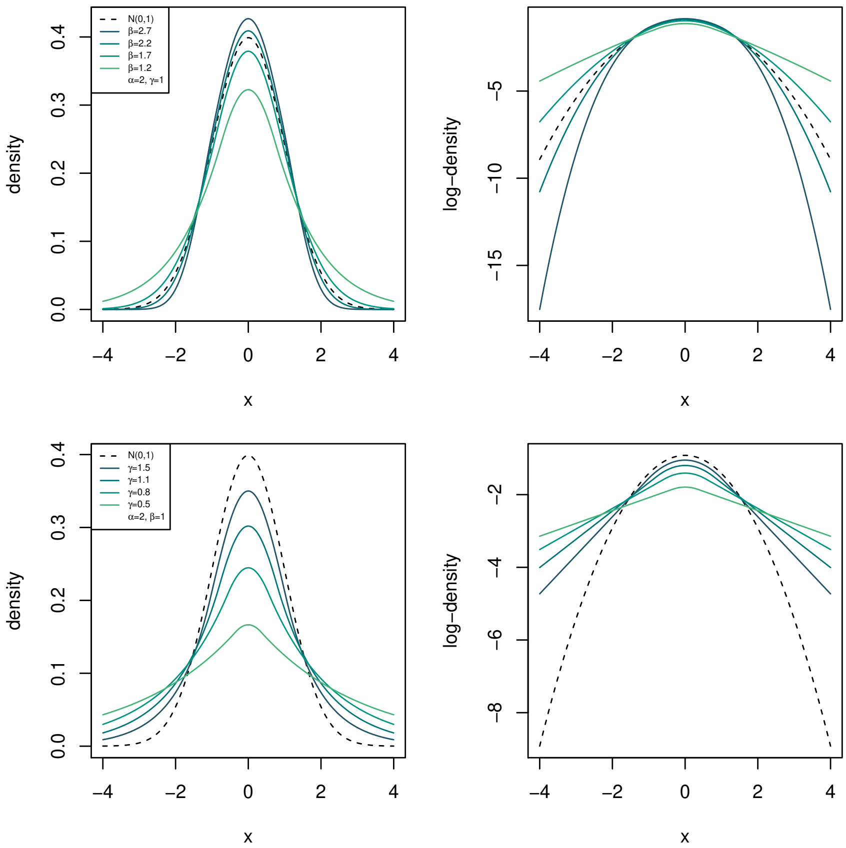

Figure 4.1: A special case of the non-standard SGN distribution that is comparable to the standard normal distribution. The normal-like shape is obtainded by a body parameter $\alpha = 2$ and a scale parameter of $\sigma = \sqrt{2}$. It is observed that varying the value of β can achieve heavy and light tails compared to the normal distribution. The γ can only achieve an increased or decreased effect of β parameter. In this case, only more or less kurtosis since $\beta = 1 < 2$.

4.2 Tail-adjusted normal probability density function

The standard TAN follows a $SGN(2, \nu, 1)$ distribution. The PDF of the standard TAN is then given by substitution of $\alpha = 1, \beta = \nu$, and $\gamma = 1$ into Equation (3.12):

$$f(z;\nu) = \begin{cases} \delta e^{-|z|^2} & |z| < z_c \\ \delta e^{-|z|^\nu + c} & z_c \leq |z| \end{cases}, \tag{4.1}$$

where ν, $c = z_c^\nu - z_c^2$, and $z_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and

$$\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, z_c^\nu\right)\right).$$

The non-standard TAN is obtained by $X = \mu + \sigma Z$, where $Z \sim TAN(\nu)$. The non-standard TAN PDF is then obtained from Equation (4.1):

$$f(x;\mu,\sigma,2,\nu) = \begin{cases} \frac{\delta}{\sigma} e^{-|\frac{x-\mu}{\sigma}|^2} & \left|\frac{x-\mu}{\sigma}\right| < x_c \\ \frac{\delta}{\sigma} e^{-|\frac{x-\mu}{\sigma}|^\nu + c} & x_c \leq \left|\frac{x-\mu}{\sigma}\right| \end{cases}, \tag{4.2}$$

where $c = x_c^\nu - x_c^2$, and $x_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and

$$\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, z_c^\nu\right)\right).$$

From here on forward a standard or non-standard TAN distribution and its parameters will be denoted by $TAN(\nu)$ and $TAN(\mu, \sigma, \nu)$.

4.3 Tail-adjusted normal distribution density function

The standard TAN follows a $SGN(2, \nu, 1)$ distribution. The CDF of the standard TAN is then given by substitution of $\alpha = 1, \beta = \nu$, and $\gamma = 1$ into Equations (3.15, 3.16, 3.17).

For $z \leq -z_c$:

$$F(z;\nu) = \delta e^{z_c^\nu - z_c^2} \frac{2\Gamma(\frac{1}{\nu})}{\nu} \Phi(z; 0, \nu) \tag{4.3}$$

For $-z_c < z \leq 0$:

$$F(z;\nu) = F(-z_c;\nu) + \delta\frac{2\Gamma(\frac{1}{2})}{2}\left(\Phi(z;0,1,2) - \Phi(-z_c;0,1,2)\right) \tag{4.4}$$

For $0 < z$:

$$F(z;\nu) = 1 - F(-z;\nu), \tag{4.5}$$

where $\nu > 0$, $z_c = \mathrm{e}^{-\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, δ is the normalising constant, and $\Phi(\cdot)$ is the CDF of a GN distribution.

The CDF of a non-standard TAN distribution, $X = \mu + \sigma Z$, $Z \sim TAN(\nu)$, is derived with the standard TAN CDF function below:

$$\begin{aligned} P(X \leq x) &= P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \\ \therefore F_X(x;\nu) &= F\left(\frac{x-\mu}{\sigma};\nu\right). \end{aligned} \tag{4.6}$$

4.4 Tail-adjusted normal quantile function

The standard TAN follows a $SGN(2,\nu,1)$ distribution. The quantile of the standard TAN is then given by substitution of $\alpha = 1, \beta = \nu$, and $\gamma = 1$ into Equations (3.5, 3.5, 3.21).

For the CDF domain of $z \leq -z_c$, the quantile function is given below:

$0 < y \leq F(-z_c;\nu)$:

$$F^{-1}(y;\nu) = \Phi^{-1}\left(\frac{y}{\delta \mathrm{e}^{z_c^{\nu} - z_c^2}}\frac{\nu}{2\Gamma(\frac{1}{\nu})};0,1,\nu\right). \tag{4.7}$$

For the CDF domain of $-z_c < z \leq 0$, the quantile function is given below:

$F(-z_c;\nu) < y \leq \frac{1}{2}$:

$$F^{-1}(y;\nu) = \Phi^{-1}\left(\frac{2}{\delta 2\Gamma(\frac{1}{2})}\left(y - F(-z_c;\nu,1) + \delta\frac{2\Gamma(\frac{1}{2})}{2}\Phi_{0,1,2}(-z_c)\right);0,1,\nu\right). \tag{4.8}$$

For the CDF domain of $0 < z$, the quantile function is given below:

$$F^{-1}(y;\nu) = -F^{-1}(1-y;\nu). \tag{4.9}$$

The quantile function of a non-standard TAN distribution, $X = \mu + \sigma Z$, $Z \sim TAN(\nu)$, is derived with the standard TAN quantile function below:

$$\begin{aligned} y &= F\left(\frac{x-\mu}{\sigma};\nu\right) \\ F^{-1}(y;\nu) &= \frac{x-\mu}{\sigma} \\ \sigma F^{-1}(y;\nu) + \mu &= x \\ \therefore F_X^{-1}(y;\mu,\sigma,\nu) &= \sigma F^{-1}(y;\nu) + \mu. \end{aligned} \tag{4.10}$$

4.5 Tail-adjusted normal moments

The standard TAN follows a $SGN(2,\nu,1)$ distribution. The moments of the standard TAN is then given by substitution of $\alpha = 1, \beta = \nu$, and $\gamma = 1$ into Equations (3.25, 3.26) the standard TAN absolute and normal moments are given below:

$$E(|Z|^r) = 2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{r+1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{r+1}{\nu}, z_c^\nu\right)\right), \tag{4.11}$$

$$E(Z^r) = \delta(1+(-1)^r)\left(\frac{1}{2}\Gamma_l\left(\frac{r+1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{r+1}{\nu}, z_c^\nu\right)\right), \tag{4.12}$$

where $\nu > 0$, $\nu \neq 2$, $z_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, $\Gamma_l(\cdot,\cdot)$ and $\Gamma_u(\cdot,\cdot)$ are the lower and upper incomplete gamma functions, and $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, z_c^\nu\right)\right)$.

The moments of a non-standard TAN distribution, $X = \mu + \sigma Z$, where $Z \sim TAN(\nu)$, is derived is derived with the standard TAN moments below:

$$\begin{aligned} E(X^r) &= E\left((\mu+\sigma Z)^r\right) \\ &= \sum_{i=0}^{r} \binom{r}{i} \mu^{r-i}\sigma^i E\left(Z^i\right) \\ &= \mu^r \sum_{i=0}^{r} \binom{r}{i} \left(\frac{\sigma}{\mu}\right)^i E\left(Z^i\right), \end{aligned} \tag{4.13}$$

where $Z \sim TAN(\nu)$. The even and odd r'th moments for the non-standard TAN distribution follows from Equations (4.13, 4.12):

$$\begin{aligned} E(X^r) = \mu^r \sum_{i=0}^{r} \binom{r}{i} \left(\frac{\sigma}{\mu}\right)^i \cdot \delta(1+(-1)^r) \\ \cdot \left(\frac{1}{2}\Gamma_l\left(\frac{i+1}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{i+1}{\nu}, z_c\right)\right). \end{aligned} \tag{4.14}$$

The first four moments of the non-standard TAN distribution is given below:

$$E(X) = \mu(1+0) = \mu, \tag{4.15}$$

$$\begin{aligned} E(X^2) &= \mu^2\left(1+0+2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right)\right) \\ &= \left(1+2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right)\right), \end{aligned} \tag{4.16}$$

$$\begin{aligned} E(X^3) &= \mu^3\left(1+0+2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right)+0\right) \\ &= \mu^3\left(1+2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right)\right), \end{aligned} \tag{4.17}$$

and

$$
\begin{aligned}
E(X^4) = \mu^4 &\left(1 + 0 + 2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right) + 0 + \right. \\
&\left. 2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{5}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{5}{\nu}, z_c\right)\right)\right) \\
= \mu^4 &\left(1 + 2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right) + \right. \\
&\left. 2\delta\left(\frac{1}{2}\Gamma_l\left(\frac{5}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{5}{\nu}, z_c\right)\right)\right). \qquad (4.18)
\end{aligned}
$$

4.6 Summary

In this chapter the TAN was motivated in Section 4.1. The following TAN statistical properties were derived in each Section: 4.2 the PDF, 4.3 the CDF, 4.4 the quantile function, and in 4.5 the moments. The TAN distribution can model different tail shapes for symmetric data. In the next chapter the TAN distribution is skewed with two-piece scaling so that different combinations of skew and symmetric data can be modelled.

Chapter 5

Two-piece tail-adjusted normal distribution

In this chapter the a skew version of the TAN is introduced, the TPTAN. The TPTAN therefore has a specific parameter for skewness. Various statistical properties of the TPTAN distribution is derived, such as the PDF, CDF, quantile function, moments, log-likelihood (LL), and Fischer information.

5.1 Two-piece tail-adjusted normal probability density function

The two-piece scaling method of obtaining a skew distributions is a common practice in flexible modelling. This method enjoys strong parameter orthogonality [23][38] and provides an *'easy and clean setup'* for modelling skew data [37]. From Equation (4.2) and [23], the standard two-piece tail-adjusted normal (TPTAN) density is given below:

$$f(z;\psi,\nu) = \begin{cases} \frac{2}{\psi+\frac{1}{\psi}} f\left(\psi z;\nu\right) & z \leq 0 \\ \frac{2}{\psi+\frac{1}{\psi}} f\left(\frac{z}{\psi};\nu\right) & 0 < z \end{cases}$$

$$f(z;\psi,\nu) = \begin{cases} \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-|\psi z|^2} & |\psi z| < z_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-|\psi z|^\nu + c} & z_c \le |\psi z| \end{cases} & z \le 0 \\ \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-|\frac{z}{\psi}|^2} & |\frac{z}{\psi}| < z_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-|\frac{z}{\psi}|^\nu + c} & z_c \le |\frac{z}{\psi}| \end{cases} & 0 < z \end{cases}$$

$$= \begin{cases} \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-(-\psi z)^2} & -z_c < \psi z < z_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-(-\psi z)^\nu + c} & \psi z \le -z_c \end{cases} & z \le 0 \\ \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{z}{\psi}\right)^2} & -z_c < \frac{z}{\psi} < z_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{z}{\psi}\right)^\nu + c} & z_c \le \frac{z}{\psi} z \end{cases} & 0 < z \end{cases}$$

$$= \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-(-\psi z)^\nu + c} & z \le -\frac{z_c}{\psi} \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-(\psi z)^2} & -\frac{z_c}{\psi} < z \le 0 \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{z}{\psi}\right)^2} & 0 < z < \psi z_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{z}{\psi}\right)^\nu + c} & \psi z_c \le z \end{cases} \tag{5.1}$$

where $\nu > 0$, $c = z_c^\nu - z_c^2$, and $z_c = \mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and

$$\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, z_c^\nu\right)\right).$$

The non-standard TPTAN is obtained by $X = \mu + \sigma Z$, where $Z \sim TPTAN(\nu)$. The non-standard TPTAN PDF is then obtained from Equation (5.1):

$$f(x;\mu,\sigma,\psi,\nu) = \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu + c} & \frac{x-\mu}{\sigma} \le -\frac{x_c}{\psi} \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{x-\mu}{\sigma/\psi}\right)^2} & -\frac{x_c}{\psi} < \frac{x-\mu}{\sigma} \le 0 \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{x-\mu}{\sigma\psi}\right)^2} & 0 < \frac{x-\mu}{\sigma} < \psi x_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} \mathrm{e}^{-\left(\frac{x-\mu}{\sigma\psi}\right)^\nu + c} & \psi x_c \le \frac{x-\mu}{\sigma} \end{cases} \tag{5.2}$$

$$f(x;\mu,\sigma,\psi,\nu) = \begin{cases} \frac{2\delta}{\psi+\frac{1}{\psi}} e^{-\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu}+c} & x \leq \mu - \frac{x_c}{\psi/\sigma} \\ \frac{2\delta}{\psi+\frac{1}{\psi}} e^{-\left(\frac{x-\mu}{\sigma/\psi}\right)^{2}} & \mu - \frac{x_c}{\psi/\sigma} < x \leq \mu \\ \frac{2\delta}{\psi+\frac{1}{\psi}} e^{-\left(\frac{x-\mu}{\sigma\psi}\right)^{2}} & \mu < x < \mu + \sigma\psi x_c \\ \frac{2\delta}{\psi+\frac{1}{\psi}} e^{-\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}+c} & \mu + \sigma\psi x_c \leq x \end{cases} \tag{5.3}$$

where $\sigma, \psi, \nu > 0$, $c = x_c^{\nu} - x_c^2$, $x_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and

$$\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, x_c\right) + \frac{e^{x_c^{\nu}-x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, x_c\right)\right).$$

From here on forward a standard or non-standard TPTAN distribution and its parameters will be denoted by $TPTAN(\psi,\nu)$ and $TPTAN(\mu,\sigma,\psi,\nu)$.

The function of the additional skewing parameter, ψ, in TPTAN PDF is shown in Figure (5.1). The value of ψ has the following interpretation: if $\psi < 1$ the PDF is skewed to the left, if $\psi = 1$ the PDF is symmetric, and if $\psi > 1$ the PDF is skewed to the right.

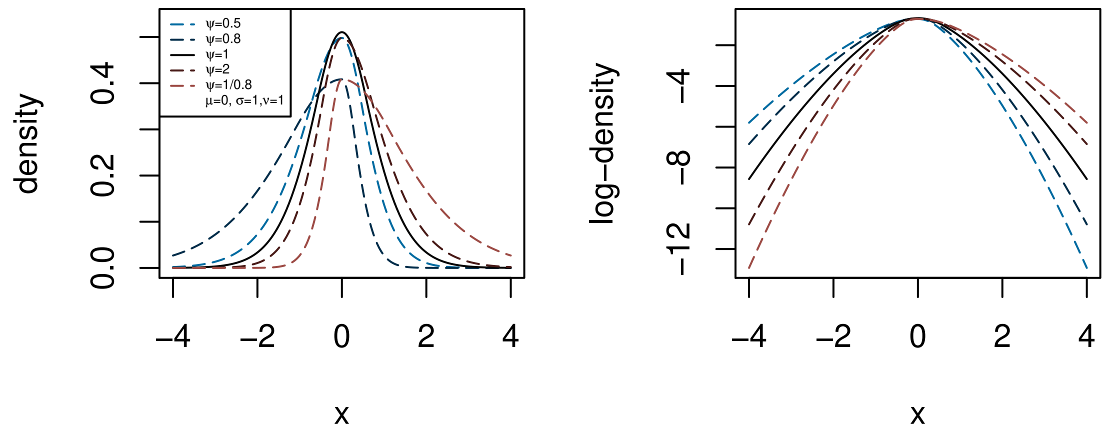

Figure 5.1: The PDF of the TPTAN distribution and the effect of the skewing parameter ψ. The TPTAN PDF is more skewed to the right for larger values of ψ and symmetric for $\psi = 1$.

5.2 Two-piece tail-adjusted normal cumulative distribution function

The standard TPTAN PDF is constructed from the TAN PDF, as such the standard TPTAN CDF is a function of the TAN CDF. The TPTAN CDF is derived with the definition of a CDF, Equation (5.1), and the transformations $l = \psi s$ and $m = t/\psi$:

For $z \leq 0$:

$$\begin{aligned}
F(z;\psi,\nu) &= \int_{-\infty}^{z} f(s;\psi,\nu)ds \\
&= \frac{2}{\psi+\frac{1}{\psi}} \int_{-\infty}^{z} f^{*}(\psi s;\nu)\,ds \\
&= \frac{2}{\psi+\frac{1}{\psi}} \cdot \frac{1}{\psi} \int_{-\infty}^{z} f^{*}(l;\nu)\,dl \\
&= \frac{2/\psi}{\psi+\frac{1}{\psi}} F^{*}(z;\nu)
\end{aligned} \tag{5.4}$$

For $0 < z$:

$$\begin{aligned}
F(z;\psi,\nu) &= \int_{-\infty}^{z} f(t;\psi,\nu)dt \\
&= \int_{-\infty}^{0} f(t;\nu)\,dt + \frac{2}{\psi+\frac{1}{\psi}} \int_{-\infty}^{z} f^{*}(t/\psi;\nu)\,dt \\
&= F(0;\psi,\nu) + \frac{2}{\psi+\frac{1}{\psi}} \cdot \psi \int_{-\infty}^{z} f^{*}(m;\nu)\,dm \\
&= F(0;\psi,\nu) + \frac{2\psi}{\psi+\frac{1}{\psi}} F^{*}(z;\nu),
\end{aligned} \tag{5.5}$$

where $f^{*}(\cdot)$ and $F^{*}(\cdot)$ are the TAN PDF and CDF functions, see Equations (4.1, 4.3-4.5).

The CDF of a non-standard TPTAN distribution, $X = \mu + \sigma Z$, $Z \sim TPTAN(\nu)$, is derived with the standard TAN CDF function below:

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$
$$\therefore F_X(x;\psi,\nu) = F\left(\frac{x-\mu}{\sigma};\psi,\nu\right). \tag{5.6}$$

5.3 Two-piece tail-adjusted normal quantile function

The standard TPTAN CDF is a function of the TAN CDF, as such the standard TPTAN quantile function is a function of the TAN quantile function.

For the CDF domain of $z \leq 0$, the corresponding quantile domain is $0 < y \leq F(0;\psi,\nu)$. The quantile function is then given by:

For: $0 < y \leq F(0;\psi,\nu)$

$$y = \frac{2/\psi}{\psi+\frac{1}{\psi}}F^*(z;\nu)$$
$$\therefore F^{-1}(y;\psi,\nu) = \frac{\psi+1/\psi}{2/\psi}F^{*^{-1}}(y;\nu), \tag{5.7}$$

where $F^{*^{-1}}$ is the quantile function of the TAN, see Equations (4.7-4.9).

For the CDF domain of $0 < z$, the corresponding quantile domain is $F(0;\psi,\nu) < y < 1$. The quantile function is then given by:

$$y = F(0;\psi,\nu) + \frac{2\psi}{\psi+\frac{1}{\psi}}F^*(z;\nu)$$
$$\frac{\psi+\frac{1}{\psi}}{2\psi}(y - F(0;\psi,\nu)) = F^*(z;\nu)$$
$$\therefore F^{-1}(y;\psi,\nu) = F^{*^{-1}}\left(\frac{\psi+\frac{1}{\psi}}{2\psi}(y - F(0;\psi,\nu))\right) \tag{5.8}$$

The quantile function of a non-standard TAN distribution, $X = \mu + \sigma Z$, $Z \sim$

$TAN(\nu)$, is derived with the standard TAN quantile function below:

$$\begin{aligned} y &= F\left(\frac{x-\mu}{\sigma}; \psi, \nu)\right) \\ F^{-1}(y; \psi, \nu)) &= \frac{x-\mu}{\sigma} \\ \sigma F^{-1}(y; \psi, \nu)) + \mu &= x \\ \therefore F_X^{-1}(y; \mu, \sigma, \nu) &= \sigma F^{-1}(y; \psi, \nu) + \mu. \end{aligned} \tag{5.9}$$

5.4 Two-piece tail-adjusted normal moments

The moments of the standard TPTAN distribution follow the form in Equation (B.2), since the baseline standard TAN distribution is symmetric around zero. Therefore it is given that:

$$E(Z_{TP}^r) = \frac{\psi^{r+1} + (-1)^r \psi^{-(r+1)}}{\psi + \frac{1}{\psi}} E(|Z|^r), \tag{5.10}$$

where $Z \sim TAN(\nu)$.

The moments of the standard TPTAN distribution is derived by substitution of Equation (5.10) into Equation (4.11):

$$E(Z^r) = \frac{\psi^{r+1} + (-1)^r \psi^{-(r+1)}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{r+1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{r+1}{\nu}, z_c^\nu\right)\right), \tag{5.11}$$

where $\nu > 0$, $\nu \neq 2$, $z_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, $\Gamma_l(\cdot,\cdot)$ and $\Gamma_u(\cdot,\cdot)$ are the lower and upper incomplete gamma functions, and $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, z_c^\nu\right)\right)$.

