# Contributions to $\kappa - \mu$ models in wireless systems

by

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# Declaration

I, Priyanka Nagar, declare that the mini-dissertation, which I hereby submit for the degree MSc (Mathematical Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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DATE:.....................................................

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# Abstract

The $\kappa - \mu$ distribution is a popular model for the fading observed due to clusters of obstacles between a transmitter and a receiver in wireless communication channels. This model includes various classical fading distributions as special cases. The basic $\kappa - \mu$ distribution is used to model the signal strength in communication channels under the assumption that the contribution to the signal strength arising from each cluster follows a normal distribution. This study extends the model by relaxing the assumption of normality, instead these contributions are assumed to follow an elliptical law. The t-distribution plays an important role in this dissertation in that it is presented as an alternative to the normal distribution.

Three extended $\kappa - \mu$ models are presented in this study, in each case the model is extended by generalizing the underlying distributional assumption to that of an elliptical distribution. The models include a univariate model, a bivariate model (which is useful when one wants model the joint behaviour of communication channels) and a composite model (which is used to model shadowing as well). As a performance measure for the degradation in communication channels resulting from fading and shadowing, the outage probability is investigated for special cases of the extended models.

The proposed extensions to the $\kappa - \mu$ model result in more flexible distributions that can be used to model wireless communication channels.

**Keywords**: $\kappa - \mu$ model, elliptical class, fading, outage probability, shadowing, t-distribution.

Supervisor: Prof A Bekker
Co-supervisor: Prof M Arashi

# Contents

# Chapter 1

# Introduction

## 1.1 Background and motivation

With the increasing popularity of wireless technologies, wireless communication systems are abundant. The vast networks of wireless channels are important tools for a fast and efficient means of data transmission. Hence, a careful study of the characteristics of these channels is required in an attempt to ensure that communication can be maintained with reliability. The signal transmission and the degradation of the signal as they pass through these channels are of interest to practitioners.

Consider the case where a channel is transmitted from a single transmitter to a single receiver. For engineering purposes, it is very important to be able to accurately model the strength of the channel. A transmission of this kind is typically subject to various disturbances. For example, there might be several large buildings, or even several clusters of buildings, between the transmitter and the receiver. On the one hand, these obstacles will impede the channels transmission by partially blocking the most direct route between the transmitter and the receiver. On the other hand, some of the obstacles might enhance the magnitude by reflecting the channel in the direction of the receiver. However, the resulting channels will have a reduced strength compared to the original channel because of the imperfections introduced by the reflection. These interferences are jointly referred to as fading and shadowing.

There is a wide variety of models available in literature to describe the statistical fluctuations in channels in fading or shadowing, fading models. The reader is referred to Simon and Alouini, (2005) [29], Shankar (2011) [28, page 247], Paris (2014) [23] and Moreno-Pozas et al (2016) [22].

Various models, with varying degrees of complexity, have been proposed in order to model the strength in a wireless channel. An appropriate model for this should be able to allow for fading and shadowing. The basic model considered in this dissertation is known as the $\kappa - \mu$ model. In this model and its extensions, fading and shadowing are taken into account by considering the contribution of each cluster of obstacles to the received channel strength separately. It is assumed that a large number of reflections occur within each cluster and that each of these reflections add a random element to the resulting contribution of this cluster. The contributions of the various clusters are then aggregated in order to arrive at the strength of the channel received.

Under the standard $\kappa - \mu$ model the contribution associated with each cluster of obstacles is assumed to have an underlying model. A question that arises naturally is how sensitive the results of analyses on wireless channel models are to various departures from normality. This study will generalize the $\kappa - \mu$ model used in the communication system field by replacing the assumption of normality in the individual contributions by the assumption that these contributions follow an elliptical law. Ollila et al (2011)[21] pointed out that a more general assumption than the normal may not be far from reality. The main contribution of this study lies in relaxing the assumption of normality to that of an elliptical distribution. Different members of the elliptical class may be explored as possible candidates. Yacoub (2007) [35] and Stein (1987) [32] raised the inadequacy of the tails of some distributions. Therefore, the introduction of the t-distribution as an alternative to normality will play a central role in this study. Against this backdrop the study of various distributions within the elliptical class is of practical as well as theoretical interest.

The $\kappa - \mu$ model originated due to the fundamental differences in the underlying assumptions of this model when compared to classical fading models. The classical models assume that the received channel can be represented as a vector sum of scattered components coming from individual obstacles, while under the $\kappa - \mu$ model the assumption is that the received channel power arrives from obstacles that are scattering clusters. The underlying assumption of this model remains the normality of the individual contributions. In this study the new univariate $\kappa - \mu$ *type* model (that arise from the elliptical assumption) is proposed as an alternative in the communications environment to model and study the performance of the channel transmission through wireless channels.

In communication engineering fading distributions that are associated with multi-antenna wireless communication systems operating over *correlated* branches are also of interest. If the antennas are sufficiently separated, it is reasonable to assume that channels received by different antennas are independent. However, this assumption is unsuitable for systems with closely spaced antennas, such as mobile phones. Thus, in practical scenarios, the channels may present some sort of dependence. This is the motivation why *bivariate* distributions have been extensively explored in the context of wireless systems; see for example Lopez-Martinez et al (2013) [15]. Ermolova and Tirkkonen (2015) [10], Reig et al (2014) [25], Mendes and Yacoub (2007) [18], de Souza et al (2012) [9] and de Souza and Yacoub (2008) [8]. For that reason, the *bivariate* $\kappa - \mu$ model will be investigated and extended to the elliptical case.

Another model of interest is the *shadowed fading* model (composite model) which simultaneously accounts for fading and shadowing in wireless systems. Shadowing can be incorporated in multipath fading models in various ways. Different approaches and models have been explored in the context of wireless systems to model random fluctuations in the received channel caused by short term fading and shadowing; see Cotton (2015) [7], Lopez-Fernandez et al (2017) [14], Lopez-Martinez et al (2016) [16], Moreno-Pozas et al (2016) [22], Paris (2014) [23], Reig et al (2014) [25], Shankar (2005) [27], Sofotasios and Freear (2013) [31], Sofotasios and Freear (2015) [30], Vural et al (2015) [34] and Yoo and Cotton (2015) [36]. In this study the focus will also be on the $\kappa - \mu$ *shadowed fading* model.

Each model is characterized in terms of quantitative measures such as the amount of fading, and the outage probability. Therefore these metrics will recieve attention for special cases of the extended models. The new results in this study give a more general approach with additional flexibility since the assumption of normality is relaxed. The author trusts that the extensions obtained by assuming elliptical distributions will enrich the models within the communication systems and stimulate research.

## 1.2 Elements for study

Below we consider various important concepts that will be used throughout this dissertation.

### 1.2.1 Elliptical Class

A random variable is said to belong to the elliptical class if its probability density function (pdf) is a function of a quadratic form. This class includes the normal, Student's t, Cauchy, Pearson VII, Bessel and many other distributions. The elliptical class is useful as it includes distributions with heavier tails than the normal.

*Description 1* [2]

A random variable $X$ has an elliptical distribution with parameters $\mu$ and $\sigma^2$, $\mathrm{E}(\mu, \sigma^2)$, if its pdf is of the form

$$f_X(x) = h\left[\frac{(x-\mu)^2}{2\sigma^2}\right]$$

for some non-negative function $h[\cdot]$, say density generator. The expected value is $E(X) = \mu$ and the $var(X) = k\sigma^2$, where $k$ is a constant given by $k = -2\left[\frac{d}{dy}\psi(y)\right]_{y=0} = -2\psi'(0)$, where $\psi(\cdot)$ is defined as some characteristic generator function. Chu (1973) [6] showed that the pdf of the elliptical distribution can be expressed as integrals of a set of normal pdfs. There is a scalar fucntion, $W(t)$, defined on $0 < t < \infty$ such that

$$f_X(x) = \int_0^\infty W(t)\, f_{N(\mu, t^{-1}\sigma^2)}(x)\, dt$$

where $N(\mu, t^{-1}\sigma^2)$ is the normal pdf with mean $\mu$ and variance $t^{-1}\sigma^2$. This analytical framework provides a computationally convenient form and will be demonstrated in the subsequent chapters.

### 1.2.2 Composite model

*Description 2*

A random variable $X$ with pdf $f(x)$ is said to have a compound or composite pdf if it has the general form

$$f(x) = \int f(x|\theta)\, h(\theta)\, d\theta$$

where $f(x|\theta)$ is a conditional pdf depending on the parameter $\theta$, itself subject to variation described by the pdf $h(\theta)$, the compounding pdf. In literature relevant to the communication systems the compound model is referred to as a composite model.

### 1.2.3 Concepts in communication systems

#### 1.2.3.1 Fading and Shadowing

As mentioned before, in wireless communications, signals transmitted do not often reach the receiver directly. The signal reaches the receiver after undergoing scattering, diffraction, reflection, etc. from the buildings, trees, and other structures in the channel between the transmitter and the receiver. As a result, there are multiple paths available for the signal to reach the receiver and the received signals have in-phase added (see Figure 1.1). The envelope (or amplitude/magnitude) of the signals from these paths are considered to be random variables, thus, the received power (square of the magnitude) is also considered random.

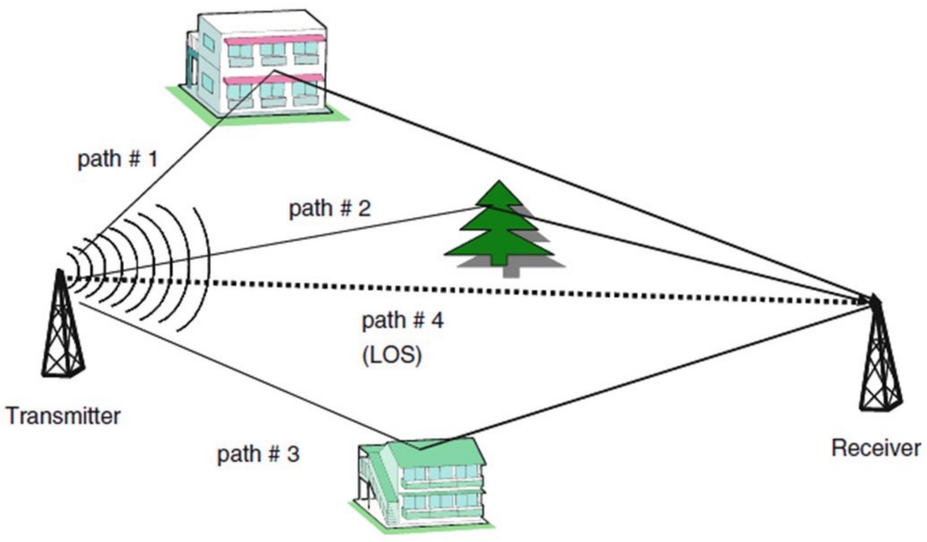



Figure 1.1: Multipath transmission of signals [28]

The random variation of the power is classified as “fading” in wireless systems. If the variation has a short period, it is known as short-term fading. Signals encountering multiple scattering or variation for long periods is known as long-term fading or “shadowing”.

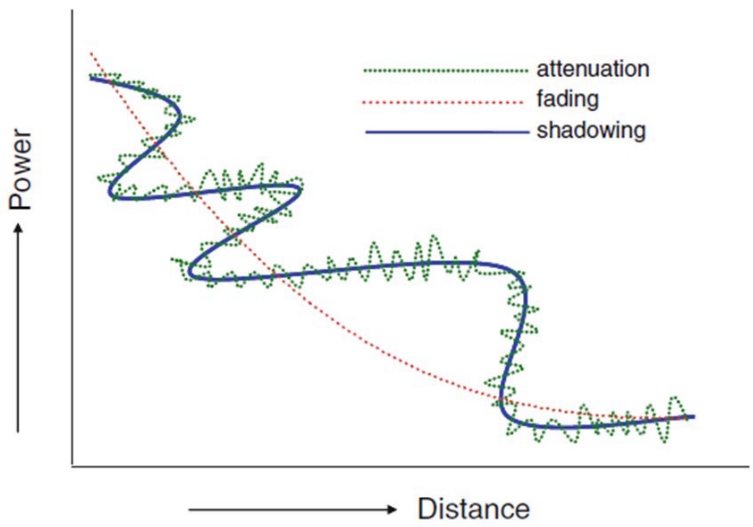



Figure 1.2: Fading and shadowing in wireless channels [28]

Signal degradation results from both the individual and/or simultaneous presence of fading and shadowing (see Figure 1.2). Several models have been used to describe fading in wireless systems, these include the simple models such as Rayleigh, Rician, Nakagami, gamma and Weibull (Simon and Alouini (2005) [29] and Shankar (2011) [28]), and complex models such as the $\kappa-\mu$ and $\eta-\mu$ (Yacoub (2007) [35] ) and $\alpha-\eta-\mu$ models (Badarneh and Aloqlah (2016)[5]). The lognormal and normal models are most commonly used to describe shadowing. Composite models are used to account for the simultaneous presence of fading and shadowing.

#### 1.2.3.2 Metrics

Comparison of these models is based on the quantitative metrics of the statistical characteristics of the models which include amongst others the amount of fading and outage probability.

**Amount of fading**

The amount of fading (AF) is a metric used to measure the severity of the fading in the wireless system. The AF is defined in terms of the first and second moments of the power of the fading signal, say $\Omega$. The AF is expressed as (Simon and Alouini, 2005.[29, equation (2.5), page 18])

$$AF = \frac{var\left(\Omega\right)}{E\left(\Omega\right)^2} = \frac{E\left(\Omega^2\right) - \left[E\left(\Omega\right)\right]^2}{\left[E\left(\Omega\right)\right]^2} = \frac{E\left(\Omega^2\right)}{\left[E\left(\Omega\right)\right]^2} - 1 \tag{1.1}$$

where $E\left(\cdot\right)$ denotes the expected value and $var\left(\cdot\right)$ the variance. The AF for some fading models is given by Shankar (2011) [28, page 247].

**Outage**

The outage probability is a metric used to quantify the performance of the wireless communication systems in different channels. It is defined as the probability that the received signal falls below a certain threshold or equivalently, the probability that the signal-to-noise ratio (SNR) or power falls below a certain specified threshold. Whenever the signal power goes below the set threshold the channel goes into outage. The outage probability can be expressed as (Simon and Alouini, 2005.[29, equation (2.5), page 18])

$$P_{out} = \int_0^{Z_T} f\left(\omega\right) d\omega = F_\omega\left(Z_T\right) \tag{1.2}$$

where $Z_T$ is the threshold SNR and $W$ can denote either the instantaneous SNR, $W = \frac{R^2 E_s}{N_0}$, or the normalized power, $\Omega = R^2$, where $R$ denotes the envelope ($E_s$ is the energy per symbol and $N_0$ is the noise power, see Simon and Alouini, 2005.[29, equation (2.5), page 18]). The pdf of the SNR is $f(\omega)$ and $F_\omega(Z_T)$ is the cdf of the SNR evaluated at $\omega = Z_T$.

## 1.3 Outline

- In Chapter 2, a collection of some fundamental mathematical results are given for use in later sections. Notation and abbreviations are also defined.
- In Chapter 3, firstly, the univariate $\kappa - \mu$ distribution with underlying normal model is reviewed, and thereafter, the univariate $\kappa - \mu$ type distribution with underlying elliptical model is derived. Closed form expressions for the envelope pdf, power pdfs, power cdfs and moments are derived.
- Chapter 4 expands the univariate cases in chapter 3 to the bivariate environment specifically to the elliptical case. Closed form expressions for the envelope pdf, envelope cdf, joint moments and outage probability are derived.
- Chapter 5, considers the composite model where the univariate $\kappa - \mu$ fading channel is subject to shadowing. An extended model is developed emanating from the elliptical assumption. Closed form expressions for the power pdf, power cdf and moments are derived.

These models originated from a underlying physical model. The focus of this study is to revisit the existing models and extend it from an elliptical viewpoint. Note that in the case of the elliptical assumption, the distributions are referred to as *type*. Some performance analysis is illustrated to justify the elliptical extension and the value added is specifically demonstrated by assuming the t-distribution

# Chapter 2

# Special functions and notation

A collection of some fundamental mathematical results are given in this chapter. These results will be used in later chapters.

## 2.1 Abbreviations and notation

| | |
|---|---|
| AF | Amount of fading |
| cdf | Cumulative distribution function |
| jcdf | Joint cumulative distribution function |
| jpdf | Joint probability density function |
| pdf | Probability density function |
| SNR | Signal-to-Noise Ratio |
| $A > 0$ | $A$ is a positive definite matrix |
| $A^{\frac{1}{2}}$ | Unique positive definite square root of $A$ |
| $\det(A)$ | Determinant of the square matrix $A$ |
| $\|A\|$ | Norm of vector $A$ |
| $A^{-1}$ | Inverse of a square matrix $A$ |
| $A^T$ | Transpose of matrix $A$ |
| $tr(A)$ | Trace of the square matrix $A$ |
| $J(X \to f(X))$ | The Jacobian of the matrix transformation $f$ |
| $R^+$ | Set of positive real numbers |
| $N$ | Set of natural numbers |
| Re | Real part of a number |
| $R^p$ | Set of real vectors of size $p$ |
| $C_k^v(t)$ | Gegenbauer function |

| | |
|---|---|
| $\Gamma(\cdot)$ | Gamma function |
| $\gamma(\cdot)$ | Incomplete gamma function |
| $B(\cdot)$ | Beta function |
| $(\alpha)_k$ | Pochhammer coefficient |
| ${}_rF_s(\cdot)$ | Hypergeometric series with $r$ upper parameters and $s$ lower parameters |
| $I_v(\cdot)$ | Modified Bessel function of the first kind and order $v$ |
| $K_v(\cdot)$ | Modified Bessel function of the second kind and order $v$ |
| $N(\mu,\sigma^2)$ | Normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $\chi^2(v)$ | Chi-squared distribution with $v$ degrees of freedom |
| $\chi^2_\lambda(v)$ | Non-central chi-squared distribution with $v$ degrees of freedom and noncentrality parameter $\lambda$ |
| $E(\mu,\sigma^2)$ | Elliptical distribution with parameters $\mu$ and $\sigma^2$ |

## 2.2 Special functions and theory

**Result 1** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 6.1.1]*

The gamma function, denoted $\Gamma(\alpha)$, is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\exp(-t)\,dt, \tag{2.1}$$

where $\operatorname{Re}(\alpha) > 0$.

Let $\alpha \in R^+$ then

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \tag{2.2}$$

and if $\alpha \in N$, then

$$\Gamma(\alpha) = (\alpha-1)! \tag{2.3}$$

**Result 2** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 6.2.1]*

The beta function, denoted $B(\alpha,\beta)$, is defined as

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.4}$$

where $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$.

**Result 3** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 6.5.2]*

The incomplete gamma function, denoted $\gamma(\alpha,x)$, is defined as

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}\exp(-t)dt, \tag{2.5}$$

where $\text{Re}\,(\alpha) > 0$.

**Result 4** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007, equation 8.354.1]*

An infinite series approximation for the incomplete gamma function is expressed as

$$\gamma\,(\alpha, x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{\alpha+k}}{k!\,(\alpha+k)}. \tag{2.6}$$

**Result 5** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 4.2.1]*

The series expansion for the exponential function is defined as

$$\exp\,(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{k!}. \tag{2.7}$$

**Result 6** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 6.1.22]*

The Pochhammer coefficient is defined as

$$(\alpha)_k = \alpha\,(\alpha+1)\ldots(\alpha+k-1) = \frac{\Gamma\,(\alpha+k)}{\Gamma\,(\alpha)}, \tag{2.8}$$

where $k = 1, 2, \ldots$, $(\alpha)_0 = 1$, $\alpha \neq 0$, $\text{Re}\,(\alpha) > 0$, $\text{Re}\,(\alpha+j) > 0$ and $\Gamma\,(\cdot)$ is the gamma function.

**Result 7** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007, equation 9.14.1]*

The hypergeometric series with $r$ upper parameters and $s$ lower parameters is defined for $|x| < 1$ as

$${}_rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_r)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{x^k}{k!}, \tag{2.9}$$

where $(\alpha)_k$ is the Pochhammer symbol (2.8).

**Result 8** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007]*

The confluent hypergeometric function is defined for $|x| < 1$ as

$${}_1F_1\,(\alpha; \beta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{x^k}{k!}. \tag{2.10}$$

**Result 9** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007]*

The Gaussian hypergeometric function is defined as

$$ {}_2F_1\left(\alpha_1,\alpha_2;\beta;x\right)=\sum_{k=0}^{\infty}\frac{\left(\alpha_1\right)_k\left(\alpha_2\right)_k}{\left(\beta\right)_k}\frac{x^k}{k!}, \tag{2.11}$$

where $|x|<1$.

Note that

$$ {}_2F_1\left(\alpha_1,\alpha_2,\beta;x\right)=\left(1-x\right)^{-\alpha_1}\ {}_2F_1\left(\alpha_1,\beta-\alpha_2,\beta;\frac{x}{x-1}\right). \tag{2.12}$$

**Result 10** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 9.6.10]*

The modified Bessel function of the first kind and order $v$ can be expressed as an infinite series

$$I_v\left(x\right)=\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^v\frac{\left(\frac{1}{4}x^2\right)^k}{k!\Gamma\left(v+k+1\right)}. \tag{2.13}$$

**Result 11** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 9.6.7]*

The modified Bessel function of the first kind and order $v$ can be approximated as

$$I_v\left(x\right)\simeq\frac{\left(\frac{x}{2}\right)^v}{\Gamma\left(v+1\right)}. \tag{2.14}$$

**Result 12** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007, equation 8.351.2]*

The incomplete gamma function can be expressed as follows

$$\gamma\left(\alpha,\beta\right)=\frac{\beta^\alpha}{\alpha}\exp\left(-\beta\right)\ {}_1F_1\left(1;1+\alpha;\beta\right) \tag{2.15}$$

where ${}_1F_1\left(\cdot\right)$ is defined in (2.10).

**Result 13** *[19, Miller, K., 1964, equation 2.2.23]*

The generalized Neumann addition formula

$$\begin{aligned}&\left|a\Xi-b\Omega\right|^{-v}I_v(\left|a\Xi-b\Omega\right|)\\=\ &2^v\Gamma(v)\sum_{k=0}^{\infty}(-1)^k(v+k)\frac{I_{v+k}(a\left|\Xi\right|)}{a^v\left|\Xi\right|^v}\frac{I_{v+k}(b\left|\Omega\right|)}{b^v\left|\Omega\right|^v}C_k^v(\cos\phi)\end{aligned} \tag{2.16}$$

where $\phi$ is the angle between $\Xi$ and $\Omega$ and $a$ and $b$ are real, $I_v\left(\cdot\right)$ is the Bessel function of the first kind (2.13) and $C_k^v(t)$ is the Gegenbauer function. The Gegenbauer function is defined by [13, Gradshteyn, I. S. and Ryzhik, I. M., 2007, equation 8.930] as follows: The polynomials $C_k^v(t)$ of degree $k$ are the coefficients of the $\alpha^k$ in the power-series expansion of the function

$$\left(1-2t\alpha+\alpha^2\right)^{-v}=\sum_{k=0}^{\infty}C_k^v(t)\alpha^k$$

**Result 14** *[19, Miller, K., 1964, equation 2.2.25]*

The Gegenbauer's generalization of the Poisson formula

$$\int_0^\pi \exp\left(\omega_1 \cos\phi\right) C_k^v\left(\cos\phi\right) \sin^{2v}\phi d\phi = \frac{2^v \Gamma\left(v+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(2v+2\right)}{b^v n!\Gamma\left(2v\right)} I_{v+n}\left(b\right) \tag{2.17}$$

where $b$ is real and $I_v\left(\cdot\right)$ is the modified Bessel function of the first kind $(2.13)$.

**Result 15** *[19, Miller, K., 1964, equation 1.5.19]*

Evaluation of the expression in terms of generalized spherical co-ordinates results in

$$R_1^{2\mu-1}\int_0^{2\pi} d\theta \prod_{k=2}^{2\mu-2}\int_0^\pi \sin^{2\mu-1-k}\phi_k d\phi_k = \frac{2\pi^{\frac{1}{2}(2\mu-1)}R_1^{2\mu-1}}{\Gamma(\frac{1}{2}(2\mu-1))}. \tag{2.18}$$

**Result 16** *[1, Abramowitz, M. and Stegun, I.A.,1964, equation 6.1.1]*

The following integral can be evaluated as

$$\int_0^\infty x^{\alpha-1}\exp\left(-\beta x\right) dx = \frac{\Gamma\left(\alpha\right)}{\beta^\alpha}, \tag{2.19}$$

where $\mathrm{Re}\left(\alpha\right) > 0$ and $\mathrm{Re}\left(\beta\right) > 0$.

**Result 17** *[13, Gradshteyn, I. S. and Ryzhik, I. M., 2007, equation 3.381.8]*

The integral can be evaluated as

$$\int_0^u x^m \exp\left(-\beta x^n\right) dx = \frac{\gamma\left(v, \beta u^n\right)}{n\beta^v}, \ v = \frac{m+1}{n}, \tag{2.20}$$

where $\mathrm{Re}\left(v\right) > 0$, $\mathrm{Re}\left(\beta\right) > 0$, $\mathrm{Re}\left(n\right) > 0$ and $u > 0$.

**Result 18** *[13, Gradshteyn, I. S. and Ryzhik, I. M. , 2007, equation 6.455.2]*

The following integral can be evaluated as

$$\int_0^\infty x^{\mu-1}\exp\left(-\beta x\right)\gamma\left(v, \alpha x\right) dx = \frac{\alpha^v \Gamma\left(\mu+v\right)}{v\left(\alpha+\beta\right)^{\mu+v}} {}_2F_1\left(1; \mu+v; v+1; \frac{\alpha}{\alpha+\beta}\right), \tag{2.21}$$

where $\mathrm{Re}\left(\alpha+\beta\right) > 0$, $\mathrm{Re}\left(\beta\right) > 0$, $\mathrm{Re}\left(v+\mu\right) > 0$ and ${}_2F_1\left(\cdot\right)$ is defined in $(2.11)$.

**Result 19** *[24, Prudnikov, A.P., Brychkov, Y.A, et. al, 1988, equation 2.15.5.4]*

The following integral can be evaluated as

$$\int_0^\infty x^{\alpha-1}\exp\left(-\beta x^2\right) I_v(\gamma x)\,dx = 2^{-v-1}\gamma^v\beta^{-\frac{(\alpha+v)}{2}}\Gamma\begin{bmatrix}\frac{\alpha+v}{2}\\ v+1\end{bmatrix}\ {}_1F_1\left(\frac{\alpha+v}{2}; v+1; \frac{\gamma^2}{4\beta}\right), \tag{2.22}$$

where $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\alpha+v)>0$, $|\arg(\gamma)|<\pi$, $\Gamma\begin{bmatrix}a,b\\ a+b\end{bmatrix}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=B(a,b)$ and ${}_1F_1(\cdot)$ is defined in (2.10). $\left(\text{Note that } \Gamma\begin{bmatrix}a\\ a+b\end{bmatrix}=\frac{\Gamma(a)}{\Gamma(a+b)}.\right)$

**Result 20** *[13, Gradshteyn, I. S. and Ryzhik, I. M. , 2007, equation 7.621.4]*

The following integral can be evaluated as

$$\int_0^\infty x^{\beta-1}\exp(-sx)\ {}_1F_1(\alpha;\lambda;kx)\,dx = \Gamma(\beta)\,s^{-\beta}\ {}_2F_1\left(\alpha,\beta;\lambda;ks^{-1}\right), \tag{2.23}$$

where $|s|>|k|$, $\operatorname{Re}(\beta)>0$ and ${}_1F_1(\cdot)$ is defined in (2.10).

**Result 21** *[13, Gradshteyn, I. S. and Ryzhik, I. M. , 2007, equation 3.194.2]*

The following integral can be evaluated as

$$\int_0^u \frac{x^{\alpha-1}}{(1+\beta x)^v}dx = \frac{u^\alpha}{\alpha}\ {}_2F_1(v,\alpha;1+\alpha;-\beta u), \tag{2.24}$$

where $\operatorname{Re}(\alpha)>0$ and $|\arg(1+\beta u)|<\pi$.

**Result 22** *[1, Abramowitz, M. and Stegun,I.A.,1964, equation 29.1.1]*

The Laplace transform is defined as

$$f(s) = L\{F(t)\} = \int_0^\infty \exp(-st)\,F(t)\,dt, \tag{2.25}$$

where $F(t)$ is a real valued function and $s$ a complex random variable. The function $F(t)$ is called the orginial function and $f(s)$ the image function.

**Result 23** *[1, Abramowitz, M. and Stegun,I.A.,1964, equation 29.3.81]*

Given the image function

$$f(s) = \frac{1}{s^{\mu}} \exp\left\{\frac{k}{s}\right\}, \quad [\mu > 0],$$

the inverse Laplace transform of the original function is

$$F(t) = \left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}\left(2\sqrt{kt}\right), \tag{2.26}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind (2.13).

**Result 24** *[19, Miller, K., 1964, equation 2.1.1]*

Let $X \in R^p$ follow the multivariate normal distribution with mean parameter $\mu \in R^p$ and covariance matrix $\Sigma > 0$, then the pdf is given as

$$f_X(x) = \frac{1}{(2\pi)^{\frac{p}{2}} \det(\Sigma)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}. \tag{2.27}$$

**Result 25** *[6, Chu, K, 1973]*

If $X$ is an elliptical random variable with parameters $\mu$ and $\sigma^2$ and pdf $f(X)$, then there is a scalar function $W(t)$ defined on $0 < t < \infty$ such that

$$f_X(x) = \int_0^{\infty} W(t) f_{N(\mu, t^{-1}\sigma^2)}(x)\, dt, \tag{2.28}$$

where $f_{N(\mu,t^{-1}\sigma^2)}(x)$ is the normal pdf with mean $\mu$ and variance $t^{-1}\sigma^2$.

**Result 26** *[6, Chu, K, 1973]*

Let $X \in R^p$ follow the multivariate elliptical distribution with mean parameter $\mu \in R^p$ and covariance matrix $\Sigma > 0$. Then for a scalar function $W(t)$ defined on $0 < t < \infty$ it follows that

$$f_X(x) = \int_0^{\infty} \frac{1}{(2\pi)^{\frac{p}{2}} \det(t^{-1}\Sigma)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T \left(t^{-1}\Sigma\right)^{-1}(x-\mu)\right\} W(t)\, dt, \tag{2.29}$$

where $f_{N(\mu,t^{-1}\Sigma)}(x)$ is the multivariate normal pdf with expected value $\mu \in R^p$ and covariance matrix $t^{-1}\Sigma$.