The moments of a non-standard TPTAN distribution $X = \mu + \sigma Z$, where $Z \sim TPTAN(\psi, \nu)$, is derived below:

$$\begin{aligned} E(X^r) &= E\left((\mu + \sigma Z)^r\right) \\ &= \sum_{i=0}^{r} \binom{r}{i} \mu^{r-i} \sigma^i E\left(Z^i\right) \\ &= \mu^r \sum_{i=0}^{r} \binom{r}{i} \left(\frac{\sigma}{\mu}\right)^i E\left(Z^i\right) \end{aligned} \tag{5.12}$$

The distribution of Z is a standard TPTAN distribution. The even and odd moments r'th moments then follow from Equations (5.12,5.11):

$$E(X^r) = \mu^r \sum_{i=0}^{r} \binom{r}{i} \left(\frac{\sigma}{\mu}\right)^i \cdot \frac{\psi^{r+1} + (-1)^r \psi^{-(r+1)}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(\frac{r+1}{2}, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{r+1}{\nu}, z_c^\nu\right)\right). \tag{5.13}$$

The first four moments of the non-standard TAN distribution is given below:

$$\begin{aligned} E(X) &= \mu\left(\frac{\psi^1 - \psi^{-1}}{\psi + \frac{1}{\psi}} + \frac{\psi^2 - \psi^{-2}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(1, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{2}{\nu}, z_c^\nu\right)\right)\right) \\ &= \mu\left(1 + \frac{\psi^2 - \psi^{-2}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(1, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{2}{\nu}, z_c^\nu\right)\right)\right) \end{aligned} \tag{5.14}$$

$$\begin{aligned} E(X^2) = \mu^2 \Bigg(1 &+ \frac{\psi^2 - \psi^{-2}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(1, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{2}{\nu}, z_c^\nu\right)\right) + \\ &\frac{\psi^3 - \psi^{-3}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right)\Bigg), \end{aligned} \tag{5.15}$$

$$\begin{aligned} E(X^3) = \mu^3 \Bigg(1 &+ \frac{\psi^2 - \psi^{-2}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(1, z_c^2\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{2}{\nu}, z_c^\nu\right)\right) + \\ &\frac{\psi^3 - \psi^{-3}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right) + \\ &\frac{\psi^4 - \psi^{-4}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{4}{2}, z_c\right) + \frac{\mathrm{e}^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{4}{\nu}, z_c\right)\right)\Bigg), \end{aligned} \tag{5.16}$$

and

$$
\begin{aligned}
E(X^4) = \mu^4 \Bigg(1 + \frac{\psi^2 - \psi^{-2}}{\psi + \frac{1}{\psi}} \cdot 2\delta \cdot \left(\frac{1}{2}\Gamma_l\left(1, z_c^2\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{2}{\nu}, z_c^\nu\right)\right) + \\
\frac{\psi^3 - \psi^{-3}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{3}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{3}{\nu}, z_c\right)\right) + \\
\frac{\psi^4 - \psi^{-4}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(2, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{4}{\nu}, z_c\right)\right) + \\
\frac{\psi^5 - \psi^{-5}}{\psi + \frac{1}{\psi}} 2\delta \left(\frac{1}{2}\Gamma_l\left(\frac{5}{2}, z_c\right) + \frac{e^{z_c^\nu - z_c^2}}{\nu}\Gamma_u\left(\frac{5}{\nu}, z_c\right)\right)\Bigg)
\end{aligned}
\tag{5.17}
$$

5.5 Two-piece tail-adjusted normal log-likelihood function

The likelihood for a random sample of $x_1, x_2, \ldots, x_n$ independent $TPTAN(\mu, \sigma, \psi, \nu)$ distributions is given as the product of a TPTAN PDF, Equation (5.3), as given below:

$$
\begin{aligned}
L(x; \mu, \sigma, \psi, \nu) &= \prod_{k=1}^{n} f(x; \mu, \sigma, \psi, \nu) \\
&= \prod_{x \in B} \frac{2\delta}{\sigma\left(\psi + \frac{1}{\psi}\right)} e^{-\left(\frac{\mu - x}{\sigma/\psi}\right)^2} \prod_{x \in A} \frac{2\delta}{\sigma\left(\psi + \frac{1}{\psi}\right)} e^{-\left(\frac{\mu - x}{\sigma/\psi}\right)^\nu + c} \\
&\quad \prod_{x \in C} \frac{2\delta}{\sigma\left(\psi + \frac{1}{\psi}\right)} e^{-\left(\frac{x - \mu}{\sigma\psi}\right)^2} \prod_{x \in D} \frac{2\delta}{\sigma\left(\psi + \frac{1}{\psi}\right)} e^{-\left(\frac{x - \mu}{\sigma\psi}\right)^\nu + c},
\end{aligned}
\tag{5.18}
$$

where $\sigma, \nu > 0$, $c = x_c^\nu - x_c^2$, $x_c = e^{\frac{-\ln(2) + \ln(\nu)}{2-\nu}}$, $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, x_c\right) + \frac{e^{x_c^\nu - x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, x_c\right)\right)$, $A = \left\{x_i | x_i \leq \mu - \frac{x_c}{\psi/\sigma}\right\}$, $B = \left\{x_j | \mu - \frac{x_c}{\psi/\sigma} < x_j \leq \mu\right\}$, $C = \{x_k | \mu < x_k < \mu + \psi\sigma x_c\}$, and $D = \{x_l | \mu + \psi\sigma x_c \leq x_l\}$.

The LL is then given as the natural log of Equation (5.19):

$$\begin{aligned}
LL(x;\mu,\sigma,\psi,\nu) &= \ln\left(L(x;\mu,\sigma,\psi,\nu)\right) \\
&= \sum_{x\in B}\ln\left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\mathrm{e}^{-\left(\frac{\mu-x}{\sigma/\psi}\right)^2}\right) + \sum_{x\in A}\ln\left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\mathrm{e}^{-\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu + x_c^\nu - x_c^2}\right) \\
&\quad + \sum_{x\in C}\ln\left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\mathrm{e}^{-\left(\frac{x-\mu}{\sigma\psi}\right)^2}\right) + \sum_{x\in D}\ln\left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\mathrm{e}^{-\left(\frac{x-\mu}{\sigma\psi}\right)^\nu + x_c^\nu - x_c^2}\right) \\
&= n\ln\left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\right) - \sum_{x\in B}\left(\frac{\mu-x}{\sigma/\psi}\right)^2 + \sum_{x\in A}\left(-\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu + x_c^\nu - x_c^2\right) \\
&\quad - \sum_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 + \sum_{x\in D}\left(-\left(\frac{x-\mu}{\sigma\psi}\right)^\nu + x_c^\nu - x_c^2\right) \\
&= n\ln(2) + n\ln(\delta) - n\ln(\sigma) - n\ln\left(\psi+\frac{1}{\psi}\right) + \sum_{x\in A\cup D} x_c^\nu - x_c^2 \\
&\quad - \sum_{x\in B}\left(\frac{\mu-x}{\sigma/\psi}\right)^2 - \sum_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu - \sum_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 \\
&\quad - \sum_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu,
\end{aligned} \tag{5.19}$$

where $\sigma,\nu>0$, $c=x_c^\nu-x_c^2$, $x_c=\mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, $\delta^{-1}=2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2},x_c\right)+\frac{\mathrm{e}^{x_c^\nu-x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu},x_c\right)\right)$, $A=\left\{x_i|x_i\leq\mu-\frac{x_c}{\psi/\sigma}\right\}$, $B=\left\{x_j|\mu-\frac{x_c}{\psi/\sigma}<x_j\leq\mu\right\}$, $C=\{x_k|\mu<x_k<\mu+\psi\sigma x_c\}$, and $D=\{x_l|\mu+\psi\sigma x_c\leq x_l\}$.

5.6 Two-piece tail-adjusted normal Fischer information

The Fischer information is defined as the matrix with elements

$$E\left(-\frac{\partial^2 LL}{\partial\theta_i\theta_j}\right), \tag{5.20}$$

where θ_i is the i'th parameter of the distribution in question. The Fischer information matrix for the TPTAN distribution contains 10 expectations. Only one of these derivations are shown here with the others available in the Appendix C, since they all follow a similar derivation.

For this section the following is defined:

$$A = \left\{ x_i | x_i \leq \mu - \frac{x_c}{\psi/\sigma} \right\}, B = \left\{ x_j | \mu - \frac{x_c}{\psi/\sigma} < x_j \leq \mu \right\},$$

$$C = \left\{ x_k | \mu < x_k < \mu + \psi\sigma x_c \right\}, D = \left\{ x_l | \mu + \psi\sigma x_c \leq x_l, \right\}$$

and

$$\mathbb{I}_{x \in X} = \begin{cases} 0 & x \notin X \\ 1 & x \in X \end{cases}.$$

For a single observation, $n = 1$, the summations in Equation (5.19) are simply replaced with indicator variables:

$$\begin{aligned} LL(x;\mu,\sigma,\psi,\nu) &= \ln\left(L(x;\mu,\sigma,\psi,\nu)\right) \\ &= n\ln(2) + n\ln(\delta) - n\ln(\sigma) - n\ln\left(\psi + \frac{1}{\psi}\right) + \mathbb{I}_{x \in A \cup D} x_c^{\nu} - x_c^2 \\ &\quad - \mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 - \mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} - \mathbb{I}_{x \in C}\left(\frac{x - \mu}{\sigma\psi}\right)^2 \\ &\quad - \mathbb{I}_{x \in D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu}, \end{aligned} \tag{5.21}$$

where $\sigma, \nu > 0$, $c = x_c^{\nu} - x_c^2$, $x_c = \mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, x_c\right) + \frac{\mathrm{e}^{x_c^{\nu} - x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, x_c\right)\right)$.

The first-order derivative of the TPTAN LL, Equation (5.21), with respect to μ is:

$$\begin{aligned} \frac{\partial LL}{\partial \mu} &= -\frac{2}{\sigma/\psi}\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right) - \frac{\nu}{\sigma/\psi}\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu - 1} \\ &\quad + \frac{2}{\sigma\psi}\mathbb{I}_{x \in C}\left(\frac{x - \mu}{\sigma\psi}\right) + \frac{\nu}{\sigma\psi}\mathbb{I}_{x \in D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu - 1}. \end{aligned} \tag{5.22}$$

The second-order derivative of the TPTAN LL function, Equation (5.21), with respect to μ is:

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial \mu^2} = \frac{\partial}{\partial \mu}\Bigg(&-\frac{2}{\sigma/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right)-\frac{\nu}{\sigma/\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\\
&+\frac{2}{\sigma\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)+\frac{\nu}{\sigma\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\Bigg)\\
= &-\frac{2}{(\sigma/\psi)^2}\mathbb{I}_{x\,\in\,B}-\frac{\nu(\nu-1)}{(\sigma/\psi)^2}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-2}\\
&-\frac{2}{(\sigma\psi)^2}\mathbb{I}_{x\,\in\,C}-\frac{\nu(\nu-1)}{(\sigma\psi)^2}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-2}
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \mu^2}\right) = &-\frac{2}{(\sigma/\psi)^2}\int_{x\in B} f(x;\mu,\sigma,\psi,\nu)dx\\
&-\frac{\nu(\nu-1)}{(\sigma/\psi)^2}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-2} f(x;\mu,\sigma,\psi,\nu)dx\\
&-\frac{2}{(\sigma\psi)^2}\int_{x\in C} f(x;\mu,\sigma,\psi,\nu)dx\\
&-\frac{\nu(\nu-1)}{(\sigma\psi)^2}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-2} f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
\tag{5.24}
$$

From the set of integrals in Equation (B.8), it follows that:

$$\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu^2}\right) &= -\frac{2}{(\sigma/\psi)^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(\frac{1}{2},x_c^2\right)\\
&\quad-\frac{\nu(\nu-1)}{(\sigma/\psi)^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\\
&\quad-\frac{2}{(\sigma\psi)^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(\frac{1}{2},x_c^2\right)\\
&\quad-\frac{\nu(\nu-1)}{(\sigma\psi)^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\\
&= -\frac{1}{\sigma/\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{1}{2},x_c^2\right)\\
&\quad-\frac{1}{\sigma\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{1}{2},x_c^2\right)\\
&\quad-\frac{\nu-1}{\sigma/\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\\
&\quad-\frac{\nu-1}{\sigma\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu-1}{1},x_c^\nu\right)\\
&= -\frac{2\delta}{\sigma^2}\left(\frac{\psi+\frac{1}{\psi}}{\psi+\frac{1}{\psi}}\right)\left(\Gamma_l\left(\frac{1}{2},x_c^2\right)+(\nu-1)\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\right)\\
&= -\frac{2\delta}{\sigma^2}\left(\Gamma_l\left(\frac{1}{2},x_c^2\right)+(\nu-1)\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\right) \qquad (5.25)
\end{aligned}$$

The first element of the TPTAN Fischer information matrix is therefore given by:

$$E\left(-\frac{\partial^2 LL}{\partial\mu^2}\right) = \frac{2\delta}{\sigma^2}\left(\Gamma_l\left(\frac{1}{2},x_c^2\right)+(\nu-1)\Gamma_u\left(\frac{\nu-1}{2},x_c^\nu\right)\right). \qquad (5.26)$$

Following in the fashion above, the rest of the TPTAN Fischer information matrix elements are derived in Equations (C.10-C.25) in Appendix C. The equations for $\frac{\partial\delta}{\partial\nu}$ and $\frac{\partial\delta}{\partial\nu}x_c$ are discussed in Appendix C Subsection C.2.4. The results of these derivations are given below.

$$E\left(-\frac{\partial^2 LL}{\partial\mu\sigma}\right) = 0 \tag{5.27}$$

$$E\left(-\frac{\partial^2 LL}{\partial\mu\psi}\right) = \frac{2\delta}{\sigma}\left(2\Gamma_l\left(1, x_c^2\right) + \nu\Gamma_u\left(\frac{\nu}{2}, x_c^\nu\right)\right) \tag{5.28}$$

$$E\left(-\frac{\partial^2 LL}{\partial\mu\nu}\right) = 0 \tag{5.29}$$

$$E\left(-\frac{\partial^2 LL}{\partial\sigma^2}\right) = \frac{1}{\sigma^2} - \frac{2\delta}{\sigma^2}\left(3\Gamma_l\left(\frac{3}{2}, x_c^2\right) + (\nu+1)\Gamma_u\left(\frac{\nu+1}{2}, x_c^\nu\right)\right) \tag{5.30}$$

$$E\left(-\frac{\partial^2 LL}{\partial\sigma\psi}\right) = \frac{2\delta}{\sigma}\left(\frac{\frac{1}{\psi^2}-1}{\psi+\frac{1}{\psi}}\right)\left(2\Gamma_l\left(\frac{3}{2}, x_c^2\right) - \nu\Gamma_u\left(\frac{\nu+1}{2}, x_c^\nu\right)\right) \tag{5.31}$$

$$E\left(-\frac{\partial^2 LL}{\partial\sigma\nu}\right) = -\frac{2\delta}{\sigma\nu}\left(\Gamma_u\left(\frac{\nu+1}{\nu}, x_c^\nu\right) + \Gamma_u^{(1)}\left(\frac{\nu+1}{\nu}, x_c^\nu\right)\right) \tag{5.32}$$

$$\begin{aligned} E\left(-\frac{\partial^2 LL}{\partial\psi^2}\right) = {} & \frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3} - \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2}\left(1-\frac{1}{\psi^2}\right)^2 \\ & + \frac{2\delta}{\psi+\frac{1}{\psi}}\left(\left(\frac{1}{\psi^3}+\frac{3}{\psi}\right)\Gamma_l\left(\frac{3}{2}, x_c^2\right) + \left(\frac{\nu-1}{\psi^3}+\frac{\nu+1}{\psi}\right)\Gamma_u\left(\frac{\nu+1}{2}, x_c^\nu\right)\right) \end{aligned} \tag{5.33}$$

$$E\left(-\frac{\partial^2 LL}{\partial\psi\nu}\right) = \left(\frac{1-\frac{1}{\psi^2}}{\psi+\frac{1}{\psi}}\right)\frac{2\delta}{\nu}\left(\Gamma_u\left(\frac{\nu+1}{\nu}, x_c^\nu\right) + \Gamma_u^{(1)}\left(\frac{\nu+1}{\nu}, x_c^\nu\right)\right) \tag{5.34}$$

$$\begin{aligned} E\left(-\frac{\partial^2 LL}{\partial\nu^2}\right) = {} & \frac{2\delta}{\nu^3}\Gamma_u^{(2)}\left(\frac{\nu+1}{\nu}, x_c^\nu\right) - \frac{\partial}{\partial\nu}\left(\left(\nu x_c^{\nu-1} + 2x_c\right)\cdot\frac{\partial}{\partial\nu}x_c\right)\cdot\frac{2\delta}{\nu}\cdot\Gamma_u\left(\frac{1}{2}, x_c^\nu\right) \\ & - \left(\frac{1}{\delta}\cdot\frac{\partial\delta}{\partial\nu}\right) \end{aligned} \tag{5.35}$$

The Equation (5.28) shows that the maximum likelihood estimates (MLE) of location, μ, and skewness, ψ, depend on each other. The Equations (5.27, 5.29) show that the MLEs of the location parameter, μ, is independent of the scale, σ, and kurtosis, ν, parameters. Equations (5.31, 5.32, 5.34) show that the remaining MLE are dependent on each other. This is mostly due to the introduction of the skewness parameter, ψ.

5.7 Summary

In this chapter the following TPTAN statistical properties were derived in each Section: 5.1 the PDF, 5.2 the CDF, 5.3 the quantile function, and in 5.4 the moments. The TPTAN distribution can therefore now model different combinations of skewness and heavy tails. In the next chapter the use of the use of the TPTAN is practically demonstrated by fitting real data and simulation studies.

Chapter 6

Applications

In this chapter the real world performance of the TPTAN is compared to other flexible models on real and simulated data. More specifically, the two data-sets are light and heavy-tailed and the simulation study focusses on small and large sample behaviour.

6.1 Preliminaries

This section gives an overview of the necessary background information for the applications. The relevant models, information criteria, and hypothesis tests are defined and discussed.

6.1.1 Comparative models

The main competing models for the applications is the skew t (ST) [5], hyperbolic (HYP) [9], and normal inverse Gaussian (NIG) [9] distributions. These model are also highly flexible capable with a wide range of skewness and kurtosis combinations. The similar properties of the TPTAN make it a direct competitor of these competing models. The generalised hyperbolic distribution (GHYP) is used in the simulation study, since it contains all of the comparative models as sub-models and implicitly the skew normal (SN) [12]. A table of the relevant distribution PDFs are given in Table (6.1).

Distiribution	PDF	Denotation
ST	$f(x;\mu,\sigma,\psi,\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sigma\sqrt{(\nu\pi)}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\left(\frac{x-\mu}{\sigma\sqrt{\nu}}\right)^2\right)^{-(\nu+1)/2}\xi\left(\psi\left(\frac{x-\mu}{\sigma}\right)\right)$	$ST(\mu,\sigma,\psi,\nu)$
HYP	$f(x;\mu,\sigma,\psi,\nu)=\frac{\nu\sigma K_1\left(\nu\sqrt{\nu^2-\psi^2}\right)}{\pi\sqrt{\sigma^2+(x-\mu)^2}}\mathrm{e}^{\sigma\sqrt{\nu^2-\psi^2}+\psi(x-\mu)}$	$HYP(\mu,\sigma,\psi,\nu)$
NIG	$f(x;\mu,\sigma,\psi,\nu)=\frac{\sqrt{\nu^2+\psi^2}}{2\nu\sigma K_1\left(\sigma\sqrt{\nu^2-\psi^2}\right)}\mathrm{e}^{-\nu\sqrt{\sigma^2+(x-\mu)^2}+\psi(x-\mu)}$	$NIG(\mu,\sigma,\psi,\nu)$
GHYP	$f(x;\mu,\sigma,\psi,\nu,\lambda)=\left(\frac{\frac{\sqrt{\nu^2+\psi^2}}{\sigma}}{\sqrt{2\pi}K_\lambda\left(\sigma\sqrt{\nu^2-\psi^2}\right)}\right)^\lambda\frac{K_{\lambda-1/2}\left(\nu\sqrt{\sigma^2+(x-\mu)^2}\right)}{\left(\frac{\sqrt{\sigma^2+(x-\mu)^2}}{\nu}\right)^{1/2-\lambda}}\mathrm{e}^{\psi(x-\mu)}$	$GHYP(\mu,\sigma,\psi,\nu,\lambda)$

*where $\xi(\cdot)$ is the CDF of the normal distribution, and $K_\alpha(\cdot)$ is the modified Bessel function of the third kind.

Table 6.1: The PDFs of the comparative models in the study.

6.1.2 Goodness of fit criterion

The goodness of fit criterion are based on the fitted LL of different models. A smaller criterion value indicates better model fitting.

Criterion	Formula	Reference
AIC	$2(p-LL)$	[1]
BIC	$p\ln(n)-2LL$	[55]
CAIC	$p(\ln(n)+1)-2LL$	[15]
HQC	$2(p\ln(\ln(n))-LL)$	[29]

*where LL is the fitted maximum LL value, p the number of parameters, and n the sample size.

Table 6.2: Application goodness of fit criterion.

6.1.3 Goodness of fit tests

Kolmogorov-Smirnov test

For an independent sample of size n, the hypothesis test is specified as follows.

H_0 : The data follows the fitted distribution.

H_1 : The data does not follow the fitted distribution.

The Kolmogorov-Smirnov (KS) test statistic is given by:

$$D_n = \max_{j=1,2,\ldots,n} \left\{ \frac{j}{n} - \hat{F}(x_j), \hat{F}(x_j) - \frac{j-1}{n} \right\}, \tag{6.1}$$

where $\hat{F}(\cdot)$ is the fitted CDF of the hypothesized distribution. The distribution of the test statistic D_n can be simulated with the algorithm below:

1: Simulate n values from a uniform distribution between 0 and 1.
2: Order the values from smallest to largest, giving. $u_1 \leq u_2 \leq \cdots \leq u_n$.
3: The simulated statistic value is then, $d_{i,n} = \max_{j=1,2,\ldots,n} \left\{ \frac{j}{n} - u_j, u_j - \frac{j-1}{n} \right\}$.
4: Repeat steps (1-3) until the desired values $d_{1,n}, d_{2,n}, d_{2,n}, \ldots$ are generated.

Algorithm 6.1: KS test statistic simulation.

The p-value in support of the null hypothesis is given by $P(D_n > d_n)$, where d_n is the observed KS value. The count of the values $d_{1,n}, d_{2,n}, d_{2,n}, \ldots$ greater than d_n gives a simulated estimate of the p-value. In the hypothesis tests 10000 replicates of $d_{1,n}, d_{2,n}, d_{2,n}, \ldots$ are generated and used to calculate the p-value in the following sections.

Bayes' factor

Definition: Assuming that the data, D, has arisen under one of two hypothetical models H_i and H_1. The priori probabilities $P(D|H_0)$ and $P(D|H_1) = 1 - P(D|H_0)$ produce posterior probabilities $P(H_0)|D)$ and $P(H_1|D) = 1 - P(H_0|D)$, respectively. The Bayes' Factor (BF) is given below:

$$B_{01} = \frac{\int L(D|H_0)P(D|\theta_0, H_0)d\theta_0}{\int L(D|H_1)P(D|\theta_1, H_1)d\theta_1}, \tag{6.2}$$

where $L(D|H_0)$ and $L(D|H_0)$ are the respective likelihood functions.

Approximate Bayes' factor: The BF can be approximated by,

$$2\log(B_{01}) \approx 2\left(\log\left(P(D|\hat{\theta}_0, H_0)\right) - \log\left(P(D|\hat{\theta}_1, H_1)\right)\right) - (p_1 - p_2)\log(n), \tag{6.3}$$

where $\hat{\theta}_i$ is the MLE's under H_i, p_i is the dimension of θ_i, for $i = 1, 2$, and sample size n.

In general the BF, and its approximation, can be used to inform on model superiority with ranges that have been provided by [40] and is summarised in Table 6.3.

Table 6.3: The interpretation of the BF values as given by [40].

B_{12}	$2\ln(B_{12})$	Evidence in favour of H_0
< 1	< 0	Negative (H_1 is accepted)
$[1, 3]$	$[0, 2]$	Weak
$[3, 20]$	$[2, 6]$	Positive
$[20, 150]$	$[6, 10]$	Strong
> 150	> 10	Very strong

6.2 Data modelling

The ML and KS distance estimates are obtained using the optim subroutine in R. The model fit is evaluated by various LL criterion, the KS distance, and the BF hypothesis tests.