**Result 27** *[28, Shankar, P.M., 2011, equation 2.83]*

A random variable $X$ is said to have a non-central chi-squared distribution with $k$ degrees of freedom and non-centrality parameter $\lambda$, denoted as $X \sim \chi^2_{\lambda}(k)$, if $X$ has pdf

$$f_X(x) = \frac{1}{2} \exp\left\{-\frac{x+\lambda}{2}\right\} \left(\frac{x}{\lambda}\right)^{\frac{k-2}{4}} I_{\frac{k-2}{2}}\left(\sqrt{\lambda x}\right), \quad x > 0, \tag{2.30}$$

where $k > 0$ and $I_v(\cdot)$ is the modified Bessel function of the first kind (2.13). The expected value is given by

$$E(X) = k + \lambda. \tag{2.31}$$

**Result 28** *[28, Shankar, P.M., 2011, equation 2.72]*

A random variable $X$ is said to have the Nakagami-$m$ distribution if $X$ has pdf

$$f_X(x) = \frac{2m^m x^{2m-1}}{\Omega^m \Gamma(m)} \exp\left\{-\frac{mx^2}{\Omega}\right\}, \ x > 0, \tag{2.32}$$

where $m \geq \frac{1}{2}$ with $m$ the Nakagami-parameter and $\Omega > 0$.

**Result 29** *[28, Shankar, P.M., 2011, equation 2.94]*

A random variable $X$ is said to have the Rayleigh distribution if $X$ has pdf

$$f_X(x) = \frac{x}{\beta} \exp\left\{-\frac{x^2}{2\beta}\right\}, \ x > 0, \tag{2.33}$$

where $\beta > 0$.

**Result 30** *[28, Shankar, P.M., 2011, equation 4.185]*

A random variable $X$ is said to have the Rice distribution if $X$ has pdf

$$f_X(x) = \frac{1}{2\sigma^2} \exp\left\{-\frac{x + d^2}{2\sigma^2}\right\} I_0\left(\frac{d\sqrt{x}}{\sigma^2}\right), \ x > 0, \tag{2.34}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and order zero.

**Result 31** *[6, Chu, K, 1973]*

With reference to (2.28): The weight function for the normal distribution is defined as the dirac delta function,

$$W(t) = \delta(t-1). \tag{2.35}$$

**Result 32** *[6, Chu, K, 1973]*

With reference to (2.28): The weight function for the t-distribution with $v$ degrees of freedom is defined

$$W(t) = \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)}. \tag{2.36}$$

**Result 33** *[2, Arashi, M. and Nadarajah, S., 2016]*

Based on the assumption that the weighting function admits the expansion, a weight approximation function is given as

$$W(t) = \sum_{k=1}^{\infty} a_k t^k. \tag{2.37}$$

**Theorem 2.1** *[19, Miller, K., 1964] Let $X_n$ be normally distributed with mean vector $A_n$ and positive covariance matrix $M_n$. Then $|X_n|$ has a one-dimensional density function. If $M_n = \psi_0 I_n$ where $\psi_0$ is a positive constant, then $r = |X_n|$ is called the Rayleigh variate. Then the density function of $r = |X_n|$ is*

$$g(r) = \frac{a}{\psi_0}\left(\frac{r}{a}\right)^{\frac{1}{2}n} e^{-(r^2+a^2)/2\psi_0} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right), \qquad r \geq 0 \tag{2.38}$$

*where $a = |A_n|$ and $I_v$ is the modified Bessel function of the first kind and order $v$.*

***Proof.*** *The density function of $X_n$ is*

$$\begin{aligned}
f_{X_n}(x_n) &= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}(X_n - A_n)'(X_n - A_n)\right\} \\
&= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}\left[X_n'X_n - X_n'A_n - A_n'X_n + A_n'A_n\right]\right\} \\
&= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}\left[|X_n|^2 + |A_n|^2 - 2X_n'A_n\right]\right\} \\
&= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}\left[|X_n|^2 + |A_n|^2\right]\right\} \exp\left\{-\frac{1}{2\psi_0}\left[-2X_n'A_n\right]\right\} \\
&= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}\left[r^2 + a^2\right]\right\} \exp\left\{\frac{X_n'A_n}{\psi_0}\right\}, \qquad (2.39)
\end{aligned}$$

*where $r = |X_n|$ and $a = |A_n|$.*
*Introduce a complete orthonormal set of vectors $E_n^{(1)}, E_n^{(2)}, ..., E_n^{(n)}$, with $E_n^{(1)}$ in the direction $A_n$. Then $A_n$ is the polar axis. If we make the generalized spherical coordinate change of variable*

$$\begin{aligned}
x_j &= r\left(\prod_{k=1}^{j-1}\sin\phi_k\right)\cos\phi_j, \qquad 1 \leq j \leq n-2, \\
x_{n-1} &= r\left(\prod_{k=1}^{n-2}\sin\phi_k\right)\cos\theta, \\
x_n &= r\left(\prod_{k=1}^{n-2}\sin\phi_k\right)\sin\theta,
\end{aligned}$$

*where $0 \leq \phi_k \leq \pi,\ 1 \leq k \leq n-2,\ 0 \leq \theta \leq 2\pi$ and $0 \leq r \leq \infty$, we obtain*

$$g(r) = \int_{|X_n|=r} f_n(X_n)d\sigma, \qquad r \geq 0 \tag{2.40}$$

*as the density function of $r$. Substituting (2.39) in (2.40) yields*

$$g(r) = \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\psi_0}\left[r^2 + a^2\right]\right\} \int_{|X_n|=r} \exp\left\{\frac{X_n'A_n}{\psi_0}\right\} d\sigma. \tag{2.41}$$

*The Jacobian of the transformation is*

$$J_n(r,\theta,\phi_1,\phi_2,...,\phi_{n-2}) = r^{n-1}\prod_{k=1}^{n-2}\sin^{n-1-k}\phi_k$$

*and since* $A_n$ *is the polar axis,* $X_n\'A_n = |X_n|\,|A_n|\cos\phi_1 = ra\cos\phi_1$. *Thus, from (2.41), we may write the integral as an iterated integral:*

$$\begin{aligned}\int\limits_{|X_n|=r}\exp\left\{\frac{X_n\'A_n}{\psi_0}\right\}d\sigma &= \int\limits_{|X_n|=r}\exp\{\frac{ra\cos\phi_1}{\psi_0}\}d\sigma\\ &= r^{n-1}\int\limits_0^{2\pi}d\theta\int\limits_0^{\pi}\exp\{\frac{ra\cos\phi_1}{\psi_0}\}\sin^{n-2}\phi_1 d\phi_1\prod_{k=1}^{n-2}\int\limits_0^{\pi}\sin^{n-1-k}\phi_k d\phi_k.\end{aligned} \tag{2.42}$$

*We recall the identity*

$$I_v(z) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(v+\frac{1}{2})}\left(\frac{1}{2}z\right)^v\int\limits_0^{\pi}\exp\{\pm z\cos\xi\}\sin^{2v}\xi d\xi, \tag{2.43}$$

*for* $\mathrm{Re}(v) > -\frac{1}{2}$, *where* $I_v$ *is the modified Bessel function of the first kind and order* $v$. *Using (2.43), the second integral in (2.42) becomes*

$$\int\limits_0^{\pi}\exp\{\frac{ra\cos\phi_1}{\psi_0}\}\sin^{n-2}\phi_1 d\phi_1 = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(n-1))\left(\frac{2\psi_0}{ra}\right)^{\frac{1}{2}(n-2)} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right).$$

*The product of integrals in (2.42) is as follows, since*

$$\int\limits_0^{\pi}\sin^{n-1-k}\phi_k d\phi_k = B\left(\frac{1}{2}(n-k),\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}(n-k)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(n-k+1)\right)},$$

*where* $B(u,w)$ *is the Beta function and* $\Gamma(u)$ *is the Gamma function. Therefore, we conclude that*

$$\prod_{k=1}^{n-2}\int\limits_0^{\pi}\sin^{n-1-k}\phi_k d\phi_k = \frac{\Gamma^{n-2}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n\right)}.$$

*Hence, (2.42) reduces to*

$$\begin{aligned}\int\limits_{|X_n|=r}\exp\left\{\frac{X_n\'A_n}{\psi_0}\right\}d\sigma &= r^{n-1}(2\pi)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(n-1))\left(\frac{2\psi_0}{ra}\right)^{\frac{1}{2}(n-2)}\\ &\quad\times I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right)\frac{\Gamma^{n-2}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n\right)}\end{aligned}$$

$$
\begin{aligned}
&= r^{n-1}(2\pi)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(n-1))\left(\frac{2\psi_0}{ra}\right)^{-1}\frac{\Gamma^{n-2}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n\right)} \\
&\quad \times \left(\frac{2\psi_0}{ra}\right)^{\frac{1}{2}n} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right) \\
&= \frac{a}{\psi_0}\left(\frac{2\pi r\psi_0}{a}\right)^{\frac{1}{2}n} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right). \qquad (2.44)
\end{aligned}
$$

*Substituting (2.44) in (2.41) we obtain the density function,*

$$
\begin{aligned}
g(r) &= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2\psi_0}\left[r^2+a^2\right]\right\}\int\limits_{|X_n|=r}\exp\left\{\frac{X_n{'}A_n}{\psi_0}\right\}d\sigma \\
&= \frac{1}{(2\pi\psi_0)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2\psi_0}\left[r^2+a^2\right]\right\}\frac{a}{\psi_0}\left(\frac{2\pi r\psi_0}{a}\right)^{\frac{1}{2}n} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right) \\
&= \frac{a}{\psi_0}\left(\frac{r}{a}\right)^{\frac{1}{2}n} e^{-(r^2+a^2)/2\psi_0} I_{\frac{1}{2}(n-2)}\left(\frac{ra}{\psi_0}\right).
\end{aligned}
$$

■

**Result 34** *Let $X_n$ follow an elliptical distribution with mean vector $A_n$ and positive covariance matrix $M_n$. Then $|X_n|$ has a one-dimensional density function. If $M_n = \psi_0 I_n$ where $\psi_0$ is a positive constant, then $r = |X_n|$ is called the Rayleigh variate. Then using* (2.28) *and* (2.38) *the density function of $r = |X_n|$ is*

$$
\begin{aligned}
g(r) &= \int_0^\infty W(t)\, g_{N(A_n,t^{-1}\psi_0)}(r|t)dt, \quad r \geq 0 \\
&= \int_0^\infty W(t)\,\frac{ta}{\psi_0}\left(\frac{r}{a}\right)^{\frac{1}{2}n} e^{-t(r^2+a^2)/2\psi_0} I_{\frac{1}{2}(n-2)}\left(\frac{tra}{\psi_0}\right)dt, \quad r \geq 0 \qquad (2.45)
\end{aligned}
$$

*where $a = |A_n|$ and $I_v$ is the modified Bessel function of the first kind and order $v$. (Proof similar to theorem 2.1).*

# Chapter 3

# Univariate Models

## 3.1 Introduction

Yacoub (2007) [35] pointed out that the well-known fading distributions have been derived assuming a homogenous, diffuse scattering field, that emanated from randomly distributed point scatters (see Figure 1.1). But this assumption is an approximation and a cluster based scattering model (see Figure 3.1) may be more relevant. The $\kappa-\mu$ model proposed by Yacoub (2007) [35] is a general physical fading model where the signal is characterised in terms of measurable physical parameters as described below in Section 3.2. The pdf of the power at the receiver and statistical properties of the $\kappa-\mu$ model will receive attention in Sections 3.2 and 3.3. The $\kappa-\mu$ type model that emanates from the elliptical assumption will be derived in Section 3.3; followed by section 3.4 where the performance measures under different members of the elliptical model will be illustrated.

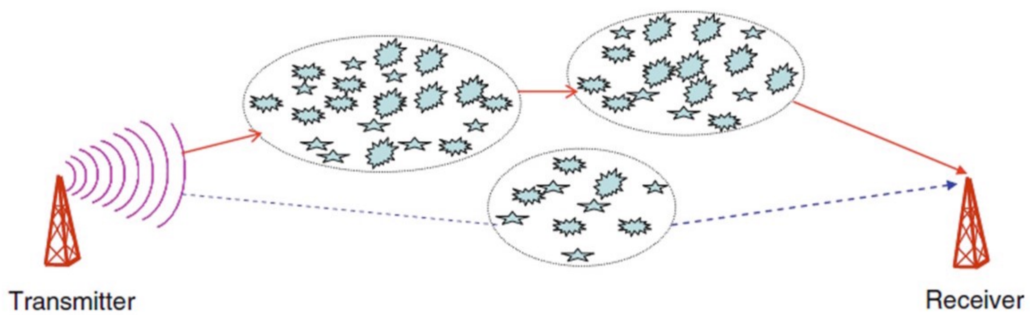



Figure 3.1: Clusters of multipath scattering [28]

## 3.2 The $\kappa - \mu$ distribution

### 3.2.1 Description

Let $X_i$ and $Y_i$ be mutually independent normal processes with $E(X_i) = E(Y_i) = 0$ and $var(X_i) = var(Y_i) = \sigma^2$. Then the envelope, $R$, of the physical model for the $\kappa - \mu$ distribution can be written in terms of the in-phase and quadrature components of the fading signal as

$$W = R^2 = \sum_{i=1}^{n} (X_i + p_i)^2 + \sum_{i=1}^{n} (Y_i + q_i)^2 , \tag{3.1}$$

where $p_i$ and $q_i$ are the mean values of the in-phase and quadrature components of the multipath waves of cluster $i$ and $n$ is the number of clusters of the multipath. Since $X_i \sim N(0, \sigma^2)$ and $Y_i \sim N(0, \sigma^2)$, we can write $(X_i + p_i) \sim N(p_i, \sigma^2)$ and $X_i^* = \left(\frac{X_i + p_i}{\sigma}\right) \sim N\left(\frac{p_i}{\sigma}, 1\right)$. Thus $(X_i^*)^2$ follows a $\chi^2_{\frac{p_i^2}{\sigma^2}}(1)$ with non-centrality parameter $\frac{p_i^2}{\sigma^2}$ (see (2.30)) and similarly we can express $Y_i^* = \left(\frac{Y_i + q_i}{\sigma}\right)$ where $(Y_i^*)^2$ follows a $\chi^2_{\frac{q_i^2}{\sigma^2}}(1)$ with non-centrality parameter $\frac{q_i^2}{\sigma^2}$ (see (2.30)).

Let $\frac{R_i^2}{\sigma^2} = \frac{W_i}{\sigma^2} = (X_i^*)^2 + (Y_i^*)^2$ then $\frac{R_i^2}{\sigma^2}$ follows a $\chi^2_\lambda(2)$ with non-centrality parameter $\lambda = \frac{d_i^2}{\sigma^2} = \frac{p_i^2 + q_i^2}{\sigma^2}$.

For $R^2 = \sum_{i=1}^{n} R_i^2$, the power of the fading signal is $W = \sum_{i=1}^{n} W_i$, and we are interested in the pdf of the envelope $R$ and the power $W$ respectively.

The following is a schematic outline of the mathematical algorithm that will be followed to obtain the necessary distributions.

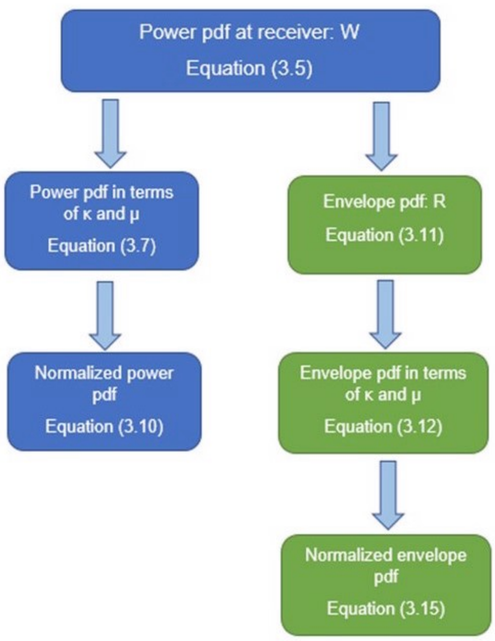



Figure 3.2

### 3.2.2 Derivation

Since the distribution of $\left(\frac{W_i}{\sigma^2}\right)$ is a $\chi^2_{\frac{d_i^2}{\sigma^2}}(2)$ distribution with non-centrality parameter $\frac{d_i^2}{\sigma^2}$, by using (2.30) the pdf of $W_i$ is then expressed as

$$f_{W_i}(w)=\frac{1}{2\sigma^2}\exp\left(-\frac{(w+d_i^2)}{2\sigma^2}\right)I_0\left(\frac{d_i\sqrt{w}}{\sigma^2}\right), \tag{3.2}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and order zero (2.13).

**Remark 3.1** *Note that equation* (3.2) *is the pdf of the Rice distribution (see* (2.34)*)*

By using (2.13) to express the Bessel function in (3.2) as an infinite series we obtain the pdf as

$$\begin{aligned} f_{W_i}(w) &= \frac{1}{2\sigma^2}\exp\left(-\frac{(w+d_i^2)}{2\sigma^2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{d_i^2 w}{4\sigma^4}\right)^k}{k!\Gamma(k+1)} \\ &= \frac{1}{2\sigma^2}\exp\left(-\frac{d_i^2}{2\sigma^2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{d_i^2}{4\sigma^4}\right)^k}{(k!)^2}w^k\exp\left(-\frac{w}{2\sigma^2}\right). \end{aligned} \tag{3.3}$$

The Laplace transform (2.25) of (3.3) is given by

$$\begin{aligned} f(s) &= L\{f_{W_i}(w)\}=\int_0^{\infty}f_{W_i}(w)\exp(-sw)\,dw \\ &= \int_0^{\infty}\frac{1}{2\sigma^2}\exp\left(-\frac{d_i^2}{2\sigma^2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{d_i^2}{4\sigma^4}\right)^k}{(k!)^2}w^k\exp\left(-\frac{w}{2\sigma^2}\right)\exp(-sw)\,dw \\ &= \frac{1}{2\sigma^2}\exp\left(-\frac{d_i^2}{2\sigma^2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{d_i^2}{4\sigma^4}\right)^k}{(k!)^2}\int_0^{\infty}w^k\exp\left\{-w\left(\frac{1}{2\sigma^2}+s\right)\right\}dw. \end{aligned}$$

Using (2.19), we obtain that

$$\begin{aligned} f(s) &= \frac{1}{2\sigma^2}\exp\left\{-\frac{d_i^2}{2\sigma^2}\right\}\sum_{k=0}^{\infty}\frac{\left(\frac{d_i^2}{4\sigma^4}\right)^k}{(k!)^2} \\ &\quad\times\frac{\Gamma(k+1)}{\left(\frac{1}{2\sigma^2}+s\right)^{k+1}}, \end{aligned}$$

and from (2.3), follows that

$$
\begin{aligned}
f(s) &= \frac{1}{2\sigma^2}\exp\left\{-\frac{d_i^2}{2\sigma^2}\right\}\sum_{k=0}^{\infty}\frac{d_i^{2k}}{(4\sigma^4)^k (k!)^2}\frac{k!}{\left(\frac{1}{2\sigma^2}+s\right)^{k+1}} \\
&= \frac{1}{2\sigma^2\left(\frac{1}{2\sigma^2}+s\right)}\exp\left\{-\frac{d_i^2}{2\sigma^2}\right\}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d_i^2}{4\sigma^4\left(\frac{1}{2\sigma^2}+s\right)}\right)^k \\
&= \frac{1}{(1+2\sigma^2 s)}\exp\left\{-\frac{d_i^2}{2\sigma^2}\right\}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d_i^2}{2\sigma^2+4\sigma^4 s}\right)^k.
\end{aligned}
$$

Using the series expansion formula (2.7), we have that

$$
\begin{aligned}
f(s) &= \frac{1}{(1+2\sigma^2 s)}\exp\left(-\frac{d_i^2}{2\sigma^2}\right)\exp\left(\frac{d_i^2}{2\sigma^2+4\sigma^4 s}\right) \\
&= \frac{1}{(1+2\sigma^2 s)}\exp\left(-\frac{d_i^2}{2\sigma^2}+\frac{d_i^2}{2\sigma^2+4\sigma^4 s}\right) \\
&= \frac{1}{(1+2\sigma^2 s)}\exp\left\{-\frac{s d_i^2}{(1+2\sigma^2 s)}\right\}.
\end{aligned}
$$

Since the $W_i,\ i=1,2,...,n$ are independent random variables it follows that the Laplace transformation, $L\{f_W(w)\}$, of $f_W(w)$ is found to be

$$
\begin{aligned}
L\{f_W(w)\} &= \prod_{i=1}^{n} L\{f_{W_i}(t)\} \\
&= \prod_{i=1}^{n}\frac{1}{(1+2\sigma^2 s)}\exp\left\{-\frac{s d_i^2}{(1+2\sigma^2 s)}\right\} \\
&= \frac{1}{(1+2\sigma^2 s)^n}\exp\left\{-\sum_{i=1}^{n}\frac{s d_i^2}{(1+2\sigma^2 s)}\right\} \\
&= \frac{1}{(1+2\sigma^2 s)^n}\exp\left\{-\frac{s d^2}{(1+2\sigma^2 s)}\right\}, \qquad (3.4)
\end{aligned}
$$

where $d^2=\sum_{i=1}^{n} d_i^2$. Hence, using (2.26) the pdf of the power is obtained from equation (3.4) as

$$
f_W(w)=\frac{1}{2\sigma^2}\left(\frac{w}{d^2}\right)^{\frac{n-1}{2}}\exp\left\{-\frac{(w+d^2)}{2\sigma^2}\right\} I_{n-1}\left(\frac{d\sqrt{w}}{\sigma^2}\right),\ w>0, \qquad (3.5)
$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind and order $v$ (2.13). Since $\frac{W}{\sigma^2}$ follows a $\chi^2_{\frac{d^2}{\sigma^2}}(n)$ distribution with non-centrality parameter $\frac{d^2}{\sigma^2}$, the expected value of $W$ is given by (2.31) as

$$
E(W)=2n\sigma^2+d^2. \qquad (3.6)
$$

**Remark 3.2** *Let* $U = \frac{W}{\sigma^2}$ *then* $W = \sigma^2 U$ *and* $\frac{dW}{dU} = \sigma^2$. *The pdf of* $U$ *is given by*

$$f_U(u) = \frac{1}{2}\exp\left(-\frac{u + \frac{d^2}{\sigma^2}}{2}\right)\left(\frac{\sigma^2 u}{d^2}\right)^{\frac{n-1}{2}} I_{n-1}\left(\frac{d\sqrt{u}}{\sigma}\right).$$

*This is the pdf of* $\chi^2_{\frac{d^2}{\sigma^2}}(2n)$ *with non-centrality parameter* $\frac{d^2}{\sigma^2}$. *Then the* $E(U) = 2n + \frac{d^2}{\sigma^2}$ *thus,* $E(W) = 2n\sigma^2 + d^2$.

Re-parameterization of the pdf of the power (3.5) in terms of $\kappa$ and $\mu$. Let $\mu = n$ and define $\kappa = \frac{d^2}{2\mu\sigma^2}$, then $\frac{d}{\sqrt{2\sigma^2}} = \sqrt{\kappa\mu}$, and the pdf of the power becomes

$$\begin{aligned} f_W(w) &= \frac{1}{2\sigma^2}\left(\frac{w}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{(w + 2\sigma^2\kappa\mu)}{2\sigma^2}\right\} I_{\mu-1}\left(\frac{\sqrt{2\sigma^2\kappa\mu}\sqrt{w}}{\sigma^2}\right) \\ &= \frac{1}{2\sigma^2}\left(\frac{w}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{w}{2\sigma^2} - \kappa\mu\right\} I_{\mu-1}\left(\frac{\sqrt{2\sigma^2\kappa\mu}\sqrt{w}}{\sigma^2}\right), \quad w > 0. \end{aligned} \tag{3.7}$$

From (3.7) the $\kappa - \mu$ pdf of the power can be written as, with $\bar{w}$ given by (see equation (3.6))

$$\bar{w} = E(W) = 2\sigma^2\mu(1+\kappa), \tag{3.8}$$

$$\begin{aligned} \bar{w}\ f_W(w) &= \frac{\bar{w}}{2\sigma^2}\left(\frac{\bar{w}\frac{w}{\bar{w}}}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{\bar{w}\frac{w}{\bar{w}}}{2\sigma^2} - \kappa\mu\right\} I_{\mu-1}\left(\frac{\sqrt{2\sigma^2\kappa\mu}\sqrt{\bar{w}\frac{w}{\bar{w}}}}{\sigma^2}\right) \\ &= \frac{2\sigma^2\mu(1+\kappa)}{2\sigma^2}\left(\frac{(2\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}}}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{(2\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}}}{2\sigma^2} - \kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(\frac{\sqrt{2\sigma^2\kappa\mu}\sqrt{(2\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}}}}{\sigma^2}\right) \\ &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\frac{w}{\bar{w}}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{(\mu(1+\kappa))w}{\bar{w}} - \kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\frac{w}{\bar{w}}}\right). \end{aligned} \tag{3.9}$$

For the normalized power $\Omega = \frac{W}{\bar{w}}$ (see Simon and Alouini, 2005 [29]), the pdf of $\Omega$ can be obtained from (3.9) as

$$\begin{aligned} f_\Omega(\omega) &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}(\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)\omega - \kappa\mu\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\omega}\right). \end{aligned} \tag{3.10}$$

The pdf of the envelope can be obtained from (3.5). Since $W = R^2$ we can express the pdf of the envelope as

$$\begin{aligned} f_R(r) &= \frac{2r}{2\sigma^2}\left(\frac{r^2}{d^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{(r^2+d^2)}{2\sigma^2}\right\} I_{n-1}\left(\frac{dr}{\sigma^2}\right) \\ &= \frac{r^n}{\sigma^2 d^{n-1}} \exp\left\{-\frac{(r^2+d^2)}{2\sigma^2}\right\} I_{n-1}\left(\frac{d}{\sigma^2}r\right). \end{aligned} \tag{3.11}$$

Using (3.7), the pdf of the envelope can be expressed in terms of $\kappa$ and $\mu$ as

$$\begin{aligned} f_R(r) &= \frac{2r}{2\sigma^2}\left(\frac{r^2}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{r^2}{2\sigma^2}-\kappa\mu\right\} I_{\mu-1}\left(\frac{\sqrt{2\sigma^2\kappa\mu}\sqrt{r^2}}{\sigma^2}\right) \\ &= \frac{1}{\sigma^2}\left(\frac{1}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} r^{\mu} \exp\left\{-\frac{r^2}{2\sigma^2}-\kappa\mu\right\} I_{\mu-1}\left(\frac{r\sqrt{2\sigma^2\kappa\mu}}{\sigma^2}\right). \end{aligned} \tag{3.12}$$

Let

$$\hat{r} = \sqrt{E(R^2)} = \sqrt{E(W)} = \sqrt{2\sigma^2\mu(1+\kappa)}, \tag{3.13}$$

then from (3.12) it follows that

$$\begin{aligned} \hat{r} f_R(r) &= \frac{\hat{r}}{\sigma^2}\left(\frac{1}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \left(\frac{r}{\hat{r}}\hat{r}\right)^{\mu} \exp\left\{-\frac{\left(\frac{r}{\hat{r}}\hat{r}\right)^2}{2\sigma^2}-\kappa\mu\right\} I_{\mu-1}\left(\frac{\left(\frac{r}{\hat{r}}\hat{r}\right)\sqrt{2\sigma^2\kappa\mu}}{\sigma^2}\right) \\ &= \frac{\left(\sqrt{2\sigma^2\mu(1+\kappa)}\right)^{\mu+1}}{\sigma^2}\left(\frac{1}{2\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \left(\frac{r}{\hat{r}}\right)^{\mu} \exp\left\{-\frac{\left(\sqrt{2\sigma^2\mu(1+\kappa)}\right)^2\left(\frac{r}{\hat{r}}\right)^2}{2\sigma^2}-\kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(\frac{\left(\frac{r}{\hat{r}}\right)\left(\sqrt{2\sigma^2\mu(1+\kappa)}\right)\sqrt{2\sigma^2\kappa\mu}}{\sigma^2}\right) \\ &= \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\frac{r}{\hat{r}}\right)^{\mu} \exp\left\{-\mu(1+\kappa)\left(\frac{r}{\hat{r}}\right)^2-\kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\left(\frac{r}{\hat{r}}\right)\right). \end{aligned} \tag{3.14}$$

For the normalized envelope $P = \frac{R}{\hat{r}}$, the pdf can be obtained from (3.14) and is given as

$$f_P(\rho) = \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\rho^{\mu} \exp\left\{-\mu\kappa-\mu(1+\kappa)\rho^2\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho\right), \tag{3.15}$$

where $\rho > 0$.

**Remark 3.3** $\kappa = \frac{d^2}{2n\sigma^2}$ *is defined as the ratio between the total power of the dominant components and the total power of the scattered waves (see Yacoub (2007) [35]).*

**Remark 3.4** *Let* $\mu = \frac{1}{var(\rho^2)} \frac{1+2\kappa}{(1+\kappa)^2}$ *with* $\mu$ *being the real extension of* $n$. *Thus,* $\mu$ *can take on non-integer values which may account for (a) non-zero correlation among the clusters of multipath components, (b) non-zero correlation between in-phase and quadrature components within each cluster, (c) the non-normal nature of the in-phase and quadrature components of each cluster of the fading signal. (See Yacoub (2007) [35]).*

**Remark 3.5** *Equations* (3.7), (3.9), (3.10), (3.14) *and* (3.15) *correspond to equations* (7), (10), (2), (11) *and* (1) *of Yacoub (2007) [35].*

**Remark 3.6** *Note that equation* (3.10) *can be approximated using the mixture gamma model which facilitates complicated or intractable performance analysis functions (see Atapattu (2011) [4]).*

### 3.2.3 Characteristics

Statistical characteristics for the $\kappa - \mu$ fading model is now discussed in this Section.

#### 3.2.3.1 Cumulative distribution function

Consider the $\kappa - \mu$ pdf for the normalized power (3.10), $\Omega$. Using (2.13), the pdf can be written as

$$\begin{aligned} f_{\Omega}(\omega) &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)\omega - \kappa\mu\} \sum_{l=0}^{\infty} \left( \frac{2\mu\sqrt{\kappa(1+\kappa)\omega}}{2} \right)^{\mu-1} \\ &\times \frac{\left( \frac{1}{4} \left( 2\mu\sqrt{\kappa(1+\kappa)\omega} \right)^2 \right)^l}{l!\Gamma(\mu+l)}. \end{aligned}$$

The cdf of $\Omega$ with $Z_T$ the specified threshold (see (1.2)), is

$$\begin{aligned} F(Z_T) &= \int_0^{Z_T} f_{\Omega}(\omega) \, d\omega \\ &= \int_0^{Z_T} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)\omega - \kappa\mu\} \sum_{l=0}^{\infty} \left( \frac{2\mu\sqrt{\kappa(1+\kappa)\omega}}{2} \right)^{\mu-1} \\ &\times \frac{\left( \frac{1}{4} \left( 2\mu\sqrt{\kappa(1+\kappa)\omega} \right)^2 \right)^l}{l!\Gamma(\mu+l)} d\omega \end{aligned}$$

$$
\begin{aligned}
F(Z_T) &= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\, l!\,\Gamma(\mu+l)} \\
&\quad \times \int_0^{Z_T} \omega^{\mu-1}\exp\{-\mu(1+\kappa)\omega\}\left(\mu^2\kappa(1+\kappa)\omega\right)^l d\omega \\
&= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\, l!\,\Gamma(\mu+l)} \\
&\quad \times \int_0^{Z_T} \omega^{\mu+l-1}\exp\{-\mu(1+\kappa)\omega\}\, d\omega. \qquad (3.16)
\end{aligned}
$$

Using (2.20), the cdf can be written as

$$
\begin{aligned}
F(Z_T) &= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\, l!\,\Gamma(\mu+l)} \\
&\quad \times [\mu(1+\kappa)]^{-(\mu+l-1)-1}\gamma(\mu+l-1+1,\mu(1+\kappa)Z_T) \\
&= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\, l!\,\Gamma(\mu+l)} \\
&\quad \times [\mu(1+\kappa)]^{-\mu-l}\gamma(\mu+l,\mu(1+\kappa)Z_T) \\
&= \sum_{l=0}^{\infty} \frac{(\mu\kappa)^l\gamma(\mu+l,\mu(1+\kappa)Z_T)}{l!\,\Gamma(\mu+l)\exp(\mu\kappa)}, \qquad (3.17)
\end{aligned}
$$

for $\mu > 0$, index $l$ where $\gamma(\cdot)$ is the incomplete gamma function (2.5).