6.2.1 Bitcoin daily returns data

The price of Bitcoin is shown in Figure (6.1) for the period 2013/04/29-2019/06/23 as retrieved from the Coinmetrics API, *https://community-api.coinmetrics.io/v2*. The data represents the daily returns of Bitcoin during the time period of 2013/07/07-2018/12/17

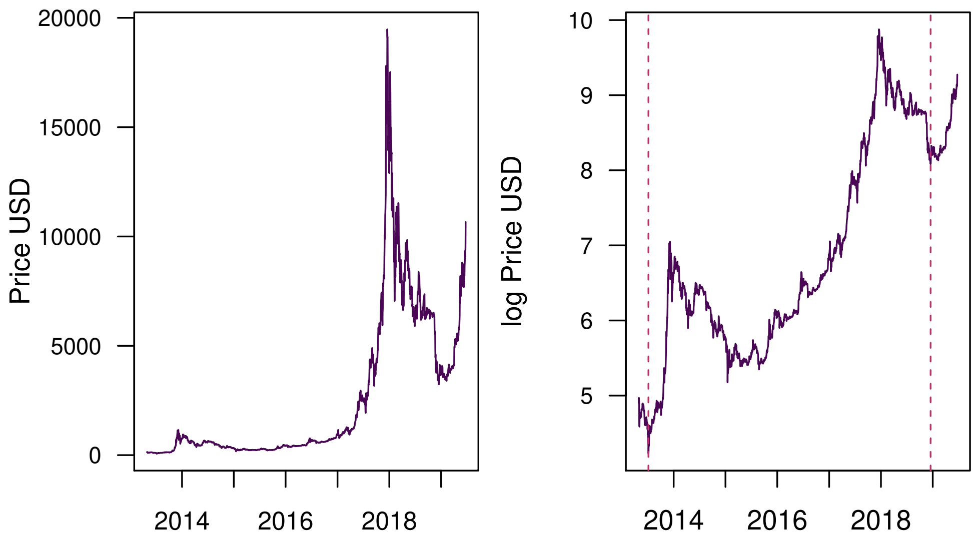

Figure 6.1: The price of Bitcoin between 2013/04/29-2019/06/23 (2247 days). The data is given as the log returns of the price of Bitcoion between the two local minima, 2013/07/07-2018/12/17 (1989 days).

(1989 days). This period is presumed two be two cycles of boom and bust and is delimited as the period between the two local minima shown by the dotted lines.

The data is heavy-tailed relative to the normal distribution as can be seen is the fitted PDFs in Figure (6.2). This is also reflected in the kurtosis parameter estimates for the ST and TPTAN, which are less than two and less than one, see Tables 6.4 and 6.6 respectively. The fitted ST distribution is problematic for inferential purposes due to the kurtosis parameter being so small that the moments of two and higher do not exist [12]. Therefore the measures of variance, skewness, and kurtosis do not exist for the fitted ST distribution. On the other hand similar to the HYP and NIG distributions, the TPTAN has closed form analytical expressions of its statistical properties. By comparing the ML and KS distance estimates in Tables 6.4 and 6.6; it is clear that the estimates for the TPTAN are the only estimates that are very similar to each other with the ML and KS distance estimates of the HYP and NIG distributions differing the most. The values in Table 6.4 show very strong evidence in favour of the TPTAN distribution. In Table 6.5 the TPTAN model has the lowest criterion values of all the models indicating a better fit. Lastly, the p-value in Table 6.6 is the highest for the TPTAN distribution. These statistics support the conclusion that the TPTAN model has the best model fit for the data.

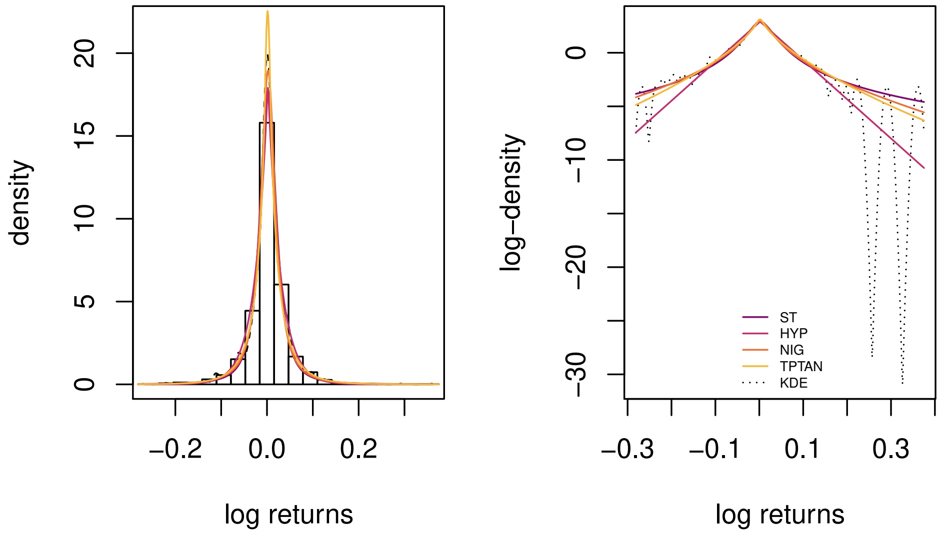

Figure 6.2: Fitted PDFs for the ST, HYP, NIG and TPTAN on the Bitcoin returns data set. The SN fit is omitted for clarity of the other better fitting models.

Table 6.4: MLEs and BF statistics for the Bitcoin returns data.

	μ	σ	ψ	ν	LL	$2\ln(B_{12})$
SN	0.002	0.043	-0.115	$\cdot$	$3,419.423$	835.798
ST	0.002	0.020	0.006	1.831	$3,805.302$	71.635
HYP	0.002	0.01	36.571	-0.009	$3,792.166$	97.907
NIG	0.002	0.019	9.350	-0.209	$3,826.911$	28.416
TPTAN	0.001	0.011	1.018	0.659	$3,841.119$	H_o

Table 6.5: Goodness-of-fit Statistics for the Bitcoin returns data.

	AIC	BIC	CAIC	HQIC
	Relative difference to Base value.			
Base	-6,832.845	-6,816.059	-6,813.059	-6,826.680
SN	0	0	0	0
ST	-769.758	-764.163	-763.163	-767.703
HYP	-743.486	-737.891	-736.891	-741.431
NIG	-812.977	-807.382	-806.382	-810.922
TPTAN	-841.393	-835.798	-834.798	-839.338

Table 6.6: Minimum KS Distance Estimates, KS distance, and p-values for the Bitcoin returns data.

	μ	σ	ψ	ν	KS	p-value
SN	0.005	0.028	-0.098	·	0.053	0
ST	0.003	0.017	-0.049	1.225	0.020	0.415
HYP	0.002	0.01	39.979	1.435	0.027	0.103
NIG	0.002	0.017	3.585	0.070	0.017	0.588
TPTAN	0.001	0.010	1.026	0.634	0.012	0.945

6.2.2 Munich rent data

The data represents 3082 observations of net rental price per square meter. The data comes for a rent survey in Munich in the year 1999 [21]. The data is light-tailed relative to the normal distribution as can be seen is the fitted PDFs in Figure (6.3). This is also reflected in the TPTAN kurtosis parameter that is more than 2, see Tables 6.7 and 6.9) respectively. The ST distribution is approaching the limiting distribution, $n \to \infty$, the SN distribution. The addition of the ST in this application is to emphasise the limitation of the ST as only being able to take into accout heavier tails than normal. By comparing the ML and KS distance estimates in Tables 6.7 and 6.9; the estimates for the ML and KS distance estimates are approximately the same for all distributions. The BF values in Table 6.7 show very strong evidence in favour of the TPTAN distribution. In Table 6.8 the TPTAN model has the lowest criterion values of all the models indicating better fit. Lastly, the p-value in Table 6.9 is the highest for the TPTAN distribution. These statisics support the conclusion that the TPTAN model has the best model fit for the data.

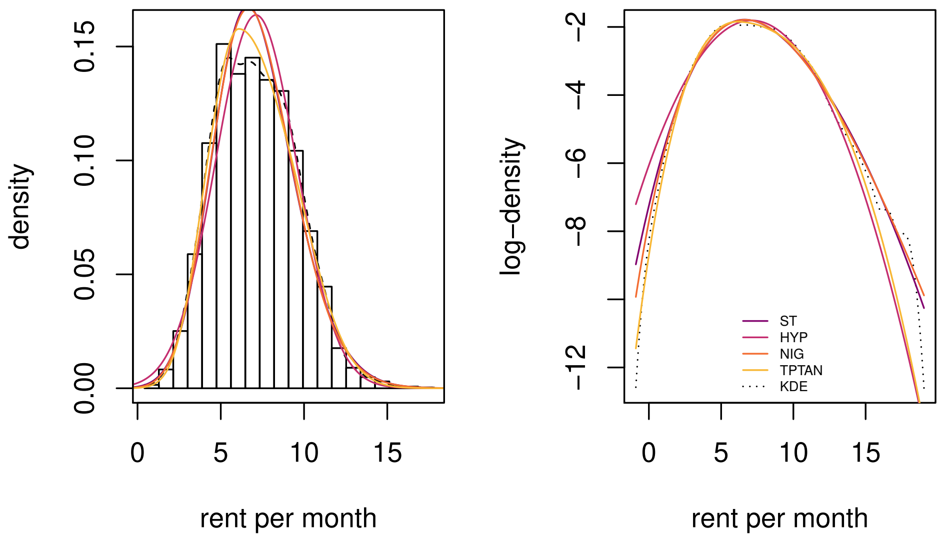

Figure 6.3: Fitted PDF for the SN, ST, and TPTAN on the Munich rent data set.

Table 6.7: MLEs and BF statistics for the Munich rent data.

	μ	σ	ψ	ν	LL	$2\ln(B_{12})$
SN	7.103	2.444	0.368	·	-7,092.876	23.622
ST	4.803	3.363	1.685	483.111	-7,093.393	32.689
HYP	7.794	64.624	10.913	-0.115	-7,117.013	79.928
NIG	-7.040	8.645	10.251	8.747	-7,088.904	23.711
TPTAN	6.083	3.626	1.304	2.446	-7,077.049	H_o

Table 6.8: Goodness-of-fit Statistics for the Munich rent data.

	AIC	BIC	CAIC	HQIC
	Relative difference to Base value.			
Base	$14,191.750$	$14,209.850$	$14,212.850$	$14,198.250$
SN	0	0	0	0
ST	3.034	9.067	10.067	5.201
HYP	50.273	56.306	57.306	52.440
NIG	-5.944	0.089	1.089	-3.777
TPTAN	-29.655	-23.622	-22.622	-27.488

Table 6.9: Minimum KS Distance Estimates, KS distance, and p-values for the Munich rent data.

	μ	σ	ψ	ν	KS	p-value
SN	7.103	2.444	0.368	·	0.023	0.066
ST	4.803	3.363	1.685	483.111	0.024	0.062
HYP	7.794	64.624	-0.115	10.913	0.023	0.066
NIG	-7.040	8.645	8.747	10.251	0.019	0.233
TPTAN	6.083	3.626	1.304	2.446	0.013	0.688

6.3 Comparative simulation study

The small and large sample performance of the TPTAN is studied by simulating values from a known GHYP distribution and fitting the TPTAN and comparative models. The convergence of distributional shape and parameters are both studied for comparison. The simulation of this section consists of a 1000 replications from a $GHYP(0.5, 10, 0.7, 0.55, -0.02)$ distribution. This procedure is is summarised in Algorithm 6.2.

1: Simulate n values the relevant $GHYP(\mu, \sigma, \psi, \nu, \lambda)$ distribution.
2: Fit the ST, NIG, TPTAN model using MLE and store the fitted parameters.
3: Repeat steps (1-2) until the desired number replications are reached.
4: Calculate the statistics in question.

Algorithm 6.2: Simulation procedure for a comparative study.

6.3.1 Parameter and shape convergence

The goal of a flexible model is to approximate the population as best as possible using a sample from the population. Naturally, the more complex the shape of the distribution, the larger sample required to model the population accurately. The process of better population shape approximation with increasing sample size is shown by in Figure (6.4). The increasing sample size usually corresponds with better parameter certainty and convergence in value. The replicate distributions of the TPTAN MLE's show this exact trend as can be seen in Figure (6.5). In this specific simulation, the fitted shape is most acceptable for sample sizes greater than $n = 100$. The root-mean-square error (RMSE) of the replicate parameters also suggest this threshold of $n = 100$ in Figure (6.6). These simulation results indicate that fitting the TPTAN to small samples, $n < 100$, must be done with caution.

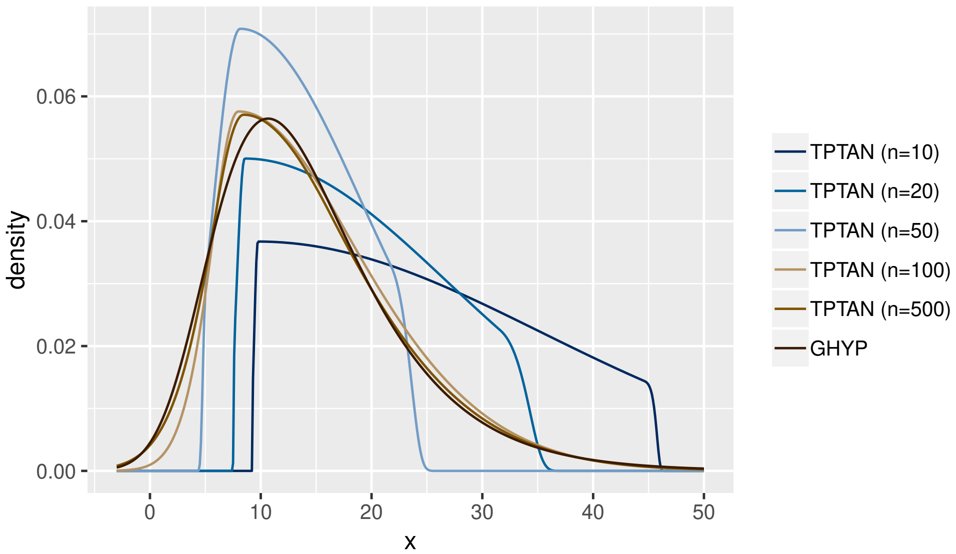

Figure 6.4: The convergence process of the fitted TPTAN distribution to the population $GHYP(0.5, 10, 0.7, 0.55, -0.02)$. The representation of typical model fit is constructed using the means of the replicate parameter distributions for each sample size, see Figure (6.5).

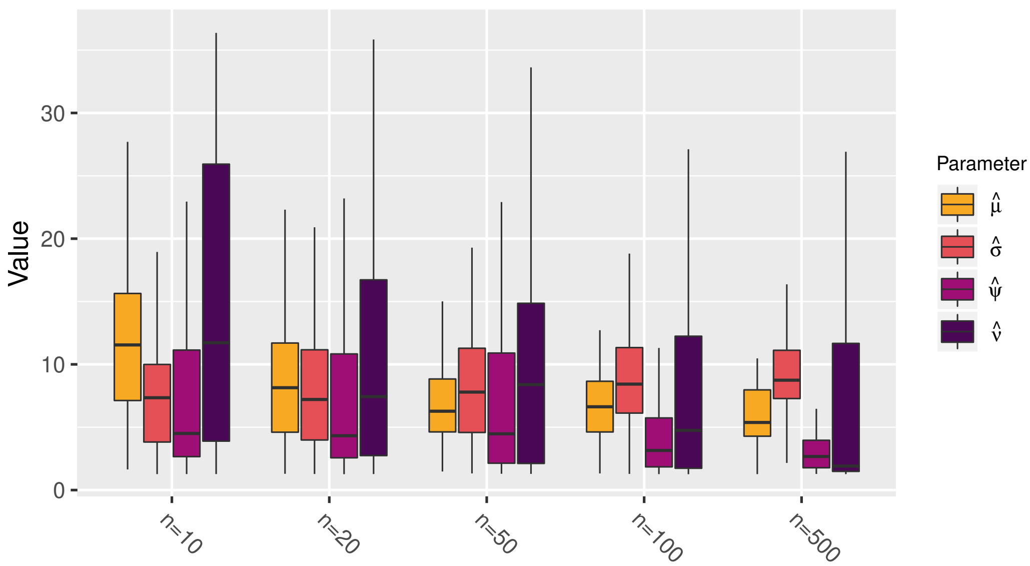

Figure 6.5: The replicate distributions of fitted MLE parameters to different sample sizes from a $GHYP(0.5, 10, 0.7, 0.55, -0.02)$ distribution.

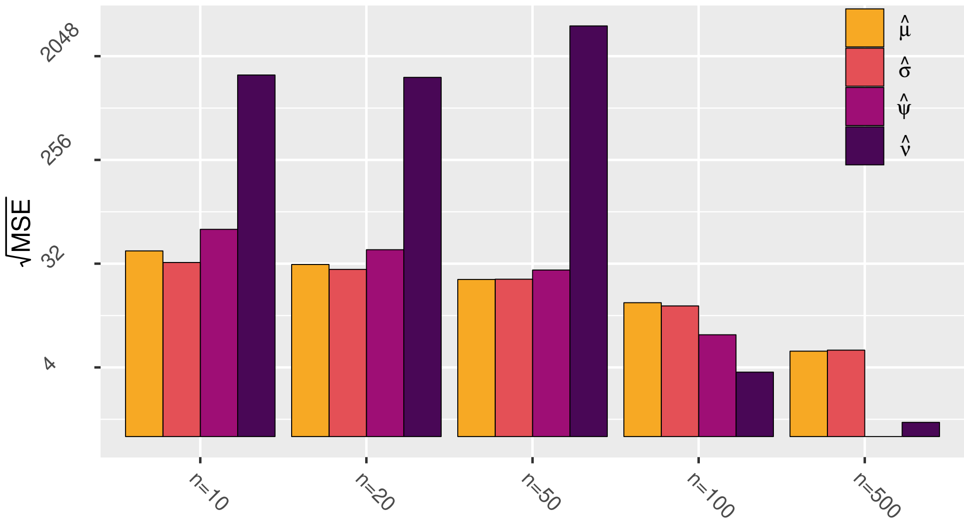

Figure 6.6: The RMSE of fitted TPTAN parameter estimates for different sample sizes from a $GHYP(0.5, 10, 0.7, 0.55, -0.02)$ distribution.

6.3.2 Distributional fit

The limited fit of the TPTAN to the small sample sizes is mostly due to the complex shape of the five-parameter GHYP distribution used in the simulation. The same behaviour is also observed in the fit of the comparative models. The estimate of a specific quantile and its error is often a quantity of interest in finance. In flexible modelling, the nature of a quantile estimate is dependent on well a distribution is approximated by the flexible model. The TPTAN performance is very similar to the competing models in this regard. The replicate error distributions for the 90th quantile estimates are similar across models, see Figure (6.7). Analysing the replicate error distributions further, the decomposition of the mean-squared-error (MSE) are given in Figure (6.8). Across the models a decreasing trend in bias and variance is observed for larger samples. This once again corresponds to distributional shape approximation. The HYP and NIG are similar in that their quantile estimates have less variance and more bias, which corresponds to them being more closely related as hyperbolic distributions. The ST and TPTAN show less bias and more variance for their quantile estimates, which suggest that they function similarly in modelling the data. There is no particular distribution which does particularly well relative to the others in this simulation. In practice, if different models perform similarly other considerations may influence model choice such as inferential properties and parameter interpretation as discussed in Section 2.3.

6.4 Summary

In this chapter the TPTAN was shown to be a competitive flexible model in real world situations. The preliminaries for chapter is given in Section 6.1, the data modelling applications in Section 6.2, and the small and large sample simulation studies in 6.3. The next and final chapter summarises the findings in the document and makes suggestions for future work.

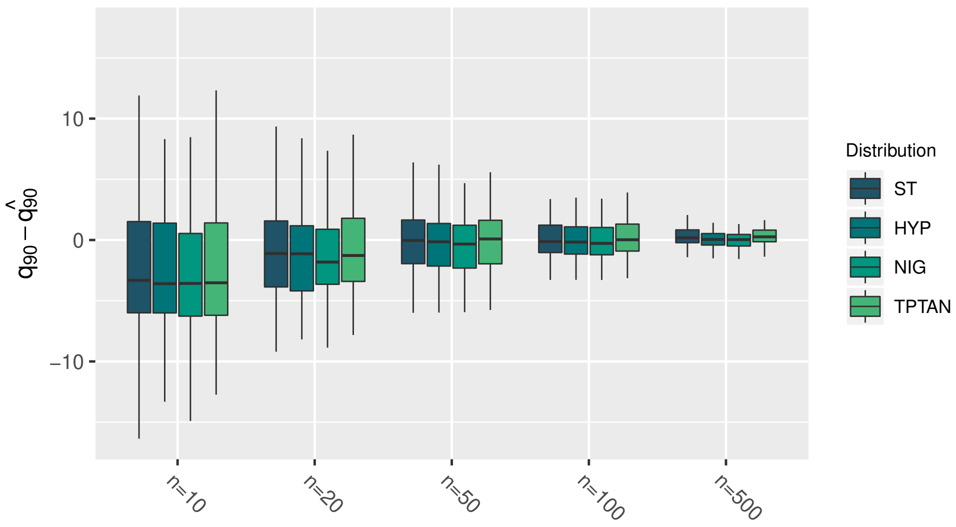

Figure 6.7: The distribution of error for a 1000 replications of the 90th quantile estimate for different distributions and different sample sizes from a $GHYP$ $(0.5, 10, 0.7, 0.55, -0.02)$ distribution.

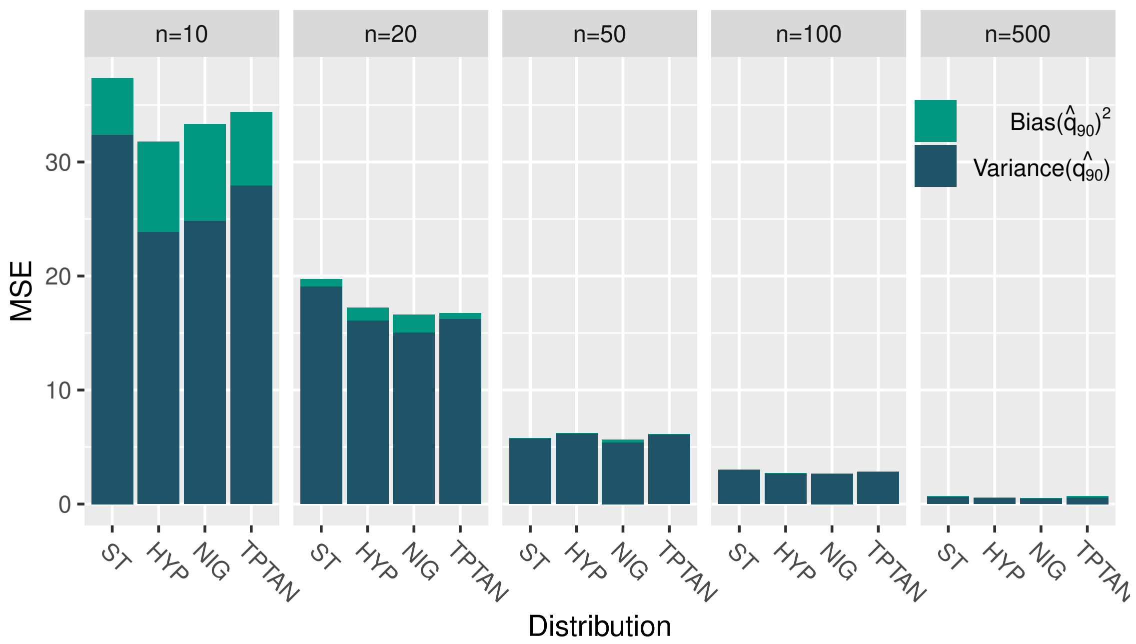

Figure 6.8: Decomposed MSE for a 1000 replications of the 90th quantile estimate for different distributions and different sample sizes from a $GHYP(0.5, 10, 0.7\ , 0.55, -0.02)$ distribution.

Chapter 7

Conclusions and future work

This chapter summarises the conclusions of this dissertation and considers future directions for further research in Sections 7.1 and 7.2 respectively.

7.1 Summary of conclusions

The dissertation sets out to demonstrate what a deliberate derivation of a flexible model could look like. In chapter 2 important traits for flexible models were identified such as skewness and kurtosis. Additionally a functional view of flexible modelling is provided for the problem at hand.

In chapter 3 the SGN distribution is introduced as a consequence of this functional outlook and the relationship log-splines is discussed. The SGN distribution is shown to be a highly tractable distribution with relatively simple derivations of the PDF, CDF, quantile function, and moments. The SGN makes for a suitable flexible modelling building block which is used in the following chapters.

In chapter 4 the TAN distribution, a sub-model of the SGN, shares the same tractable properties of the SGN from which the PDF, CDF, quantile function, and moments are derived. The TAN distribution generalises the tail shape of the normal distribution which addresses the kurtosis property of flexible modelling discussed in chapter 2.