**Remark 3.7** *The outage probability can be calculated from* (3.17) *for a specificied threshold* $Z_T$.

#### 3.2.3.2 Moments

From (3.15) the $j^{th}$ moment of the normalized envelope, $P$, is

$$
\begin{aligned}
E\left(P^j\right) &= \int_0^{\infty} \rho^j f_P(\rho)\, d\rho \\
&= \int_0^{\infty} \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\rho^{\mu+j}\exp\left\{-\mu\kappa-\mu(1+\kappa)\rho^2\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}\rho\right) d\rho.
\end{aligned}
$$

Using (2.22), the $j^{th}$ moment is

$$\begin{aligned}
E\left(P^j\right) &= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left(\mu\kappa\right)}2^{-\mu+1-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\right)^{\mu-1}\left(\mu\left(1+\kappa\right)\right)^{-\frac{(\mu+j+1+\mu-1)}{2}}\Gamma\left[\begin{array}{c}\frac{\mu+j+1+\mu-1}{2}\\ \mu-1+1\end{array}\right]\\
&\times{}_1F_1\left(\frac{\mu+j+1+\mu-1}{2};\mu-1+1;\frac{\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\right)^2}{4\mu\left(1+\kappa\right)}\right)\\
&= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left(\mu\kappa\right)}2^{-\mu}\left(2\mu\right)^{\mu-1}\left(\kappa\left(1+\kappa\right)\right)^{\frac{\mu-1}{2}}\left(\mu\left(1+\kappa\right)\right)^{-\mu-\frac{j}{2}}\frac{\Gamma\left(\mu+\frac{j}{2}\right)}{\Gamma\left(\mu\right)}\\
&\times{}_1F_1\left(\mu+\frac{j}{2};\mu;\frac{4\mu^2\kappa\left(1+\kappa\right)}{4\mu\left(1+\kappa\right)}\right)\\
&= \frac{\Gamma\left(\mu+\frac{j}{2}\right)}{\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}\,{}_1F_1\left(\mu+\frac{j}{2};\mu;\mu\kappa\right), \qquad (3.18)
\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function (see (2.1)) and ${}_1F_1(\cdot)$ is the confluent hypergeometric series (see (2.10)).

From (3.10) the $j^{th}$ moment of the normalized power, $\Omega$, is

$$\begin{aligned}
E\left(\Omega^j\right) &= \int_0^\infty \omega^j f_\Omega\left(\omega\right)d\omega\\
&= \int_0^\infty \frac{\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\omega\right)^{\frac{\mu-1}{2}+j}\exp\left\{-\mu\left(1+\kappa\right)\omega-\kappa\mu\right\}\\
&\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)\omega}\right)d\omega.
\end{aligned}$$

Let $\alpha=\sqrt{\omega}$, then

$$\begin{aligned}
E\left(\Omega^j\right) &= \int_0^\infty \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\alpha\right)^{\mu+1+2j-1}\exp\left\{-\mu\left(1+\kappa\right)\alpha^2-\kappa\mu\right\}\\
&\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\alpha\right)d\alpha.
\end{aligned}$$

Using (2.22), we obtain the $j^{th}$ moment as

$$\begin{aligned}
E\left(\Omega^j\right) &= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left(\mu\kappa\right)}2^{-\mu+1-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\right)^{\mu-1}\left(\mu\left(1+\kappa\right)\right)^{-\frac{(\mu+1+2j+\mu-1)}{2}}\Gamma\left[\begin{array}{c}\frac{\mu+1+2j+\mu-1}{2}\\ \mu-1+1\end{array}\right]\\
&\times\ {}_1F_1\left(\frac{\mu+1+2j+\mu-1}{2};\mu-1+1;\frac{\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\right)^2}{4\mu\left(1+\kappa\right)}\right)
\end{aligned}$$

$$\begin{aligned} E\left(\Omega^j\right) &= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left(\mu\kappa\right)} 2^{-\mu}\left(2\mu\right)^{\mu-1}\left(\kappa\left(1+\kappa\right)\right)^{\frac{\mu-1}{2}}\left(\mu\left(1+\kappa\right)\right)^{-\mu-j}\frac{\Gamma\left(\mu+j\right)}{\Gamma\left(\mu\right)} \\ &\times {}_1F_1\left(\frac{\mu+1+2j+\mu-1}{2};\mu-1+1;\frac{4\mu^2\kappa\left(1+\kappa\right)}{4\mu\left(1+\kappa\right)}\right) \\ &= \frac{\Gamma\left(\mu+j\right)}{\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^j}\ {}_1F_1\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right). \end{aligned} \tag{3.19}$$

#### 3.2.3.3 Amount of fading

The amount of fading for the $\kappa - \mu$ model is given as (see (1.1))

$$\begin{aligned} AF &= \frac{E\left(\Omega^2\right)}{E\left(\Omega\right)^2} - 1 \\ &= \frac{\left[\frac{\Gamma(\mu+2)}{\Gamma(\mu)\exp\{\mu\kappa\}[\mu(1+\kappa)]^2}\ {}_1F_1\left(\mu+2\ ;\ \mu\ ;\mu\kappa\right)\right]}{\left[\frac{\Gamma(\mu+1)}{\Gamma(\mu)\exp\{\mu\kappa\}[\mu(1+\kappa)]}\ {}_1F_1\left(\mu+1\ ;\ \mu\ ;\mu\kappa\right)\right]^2} - 1 \\ &= \left(\frac{\Gamma\left(\mu+2\right)\left(2\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}\left[\mu\left(1+\kappa\right)\right]\right)^2}{\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}\left[\mu\left(1+\kappa\right)\right]^2\left(\Gamma\left(\mu+1\right)\right)^2}\frac{{}_1F_1\left(\mu+2\ ;\ \mu\ ;\mu\kappa\right)}{\left({}_1F_1\left(\mu+1\ ;\ \mu\ ;\mu\kappa\right)\right)^2}\right) - 1 \\ &= \left(\frac{\Gamma\left(\mu+2\right)\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}}{\left(\Gamma\left(\mu+1\right)\right)^2}\frac{{}_1F_1\left(\mu+2\ ;\ \mu\ ;\mu\kappa\right)}{\left({}_1F_1\left(\mu+1\ ;\ \mu\ ;\mu\kappa\right)\right)^2}\right) - 1 \\ &= \left(\frac{\left(\mu+1\right)\exp\left\{\mu\kappa\right\}}{\mu}\frac{{}_1F_1\left(\mu+2\ ;\ \mu\ ;\mu\kappa\right)}{\left({}_1F_1\left(\mu+1\ ;\ \mu\ ;\mu\kappa\right)\right)^2}\right) - 1. \end{aligned} \tag{3.20}$$

### 3.2.4 Special cases

#### 3.2.4.1 The $\kappa - \mu$ distribution for $\kappa \to 0$ (Nakagami - $m$)

From (3.15) and (2.14) it follows

$$\begin{aligned} f_P\left(\rho\right) &= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left\{\mu\kappa\right\}}\rho^\mu\exp\left\{-\mu\left(1+\kappa\right)\rho^2\right\}\frac{\left(\mu\sqrt{\kappa\left(1+\kappa\right)}\rho\right)^{\mu-1}}{\Gamma\left(\mu\right)} \\ &= \frac{2\mu^\mu\left(1+\kappa\right)^\mu}{\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}}\rho^{2\mu-1}\exp\left\{-\mu\left(1+\kappa\right)\rho^2\right\}. \end{aligned} \tag{3.21}$$

As $\kappa \to 0$, the pdf (3.21) tends to

$$f_P\left(\rho\right) = \frac{2\mu^\mu}{\Gamma\left(\mu\right)}\rho^{2\mu-1}\exp\left\{-\mu\rho^2\right\},\ \ \rho > 0,$$

which has the form of a Nakagami-$m$ distribution (see (2.32) with $\mu = m$ and $\Omega = 1$).

Table 1 gives other special cases of the $\kappa - \mu$ model

| Distribution | Parameters |
|---|---|
| Rice | $\mu = 1$ |
| Rayleigh | $\mu = 1,\ \kappa = 0$ |
| One-sided Gaussian | $\mu = 0.5,\ \kappa \to 0$ |

Table 1: Special cases

## 3.3 The $\kappa - \mu$ type distribution

### 3.3.1 Description

Let $X_i$ and $Y_i$ be mutually independent elliptical distributed with $E(X_i) = E(Y_i) = 0$ and $var(X_i) = var(Y_i) = -2\sigma^2\Psi'(0)$. Then the envelope, $R$, of the physical model for the $\kappa - \mu$ distribution can be written in terms of the in-phase and quadrature components of the fading signal as

$$W = R^2 = \sum_{i=1}^{n}(X_i + p_i)^2 + \sum_{i=1}^{n}(Y_i + q_i)^2, \tag{3.22}$$

where $p_i$ and $q_i$ are the mean values of the in-phase and quadrature components of the multipath waves of cluster $i$ and $n$ is the number of clusters of multipath.

Since $X_i \sim E(0,\sigma^2)$ and $Y_i \sim E(0,\sigma^2)$, therefore from (2.28), $X_i|t \sim N(0,a_t)$ and $Y_i|t \sim N(0,a_t)$ where $a_t = t^{-1}\sigma^2$. Hence, $(X_i + p_i)|t \sim N(p_i, a_t)$ and $X_i^*|t = \left(\frac{X_i+p_i}{\sqrt{a_t}}|t\right) \sim N\left(\frac{p_i}{\sqrt{a_t}}, 1\right)$. Thus $(X_i^*)^2|t$ follows a $\chi^2_{\frac{p_i^2}{a_t}}(1)$ with non-centrality parameter $\frac{p_i^2}{a_t}$ and similarly we can express $Y_i^*|t = \left(\frac{Y_i+q_i}{\sqrt{a_t}}|t\right) \sim N\left(\frac{q_i}{\sqrt{a_t}}, 1\right)$ and $(Y_i^*)^2|t$ follows a $\chi^2_{\frac{q_i^2}{a_t}}(1)$ with non-centrality parameter $\frac{q_i^2}{a_t}$.

Let $\frac{R_i^2|t}{a_t} = \frac{W_i|t}{a_t} = (X_i^*)^2|t + (Y_i^*)^2|t$ then $\frac{R_i^2|t}{a_t}$ follows a $\chi^2_{\frac{d_i^2}{a_t}}(2)$ with non-centrality parameter $\frac{d_i^2}{a_t} = \frac{p_i^2+q_i^2}{a_t}$.

For $R^2|t = \sum_{i=1}^{n} R_i^2|t$, the power of the fading signal is $W|t = \sum_{i=1}^{n} W_i|t$, and we are interested in the pdf of the envelope $R$ and the power $W$ respectively.

The following is a schematic outline of the mathematical algorithm that will be followed to obtain the necessary distributions.

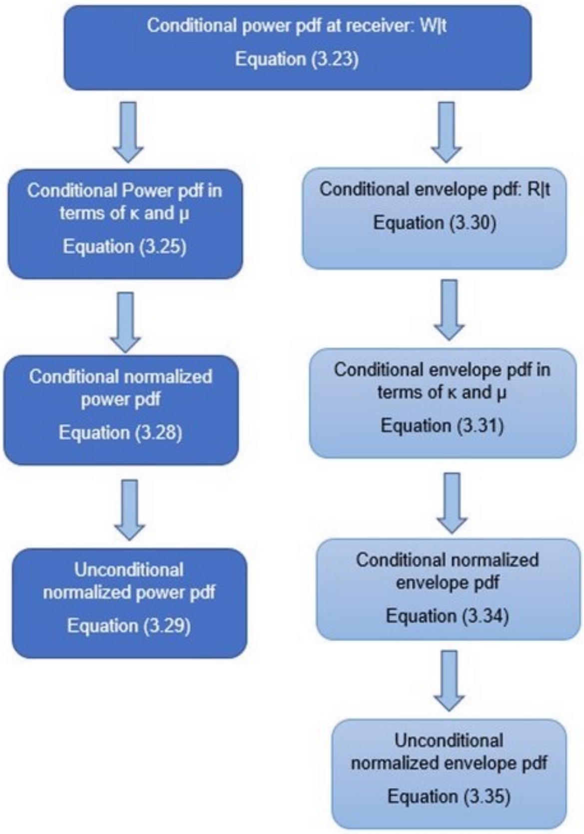



Figure 3.3

### 3.3.2 Derivation

The pdf of $\left(\frac{W_i|t}{a_t}\right)$ is a $\chi^2_{\frac{d_i^2}{a_t}}(2)$ distribution with non-centrality parameter $\frac{d_i^2}{a_t}$. (Note for the elliptical case for simplicity, $f_{W_i|t}(w_i|t) \equiv f(w_i|t)$). The conditional pdf of $W_i$ is then

$$f(w_i|t) = \frac{1}{2a_t} \exp\left\{-\frac{(w_i + d_i^2)}{2a_t}\right\} I_0\left(\frac{d_i\sqrt{w_i}}{a_t}\right).$$

Using the Laplace transform method as in Section 3.2, the conditional pdf of the power, $W$, is

$$f(w|t) = \frac{1}{2a_t}\left(\frac{w}{d^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{(w+d^2)}{2a_t}\right\} I_{n-1}\left(\frac{d\sqrt{w}}{a_t}\right), \quad w > 0. \tag{3.23}$$

Since $\frac{W|t}{a_t}$ follows a $\chi^2_{\frac{d^2}{a_t}}(n)$ distribution with non-centrality parameter $\frac{d^2}{a_t}$, the expected value of $W|t$ is given by (2.31) as

$$E(W|t) = 2na_t + d^2. \tag{3.24}$$

Re-parameterization of the conditional pdf of the power in terms of $\kappa$ and $\mu$ and let $\mu=n$ with $\kappa=\frac{d^2}{2\mu t^{-1}\sigma^2}$, then $\frac{d}{\sqrt{2t^{-1}\sigma^2}}=\sqrt{\kappa\mu}$. The conditional pdf of the power becomes

$$\begin{aligned} f(w|t) &= \frac{1}{2t^{-1}\sigma^2}\left(\frac{w}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}}\exp\left\{-\frac{(w+2t^{-1}\sigma^2\kappa\mu)}{2t^{-1}\sigma^2}\right\}I_{\mu-1}\left(\frac{\sqrt{2t^{-1}\sigma^2\kappa\mu}\sqrt{w}}{t^{-1}\sigma^2}\right) \\ &= \frac{1}{2t^{-1}\sigma^2}\left(\frac{w}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}}\exp\left\{-\frac{w}{2t^{-1}\sigma^2}-\kappa\mu\right\}I_{\mu-1}\left(\frac{\sqrt{2t^{-1}\sigma^2\kappa\mu}\sqrt{w}}{t^{-1}\sigma^2}\right) \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) the conditional pdf of the power can be written as follows, with (see equation (3.24))

$$\bar{w}_t \equiv E(W|t) = 2a_t\mu(1+\kappa) = 2t^{-1}\sigma^2\mu(1+\kappa) = t^{-1}\omega, \tag{3.26}$$

$$\begin{aligned} \bar{w}_t\, f(w|t) &= \frac{\bar{w}_t}{2t^{-1}\sigma^2}\left(\frac{\bar{w}_t\frac{w}{\bar{w}_t}}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}}\exp\left\{-\frac{\bar{w}_t\frac{w}{\bar{w}_t}}{2t^{-1}\sigma^2}-\kappa\mu\right\}I_{\mu-1}\left(\frac{\sqrt{2t^{-1}\sigma^2\kappa\mu}\sqrt{\bar{w}_t\frac{w}{\bar{w}_t}}}{t^{-1}\sigma^2}\right) \\ &= \frac{2t^{-1}\sigma^2\mu(1+\kappa)}{2t^{-1}\sigma^2}\left(\frac{(2t^{-1}\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}_t}}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}}\exp\left\{-\frac{(2t^{-1}\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}_t}}{2t^{-1}\sigma^2}-\kappa\mu\right\} \\ &\quad\times I_{\mu-1}\left(\frac{\sqrt{2t^{-1}\sigma^2\kappa\mu}\sqrt{(2t^{-1}\sigma^2\mu(1+\kappa))\frac{w}{\bar{w}_t}}}{t^{-1}\sigma^2}\right) \\ &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\frac{w}{\bar{w}_t}\right)^{\frac{\mu-1}{2}}\exp\left\{-\frac{(\mu(1+\kappa))w}{\bar{w}_t}-\kappa\mu\right\} \\ &\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\frac{w}{\bar{w}_t}}\right). \end{aligned} \tag{3.27}$$

For the pdf of the normalized power $\Omega_t \equiv \Omega|t = \frac{W|t}{\bar{w}_t} = \frac{\omega_t}{\bar{\omega}_t}$, the pdf can be obtained from (3.27) as

$$\begin{aligned} f_{\Omega_t}(\omega_t) &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}(\omega_t)^{\frac{\mu-1}{2}}\exp\{-\mu(1+\kappa)\omega_t-\kappa\mu\} \\ &\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\omega_t}\right). \end{aligned} \tag{3.28}$$

Hence the unconditional pdf of the normalized power, $\Omega=\frac{W}{t^{-1}\bar{w}}=t\omega$ (see (3.26)), is given by

$$f_\Omega(\omega) = \int_0^\infty W(t)\, f_{\Omega_t}(\omega_t)\, dt$$

$$\begin{aligned} f_{\Omega}(\omega) &= \int_0^{\infty} W(t) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)t\omega - \kappa\mu\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt. \end{aligned} \tag{3.29}$$

The conditional pdf of the envelope can be obtained from (3.23). Since $W = R^2$, we can express the conditional pdf of the envelope as

$$f(r|t) = \frac{1}{2a_t}\left(\frac{r^2}{d^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{(r^2+d^2)}{2a_t}\right\} I_{n-1}\left(\frac{dr}{a_t}\right). \tag{3.30}$$

Using (3.25), the conditional pdf of the envelope can be expressed in terms of $\kappa$ and $\mu$, as follows

$$\begin{aligned} f(r|t) &= \frac{2r}{2t^{-1}\sigma^2}\left(\frac{r^2}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \exp\left\{-\frac{r^2}{2t^{-1}\sigma^2} - \kappa\mu\right\} I_{\mu-1}\left(\frac{\sqrt{2t^{-1}\sigma^2\kappa\mu}\sqrt{r^2}}{t^{-1}\sigma^2}\right) \\ &= \frac{1}{t^{-1}\sigma^2}\left(\frac{1}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} r^{\mu} \exp\left\{-\frac{r^2}{2t^{-1}\sigma^2} - \kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(\frac{r\sqrt{2t\kappa\mu}}{\sigma}\right). \end{aligned} \tag{3.31}$$

The conditional $\kappa - \mu$ pdf of the envelope is then obtained from (3.31), with (see (3.13) and (3.26))

$$\hat{r}_t \equiv \sqrt{E(R^2|t)} = \sqrt{E(W|t)} = \left(\sqrt{2t^{-1}\sigma^2\mu(1+\kappa)}\right), \tag{3.32}$$

from equation (3.26), as follows

$$\begin{aligned} \hat{r}_t f(r|t) &= \frac{\hat{r}_t}{t^{-1}\sigma^2}\left(\frac{1}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}} \left(\frac{r}{\hat{r}_t}\hat{r}_t\right)^{\mu} \exp\left\{-\frac{\left(\frac{r}{\hat{r}_t}\hat{r}_t\right)^2}{2t^{-1}\sigma^2} - \kappa\mu\right\} \\ &\quad \times I_{\mu-1}\left(\frac{\left(\frac{r}{\hat{r}_t}\hat{r}_t\right)\sqrt{2t\kappa\mu}}{\sigma}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\sqrt{2t^{-1}\sigma^2\mu\left(1+\kappa\right)}\right)^{\mu+1}}{t^{-1}\sigma^2}\left(\frac{1}{2t^{-1}\sigma^2\kappa\mu}\right)^{\frac{\mu-1}{2}}\left(\frac{r}{\hat{r}_t}\right)^{\mu} \\
&\quad\times\exp\left\{-\frac{\left(\sqrt{2t^{-1}\sigma^2\mu\left(1+\kappa\right)}\right)^2\left(\frac{r}{\hat{r}_t}\right)^2}{2t^{-1}\sigma^2}-\kappa\mu\right\} \\
&\quad\times I_{\mu-1}\left(\frac{\left(\frac{r}{\hat{r}_t}\right)\left(\sqrt{2t^{-1}\sigma^2\mu\left(1+\kappa\right)}\right)\sqrt{2t\kappa\mu}}{\sigma}\right) \\
&= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\frac{r}{\hat{r}_t}\right)^{\mu}\exp\left\{-\mu\left(1+\kappa\right)\left(\frac{r}{\hat{r}_t}\right)^2-\kappa\mu\right\} \\
&\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\left(\frac{r}{\hat{r}_t}\right)\right). \qquad (3.33)
\end{aligned}$$

From (3.33) the pdf for the normalized envelope $P_t = P|t = \frac{R|t}{\hat{r}_t}$, is

$$f_{P_t}\left(\rho_t\right) = \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\rho_t^{\mu}\exp\left\{-\mu\kappa-\mu\left(1+\kappa\right)\rho_t^2\right\}\; I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\rho_t\right). \qquad (3.34)$$

As before the unconditional pdf of the normalized envelope is given by

$$\begin{aligned}
f_P\left(\rho\right) &= \int_0^{\infty} W\left(t\right)\frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(t\rho\right)^{\mu}\exp\left\{-\mu\kappa-\mu\left(1+\kappa\right)\left(t\rho\right)^2\right\} \\
&\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}t\rho\right)dt, \qquad (3.35)
\end{aligned}$$

where $\rho > 0$.

**Remark 3.8** *For the normal distribution the weight function is* (2.35). *The unconditional pdf of the normalized envelope follows from* (3.35) *as*

$$\begin{aligned}
f_P\left(\rho\right) &= \int_0^{\infty} \delta\left(t-1\right)\frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(t\rho\right)^{\mu}\exp\left\{-\mu\kappa-\mu\left(1+\kappa\right)\left(t\rho\right)^2\right\} \\
&\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}t\rho\right)dt. \\
&= \frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\rho^{\mu}\exp\left\{-\mu\kappa-\mu\left(1+\kappa\right)\rho^2\right\} \\
&\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)}\rho\right). \qquad (3.36)
\end{aligned}$$

*The unconditional pdf of the normalized power follows from* (3.29)

$$\begin{aligned} f_{\Omega}(\omega) &= \int_0^{\infty} \delta(t-1) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)t\omega - \kappa\mu\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt. \\ &\quad \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} \omega^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)\omega - \kappa\mu\} \\ &\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\omega}\right) \\ &= (3.10) \end{aligned} \tag{3.37}$$

**Remark 3.9** *For the t-distribution the weight function is* (2.36). *The unconditional pdf of the normalized envelope follows from* (3.35) *as*

$$\begin{aligned} f_P(\rho) &= \int_0^{\infty} \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)} \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\rho)^{\mu} \\ &\quad \times \exp\left\{-\mu\kappa - \mu(1+\kappa)(t\rho)^2\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}t\rho\right) dt \\ &= \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)} \rho^{\mu} \left(\frac{v}{2}\right)^{\frac{v}{2}} \frac{1}{2\Gamma\left(\frac{v}{2}\right)} \int_0^{\infty} t^{\mu+\frac{v}{2}-1} \\ &\quad \times \exp\left\{-\mu(1+\kappa)t^2\rho^2 - \frac{vt}{2}\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}t\rho\right) dt \\ &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\frac{v}{2}\right)^{\frac{v}{2}}\rho^{\mu}}{\kappa^{\frac{\mu-1}{2}}\Gamma\left(\frac{v}{2}\right)\exp(\mu\kappa)} \sum_{z=0}^{\infty} \frac{(-1)^z}{z!} \left(\frac{v}{2}\right)^z \int_0^{\infty} t^{\mu+\frac{v}{2}+z-1} \\ &\quad \times \exp\left\{-\mu(1+\kappa)t^2\rho^2\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}t\rho\right) dt. \end{aligned}$$

*Using* (2.22), *we get*

$$\begin{aligned} f_P(\rho) &= \sum_{z=0}^{\infty} \frac{(-1)^z \mu(1+\kappa)^{\frac{\mu+1}{2}} \left(\frac{v}{2}\right)^{\frac{v}{2}+z} \rho^{\mu}}{z!\kappa^{\frac{\mu-1}{2}}\Gamma\left(\frac{v}{2}\right)\exp(\mu\kappa)} 2^{-\mu+1-1}\left[2\mu\sqrt{\kappa(1+\kappa)}\rho\right]^{\mu-1} \\ &\quad \times \left[\mu(1+\kappa)\rho^2\right]^{-\frac{1}{2}\left(\mu+\frac{v}{2}+z+\mu-1\right)} \Gamma\left[\begin{array}{c} \frac{1}{2}\left(\mu+\frac{v}{2}+z+\mu-1\right) \\ \mu-1+1 \end{array}\right] \\ &\quad \times {}_1F_1\left(\frac{1}{2}\left(\mu+\frac{v}{2}+z+\mu-1\right); \mu-1+1; \frac{4\mu^2\kappa(1+\kappa)\rho^2}{4\mu(1+\kappa)\rho^2}\right) \end{aligned}$$

$$f_P(\rho) = \sum_{z=0}^{\infty} \frac{(-1)^z \kappa^{\frac{\mu}{2}-\frac{1}{2}} (1+\kappa)^{-\frac{1}{2}\left(\frac{v}{2}+z-1\right)} \mu^{\frac{v}{2}} v^z \rho^{-\frac{v}{2}-z}}{2^{\frac{v}{2}+z+1} z! \Gamma\left(\frac{v}{2}\right) \exp(\mu\kappa)} \times \Gamma\left[\begin{array}{c} \mu+\frac{v}{4}+\frac{z}{2}-\frac{1}{2} \\ \mu \end{array}\right] {}_1F_1\left(\mu+\frac{v}{4}+\frac{z}{2}-\frac{1}{2}; \mu; \mu\kappa\right), \quad (3.38)$$

*where* $\rho > 0$.

*The unconditional pdf of the normalized power follows from* (3.29) *as*

$$\begin{aligned} f_\Omega(\omega) &= \int_0^\infty \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \\ &\quad \times \exp\{-\mu(1+\kappa)t\omega - \kappa\mu\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt \\ &= \int_0^\infty \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}} v^{\frac{v}{2}} \omega^{\frac{\mu-1}{2}}}{2^{\frac{v}{2}} \Gamma\left(\frac{v}{2}\right) \kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} t^{\frac{v}{2}-1+\frac{\mu}{2}-\frac{1}{2}} \\ &\quad \times \exp\left\{-\mu(1+\kappa)t\omega - \frac{vt}{2}\right\} I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt. \end{aligned}$$

*Let* $q = \sqrt{t}$ *then* $dt = 2qdq$. *The pdf can be written as*

$$\begin{aligned} f_\Omega(\omega) &= \int_0^\infty \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}} v^{\frac{v}{2}} \omega^{\frac{\mu-1}{2}}}{2^{\frac{v}{2}} \Gamma\left(\frac{v}{2}\right) \kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} q^{v+\mu-2} \\ &\quad \times \exp\left\{-q^2\left[\mu(1+\kappa)\omega - \frac{v}{2}\right]\right\} I_{\mu-1}\left(q2\mu\sqrt{\kappa(1+\kappa)\omega}\right) dq. \end{aligned}$$

*Using* (2.22), *we get*

$$\begin{aligned} f_\Omega(\omega) &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}} v^{\frac{v}{2}} \omega^{\frac{\mu-1}{2}}}{2^{\frac{v}{2}-1} \Gamma\left(\frac{v}{2}\right) \kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} 2^{-\mu+1-1} \left[2\mu\sqrt{\kappa(1+\kappa)\omega}\right]^{\mu-1} \\ &\quad \times \left[\mu(1+\kappa)\omega - \frac{v}{2}\right]^{-\frac{(v+\mu-1+\mu-1)}{2}} \Gamma\left[\begin{array}{c} \frac{(v+\mu-1+\mu-1)}{2} \\ \mu-1+1 \end{array}\right] \\ &\quad \times {}_1F_1\left(\frac{(v+\mu-1+\mu-1)}{2}; \mu-1+1; \frac{\left[2\mu\sqrt{\kappa(1+\kappa)\omega}\right]^2}{4\left[\mu(1+\kappa)\omega - \frac{v}{2}\right]}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu^{\mu}(1+\kappa)^{\mu} v^{\frac{v}{2}} \omega^{\mu-1}}{2^{\frac{v}{2}} \Gamma\left(\frac{v}{2}\right) \exp(\mu\kappa)} \left[\mu(1+\kappa)\omega - \frac{v}{2}\right]^{-\left(\frac{v}{2}+\mu-1\right)} \Gamma\left[\begin{array}{c} \frac{v}{2}+\mu-1 \\ \mu \end{array}\right] \\
&\times {}_1F_1\left(\frac{v}{2}+\mu-1; \mu; \frac{\mu^2\kappa(1+\kappa)\omega}{\left[\mu(1+\kappa)\omega - \frac{v}{2}\right]}\right), \ \omega > 0. \qquad (3.39)
\end{aligned}
$$

In Table 2 below, particular cases of the pdf's are focused on, since they form part of the investigation in section 3.4.

| Weight function $W(t)$ | normalized pdf | |
|---|---|---|
| normal weight (2.35) | Envelope | equation (3.36) |
| | Power | equation (3.37) |
| t-distribution weight (2.36) | Envelope | equation (3.38) |
| | Power | equation (3.39) |

Table 2: Particular cases of (3.29) and (3.35)

### 3.3.3 Characteristics

Subsequently the statistical characteristics for the $\kappa-\mu$ type fading model is derived.

#### 3.3.3.1 Cumulative distribution function

Consider the $\kappa-\mu$ type pdf for normalized power, $\Omega$ (3.29), and using (2.13) it follows that

$$
\begin{aligned}
f_{\Omega}(\omega) &= \int_0^{\infty} W(t) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)t\omega - \kappa\mu\} \sum_{l=0}^{\infty} \left(\frac{2\mu\sqrt{\kappa(1+\kappa)t\omega}}{2}\right)^{\mu-1} \\
&\times \frac{\left(\frac{1}{4}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right)^2\right)^l}{l!\Gamma(\mu+l)} dt.
\end{aligned}
$$

The cdf of $\Omega$, is given by

$$
\begin{aligned}
F(Z_T) &= \int_0^{Z_T} f_{\Omega}(\omega)\, d\omega \\
&= \int_0^{Z_T}\int_0^{\infty} W(t) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \exp\{-\mu(1+\kappa)t\omega - \kappa\mu\} \sum_{l=0}^{\infty} \left(\frac{2\mu\sqrt{\kappa(1+\kappa)t\omega}}{2}\right)^{\mu-1} \\
&\times \frac{\left(\frac{1}{4}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right)^2\right)^l}{l!\Gamma(\mu+l)} dt\, d\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{W(t)\,\mu(1+\kappa)^{\frac{\mu+1}{2}}(t)^{\frac{\mu-1}{2}}\left(\mu\sqrt{t\kappa(1+\kappa)}\right)^{\mu-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\omega^{\mu-1}\exp\{-\mu(1+\kappa)t\omega\}\left(\mu^2\kappa(1+\kappa)t\omega\right)^l dt\ d\omega \\
&= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{W(t)\,\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l(t)^{\mu+l-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,dt\ d\omega, \qquad (3.40)
\end{aligned}
$$

where $Z_T$ is the specified threshold (see 1.2).

Subsequently special cases (normal distribution and t-distribution) for the outage probability will be derived.

**Remark 3.10** *For the normal distribution the weight function is* (2.35). *The outage probability follows from* (3.40) *as*

$$
\begin{aligned}
F(Z_T) &= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{\delta(t-1)\,\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l(t)^{\mu+l-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,dt\ d\omega.
\end{aligned}
$$

*Consider the result,*

$$
\begin{aligned}
F(Z_T,t) &= \int_0^{\infty}\delta(t)\,F(Z_T,t+1)\,dt \\
&= F(Z_T,1).
\end{aligned}
$$

*Let $q=t-1$ then $t=q+1$, the outage probability is*

$$
\begin{aligned}
F(Z_T,t) &= F(Z_T,q+1) \\
&= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{\delta(q)\,\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l(q+1)^{\mu+l-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,dt\ d\omega \\
&= F(Z_T,1).
\end{aligned}
$$

*thus,*

$$
\begin{aligned}
F(Z_T) &= \int_0^{Z_T}\sum_{l=0}^{\infty}\frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^l(1+\kappa)^l}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,d\omega.
\end{aligned}
$$

*Using* (2.20) *the outage is*

$$
\begin{aligned}
F(Z_T) &= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^{l}(1+\kappa)^{l}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times[\mu(1+\kappa)]^{-(\mu+l-1)-1}\gamma(\mu+l-1+1,\mu(1+\kappa)Z_T) \\
&= \sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^{l}(1+\kappa)^{l}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times[\mu(1+\kappa)]^{-\mu-l}\gamma(\mu+l,\mu(1+\kappa)Z_T) \\
&= \sum_{l=0}^{\infty} \frac{(\mu\kappa)^{l}\gamma(\mu+l,\mu(1+\kappa)Z_T)}{l!\Gamma(\mu+l)\exp(\mu\kappa)} \qquad (3.41) \\
&= (3.17)\,.
\end{aligned}
$$

**Remark 3.11** *For the t-distribution the weight function is* (2.36). *The outage probability follows from* (3.40) *as*

$$
\begin{aligned}
F(Z_T) &= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^{l}(1+\kappa)^{l}(t)^{\mu+l-1}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times\frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left\{-\frac{vt}{2}\right\}}{2\Gamma\left(\frac{v}{2}\right)}\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,dt\ d\omega \\
&= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty} \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}\left(\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1}\mu^{2l}\kappa^{l}(1+\kappa)^{l}v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}}{2\Gamma\left(\frac{v}{2}\right)\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\,l!\Gamma(\mu+l)} \\
&\quad \times(t)^{\mu+l+\frac{v}{2}-2}\exp\left\{-\frac{vt}{2}\right\}\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)t\omega\}\,dt\ d\omega \\
&= \sum_{l=0}^{\infty}\frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l}v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}}{2\Gamma\left(\frac{v}{2}\right)\exp(\mu\kappa)\,l!\Gamma(\mu+l)}\int_0^{Z_T}\omega^{\mu+l-1} \\
&\quad \times\int_0^{\infty}(t)^{\mu+l+\frac{v}{2}-2}\exp\left\{-t\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]\right\}dt\ d\omega.
\end{aligned}
$$

*Using* (2.19),

$$
\begin{aligned}
F(Z_T) &= \sum_{l=0}^{\infty}\frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l}v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}}{2\Gamma\left(\frac{v}{2}\right)\exp(\mu\kappa)\,l!\Gamma(\mu+l)}\int_0^{Z_T}\omega^{\mu+l-1} \\
&\quad \times\frac{\Gamma\left(\mu+l+\frac{v}{2}-1\right)}{\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]^{\mu+l+\frac{v}{2}-1}}
\end{aligned}
$$

$$= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{\frac{v}{2}} \Gamma\left(\mu+l+\frac{v}{2}-1\right)}{2^{\frac{v}{2}} l! \Gamma(\mu+l) \Gamma\left(\frac{v}{2}\right) \exp(\mu\kappa)}$$

$$\times \int_{0}^{Z_T} \frac{\omega^{\mu+l-1}}{\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]^{\mu+l+\frac{v}{2}-1}} d\omega$$

$$= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{\frac{v}{2}}}{2^{\frac{v}{2}} l! B\left(\frac{v}{2},\mu+l\right)\left(\mu+l+\frac{v}{2}-1\right) \exp(\mu\kappa)}$$

$$\times \int_{0}^{Z_T} \frac{\omega^{\mu+l-1}}{\left[\frac{v}{2}\left\{1+\frac{2}{v}\mu(1+\kappa)\omega\right\}\right]^{\mu+l+\frac{v}{2}-1}} d\omega$$

$$= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{\frac{v}{2}}}{2^{\frac{v}{2}} l! B\left(\frac{v}{2},\mu+l\right)\left(\mu+l+\frac{v}{2}-1\right) \exp(\mu\kappa)}$$

$$\times \left(\frac{v}{2}\right)^{-\left(\mu+l+\frac{v}{2}-1\right)} \int_{0}^{Z_T} \frac{\omega^{\mu+l-1}}{\left[1+\frac{2}{v}\mu(1+\kappa)\omega\right]^{\mu+l+\frac{v}{2}-1}} d\omega.$$

*Using* (2.24) *and* (2.12), *follows that*

$$\begin{aligned} F(Z_T) &= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{\frac{v}{2}} \left(\frac{v}{2}\right)^{-\left(\mu+l+\frac{v}{2}-1\right)}}{2^{\frac{v}{2}} l! B\left(\frac{v}{2},\mu+l\right)\left(\mu+l+\frac{v}{2}-1\right) \exp(\mu\kappa)} \\ &\quad \times \frac{(Z_T)^{\mu+l}}{(\mu+l)} \, {}_2F_1\left(\mu+l+\frac{v}{2}-1, \mu+l; \mu+l+1; -\mu\left(\frac{2}{v}\right)(1+\kappa) Z_T\right) \\ &= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{-\mu-l+1} (Z_T)^{\mu+l}}{2^{-(\mu+l-1)} l! B\left(\frac{v}{2},\mu+l\right)\left(\mu+l+\frac{v}{2}-1\right)(\mu+l) \exp(\mu\kappa)} \\ &\quad \times {}_2F_1\left(\mu+l+\frac{v}{2}-1, \mu+l; \mu+l+1; -\frac{2\mu(1+\kappa) Z_T}{v}\right) \\ &= \sum_{l=0}^{\infty} \frac{\mu^{\mu+2l}\kappa^{l}(1+\kappa)^{\mu+l} v^{-\mu-l+1} (Z_T)^{\mu+l}}{2^{-(\mu+l-1)} l! B\left(\frac{v}{2},\mu+l\right)\left(\mu+l+\frac{v}{2}-1\right)(\mu+l) \exp(\mu\kappa)} \\ &\quad \times \left(1-\frac{2\mu(1+\kappa) Z_T}{v}\right)^{-\left(\mu+l+\frac{v}{2}-1\right)} \\ &\quad \times {}_2F_1\left(\mu+l+\frac{v}{2}-1, 1; \mu+l+1; -\frac{2\mu(1+\kappa) Z_T}{2\mu(1+\kappa) Z_T+v}\right), \end{aligned} \tag{3.42}$$

*where* $(\mu+l)>0$ *and* ${}_2F_1(\cdot)$ *is the Gaussian hypergeometric function* (2.11).

#### 3.3.3.2 Moments

From (3.35) the $j^{th}$ moment of the normalized envelope, $P$, is

$$\begin{aligned}
E\left(P^j\right) &= \int_0^\infty \rho^j f_P(\rho)\, d\rho \\
&= \int_0^\infty \int_0^\infty \rho^j W(t) \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\rho)^\mu \exp\left\{-\mu\kappa - \mu(1+\kappa)(t\rho)^2\right\} \\
&\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}t\rho\right) dt\, d\rho \\
&= \int_0^\infty \int_0^\infty W(t) \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}} t^\mu}{\kappa^{\frac{\mu-1}{2}}} \rho^{\mu+j} \exp\left\{-\mu\kappa - \mu(1+\kappa)t^2\rho^2\right\} \\
&\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)}t\rho\right) dt\, d\rho.
\end{aligned}$$

Using (2.22) the $j^{th}$ moment is

$$\begin{aligned}
E\left(P^j\right) &= \int_0^\infty W(t) \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}} t^\mu}{\kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} 2^{-\mu+1-1} \left(2t\mu\sqrt{\kappa(1+\kappa)}\right)^{\mu-1} \left(\mu(1+\kappa)t^2\right)^{-\frac{(\mu+j+1+\mu-1)}{2}} \\
&\quad \times \Gamma\left[\frac{\frac{\mu+j+1+\mu-1}{2}}{\mu-1+1}\right] {}_1F_1\left(\frac{\mu+j+1+\mu-1}{2}; \mu-1+1; \frac{\left(2t\mu\sqrt{\kappa(1+\kappa)}\right)^2}{4\mu(1+\kappa)t^2}\right) dt \\
&= \int_0^\infty W(t) \frac{\Gamma\left(\mu+\frac{j}{2}\right)}{t^{j+1}\left[\mu(1+\kappa)\right]^{\frac{j}{2}} \Gamma(\mu) \exp(\mu\kappa)} {}_1F_1\left(\mu+\frac{j}{2}; \mu; \mu\kappa\right) dt. \qquad (3.43)
\end{aligned}$$

From (3.29) the $j^{th}$ moment of the normalized power, $\Omega$, is

$$\begin{aligned}
E\left(\Omega^j\right) &= \int_0^\infty \omega^j f_\Omega(\omega)\, d\omega \\
&= \int_0^\infty \int_0^\infty \omega^j W(t) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (t\omega)^{\frac{\mu-1}{2}} \exp\left\{-\mu(1+\kappa)t\omega - \kappa\mu\right\} \\
&\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt\, d\omega \\
&= \int_0^\infty \int_0^\infty W(t) \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}} (t)^{\frac{\mu-1}{2}}}{\kappa^{\frac{\mu-1}{2}}} (\omega)^{\frac{\mu-1}{2}+j} \exp\left\{-\mu(1+\kappa)t\omega - \kappa\mu\right\} \\
&\quad \times I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)t\omega}\right) dt\, d\omega.
\end{aligned}$$

Let $\alpha=\sqrt{\omega}$, then

$$\begin{aligned} E\left(\Omega^{j}\right) &= \int_{0}^{\infty}\int_{0}^{\infty} W\left(t\right)\frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}\left(t\right)^{\frac{\mu-1}{2}}}{\kappa^{\frac{\mu-1}{2}}}\left(\alpha\right)^{\mu+1+2j-1}\exp\left\{-\mu\left(1+\kappa\right)t\alpha^{2}-\kappa\mu\right\} \\ &\quad\times I_{\mu-1}\left(2\mu\sqrt{\kappa\left(1+\kappa\right)t}\alpha\right)d\alpha\ dt. \end{aligned}$$

Using (2.22), we obtain the $j^{th}$ moment as

$$\begin{aligned} E\left(\Omega^{j}\right) &= \int_{0}^{\infty} W\left(t\right)\frac{2\mu\left(1+\kappa\right)^{\frac{\mu+1}{2}}\left(t\right)^{\frac{\mu-1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp\left(\mu\kappa\right)}2^{-\mu+1-1}\left(2\mu\sqrt{t\kappa\left(1+\kappa\right)}\right)^{\mu-1}\left(\mu\left(1+\kappa\right)t\right)^{-\frac{\left(\mu+1+2j+\mu-1\right)}{2}} \\ &\quad\times\Gamma\left[\frac{\frac{\mu+1+2j+\mu-1}{2}}{\mu-1+1}\right]{}_{1}F_{1}\left(\frac{\mu+1+2j+\mu-1}{2};\mu-1+1;\frac{\left(2\mu\sqrt{t\kappa\left(1+\kappa\right)}\right)^{2}}{4\mu\left(1+\kappa\right)t}\right)dt \end{aligned}$$

$$E\left(\Omega^{j}\right)=\int_{0}^{\infty} W\left(t\right)\frac{\Gamma\left(\mu+j\right)}{t^{j+1}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^{j}}\,{}_{1}F_{1}\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)dt. \tag{3.44}$$

**Remark 3.12** *Substituting the weight functions* (2.35) *and* (2.36) *into* (3.43) *and* (3.44) *respectively. The following expressions are derived:*

*(i) For the normal distribution, the $j^{th}$ moment of the normalized envelope, $P$, is*

$$E\left(P^{j}\right)=\int_{0}^{\infty}\delta\left(t-1\right)\frac{\Gamma\left(\mu+\frac{j}{2}\right)}{t^{j+1}\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}\,{}_{1}F_{1}\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)dt.$$

*Consider the result,*

$$\begin{aligned} F\left(Z_{T},t\right) &= \int_{0}^{\infty}\delta\left(t\right)F\left(Z_{T},t+1\right)dt \\ &= F\left(Z_{T},1\right). \end{aligned} \tag{3.45}$$

*Let $q=t-1$ then $t=q+1$, the $j^{th}$ moment is*

$$\begin{aligned} E\left(P^{j}\right) &= \int_{0}^{\infty}\delta\left(q\right)\frac{\Gamma\left(\mu+\frac{j}{2}\right)}{\left(q+1\right)^{j+1}\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}\,{}_{1}F_{1}\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)dq \\ &= \frac{\Gamma\left(\mu+\frac{j}{2}\right)}{\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}\,{}_{1}F_{1}\left(\mu+\frac{j}{2};\mu;\mu\kappa\right). \\ &\equiv (3.18) \end{aligned}$$

*Using* (3.44) *, the* $j^{th}$ *moment of the normalized power,* $\Omega$*, is*

$$
\begin{aligned}
E\left(\Omega^{j}\right) &= \int_{0}^{\infty}\delta\left(t-1\right)\frac{\Gamma\left(\mu+j\right)}{t^{j+1}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^{j}}\ {}_1F_1\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)dt \\
&= \frac{\Gamma\left(\mu+j\right)}{\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^{j}}\ {}_1F_1\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right). \\
&\equiv (3.19)
\end{aligned}
$$

*(ii) For the t-distribution, the* $j^{th}$ *moment of the normalized envelope,* $P$*, is*

$$
\begin{aligned}
E\left(P^{j}\right) &= \int_{0}^{\infty}\frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)}\frac{\Gamma\left(\mu+\frac{j}{2}\right)}{t^{j+1}\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)} \\
&\quad\times {}_1F_1\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)dt \\
&= \frac{v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}\Gamma\left(\mu+\frac{j}{2}\right)\ {}_1F_1\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)}{2\Gamma\left(\frac{v}{2}\right)\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)} \\
&\quad\times\int_{0}^{\infty}t^{\frac{v}{2}-j-2}\exp\left(-\frac{vt}{2}\right)dt.
\end{aligned}
$$

*Using* (2.19) *, the* $j^{th}$ *moment simplifies to*

$$
\begin{aligned}
E\left(P^{j}\right) &= \frac{v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}\Gamma\left(\mu+\frac{j}{2}\right)\ {}_1F_1\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)}{2\Gamma\left(\frac{v}{2}\right)\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}\frac{\Gamma\left(\frac{v}{2}-j-1\right)}{\left(\frac{v}{2}\right)^{\frac{v}{2}-j-1}} \\
&= \frac{2^{-j-1}v^{j+1}\Gamma\left(\mu+\frac{j}{2}\right)\Gamma\left(\frac{v}{2}-j-1\right)\ {}_1F_1\left(\mu+\frac{j}{2};\mu;\mu\kappa\right)}{\Gamma\left(\frac{v}{2}\right)\left[\mu\left(1+\kappa\right)\right]^{\frac{j}{2}}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)}. \qquad (3.46)
\end{aligned}
$$

*The* $j^{th}$ *moment of the normalized power,* $\Omega$*, is*

$$
\begin{aligned}
E\left(\Omega^{j}\right) &= \int_{0}^{\infty}\frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)}\frac{\Gamma\left(\mu+j\right)}{t^{j+1}\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^{j}} \\
&\quad\times {}_1F_1\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)dt \\
&= \frac{v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}\Gamma\left(\mu+j\right)\ {}_1F_1\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)}{2\Gamma\left(\frac{v}{2}\right)\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}\left[\mu\left(1+\kappa\right)\right]^{j}} \\
&\quad\times\int_{0}^{\infty}t^{\frac{v}{2}-j-2}\exp\left(-\frac{vt}{2}\right)dt.
\end{aligned}
$$

*Using* (2.19) , *the* $j^{th}$ *moment simplifies to*

$$\begin{aligned} E\left(\Omega^{j}\right) &= \frac{v\left(\frac{v}{2}\right)^{\frac{v}{2}-1}\Gamma\left(\mu+j\right)\ {}_{1}F_{1}\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)}{2\Gamma\left(\frac{v}{2}\right)\Gamma\left(\mu\right)\exp\left\{\mu\kappa\right\}\left[\mu\left(1+\kappa\right)\right]^{j}}\frac{\Gamma\left(\frac{v}{2}-j-1\right)}{\left(\frac{v}{2}\right)^{\frac{v}{2}-j-1}} \\ &= \frac{2^{j+1}v^{j+1}\Gamma\left(\mu+j\right)\Gamma\left(\frac{v}{2}-j-1\right)\ {}_{1}F_{1}\left(\mu+j\ ;\ \mu\ ;\mu\kappa\right)}{\Gamma\left(\frac{v}{2}\right)\Gamma\left(\mu\right)\exp\left(\mu\kappa\right)\left[\mu\left(1+\kappa\right)\right]^{j}}. \end{aligned}$$

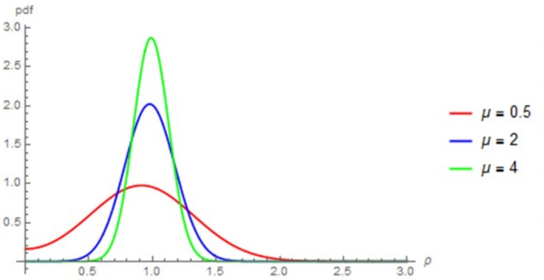
#### 3.3.3.3 Amount of fading

The amount of fading for the $\kappa-\mu$ type model for the weight function (2.36) is given as (see (1.1))

$$\begin{aligned} AF &= \frac{E\left(\Omega^{2}\right)}{E\left(\Omega\right)^{2}}-1 \\ &= \frac{\frac{2^{3}v^{3}\Gamma(\mu+2)\Gamma\left(\frac{v}{2}-3\right)\ {}_{1}F_{1}(\mu+2\ ;\ \mu\ ;\mu\kappa)}{\Gamma\left(\frac{v}{2}\right)\Gamma(\mu)\exp(\mu\kappa)[\mu(1+\kappa)]^{2}}}{\left[\frac{2^{2}v^{2}\Gamma(\mu+1)\Gamma\left(\frac{v}{2}-2\right)\ {}_{1}F_{1}(\mu+1\ ;\ \mu\ ;\mu\kappa)}{\Gamma\left(\frac{v}{2}\right)\Gamma(\mu)\exp(\mu\kappa)[\mu(1+\kappa)]}\right]^{2}}-1 \\ &= \frac{\left(\mu+1\right)\left(\frac{v}{2}-2\right)\exp\left(\mu\kappa\right)\ {}_{1}F_{1}\left(\mu+2\ ;\ \mu\ ;\mu\kappa\right)}{2v\mu\left(\frac{v}{2}-1\right)\left(\frac{v}{2}\right)\left[\ {}_{1}F_{1}\left(\mu+1\ ;\ \mu\ ;\mu\kappa\right)\right]^{2}}-1. \end{aligned} \quad (3.47)$$

## 3.4 Performance measures

In this chapter the $\kappa-\mu$ model is revisited and the $\kappa-\mu$ *type* model is derived. This section has graphical displays of the $\kappa-\mu$ type model and corresponding performance metrics. The plots of the $\kappa-\mu$ type model for the envelope (3.35) and power (3.29) will be illustrated for the weight functions (2.35) and (2.36) respectively. Similarly the outage probability as well as the amount of fading are plotted.



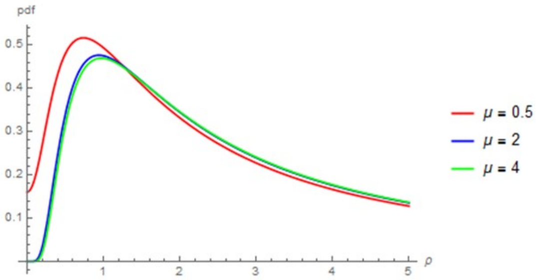




Figure 3.4.1 shows the pdf of the envelope if $\kappa=5$ for the normal case (left) and t-distribution, where $v=3$ degrees of freedom was considered, (right).

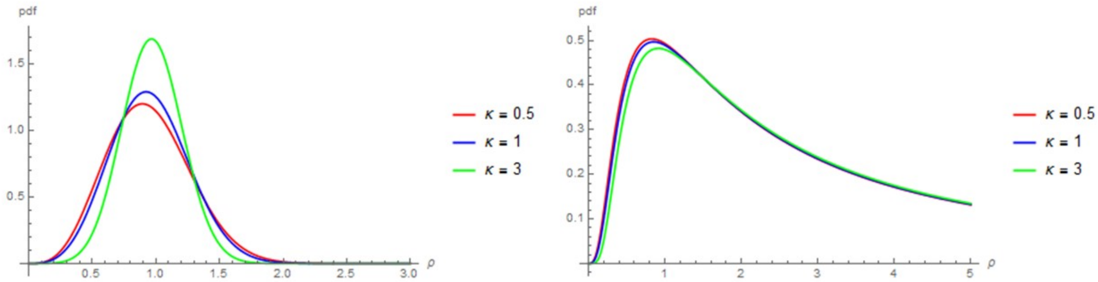



Figure 3.4.2 shows the pdf of the envelope if $\mu = 2$ for the normal case (left) and t-distribution, where $v = 3$ degrees of freedom was considered, (right) .

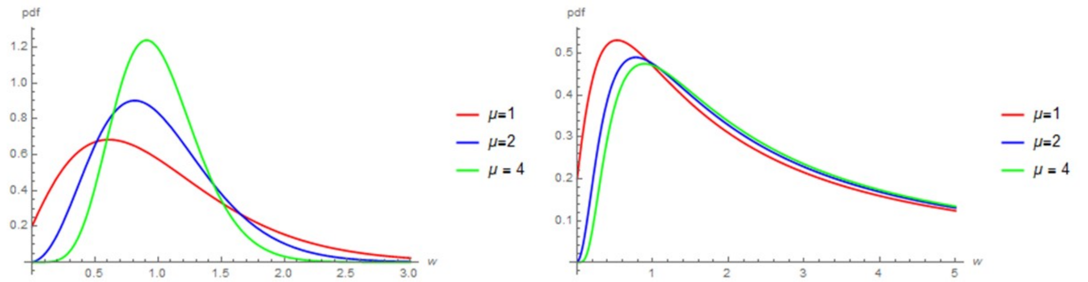



Figure 3.4.3 shows the pdf of the power if $\kappa = 3$ for the normal case (left) and t-distribution, where $v = 3$ degrees of freedom was considered, (right) .

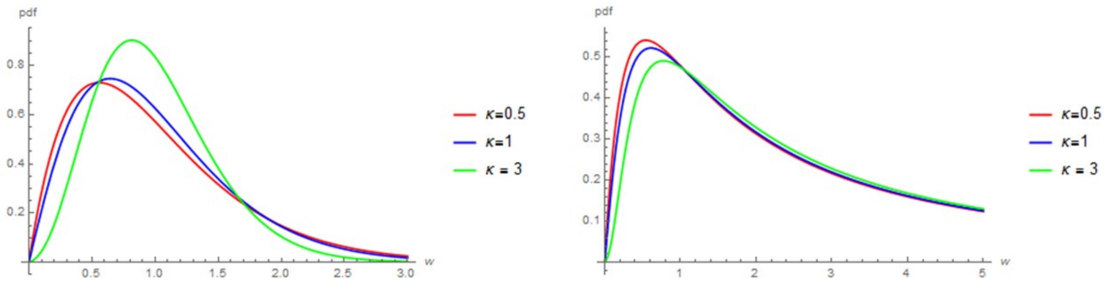



Figure 3.4.4 shows the pdf of the power if $\mu = 2$ for the normal case (left) and t-distribution, where $v = 3$ degrees of freedom was considered, (right) .

The outage probability (3.40) is plotted for the weight function (2.35) and (2.36) in Figure 3.4.5 for $\kappa = 5$ and in Figure 3.4.6 for $\mu = 2$. (Note that for the t-distribution, $v = 3$ degrees of freedom was considered.). The assumption of the t-distribution as the underlying model is more appropriate for larger specified threshold values since the t-distribution outperrforms the normal distribution only after a certain threshold.

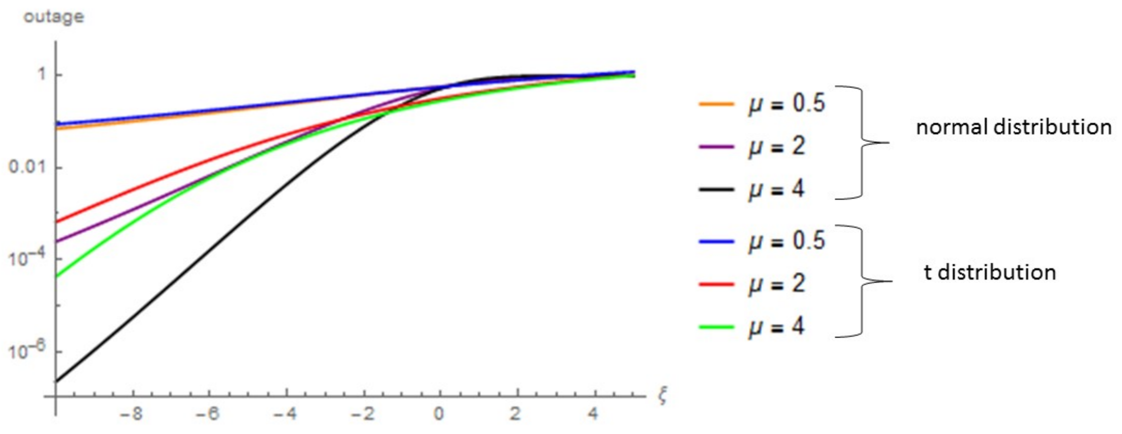



Figure 3.4.5

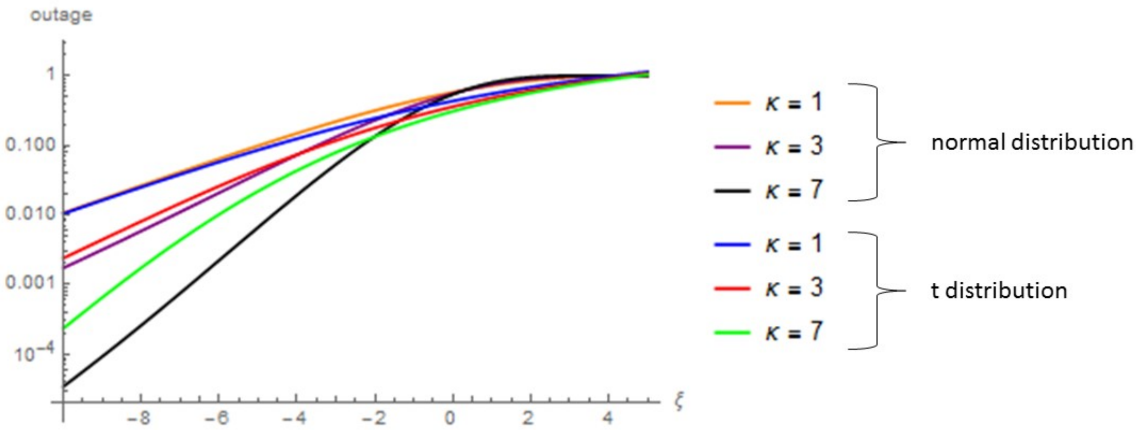



Figure 3.4.6

The amount of fading in the $\kappa - \mu$ channel is shown in Figure 3.4.7 for the normal ((3.20), on left) and t-distribution, where $v = 3$ degrees of freedom was considered, ((3.47), on right) cases respectively. The amount of fading for the t-distribution is lower than the normal distribution which indicates that t-distribution more suitable.

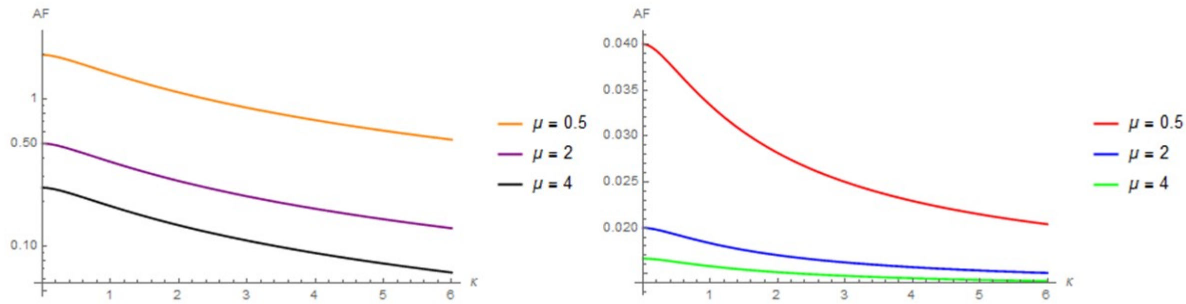



Figure 3.4.7

# Chapter 4

# Bivariate Models

## 4.1 Introduction

Villavicencio, et al.,(2016) [33] presented the bivariate $\kappa - \mu$ fading model used to assess the performance of a dual-branch selection combining (SC) scheme by using the framework developed by [19] for a general multidimensional normal distribution. According to Miller (1964) [19] each random process is considered to be normally distributed in $n = \mu$ (integer) dimensions. The approach of Miller (1964) [19] is followed by Villavicencio, et al.,(2016) [33] and generalized to accommodate a real extension of $\mu$ and outline in Section 4.2. In Section 4.2 the statistical properties of the bivariate $\kappa - \mu$ model will be given. The bivariate $\kappa - \mu$ type model that emanates from the elliptical assumption will then be derived in Section 4.3; followed by Section 4.5 where the performance measures under the different members of the elliptical model will be illustrated.