In chapter 5 the TPTAN is introduced to address the skewness property of flexible modelling discussed in chapter 2. The TPTAN is the capstone of the dissertation and builds on the tractable statistical properties of the TAN distribution. Importantly, the TPTAN is shown to have interpretable parameters as well as simple derivations of the PDF, CDF, quantile function, and moments.

In chapter 6 the TPTAN verified to be useful in a practical sense. It is shown that the TPTAN is a highly competitive model through the modelling of real data and simulation studies. Overall, it can be concluded that following a well informed and structured approach to flexible modelling is essential for success of a theorist and practitioner in flexible modelling.

7.2 Future work

The functional paradigm for investigating flexible models open a multitude of future work for new developments, more thorough research that were not focussed on specifically, and possible extensions.

- Other skewing mechanisms than two-piece scaling can be investigated and compared for the TAN such as the Azzalini skewing [5].
- Although the TPTAN is highly tractable in the univariate space, the mathematics can be improved in such a way that it can be extended to the multivariate space more easily. This would entail achieving the same body and tail shape functions without the use of indicator functions.
- A study evaluating known models according to the desirable traits in Section 2.3 could be a considerable help to practitioners in choosing a "good" models for their purposes.
- For practical and inferential purposes existing generalisations of known distributions can be redefined or adjusted keeping the desirable traits in Section 2.3 in mind.
- A PDF is a function with an input and output, the two-piece scaling changes the input to a PDF. In other words this is a form of pre-processing which can be

studied on an analytic and stochastic level. Likewise, the same can be said for post-processing.

- The manner in which measures of skewness and kurtosis work can give different and insightful view of flexible modelling and the shortcomings in of existing distributions. Using this knowledge gained can lead to new measures as well as distributions addressing these shortcomings.

- Since the entropy view of PDFs has been proved to be useful in a univariate setting extending this principle of analysis in a multivariate environment would be a natural next step. This view could be particularly beneficial to characterisation of skewness and kurtosis in higher dimensions.

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Appendix A

Acronyms

This appendix provides a list of the important acronyms used or newly-defined in the dissertation, as well as their definitions.

AIC	Akaike information criterion
BF	Bayes' Factor
BIC	Bayesian information criterion
CAIC	consistent Akaike information
CDF	cumulative distribution function criterion
HQIC	Hannan-Quinn information criterion
HYP	hyperbolic
GHYP	generalised hyperbolic
GN	generalised normal
KDE	kernel density estimate
KS	Kolmogorov-Smirnov
LL	log-likelihood
MLE	maximum likelihood estimate
MSE	mean-squared-error
NIG	normal inverse Gaussian
PDF	probability density function

RMSE	root-mean-squared-error
SGN	spliced generalised normal
SN	skew normal
ST	skew t
TAN	tail-adjusted normal
TPTAN	two-piece tail-adjusted normal

Appendix B

Important integrals and expectations properties

In this appendix important properties and integrals are dealt with that pertain to the expectation integrals of the TPTAN distribution.

B.1 Incomplete gamma derivative identities

The derivatives of the upper and lower incomplete gamma functions with respect to α has the following integral representation according to [60]:

$$\Gamma_u^{(n)}(\alpha, y) = \int_x^{\infty} y^{\alpha-1}\mathrm{e}^{-m}\ln(y)^n dy, \tag{B.1}$$

$$\Gamma_l^{(n)}(\alpha, x) = \int_0^{x} y^{\alpha-1}\mathrm{e}^{-m}\ln(x)^n dy.$$

B.2 Two-piece distribution moments property

A two-piece scaled distribution, X, constructed as in Equation (5.1) with symmetric baseline distribution, Z, has non-zero r'th moments of the form:

$$E(X^r) = \frac{\psi^{r+1} + (-1)^r \psi^{-(r+1)}}{\psi + \frac{1}{\psi}} E(|Z|^r). \tag{B.2}$$

Proof:

Let $g(x)$ be the PDF of X and $f(z)$ be the PDF of the symmetric baseline distribution Z.

For odd r'th moments, it is given that

$$\begin{aligned} E(X^r) &= \int_{\Re} z^r g(x) dx \\ &= \int_{-\infty}^{0} \frac{2}{\psi + \frac{1}{\psi}} s^r f(\psi s) ds + \int_{0}^{\infty} \frac{2}{\psi + \frac{1}{\psi}} t^r f\left(\frac{t}{\psi}\right) dt. \end{aligned} \tag{B.3}$$

Let $l = \psi s$ and $m = \frac{t}{\psi}$ in, which implies $s = \frac{l}{\psi}$ and $t = \psi m$.

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{0} \frac{2}{\psi + \frac{1}{\psi}} \left(\frac{l}{\psi}\right)^r f(l) \frac{1}{\psi} dl + \int_{0}^{\infty} \frac{2}{\psi + \frac{1}{\psi}} (\psi m)^r f(m) \psi dm \\ &= \frac{2}{\psi + \frac{1}{\psi}} \left(\frac{1}{\psi^{r+1}} \int_{-\infty}^{0} l^r f(l) dl + \psi^{r+1} \int_{0}^{\infty} m^r f(m) dm \right) \\ &= \frac{2}{\psi + \frac{1}{\psi}} \left(\left(\psi^{r+1} - \frac{1}{\psi^{r+1}} \right) \int_{0}^{\infty} t^r f(m) dm \right) \\ &= \frac{\psi^{r+1} - \psi^{-(r+1)}}{\psi + \frac{1}{\psi}} E(|Z|^r) \end{aligned} \tag{B.4}$$

For even r'th moments, it is given that:

$$E(X^r) = \int_{-\infty}^{0} \frac{2}{\psi + \frac{1}{\psi}} s^r f(\psi s) ds + \int_{0}^{\infty} \frac{2}{\psi + \frac{1}{\psi}} t^r f\left(\frac{t}{\psi}\right) dt.$$

Let $l = \psi s$ and $m = \frac{t}{\psi}$ in, which implies $s = \frac{l}{\psi}$ and $t = \psi m$.

$$
\begin{aligned}
E(X^r) &= \int_{-\infty}^{0} \frac{2}{\psi+\frac{1}{\psi}}\left(\frac{l}{\psi}\right)^r f(l)\frac{1}{\psi}dl + \int_0^\infty \frac{2}{\psi+\frac{1}{\psi}}(\psi m)^r f(m)\,\psi dm \\
&= \frac{2}{\psi+\frac{1}{\psi}}\left(\frac{1}{\psi^{r+1}}\int_{-\infty}^{0} l^r f(l)dl + \psi^{r+1}\int_0^\infty m^r f(m)dm\right) \\
&= \frac{2}{\psi+\frac{1}{\psi}}\left(\left(\psi^{r+1}+\frac{1}{\psi^{r+1}}\right)\int_0^\infty t^r f(m)dm\right) \\
&= \frac{\psi^{r+1}+\psi^{-(r+1)}}{\psi+\frac{1}{\psi}}E(Z^r) \qquad \text{(B.5)}
\end{aligned}
$$

Combining Equations (B.4, B.5) gives:

$$
E(X) = \frac{\psi^{r+1}+(-1)^r\psi^{-(r+1)}}{\psi+\frac{1}{\psi}}E(|Z|^r). \quad \square
$$

B.3 Expectation integral type I

For this section the following is defined: $A=\left\{x|x\le\mu-\frac{\sigma}{\psi}x_c\right\}$, $B=\left\{x|\mu-\frac{\sigma}{\psi}x_c<x\le\mu\right\}$, $C=\{x|\mu<x<\mu+\sigma\psi x_c\}$, $D=\{x|\mu+\sigma\psi x_c\le x\}$, and $\mathbb{I}_{x\,\in\,X}=\begin{cases}0 & x\notin X\\ 1 & x\in X\end{cases}$.

Consider the following transformation $l=\left(\frac{\mu-s}{\sigma/\psi}\right)^x$ and $m=\left(\frac{t-\mu}{\sigma\psi}\right)^y$, which implies $s=\mu-\sigma/\psi l^{\frac{1}{x}}$ and $t=\mu+\sigma\psi m^{\frac{1}{y}}$ on the integrals below:

$$
\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2} ds &= \int_{x_c^2}^{0} l^{\frac{r}{2}}\mathrm{e}^{-l}\cdot\left(-\frac{\sigma/\psi}{2}l^{\frac{1}{2}-1}\right)dl \\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2} ds &= \int_0^{x_c^2} m^{\frac{r}{2}}\mathrm{e}^{-m}\cdot\frac{\sigma\psi}{2}m^{\frac{1}{2}-1}dm \\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu} dt &= \int_{\infty}^{x_c^\nu} l^{\frac{r}{\nu}}\mathrm{e}^{-l}\cdot\left(-\frac{\sigma/\psi}{\nu}l^{\frac{1}{\nu}-1}\right)dl \\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu} dt &= \int_{x_c^\nu}^{\infty} m^{\frac{r}{\nu}}\mathrm{e}^{-m}\cdot\frac{\sigma\psi}{\nu}m^{\frac{1}{\nu}-1}dm \qquad \text{(B.6)}
\end{aligned}
$$

$$\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2}ds &= -\frac{\sigma/\psi}{2}\int_{x_c^2}^{0} l^{\frac{r+1}{2}-1}\mathrm{e}^{-l}dl\\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2}ds &= \frac{\sigma\psi}{2}\int_{0}^{x_c^2} m^{\frac{r+1}{2}-1}\mathrm{e}^{-m}dm\\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu}dt &= -\frac{\sigma/\psi}{\nu}\int_{\infty}^{x_c^\nu} l^{\frac{r+1}{\nu}-1}\mathrm{e}^{-l}dl\\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu}dt &= \frac{\sigma\psi}{\nu}\int_{x_c^\nu}^{\infty} m^{\frac{r+1}{\nu}-1}\mathrm{e}^{-m}dm
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2}ds &= \frac{\sigma/\psi}{2}\Gamma_l\left(\frac{r+1}{2},x_c^2\right)\\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2}ds &= \frac{\sigma\psi}{2}\Gamma_l\left(\frac{r+1}{2},x_c^2\right)\\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu}dt &= \frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{r+1}{2},x_c^\nu\right)\\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu}dt &= \frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{r+1}{2},x_c^\nu\right)
\end{aligned} \tag{B.8}$$

B.4 Expectation integral type II

For this section the following is defined: $A=\left\{x|x\leq\mu-\frac{\sigma}{\psi}x_c\right\}$, $B=\left\{x|\mu-\frac{\sigma}{\psi}x_c<x\leq\mu\right\}$, $C=\{x|\mu<x<\mu+\sigma\psi x_c\}$, $D=\{x|\mu+\sigma\psi x_c\leq x\}$, and $\mathbb{I}_{x\,\in\,X}=\begin{cases}0 & x\notin X\\ 1 & x\in X\end{cases}$.

Consider the following transformation $l=\left(\frac{\mu-s}{\sigma/\psi}\right)^x$ and $m=\left(\frac{t-\mu}{\sigma\psi}\right)^y$, which implies $s=\mu-\sigma/\psi l^{\frac{1}{x}}$ and $t=\mu+\sigma\psi m^{\frac{1}{y}}$ on the integrals below:

$$\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \ln\left(\frac{\mu-s}{\sigma/\psi}\right)^p \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2} ds &= \int_{x_c^2}^{0} l^{\frac{r}{2}}\mathrm{e}^{-l}\frac{1}{2^p}\ln(l)^p\cdot\left(-\frac{\sigma/\psi}{2}l^{\frac{1}{2}-1}\right)dl\\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \ln\left(\frac{t-\mu}{\sigma\psi}\right)^p \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2} ds &= \int_{0}^{x_c^2} m^{\frac{r}{2}}\mathrm{e}^{-m}\frac{1}{2^p}\ln(m)^p\cdot\frac{\sigma\psi}{2}m^{\frac{1}{2}-1}dm\\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \ln\left(\frac{\mu-s}{\sigma/\psi}\right)^p \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu} dt &= \int_{\infty}^{x_c^\nu} l^{\frac{r}{\nu}}\mathrm{e}^{-l}\frac{1}{\nu^p}\ln(l)^p\cdot\left(-\frac{\sigma/\psi}{\nu}l^{\frac{1}{\nu}-1}\right)dl\\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \ln\left(\frac{t-\mu}{\sigma\psi}\right)^p \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu} dt &= \int_{x_c^\nu}^{\infty} m^{\frac{r}{\nu}}\mathrm{e}^{-m}\frac{1}{\nu^p}\ln(m)^p\cdot\frac{\sigma\psi}{\nu}m^{\frac{1}{\nu}-1}dm
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2} ds &= -\frac{\sigma/\psi}{2^{p+1}}\int_{x_c^2}^{0} l^{\frac{r+1}{2}-1}\mathrm{e}^{-l}\ln(l)^p dl\\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2} ds &= \frac{\sigma\psi}{2^{p+1}}\int_{0}^{x_c^2} m^{\frac{r+1}{2}-1}\mathrm{e}^{-m}\ln(m)^p dm\\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu} dt &= -\frac{\sigma/\psi}{\nu^{p+1}}\int_{\infty}^{x_c^\nu} l^{\frac{r+1}{\nu}-1}\mathrm{e}^{-l}\ln(l)^p dl\\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu} dt &= \frac{\sigma\psi}{\nu^{p+1}}\int_{x_c^\nu}^{\infty} m^{\frac{r+1}{\nu}-1}\mathrm{e}^{-m}\ln(m)^p dm
\end{aligned} \tag{B.10}$$

Finally it follows from Equations (B.1) and (B.2) that:

$$\begin{aligned}
\int_{s\in B}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^2} ds &= \frac{\sigma/\psi}{2^{p+1}}\Gamma_l^{(p)}\left(\frac{r+1}{2},x_c^2\right)\\
\int_{t\in C}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^2} ds &= \frac{\sigma\psi}{2^{p+1}}\Gamma_l^{(p)}\left(\frac{r+1}{2},x_c^2\right)\\
\int_{s\in A}\left(\frac{\mu-s}{\sigma/\psi}\right)^r \mathrm{e}^{-\left(\frac{\mu-s}{\sigma/\psi}\right)^\nu} dt &= \frac{\sigma/\psi}{\nu^{p+1}}\Gamma_u^{(p)}\left(\frac{r+1}{\nu},x_c^\nu\right)\\
\int_{t\in D}\left(\frac{t-\mu}{\sigma\psi}\right)^r \mathrm{e}^{-\left(\frac{t-\mu}{\sigma\psi}\right)^\nu} dt &= \frac{\sigma\psi}{\nu^{p+1}}\Gamma_u^{(p)}\left(\frac{r+1}{\nu},x_c^\nu\right)
\end{aligned} \tag{B.11}$$

B.5 Summary

In this appendix important integrals for the TPTAN distribution are derived for expectation integrals of the TPTAN. In the next appendix some of these results will be extended and used to derive the TPTAN Fischer information.

Appendix C

The two-piece tail-adjusted normal Fischer information

In this appendix the remaining TPTAN Fischer information derivation of Section 5.6 is shown.

C.1 Two-piece tail-adjusted normal log-likelihood

For this section the following is defined: $A = \{x | x \leq \mu - \sigma/\psi x_c\}$, $B = \{x | \mu - \sigma/\psi x_c < x \leq \mu\}$, $C = \left\{x | \mu < x < \mu + \frac{\sigma}{\psi} x_c\right\}$, $D = \left\{x | \mu + \frac{\sigma}{\psi} x_c \leq x\right\}$, and $\mathbb{I}_{x \in X} = \begin{cases} 0 & x \notin X \\ 1 & x \in X \end{cases}$.

The LL function for a single observation, $n = 1$, in Equation (5.21), is repeated below:

$$\begin{aligned} LL(x; \mu, \sigma, \psi, \nu) &= \ln\left(L(x; \mu, \sigma, \psi, \nu)\right) \\ &= n\ln(2) + n\ln(\delta) - n\ln(\sigma) - n\ln\left(\psi + \frac{1}{\psi}\right) + \mathbb{I}_{x \in A \cup D} x_c^{\nu} - x_c^2 \\ &\quad - \mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 - \mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} - \mathbb{I}_{x \in C}\left(\frac{x - \mu}{\sigma\psi}\right)^2 \\ &\quad - \mathbb{I}_{x \in D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu}, \end{aligned} \tag{C.1}$$

where $\sigma, \nu > 0$, $c = x_c^{\nu} - x_c^2$, $x_c = \mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, x_c\right) + \frac{\mathrm{e}^{x_c^{\nu} - x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, x_c\right)\right)$.

C.1.1 Partial derivatives

The partial derivatives with respect to the TPTAN parameters in Equation (C.1) follow.

$$\begin{aligned}\frac{\partial LL}{\partial \mu} &= -\frac{2}{\sigma/\psi}\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right) - \frac{\nu}{\sigma/\psi}\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu-1} \\ &\quad + \frac{2}{\sigma\psi}\mathbb{I}_{x \in C}\left(\frac{x-\mu}{\sigma\psi}\right) + \frac{\nu}{\sigma\psi}\mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\end{aligned} \tag{C.2}$$

$$\begin{aligned}\frac{\partial LL}{\partial \sigma} &= -\frac{n}{\sigma} - \frac{2}{\sigma}\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)\left(-\frac{\mu - x}{\sigma^2/\psi}\right) - \nu\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu-1}\left(-\frac{\mu - x}{\sigma^2/\psi}\right) \\ &\quad - \frac{2}{\sigma}\mathbb{I}_{x \in C}\left(\frac{x-\mu}{\sigma\psi}\right)\left(-\frac{x-\mu}{\sigma^2\psi}\right) - \nu\mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\left(-\frac{x-\mu}{\sigma^2\psi}\right) \\ &= -\frac{n}{\sigma} + \frac{2}{\sigma}\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} \\ &\quad + \frac{2}{\sigma}\mathbb{I}_{x \in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\end{aligned} \tag{C.3}$$

$$\begin{aligned}\frac{\partial LL}{\partial \psi} &= -\frac{n}{\psi + \frac{1}{\psi}}\left(1 - \frac{1}{\psi^2}\right) \\ &\quad - 2\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)\left(\frac{\mu - x}{\sigma}\right) - \nu\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu-1}\left(\frac{\mu - x}{\sigma}\right) \\ &\quad - 2\mathbb{I}_{x \in C}\left(\frac{x-\mu}{\sigma\psi}\right)\left(-\frac{x-\mu}{\sigma\psi^2}\right) - \nu\mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\left(-\frac{x-\mu}{\sigma\psi^2}\right) \\ &= -\frac{n}{\psi + \frac{1}{\psi}}\left(1 - \frac{1}{\psi^2}\right) - \frac{2}{\psi}\mathbb{I}_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 - \frac{\nu}{\psi}\mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} \\ &\quad + \frac{2}{\psi}\mathbb{I}_{x \in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 + \frac{\nu}{\psi}\mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\end{aligned} \tag{C.4}$$

$$\begin{aligned}\frac{\partial LL}{\partial \nu} &= \frac{n}{\delta}\cdot\frac{\partial \delta}{\partial \nu} + \mathbb{I}_{x \in A \cup D}\left(\nu x_c^{\nu-1} + 2x_c\right)\cdot\frac{\partial}{\partial \nu}x_c - \mathbb{I}_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu}\ln\left(\frac{\mu - x}{\sigma/\psi}\right) \\ &\quad - \mathbb{I}_{x \in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\ln\left(\frac{x-\mu}{\sigma\psi}\right)\end{aligned} \tag{C.5}$$

$x_c = \mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$, and $\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2}, x_c\right) + \frac{\mathrm{e}^{x_c^{\nu} - x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu}, x_c\right)\right)$.

C.2 Two-piece tail-adjusted normal Fischer elements

In this section the remaining TPTAN Fischer elements are derived.

C.2.1 Parameter μ:

$$\begin{aligned}
\frac{\partial^2 LL}{\partial\mu\sigma} &= \frac{\partial}{\partial\sigma}\left(-\frac{2}{\sigma/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right)-\frac{\nu}{\sigma/\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\right.\\
&\qquad\left.+\frac{2}{\sigma\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)+\frac{\nu}{\sigma\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\right)\\
&= \frac{\partial}{\partial\sigma}\left(-\frac{2}{\sigma^2/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\psi}\right)-\frac{\nu}{\sigma^\nu\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\psi}\right)^{\nu-1}\right.\\
&\qquad\left.+\frac{2}{\sigma^2\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{1/\psi}\right)+\frac{\nu}{\sigma^\nu/\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{1/\psi}\right)^{\nu-1}\right)\\
&= \frac{4}{\sigma^2/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right)+\frac{\nu^2}{\sigma^2/\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\\
&\quad-\frac{4}{\sigma^2\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)-\frac{\nu^2}{\sigma^2\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu\sigma}\right) &= \frac{4}{\sigma^2/\psi}\int_{x\in B}\left(\frac{\mu-x}{\sigma/\psi}\right)f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad+\frac{\nu^2}{\sigma^2/\psi}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad-\frac{4}{\sigma^2\psi}\int_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad-\frac{\nu^2}{\sigma^2\psi}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned} \tag{C.7}$$

From the set of integrals in Equation (B.8), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu\sigma}\right) &= \frac{4}{\sigma^2/\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(1,x_c^2\right) \\
&+\frac{\nu^2}{\sigma^2/\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu}{2},x_c^\nu\right) \\
&-\frac{4}{\sigma^2\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(1,x_c^2\right) \\
&-\frac{\nu^2}{\sigma^2\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu}{2},x_c^\nu\right) \\
&=\frac{4\delta}{\sigma^2\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(1,x_c^2\right) \\
&-\frac{4\delta}{\sigma^2\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(1,x_c^2\right) \\
&+\frac{2\delta\nu}{\sigma^2\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu}{2},x_c^\nu\right) \\
&-\frac{2\delta\nu}{\sigma^2\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu}{2},x_c^\nu\right) \\
&=0
\end{aligned}
\tag{C.8}
$$

$$\begin{aligned}
\frac{\partial^2 LL}{\partial\mu\psi} &= \frac{\partial}{\partial\psi}\left(-\frac{2}{\sigma/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right) - \frac{\nu}{\sigma/\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\right.\\
&\qquad\left.+\frac{2}{\sigma\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right) + \frac{\nu}{\sigma\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\right)\\
&= \frac{\partial}{\partial\psi}\left(-\frac{2}{\sigma/\psi^2}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma}\right) - \frac{\nu}{\sigma/\psi^\nu}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma}\right)^{\nu-1}\right.\\
&\quad\left.+\frac{2}{\sigma\psi^2}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma}\right) + \frac{\nu}{\sigma\psi^\nu}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma}\right)^{\nu-1}\right)\\
&= -\frac{4}{\sigma}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right) - \frac{\nu^2}{\sigma}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\\
&\quad - \frac{4}{\sigma\psi^2}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right) - \frac{\nu^2}{\sigma\psi^2}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu\psi}\right) &= -\frac{4}{\sigma}\int_{x\in B}\left(\frac{\mu-x}{\sigma}\right)f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad - \frac{\nu^2}{\sigma}\int_{x\in A}\left(\frac{\mu-x}{\sigma}\right)^{\nu-1}f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad - \frac{4}{\sigma\psi^2}\int_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad - \frac{\nu^2}{\sigma\psi^2}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}$$

From the set of integrals in Equation (B.8), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \mu \psi}\right) &= -\frac{4}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(1,x_c^2\right) \\
&\quad -\frac{\nu^2}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu}{2},x_c^{\nu}\right) \\
&\quad -\frac{4}{\sigma\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(1,x_c^2\right) \\
&\quad -\frac{\nu^2}{\sigma\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu}{2},x_c^{\nu}\right) \\
&= -2\psi\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_l\left(1,x_c^2\right) \\
&\quad -\nu\psi\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_u\left(\frac{\nu}{2},x_c^{\nu}\right) \\
&\quad -\frac{2}{\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_l\left(1,x_c^2\right) \\
&\quad -\frac{\nu}{\psi}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_u\left(\frac{\nu}{2},x_c^{\nu}\right) \\
&= -\frac{2\delta}{\sigma}\left(2\Gamma_l\left(1,x_c^2\right)+\nu\Gamma_u\left(\frac{\nu}{2},x_c^{\nu}\right)\right)
\end{aligned} \tag{C.10}
$$