## 4.2 The bivariate $\kappa - \mu$ distribution

The aim of this section is to study a bivariate $\kappa - \mu$ distribution with as the underlying model, the normal distribution. Exact formulae for the jpdf, jcdf and joint arbitrary moments will be given. The outage probability is the primary metric for the analysis of diversity schemes in wireless communications systems and will also receive attention.

### 4.2.1 Description

Let $R_1$ and $R_2$ be two $\kappa - \mu$ envelopes then,

$$R_1^2 = \sum_{i=1}^{2\mu} X_i^2$$

$$R_2^2 = \sum_{i=1}^{2\mu} Y_i^2,$$

where $X_i$ and $Y_i$ are mutually independent normal processes with mean equal to $a$ and variances $\sigma_1^2$ and $\sigma_2^2$ respectively. [Note: $2\mu$ = number of clusters of multipath].

If $\{X, Y, A\}$ be $2\mu$- dimensional vectors where $X = [X_1 \quad X_2 \quad ... \quad X_{2\mu}]^T$, $Y = [Y_1 \quad Y_2 \quad ... \quad Y_{2\mu} \quad ]^T$ and $A = [a \quad a \quad ... \quad a \quad ]^T$. Let $V_i = [X_i \quad Y_i]^T$ where $X_i$ and $Y_i$ are correlated if and only if $i = j$, $i = 1, 2, ..., 2\mu$. Note that $A$ is the vector of the dominant component of the cluster. The interest is the jpdf of the envelope $R_1$ and $R_2$, as well as, the normalized envelopes, $P_1$ and $P_2$ with $P_i = \frac{R_i}{\sqrt{\hat{r}_i}}$ and $\hat{r}_i$ as defined in (3.13).

### 4.2.2 Derivation

The covariance matrix, $\Sigma > 0$, is then

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \delta\sigma_1\sigma_2 \\ \delta\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where $\delta$ is the correlation coefficient of the normal component,

$$\delta = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} = \frac{\Sigma_{12}}{\sqrt{\sigma_1^2\sigma_2^2}}$$
$$\therefore \quad \Sigma_{12} = \delta\sqrt{\sigma_1^2\sigma_2^2} = \delta\sigma_1\sigma_2.$$

The inverse of the covariance matrix is:

$$\begin{aligned} \Sigma^{-1} &= \begin{bmatrix} \sigma_1^2 & \delta\sigma_1\sigma_2 \\ \delta\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}^{-1} \\ &= \frac{1}{\sigma_1^2\sigma_2^2 - \delta^2\sigma_1^2\sigma_2^2} \begin{bmatrix} \sigma_2^2 & -\delta\sigma_1\sigma_2 \\ -\delta\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{\sigma_1^2\sigma_2^2\left(1-\delta^2\right)} \begin{bmatrix} \sigma_2^2 & -\delta\sigma_1\sigma_2 \\ -\delta\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{\left(1-\delta^2\right)} \begin{bmatrix} \sigma_1^{-2} & -\delta\sigma_1^{-1}\sigma_2^{-1} \\ -\delta\sigma_1^{-1}\sigma_2^{-1} & \sigma_2^{-2} \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{11}^{-1} & \Sigma_{21}^{-1} \\ \Sigma_{21}^{-1} & \Sigma_{22}^{-1} \end{bmatrix}. \end{aligned} \tag{4.1}$$

The pdf of the multivariate normal distribution is given in (2.27). Thus, the jpdf of $X$ and $Y$ is as follows

$$\begin{aligned} f_{X,Y}(X,Y) &= \prod_{i=1}^{2\mu} \frac{1}{(2\pi)\det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left[(X_i - a)(Y_i - a)\right]^T \Sigma^{-1} \begin{bmatrix} (X_i - a) \\ (Y_i - a) \end{bmatrix}\right) \\ &= \prod_{i=1}^{2\mu} \frac{1}{(2\pi)\det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left[V_i - C_i\right]^T \Sigma^{-1}\left[V_i - C_i\right]\right) \\ &= \prod_{i=1}^{2\mu} f_i(V_i) \end{aligned}$$

$$f_{X,Y}(X,Y)=\frac{1}{(2\pi)^{2\mu}\det(\Sigma)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\sum_{i=1}^{2\mu}[V_i-C_i]^T\Sigma^{-1}[V_i-C_i]\right), \tag{4.2}$$

where $V_i=[X_i\ \ Y_i]^T$ and $C_i=[a\ a]^T$.

Expanding the quadratic form and replacing the $V_i$ by $X_i$ and $Y_i$ it follows that

$$\begin{aligned}&\sum_{i=1}^{2\mu}[V_i-C_i]^T\Sigma^{-1}[V_i-C_i] \\ =&\sum_{i=1}^{2\mu}[(X_i-a)\,(Y_i-a)]\,\Sigma^{-1}\left[\begin{array}{c}(X_i-a)\\(Y_i-a)\end{array}\right].\end{aligned} \tag{4.3}$$

Then from above and (4.1) it follows that

$$\begin{aligned}&\sum_{i=1}^{2\mu}[(X_i-a)\,(Y_i-a)]\left[\begin{array}{cc}\Sigma_{11}^{-1}&\Sigma_{12}^{-1}\\\Sigma_{21}^{-1}&\Sigma_{22}^{-1}\end{array}\right]\left[\begin{array}{c}(X_i-a)\\(Y_i-a)\end{array}\right]\\
=&\sum_{i=1}^{2\mu}\left[(X_i-a)\Sigma_{11}^{-1}+(Y_i-a)\Sigma_{21}^{-1}\right](X_i-a)+\left[(X_i-a)\Sigma_{21}^{-1}+(Y_i-a)\Sigma_{22}^{-1}\right](Y_i-a)\\
=&\sum_{i=1}^{2\mu}(X_i-a)\Sigma_{11}^{-1}(X_i-a)+(Y_i-a)\Sigma_{21}^{-1}(X_i-a)+(X_i-a)\Sigma_{21}^{-1}(Y_i-a)\\
&+(Y_i-a)\Sigma_{22}^{-1}(Y_i-a)\\
=&\sum_{i=1}^{2\mu}X_i\Sigma_{11}^{-1}X_i-X_i\Sigma_{11}^{-1}a-a\Sigma_{11}^{-1}X_i+a\Sigma_{11}^{-1}a+Y_i\Sigma_{21}^{-1}X_i-Y_i\Sigma_{21}^{-1}a-a\Sigma_{21}^{-1}X_i+\\
&a\Sigma_{21}^{-1}a+X_i\Sigma_{21}^{-1}Y_i-X_i\Sigma_{21}^{-1}a-a\Sigma_{21}^{-1}X_i+a\Sigma_{21}^{-1}a+Y_i\Sigma_{22}^{-1}Y_i-Y_i\Sigma_{22}^{-1}a-a\Sigma_{22}^{-1}Y_i+a\Sigma_{22}^{-1}a\\
=&\sum_{i=1}^{2\mu}a\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)a-a\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X_i-X_i\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)a-\\
&a\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)Y_i-Y_i\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)a+X_i\Sigma_{11}^{-1}X_i+X_i\Sigma_{21}^{-1}Y_i+Y_i\Sigma_{21}^{-1}X_i+Y_i\Sigma_{22}^{-1}Y_i.\end{aligned}$$

Since the inverse of the covariance matrix are constant, the above expression simplifies to

$$\begin{aligned}&\sum_{i=1}^{2\mu}a\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)a-a\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X_i-X_i\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)a-\\
&a\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)Y_i-Y_i\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)a+X_i\Sigma_{11}^{-1}X_i+X_i\Sigma_{21}^{-1}Y_i+Y_i\Sigma_{21}^{-1}X_i+Y_i\Sigma_{22}^{-1}Y_i\end{aligned}$$

$$
\begin{aligned}
&= \left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2+\\
&\quad \sum_{i=1}^{2\mu}[X_i\Sigma_{21}^{-1}Y_i+Y_i\Sigma_{21}^{-1}X_i-a\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X_i-X_i\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)a\\
&\quad -a\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)Y_i-Y_i\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)a]\\
&= \left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2+2\Sigma_{21}^{-1}XY-2\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)AX\\
&\quad -2\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)AY \qquad (4.4)
\end{aligned}
$$

with $|A| = a = \sqrt{a_1^2+a_2^2+...+a_{2\mu}^2}$. Substituting (4.4) into (4.2), it follows that

$$
\begin{aligned}
&\frac{1}{(2\pi)^{2\mu}\det(\Sigma)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\sum_{i=1}^{2\mu}[V_i-C_i]^T\Sigma^{-1}[V_i-C_i]\right)\\
&= \frac{1}{(2\pi)^{2\mu}\det(\Sigma)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2+2\Sigma_{21}^{-1}XY\right]\right)\\
&\quad \exp\left(-\frac{1}{2}\left[-2\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)AX-2\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)AY\right]\right)\\
&= \frac{1}{(2\pi)^{2\mu}|\det(\Sigma)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\quad \exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)AX+\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)AY-\Sigma_{21}^{-1}XY\right]\\
&= \frac{1}{(2\pi)^{2\mu}\det(\Sigma)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\quad \exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)AX+Y\left(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}X\right)\right].
\end{aligned}
$$

Hence, the jpdf can be written as follows

$$
\begin{aligned}
f_{X,Y}(X,Y) &= \frac{1}{(2\pi)^{2\mu}\det(\Sigma)^{\mu}}\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\quad \times\exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)AX+Y\left(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}X\right)\right].
\end{aligned}
$$

In order to obtain the jpdf of the envelopes, an integration with respect to $X$ and $Y$ subject to the constraint $|X| = R_1$, and $|Y| = R_2$ is required (see Miller (1964) [19, page 33]). That is

$$
\begin{aligned}
f_{X,Y}(X,Y) &= \frac{\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}|X|^2+\Sigma_{22}^{-1}|Y|^2\right]\right)}{(2\pi)^{2\mu}\det(\Sigma)^{\mu}}\\
&\quad \times\int_{|X|=R_1}\exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X^TA\right]ds_1\\
&\quad \times\int_{|Y|=R_2}\exp\left[Y^T\left(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}X\right)\right]ds_2, \qquad (4.5)
\end{aligned}
$$

where $ds_1$ and $ds_2$ are the elements of the integration surface area (see theorem 2.1). The second integral in the (4.5) can be expressed as

$$\begin{aligned} & \int_{|Y|} \exp\left[Y^T\left(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}X\right)\right]ds_2 \\ = & (2\pi R_2)^{\frac{2\mu}{2}}\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|^{1-\mu} \\ & \times I_{\frac{1}{2}(2\mu-2)}(R_2\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|), \end{aligned} \tag{4.6}$$

Using (2.16), the expression (4.6) can be written as

$$\begin{aligned} & (2\pi R_2)^{\frac{2\mu}{2}}\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|^{1-\mu} I_{\frac{1}{2}(2\mu-2)}(R_2\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|) \\ = & \frac{(2\pi R_2)^{\mu}}{(R_2)^{1-\mu}}(R_2\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|)^{1-\mu} I_{\frac{1}{2}(2\mu-2)}(R_2\left|\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)A-\Sigma_{21}^{-1}R_1\right|) \\ = & \frac{(2\pi R_2)^{\mu}}{(R_2)^{1-\mu}}2^{\mu-1}\Gamma(\mu-1)\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\frac{I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|\,R_2)}{(R_2)^{\mu-1}\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)^{\mu-1}|A|^{\mu-1}} \\ & \times\frac{I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2)}{\left(\Sigma_{21}^{-1}\right)^{\mu-1}(R_1R_2)^{\mu-1}}C_k^{\mu-1}(\cos\phi_1) \\ = & \frac{2^{2\mu-1}(\pi R_2)^{\mu}}{\left[\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)\left(\Sigma_{21}^{-1}\right)R_2\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}}\Gamma(\mu-1)\sum_{k=0}^{\infty}(-1)^k(\mu-1+k) \\ & \times I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|\,R_2)I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2)C_k^{\mu-1}(\cos\phi_1), \end{aligned}$$

where $\phi_1$ is the angle between $X$ and $A$.

Thus, the jpdf of the envelopes is given as

$$\begin{aligned} & \frac{2^{2\mu-1}(\pi R_2)^{\mu}\Gamma(\mu-1)\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{(2\pi)^{2\mu}\det(\Sigma)^{\mu}\left[\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)\left(\Sigma_{21}^{-1}\right)R_2\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}} \\ & \times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|\,R_2)I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2) \\ & \times\int_{|X|}\exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X^TA\right]C_k^{\mu-1}(\cos\phi_1)ds_1. \end{aligned} \tag{4.7}$$

To evaluate the remaining integral in equation (4.7), express it in generalized spherical coordinates

$$\begin{aligned} & \int_{|X|}\exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X^TA\right]C_k^{\mu-1}(\cos\phi_1)ds_1 \\ = & \int_{|X|}\exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1A\cos\phi_1\right]C_k^{\mu-1}(\cos\phi_1)ds_1 \end{aligned}$$

$$= R_1^{2\mu-1}\int_0^{2\pi} d\theta \int_0^{\pi} \exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A\cos\phi_1\right] C_k^{\mu-1}(\cos\phi_1)\sin^{2\mu-2} d\phi_1$$
$$\times \prod_{k=2}^{2\mu-2}\int_0^{\pi}\sin^{2\mu-1-k}\phi_k d\phi_k. \tag{4.8}$$

Applying (2.17) then it follows that

$$\int_0^{\pi} \exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A\cos\phi_1\right] C_k^{\mu-1}(\cos\phi_1)\sin^{2\mu-2} d\phi_1$$
$$= \frac{2^{\mu-1}\Gamma(\mu-1+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2(\mu-1)+k)}{(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)^{\mu-1}k!\Gamma(2(\mu-1))}$$
$$\times I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A), \tag{4.9}$$

and using (2.18),

$$R_1^{2\mu-1}\int_0^{2\pi} d\theta \prod_{k=2}^{2\mu-2}\int_0^{\pi}\sin^{2\mu-1-k}\phi_k d\phi_k$$
$$= \frac{2\pi^{\frac{1}{2}(2\mu-1)}R_1^{2\mu-1}}{\Gamma(\frac{1}{2}(2\mu-1))}. \tag{4.10}$$

Therefore from (4.8) , (4.9) and (4.10) follows that

$$\int_{|X|} \exp\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)X^T A\right] C_k^{\mu-1}(\cos\phi_1) ds_1$$
$$= \frac{2\pi^{\frac{1}{2}(2\mu-1)}R_1^{2\mu-1}}{\Gamma(\frac{1}{2}(2\mu-1))}\frac{2^{\mu-1}\Gamma(\mu-1+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2(\mu-1)+k)}{(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)^{\mu-1}k!\Gamma(2(\mu-1))} I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)$$
$$= \frac{2^{\mu}\pi^{\mu-\frac{1}{2}}\Gamma(\frac{1}{2})R_1^{2\mu-1}}{(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)^{\mu-1}}\frac{\Gamma(2\mu-2+k)}{k!\Gamma(2\mu-2)} I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)$$
$$= \frac{2^{\mu}\pi^{\mu}R_1^{2\mu-1}}{(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A)^{\mu-1}}\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix} I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1 A), \tag{4.11}$$

where $\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix}$ is the binomial coefficient for $k\geq 0$. Substituting (4.11) into (4.7) the jpdf of the envelopes is

$$\begin{aligned} f_{R_1,R_2}(R_1,R_2) &= \frac{2^{2\mu-1}R_2\Gamma(\mu-1)\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\pi^{\mu}\det(\Sigma)^{\mu}\left[\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)\left(\Sigma_{21}^{-1}\right)\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}} \\ &\times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|R_2)I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2) \\ &\times\frac{2^{\mu}\pi^{\mu}R_1^{2\mu-1}}{(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}}\left(\begin{array}{c}2\mu+k-3\\2\mu-3\end{array}\right)I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1A) \\ &= \frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\det(\Sigma)^{\mu}\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)\left(\Sigma_{21}^{-1}\right)|A|^2\right]^{\mu-1}} \\ &\times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\left(\begin{array}{c}2\mu+k-3\\2\mu-3\end{array}\right)I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|R_2) \\ &\times I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2)I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1A). \end{aligned} \tag{4.12}$$

This expression (4.12) can be written in terms of the coefficients of the inverse of the covariance matrix by using (4.1). Thus, the jpdf is

$$\begin{aligned} f_{R_1,R_2}(R_1,R_2) &= \frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)\exp\left(-\frac{1}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\det(\Sigma)^{\mu}\left[\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)\left(\Sigma_{21}^{-1}\right)|A|^2\right]^{\mu-1}} \\ &\times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\left(\begin{array}{c}2\mu+k-3\\2\mu-3\end{array}\right)I_{\mu-1+k}(\left(\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|R_2) \\ &\times I_{\mu-1+k}(\Sigma_{21}^{-1}R_1R_2)I_{\mu-1+k}(\left(\Sigma_{11}^{-1}+\Sigma_{21}^{-1}\right)R_1A) \\ &= \sum_{k=0}^{\infty}\frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)(-1)^k(\mu-1+k)\left(\begin{array}{c}2\mu+k-3\\2\mu-3\end{array}\right)}{\left[\sigma_1^2\sigma_2^2(1-\delta^2)\right]^{\mu}\left[\frac{|A|\delta(1-\delta\frac{\sigma_2}{\sigma_1})(1-\delta\frac{\sigma_1}{\sigma_2})}{-(1-\delta^2)^3\sigma_1^3\sigma_2^3}\right]^{\mu-1}} \\ &\times\exp\left\{-\frac{1}{2}\left(\frac{1}{(1-\delta^2)}\left[\frac{R_1^2}{\sigma_1^2}+\frac{R_2^2}{\sigma_2^2}+\left(\frac{\sigma_1^2+\sigma_2^2-2\delta\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)|A|^2\right]\right)\right\} \\ &\times I_{\mu-1+k}\left(\frac{|A|R_2(1-\delta\frac{\sigma_2}{\sigma_1})}{(1-\delta^2)\sigma_2^2}\right)I_{\mu-1+k}\left(\frac{|A|R_1(1-\delta\frac{\sigma_1}{\sigma_2})}{(1-\delta^2)\sigma_1^2}\right)I_{\mu-1+k}\left(\frac{R_1R_2\delta}{(1-\delta^2)\sigma_1\sigma_2}\right) \\ &= \sum_{k=0}^{\infty}\frac{R_1R_2\Gamma(\mu-1)}{(\sigma_1^2\sigma_2^2)^{\mu}(1-\delta^2)^{\mu}}\left[\frac{-2(1-\delta^2)^3\sigma_1^3\sigma_2^3}{|A|^2\delta(1-\delta\frac{\sigma_2}{\sigma_1})(1-\delta\frac{\sigma_1}{\sigma_2})}\right]^{\mu-1} \\ &\times\exp\left\{-\frac{1}{2}\left(\frac{1}{(1-\delta^2)}\left[\frac{R_1^2}{\sigma_1^2}+\frac{R_2^2}{\sigma_2^2}+\left(\frac{\sigma_1^2+\sigma_2^2-2\delta\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)|A|^2\right]\right)\right\} \\ &\times(-1)^k(\mu-1+k)\left(\begin{array}{c}2\mu+k-3\\2\mu-3\end{array}\right)I_{\mu-1+k}\left(\frac{R_1R_2\delta}{(1-\delta^2)\sigma_1\sigma_2}\right) \\ &\times I_{\mu-1+k}\left(\frac{|A|R_2(1-\delta\frac{\sigma_2}{\sigma_1})}{(1-\delta^2)\sigma_2^2}\right)I_{\mu-1+k}\left(\frac{|A|R_1(1-\delta\frac{\sigma_1}{\sigma_2})}{(1-\delta^2)\sigma_1^2}\right). \end{aligned} \tag{4.13}$$

Define $\kappa_i = \frac{|A|^2}{2\mu\sigma_i^2} = \frac{a^2}{\sigma_i^2}, i = 1, 2$. Then by normalizing the two $\kappa - \mu$ envelopes, $R_1$ and $R_2$ (see equation (3.13)), where $P_i = \frac{R_i}{\sqrt{\hat{r}_i}}$. The Jacobian of the transformation from $(R_1, R_2) \longrightarrow (P_1, P_2)$ is

$$\begin{aligned} |J((R_1, R_2) \longrightarrow (P_1, P_2))| &= \begin{vmatrix} \sigma_1\sqrt{2\mu(1+\kappa_1)} & 0 \\ 0 & \sigma_2\sqrt{2\mu(1+\kappa_2)} \end{vmatrix} \\ &= \sigma_1\sigma_2 2\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}. \end{aligned}$$

The final expression for the normalized jpdf of the envelopes is

$$\begin{aligned} f_{P_1,P_2}(\rho_1, \rho_2) &= \sum_{k=0}^{\infty} \frac{\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3} 2\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(\sigma_1\sigma_2)\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\ &\quad \times \exp\left\{\frac{-2\mu}{2\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\ &\quad \times (-1)^k(\mu+k-1)\begin{pmatrix} 2\mu+k-3 \\ 2\mu-3 \end{pmatrix} I_{\mu-1+k}\left(\frac{2\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right) \\ &\quad \times I_{\mu-1+k}\left(\frac{2\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right) I_{\mu-1+k}\left(\frac{2\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right) \\ &\quad \times 2\mu\sigma_1\sigma_2\sqrt{(1+\kappa_1)(1+\kappa_2)} \\ &= \frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\ &\quad \times \exp\left\{\frac{-\mu}{(1-\delta^2)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(\mu+k-1)(2\mu-2)_k}{k!} I_{\mu-1+k}\left(\frac{2\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right) \\ &\quad \times I_{\mu-1+k}\left(\frac{2\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right) \\ &\quad \times I_{\mu-1+k}\left(\frac{2\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right). \end{aligned} \tag{4.14}$$

### 4.2.3 Characteristics

#### 4.2.3.1 Joint cumulative distribution function

The jpdf in (4.14) is represented as follows in order to get an expression for the jcdf, that will be used to obtain the outage probability. Expand the Bessel functions as infinite series by using (2.13) and then integrate term-wise, the jpdf (4.14) can be written as

$$
\begin{aligned}
f_{P_1,P_2}(\rho_1,\rho_2) &= \frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \exp\left\{\frac{-2\mu}{2\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\
&\times \sum_{k=0}^{\infty}\frac{(\mu+k-1)(2\mu-2)_k}{k!} \\
&\times \sum_{z=0}^{\infty}\left(\frac{2\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{2(1-\delta^2)}\right)^{\mu-1+k}\frac{\left(\frac{1}{4}\left(\frac{2\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^2\right)^z}{z!\Gamma(\mu+k+z)} \\
&\times \sum_{w=0}^{\infty}\left(\frac{2\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{2(1-\delta^2)}\right)^{\mu-1+k}\frac{\left(\frac{1}{4}\left(\frac{2\mu\rho_2\sqrt{1+\kappa_2}(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1})}{(1-\delta^2)}\right)^2\right)^w}{w!\Gamma(\mu+k+w)} \\
&\times \sum_{l=0}^{\infty}\left(\frac{2\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{2(1-\delta^2)}\right)^{\mu-1+k}\frac{\left(\frac{1}{4}\left(\frac{2\mu\rho_1\sqrt{1+\kappa_1}(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2})}{(1-\delta^2)}\right)^2\right)^l}{l!\Gamma(\mu+k+l)} \\
&= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\times \frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \exp\left\{\frac{-\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{\rho_1^2\mu(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_1^2\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\times \exp\left\{\frac{-\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{\rho_2^2\mu(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_2^2\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1} \\
&\times \left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l \\
&\times \left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w \\
&\times \left(\frac{\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z. \qquad (4.15)
\end{aligned}
$$

Thus from (4.15), the jcdf

$$
\begin{aligned}
F_{P_1,P_2}(p_1,p_2) &= \int_0^{p_1}\int_0^{p_2} f_{P_1,P_2}(\rho_1,\rho_2)\,d\rho_1 d\rho_2 \\
&= \int_0^{q_1}\int_0^{q_2}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\quad\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\quad\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad\times\exp\left\{\frac{-\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{\rho_1^2\mu(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_1^2\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\quad\times\exp\left\{\frac{-\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{\rho_2^2\mu(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_2^2\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1} \\
&\quad\times\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l} \\
&\quad\times\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
&\quad\times\left(\frac{\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z} d\rho_1 d\rho_2 \\
&= \int_0^{q_1}\int_0^{q_2}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\quad\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\quad\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad\times\left(\frac{1-\delta^2}{\rho_1^2\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_2^2\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1} \\
&\quad\times\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l} \\
&\quad\times\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}
\end{aligned}
$$

$$\times \left(\frac{\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z$$
$$\times \exp\left\{-\frac{\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1}$$
$$\times \exp\left\{-\frac{\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1} d\rho_1 d\rho_2.$$

After simplifying it follows,

$$\begin{aligned}
F_{P_1,P_2}(p_1,p_2) &= \int_0^{p_1}\int_0^{p_2}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{4\mu^2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\times \frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \left(\frac{1-\delta^2}{\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1}\left(\frac{1-\delta^2}{\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1} \\
&\times \left(\frac{\mu\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l \\
&\times \left(\frac{\mu\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w \\
&\times \left(\frac{\delta\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\delta^2\mu^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z \\
&\times \left(\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1}\left(\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1} \\
&\times \rho_1^{2\mu+2k+2z+2l-1}\rho_2^{2\mu+2k+2z+2w-1}\exp\left\{-\frac{\rho_1^2\mu(1+\kappa_1)}{(1-\delta^2)}\right\} \\
&\times \exp\left\{-\frac{\rho_2^2\mu(1+\kappa_2)}{(1-\delta^2)}\right\} d\rho_1 d\rho_2.
\end{aligned}$$

Using (2.20), the jcdf is

$$
\begin{aligned}
F_{P_1,P_2}(\rho_1,\rho_2) &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{\mu^2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{1-\delta^2}{\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\times\left(\frac{1-\delta^2}{\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1}\left(\frac{\mu\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{\mu^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{\mu\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{\mu^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}\left(\frac{\delta\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{\delta^2\mu^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z}\left(\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1} \\
&\times\left(\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1}\left[\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right]^{-(\mu+k+z+l)}\left[\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right]^{-(\mu+k+z+w)} \\
&\times\gamma\left(\mu+k+z+l,\frac{\mu(1+\kappa_1)\rho_1^2}{(1-\delta^2)}\right)\gamma\left(\mu+k+z+w,\frac{\mu(1+\kappa_2)\rho_2^2}{(1-\delta^2)}\right)dt.
\end{aligned}
$$

Further simplification results in

$$
\begin{aligned}
F_{P_1,P_2}(\rho_1,\rho_2) &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^{\mu}\delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\times\gamma\left(l+k+z+\mu,\frac{\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right)\gamma\left(w+k+z+\mu,\frac{\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right) \\
&\times\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}}\left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}}. \qquad (4.16)
\end{aligned}
$$

#### 4.2.3.2 Moments

Using the jpdf (4.14), the joint arbitrary moments can be found as

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] &= \int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2} f_{\rho_1,\rho_2}(\rho_1,\rho_2)d\rho_1 d\rho_2\\
&= \int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2}\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\\
&\quad\times\exp\left\{\frac{-2\mu}{2\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}\\
&\quad\times\sum_{k=0}^\infty\frac{(\mu+k-1)(2\mu-2)_k}{k!}I_{\mu-1+k}\left(\frac{2\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)\\
&\quad\times I_{\mu-1+k}\left(\frac{2\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)\\
&\quad\times I_{\mu-1+k}\left(\frac{2\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)d\rho_1 d\rho_2\\
&= \int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2}\sum_{z=0}^\infty\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{w=0}^\infty\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\\
&\quad\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu-1+k+l+1)\Gamma(\mu-1+k+w+1)\Gamma(\mu-1+k+z+1)}\\
&\quad\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}\exp\left\{\frac{-\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\exp\left\{\frac{-\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\\
&\quad\times\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l\\
&\quad\times\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w\\
&\quad\times\left(\frac{\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z d\rho_1 d\rho_2.
\end{aligned}
$$

Following a method similar to the derivation of the jcdf the joint arbitrary moments are obtained. Using (2.13) and interchange the summation and integral it follows that

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] = & \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{4\mu^2\Gamma(\mu-1)\left(\mu+k-1\right)\left(2\mu-2\right)_k\left(1-\delta^2\right)^{2\mu-3}}{k!z!w!l!\Gamma\left(\mu+k+l\right)\Gamma\left(\mu+k+w\right)\Gamma\left(\mu+k+z\right)} \\
& \times\frac{\left(1+\kappa_1\right)\left(1+\kappa_2\right)\exp\left\{\frac{-\mu}{\left(1-\delta^2\right)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
& \times\rho_1^{n_1+1}\rho_2^{n_2+1}\exp\left\{\frac{-\mu\rho_1^2\left(1+\kappa_1\right)}{\left(1-\delta^2\right)}\right\}\exp\left\{\frac{-\mu\rho_2^2\left(1+\kappa_2\right)}{\left(1-\delta^2\right)}\right\} \\
& \times\left(\frac{\delta\mu\rho_1\rho_2\sqrt{\left(1+\kappa_1\right)\left(1+\kappa_2\right)}}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k} \\
& \times\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2\left(1+\kappa_2\right)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
& \times\left(\frac{\mu^2\rho_1^2\left(1+\kappa_1\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2\left(1+\kappa_1\right)\left(1+\kappa_2\right)}{(1-\delta^2)^2}\right)^{z}d\rho_1 d\rho_2.
\end{aligned}
$$

Further simplification results in

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] = & \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{4\mu^2\Gamma(\mu-1)\left(\mu+k-1\right)\left(2\mu-2\right)_k\left(1-\delta^2\right)^{2\mu-3}}{k!z!w!l!\Gamma\left(\mu+k+l\right)\Gamma\left(\mu+k+w\right)\Gamma\left(\mu+k+z\right)} \\
& \times\frac{\left(1+\kappa_1\right)\left(1+\kappa_2\right)\exp\left\{\frac{-\mu}{\left(1-\delta^2\right)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\exp\left\{\frac{-\mu\rho_1^2\left(1+\kappa_1\right)}{\left(1-\delta^2\right)}\right\} \\
& \times\rho_1^{n_1+1}\rho_2^{n_2+1}\exp\left\{\frac{-\mu\rho_2^2\left(1+\kappa_2\right)}{\left(1-\delta^2\right)}\right\}\left(\frac{\mu\rho_1^2\left(1+\kappa_1\right)}{1-\delta^2}\right)^{l+k+z+\mu+\frac{n_1}{2}-1} \\
& \times\left(\frac{1-\delta^2}{\mu\rho_1^2\left(1+\kappa_1\right)}\right)^{l+k+z+\mu+\frac{n_1}{2}-1}\left(\frac{\mu\rho_2^2\left(1+\kappa_2\right)}{1-\delta^2}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} \\
& \times\left(\frac{1-\delta^2}{\mu\rho_2^2\left(1+\kappa_2\right)}\right)^{w+k+z+\mu+\frac{n_2}{2}-1}\left(\frac{\delta\mu\rho_1\rho_2\sqrt{\left(1+\kappa_1\right)\left(1+\kappa_2\right)}}{\left(1-\delta^2\right)}\right)^{\mu-1+k} \\
& \times\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k} \\
& \times\left(\frac{\mu^2\rho_2^2\left(1+\kappa_2\right)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}\left(\frac{\mu^2\rho_1^2\left(1+\kappa_1\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l} \\
& \times\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2\left(1+\kappa_1\right)\left(1+\kappa_2\right)}{(1-\delta^2)^2}\right)^{z}d\rho_1 d\rho_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] \;=\; & \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\Gamma(\mu-1)\,(\mu+k-1)\,(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\,\Gamma(\mu+k+w)\,\Gamma(\mu+k+z)} \\
& \times\left(\frac{1-\delta^2}{\mu\rho_1^2(1+\kappa_1)}\right)^{l+k+z+\mu+\frac{n_1}{2}-1}\left(\frac{1-\delta^2}{\mu\rho_2^2(1+\kappa_2)}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} \\
& \times\left(\frac{\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
& \times\left(\frac{\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
& \times\left(\frac{\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z} \\
& \times\exp\left\{\frac{-\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{\mu\rho_1^2(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu+\frac{n_1}{2}-1} \\
& \times\exp\left\{\frac{-\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{\mu\rho_2^2(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} d\rho_1 d\rho_2.
\end{aligned}
$$

Simplifying the above expression and using (2.1) the joint arbitrary moments are,

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] \;=\; & \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{\Gamma(\mu-1)\,(\mu+k-1)\,(2\mu-2)_k\,(1-\delta^2)^{\mu}\,\delta^{k+2z}}{k!z!w!l!\Gamma(\mu+k+l)\,\Gamma(\mu+k+w)\,\Gamma(\mu+k+z)} \\
& \times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\
& \times\left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w+\frac{k}{2}}\left(\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{-\frac{n_1}{2}}\left(\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{-\frac{n_2}{2}} \\
& \times\Gamma\left(l+k+z+\mu+\frac{n_1}{2}\right)\Gamma\left(w+k+z+\mu+\frac{n_2}{2}\right). \qquad (4.17)
\end{aligned}
$$

### 4.2.4 Outage Probability

The parameter of interest is the power, $\Omega_i$, at each diversity branch $i$. For the dual branch selection combinig (SC) scheme the outage probability is the jcdf of the power evaluated at $\vartheta_i=\vartheta$, with $\vartheta=\Omega_i$ and $\bar{\vartheta}_i=\frac{E_b}{N_0}E\left[R_i^2\right]$ being the average power (see Section 4.2.1) . By expressing $P_i$ in terms of $R_i$,

$$
P_i^2=\frac{R_i^2}{E\left[R_i^2\right]}=\frac{R_i^2\frac{E_b}{N_0}}{E\left[R_i^2\right]\frac{E_b}{N_0}}=\frac{\vartheta}{\bar{\vartheta}_i}.
$$

Thus, from equation (4.16) the outage probability is defined as

$$\begin{aligned}F_{SC}(\vartheta) &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^{\mu}\delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\ &\quad\times\exp\left\{\frac{-\mu}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\ &\quad\times\left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}}\gamma\left(l+k+z+\mu,\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right) \\ &\quad\times\gamma\left(w+k+z+\mu,\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right). \end{aligned} \tag{4.18}$$

**Remark 4.1** *This equation* (4.18) *corresponds to equation* (21) *of Villavicencio, et al.,(2016) [33].*

# 4.3 The bivariate $\kappa-\mu$ type distribution

The aim of this section is to derive a bivariate $\kappa-\mu$ type distribution with as the underlying model, the elliptical class. Exact formulae for the jpdf, jcdf and joint arbitrary moments will be explored. Special cases of the elliptical model will receive attention. A weight approximation as outlined by Arashi and Nadarajah (2016) [2] will also receive attention.