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial \mu \nu} &= \frac{\partial}{\partial \nu}\left(-\frac{2}{\sigma/\psi}\mathbb{I}_{x\in B}\left(\frac{\mu-x}{\sigma/\psi}\right)-\frac{\nu}{\sigma/\psi}\mathbb{I}_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\right. \\
&\quad \left. +\frac{2}{\sigma\psi}\mathbb{I}_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)+\frac{\nu}{\sigma\psi}\mathbb{I}_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\right) \\
&= -\mathbb{I}_{x\in A}\left(\frac{1}{\sigma/\psi}\cdot\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}+\frac{\nu}{\sigma/\psi}\cdot\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\ln\left(\frac{\mu-x}{\sigma/\psi}\right)\right) \\
&\quad +\mathbb{I}_{x\in D}\left(\frac{1}{\sigma\psi}\cdot\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}+\frac{\nu}{\sigma\psi}\cdot\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\ln\left(\frac{x-\mu}{\sigma\psi}\right)\right)
\end{aligned} \tag{C.11}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu\nu}\right) = &-\frac{1}{\sigma/\psi}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1} f(x;\mu,\sigma,\psi,\nu)dx \\
&-\frac{\nu}{\sigma/\psi}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu-1}\ln\left(\frac{\mu-x}{\sigma/\psi}\right) f(x;\mu,\sigma,\psi,\nu)dx \\
&+\frac{1}{\sigma\psi}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1} f(x;\mu,\sigma,\psi,\nu)dx \\
&+\frac{\nu}{\sigma\psi}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu-1}\ln\left(\frac{x-\mu}{\sigma\psi}\right) f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
$$

From the set of integrals in Equation (B.8) and Equation (B.11), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\mu\nu}\right) &= -\frac{1}{\sigma/\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(\frac{\nu}{2},x_c^2\right) \\
&\quad+\frac{1}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(\frac{\nu}{2},x_c^2\right) \\
&\quad-\frac{\nu}{\sigma/\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu}{2},x_c^\nu\right) \\
&\quad+\frac{\nu}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu}{2},x_c^\nu\right) \\
&= -\frac{\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{\nu}{2},x_c^2\right) \\
&\quad+\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{\nu}{2},x_c^2\right) \\
&\quad-\nu\frac{\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{1}{\nu}\Gamma_u^{(1)}\left(\frac{\nu}{2},x_c^\nu\right) \\
&\quad+\nu\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{1}{\nu}\Gamma_u^{(1)}\left(\frac{\nu}{2},x_c^\nu\right) \\
&= 0 \qquad \text{(C.12)}
\end{aligned}
$$

C.2.2 Parameter σ:

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma}\left(-\frac{1}{\sigma} + \frac{2}{\sigma}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu - x}{\sigma/\psi}\right)^\nu \right.\\
&\qquad \left. + \frac{2}{\sigma}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu\right)\\
&= \frac{\partial}{\partial \sigma}\left(-\frac{1}{\sigma} + \frac{2}{\sigma^3}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu - x}{\psi}\right)^2 + \frac{\nu}{\sigma^{\nu+1}}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu - x}{\psi}\right)^\nu \right.\\
&\qquad \left. + \frac{2}{\sigma^3}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{1/\psi}\right)^2 + \frac{\nu}{\sigma^{\nu+1}}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{1/\psi}\right)^\nu\right)\\
&= \frac{1}{\sigma^2} - \frac{6}{\sigma^2}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 - \frac{\nu(\nu+1)}{\sigma^2}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu - x}{\sigma/\psi}\right)^\nu\\
&\quad - \frac{6}{\sigma^2}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 - \frac{\nu(\nu+1)}{\sigma^2}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu
\end{aligned} \tag{C.13}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \sigma^2}\right) = \frac{1}{\sigma^2} - \frac{6}{\sigma^2}\int_{x\in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx\\
- \frac{\nu(\nu+1)}{\sigma^2}\int_{x\in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^\nu f(x;\mu,\sigma,\psi,\nu)dx\\
- \frac{6}{\sigma^2}\int_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx\\
- \frac{\nu(\nu+1)}{\sigma^2}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
$$

From the set of integrals in Equation (B.8), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \sigma^2}\right) &= \frac{1}{\sigma^2} - \frac{6}{\sigma^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu(\nu+1)}{\sigma^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&\quad - \frac{6}{\sigma^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu(\nu+1)}{\sigma^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&= \frac{1}{\sigma^2} - \frac{3/\psi}{\sigma^2}\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{3\psi}{\sigma^2}\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{(\nu+1)/\psi}{\sigma^2}\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&\quad - \frac{(\nu+1)\psi}{\sigma^2}\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&= \frac{1}{\sigma^2} - \frac{2\delta}{\sigma^2}\left(\frac{\psi+\frac{1}{\psi}}{\psi+\frac{1}{\psi}}\right)\left(3\Gamma_l\left(\frac{3}{2},x_c^2\right)+(\nu+1)\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\right) \\
&= \frac{1}{\sigma^2} - \frac{2\delta}{\sigma^2}\left(3\Gamma_l\left(\frac{3}{2},x_c^2\right)+(\nu+1)\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\right) \qquad \text{(C.14)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial \sigma \psi} &= \frac{\partial}{\partial \psi}\left(-\frac{1}{\sigma} + \frac{2}{\sigma}\mathbb{I}_{x \,\in\, B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x \,\in\, A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu}\right. \\
&\qquad \left. + \frac{2}{\sigma}\mathbb{I}_{x \,\in\, C}\left(\frac{x - \mu}{\sigma\psi}\right)^2 + \frac{\nu}{\sigma}\mathbb{I}_{x \,\in\, D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu}\right) \\
&= \frac{\partial}{\partial \psi}\left(-\frac{1}{\sigma} + \frac{2}{\sigma/\psi^2}\mathbb{I}_{x \,\in\, B}\left(\frac{\mu - x}{\sigma}\right)^2 + \frac{\nu}{\sigma/\psi^{\nu}}\mathbb{I}_{x \,\in\, A}\left(\frac{\mu - x}{\sigma}\right)^{\nu}\right. \\
&\qquad \left. + \frac{2}{\sigma\psi^2}\mathbb{I}_{x \,\in\, C}\left(\frac{x - \mu}{\sigma}\right)^2 + \frac{\nu}{\sigma\psi^{\nu}}\mathbb{I}_{x \,\in\, D}\left(\frac{x - \mu}{\sigma}\right)^{\nu}\right) \\
&= \frac{4}{\sigma\psi}\mathbb{I}_{x \,\in\, B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 + \frac{\nu^2}{\sigma\psi}\mathbb{I}_{x \,\in\, A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} \\
&\quad - \frac{4}{\sigma\psi}\mathbb{I}_{x \,\in\, C}\left(\frac{x - \mu}{\sigma\psi}\right)^2 - \frac{\nu^2}{\sigma\psi}\mathbb{I}_{x \,\in\, D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu}
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \sigma/\psi}\right) &= \frac{4}{\sigma\psi}\int_{x \in B}\left(\frac{\mu - x}{\sigma/\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx \\
&\quad + \frac{\nu^2}{\sigma\psi}\int_{x \in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu} f(x;\mu,\sigma,\psi,\nu)dx \\
&\quad - \frac{4}{\sigma\psi}\int_{x \in C}\left(\frac{x - \mu}{\sigma\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx \\
&\quad - \frac{\nu^2}{\sigma\psi}\int_{x \in D}\left(\frac{x - \mu}{\sigma\psi}\right)^{\nu} f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
$$

From the set of integrals in Equation (B.8), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\sigma\psi}\right) &= \frac{4}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right)\\
&+\frac{\nu^2}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&-\frac{4}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right)\\
&-\frac{\nu^2}{\sigma\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&=\frac{2}{\psi^2}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{3}{2},x_c^2\right)\\
&+\frac{\nu}{\psi^2}\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&-\frac{4\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_l\left(\frac{3}{2},x_c^2\right)\\
&-\frac{2\delta\nu}{\sigma\left(\psi+\frac{1}{\psi}\right)}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&=-\frac{2\delta}{\sigma}\left(\frac{\frac{1}{\psi^2}-1}{\psi+\frac{1}{\psi}}\right)\left(2\Gamma_l\left(\frac{3}{2},x_c^2\right)-\nu\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\right) \qquad \text{(C.16)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial\sigma\nu} &= \frac{\partial^2}{\partial\nu}\left(-\frac{1}{\sigma}+\frac{2}{\sigma}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^2+\frac{\nu}{\sigma}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\right.\\
&\left.+\frac{2}{\sigma}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma/\psi}\right)^2+\frac{\nu}{\sigma}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma/\psi}\right)^\nu\right)\\
&=\mathbb{I}_{x\,\in\,A}\left(\frac{1}{\sigma}\cdot\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu+\frac{\nu}{\sigma}\cdot\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\cdot\ln\left(\frac{\mu-x}{\sigma/\psi}\right)\right)\\
&+\mathbb{I}_{x\,\in\,D}\left(\frac{1}{\sigma}\cdot\left(\frac{x-\mu}{\sigma\psi}\right)^\nu+\frac{\nu}{\sigma}\cdot\left(\frac{x-\mu}{\sigma\psi}\right)^\nu\cdot\ln\left(\frac{x-\mu}{\sigma\psi}\right)\right) \qquad \text{(C.17)}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\sigma\nu}\right) &= \frac{1}{\sigma}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu} f(x;\mu,\sigma,\psi,\nu)dx \\
&+ \frac{\nu}{\sigma}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^{\nu}\ln\left(\frac{\mu-x}{\sigma/\psi}\right) f(x;\mu,\sigma,\psi,\nu)dx \\
&+ \frac{1}{\sigma}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu} f(x;\mu,\sigma,\psi,\nu)dx \\
&+ \frac{\nu}{\sigma}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\ln\left(\frac{x-\mu}{\sigma\psi}\right) f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
$$

From the set of integrals in Equation (B.8) and Equation (B.11), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\sigma\nu}\right) &= \frac{1}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{\nu}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{1}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{\nu}{\sigma}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&= \frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{\psi}{\nu}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{1}{\psi\nu}\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&+ \frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\frac{1}{\psi\nu}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&= \frac{2\delta}{\sigma\nu}\left(\frac{\psi+\frac{1}{\psi}}{\psi+\frac{1}{\psi}}\right)\left(\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right)+\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right)\right) \\
&= \frac{2\delta}{\sigma\nu}\left(\Gamma_u\left(\frac{\nu+1}{\nu},x_c^{\nu}\right)+\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right)\right) \qquad \text{(C.18)}
\end{aligned}
$$

C.2.3 Parameter ψ:

$$\begin{aligned}
\frac{\partial^2 LL}{\partial \psi^2} &= \frac{\partial}{\partial \psi}\left(-\frac{1}{\psi+\frac{1}{\psi}}\left(1-\frac{1}{\psi^2}\right)-\frac{2}{\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right)^2-\frac{\nu}{\psi}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\right.\\
&\qquad\left.+\frac{2}{\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)^2+\frac{\nu}{\psi}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu\right)\\
&= \frac{\partial}{\partial \psi}\left(-\frac{1}{\psi+\frac{1}{1\psi}}\left(1-\frac{1}{\psi^2}\right)-\frac{2}{1/\psi}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma}\right)^2-\frac{\nu}{1/\psi^{\nu-1}}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma}\right)^\nu\right.\\
&\qquad\left.+\frac{2}{\psi}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma}\right)^2+\frac{\nu}{\psi^{\nu+1}}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma}\right)^\nu\right)\\
&= \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2}\left(1-\frac{1}{\psi^2}\right)^2-\frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3}\\
&\quad-\frac{2}{\psi^2}\mathbb{I}_{x\,\in\,B}\left(\frac{\mu-x}{\sigma/\psi}\right)^2-\frac{\nu(\nu-1)}{\psi^2}\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\\
&\quad-\frac{6}{\psi^2}\mathbb{I}_{x\,\in\,C}\left(\frac{x-\mu}{\sigma\psi}\right)^2-\frac{\nu(\nu+1)}{\psi^2}\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu
\end{aligned} \tag{C.19}$$

$$\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \psi^2}\right) &= \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2}\left(1-\frac{1}{\psi^2}\right)^2-\frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3}\\
&\quad-\frac{2}{\psi^2}\int_{x\in B}\left(\frac{\mu-x}{\sigma/\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad-\frac{\nu(\nu-1)}{\psi^2}\int_{x\in A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad-\frac{6}{\psi^2}\int_{x\in C}\left(\frac{x-\mu}{\sigma\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx\\
&\quad-\frac{\nu(\nu+1)}{\psi^2}\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}$$

From the set of integrals in Equation (B.8), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \psi^2}\right) &= \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2}\left(1-\frac{1}{\psi^2}\right)^2 - \frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3} \\
&\quad - \frac{2}{\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu(\nu-1)}{\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&\quad - \frac{6}{\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{2}\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu(\nu+1)}{\psi^2}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&= \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2}\left(1-\frac{1}{\psi^2}\right)^2 - \frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3} \\
&\quad - \frac{1}{\psi^3}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu-1}{\psi^3}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&\quad - \frac{3}{\psi}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_l\left(\frac{3}{2},x_c^2\right) \\
&\quad - \frac{\nu+1}{\psi}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right) \\
&= \frac{1}{\left(\psi+\frac{1}{\psi}\right)^2} - \frac{1}{\psi+\frac{1}{\psi}}\cdot\frac{2}{\psi^3}\left(1-\frac{1}{\psi^2}\right)^2 \\
&\quad - \frac{2\delta}{\psi+\frac{1}{\psi}}\left(\left(\frac{1}{\psi^3}+\frac{3}{\psi}\right)\Gamma_l\left(\frac{3}{2},x_c^2\right) + \right. \\
&\qquad \left.\left(\frac{\nu-1}{\psi^3}+\frac{\nu+1}{\psi}\right)\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\right) \qquad \text{(C.20)}
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^2 LL}{\partial \psi \nu} = \frac{\partial^2}{\partial \nu} \Bigg(& -\frac{1}{\psi + \frac{1}{\psi}} \left(1 - \frac{1}{\psi^2}\right) - \frac{2}{\psi} \mathbb{I}_{x \in B} \left(\frac{\mu - x}{\sigma/\psi}\right)^2 - \frac{\nu}{\psi} \mathbb{I}_{x \in A} \left(\frac{\mu - x}{\sigma/\psi}\right)^\nu \\
& + \frac{2}{\psi} \mathbb{I}_{x \in C} \left(\frac{x - \mu}{\sigma\psi}\right)^2 + \frac{\nu}{\psi} \mathbb{I}_{x \in D} \left(\frac{x - \mu}{\sigma\psi}\right)^\nu \Bigg) \\
= & -\mathbb{I}_{x \in A} \left(\frac{1}{\psi} \cdot \left(\frac{\mu - x}{\sigma/\psi}\right)^\nu + \frac{\nu}{\psi} \cdot \left(\frac{\mu - x}{\sigma/\psi}\right)^\nu \cdot \ln\left(\frac{\mu - x}{\sigma/\psi}\right)\right) \\
& + \mathbb{I}_{x \in D} \left(\frac{1}{\psi} \cdot \left(\frac{x - \mu}{\sigma\psi}\right)^\nu + \frac{\nu}{\psi} \cdot \left(\frac{x - \mu}{\sigma\psi}\right)^\nu \cdot \ln\left(\frac{x - \mu}{\sigma\psi}\right)\right) \qquad \text{(C.21)}
\end{aligned}$$

(C.22)

$$\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \sigma \nu}\right) = & -\frac{1}{\psi} \int_{x \in A} \left(\frac{\mu - x}{\sigma/\psi}\right)^\nu f(x; \mu, \sigma, \psi, \nu) dx \\
& - \frac{\nu}{\psi} \int_{x \in A} \left(\frac{\mu - x}{\sigma/\psi}\right)^\nu \ln\left(\frac{\mu - x}{\sigma/\psi}\right) f(x; \mu, \sigma, \psi, \nu) dx \\
& + \frac{1}{\psi} \int_{x \in D} \left(\frac{x - \mu}{\sigma\psi}\right)^\nu f(x; \mu, \sigma, \psi, \nu) dx \\
& + \frac{\nu}{\psi} \int_{x \in D} \left(\frac{x - \mu}{\sigma\psi}\right)^\nu \ln\left(\frac{x - \mu}{\sigma\psi}\right) f(x; \mu, \sigma, \psi, \nu) dx
\end{aligned}$$

From the set of integrals in Equation (B.8) and Equation (B.11), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial\psi\nu}\right) &= -\frac{1}{\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&\quad -\frac{\nu}{\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^\nu\right)\\
&\quad +\frac{1}{\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&\quad +\frac{\nu}{\psi}\cdot\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu^2}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^\nu\right)\\
&= -\frac{1}{\psi^2}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{1}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&\quad -\frac{1}{\psi^2}\cdot\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{1}{\nu}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^\nu\right)\\
&\quad +\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{1}{\nu}\Gamma_u\left(\frac{\nu+1}{2},x_c^\nu\right)\\
&\quad +\frac{2\delta}{\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{1}{\nu}\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^\nu\right)\\
&= \left(\frac{1-\frac{1}{\psi^2}}{\psi+\frac{1}{\psi}}\right)\frac{2\delta}{\nu}\left(\Gamma_u\left(\frac{\nu+1}{\nu},x_c^\nu\right)+\Gamma_u^{(1)}\left(\frac{\nu+1}{\nu},x_c^\nu\right)\right) \qquad \text{(C.23)}
\end{aligned}
$$

C.2.4 Parameter ν :

$$
\begin{aligned}
\frac{\partial^2 LL}{\partial\nu^2} &= \frac{\partial}{\partial\nu}\left(\frac{n}{\delta}\cdot\frac{\partial\delta}{\partial\nu}+\mathbb{I}_{x\,\in\,A\,\cup\,D}\left(\nu x_c^{\nu-1}+2x_c\right)\cdot\frac{\partial}{\partial\nu}x_c-\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\ln\left(\frac{\mu-x}{\sigma/\psi}\right)\right.\\
&\quad\left.-\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu\ln\left(\frac{x-\mu}{\sigma\psi}\right)\right)\\
&= \left(\frac{n}{\delta}\cdot\frac{\partial\delta}{\partial\nu}\right)+\frac{\partial}{\partial\nu}\left(\left(\nu x_c^{\nu-1}+2x_c\right)\cdot\frac{\partial}{\partial\nu}x_c\right)\mathbb{I}_{x\,\in\,A\,\cup\,D}\\
&\quad -\mathbb{I}_{x\,\in\,A}\left(\frac{\mu-x}{\sigma/\psi}\right)^\nu\ln\left(\frac{\mu-x}{\sigma/\psi}\right)^2-\mathbb{I}_{x\,\in\,D}\left(\frac{x-\mu}{\sigma\psi}\right)^\nu\ln\left(\frac{x-\mu}{\sigma\psi}\right)^2\Bigg) \qquad \text{(C.24)}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \nu^2}\right) = &\left(\frac{n}{\delta}\cdot\frac{\partial\delta}{\partial\nu}\right) + \frac{\partial}{\partial\nu}\left(\left(\nu x_c^{\nu-1} + 2x_c\right)\cdot\frac{\partial}{\partial\nu}x_c\right)\cdot \\
&\int_{x\in A\cup D} 1\cdot f(x;\mu,\sigma,\psi,\nu)dx \\
&-\int_{x\in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu}\ln\left(\frac{\mu - x}{\sigma/\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx \\
&-\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\ln\left(\frac{x-\mu}{\sigma\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx \\
= &\left(\frac{n}{\delta}\cdot\frac{\partial\delta}{\partial\nu}\right) + \frac{\partial}{\partial\nu}\left(\left(\nu x_c^{\nu-1} + 2x_c\right)\cdot\frac{\partial}{\partial\nu}x_c\right)\cdot \\
&\left(\int_{x\in A} 1\cdot f(x;\mu,\sigma,\psi,\nu)dx + \int_{x\in D} 1\cdot f(x;\mu,\sigma,\psi,\nu)dx\right) \\
&-\int_{x\in A}\left(\frac{\mu - x}{\sigma/\psi}\right)^{\nu}\ln\left(\frac{\mu - x}{\sigma/\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx \\
&-\int_{x\in D}\left(\frac{x-\mu}{\sigma\psi}\right)^{\nu}\ln\left(\frac{x-\mu}{\sigma\psi}\right)^2 f(x;\mu,\sigma,\psi,\nu)dx
\end{aligned}
$$

From the set of integrals in Equation (B.8) and Equation (B.11), it follows that:

$$
\begin{aligned}
E\left(\frac{\partial^2 LL}{\partial \nu^2}\right) &= \left(\frac{1}{\delta}\cdot\frac{\partial \delta}{\partial \nu}\right) + \frac{\partial}{\partial \nu}\left(\left(\nu x_c^{\nu-1}+2x_c\right)\cdot\frac{\partial}{\partial \nu}x_c\right)\cdot \\
&\quad \left(\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu}\Gamma_u\left(\frac{1}{2},x_c^{\nu}\right)\right. \\
&\quad \left. +\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu}\Gamma_u\left(\frac{1}{2},x_c^{\nu}\right)\right) \\
&\quad -\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma/\psi}{\nu^3}\Gamma_u^{(2)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&\quad -\frac{2\delta}{\sigma\left(\psi+\frac{1}{\psi}\right)}\cdot\frac{\sigma\psi}{\nu^3}\Gamma_u^{(2)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&= \left(\frac{1}{\delta}\cdot\frac{\partial \delta}{\partial \nu}\right) + \frac{\partial}{\partial \nu}\left(\left(\nu x_c^{\nu-1}+2x_c\right)\cdot\frac{\partial}{\partial \nu}x_c\right)\cdot \\
&\quad \frac{2\delta}{\nu}\cdot\left(\frac{\psi+\frac{1}{\psi}}{\psi+\frac{1}{\psi}}\right)\Gamma_u\left(\frac{1}{2},x_c^{\nu}\right) \\
&\quad -\frac{2\delta}{\nu^3}\left(\frac{\psi+\frac{1}{\psi}}{\psi+\frac{1}{\psi}}\right)\Gamma_u^{(2)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right) \\
&= \left(\frac{1}{\delta}\cdot\frac{\partial \delta}{\partial \nu}\right) + \frac{\partial}{\partial \nu}\left(\left(\nu x_c^{\nu-1}+2x_c\right)\cdot\frac{\partial}{\partial \nu}x_c\right)\cdot \\
&\quad \frac{2\delta}{\nu}\cdot\Gamma_u\left(\frac{1}{2},x_c^{\nu}\right)-\frac{2\delta}{\nu^3}\Gamma_u^{(2)}\left(\frac{\nu+1}{\nu},x_c^{\nu}\right)
\end{aligned} \tag{C.25}
$$

Derivatives of δ

The normalising constant, δ, from Equation (3.11) is repeated below:

$\delta^{-1} = 2\left(\frac{1}{2}\Gamma_l\left(\frac{1}{2},x_c\right)+\frac{e^{x_c^{\nu}-x_c^2}}{\nu}\Gamma_u\left(\frac{1}{\nu},x_c\right)\right)$.

The first derivative of δ is given below:

$$
\frac{\partial \delta}{\partial \nu} = -2\delta^2\left(\frac{1}{2}\frac{\partial}{\partial \nu}\Gamma_l\left(\frac{1}{2},x_c\right)+\left(\frac{e^{x_c^{\nu}-x_c^2}}{\nu}\cdot\frac{\partial}{\partial \nu}\Gamma_u\left(\frac{1}{\nu},x_c\right)+\frac{\partial}{\partial \nu}\frac{e^{x_c^{\nu}-x_c^2}}{\nu}\cdot\Gamma_u\left(\frac{1}{\nu},x_c\right)\right)\right), \tag{C.26}
$$

where $x_c = e^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}$.