## 4.3.1 Description

Let $R_1$ and $R_2$ be 2 $\kappa-\mu$ envelopes then,

$$R_1^2=\sum_{i=1}^{2\mu}X_i^2$$

$$R_2^2=\sum_{i=1}^{2\mu}Y_i^2,$$

where $X_i$ and $Y_i$ are mutually independent elliptical random variables with mean equal to $a$ and variances $\sigma_1^2$ and $\sigma_2^2$ respectively. [Note: $2\mu=$ number of clusters of multipath]. From (2.28) $X_i|t\sim N(0,a_t)$ and $Y_i|t\sim N(0,a_t)$ where $a_t=t^{-1}\sigma^2$. If $\{X|t,Y|t,A\}$ be $2\mu$- dimensional vectors then $X|t=[X_1|t \;\; X_2|t \;\; \ldots \;\; X_{2\mu}|t\;]^T$, $Y|t=[Y_1|t \;\; Y_2|t \;\; \ldots \;\; Y_{2\mu}|t\;]^T$ and $A=[\;a \;\; a \;\; \ldots \;\; a\;]^T$. Let $V_i|t=[X_i|t \;\; Y_i|t]^T$ where $X_i|t$ and $Y_i|t$ are correlated if and only if $i=j$, $i=1,2,\ldots,2\mu$. Note that $A$ is the vector of the dominant component of the cluster. The interest is the jpdf of the envelope $R_1$ and $R_2$, as well as, the normalized envelopes, $P_1$ and $P_2$ with $P_i=\frac{R_i}{\sqrt{\hat{r}_i}}$ and $\hat{r}_i$ as defined in (3.13).

### 4.3.2 Derivation

The positive definite covariance matrix is

$$t^{-1}\Sigma = \begin{bmatrix} t^{-1}\Sigma_{11} & t^{-1}\Sigma_{12} \\ t^{-1}\Sigma_{21} & t^{-1}\Sigma_{22} \end{bmatrix} = \begin{bmatrix} t^{-1}\sigma_1^2 & \delta t^{-1}\sigma_1\sigma_2 \\ \delta t^{-1}\sigma_1\sigma_2 & t^{-1}\sigma_2^2 \end{bmatrix},$$

where $\delta$ is the correlation coefficient of the elliptical component,

$$\begin{aligned} \delta &= \frac{t^{-1}\Sigma_{12}}{\sqrt{\left(t^{-1}\Sigma_{11}\right)\left(t^{-1}\Sigma_{22}\right)}} = \frac{\Sigma_{12}}{\sigma_1\sigma_2} \\ \therefore & \quad \Sigma_{12} = \delta\sigma_1\sigma_2. \end{aligned}$$

The inverse of the covariance matrix is

$$\begin{aligned} \left(t^{-1}\Sigma\right)^{-1} &= \begin{bmatrix} \left(t^{-1}\Sigma_{11}\right)^{-1} & \left(t^{-1}\Sigma_{12}\right)^{-1} \\ \left(t^{-1}\Sigma_{21}\right)^{-1} & \left(t^{-1}\Sigma_{22}\right)^{-1} \end{bmatrix} = \begin{bmatrix} t^{-1}\sigma_1^2 & \delta t^{-1}\sigma_1\sigma_2 \\ \delta t^{-1}\sigma_1\sigma_2 & t^{-1}\sigma_2^2 \end{bmatrix}^{-1} \\ &= \frac{1}{t^{-2}\sigma_1^2\sigma_2^2 - t^{-2}\delta^2\sigma_1^2\sigma_2^2} \begin{bmatrix} t^{-1}\sigma_2^2 & -\delta t^{-1}\sigma_1\sigma_2 \\ -\delta t^{-1}\sigma_1\sigma_2 & t^{-1}\sigma_1^2 \end{bmatrix} \\ &= \frac{1}{t^{-2}\sigma_1^2\sigma_2^2\left(1-\delta^2\right)} \begin{bmatrix} t^{-1}\sigma_2^2 & -\delta t^{-1}\sigma_1\sigma_2 \\ -\delta t^{-1}\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \\ \Sigma^{-1} &= \frac{1}{\left(1-\delta^2\right)} \begin{bmatrix} t\sigma_1^{-2} & -\delta t\sigma_1^{-1}\sigma_2^{-1} \\ -\delta t\sigma_1^{-1}\sigma_2^{-1} & t\sigma_2^{-2} \end{bmatrix} \end{aligned}$$

$$\left(t^{-1}\Sigma\right)^{-1} = \begin{bmatrix} t\Sigma_{11}^{-1} & t\Sigma_{21}^{-1} \\ t\Sigma_{21}^{-1} & t\Sigma_{22}^{-1} \end{bmatrix}. \tag{4.19}$$

The pdf of the multivariate elliptical distribution is given by (2.29). Thus, the jpdf of $X$ and $Y$ is given by

$$\begin{aligned} f_{X,Y}(X,Y) &= \prod_{i=1}^{2\mu} \int_0^\infty \frac{W(t)}{(2\pi)\det\left(t^{-1}\Sigma\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left[\left(X_i-\mu\right)\left(Y_i-\mu\right)\right]^T \left(t^{-1}\Sigma\right)^{-1} \begin{bmatrix} \left(X_i-\mu\right) \\ \left(Y_i-\mu\right) \end{bmatrix}\right) dt \\ &= \prod_{i=1}^{2\mu} \int_0^\infty \frac{W(t)}{(2\pi)\det\left(t^{-1}\Sigma\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left[V_i-C_i\right]^T \left(t^{-1}\Sigma\right)^{-1}\left[V_i-C_i\right]\right) dt \\ &= \prod_{i=1}^{2\mu} f_i\left(V_i\right) \\ &= \int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\frac{2\mu}{2}}} \exp\left(-\frac{1}{2}\sum_{i=1}^{2\mu}\left[V_i-C_i\right]^T \left(t^{-1}\Sigma\right)^{-1}\left[V_i-C_i\right]\right) dt, \end{aligned} \tag{4.20}$$

where $V_i = [X_i \ \ Y_i]^T$ and $C_i = [a \ \ a]^T$.

Expanding the quadratic form and replacing the $V_i$ by $X_i$ and $Y_i$ it follows that

$$\begin{aligned}
&\sum_{i=1}^{2\mu} [V_i - C_i]^T \left(t^{-1}\Sigma\right)^{-1} [V_i - C_i] \\
= &\sum_{i=1}^{2\mu} [(X_i - a)\,(Y_i - a)] \left(t^{-1}\Sigma\right)^{-1} \left[\begin{array}{c} (X_i - a) \\ (Y_i - a) \end{array}\right].
\end{aligned}$$

Then from the above and (4.19) it follows

$$\begin{aligned}
&\sum_{i=1}^{2\mu} [(X_i - a)\,(Y_i - a)] \left[\begin{array}{cc} t\Sigma_{11}^{-1} & t\Sigma_{12}^{-1} \\ t\Sigma_{21}^{-1} & t\Sigma_{22}^{-1} \end{array}\right] \left[\begin{array}{c} (X_i - a) \\ (Y_i - a) \end{array}\right] \\
= &\sum_{i=1}^{2\mu} \left[(X_i - a)\, t\Sigma_{11}^{-1} + (Y_i - a)\, t\Sigma_{21}^{-1}\right](X_i - a) + \left[(X_i - a)\, t\Sigma_{21}^{-1} + (Y_i - a)\, t\Sigma_{22}^{-1}\right](Y_i - a) \\
= &\sum_{i=1}^{2\mu} (X_i - a)\, t\Sigma_{11}^{-1}\,(X_i - a) + (Y_i - a)\, t\Sigma_{21}^{-1}\,(X_i - a) + (X_i - a)\, t\Sigma_{21}^{-1}\,(Y_i - a) \\
&+ (Y_i - a)\, t\Sigma_{22}^{-1}\,(Y_i - a) \\
= &\sum_{i=1}^{2\mu} X_i t\Sigma_{11}^{-1} X_i - X_i t\Sigma_{11}^{-1} a - a t\Sigma_{11}^{-1} X_i + a t\Sigma_{11}^{-1} a + Y_i t\Sigma_{21}^{-1} X_i - Y_i t\Sigma_{21}^{-1} a - a t\Sigma_{21}^{-1} X_i \\
&+ a t\Sigma_{21}^{-1} a + X_i t\Sigma_{21}^{-1} Y_i - X_i t\Sigma_{21}^{-1} a - a t\Sigma_{21}^{-1} X_i + a t\Sigma_{21}^{-1} a + Y_i t\Sigma_{22}^{-1} Y_i - Y_i t\Sigma_{22}^{-1} a \\
&- a t\Sigma_{22}^{-1} Y_i + a t\Sigma_{22}^{-1} a \\
= &\sum_{i=1}^{2\mu} a\left(t\Sigma_{11}^{-1} + 2t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) a - a\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) X_i - X_i\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) a \\
&- a\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) Y_i - Y_i\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) a + X_i t\Sigma_{11}^{-1} X_i + X_i t\Sigma_{21}^{-1} Y_i + Y_i t\Sigma_{21}^{-1} X_i + Y_i t\Sigma_{22}^{-1} Y_i.
\end{aligned}$$

Since the inverse of the covariance matrix are constant the above simplifies to

$$\begin{aligned}
&\sum_{i=1}^{2\mu} a\left(t\Sigma_{11}^{-1} + 2t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) a - a\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) X_i - X_i\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) a \\
&- a\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) Y_i - Y_i\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) a + X_i t\Sigma_{11}^{-1} X_i + X_i t\Sigma_{21}^{-1} Y_i + Y_i t\Sigma_{21}^{-1} X_i + Y_i t\Sigma_{22}^{-1} Y_i \\
= &\left(t\Sigma_{11}^{-1} + 2t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right)|A|^2 + t\Sigma_{11}^{-1}|X|^2 + t\Sigma_{22}^{-1}|Y|^2 \\
&+ \sum_{i=1}^{2\mu} [X_i t\Sigma_{21}^{-1} Y_i + Y_i t\Sigma_{21}^{-1} X_i - a\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) X_i - X_i\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) a \\
&- a\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) Y_i - Y_i\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) a] \\
= &\left(t\Sigma_{11}^{-1} + 2t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right)|A|^2 + t\Sigma_{11}^{-1}|X|^2 + t\Sigma_{22}^{-1}|Y|^2 + 2t\Sigma_{21}^{-1} XY \\
&- 2\left(t\Sigma_{11}^{-1} + t\Sigma_{21}^{-1}\right) AX - 2\left(t\Sigma_{21}^{-1} + t\Sigma_{22}^{-1}\right) AY. \qquad (4.21)
\end{aligned}$$

Substituting (4.21) into (4.20) gives

$$\begin{aligned}
&\int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\sum_{i=1}^{2\mu}\left[V_i-C_i\right]^T\left(t^{-1}\Sigma\right)^{-1}\left[V_i-C_i\right]\right)dt\\
=&\int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(t\Sigma_{11}^{-1}+2t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|^2+t\Sigma_{11}^{-1}|X|^2+t\Sigma_{22}^{-1}|Y|^2+2t\Sigma_{21}^{-1}XY\right]\right.\\
&\times\exp\left(-\frac{1}{2}\left[-2\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)AX-2\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)AY\right]\right)dt\\
=&\int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(t\Sigma_{11}^{-1}+2t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|^2+t\Sigma_{11}^{-1}|X|^2+t\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\times\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)AX+\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)AY-t\Sigma_{21}^{-1}XY\right]dt\\
=&\int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\frac{2\mu}{2}}}\exp\left(-\frac{1}{2}\left[\left(t\Sigma_{11}^{-1}+2t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|^2+t\Sigma_{11}^{-1}|X|^2+t\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\times\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)AX+Y\left(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}X\right)\right]dt.
\end{aligned}$$

Hence, the jpdf can be written as follows

$$\begin{aligned}
f_{X,Y}(X,Y) =& \int_0^\infty \frac{W(t)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\mu}}\exp\left(-\frac{1}{2}\left[\left(t\Sigma_{11}^{-1}+2t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|^2+t\Sigma_{11}^{-1}|X|^2+t\Sigma_{22}^{-1}|Y|^2\right]\right)\\
&\times\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)AX+Y\left(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}X\right)\right]dt.
\end{aligned}$$

As before in order to obtain the jpdf of the envelopes, an integration with respect to $X$ and $Y$ subject to the constraint $|X|=R_1$, and $|Y|=R_2$, is required (see (2.45)). That is

$$\begin{aligned}
f_{X,Y}(X,Y) =& \int_0^\infty W(t)\frac{\exp\left(-\frac{1}{2}\left[\left(t\Sigma_{11}^{-1}+2t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|^2+t\Sigma_{11}^{-1}|X|^2+t\Sigma_{22}^{-1}|Y|^2\right]\right)}{(2\pi)^{2\mu}\det\left(t^{-1}\Sigma\right)^{\mu}}\\
&\times\int_{|X|=R_1}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)X^TA\right]ds_1\\
&\times\int_{|Y|=R_2}\exp\left[Y^T\left(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}X\right)\right]ds_2dt. \qquad (4.22)
\end{aligned}$$

where $ds_1$ and $ds_2$ are the elements of the integration surface area (see (2.45). The second integral in equation (4.22) can be expressed as

$$\begin{aligned}
&\int_{|Y|}\exp\left[Y^T\left(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}X\right)\right]ds_2\\
=&\left(2\pi R_2\right)^{\frac{2\mu}{2}}\left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}R_1\right|^{1-\mu}\\
&\times I_{\frac{1}{2}(2\mu-2)}\left(R_2\left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A-t\Sigma_{21}^{-1}R_1\right|\right). \qquad (4.23)
\end{aligned}$$

Using (2.16), the expression (4.23) can be written as

$$
\begin{aligned}
&(2\pi R_2)^{\frac{2\mu}{2}} \left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A - t\Sigma_{21}^{-1}R_1\right|^{1-\mu} I_{\frac{1}{2}(2\mu-2)}(R_2\left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A - t\Sigma_{21}^{-1}R_1\right|)\\
=\ & \frac{(2\pi R_2)^{\mu}}{(R_2)^{1-\mu}}(R_2\left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A - t\Sigma_{21}^{-1}R_1\right|)^{1-\mu} I_{\frac{1}{2}(2\mu-2)}(R_2\left|\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)A - t\Sigma_{21}^{-1}R_1\right|)\\
=\ & \frac{(2\pi R_2)^{\mu}}{(R_2)^{1-\mu}}2^{\mu-1}\Gamma(\mu-1)\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\frac{I_{\mu-1+k}(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|R_2)}{(R_2)^{\mu-1}\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)^{\mu-1}|A|^{\mu-1}}\\
&\times\frac{I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)}{\left(t\Sigma_{21}^{-1}\right)^{\mu-1}(R_1R_2)^{\mu-1}}C_k^{\mu-1}(\cos\phi_1)\\
&\times\frac{2^{2\mu-1}(\pi R_2)^{\mu}}{\left[\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)\left(t\Sigma_{21}^{-1}\right)R_2\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}}\Gamma(\mu-1)\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\\
&\times I_{\mu-1+k}(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|R_2)I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)C_k^{\mu-1}(\cos\phi_1).
\end{aligned}
$$

where $\phi_1$ is the angle between $X$ and $A$.
Thus, the jpdf of the envelopes is

$$
\begin{aligned}
&\int_0^{\infty}\frac{2^{2\mu-1}(\pi R_2)^{\mu}\Gamma(\mu-1)\exp\left(-\frac{t}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{(2\pi)^{2\mu}|t^{-1}\Sigma|^{\mu}\left[\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)\left(t\Sigma_{21}^{-1}\right)R_2\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}}\\
&\times W(t)\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)I_{\mu-1+k}(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|R_2)I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)\\
&\times\int_{|X|}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)X'A\right]C_k^{\mu-1}(\cos\phi_1)ds_1dt. \qquad (4.24)
\end{aligned}
$$

To evaluate the remaining integral in equation (4.24) express the integral in generalized spherical coordinates

$$
\begin{aligned}
&\int_{|X|}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)X'A\right]C_k^{\mu-1}(\cos\phi_1)ds_1\\
=\ &\int_{|X|}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A\cos\phi_1\right]C_k^{\mu-1}(\cos\phi_1)ds_1\\
=\ &R_1^{2\mu-1}\int_0^{2\pi}d\theta\int_0^{\pi}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A\cos\phi_1\right]C_k^{\mu-1}(\cos\phi_1)\sin^{2\mu-2}d\phi_1\\
&\times\prod_{k=2}^{2\mu-2}\int_0^{\pi}\sin^{2\mu-1-k}\phi_k d\phi_k. \qquad (4.25)
\end{aligned}
$$

Applying (2.17)

$$\begin{aligned}&\int_0^\pi \exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A\cos\phi_1\right]C_k^{\mu-1}(\cos\phi_1)\sin^{2\mu-2}d\phi_1\\ &= \frac{2^{\mu-1}\Gamma(\mu-1+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2(\mu-1)+k)}{(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}k!\Gamma(2(\mu-1))}I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A) \qquad (4.26)\end{aligned}$$

and using (2.18),

$$\begin{aligned}&R_1^{2\mu-1}\int_0^{2\pi}d\theta\prod_{k=2}^{2\mu-2}\int_0^\pi \sin^{2\mu-1-k}\phi_k d\phi_k\\ &= \frac{2\pi^{\frac{1}{2}(2\mu-1)}R_1^{2\mu-1}}{\Gamma(\frac{1}{2}(2\mu-1))}, \qquad (4.27)\end{aligned}$$

it follows from (4.25) , (4.26) and (4.27) that

$$\begin{aligned}&\int_{|X|}\exp\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)X\acute{}A\right]C_k^{\mu-1}(\cos\phi_1)ds_1\\ &= \frac{2\pi^{\frac{1}{2}(2\mu-1)}R_1^{2\mu-1}}{\Gamma(\frac{1}{2}(2\mu-1))}\frac{2^{\mu-1}\Gamma(\mu-1+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2(\mu-1)+k)}{(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}k!\Gamma(2(\mu-1))}I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)\\ &= \frac{2^\mu\pi^{\mu-\frac{1}{2}}\Gamma(\frac{1}{2})R_1^{2\mu-1}}{(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}}\frac{\Gamma(2\mu-2+k)}{k!\Gamma(2\mu-2)}I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)\\ &= \frac{2^\mu\pi^\mu R_1^{2\mu-1}}{(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}}\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix}I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A). \qquad (4.28)\end{aligned}$$

Substituting (4.28) into (4.24) the jpdf of the envelopes is

$$\begin{aligned}f_{R_1,R_2}(R_1,R_2) &= \int_0^\infty \frac{2^{2\mu-1}R_2\Gamma(\mu-1)\exp\left(-\frac{t}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\pi^\mu\det\left(t^{-1}\Sigma\right)^\mu\left[\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)\left(t\Sigma_{21}^{-1}\right)\right]^{\mu-1}|A|^{\mu-1}R_1^{\mu-1}}\\ &\times\sum_{k=0}^\infty(-1)^k(\mu-1+k)I_{\mu-1+k}(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|R_2)I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)\\ &\times\frac{2^\mu\pi^\mu R_1^{2\mu-1}}{(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)^{\mu-1}}\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix}I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)\,W(t)\;dt\\ &= \int_0^\infty \frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)\exp\left(-\frac{t}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\det\left(t^{-1}\Sigma\right)^\mu\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)\left(t\Sigma_{21}^{-1}\right)|A|^2\right]^{\mu-1}}\\ &\times\sum_{k=0}^\infty(-1)^k(\mu-1+k)\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix}I_{\mu-1+k}(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|R_2)\\ &\times I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)I_{\mu-1+k}(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A)\,W(t)\;dt. \qquad (4.29)\end{aligned}$$

Expression (4.29) can be written in terms of the coefficients of the inverse of the covariance matrix by using (4.19). Thus, the jpdf is

$$\begin{aligned}
f_{R_1,R_2}(R_1,R_2) &= \int_0^\infty \frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)\exp\left(-\frac{t}{2}\left[\left(\Sigma_{11}^{-1}+2\Sigma_{21}^{-1}+\Sigma_{22}^{-1}\right)|A|^2+\Sigma_{11}^{-1}R_1^2+\Sigma_{22}^{-1}R_2^2\right]\right)}{\det\left(t^{-1}\Sigma\right)^{\mu}\left[\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)\left(t\Sigma_{21}^{-1}\right)|A|^2\right]^{\mu-1}} \\
&\quad \times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\begin{pmatrix} 2\mu+k-3 \\ 2\mu-3\end{pmatrix} I_{\mu-1+k}\left(\left(t\Sigma_{21}^{-1}+t\Sigma_{22}^{-1}\right)|A|\,R_2\right) \\
&\quad \times I_{\mu-1+k}(t\Sigma_{21}^{-1}R_1R_2)I_{\mu-1+k}\left(\left(t\Sigma_{11}^{-1}+t\Sigma_{21}^{-1}\right)R_1A\right)W(t)\ dt \\
&= \int_0^\infty\sum_{k=0}^{\infty}W(t)\frac{2^{\mu-1}R_1R_2\Gamma(\mu-1)(-1)^k(\mu-1+k)\begin{pmatrix} 2\mu+k-3 \\ 2\mu-3\end{pmatrix}}{\left[t^{-2}\sigma_1^2\sigma_2^2(1-\delta^2)\right]^{\mu}\left[\frac{|A|\delta(1-\delta\frac{\sigma_2}{\sigma_1})(1-\delta\frac{\sigma_1}{\sigma_2})}{-(1-\delta^2)^3t^3\sigma_1^3\sigma_2^3}\right]^{\mu-1}} \\
&\quad \times\exp\left\{-\frac{1}{2}\left(\frac{1}{(1-\delta^2)}\left[\frac{tR_1^2}{\sigma_1^2}+\frac{tR_2^2}{\sigma_2^2}+\left(\frac{t\sigma_1^2+t\sigma_2^2-2\delta t\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)|A|^2\right]\right)\right\} \\
&\quad \times I_{\mu-1+k}\left(\frac{t|A|\,R_2(1-\delta\frac{\sigma_2}{\sigma_1})}{(1-\delta^2)\sigma_2^2}\right)I_{\mu-1+k}\left(\frac{t|A|\,R_1(1-\delta\frac{\sigma_1}{\sigma_2})}{(1-\delta^2)\sigma_1^2}\right) \\
&\quad \times I_{\mu-1+k}\left(\frac{tR_1R_2\delta}{(1-\delta^2)\sigma_1\sigma_2}\right)dt \\
&= \int_0^\infty\frac{W(t)\,R_1R_2\Gamma(\mu-1)}{t^{-2\mu}\left(\sigma_1^2\sigma_2^2\right)^{\mu}(1-\delta^2)^{\mu}}\left[\frac{-2(1-\delta^2)^3\sigma_1^3\sigma_2^3}{|A|^2\,\delta t^3(1-\delta\frac{\sigma_2}{\sigma_1})(1-\delta\frac{\sigma_1}{\sigma_2})}\right]^{\mu-1} \\
&\quad \times\exp\left\{-\frac{1}{2}\left(\frac{1}{(1-\delta^2)}\left[\frac{tR_1^2}{\sigma_1^2}+\frac{tR_2^2}{\sigma_2^2}+\left(\frac{t\sigma_1^2+t\sigma_2^2-2\delta t\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)|A|^2\right]\right)\right\} \\
&\quad \times\sum_{k=0}^{\infty}(-1)^k(\mu-1+k)\begin{pmatrix} 2\mu+k-3 \\ 2\mu-3\end{pmatrix} I_{\mu-1+k}\left(\frac{tR_1R_2\delta}{(1-\delta^2)\sigma_1\sigma_2}\right) \\
&\quad \times I_{\mu-1+k}\left(\frac{t|A|\,R_2(1-\delta\frac{\sigma_2}{\sigma_1})}{(1-\delta^2)\sigma_2^2}\right)I_{\mu-1+k}\left(\frac{t|A|\,R_1(1-\delta\frac{\sigma_1}{\sigma_2})}{(1-\delta^2)\sigma_1^2}\right)dt. \qquad (4.30)
\end{aligned}$$

Define $\kappa_i=\frac{|A|^2}{2\mu\sigma_i^2}=\frac{a^2}{\sigma_i^2}$, $i=1,2$. Then by normalizing the two $\kappa-\mu$ envelopes, $R_1$ and $R_2$ (see equation (3.13)) where $P_i=\frac{R_i}{\sqrt{\hat{r}_i}}$ . The Jacobian of the transformation from $(R_1,R_2)\longrightarrow(P_1,P_2)$ is

$$\begin{aligned}
|J\left((R_1,R_2)\longrightarrow(P_1,P_2)\right)| &= \begin{vmatrix} \sigma_1\sqrt{2\mu(1+\kappa_1)} & 0 \\ 0 & \sigma_2\sqrt{2\mu(1+\kappa_2)}\end{vmatrix} \\
&= \sigma_1\sigma_2 2\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}.
\end{aligned}$$

The final expression for the normalized jpdf of the envelope is

$$\begin{aligned}
f_{P_1,P_2}(\rho_1,\rho_2) &= \int_0^\infty W(t)\frac{\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}2\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{t^{\mu-3}(\sigma_1\sigma_2)\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times\exp\left\{\frac{-2\mu t}{2\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\
&\times\sum_{k=0}^{\infty}(-1)^k(\mu+k-1)\begin{pmatrix}2\mu+k-3\\2\mu-3\end{pmatrix}I_{\mu-1+k}\left(\frac{2t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right) \\
&\times I_{\mu-1+k}\left(\frac{2t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)I_{\mu-1+k}\left(\frac{2t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right) \\
&\times 2\mu\sigma_1\sigma_2\sqrt{(1+\kappa_1)(1+\kappa_2)}dt \\
&= \int_0^\infty W(t)\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times\exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\
&\times\sum_{k=0}^{\infty}\frac{(\mu+k-1)(2\mu-2)_k}{k!}I_{\mu-1+k}\left(\frac{2t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right) \\
&\times I_{\mu-1+k}\left(\frac{2t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right) \\
&\times I_{\mu-1+k}\left(\frac{2t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)dt. \qquad (4.31)
\end{aligned}$$