The elements of the partial derivatives in Equation (C.26) are given next.

$$\begin{aligned}\frac{\partial}{\partial\nu}\Gamma_l\left(\frac{1}{2},x_c\right) &= \frac{\partial}{\partial\nu}\int_0^{x_c} t^{\frac{1}{2}-1}\mathrm{e}^{-t}dt \\ &= x_c^{-\frac{1}{2}}\mathrm{e}^{-x_c}\cdot\frac{\partial}{\partial\nu}\mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}} \\ &= x_c^{-\frac{1}{2}}\mathrm{e}^{-x_c}\cdot\mathrm{e}^{\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\frac{\partial}{\partial\nu}\frac{-\ln(2)+\ln(\nu)}{2-\nu} \\ &= x_c^{\frac{1}{2}}\mathrm{e}^{-x_c}\cdot\frac{\frac{1}{\nu}(2-\nu)-\ln(2)+\ln(\nu)}{(2-\nu)^2}\end{aligned} \tag{C.27}$$

$$\frac{\partial}{\partial\nu}\Gamma_u\left(\frac{1}{\nu},x_c\right) = \frac{\partial}{\partial\nu}\int_{x_c}^{\infty} t^{\frac{1}{\nu}-1}\mathrm{e}^{-t}dt \tag{C.28}$$

$$\begin{aligned}\frac{\partial}{\partial\nu}\frac{\mathrm{e}^{x_c^{\nu}-x_c^2}}{\nu} &= \frac{\nu\cdot\frac{\partial}{\partial\nu}\mathrm{e}^{x_c^{\nu}-x_c^2}-\mathrm{e}^{x_c^{\nu}-x_c^2}}{\nu^2} \\ &= \frac{\nu\left(\mathrm{e}^{x_c^{\nu}-x_c^2}\cdot\frac{\partial}{\partial\nu}\left(\mathrm{e}^{\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}}-\mathrm{e}^{2\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\right)\right)-\mathrm{e}^{x_c^{\nu}-x_c^2}}{\nu^2} \\ &= \frac{\nu\mathrm{e}^{x_c^{\nu}-x_c^2}\left(\frac{\partial}{\partial\nu}\left(\mathrm{e}^{\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}}-\mathrm{e}^{2\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\right)-\nu^{-1}\right)}{\nu^2} \\ &= \nu^{-1}\mathrm{e}^{x_c^{\nu}-x_c^2}\left(\left(\mathrm{e}^{\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot\frac{\partial}{\partial\nu}\left(\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}\right)-\mathrm{e}^{2\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot\right.\right. \\ &\qquad\left.\left.\frac{\partial}{\partial\nu}\left(2\frac{-\ln(2)+\ln(\nu)}{2-\nu}\right)\right)-\nu^{-1}\right) \\ &= \nu^{-1}\mathrm{e}^{x_c^{\nu}-x_c^2}\left(\left(\mathrm{e}^{\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot\left(\frac{-\ln(2)+\ln(\nu)}{2-\nu}+\nu\frac{\partial}{\partial\nu}\frac{-\ln(2)+\ln(\nu)}{2-\nu}\right)\right.\right. \\ &\qquad\left.\left.-\mathrm{e}^{2\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot 2\frac{\frac{1}{\nu}(2-\nu)+(-\ln(2)+\ln(\nu))}{(2-\nu)^2}\right)-\nu^{-1}\right) \\ &= \mathrm{e}^{x_c^{\nu}-x_c^2}\left(\nu^{-1}\left(\mathrm{e}^{\nu\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot\left(\frac{-\ln(2)+\ln(\nu)}{2-\nu}+\nu\frac{\frac{1}{\nu}(2-\nu)-\ln(2)+\ln(\nu)}{(2-\nu)^2}\right)\right.\right. \\ &\qquad\left.\left.-\mathrm{e}^{2\frac{-\ln(2)+\ln(\nu)}{2-\nu}}\cdot 2\frac{\frac{1}{\nu}(2-\nu)-\ln(2)+\ln(\nu)}{(2-\nu)^2}\right)-1\right)\end{aligned} \tag{C.29}$$

The derivatives of δ are analytically tractable as shown above. The second derivative of δ is not given here since it is easier to simply use a numerical approximation for the derivatives of δ.

C.3 Summary

In this appendix the remaining TPTAN Fischer elements were derived for Section 5.6.

Appendix D

Program code

In this appendix the R code of each chapter is given. For the version of R see [?].

D.1 User-defined functions

```
require(colorspace)
require(zipfR)
require(VGAM)
require(sn)
require(GeneralizedHyperbolic)
################################################################
#Color palettes
o2 <- sequential_hcl(3, "Oranges")[1:2]
o3 <- sequential_hcl(4, "Oranges")[1:3]
o4 <- sequential_hcl(5, "Oranges")[1:4]
o5 <- sequential_hcl(6, "Oranges")[1:5]
ber4 <- diverging_hcl(6, "Berlin")[2:5]
ber5 <- diverging_hcl(7, "Berlin")[2:6]
ber6 <- diverging_hcl(8, "Berlin")[2:7]
laj4 <- sequential_hcl(6, "Lajolla")[3:6]
laj5 <- sequential_hcl(7, "Lajolla")[3:7]
```

```
gy3 <- sequential_hcl(5, "ag_GrnYl")[1:3]
gy4 <- sequential_hcl(6, "ag_GrnYl")[1:4]
gy5 <- sequential_hcl(7, "ag_GrnYl")[1:5]
inf3<- sequential_hcl(5, "Inferno")[2:4]
inf4<- sequential_hcl(6, "Inferno")[2:5]
inf5<- sequential_hcl(7, "Inferno")[2:6]
inf6<- sequential_hcl(8, "Inferno")[2:7]
pb4<- diverging_hcl(5, "Purple-Brown")[c(1,2,4,5)]
pb6<- diverging_hcl(7, "Purple-Brown")[c(1,2,3,5,6,7)]
vk4<- diverging_hcl(5, "Vik")[c(1,2,4,5)]
vk4b<- diverging_hcl(4, "Vik")
vk6<- diverging_hcl(7, "Vik")[c(1,2,3,5,6,7)]
vkb6<- diverging_hcl(8, "Vik")[c(1,2,3,6,7,8,9)]
br4<- diverge_hcl(5,'Blue-Red 3')[c(1,2,4,5)]
crk4<-diverge_hcl(5,'Cork')[c(1,2,4,5)]
cmp4<-c(sequential_hcl(5, "Peach")[1:2]
        ,sequential_hcl(5, "Red-Blue")[c(2,1)])
cur_dir<-'/home/matthias/Desktop/Link to mast/
          Report_march/dissertation_march/Graphs'
##########################################################################
# Generalised Phi functions
GNphi <- function(z,s)
{
  if(s>0)
  {
    T1<-s/(2*gamma(1/s))
    T2<-exp(-abs(z)^s)
    d<-T1*T2
  }
  else
  {
```

```
    d<-rep(0,length(z))
  }

  return(d)
}

GNPhi <- function(z,s)
{

  if(s>0)
  {
    pos<-z<=0
    neg<-z>0
    z[pos]<-Igamma(1/s,(-z[pos])^s
                        ,lower = FALSE)/(2*gamma(1/s))
    # for z elements negative
    z[neg]<- 1-Igamma(1/s,z[neg]^s
                        ,lower = FALSE)/(2*gamma(1/s))
    # for z elements positive
    phi<-z
  }
  else
  {
    phi<-rep(0,length(z))
  }

  return(phi)
}
#Generalised normal density function

dGN <- function(x,par)
```

```
{
  mu     <-par[1]
  sigma <-par[2]
  s      <-par[3]
  z      <-(x-mu)/sigma
  if(sigma>0 & s>0)
  {
    d  <- GNphi(z,s)/sigma
  }
  else
  {
    d<-rep(0,length(x))
  }

  return(d)
}
##################################################################
#Generalised normal cumulative density function

cdGN <- function(x,par)
{
  mu     <-par[1]
  sigma <-par[2]
  s      <-par[3]
  z      <-(x-mu)/sigma
  if(sigma>0 & s>0)
  {
    cd <- GNPhi(z,s)
  }
  else
  {
```

```
    cd<-rep(0,length(x))
  }

  return(cd)
}
##################################################################
#Quantiles of Generalised Normal
qGN<-function(y,par)
{
  mu     <-par[1]
  sigma  <-par[2]
  s      <-par[3]
  z<-rep(0,length(y))
  if(all(0<=y & y<=1))
  {
    neg<-y<=0.5
    pos<-y>0.5
    z[neg]<--Igamma.inv(1/s,((y[neg])*2*gamma(1/s))
                              ,lower = FALSE)^(1/s)
    z[pos]<-Igamma.inv(1/s,((1-y[pos])*2*gamma(1/s))
                             ,lower = FALSE)^(1/s)
  }
  x<-sigma*z+mu
  return(x)
}
##################################################################
#Normalising constant Spliced Generalised Normal
normSGN<-function(par,area=FALSE)
{
  mu    <-par[1]
  sigma<-par[2]
```

```
  alpha<-par[3]
  beta <-par[4]
  gam  <-par[5]

  xc<-exp((-log(alpha)+log(beta)+log(gam))/(alpha-beta))
  a<-gamma(1/alpha)/alpha*(0.5-GNPhi(-xc,alpha))
  b<-exp(gam*xc^beta-xc^alpha)*gam^(-1/beta)*gamma(1/beta)/
    beta*GNPhi(-xc/gam^(-1/beta),beta)
  k<-1/(4*(a+b)*sigma)
  if(area)
    return(1/k)
  else
    return(k)
}

###### The PDF of proposed distribution ##
dSGN<-function(x, par)
{
  mu   <-par[1]
  sigma<-par[2]
  alpha<-par[3]
  beta <-par[4]
  gam  <-par[5]
  z<-abs((x-mu)/sigma)
  d<-rep(0,length(x))
  if(sigma>0 & alpha>0 & beta>0 & gam>0 & alpha!=beta )
  {
    zc<-exp((-log(alpha)+log(beta)+log(gam))/(alpha-beta))
    d[z<=zc]<-exp(-z[z<=zc]^alpha)
    d[z>zc]<-exp(-gam*z[z>zc]^beta + gam*zc^beta-zc^alpha)
    d<-normSGN(par)*d
```

```
  }else if(sigma>0 & alpha>0 & beta>0 & gam>0 & gam!=1
            & alpha==beta)
  {
    zc<-exp((-log(alpha)+log(beta)+log(gam))/(alpha-beta))
    d[z<=zc]<-exp(-z[z<=zc]^alpha)
    d[z>zc]<-exp(-gam*z[z>zc]^beta + gam*zc^beta-zc^alpha)
    d<-normSGN(par)*d
  }else if(sigma>0 & alpha>0 & gam==1 & alpha==beta)
  {
    d<-dGN(x,c(mu,sigma,alpha))
  }
  return(d)
}

#####################################################################
#Splined Generalised Normal Cumulative density
cdSGN<-function(x,par)
{
  mu    <-par[1]
  sigma<-par[2]
  alpha<-par[3]
  beta  <-par[4]
  gam   <-par[5]
  z<-(x-mu)/sigma
  cd<-rep(0,length(x))
  if(sigma>0 & alpha>0 & beta>0 & gam>0 & alpha!=beta  )
  {
    zc<-exp((-log(alpha)+log(beta)+log(gam))/(alpha-beta))
    ltail<-z< -zc
    rtail<-zc<z
    cd[ltail]<-(normSGN(c(0,1,tail(par,3)))
```

```
                          *exp(gam*zc^beta-zc^alpha)
                          *2*gam^(-1/beta)*gamma(1/beta)/beta
                          *cdGN(z[ltail],c(0,gam^(-1/beta),beta)))
    cd[rtail]<-1-(normSGN(c(0,1,tail(par,3)))
                            *exp(gam*zc^beta-zc^alpha)
                            *2*gam^(-1/beta)*gamma(1/beta)/beta
                            *cdGN(-z[rtail],c(0,gam^(-1/beta),beta)))
    gc<-(normSGN(c(0,1,tail(par,3)))
             *exp(gam*zc^beta-zc^alpha)
             *2*gam^(-1/beta)*gamma(1/beta)/beta
             *cdGN(-zc,c(0,gam^(-1/beta),beta)))
    lmid<--zc<=z & z<=0
    rmid<-z<=zc & 0<z
    cd[lmid]<-(gc+normSGN(c(0,1,tail(par,3)))
                         *2*gamma(1/alpha)/alpha
                         *(cdGN(z[lmid],c(0,1,alpha))
                           -cdGN(-zc,c(0,1,alpha))))
    cd[rmid]<-1-(gc+normSGN(c(0,1,tail(par,3)))
                           *2*gamma(1/alpha)/alpha
                           *(cdGN(-z[rmid],c(0,1,alpha))
                             -cdGN(-zc,c(0,1,alpha))))
  }
  return(cd)
}

##########################################################################
#Quantiles of the Spliced Generalised Normal
hqSGN<-function(y,par)
{
  mu    <-par[1]
  sigma<-par[2]
```

```
  alpha<-par[3]
  beta <-par[4]
  gam  <-par[5]
  zc<-exp((-log(alpha)+log(beta)+log(gam))/(alpha-beta))
  yc<-cdSGN(-zc,c(0,1,par[3:5]))
  ltail<-y<=yc
  mid<- yc<y & y< 1-yc
  rtail<- 1-yc<=y
  z<-rep(NA,length(y))
  z[ltail]<-qGN((y[ltail]
                  *normSGN(c(0,1,par[3:5]), area = TRUE)
                  *1/exp(gam*zc^beta-zc^alpha)
                  *beta/(2*gam^(-1/beta)*gamma(1/beta)))
                 ,c(0,gam^(-1/beta),beta))
  z[rtail]<- -qGN(((1-y[rtail])
                   *normSGN(c(0,1,par[3:5]), area = TRUE)
                   *1/exp(gam*zc^beta-zc^alpha)
                   *beta/(2*gam^(-1/beta)*gamma(1/beta)))
                  ,c(0,gam^(-1/beta),beta))
  z[mid]<-qGN((normSGN(c(0,1,par[3:5]), area = TRUE)
                *alpha/(2*gamma(1/alpha))
                *(y[mid]-yc + normSGN(c(0,1,par[3:5])
                 ,area=FALSE)*2*gamma(1/alpha)/alpha
                *cdGN(-zc,c(0,1,alpha))))
               ,c(0,1,alpha))
  return(sigma*z+mu)
}
###################################################################
#Simulation of Splised Generalised Normal
rSGN<-function(par,n,seedval)
{
```

```
  set.seed(seedval)
  y<-runif(n)
  return(hqSGN(y,par))
}

###########################################################################
minLL_SGN<-function(x,par)
{
  return(minLL_func(dSGN,x,par))
}
###########################################################################
# Tail adjusted normal distriubtion
dTAN<-function(x, par)
{
  mu    <-par[1]
  sigma<-par[2]
  beta <-par[3]
  d<-dSGN(x, c(mu,sigma,2,beta,1))
  return(d)
}
###########################################################################
# Tail adjusted normal distriubtion
cdTAN<-function(x, par)
{
  mu    <-par[1]
  sigma<-par[2]
  beta <-par[3]
  cd<-cdSGN(x, c(mu,sigma,2,beta,1))
  return(cd)
}
###########################################################################
```

```
#Simulation of tail adjusted normal

rTAN<-function(par,n,seedval)
{
  mu    <-par[1]
  sigma<-par[2]
  beta <-par[3]
  x<-rSGN(c(mu,sigma,2,beta,1),n,seedval)
  return(x)
}

######################################################################
minLL_TAN<-function(x,par)
{
  return(minLL_func(dTAN,x,par))
}


######################################################################
# Skewed Tail adjusted normal distriubtion
dSTAN<-function(x, par)
{
  mu<-par[1]
  sigma<-par[2]
  psi<-par[3]
  k<-par[4]
  z<-(x-mu)/sigma
  d<-rep(0,length(x))
  d[z<0]<-dTAN(z[z<0]*psi,c(0,1,k))
  d[0<=z]<-dTAN(z[0<=z]/psi,c(0,1,k))
  d<-2/(psi+1/psi)*d/sigma
```

```
  return(d)
}
##################################################################
# Skewed Tail adjusted normal distriubtion
cdSTAN<-function(x, par)
{
  mu<-par[1]
  sigma<-par[2]
  psi<-par[3]
  k<-par[4]
  z<-(x-mu)/sigma
  cd<-rep(0,length(x))
  c1<-2/(psi+1/psi)
  cd[z<0]<-c1/psi*cdTAN(z[z<0]*psi,c(0,1,k))
  F0<-c1/psi*cdTAN(0,c(0,1,k))
  cd[0<=z]<-F0+c1*psi*(cdTAN(z[0<=z]/psi,c(0,1,k))-0.5)
  return(cd)
}
##################################################################
# Skewed Tail adjusted normal distriubtion
qSTAN<-function(y, par)
{
  mu<-par[1]
  sigma<-par[2]
  psi<-par[3]
  k<-par[4]
  z<-rep(0,length(y))
  F0<-2/(psi+1/psi)/psi*cdTAN(0,c(0,1,k))
  lind<-0<y&y<F0
  rind<-F0<=y& y<1
  c1<-2/(psi+1/psi)
```

```
  z[lind]<-1/psi*hqSGN(psi/c1*y[lind],c(0,1,2,k,1))
  z[rind]<-psi*hqSGN((y[rind]-F0)/(c1*psi)+0.5,c(0,1,2,k,1))
  x<-mu+sigma*z
  return(x)
}
###################################################################
#Simulation of Splised Generalised Normal
rSTAN<-function(par,n,seedval)
{
  set.seed(seedval)
  y<-runif(n)
  return(qSTAN(y,par))
}
###################################################################
minLL_STAN<-function(x,par)
{
  return(minLL_func(dSTAN,x,par))
}
###################################################################
minKS_STAN<-function(x,cdx,par)
{
  return(minKS_func(cdSTAN,x,cdx,par))
}
###################################################################
cdsn<-function(x,par)
{
  return(psn(x,par[1],par[2],par[3]))
}
###################################################################
minKS_sn<-function(x,cdx,par)
{
```

```
  return(minKS_func(cdsn,x,cdx,par))
}
##########################################################################
ddst<-function(x,par)
{
  return(dst(x,par[1],par[2],par[3],par[4]))
}
##########################################################################
#t distribution
ddt<-function(x,par)
{
  mu    <-par[1]
  sigma<-par[2]
  kappa<-par[3]
  d<-rep(0,length(x))
  if(sigma>0 & kappa>0)
  {
    d<-dt((x-mu)/sigma,kappa)/sigma
  }
  return(d)
}
##########################################################################

minLL_t<-function(x,par)
{
  return(minLL_func(ddt,x,par))
}

##########################################################################
cdst<-function(x,par)
{
```

```
  return(pst(x,par[1],par[2],par[3],par[4]))
}
######################################################################
qqst<-function(x,par)
{
  return(qst(x,par[1],par[2],par[3],par[4]))
}
######################################################################
minLL_st<-function(x,par)
{
  return(minLL_func(ddst,x,par))
}
######################################################################
minKS_st<-function(x,cdx,par)
{
  return(minKS_func(cdst,x,cdx,par))
}
######################################################################
#KDE CDF
cdensity <- function(x,xdat)
{
  bw<-density(xdat)$bw
  rowMeans(pnorm(outer(x,xdat,"-"),0,bw))
}
######################################################################
# minLikelihood function for given
# density_func and parameters
minLL_func <- function(density_func,x,parameters)
{
  maxLL <- -Inf
  LLvec <- density_func(x,parameters)
```

```
  if(all(is.finite(log(LLvec))))
  {
      LLvec <- log(LLvec)
      maxLL <- sum(LLvec)
  }
  minLL <- -maxLL
  return(minLL)
}
######################################################################
# Likelihood criteria
criterion<-function(LL,n,p)
{
  AIC<-2*(p-LL)
  BIC<-p*log(n)-2*LL
  CAIC<-p*(log(n)+1)-2*LL
  HQC<-2*(-LL+p*log(log(n)))
  return(c(AIC,BIC,CAIC,HQC))
}
######################################################################
# Likelihood criteria
bf<-function(LL1,LL2,p1,p2,n)
{
  f<-2*(LL1-LL2)-(p1-p2)*log(n)
  return(f)
}
######################################################################
# minKS function for given parameters cdf_func and parameters

minKS_func<-function(cdf_func,x,cdfvalues,parameters)
{
  n<-length(x)
```

```
  statistic1 <- max(as.vector(abs(cdf_func(x,parameters)
                                   -cdfvalues)))
  statistic2 <- max(as.vector(abs(cdf_func(x,parameters)
                                   +1/n-cdfvalues)))
  statistic <- max(statistic1,statistic2)
  return(statistic)
}
##################################################################
rks<-function(n,m=10000)
{
  u<-matrix(runif(n*m),m,n)
  u<-t(apply(u,1,sort))
  up<-matrix(rep(1:n,m),m,n, byrow=TRUE)/n-u
  lw<-u-matrix(rep(0:(n-1),m),m,n, byrow=TRUE)/n
  ks<-cbind(up,lw)
  ks<-t(apply(ks,1,max))
  return(ks)
}

##################################################################
#KS HYP
cdhyp<-function(x,par)
{
  return(phyperb(x,param = par))
}
minKS_hyp<-function(x,cdx,par)
{
  ks<-tryCatch(minKS_func(cdhyp,x,cdx,par),error= function(e) 1)
  return(ks)
  #return(minKS_func(cdnig,x,cdx,par))
}
```

```
###########################################################################
#KS NIG
cdnig<-function(x,par)
{
  return(pnig(x,param = par))
}
minKS_nig<-function(x,cdx,par)
{
    ks<-tryCatch(minKS_func(cdnig,x,cdx,par)
                 ,error= function(e) 1)
    return(ks)
    #return(minKS_func(cdnig,x,cdx,par))
}
###########################################################################
#KS normal
cdnorm<-function(x,par)
{
  return(pnorm(x))
}
minKS_norm<-function(x)
{
  ecd<-ecdf(x)
  return(minKS_func(cdnorm,x,ecd(x),par))
}
###########################################################################
dfgm<-function(x,par)
{
  d<-apply(par[,2:3],1,function(y) dgamma(x,y[1],scale = y[2]))
  return(d%*%par[1:nrow(par),1])
}
###########################################################################
```

```
dfnm<-function(x,par)
{
  d<-apply(par[,2:3],1,function(y) dnorm(x,y[1], y[2]))
  return(d%*%par[1:nrow(par),1])
}
##########################################################################
dfsnm<-function(x,par)
{
  d<-apply(par[,2:4],1,function(y) dsn(x,y[1], y[2], y[3]))
  return(d%*%par[1:nrow(par),1])
}
```

D.2 Generalised normal density

```
library(colorspace)
source("UserFunc19.R")

######################################################################
b4 <- diverging_hcl(6, "Berlin")[2:5]
setEPS()
mypath <- file.path('Graphs',paste('Spliced_', 'GN_both', ".eps"
                                        , sep = ''))
postscript(mypath, width = 6, height = 3)
par(mar=c(5,5,0.5,0.5))
###
par(mfrow=c(1,2))
xr<-seq(-4.5,4.5,0.01)
plot(xr,dGN(xr,c(0,1,2)), type = 'l', col = b4[1], ylab = 'density'
      , xlab ='x')
points(xr,dGN(xr,c(0,1,1)), col=b4[2], type = 'l')
points(xr,dGN(xr,c(0,1,5)), col = b4[3], type = 'l')
points(xr,(dGN(xr,c(0,1,0.5))), col = b4[4], type = 'l')
```

```
legend('topleft',legend = c('s=0.5','s=1','s=2','s=5'), col = b4
       , lty = c(1,1,1,1),cex = 0.6)

plot(xr,log(dGN(xr,c(0,1,2))), type = 'l', col = b4[1]
     , ylab = 'log-density', xlab ='x')
points(xr,log(dGN(xr,c(0,1,1))), col=b4[2], type = 'l')
points(xr,log(dGN(xr,c(0,1,5))), col = b4[3], type = 'l')
points(xr,log(dGN(xr,c(0,1,0.5))), col = b4[4], type = 'l')
###
dev.off()
```

D.3 Spliced generalised normal derivation

```
library(colorspace)
library(pBrackets)

or2 <- sequential_hcl(3, "Oranges")[1:2]
small<-function(x,a,b)
{
  xc<-exp((-log(a)+log(b)+log(1))/(a-b))
  y<-rep(1,length(x))
  y[x<=xc]<--x[x<=xc]^a
  y[x>xc]<--x[x>xc]^b-xc^a+xc^b
  return(y)
}

############################################################################
#setEPS()
mypath <- file.path('Graphs',paste('gen_kern_', 'der', ".eps"
                                   , sep = ''))
postscript(mypath, width = 6, height = 3)
par(mar=c(5,5,0.5,0.5))
```

```
###
par(mfrow=c(1,2))
a<-2
b<-1/a
xc<-exp((-log(a)+log(b)+log(1))/(a-b))
x<-seq(0,1.3,0.001)

plot(x,-x^a, type = 'l', col = or2[1]
     , ylab = 'log-kernel components', xlab='z')
lines(x,-x^b, col = or2[2] )
abline(v=xc, lty = 4, col = 'black')
abline(a=-xc^a+a*xc^(a-1)*xc,b=-a*xc^(a-1), lty = 3)
abline(a=-xc^b+a*xc^(a-1)*xc,b=-a*xc^(a-1), lty = 3)
brackets(xc, -xc^a, xc, -xc^b, lwd=1)
text(0.5,-(xc^a+xc^b)/2-0.015, labels= 'c', adj=c(0,0))
text(0.5,-1.7, labels= expression('z'[c]), adj=c(0,0))
###
plot(x,small(x,a,b), type = 'l', col = or2[1]
     , ylab = 'log-kernel', xlab='z')
lines(x,-x^a, col = or2[1], lty = 3)
lines(x[x>xc],small(x[x>xc],a,b), col = or2[2])
abline(v=xc, lty = 4, col = 'black')
text(0.5,-0.67, labels= expression('z'[c]), adj=c(0,0))
legend(legend = c(expression(paste('-z'^alpha))
                  ,expression(paste('-'*gamma*'z'^beta)))
       ,col = or2
       ,x ="topright"
       ,lty = c(1,1)
       ,cex = 0.9
       ,bty = "n" )
```

```
par(mfrow=c(1,1))
###
dev.off()
```

D.4 Spliced generalised normal density

```
source("UserFunc19.R")

v3 <- sequential_hcl(4, "Oranges")[1:3]
b4 <- diverging_hcl(6, "Berlin")[2:5]
l4 <- sequential_hcl(6, "Lajolla")[3:6]
gy4 <- sequential_hcl(6, "ag_GrnYl")[1:4]


##################################################################
#1
setEPS()
mypath <- file.path(cur_dir,paste('SGN', 'PDF1', ".eps", sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))
###

xr<-seq(-6,6,0.01)
cx<-0.55
plot(xr,(dSGN(xr,c(0,1,1,0.9,1))), type = 'l', col = b4[1]
     , ylab = 'density', xlab ='x')
points(xr,(dSGN(xr,c(0,1,1.5,0.9,1))), col = b4[2], type = 'l')
points(xr,(dSGN(xr,c(0,1,2,0.9,1))), col = b4[3], type = 'l')
points(xr,(dSGN(xr,c(0,1,3,0.9,1))), col = b4[4], type = 'l')
legend('topleft',legend = c(expression(alpha*'=0.5')
                             ,expression(alpha*'=1.5')
```

```
                                        ,expression(alpha*'=2')
                                        ,expression(alpha*'=3')
                                        ,expression(
                                          beta*'=0.9, '*gamma*'=1'))
            , col = c(b4,'#ffffff'), lty = c(1,1,1,1),cex = cx)
xr<-seq(-2,2,0.01)
plot(xr,log(dSGN(xr,c(0,1,1,0.9,1))), type = 'l', col = b4[1]
      , ylab = 'log-density', xlab ='x')
points(xr,log(dSGN(xr,c(0,1,1.5,0.9,1))), col=b4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,2,0.9,1))), col = b4[3], type = 'l')
points(xr,log(dSGN(xr,c(0,1,3,0.9,1))), col = b4[4], type = 'l')
###
dev.off()
###########################################################################
#1
setEPS()
mypath <- file.path('Graphs',paste('SGN', 'PDF2', ".eps"
                                        , sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))
###
xr<-seq(-6,6,0.01)
plot(xr,dSGN(xr,c(0,1,1.5,2,1)), type = 'l', col = gy4[1]
      , ylab = 'density', xlab ='x')
points(xr,dSGN(xr,c(0,1,1.5,1.5,1)), col=gy4[2], type = 'l')
points(xr,dSGN(xr,c(0,1,1.5,1,1)), col = gy4[3], type = 'l')
points(xr,dSGN(xr,c(0,1,1.5,0.75,1)), col = gy4[4], type = 'l')
legend('topleft',legend = c(expression(beta*'=2')
                                        ,expression(beta*'=1.5')
                                        ,expression(beta*'=1')
```

```
                                    ,expression(beta*'=0.75')
                                    ,expression(
                                      alpha*'=1.5', '*gamma*'=1'))
         , col = c(gy4,'#ffffff'), lty = c(1,1,1,1),cex = cx)
xr<-seq(-2,2,0.01)
plot(xr,log(dSGN(xr,c(0,1,1.5,2,1))), type = 'l', col = gy4[1]
       , ylab = 'density', xlab ='x')
points(xr,log(dSGN(xr,c(0,1,1.5,1.5,1))), col=gy4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.5,1,1))), col = gy4[3], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.5,0.75,1))), col = gy4[4]
         , type = 'l')
###
dev.off()
############################################################################
#3
setEPS()
mypath <- file.path(cur_dir,paste('SGN', 'PDF3', ".eps", sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))

###

xr<-seq(-5,5,0.01)
plot(xr,dSGN(xr,c(0,1,2,1.2,0.9)), type = 'l', col = 14[1]
       , ylab = 'density', xlab ='x', ylim=c(0,0.55))
points(xr,dSGN(xr,c(0,1,2,1.2,1.2)), col=14[2], type = 'l')
points(xr,dSGN(xr,c(0,1,2,1.2,1.5)), col = 14[3], type = 'l')
legend('topleft',legend = c(expression(gamma*'=0.9')
                                    ,expression(gamma*'=1.2')
                                    ,expression(gamma*'=1.5')
```

```
                                    ,expression(
                                        alpha*'=2, '*beta*'=1.2'))
        , col = c(l4[1:3], '#ffffff'), lty = c(1,1,1,1),cex = cx)

xr<-seq(-3,3,0.01)
r<-range(log(dSGN(xr,c(0,1,1.2,1,2))))
plot(xr,log(dSGN(xr,c(0,1,1.2,1,0.9))), type = 'l', col = l4[1]
     , ylab = 'density', xlab ='x', ylim=(r+0.05))
points(xr,log(dSGN(xr,c(0,1,1.2,1,1.2))), col=l4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.2,1,1.5))), col = l4[3], type = 'l')

###
dev.off()

v3 <- sequential_hcl(4, "Oranges")[1:3]
b4 <- diverging_hcl(6, "Berlin")[2:5]
l4 <- sequential_hcl(6, "Lajolla")[3:6]
gy4 <- sequential_hcl(6, "ag_GrnYl")[1:4]


###########################################################
#1
setEPS()
mypath <- file.path(cur_dir,paste('SGN', 'PDF1', ".eps", sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))
###

xr<-seq(-6,6,0.01)
cx<-0.55
```

```
plot(xr,(dSGN(xr,c(0,1,1,0.9,1))), type = 'l', col = b4[1]
      , ylab = 'density', xlab ='x')
points(xr,(dSGN(xr,c(0,1,1.5,0.9,1))), col = b4[2], type = 'l')
points(xr,(dSGN(xr,c(0,1,2,0.9,1))), col = b4[3], type = 'l')
points(xr,(dSGN(xr,c(0,1,3,0.9,1))), col = b4[4], type = 'l')
legend('topleft',legend = c(expression(alpha*'=0.5')
                            ,expression(alpha*'=1.5')
                            ,expression(alpha*'=2')
                            ,expression(alpha*'=3')
                            ,expression(
                              beta*'=0.9, '*gamma*'=1'))
        , col = c(b4,'#ffffff'), lty = c(1,1,1,1),cex = cx)
xr<-seq(-2,2,0.01)
plot(xr,log(dSGN(xr,c(0,1,1,0.9,1))), type = 'l', col = b4[1]
      , ylab = 'log-density', xlab ='x')
points(xr,log(dSGN(xr,c(0,1,1.5,0.9,1))), col=b4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,2,0.9,1))), col = b4[3], type = 'l')
points(xr,log(dSGN(xr,c(0,1,3,0.9,1))), col = b4[4], type = 'l')
###
dev.off()
##########################################################################
#1
setEPS()
mypath <- file.path('Graphs',paste('SGN', 'PDF2', ".eps"
                                   , sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))
###
xr<-seq(-6,6,0.01)
plot(xr,dSGN(xr,c(0,1,1.5,2,1)), type = 'l', col = gy4[1]
```

```
      , ylab = 'density', xlab ='x')
points(xr,dSGN(xr,c(0,1,1.5,1.5,1)), col=gy4[2], type = 'l')
points(xr,dSGN(xr,c(0,1,1.5,1,1)), col = gy4[3], type = 'l')
points(xr,dSGN(xr,c(0,1,1.5,0.75,1)), col = gy4[4], type = 'l')
legend('topleft',legend = c(expression(beta*'=2')
                              ,expression(beta*'=1.5')
                              ,expression(beta*'=1')
                              ,expression(beta*'=0.75')
                              ,expression(
                                alpha*'=1.5, '*gamma*'=1'))
        , col = c(gy4,'#ffffff'), lty = c(1,1,1,1),cex = cx)
xr<-seq(-2,2,0.01)
plot(xr,log(dSGN(xr,c(0,1,1.5,2,1))), type = 'l', col = gy4[1]
      , ylab = 'density', xlab ='x')
points(xr,log(dSGN(xr,c(0,1,1.5,1.5,1))), col=gy4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.5,1,1))), col = gy4[3], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.5,0.75,1))), col = gy4[4]
        , type = 'l')
###
dev.off()
############################################################################
#3
setEPS()
mypath <- file.path(cur_dir,paste('SGN', 'PDF3', ".eps"
                                  , sep = ''))
postscript(mypath, width = 6.5, height = 3)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))

###
```

```
xr<-seq(-5,5,0.01)
plot(xr,dSGN(xr,c(0,1,2,1.2,0.9)), type = 'l', col = l4[1]
      , ylab = 'density', xlab ='x', ylim=c(0,0.55))
points(xr,dSGN(xr,c(0,1,2,1.2,1.2)), col=l4[2], type = 'l')
points(xr,dSGN(xr,c(0,1,2,1.2,1.5)), col = l4[3], type = 'l')
legend('topleft',legend = c(expression(gamma*'=0.9')
                               ,expression(gamma*'=1.2')
                               ,expression(gamma*'=1.5')
                               ,expression(
                                 alpha*'=2, '*beta*'=1.2'))
        , col = c(l4[1:3],'#ffffff'), lty = c(1,1,1,1),cex = cx)

xr<-seq(-3,3,0.01)
r<-range(log(dSGN(xr,c(0,1,1.2,1,2))))
plot(xr,log(dSGN(xr,c(0,1,1.2,1,0.9))), type = 'l', col = l4[1]
      , ylab = 'density', xlab ='x', ylim=(r+0.05))
points(xr,log(dSGN(xr,c(0,1,1.2,1,1.2))), col=l4[2], type = 'l')
points(xr,log(dSGN(xr,c(0,1,1.2,1,1.5))), col = l4[3], type = 'l')

###
dev.off()
par(mfrow=c(1,1))
```

D.5 Tail-adjusted normal density

```
source("UserFunc19.R")

v3 <- sequential_hcl(4, "Oranges")[1:3]
b4 <- diverging_hcl(6, "Berlin")[2:5]
l4 <- sequential_hcl(6, "Lajolla")[3:6]
gy4 <- sequential_hcl(6, "ag_GrnYl")[1:4]
```

```
################################################################
#1
setEPS()
mypath <- file.path(cur_dir,paste('TAN_', 'DERI', ".eps", sep = ''))
postscript(mypath, width = 6.5, height = 6.5)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(2,2))
###
xr<-seq(-4,4,0.01)
  plot(xr,(dSGN(xr,c(0,sqrt(2),2,2.7,1))), type = 'l'
        , col = gy4[1], ylab = 'density', xlab ='x')
lines(xr,dnorm(xr), lty = 2)
points(xr,(dSGN(xr,c(0,sqrt(2),2,2.2,1))), col=gy4[2]
        , type = 'l')
points(xr,(dSGN(xr,c(0,sqrt(2),2,1.7,1))), col = gy4[3]
        , type = 'l')
points(xr,(dSGN(xr,c(0,sqrt(2),2,1.2,1))), col = gy4[4]
        , type = 'l')

legend('topleft',legend = c('N(0,1)'
                              ,expression(beta*'=2.7')
                              ,expression(beta*'=2.2')
                              ,expression(beta*'=1.7')
                              ,expression(beta*'=1.2')
                              ,expression(alpha*'=2, '*gamma*'=1'))
, col = c('#000000',gy4,'#ffffff'), lty = c(2,1,1,1,1),cex = cx)

  plot(xr,log(dSGN(xr,c(0,sqrt(2),2,2.7,1))), type = 'l'
        , col = gy4[1], ylab = 'log density', xlab ='x')
 lines(xr,log(dnorm(xr)), lty = 2, col = 'black')
```

```
points(xr,log(dSGN(xr,c(0,sqrt(2),2,2.2,1))), col=gy4[2]
          , type = 'l')
points(xr,log(dSGN(xr,c(0,sqrt(2),2,1.7,1))), col = gy4[3]
          , type = 'l')
points(xr,log(dSGN(xr,c(0,sqrt(2),2,1.2,1))), col = gy4[4]
          , type = 'l')

###
#2

plot(xr,dnorm(xr), type = 'l', lty = 2, col = 'black'
       , ylab = 'density', xlab ='x')
lines(xr,(dSGN(xr,c(0,sqrt(2),2,1,1.5))), col = gy4[1])
points(xr,(dSGN(xr,c(0,sqrt(2),2,1,1.1))), col=gy4[2]
          , type = 'l')
points(xr,(dSGN(xr,c(0,sqrt(2),2,1,0.8))), col = gy4[3]
          , type = 'l')
points(xr,(dSGN(xr,c(0,sqrt(2),2,1,0.5))), col = gy4[4]
          , type = 'l')

legend('topleft',legend = c('N(0,1)'
                            ,expression(gamma*'=1.5')
                            ,expression(gamma*'=1.1')
                            ,expression(gamma*'=0.8')
                            ,expression(gamma*'=0.5')
                            ,expression(alpha*'=2, '*beta*'=1'))
, col = c('#000000',gy4,'#ffffff'), lty = c(2,1,1,1,1),cex = cx)

plot(xr,log(dnorm(xr)), type = 'l', lty = 2, col = 'black'
       , ylab = 'log-density', xlab ='x')
lines(xr,log(dSGN(xr,c(0,sqrt(2),2,1,1.5))), col = gy4[1])
```

```
points(xr,log(dSGN(xr,c(0,sqrt(2),2,1,1.1))), col=gy4[2]
        , type = 'l')
points(xr,log(dSGN(xr,c(0,sqrt(2),2,1,0.8))), col = gy4[3]
        , type = 'l')
points(xr,log(dSGN(xr,c(0,sqrt(2),2,1,0.5))), col = gy4[4]
        , type = 'l')
###
dev.off()
par(mfrow=c(1,1))
```

D.6 Two-piece tail-adjusted normal density

```
source("UserFunc19.R")
##########################################################################
#1
setEPS()
mypath <- file.path(cur_dir,paste('TPTAN_', 'DERI', ".eps"
                                  , sep = ''))
postscript(mypath, width = 6, height = 2.5)
par(mar=c(5,5,0.5,0.5))
par(mfrow=c(1,2))
###
cx<-0.4
xr<-seq(-4,4,0.01)
yr<-1.1*c(0,max(dSTAN(xr,c(0,1,1/0.75,1.5))))
  plot(xr,(dSTAN(xr,c(0,1,0.8,1.5))), type = 'l', col = ber5[1]
        , ylab = 'density', xlab ='x', ylim = yr,lty =5)
lines(xr,(dSTAN(xr,c(0,1,0.5,1.5))), col=ber5[2], lty = 5)
lines(xr,(dSTAN(xr,c(0,1,1,1.5))), col = ber5[3], type = 'l'
       , lty = 1)
lines(xr,(dSTAN(xr,c(0,1,1/0.8,1.5))), col = ber5[4], lty =5)
lines(xr,(dSTAN(xr,c(0,1,1/0.5,1.5))), col = ber5[5], lty =5)
```

```
legend('topleft',legend = c(expression(psi*'=0.5')
                           ,expression(psi*'=0.8')
                           ,expression(psi*'=1')
                           ,expression(psi*'=2')
                           ,expression(psi*'=1/0.8')
                           ,expression(
                             mu*'=0, '*sigma*'=1,'*nu*'=1'))
          , col = c(ber5,'#ffffff'), lty = c(5,5,1,5,5),cex = cx)
##########################################################################
plot(xr,log(dSTAN(xr,c(0,1,0.75,1.5))), type = 'l', col = ber5[1]
     , ylab = 'log-density', xlab ='x', lty = 5)
lines(xr,log(dSTAN(xr,c(0,1,0.85,1.5))), col=ber5[2], lty = 5)
lines(xr,log(dSTAN(xr,c(0,1,1,1.5))), col = ber5[3], lty = 1)
lines(xr,log(dSTAN(xr,c(0,1,1/0.85,1.5))), col = ber5[4], lty = 5)
lines(xr,log(dSTAN(xr,c(0,1,1/0.75,1.5))), col = ber5[5], lty = 5)
##########################################################################
dev.off()
par(mfrow=c(1))
```

D.7 Bitcoin application

```
library(zipfR)
library(sn)
library(stargazer)
library(GeneralizedHyperbolic)
library(optimx)
library(tseries)


btc<- read.csv(file="/home/matthias/Desktop/Data/BTC_cur.csv"
                                  , header=TRUE
```

```
                                        , sep=",")

source("UserFunc19.R")
obs<-btc$USD
n<-length(obs)
end<-as.Date("2019-06-23")
start<-end-length(obs)+1
start
end
start-end
n
dates<-seq(start, end, 'day')
dates[c(70,2059)]
diff(dates[c(70,2059)])
##############################################################################
  setEPS()
 mypath <- file.path(cur_dir,paste('BTC_', 'USD', ".eps"
                                                   , sep = ''))
 postscript(mypath, width = 6, height = 3.5)
  par(mar=c(3,4,0.5,0.5))
  par(mfrow=c(1,2))
  ###
  plot(dates,(obs), type = 'l', ylab ='Price USD', xlab = ''
        , yaxt='n', xaxt = 'n', col = inf4[1])
  axis.Date(1,dates,cex.axis=0.9, las =1)
  axis(2,cex.axis=0.8, las =2)

  plot(dates,log(obs), type = 'l', ylab ='log Price USD'
        , xlab = '', yaxt='n', col = inf4[1])
  axis(2,cex.axis=0.8, las = 2)
  abline(v=dates[c(70,2059)], lty = 2, col = inf5[3])
```

```
  ###
  par(mfrow=c(1,1))
  dev.off()

which(obs[1:500]==max(obs[1:500]))
which(obs==max(obs))

which(obs==min(obs))
which(obs==min(obs[1695:2247]))

obs<-obs[70:2059]
###
obs<-diff(log(obs))
n<-length(obs)
summary(obs)
hist(obs, breaks = 'fd')
moments::kurtosis(obs)
moments::skewness(obs)



ptm <- proc.time()
meth<-'Nelder-Mead'
#meth<-'BFGS'
fit_sn<-selm(obs~ 1 , family="SN")
fit_st<-optim(fn = minLL_st, c(mean(obs),var(obs),0,2), x=obs
              , method = meth)
fit_hyp<- hyperbFit(obs)
fit_nig <- nigFit(obs)
fit_STAN<-optim(fn = minLL_STAN, c(mean(obs),sd(obs),2,0.5)
                , x=obs, method = meth)
```

```
fit_bt<-optim(fn = minLL_bt, c(mean(obs),sd(obs),1,1,10)
                , x=obs, method = meth)
proc.time() - ptm
#####################################################################
setEPS()
mypath <- file.path(cur_dir,paste('TPTAN_', 'BTC', ".eps"
                                        , sep = ''))
postscript(mypath, width = 6, height = 3.5)
par(mar=c(5,5,0.5,0.5))
###

par(mfrow=c(1,2))
den<-density(obs)
ylim<-c(0,1.1*max(den$y))
hist(obs,breaks = seq(min(obs), max(obs), length.out = 21)
     ,ylab = 'density', xlab = 'log returns', probability = TRUE
     , main = '', ylim = ylim)
lines(den$x,den$y, lty=2)
lines(den$x,(ddst(den$x,fit_st$par)), col=inf5[2])
lines(den$x,(dhyperb(den$x,param = fit_hyp$param)), col = inf5[3])
lines(den$x,(dnig(den$x,param = fit_nig$param)), col = inf5[4])
lines(den$x,(dSTAN(den$x,fit_STAN$par)), col= inf5[5])
box()

plot(den$x,log(den$y), type = 'l',lty = 3, xlab='log returns'
     ,ylab='log-density')
lines(den$x,log(ddst(den$x,fit_st$par)), col= inf5[2])
lines(den$x,log(dhyperb(den$x,param = fit_hyp$param))
      , col = inf5[3])
lines(den$x,log(dnig(den$x,param = fit_nig$param))
      , col = inf5[4])
```

```
lines(den$x,log(dSTAN(den$x,fit_STAN$par)),  col=inf5[5])

legend(legend = c("ST"
                  ,'HYP'
                  ,'NIG'
                  ,"TPTAN"
                  ,'KDE')
       ,col = c(inf5[2:5],'#000000')
       ,x ="bottom"
       ,lty = c(1,1,1,1,3)
       ,cex = 0.5
       ,bty = "n"  )
par(mfrow=c(1,1))
###
dev.off()


summary(obs)

#Prelim
criterion(logLik(fit_sn),  length(obs),3)
criterion(-fit_st$value,  length(obs),4)
criterion(fit_hyp$maxLik,length(obs),4)
criterion(fit_nig$maxLik,length(obs),4)
criterion(-fit_STAN$value,length(obs),4)
p1<-4
p2<-c(3,4,4,4)
LL1<--fit_STAN$value
LL2<-c(logLik(fit_sn),-fit_st$value,fit_hyp$maxLik,fit_nig$maxLik)
bf(LL1,LL2,p1,p2,n)
########################################################################
```

```
#criterion
rsn<-criterion(logLik(fit_sn), length(obs),3)
rst<-criterion(-fit_st$value, length(obs),4)
rhyp<-criterion(fit_hyp$maxLik,length(obs),4)
rnig<-criterion(fit_nig$maxLik,length(obs),4)
rstan<-criterion(-fit_STAN$value,length(obs),4)

tabl<-rbind(rsn,rst,rhyp,rnig,rstan)
tabl<-sweep(tabl,2,tabl[1,])
tabl<-rbind(rsn,tabl)
rownames(tabl)<-c('Base','SN','ST','HYP','NIG','TPTAN')
colnames(tabl)<-c("AIC","BIC","CAIC",'HQIC')
stargazer(tabl,title =
      "Goodness-of-fit Statistics for the Bitcoin Returns Data.")
####################################################################
#MLE's
rsn<-c(coef(fit_sn),0)
rst<-fit_st$par
rhyp<-fit_hyp$param
rnig<-fit_nig$param
rstan<-fit_STAN$par
#Bayes Factor
p1<-4
p2<-c(3,4,4,4)
LL1<--fit_STAN$value
LL2<-c(logLik(fit_sn),-fit_st$value,fit_hyp$maxLik,fit_nig$maxLik)
rbay<-c(bf(LL1,LL2,p1,p2,n),0)


tabl<-cbind(rbind(rsn,rst,rhyp,rnig,rstan),c(LL2,LL1),rbay)
rownames(tabl)<-c('SN','ST','HYP','NIG','TPTAN')
```

```
colnames(tabl)<-c("mu","sigma","psi","nu",'LL','Bayes Factor')
stargazer(tabl,title =
"Maximum Likelihood Estimates and Bayes Factor statistics for
the Bitcoin Returns Data.")
##################################################################
#KS
cd<-ecdf(obs)
cdobs<-cd(obs)
fit_sn_KS<-optimx(fn = minKS_sn,
                  par=c(0.005028788,0.02835759,-0.09824833)
                  ,x=obs,cdx=cdobs,itnmax=300
                  ,method = 'Nelder-Mead')
fit_st_KS<-optimx(fn = minKS_st
                  ,par=(c(0.003345542,0.01691121,-0.0502641
                        7,1.223438))
                  ,x=obs,cdx=cdobs,itnmax=300
                  ,method = 'Nelder-Mead')
fit_hyp_KS<-optimx(fn = minKS_hyp
                   ,par=c(0.001559097,3.882156e-06,39.97874
                          ,1.435293 ), x=obs,cdx=cdobs
                   ,itnmax=300, method = 'Nelder-Mead')
#487.06 seconds
fit_nig_KS<-optimx(fn = minKS_nig, par=(c(0.002407458,0.01679672
                                          ,3.584515,0.0700541))
                   , x=obs,cdx=cdobs,itnmax=150
                   , method = 'Nelder-Mead')
fit_STAN_KS<-optimr(fn = minKS_STAN, fit_STAN$par
                    , x=obs,cdx=cdobs)
alarm()

plot(cd)
```

```
lines(den$x,cdSTAN(den$x,fit_STAN_KS$par), col = 'red', lty =3)
###################################################################
#KSDE's
#KS sn
rsn<-c(0.005028788,0.02835759,-0.09824833)
rst<-c(0.00333748,0.01691756,-0.04945014,1.224695)
rhyp<-c(0.001559321,2.131801e-09,39.97878,1.435017)
rnig<-c(0.002407458,0.01679672,3.584515,0.0700541)
rstan<-fit_STAN_KS$par
tabl1<-rbind(rsn,rst,rhyp,rnig,rstan)

rsn<-c(0.05344099,0)
rst<-c(0.01970746,0.4153987)
rhyp<-c(0.02726657,0.1027905)
rnig<-c(0.01725588, 0.5879627)
rstan<-c(0.01224732, 0.9453627)
tabl2<-rbind(rsn,rst,rhyp,rnig,rstan)
tabl<-cbind(tabl1,tabl2)
rownames(tabl)<-c('SN','ST','HYP','NIG','TPTAN')
colnames(tabl)<-c("mu","sigma","psi","nu","KS","p-value")
stargazer(tabl,title =
"Minimum KS Distance Estimates, KS distance, and p-values for
the Munich Rent Data.")
###################################################################
#KS sn
#c(0.05444333, 0)
#c(0.005209598,0.02817458,-0.09972193)

#KS st
#c(0.01970746,0.4153987)
#c(0.00333748,0.01691756,-0.04945014,1.224695)
```

```
#KS hyp
#c(0.02726657,0.1027905)
#c(0.001559321,2.131801e-09,39.97878,1.435017)
#KS nig
# c(0.01725588, 0.5879627)
# c(0.002407458,0.01679672,3.584515,0.0700541)
#KS STAN
#0.01224732   0.9453627
#######################################################################

n
m<-round(100000/n*1000)
 ksdist<-rks(length(obs),m)
sum(ksdist >=0.05444333)/m
sum(ksdist >=0.01970746)/m
sum(ksdist >=0.02726657)/m
sum(ksdist >=0.01725588)/m
sum(ksdist>=minKS_STAN(obs,cdobs,fit_STAN_KS$par))/m
minKS_STAN(obs,cdobs,fit_STAN_KS$par)


sum(ksdist>=minKS_nig(obs, cdobs,fit_nig$param))/m
sum(ksdist>=minKS_STAN(obs,cdobs,fit_STAN$par))/m
```