### 4.3.3 Characteristics

#### 4.3.3.1 Joint cumulative distribution function

To find the jcdf of $P_1$ and $P_2$, we expand the Bessel functions as infinite series using (2.13) and then integrate term-wise. Hence, the jpdf can be written as

$$f_{P_1,P_2}(\rho_1,\rho_2) = \int_0^\infty W(t)\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \tag{4.32}$$

$$\times\exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \tag{4.33}$$

$$\times\sum_{k=0}^\infty\frac{(\mu+k-1)(2\mu-2)_k}{k!}\sum_{z=0}^\infty\left(\frac{2t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{2(1-\delta^2)}\right)^{\mu-1+k} \tag{4.34}$$

$$\times\frac{\left(\frac{1}{4}\left(\frac{2t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^2\right)^z}{z!\Gamma(\mu-1+k+z+1)}\sum_{w=0}^\infty\left(\frac{2t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{2(1-\delta^2)}\right)^{\mu-1+k} \tag{4.35}$$

$$\times\frac{\left(\frac{1}{4}\left(\frac{2t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^2\right)^w}{w!\Gamma(\mu-1+k+w+1)}\sum_{l=0}^\infty\left(\frac{2t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{2(1-\delta^2)}\right)^{\mu-1+k} \tag{4.36}$$

$$\times\frac{\left(\frac{1}{4}\left(\frac{2t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^2\right)^l}{l!\Gamma(\mu-1+k+l+1)}dt \tag{4.37}$$

$$= \int_0^\infty\sum_{z=0}^\infty\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{w=0}^\infty W(t)\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \tag{4.38}$$

$$\times\exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}$$

$$\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu-1+k+l+1)\Gamma(\mu-1+k+w+1)\Gamma(\mu-1+k+z+1)}$$

$$\times\exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{\left(1-\delta^2\right)}\right\}\left(\frac{\rho_1^2\mu t(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_1^2\mu t(1+\kappa_1)}\right)^{l+k+z+\mu-1}$$

$$\times\exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{\left(1-\delta^2\right)}\right\}\left(\frac{\rho_2^2\mu t(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu-1}\left(\frac{1-\delta^2}{\rho_2^2\mu t(1+\kappa_2)}\right)^{w+k+z+\mu-1}$$

$$\times\left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l$$

$$\times\left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w$$

$$\times\left(\frac{t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z dt. \tag{4.39}$$

Thus from (4.39), jcdf of $P_1$ and $P_2$ is

$$\begin{aligned}
F_{P_1,P_2}(p_1,p_2) &= \int_0^{p_1}\int_0^{p_2} f_{\rho_1,\rho_2}(\rho_1,\rho_2)\, d\rho_1 d\rho_2 \\
&= \int_0^{p_1}\int_0^{p_2}\int_0^{\infty} \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} W(t) \frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\quad \times \exp\left\{\frac{-\mu t}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\quad \times \frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{t^2\rho_1^2\mu(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu-1} \\
&\quad \times \left(\frac{1-\delta^2}{t\rho_1^2\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\} \\
&\quad \times \left(\frac{\rho_2^2\mu t(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu-1}\left(\frac{1-\delta^2}{t\rho_2^2\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1} \\
&\quad \times \left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l} \\
&\quad \times \left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
&\quad \times \left(\frac{t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z} dt d\rho_1 d\rho_2,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
F_{P_1,P_2}(p_1,p_2) &= \int_0^{p_1}\int_0^{p_2}\int_0^{\infty}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} W(t)\,\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times\exp\left\{\frac{-\mu t}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{1-\delta^2}{t\rho_1^2\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\times\left(\frac{1-\delta^2}{t\rho_2^2\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1}\left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}\left(\frac{t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z} \\
&\times\exp\left\{-\frac{t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\left(\frac{t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1} \\
&\times\exp\left\{-\frac{t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1}\qquad dt\,d\rho_1\,d\rho_2,
\end{aligned}$$

and

$$
\begin{aligned}
F_{P_1,P_2}(p_1,p_2) &= \int_0^{p_1}\int_0^{p_2}\int_0^{\infty}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} W(t)\frac{4\mu^2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\quad\times\frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad\times\exp\left\{\frac{-\mu t}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{1-\delta^2}{t\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\quad\times\left(\frac{1-\delta^2}{t\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1}\left(\frac{t\mu\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t^2\mu^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{t\mu\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t^2\mu^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}\left(\frac{t\delta\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t^2\delta^2\mu^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z}\left(\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1} \\
&\quad\times\left(\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1}\rho_1^{2\mu+2k+2z+2l-1}\rho_2^{2\mu+2k+2z+2w-1} \\
&\quad\times\exp\left\{-\frac{\rho_1^2t\mu(1+\kappa_1)}{(1-\delta^2)}\right\}\exp\left\{-\frac{\rho_2^2t\mu(1+\kappa_2)}{(1-\delta^2)}\right\}dtd\rho_1d\rho_2.
\end{aligned}
$$

If we interchange the integrals then using (2.20), the jcdf is

$$
\begin{aligned}
F_{P_1,P_2}(\rho_1,\rho_2) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty W(t) \frac{\mu^2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \exp\left\{\frac{-\mu t}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{1-\delta^2}{t\mu(1+\kappa_1)}\right)^{l+k+z+\mu-1} \\
&\times \left(\frac{1-\delta^2}{t\mu(1+\kappa_2)}\right)^{w+k+z+\mu-1}\left(\frac{t\mu\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times \left(\frac{t^2\mu^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l\left(\frac{t\mu\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times \left(\frac{t^2\mu^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w\left(\frac{t\delta\mu\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\times \left(\frac{t^2\delta^2\mu^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z\left(\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+z+\mu-1} \\
&\times \left(\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+z+\mu-1}\left[\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right]^{-(\mu+k+z+l)}\left[\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right]^{-(\mu+k+z+w)} \\
&\gamma\left(\mu+k+z+l,\frac{t\mu(1+\kappa_1)\rho_1^2}{(1-\delta^2)}\right)\gamma\left(\mu+k+z+w,\frac{t\mu(1+\kappa_2)\rho_2^2}{(1-\delta^2)}\right)dt.
\end{aligned}
$$

Further simplification results in

$$\begin{aligned}
F_{P_1,P_2}(\rho_1,\rho_2) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty W(t) \frac{t^{k+l+w}\mu^{k+z+l+w-4}\Gamma(\mu-1)\delta^{k+2z}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^{2z+k}}{\left(1-\delta^2\right)^{2k+2z+l+w-1}} \\
&\times \frac{\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^{2l+k}(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\
&\times \gamma\left(\mu+k+z+l, \frac{t\mu(1+\kappa_1)\rho_1^2}{\left(1-\delta^2\right)}\right)\gamma\left(\mu+k+z+w, \frac{t\mu(1+\kappa_2)\rho_2^2}{\left(1-\delta^2\right)}\right)dt \\
&= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty W(t) \frac{t^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^\mu\delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \exp\left\{\frac{-t\mu}{\left(1-\delta^2\right)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{\left(1-\delta^2\right)}\right)^{l+\frac{k}{2}} \\
&\times \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{\left(1-\delta^2\right)}\right)^{w+\frac{k}{2}}\gamma\left(l+k+z+\mu, \frac{t\mu\rho_1^2(1+\kappa_1)}{\left(1-\delta^2\right)}\right) \\
&\times \gamma\left(w+k+z+\mu, \frac{t\mu\rho_2^2(1+\kappa_2)}{\left(1-\delta^2\right)}\right)dt. \qquad (4.40)
\end{aligned}$$

#### 4.3.3.2 Moments

The joint arbitrary moments can be found using the jpdf (4.31) as

$$\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] &= \int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2} f_{\rho_1,\rho_2}(\rho_1,\rho_2)d\rho_1 d\rho_2 \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2} W(t)\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \exp\left\{\frac{-\mu t}{(1-\delta^2)}\left[\rho_1^2(1+\kappa_1)+\rho_2^2(1+\kappa_2)+\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\
&\times \sum_{k=0}^\infty \frac{(\mu+k-1)(2\mu-2)_k}{k!} I_{\mu-1+k}\left(\frac{2t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right) \\
&\times I_{\mu-1+k}\left(\frac{2t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right) \\
&\times I_{\mu-1+k}\left(\frac{2t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right) dt d\rho_1 d\rho_2 \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \rho_1^{n_1}\rho_2^{n_2}\sum_{z=0}^\infty\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{w=0}^\infty W(t)\frac{4\mu^2\rho_1\rho_2\Gamma(\mu-1)\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)}{t^{\mu-3}\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}} \\
&\times \frac{(\mu+k-1)(2\mu-2)_k}{k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\times \exp\left\{\frac{-\mu t}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}\exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\} \\
&\times \left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^l \\
&\times \left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^w \\
&\times \left(\frac{\delta t\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{\delta^2t^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^z dt d\rho_1 d\rho_2.
\end{aligned}$$

Following a method similar to the derivation of the jcdf the joint arbitrary moments are obtained. Using (2.13), and interchange the summation and integral, it follows that

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{W(t)\,4\mu^2\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k\left(1-\delta^2\right)^{2\mu-3}}{t^{\mu-3}k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad\times\frac{(1+\kappa_1)(1+\kappa_2)\exp\left\{\frac{-\mu t}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\times\rho_1^{n_1+1}\rho_2^{n_2+1} \\
&\quad\times\exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\}\exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{\delta t\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
&\quad\times\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z}d\rho_1 d\rho_2 dt \\
&= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{W(t)\,4\mu^2\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k\left(1-\delta^2\right)^{2\mu-3}}{t^{\mu-3}k!z!w!l!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad\times\frac{(1+\kappa_1)(1+\kappa_2)\exp\left\{\frac{-\mu t}{(1-\delta^2)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\times\exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{(1-\delta^2)}\right\} \\
&\quad\times\rho_1^{n_1+1}\rho_2^{n_2+1}\exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{(1-\delta^2)}\right\}\left(\frac{t\mu\rho_1^2(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu+\frac{n_1}{2}-1} \\
&\quad\times\left(\frac{1-\delta^2}{t\mu\rho_1^2(1+\kappa_1)}\right)^{l+k+z+\mu+\frac{n_1}{2}-1}\left(\frac{t\mu\rho_2^2(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} \\
&\quad\times\left(\frac{1-\delta^2}{t\mu\rho_2^2(1+\kappa_2)}\right)^{w+k+z+\mu+\frac{n_2}{2}-1}\left(\frac{\delta t\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{\mu-1+k}\left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{\mu-1+k} \\
&\quad\times\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w}\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l} \\
&\quad\times\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z}d\rho_1 d\rho_2 dt.
\end{aligned}
$$

Further simplification results in

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{W(t)\,\Gamma(\mu-1)\,(\mu+k-1)\,(2\mu-2)_k}{t^{\mu-3}k!z!w!l!\Gamma(\mu+k+l)\,\Gamma(\mu+k+w)\,\Gamma(\mu+k+z)} \\
&\times\frac{4\mu^2\left(1-\delta^2\right)^{2\mu-3}(1+\kappa_1)(1+\kappa_2)\,\rho_1^{n_1+1}\rho_2^{n_2+1}}{\left[\delta\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)\right]^{\mu-1}}\times\exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\} \\
&\times\left(\frac{1-\delta^2}{t\mu\rho_1^2(1+\kappa_1)}\right)^{l+k+z+\mu+\frac{n_1}{2}-1}\left(\frac{1-\delta^2}{t\mu\rho_2^2(1+\kappa_2)}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} \\
&\times\left(\frac{t\delta\mu\rho_1\rho_2\sqrt{(1+\kappa_1)(1+\kappa_2)}}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{t\mu\rho_1\sqrt{1+\kappa_1}\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k} \\
&\times\left(\frac{t\mu\rho_2\sqrt{1+\kappa_2}\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{\left(1-\delta^2\right)}\right)^{\mu-1+k}\left(\frac{t^2\mu^2\rho_2^2(1+\kappa_2)\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w} \\
&\times\left(\frac{t^2\mu^2\rho_1^2(1+\kappa_1)\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{(1-\delta^2)^2}\right)^{l}\left(\frac{t^2\delta^2\mu^2\rho_1^2\rho_2^2(1+\kappa_1)(1+\kappa_2)}{(1-\delta^2)^2}\right)^{z} \\
&\times\exp\left\{\frac{-t\mu\rho_1^2(1+\kappa_1)}{\left(1-\delta^2\right)}\right\}\left(\frac{t\mu\rho_1^2(1+\kappa_1)}{1-\delta^2}\right)^{l+k+z+\mu+\frac{n_1}{2}-1} \\
&\times\exp\left\{\frac{-t\mu\rho_2^2(1+\kappa_2)}{\left(1-\delta^2\right)}\right\}\left(\frac{t\mu\rho_2^2(1+\kappa_2)}{1-\delta^2}\right)^{w+k+z+\mu+\frac{n_2}{2}-1} d\rho_1 d\rho_2 dt.
\end{aligned}
$$

Simplifying the above expression and using (2.1) the following expression is obtained

$$
\begin{aligned}
E[P_1^{n_1}P_2^{n_2}] &= \int_0^{\infty}\sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\frac{W(t)\,\Gamma(\mu-1)\,(\mu+k-1)\,(2\mu-2)_k\left(1-\delta^2\right)^{\mu}\delta^{k+2z}}{k!z!w!l!\Gamma(\mu+k+l)\,\Gamma(\mu+k+w)\,\Gamma(\mu+k+z)} \\
&\times\exp\left\{\frac{-\mu t}{\left(1-\delta^2\right)}\left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right]\right\}t^{2\mu+2l+2z+2w+3k-1} \\
&\times\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)^2}{\left(1-\delta^2\right)}\right)^{l+\frac{k}{2}}\left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)^2}{(1-\delta^2)^2}\right)^{w+\frac{k}{2}} \\
&\times\left(\frac{\mu(1+\kappa_1)}{\left(1-\delta^2\right)}\right)^{-\frac{n_1}{2}}\left(\frac{\mu(1+\kappa_2)}{\left(1-\delta^2\right)}\right)^{-\frac{n_2}{2}} \\
&\times\Gamma\left(l+k+z+\mu+\frac{n_1}{2}\right)\Gamma\left(w+k+z+\mu+\frac{n_2}{2}\right)dt. \qquad (4.41)
\end{aligned}
$$

### 4.3.4 Outage Probability

As before (see Section (4.2.4)), the outage probability is defined, using (4.40), as

$$\begin{aligned} F_{SC}(\vartheta) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty W(t) \frac{t^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\ &\times \exp\left\{\frac{-t\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\ &\times \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right) \\ &\times\gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right) dt. \end{aligned} \tag{4.42}$$

#### 4.3.4.1 Special cases

**Normal case:**
Assume the normal distribution weight function (2.35) is substituted in the outage probability function (4.42), then the outage probability (4.42) becomes,

$$\begin{aligned} F_{SC}(\vartheta) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \delta(t-1) \frac{t^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\ &\times \exp\left\{\frac{-t\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\ &\times \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right) \\ &\times\gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right) dt. \end{aligned}$$

Consider the result,

$$\begin{aligned} F_{SC}(\vartheta, x) &= \int_0^\infty \delta(x) F_{SC}(\vartheta, x+1)\, dx \\ &= F_{SC}(\vartheta, 1). \end{aligned}$$

Let $q=t-1$ then $t=q+1$, we get that the outage probability is

$$
\begin{aligned}
F_{SC}(\vartheta,t) &= F_{SC}(\vartheta,q+1) \\
&= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \delta(q) \frac{(q+1)^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \exp\left\{\frac{-(q+1)\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\
&\quad \times \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{(q+1)\mu(1+\kappa_1)}{(1-\delta^2)}\right) \\
&\quad \times \gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{(q+1)\mu(1+\kappa_2)}{(1-\delta^2)}\right) dq \\
&= F_{SC}(\vartheta,1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F_{SC}(\vartheta) &= \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \frac{\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \exp\left\{\frac{-\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \\
&\quad \times \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right) \\
&\quad \times \gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right). \qquad (4.43)
\end{aligned}
$$

Note that (4.43) is obtained by substituting the normal distribution weight function in the elliptical model yields the same result as the case when the underlying distribution is assumed to be normal (4.18).

**t-distribution case:**
Assume the t- distribution (with $v$ degrees of freedom), therefore the weight function (2.36) is substituted in the outage probability function (4.42). Then the outage probability (4.42) becomes,

$$\begin{aligned} F_{SC}(\vartheta) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \frac{t^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\ &\times \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)} \exp\left\{\frac{-t\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\} \\ &\times \left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \\ &\times \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right)\gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right) dt \\ &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \frac{v^{\frac{v}{2}}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{2^{\frac{v}{2}}k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)\Gamma\left(\frac{v}{2}\right)} \\ &\times \left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \\ &\times t^{k+l+w+\frac{v}{2}-1}\exp\left\{-t\left[\frac{v}{2}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right]\right\} \\ &\times \gamma\left(l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1}\frac{t\mu(1+\kappa_1)}{(1-\delta^2)}\right)\gamma\left(w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2}\frac{t\mu(1+\kappa_2)}{(1-\delta^2)}\right) dt. \end{aligned} \tag{4.44}$$

Evaluating the integral in equation (4.44) using (2.6) and (2.21), we obtain

$$\begin{aligned} &\int_0^\infty t^{k+l+w+\frac{v}{2}-1}\exp\left\{-t\left[\frac{v}{2}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right]\right\} \\ &\times \gamma\left(l+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)\gamma\left(w+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right) dt \\ = &\int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n \left(t\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!(w+k+\mu+n)} t^{k+l+w+\frac{v}{2}-1} \\ &\times \exp\left\{-t\left[\frac{v}{2}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right]\right\}\gamma\left(l+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right) dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\vartheta}{\vartheta_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \int_0^{\infty} t^{2w+2k+l+\mu+z+n+\frac{v}{2}-1} \\
&\quad \times \exp\left\{-t\left[\frac{v}{2} - \frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right]\right\} \gamma\left(l+k+z+\mu, t\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right) dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\vartheta}{\vartheta_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \frac{\Gamma\left(n+\frac{v}{2}+2l+2w+3k+3\mu+3z\right)}{(l+k+\mu+z)} \\
&\quad \times \frac{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+\mu+z}}{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)} + \frac{v}{2} - \frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right)^{n+\frac{v}{2}+2l+2w+3k+3\mu+3z}} \\
&\quad \times {}_2F_1\left(1, n+\frac{v}{2}+2l+2w+3k+3\mu+3z; l+k+\mu+z+1; \tau\right),
\end{aligned}
$$

where $\tau = \frac{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)}{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}+\frac{v}{2}-\frac{\mu}{(1-\delta^2)}(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2})\right)}$. Thus, the outage probability is

$$
\begin{aligned}
F_{SC}(\vartheta) &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty} \frac{v^{\frac{v}{2}}\Gamma(\mu-1)(\mu+k-1)\,(2\mu-2)_k\,(1-\delta^2)^{\mu}\delta^{k+2z}}{2^{\frac{v}{2}}k!z!l!w!\Gamma(\mu+k+l)\,\Gamma(\mu+k+w)\,\Gamma(\mu+k+z)\,\Gamma\left(\frac{v}{2}\right)} \\
&\quad \times \left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{\vartheta}{\vartheta_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \times \frac{\Gamma\left(\frac{v}{2}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right)}{(l+k+\mu+z)} \\
&\quad \times \frac{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+\mu+z}}{\left(\frac{\vartheta}{\vartheta_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)} + \frac{v}{2} - \frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right)^{n+\frac{v}{2}+2l+2w+3k+3\mu+3z}} \\
&\quad \times {}_2F_1\left(1, n+\frac{v}{2}+2l+2w+3k+3\mu+3z; l+k+\mu+z+1; \tau\right),
\end{aligned}
\tag{4.45}
$$

where $\frac{v}{2} > \frac{\mu(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2})}{(1-\delta^2)}$.

## 4.4 Weight approximation function

An expansion for the jcdf (4.40) is given, based on the assumption that the weighting function admits the expansion. For the weight function (2.37), the outage probability (4.42) becomes,

$$
\begin{aligned}
F_{SC}(\vartheta) &= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \left( \sum_{j=1}^\infty a_j t^j \right) \frac{t^{k+l+w}\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \exp\left\{ \frac{-t\mu}{(1-\delta^2)} \left[\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right] \right\} \left( \frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)} \right)^{l+\frac{k}{2}} \\
&\quad \times \left( \frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)} \right)^{w+\frac{k}{2}} \gamma\left( l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1} \frac{t\mu(1+\kappa_1)}{(1-\delta^2)} \right) \\
&\quad \times \gamma\left( w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_{21}} \frac{t\mu(1+\kappa_2)}{(1-\delta^2)} \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{z=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{w=0}^\infty \sum_{j=1}^\infty \frac{a_j\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k (1-\delta^2)^\mu \delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \left( \frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)} \right)^{l+\frac{k}{2}} \left( \frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)} \right)^{w+\frac{k}{2}} \\
&\quad \times t^{k+l+w+j} \exp\left\{ \frac{-t\mu}{(1-\delta^2)} \left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right) \right\} \\
&\quad \times \gamma\left( l+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_1} \frac{t\mu(1+\kappa_1)}{(1-\delta^2)} \right) \gamma\left( w+k+z+\mu, \frac{\vartheta}{\bar{\vartheta}_2} \frac{t\mu(1+\kappa_2)}{(1-\delta^2)} \right) dt
\end{aligned} \tag{4.46}
$$

Evaluating the integral in equation (4.46) using (2.6) and (2.21), we obtain

$$
\begin{aligned}
&\int_0^\infty t^{k+l+w+j} \exp\left\{ -\frac{t\mu}{(1-\delta^2)} \left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right) \right\} \\
&\times \gamma\left( l+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_1} \frac{\mu(1+\kappa_1)}{(1-\delta^2)} \right) \gamma\left( w+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_2} \frac{\mu(1+\kappa_2)}{(1-\delta^2)} \right) dt \\
= &\int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n \left( t\frac{\vartheta}{\bar{\vartheta}_2} \frac{\mu(1+\kappa_2)}{(1-\delta^2)} \right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+n)} t^{k+l+w+j} \\
&\times \exp\left\{ -\frac{t\mu}{(1-\delta^2)} \left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right) \right\} \gamma\left( l+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_1} \frac{\mu(1+\kappa_1)}{(1-\delta^2)} \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \int_0^{\infty} t^{2k+2w+\mu+l+z+j+n} \\
&\quad \times \exp\left\{-\frac{t\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right\}\gamma\left(l+k+z+\mu, t\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \frac{\Gamma(3k+2w+2\mu+2l+2z+j+n+1)}{(l+k+\mu+z)} \\
&\quad \times \frac{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+\mu+z}}{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right)^{3k+2w+2\mu+2l+2z+j+n+1}} \\
&\quad \times {}_2F_1\left(1, 3k+2w+2\mu+2l+2z+j+n+1; l+k+\mu+z+1; \pi\right),
\end{aligned}
$$

where $\pi = \frac{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)}{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}-\frac{\mu}{(1-\delta^2)}(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2})\right)}$. Thus, the outage probability is

$$
\begin{aligned}
F_{SC}(\vartheta) &= \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\sum_{j=1}^{\infty} \frac{a_j\Gamma(\mu-1)(\mu+k-1)(2\mu-2)_k(1-\delta^2)^{\mu}\delta^{k+2z}}{k!z!l!w!\Gamma(\mu+k+l)\Gamma(\mu+k+w)\Gamma(\mu+k+z)} \\
&\quad \times \left(\frac{\mu\left(\sqrt{\kappa_1}-\delta\sqrt{\kappa_2}\right)}{(1-\delta^2)}\right)^{l+\frac{k}{2}} \left(\frac{\mu\left(\sqrt{\kappa_2}-\delta\sqrt{\kappa_1}\right)}{(1-\delta^2)}\right)^{w+\frac{k}{2}} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\vartheta}{\bar{\vartheta}_2}\frac{\mu(1+\kappa_2)}{(1-\delta^2)}\right)^{w+k+\mu+z+n}}{n!\,(w+k+\mu+z+n)} \frac{\Gamma(3k+2w+2\mu+2l+2z+j+n+1)}{(l+k+\mu+z)} \\
&\quad \times \frac{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}\right)^{l+k+\mu+z}}{\left(\frac{\vartheta}{\bar{\vartheta}_1}\frac{\mu(1+\kappa_1)}{(1-\delta^2)}-\frac{\mu}{(1-\delta^2)}\left(\kappa_1+\kappa_2-2\delta\sqrt{\kappa_1\kappa_2}\right)\right)^{3k+2w+2\mu+2l+2z+j+n+1}} \\
&\quad \times {}_2F_1\left(1, 3k+2w+2\mu+2l+2z+j+n+1; l+k+\mu+z+1; \pi\right),
\end{aligned} \tag{4.47}
$$

where $|\pi| < 1$.

## 4.5 Performance Measures

Following a similar approach as Villavicencio, et al.,(2016) [33], the bivariate $\kappa-\mu$ *type* model is introduced. Graphical displays of (4.31) for special cases ((2.35) and (2.36)) are shown below for $\mu = 0.5$ and $\kappa_1 = \kappa_2 = 5$, on left and $\mu = 2$ and $\kappa_1 = \kappa_2 = 1$, on right.

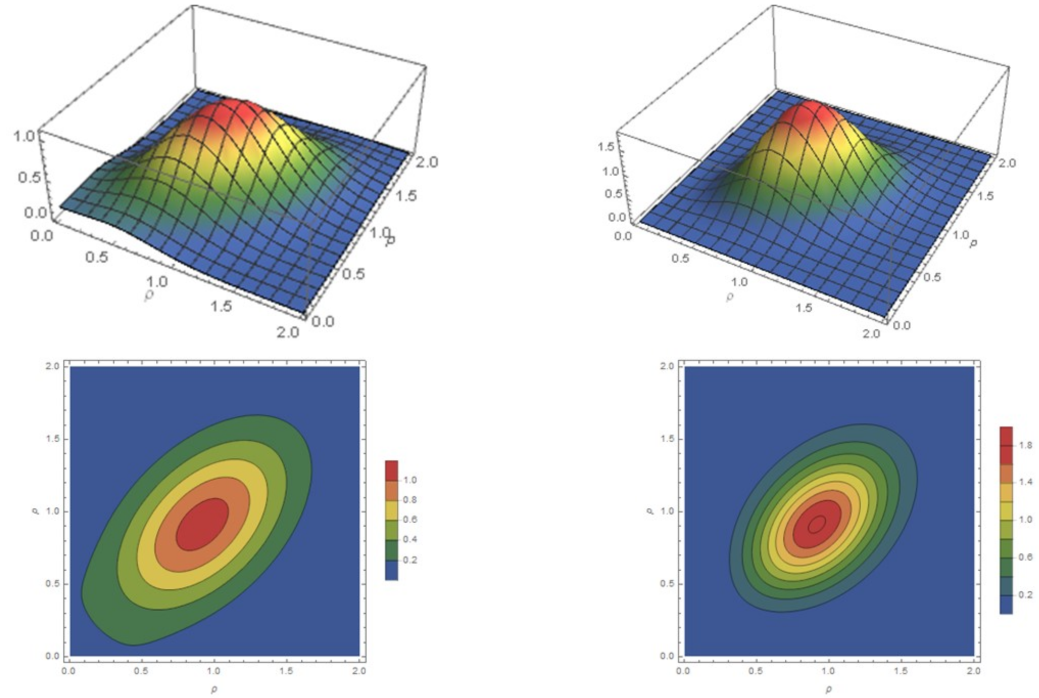


Figure 4.5.1 displays the normal case.

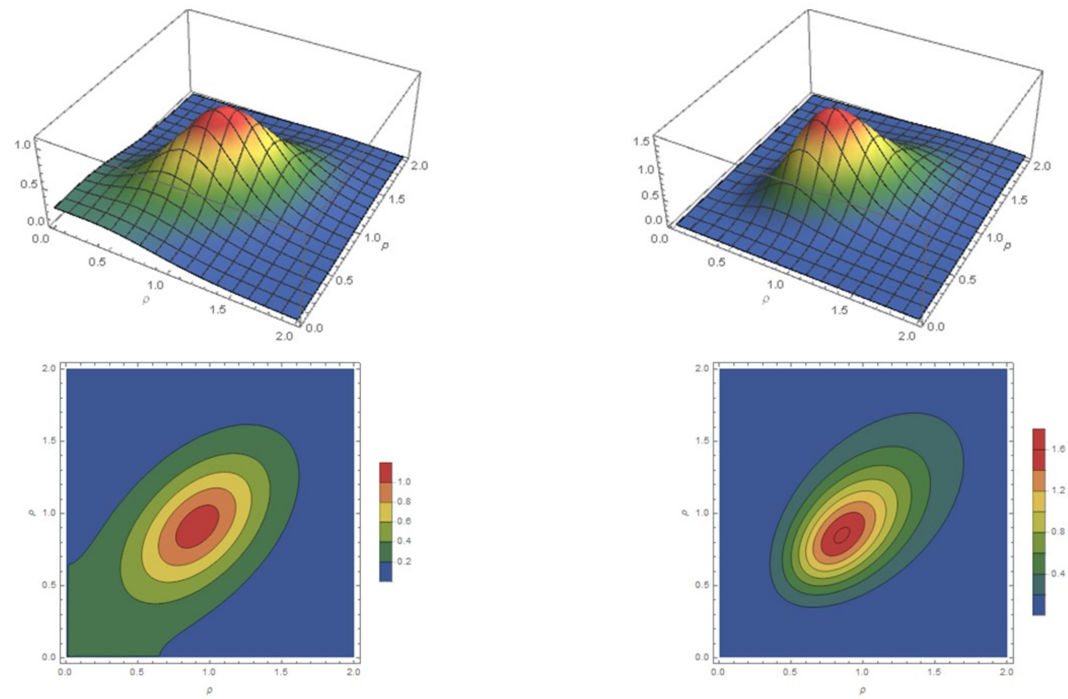


Figure 4.5.2 shows the t-distribution case, where $v = 3$ degrees of freedom were considered.

The following figures illustrate the impact of the bivariate $\kappa - \mu$ -type distribution in the context of outage probability where the two cases of the elliptical model, namely normal distribution (2.35) and t-distribution (2.36) is considered. The effect of assuming the t-distribution as the underlying model is visible in Figure 4.5.3 and Figure 4.5.4 where the t-distribution outperforms the normal distribution. In the study of [11] it was shown that superior performance in terms of increased capacity of the communication system is observed when considering other distributions than the usual normal assumption.

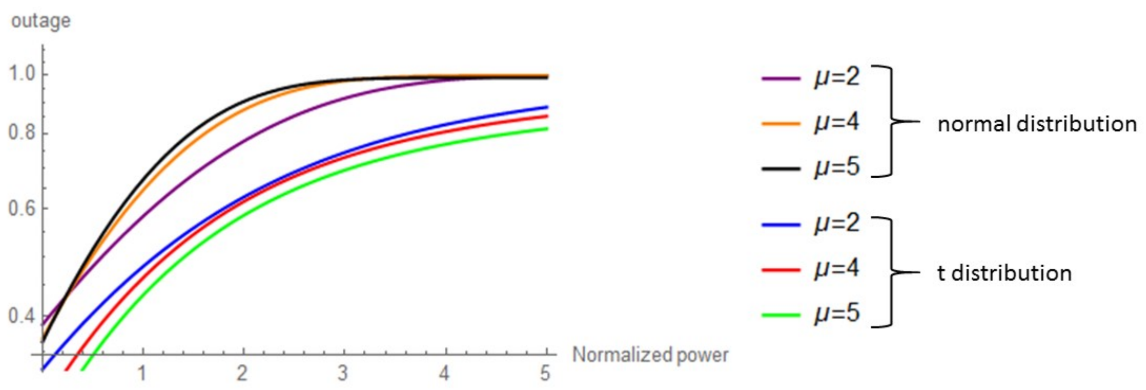



Figure 4.5.3 shows the outage probability for $\kappa_1 = \kappa_2 = 2$ and $\delta = 0.5$ for the normal case and t-distribution with $v = 3$ degrees of freedom.