D.8 Munich application

```
library(optimx)
library(zipfR)
library(VGAM)
library(gamlss.data)
library(NMOF)
```

```
library(logKDE)
library(sn)
library(stargazer)
source("UserFunc19.R")

###
head(rent99)
obs<-rent99$rentsqm
###

hist(obs, breaks = 'FD')
# hist(log(obs), breaks = 'FD')
# obs<-log(obs)
n<-length(obs)
n
meth<-"Nelder-Mead"#, "BFGS", "CG", "L-BFGS-B", "SANN","Brent"
fit_sn<-selm(obs~ 1 , family="SN")
fit_st<-optim(fn = minLL_st, c(mean(obs),sd(obs),0,10), x=obs
                    , method = meth)
fit_hyp<- hyperbFit(obs)
fit_nig <- nigFit(obs)
fit_STAN<-optim(fn = minLL_STAN, c(mean(obs),sd(obs),2,0.5)
                    , x=obs, method = meth)

########################################################################
setEPS()
mypath <- file.path(cur_dir,paste('TPTAN_', 'Munich', ".eps"
                                          , sep = ''))
postscript(mypath, width = 6, height = 3.5)
par(mar=c(5,5,0.5,0.5))
###
```

```
par(mfrow=c(1,2))
den<-density(obs)
#den<-logdensity(obs)
ylim<-c(0,1.1*max(den$y))
hist(obs,breaks = seq(min(obs), max(obs), length.out = 21)
      ,ylab = 'density', xlab = 'rent per month'
      , probability = TRUE, main = '', ylim = ylim)
lines(den$x,den$y, lty=2)
lines(den$x,(ddst(den$x,fit_st$par)), col=inf5[2])
lines(den$x,(dhyperb(den$x,param = fit_hyp$param)), col = inf5[3])
lines(den$x,(dnig(den$x,param = fit_nig$param)), col = inf5[4])
lines(den$x,(dSTAN(den$x,fit_STAN$par)), col= inf5[5])
box()

plot(den$x,log(den$y), type = 'l',lty = 3, xlab='rent per month'
      ,ylab='log-density')
lines(den$x,log(ddst(den$x,fit_st$par)), col= inf5[2])
lines(den$x,log(dhyperb(den$x,param = fit_hyp$param))
       , col = inf5[3])
lines(den$x,log(dnig(den$x,param = fit_nig$param))
       , col = inf5[4])
lines(den$x,log(dSTAN(den$x,fit_STAN$par)), col=inf5[5])

legend(legend = c("ST"
  ,'HYP'
  ,'NIG'
  ,"TPTAN"
  ,'KDE')
  ,col = c(inf5[2:5],'#000000')
  ,x ="bottom"
```

```
    ,lty = c(1,1,1,1,3)
    ,cex = 0.5
    ,bty = "n" )
par(mfrow=c(1,1))
######
dev.off()
#Prelim
criterion(logLik(fit_sn), length(obs),3)
criterion(-fit_st$value, length(obs),4)
criterion(fit_hyp$maxLik,length(obs),4)
criterion(fit_nig$maxLik,length(obs),4)
criterion(-fit_STAN$value,length(obs),4)
p1<-4
p2<-c(3,4,4,4)
LL1<--fit_STAN$value
LL2<-c(logLik(fit_sn),-fit_st$value,fit_hyp$maxLik,fit_nig$maxLik)
bf(LL1,LL2,p1,p2,n)

##########################################################################
#Criterion
rsn<-criterion(logLik(fit_sn), length(obs),3)
rst<-criterion(-fit_st$value, length(obs),4)
rhyp<-criterion(fit_hyp$maxLik,length(obs),4)
rnig<-criterion(fit_nig$maxLik,length(obs),4)
rstan<-criterion(-fit_STAN$value,length(obs),4)

tabl<-rbind(rsn,rst,rhyp,rnig,rstan)
tabl<-sweep(tabl,2,tabl[1,])
tabl<-rbind(rsn,tabl)
rownames(tabl)<-c('Base','SN','ST','HYP','NIG','TPTAN')
colnames(tabl)<-c("AIC","BIC","CAIC",'HQIC')
```

```
stargazer(tabl, title =
"Goodness-of-fit Statistics for the Munich Rent Data.")
#####################################################################
#MLE's
rsn<-c(coef(fit_sn),0)
rst<-fit_st$par
rhyp<-fit_hyp$param[c(1,2,4,3)]
rnig<-fit_nig$param[c(1,2,4,3)]
rstan<-fit_STAN$par
#Bayes Factor
p1<-4
p2<-c(3,4,4,4)
LL1<--fit_STAN$value
LL2<-c(logLik(fit_sn),-fit_st$value, fit_hyp$maxLik, fit_nig$maxLik)
rbay<-c(bf(LL1,LL2,p1,p2,n),0)


tabl<-cbind(rbind(rsn, rst, rhyp, rnig, rstan),c(LL2,LL1),rbay)
rownames(tabl)<-c('SN', 'ST', 'HYP', 'NIG', 'TPTAN')
colnames(tabl)<-c("mu","sigma","psi","nu", 'LL', 'Bayes Factor')
stargazer(tabl, title =
"Maximum Likelihood Estimates and Bayes Factor statistics for
the Munich Rent Data.")
#####################################################################
#KS
cd<-ecdf(obs)
cdobs<-cd(obs)
fit_sn_KS<-optimx(fn = minKS_sn,
par=c(0.005028788,0.02835759,-0.09824833), x=obs
,cdx=cdobs,itnmax=312, method = 'Nelder-Mead')
fit_st_KS<-optimx(fn = minKS_st,
```

```
par=c(7.434904,2.500385, -0.2193032,100), x=obs,cdx=cdobs
,itnmax=600, method = 'Nelder-Mead')
fit_hyp_KS<-optimx(fn = minKS_hyp,
par=fit_hyp$param, x=obs,cdx=cdobs,itnmax=600
, method = 'Nelder-Mead')
fit_nig_KS<-optimx(fn = minKS_nig
, par=c(0.002407458,0.01679672,3.584515,0.0700541), x=obs
,cdx=cdobs,itnmax=150, method = 'Nelder-Mead')
fit_STAN_KS<-optimr(fn = minKS_STAN, fit_STAN$par, x=obs
                    ,cdx=cdobs)
alarm()

plot(cd)
lines(den$x,cdSTAN(den$x,fit_STAN_KS$par))

####################################################################
#KSDE's
#KS sn
rsn<-c(7.434904,2.500385,-0.2193032,0.02336378)
rst<-c(7.147698,2.461011,-0.07060769,180.1513)
rhyp<-c(7.719625,70.54082,11.63761,-0.117212)
rnig<-c(-7.388795,8.391496,10.16419,8.800501)
rstan<-c(6.311423,3.866365,1.221137,2.642361)

tabl1<-rbind(rsn,rst,rhyp,rnig,rstan)
rsn<-c(0.02336378,0.06644887)
rst<-c(0.02364365,0.06185662)
rhyp<-c(0.02336102,0.06644887)
rnig<-c(0.0185411,0.2327251)
rstan<-c(0.01280533,0.6875116)
```

```
tabl2<-rbind(rsn,rst,rhyp,rnig,rstan)
tabl<-cbind(tabl1,tabl2)
rownames(tabl)<-c('SN','ST','HYP','NIG','TPTAN')
colnames(tabl)<-c("mu","sigma","psi","nu","KS","p-value")
stargazer(tabl,title =
"Minimum KS Distance Estimates, KS distance, and p-values for
the Munich Rent Data.")
####################################################################

n
m<-round(100000/n*1000)
ksdist<-rks(length(obs),m)
sum(ksdist >=0.02336378)/m
sum(ksdist >=0.02364365)/m
sum(ksdist >=0.02336102)/m
sum(ksdist >=0.0185411)/m
sum(ksdist >=0.01280533)/m
minKS_STAN(obs,cdobs,fit_STAN_KS$par)


sum(ksdist>=minKS_nig(obs, cdobs,fit_nig$param))/m
sum(ksdist>=minKS_STAN(obs,cdobs,fit_STAN$par))/m
```

D.9 Simulation study

```
library(zipfR)
library(sn)
library(stargazer)
library(GeneralizedHyperbolic)
library(optimx)
library(tseries)
library(doParallel)
```

```
library(ggplot2)
library(NMOF)
library(R.utils)
source("UserFunc19.R")
# mu Location parameter    , default is 0.
# delta Scale parameter    , default is 1.
# alpha Tail parameter    , default is 1.
# beta Skewness parameter    , default is 0.
# lambda Shape parameter    , default is 1.
perc<-c(0.01,0.05,0.1,0.25,0.5,0.75,0.90,0.95,0.99)

par<-c(0.5,10,0.7,0.55,-0.02)
obs<-rghyp(10000,param = par)
den<-density(obs)
plot(den$x,log(den$y), type = 'l',lty = 3, xlab='rent per month'
      ,ylab='log-density')

hist(obs, breaks = 'fd')
qtrue<-qghyp(perc, param = par)
abline(v=qtrue)
abline(v=quantile(obs,perc), col = 'red', lty=2)
moments::kurtosis(obs)
moments::skewness(obs)
###############################################################
obs<-rghyp(1000,param = par)
meth<-'Nelder-Mead'
#meth<-'BFGS'
fit_sn<-selm(obs~ 1 , family="SN")
fit_st<-optim(fn = minLL_st, c(mean(obs),var(obs),0,2), x=obs
                 , method = meth)
fit_hyp<- hyperbFit(obs)
```

```
fit_nig <- nigFit(obs)
fit_STAN<-optim(fn = minLL_STAN, c(mean(obs),sd(obs),1,1), x=obs
                    , method = meth)
#Prelim
p1<-4
p2<-c(4,4,4)
LL1<--fit_STAN$value
LL2<-c(-fit_st$value, fit_hyp$maxLik, fit_nig$maxLik)
bf(LL1,LL2,p1,p2,length(obs))


den<-density(obs)
plot(den$x,(den$y), type = 'l',lty = 3, xlab='rent per month'
      ,ylab='log-density')
lines(den$x,(dSTAN(den$x,fit_STAN$par)), col=inf5[5])


plot(den$x,log(den$y), type = 'l',lty = 3, xlab='rent per month'
      ,ylab='log-density')
lines(den$x,log(ddst(den$x,fit_st$par)), col= inf5[2])
lines(den$x,log(dhyperb(den$x,param = fit_hyp$param))
        , col = inf5[3])
lines(den$x,log(dnig(den$x,param = fit_nig$param))
        , col = inf5[4])
lines(den$x,log(dSTAN(den$x,fit_STAN$par)), col=inf5[5])
abline(v=qSTAN(perc,fit_STAN$par), col = 'purple')
abline(v=quantile(obs,perc), col = 'blue', lty=2)

######################################################################
```

```
funcs<-c('withTimeout({d<-unlist(optim(fn = minLL_st
          ,c(mean(obs),var(obs),0,2), x=obs)[c(1,2)])
            c(d,qqst(perc,d[1:4]))}
            ,timeout = 7, onTimeout = "error")'
         ,'withTimeout({d<-hyperbFit(obs)
            c(d$param[c(1,2,3,4)],d$maxLik,qhyperb(perc
          ,param = d$param))}
            ,timeout = 10, onTimeout = "error")'
         ,'withTimeout({d<-nigFit(obs)
            c(d$param[c(1,2,3,4)],d$maxLik,qnig(perc
          ,param = d$param))}
            ,timeout = 10, onTimeout = "error")'
         ,'withTimeout({d<-unlist(optim(fn = minLL_STAN
          , c(mean(obs),sd(obs),2,0.5), x=obs)[c(1,2)])
            c(d,qSTAN(perc,d[1:4]))},timeout = 7
          , onTimeout = "error")')
#######################################################

n<-c(10,20,50,100,500)
m<-round(1.01*100)
dat<-matrix(nrow = 0,ncol = 16)
ptm <- proc.time()
for(k in n)
{
  print(k)
  count<-0
  while(count<m)
  {
    ptmint <- proc.time()
    obs<-rghyp(k,param = par)
    # clusterExport(cl, 'obs',envir=environment())
```

```
    # registerDoParallel(cl)
    l<-foreach(i=funcs) %do%
    {
      tryCatch(eval(parse(text=i)), error = function(e) NA)
    }
    l
    if(all(!is.na(l)))
    {
    d<-cbind(k,1:4,t(matrix(unlist(l),ncol = 4)))
    dat<- rbind(dat,d)
    count<-count+1
    print(c(k,count))
    }
  }
  print((proc.time() - ptm)/60)
}
(proc.time() - ptm)/60
ptm <- proc.time()

#50  @ 1:30
#100 @ 3:20
#100 @ 3:20
#100 @ 2:50
#100 @ 2:30
#100 @ 2:45
#400 @ 13:15
#400 @ 11:40

qtrue<-qghyp(perc, param = par)
monty<-as.data.frame(dat, stringsAsFactors = FALSE)
qcols<-paste0('q',as.character(100*perc))
```

```
monty$n<-as.factor(monty$n)
colnames(monty)<-c('n','Distribution','mu','sigma','psi'
                  ,'nu','LL',qcols)
monty[,qcols]<-sweep(monty[,qcols],2,qtrue)
monty$Distribution<-as.factor(monty$Distribution)
levels(monty$Distribution)<-c('ST','HYP','NIG','TPTAN')
head(monty)
####################
#DF of t distribution
aggregate(monty[,3:6], by = list(monty$n,monty$Distribution)
          , median)
aggregate(monty[,3:6], by = list(monty$n,monty$Distribution)
          , mean)

##############################################################
#STOP DO NOT RUN
##############################################################
#Quantile plots
qsing<-'q90'
qundf<-monty[,c('Distribution','n','LL',qsing)]
qundf$Out<-TRUE
for(i in levels(qundf$Distribution))
{
  for(j in levels(qundf$n))
  {
    ind<-qundf$Distribution==i & qundf$n==j
    qundf$Out[ind]<-trm(qundf[ind,ncol(qundf)-1]^2,p=0.9
                        ,indcator = TRUE, surpress = TRUE)
  }
}
##############################################################
```

```
#STOP DO NOT RUN
samplabs <- paste0('n=',levels(qundf$n))



setEPS()
mypath <- file.path(cur_dir,paste('SIM', 'q_q', ".eps"
                                  , sep = ''))
postscript(mypath, width = 6, height = 3.5)

ggplot(qundf[qundf$Out,], aes(x=as.factor(n)
                              ,y=eval(parse(text = qsing))
                              , fill = Distribution))+
            geom_boxplot(outlier.shape=NA, size = 0.3)+
            xlab('') +
            ylab(expression(q[90]-hat(q[90])))+
            scale_fill_manual(values = gy4)+
            scale_x_discrete(labels=samplabs)+
            theme(axis.text.x = element_text(angle = -45
                                             , hjust = 0.05)
                  ,legend.title = element_text( size=8)
                  , legend.text=element_text(size=8))
dev.off()

######################################################################
#STOP DO NOT RUN
tdf<-aggregate(qundf[,qsing]
               , by= list(qundf$n,qundf$Distribution)
               , function(x) c(mean(x,na.rm=TRUE)^2
                              ,var(x, na.rm=TRUE)))
tdf<-data.frame(tdf$Group.1,tdf$Group.2,tdf$x[,1], tdf$x[,2])
colnames(tdf)<-c('n','Distribution','Bias2','Variance')
```

```
stack<-reshape2::melt(tdf,id = c('n','Distribution'))
tdf$n<-as.factor(tdf$n)
samplabs <- paste0('n=',levels(tdf$n))
names(samplabs) <- levels(tdf$n)
##################################################################
setEPS()
mypath <- file.path(cur_dir,paste('SIM', 'MSE', ".eps", sep = ''))
postscript(mypath, width = 6, height = 3.5)

ggplot(stack, aes(fill=factor(variable), y=value, x=Distribution)) +
  geom_bar( stat="identity") +
  facet_grid(~n, labeller = labeller(n =samplabs)) +
  ylab('MSE') +
  labs(fill = '') +
  scale_fill_manual(values = gy4[c(3,1)],labels =
  expression('Bias('*hat(q)[90]*')'^2,'Variance('*hat(q[90])*')'))+
  scale_color_manual(name = "Clarity"
                    , values = c("black","black","black","black"))+
  theme(axis.text.x = element_text(angle = -45, hjust = 0.05)
        ,legend.position = c(0.9, 0.86)
        ,legend.background = element_rect(fill = "transparent")
        ,legend.key = element_rect(fill = "transparent"
                                  , color = NA))

dev.off()
##################################################################
#STOP DO NOT RUN
#Density approximation plot
dendf<-monty[,1:6]
dendf<-aggregate(dendf[,3:6], by= list(dendf$n,dendf$Distribution),mean)
colnames(dendf)<-c('n','Distribution','mu','sigma','psi','nu')
```

```
xplt<-(apply(denpl,1,function(x) dSTAN(xr,c(x[1],x[2],x[3],x[4]))))
xplt<-as.data.frame(cbind(xr,xplt,dghyp(xr,param = par)))
names(xplt)<-c('x','TPTAN (n=10)','TPTAN (n=20)','TPTAN (n=50)'
              ,'TPTAN (n=100)','TPTAN (n=500)','GHYP')
xplt<-reshape2::melt(xplt, id =c('x') )
names(xplt)[c(2,3)]<-c('Sample size','density')
xplt$variable<-as.factor(xplt$`Sample size`)



setEPS()
mypath <- file.path(cur_dir,paste('SIM', 'conv', ".eps"
                                  , sep = ''))
postscript(mypath, width = 6, height = 3.5)
###
ggplot(data=xplt, aes(x=x, y=density, group=`Sample size`)) +
  geom_line(aes(color=`Sample size`)) +
  scale_color_manual(values=vkb6) +
  theme(legend.title = element_blank())

###
dev.off()
#########################################################################
head(monty)
pardf<-reshape2::melt(monty[qundf$Out,1:6]
                      ,id = list('n','Distribution'))
pardf<-na.omit(pardf)
colnames(pardf)[3]<-'Parameter'
head(pardf)
summary(pardf)
```

```
setEPS()
mypath <- file.path(cur_dir,paste('SIM', 'BARPAR'
                                  , ".eps", sep = ''))
postscript(mypath, width = 6, height = 3.5)

xlbs<-c(expression(hat(mu)), expression(hat(sigma))
        , expression(hat(psi)), expression(hat(nu)))
rang<-quantile(pardf$value,c(0.25,0.75))
rangm<-median(pardf$value)
ggplot(pardf,aes(x=as.factor(n),y= value, fill = Parameter))+
  geom_boxplot(outlier.shape=NA, size = 0.3)+
  xlab('')+
  ylab('Value')+
  scale_y_continuous(limits = c(rangm-3*rang[1]
                                ,rangm+3*rang[2]))+
  scale_fill_manual(values = inf4[4:1], labels = xlbs)+
  scale_x_discrete(labels=samplabs)+
    theme(axis.text.x = element_text(angle = -45, hjust = 0.05)
        ,legend.title = element_text( size=8)
        ,legend.text=element_text(size=8))


dev.off()
#####################################################################
#STOP DO NOT RUN
parmsdf<-aggregate(pardf$value
, by= list(pardf$n,pardf$Distribution, pardf$Parameter)
, function(x) sd(x, na.rm=TRUE))
colnames(parmsdf)<-c('n','Distribution','Parameter','SD')
min(stack$value)
```

```
parmsdf$n<-as.factor(parmsdf$n)
names(samplabs) <- levels(parmsdf$n)

setEPS()
mypath <- file.path(cur_dir,paste('SIM', 'MSEPAR', ".eps"
                                   , sep = ''))
postscript(mypath, width = 6, height = 3.5)


parmsdf$SD<-parmsdf$SD/min(parmsdf$SD)
ggplot(parmsdf[parmsdf$Distribution=='TPTAN',]
       , aes(fill=Parameter,y=SD, x=as.factor(n))) +
  geom_bar( stat="identity", position = 'dodge'
            , colour="black", size = 0.2) +
  xlab('')+
  ylab(expression(sqrt('MSE'))) +
  scale_x_discrete(labels=samplabs)+
  labs(fill = '') +
  scale_fill_manual(values = inf4[4:1],labels = xlbs)+
  scale_y_continuous(trans='log2') +
  theme(axis.text.x = element_text(angle = -45, hjust = 0.05)
        ,axis.text.y = element_text(angle = 45, hjust = 0.05)
        ,legend.position = c(0.9, 0.86)
        ,legend.background = element_rect(fill = "transparent")
        ,legend.key = element_rect(fill = "transparent"
                                   , color = NA))

dev.off()
```