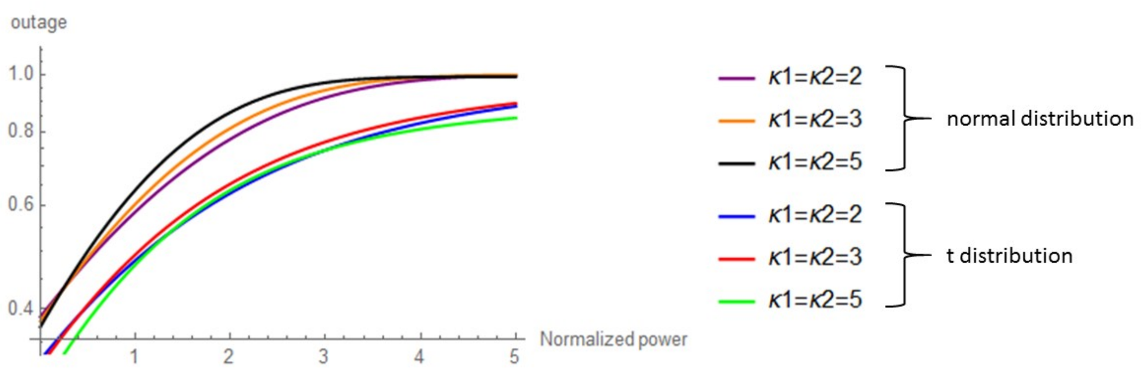



Figure 4.5.4 shows the outage probability for $\mu = 2$ and $\delta = 0.5$ for the normal case and t-distribution with $v = 3$ degrees of freedom.

# Chapter 5

# Composite models

## 5.1 Introduction

Composite models are used to account for the simultaneous presence of fading and shadowing (see Section 1.2.2). Shadowing can be incorporated in multipath fading models in various ways. The $\kappa - \mu$ shadowed fading model proposed by Paris (2014) [23] assumes the $\kappa - \mu$ distribution for the multipath fading and the Nakagami-$m$ distribution for the shadowing. Right after Paris (2014) [23] in an independent work Cotton (2015) [7] proposed the same model. Figure 5.1 illustrates the physical model that motivates the development by Cotton (2015) [7] where shadowing events are caused not only by local environment but also by the user's body. The $\kappa - \mu$ shadowed fading distribution relies on a generalization of the physical model for the $\kappa - \mu$ distribution presented by Yacoub (2007) [35]. The $\kappa - \mu$ shadowed fading distribution has been derived assuming that the signals are structured in clusters of waves propagating in a nonhomogeneous environment. The $\kappa - \mu$ model Yacoub (2007) [35] assumes a fixed dominant component within each cluster whereas, the $\kappa - \mu$ shadowed model assumes that the dominant components of all the clusters can vary randomly as a result of shadowing.

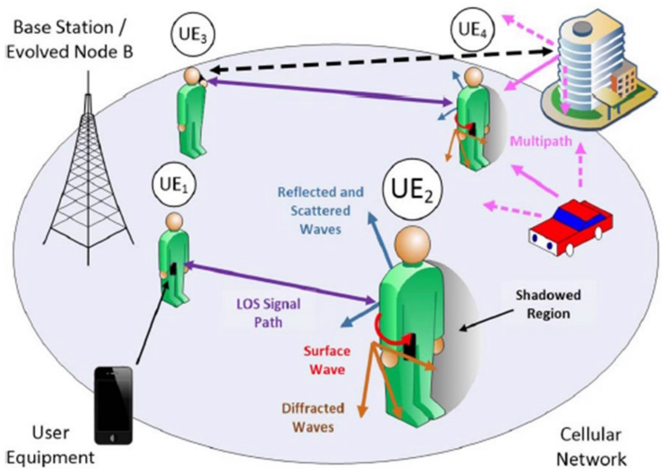



Figure 5.1: Illustration of structures associated with shadowing [7]

This chapter is organized as follows. In Section 5.2 the $\kappa-\mu$ shadowed fading model is characterised in terms of measurable physical parameters. The pdf of the SNR and the statistical properties of the $\kappa-\mu$ shadowed fading model will receive attention in Sections 5.2 and 5.3, respectively. The $\kappa-\mu$ type shadowed fading model that emanates from the elliptical assumption will be derived in Section 5.3; followed by Section 5.4 where the outage probability under the different members of the elliptical model will be compared.

## 5.2 The $\kappa-\mu$ composite distribution

### 5.2.1 Description

Let $X_i$ and $Y_i$ be mutually independent normal processes with $E(X_i)=E(Y_i)=0$ and $var(X_i)=var(Y_i)=\sigma^2$. Then the envelope, $R$, of the physical model for the $\kappa-\mu$ distribution can be written in terms of the in-phase and quadrature components of the fading signal as

$$W=R^2=\sum_{i=1}^{n}(X_i+\xi p_i)^2+\sum_{i=1}^{n}(Y_i+\xi q_i)^2, \tag{5.1}$$

where $\xi p_i$ and $\xi q_i$ are the mean values of the in-phase and quadrature components of the multipath waves of cluster $i$ and $n$ is the number of clusters of multipath, i.e.

$$W_i=R_i=(X_i+\xi p_i)^2+(Y_i+\xi q_i)^2,\ i=1,2,...,n.$$

Define $\xi^2\delta^2=\sum_{i=1}^{n}\left(\xi^2p_i^2+\xi^2q_i^2\right)$ to represent the mean power of the dominant component where we assume $\xi$ to be a Nakagami-$m$ random variable with shaping parameter $m$ and $E\left(\xi^2\right)=\bar{\xi}$. For given $\xi$ let $X_i\sim N(0,\sigma^2)$ and $Y_i\sim N(0,\sigma^2)$, thus $(X_i+\xi p_i)\sim N(\xi p_i,\sigma^2)$ and $X_i^*=\left(\frac{X_i+\xi p_i}{\sigma}\right)\sim N\left(\frac{\xi p_i}{\sigma},1\right)$. Therefore $(X_i^*)^2$ follows a $\chi^2_{\frac{\xi^2p_i^2}{\sigma^2}}(1)$ with non-centrality parameter $\frac{\xi^2p_i^2}{\sigma^2}$. Similarly, for given $\xi$ let $Y_i^*=\left(\frac{Y_i+\xi q_i}{\sigma}\right)$ and $(Y_i^*)^2$ follows a $\chi^2_{\frac{\xi^2q_i^2}{\sigma^2}}(1)$ with non-centrality parameter $\frac{\xi^2q_i^2}{\sigma^2}$. As a result for a given $\xi$, $\frac{R_i^2}{\sigma^2}$ follows a $\chi^2_{\frac{\xi^2\delta_i^2}{\sigma^2}}(2)$ with non-centrality parameter $\frac{\xi^2\delta_i^2}{\sigma^2}=\frac{\xi^2p_i^2+\xi^2q_i^2}{\sigma^2}$. The focus is to derive the pdf of the power of the fading signal.

### 5.2.2 Derivation

For the given model (5.1), the pdf of the power, given the shadowing amplitude $\xi$, follows a $\kappa-\mu$ distribution with pdf (see equation (3.5))

$$f_{W|\xi}(w|\xi)=\frac{1}{2\sigma^2}\left(\frac{w}{\xi^2\delta^2}\right)^{\frac{n-1}{2}}\exp\left\{-\frac{(w+\xi^2\delta^2)}{2\sigma^2}\right\}I_{n-1}\left(\frac{\xi\delta}{\sigma^2}\sqrt{w}\right), \tag{5.2}$$

where $w>0$.

From (5.2) and (2.32) the unconditional pdf of the power is then obtained as follows

$$\begin{aligned}
f_W(w) &= \int_0^\infty f_{W|\xi}(w|\xi)\, f_\xi(\xi)\, d\xi \\
&= \int_0^\infty \frac{1}{2\sigma^2}\left(\frac{w}{\xi^2\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{\left(w+\xi^2\delta^2\right)}{2\sigma^2}\right\} I_{n-1}\left(\frac{\xi\delta}{\sigma^2}\sqrt{w}\right) \\
&\quad \times \frac{2m^m}{\theta^m\Gamma(m)}\xi^{2m-1}\exp\left\{-\frac{m\xi^2}{\theta}\right\} d\xi \\
&= \frac{m^m}{\sigma^2\theta^m\Gamma(m)}\left(\frac{w}{\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{w}{2\sigma^2}\right\} \int_0^\infty \xi^{2m-n} \\
&\quad \exp\left\{-\xi^2\left(\frac{\delta^2}{2\sigma^2}+\frac{m}{\theta}\right)\right\} I_{n-1}\left(\frac{\xi\delta}{\sigma^2}\sqrt{w}\right) d\xi.
\end{aligned}$$

Using (2.22), the unconditional pdf of the power is

$$\begin{aligned}
f_W(w) &= \frac{m^m}{\sigma^2\theta^m\Gamma(m)}\left(\frac{w}{\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{w}{2\sigma^2}\right\} 2^{-n+1-1}\left(\frac{\delta\sqrt{w}}{\sigma^2}\right)^{n-1} \\
&\quad \times \left(\frac{\delta^2}{2\sigma^2}+\frac{m}{\theta}\right)^{-\frac{(2m-n+1+n-1)}{2}} \Gamma\left[\frac{\frac{(2m-n+1+n-1)}{2}}{n-1+1}\right] \\
&\quad \times {}_1F_1\left(\frac{(2m-n+1+n-1)}{2}; n-1+1; \frac{\left(\frac{\delta\sqrt{w}}{\sigma^2}\right)^2}{4\left(\frac{\delta^2}{2\sigma^2}+\frac{m}{\theta}\right)}\right) \\
&= \frac{m^m\Gamma(m)\, w^{n-1}}{\left(2\sigma^2\right)^n\theta^m\Gamma(m)\Gamma(n)\left(\frac{\delta^2}{2\sigma^2}+\frac{m}{\theta}\right)^m} \exp\left\{-\frac{w}{2\sigma^2}\right\} \\
&\quad \times {}_1F_1\left(m\ ;\ n\ ;\ \frac{\delta^2 w}{4\sigma^4\left(\frac{\delta^2}{2\sigma^2}+\frac{m}{\theta}\right)}\right). \qquad (5.3)
\end{aligned}$$

As before, define the parameter $\kappa = \frac{\delta^2}{2\sigma^2\mu}$ to be the ratio between the total power of the dominant components and the total power of the scattered waves (see remark 3.3). Replace $n$ with $\mu$ and let $\Omega$ represent the normalized power for the fading channel then $\Omega = \frac{W}{\bar{w}}$, where $\bar{w} = E(W) = \delta^2 + 2\sigma^2\mu = 2\sigma^2\mu(1+\kappa)$ (see (3.8)). Thus, the pdf (5.3) can be written in terms of $\kappa$ and $\mu$ as follows

$$\begin{aligned}
f_{\Omega}(\omega) &= \frac{m^m\Gamma(m)(\omega\bar{w})^{\mu-1}}{(2\sigma^2)^{\mu}\theta^m\Gamma(m)\Gamma(\mu)\left(\frac{(2\sigma^2\mu\kappa)}{2\sigma^2}+\frac{m}{\theta}\right)^m}\exp\left\{-\frac{(\omega\bar{w})}{2\sigma^2}\right\} \\
&\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{(2\sigma^2\mu\kappa)(\omega\bar{w})}{4\sigma^4\left(\frac{(2\sigma^2\mu\kappa)}{2\sigma^2}+\frac{m}{\theta}\right)}\right)\bar{w} \\
&= \frac{m^m\Gamma(m)(\omega(2\sigma^2\mu(1+\kappa)))^{\mu-1}(2\sigma^2\mu(1+\kappa))}{(2\sigma^2)^{\mu}\theta^m\Gamma(m)\Gamma(\mu)\left(\frac{(2\sigma^2\mu\kappa)}{2\sigma^2}+\frac{m}{\theta}\right)^m}\exp\left\{-\frac{(\omega(2\sigma^2\mu(1+\kappa)))}{2\sigma^2}\right\} \\
&\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{(2\sigma^2\mu\kappa)(\omega(2\sigma^2\mu(1+\kappa)))}{4\sigma^4\left(\frac{(2\sigma^2\mu\kappa)}{2\sigma^2}+\frac{m}{\theta}\right)}\right) \\
&= \frac{m^m\mu^{\mu}(1+\kappa)^{\mu}\omega^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\exp\{-\mu(1+\kappa)\omega\}\ {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right).
\end{aligned} \tag{5.4}$$

### 5.2.3 Characteristics

#### 5.2.3.1 Cumulative distribution function

The cdf of $\Omega$ is as follows

$$\begin{aligned}
F_{\Omega}(Z_T) &= \int_0^{Z_T}\frac{m^m\mu^{\mu}(1+\kappa)^{\mu}\omega^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\exp\{-\mu(1+\kappa)\omega\}\ {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)d\omega \\
&= \frac{m^m\mu^{\mu}(1+\kappa)^{\mu}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\int_0^{Z_T}\omega^{\mu-1}\exp\{-\mu(1+\kappa)\omega\}\ {}_1F_1\left(m;\mu;\frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)d\omega.
\end{aligned}$$

Using (2.10) the confluent hypergeometric function can be expressed as an infinite series. Thus, the cdf can be written as

$$\begin{aligned}
F_{\Omega}(Z_T) &= \frac{m^m\mu^{\mu}(1+\kappa)^{\mu}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\int_0^{Z_T}\omega^{\mu-1}\exp\{-\mu(1+\kappa)\omega\} \\
&\quad\times\sum_{l=0}^{\infty}\frac{(m)_l}{l!(\mu)_l}\left(\frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l d\omega \\
&= \sum_{l=0}^{\infty}\frac{m^m\mu^{\mu}(1+\kappa)^{\mu}(m)_l}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m l!(\mu)_l}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l \\
&\quad\times\int_0^{Z_T}\omega^{\mu+l-1}\exp\{-\mu(1+\kappa)\omega\}d\omega.
\end{aligned}$$

Using (2.20), the cdf becomes

$$\begin{aligned} F_{\Omega}(Z_T) &= \sum_{l=0}^{\infty} \frac{m^m \mu^{\mu} (1+\kappa)^{\mu} (m)_l}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^m l! (\mu)_l} \left( \frac{\mu^2 \kappa (1+\kappa)}{\left(\mu\kappa + \frac{m}{\theta}\right)} \right)^l \\ &\quad \times \left[\mu(1+\kappa)\right]^{-(\mu+l)} \gamma\left(\mu+l, \mu(1+\kappa) Z_T\right) \\ &= \sum_{l=0}^{\infty} \frac{m^m \mu^l \kappa^l (m)_l}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^{m+l} l! (\mu)_l} \gamma\left(\mu+l, \mu(1+\kappa) Z_T\right). \end{aligned} \tag{5.5}$$

#### 5.2.3.2 Moments

From (5.4) the $j^{th}$ moment of the normalized power, $\Omega$, is

$$\begin{aligned} E\left(\Omega^j\right) &= \int_0^{\infty} \omega^j f_{\Omega}(\omega)\, d\gamma \\ &= \int_0^{\infty} \omega^j \frac{m^m \mu^{\mu} (1+\kappa)^{\mu} \omega^{\mu-1}}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^m} \exp\left\{-\mu(1+\kappa)\omega\right\} \\ &\quad \times {}_1F_1\left(m; \mu+; \frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa + \frac{m}{\theta}\right)}\right) d\omega \\ &= \frac{m^m \mu^{\mu} (1+\kappa)^{\mu}}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^m} \int_0^{\infty} \omega^{\mu+j-1} \exp\left\{-\mu(1+\kappa)\omega\right\} \\ &\quad \times {}_1F_1\left(m; \mu; \frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa + \frac{m}{\theta}\right)}\right) d\omega. \end{aligned}$$

Using (2.23), the $j^{th}$ moment is

$$\begin{aligned} E\left(\Omega^j\right) &= \frac{m^m \mu^{\mu} (1+\kappa)^{\mu}}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^m} \times \frac{\Gamma(\mu+j)}{\left(\mu(1+\kappa)\right)^{\mu+j}} \\ &\quad \times {}_2F_1\left(m \,,\, \mu+j \,;\, \mu \,;\, \frac{\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)}{\left(\mu(1+\kappa)\right)}\right) \\ &= \frac{m^m \Gamma(\mu+j)}{\theta^m \Gamma(\mu) \left(\mu\kappa + \frac{m}{\theta}\right)^m \left(\mu(1+\kappa)\right)^j} \, {}_2F_1\left(m \,,\, \mu+j \,;\, \mu \,;\, \frac{\mu\kappa}{\mu\kappa + \frac{m}{\theta}}\right). \end{aligned} \tag{5.6}$$

#### 5.2.3.3 Amount of fading

The amount of fading is given as (see equation (1.1))

$$\begin{aligned} AF &= \frac{E\left(\Omega^2\right)}{\left[E\left(\Omega\right)\right]^2}-1 \\ &= \frac{\left[\frac{m^m\Gamma(\mu+2)}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^2}\ {}_2F_1\left(m\ ,\ \mu+2\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)\right]}{\left[\frac{m^m\Gamma(\mu+1)}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))}\ {}_2F_1\left(m\ ,\ \mu+1\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)\right]^2}-1 \\ &= \left(\frac{\theta^m\left(\mu+1\right)\left(\mu\kappa+\frac{m}{\theta}\right)^m\ {}_2F_1\left(m\ ,\ \mu+2\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)}{m^m\mu\left[\ {}_2F_1\left(m\ ,\ \mu+1\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)\right]^2}\right)-1. \end{aligned} \tag{5.7}$$

## 5.3 The $\kappa-\mu$ type composite model

### 5.3.1 Description

Let $X_i$ and $Y_i$ be mutually independent elliptical processes with $E\left(X_i\right)=E\left(Y_i\right)=0$ and $var\left(X_i\right)=var\left(Y_i\right)=-2\Psi'\left(0\right)$. Then the envelope, $R$, of the physical model for the $\kappa-\mu$ distribution can be written in terms of the in-phase and quadrature components of the fading signal as

$$W=R^2=\sum_{i=1}^{n}\left(X_i+\xi p_i\right)^2+\sum_{i=1}^{n}\left(Y_i+\xi q_i\right)^2, \tag{5.8}$$

where $\xi p_i$ and $\xi q_i$ are the mean values of the in-phase and quadrature components of the multipath waves of cluster $i$ and $n$ is the number of clusters of multipath, i.e.

$$W_i=R_i^2=\left(X_i+\xi p_i\right)^2+\left(Y_i+\xi q_i\right)^2,\ i=1,2,...,n.$$

Define $\xi^2\delta^2=\sum\limits_{i=1}^{n}\left(\xi^2p_i^2+\xi^2q_i^2\right)$ to represent the mean power of the dominant component where we assume $\xi$ to be a Nakagami-$m$ random variable with shaping parameter $m$ and $E\left(\xi^2\right)=\bar{\xi}$. For given $\xi$ let $X_i\sim E\left(0,\sigma^2\right)$ and $Y_i\sim E\left(0,\sigma^2\right)$, therefore from (2.28), $X_i|t\sim N\left(0,a_t\right)$ and $Y_i|t\sim N\left(0,a_t\right)$ where $a_t=t^{-1}\sigma^2$. Hence, $\left(X_i+\xi p_i\right)|\left(t,\xi\right)\sim N\left(\xi p_i,a_t\right)$ and $X_i^*|\left(t,\xi\right)=\left(\frac{X_i+\xi p_i}{\sqrt{a_t}}|\left(t,\xi\right)\right)\sim N\left(\frac{\xi p_i}{\sqrt{a_t}},1\right)$. Thus $\left(X_i^*\right)^2|\left(t,\xi\right)$ follows a $\chi^2_{\frac{\xi^2p_i^2}{a_t}}(1)$ with non-centrality parameter $\frac{\xi^2p_i^2}{a_t}$. Similarly $Y_i^*|\left(t,\xi\right)=\left(\frac{Y_i+\xi q_i}{\sqrt{a_t}}|\left(t,\xi\right)\right)\sim N\left(\frac{\xi q_i}{\sqrt{a_t}},1\right)$ and $\left(Y_i^*\right)^2|\left(t,\xi\right)$ follows a $\chi^2_{\frac{\xi^2q_i^2}{a_t}}(1)$ with non-centrality parameter $\frac{\xi^2q_i^2}{a_t}$. As a result $\left(\frac{R_i^2}{a_t}|\left(t,\xi\right)\right)$ follows a $\chi^2_{\frac{\xi_i^2\delta_i^2}{a_t}}(2)$ with non-centrality parameter $\frac{\xi_i^2\delta_i^2}{a_t}=\frac{\xi^2p_i^2+\xi^2q_i^2}{a_t}$. The focus is to derive the pdf of the power of the fading signal $W=\sum\limits_{i=1}^{n}W_i$.

### 5.3.2 Derivation

For the given model (5.8), the conditional pdf of the power, given the shadowing amplitude $\xi$, follows a $\kappa - \mu$ distribution with pdf (see equation (3.23))

$$f_{W|(t,\xi)}(w|t,\xi) = \frac{1}{2a_t}\left(\frac{w}{\xi^2\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{(w+\xi^2\delta^2)}{2a_t}\right\} I_{n-1}\left(\frac{\xi\delta}{a_t}\sqrt{w}\right), \tag{5.9}$$

where $w > 0$.
From (5.9) the conditional pdf of the power is obtained as follows

$$\begin{aligned} f_{W|t}(w|t) &= \int_0^\infty f_{W|(t,\xi)}(w|t,\xi) f_\xi(\xi)\, d\xi \\ &= \int_0^\infty \frac{1}{2a_t}\left(\frac{w}{\xi^2\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{(w+\xi^2\delta^2)}{2a_t}\right\} I_{n-1}\left(\frac{\xi\delta}{a_t}\sqrt{w}\right) \\ &\quad \times \frac{2m^m}{\theta^m\Gamma(m)}\xi^{2m-1}\exp\left\{-\frac{m\xi^2}{\theta}\right\} d\xi \\ &= \frac{m^m}{a_t\theta^m\Gamma(m)}\left(\frac{w}{\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{w}{2a_t}\right\} \int_0^\infty \xi^{2m-n} \\ &\quad \times \exp\left\{-\xi^2\left(\frac{\delta^2}{2a_t}+\frac{m}{\theta}\right)\right\} I_{n-1}\left(\frac{\xi\delta}{a_t}\sqrt{w}\right) d\xi. \end{aligned}$$

Using (2.22), the conditional pdf of the power is

$$\begin{aligned} f_{W|t}(w|t) &= \frac{m^m}{a_t\theta^m\Gamma(m)}\left(\frac{w}{\delta^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{w}{2a_t}\right\} 2^{-n+1-1} \\ &\quad \times \left(\frac{\delta\sqrt{w}}{a_t}\right)^{n-1}\left(\frac{\delta^2}{2a_t}+\frac{m}{\theta}\right)^{-\frac{(2m-n+1+n-1)}{2}} \Gamma\left[\frac{\frac{(2m-n+1+n-1)}{2}}{n-1+1}\right] \\ &\quad \times {}_1F_1\left(\frac{(2m-n+1+n-1)}{2}; n-1+1; \frac{\left(\frac{\delta\sqrt{w}}{a_t}\right)^2}{4\left(\frac{\delta^2}{2a_t}+\frac{m}{\theta}\right)}\right) \\ &= \frac{m^m\Gamma(m)\, w^{n-1}}{(2a_t)^n\,\theta^m\Gamma(m)\Gamma(n)\left(\frac{\delta^2}{2a_t}+\frac{m}{\theta}\right)^m} \exp\left\{-\frac{w}{2a_t}\right\} \\ &\quad \times {}_1F_1\left(m\ ;\ n\ ;\ \frac{\delta^2 w}{4a_t^2\left(\frac{\delta^2}{2a_t}+\frac{m}{\theta}\right)}\right). \end{aligned} \tag{5.10}$$

Replace $n$ with $\mu$ and define the parameter $\kappa = \frac{\delta^2}{2t^{-1}\sigma^2\mu}$. Let $\Omega_t$ represent the conditional normalized power for the fading channel then $\Omega_t \equiv \frac{\omega_t}{\bar{w}_t}$, where $\bar{w}_t \equiv E(W|t) = \delta^2 + 2t^{-1}\sigma^2\mu = 2t^{-1}\sigma^2\mu(1+\kappa) = t^{-1}\bar{\omega}$ (see (3.26)). Thus, the pdf (5.10) can be written in terms of $\kappa$ and $\mu$ as

$$\begin{aligned} f_{\Omega_t}(\omega_t) &= \frac{m^m\Gamma(m)(\omega_t\bar{w}_t)^{\mu-1}}{(2a_t)^\mu\theta^m\Gamma(m)\Gamma(\mu)\left(\frac{(2a_t\mu\kappa)}{2a_t}+\frac{m}{\theta}\right)^m}\exp\left\{-\frac{(\omega_t\bar{w}_t)}{2a_t}\right\} \\ &\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{(2a_t\mu\kappa)(\omega_t\bar{w}_t)}{4a_t^2\left(\frac{(2a_t\mu\kappa)}{2a_t}+\frac{m}{\theta}\right)}\right)\bar{w}_t \\ &= \frac{m^m\Gamma(m)(\omega_t(2a_t\mu(1+\kappa)))^{\mu-1}(2a_t\mu(1+\kappa))}{(2a_t)^\mu\theta^m\Gamma(m)\Gamma(\mu)\left(\frac{(2a_t\mu\kappa)}{2a_t}+\frac{m}{\theta}\right)^m}\exp\left\{-\frac{(\omega_t(2a_t\mu(1+\kappa)))}{2a_t}\right\} \\ &\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{(2a_t\mu\kappa)(\omega_t(2a_t\mu(1+\kappa)))}{4a_t^2\left(\frac{(2\sigma^2\mu\kappa)}{2\sigma^2}+\frac{m}{\theta}\right)}\right) \\ &= \frac{m^m\mu^\mu(1+\kappa)^\mu\omega_t^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\exp\{-\mu(1+\kappa)\omega_t\} \\ &\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)\omega_t}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right). \end{aligned} \tag{5.11}$$

Hence the unconditional pdf of the normalized power, $\Omega=\frac{W}{t^{-1}\bar{w}}=t\omega$ (see section 3.3) is given by

$$\begin{aligned} f_\Omega(\omega) &= \int_0^\infty W(t)f_{\Omega_t}(\omega_t)\,dt \\ &= \int_0^\infty W(t)\frac{m^m\mu^\mu(1+\kappa)^\mu(t\omega)^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\exp\{-\mu(1+\kappa)t\omega\} \\ &\quad\times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)dt. \end{aligned} \tag{5.12}$$

Particular cases of $W(t)$ in equation (5.12) are focused on since it forms part of the investigation in section 5.4.

**Normal case**

Assume (2.35), then (5.12) simplifies to (5.4) .

**t-distribution**

Assume (2.36), then it follows from (5.4) that

$$\begin{aligned} f_\Omega(\omega) &= \int_0^\infty \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)}\frac{m^m\mu^\mu(1+\kappa)^\mu(t\omega)^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m} \\ &\quad\times\exp\{-\mu(1+\kappa)t\omega\}\ {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)dt \end{aligned}$$

$$= \int_0^\infty \frac{v^{\frac{v}{2}} m^m \mu^\mu (1+\kappa)^\mu \omega^{\mu-1}}{2^{\frac{v}{2}} \theta^m \Gamma(\mu) \Gamma\left(\frac{v}{2}\right) \left(\mu\kappa + \frac{m}{\theta}\right)^m} \times t^{\frac{v}{2}+\mu-2} \exp\left\{-t\mu(1+\kappa)\omega - \frac{tv}{2}\right\} \times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right) dt.$$

Using (2.23), the pdf of the normalized power follows as

$$f_\Omega(\omega) = \frac{v^{\frac{v}{2}} m^m \mu^\mu (1+\kappa)^\mu \omega^{\mu-1}}{2^{\frac{v}{2}} \theta^m \Gamma(\mu) \Gamma\left(\frac{v}{2}\right) \left(\mu\kappa + \frac{m}{\theta}\right)^m} \Gamma\left(\frac{v}{2}+\mu-1\right) \times \left[\mu(1+\kappa)\omega + \frac{v}{2}\right]^{-\left(\frac{v}{2}+\mu-1\right)} \times {}_2F_1\left(m; \frac{v}{2}+\mu-1; \mu; \frac{\frac{\mu^2\kappa(1+\kappa)\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}}{\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]}\right), \tag{5.13}$$

where $\omega > 0$.

### 5.3.3 Characteristics

#### 5.3.3.1 Cumulative distribution function

The cdf of $\Omega$ is

$$F_\Omega(Z_T) = \int_0^{Z_T}\int_0^\infty W(t) \frac{m^m \mu^\mu (1+\kappa)^\mu t^{\mu-1}}{\theta^m \Gamma(\mu) \left(\mu\kappa+\frac{m}{\theta}\right)^m} \omega^{\mu-1} \exp\{-\mu(1+\kappa)t\omega\} \times {}_1F_1\left(m; \mu; \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right) dt d\omega. \tag{5.14}$$

Subsequently special cases ( (2.35) and (2.36)) for the outage probability will be derived.

**Remark 5.1** *For the normal distribution case with weight function* (2.35)*, the outage probability is given by* (5.5) *.*

**Remark 5.2** *For the t-distribution case with weight function* (2.36)*, the outage probability is obtained as follows:*

$$F_\Omega(Z_T) = \int_0^{Z_T}\int_0^\infty \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)} \frac{m^m \mu^\mu (1+\kappa)^\mu t^{\mu-1}}{\theta^m \Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\omega^{\mu-1} \times \exp\{-\mu(1+\kappa)t\omega\}\ {}_1F_1\left(m; \mu; \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right) dt d\omega.$$

*Using* (2.10) *the confluent hypergeometric function in equation can be expressed as an infinite series. Thus, the cdf can be written as*

$$\begin{aligned}
F_{\Omega}(Z_T) &= \int_0^{Z_T}\int_0^{\infty} \frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}\omega^{\mu-1}t^{\mu+\frac{v}{2}-2}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\exp\left\{-\mu(1+\kappa)t\omega-\frac{v}{2}t\right\} \\
&\times\sum_{l=0}^{\infty}\frac{(m)_l}{l!(\mu)_l}\left(\frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l d\omega \\
&= \int_0^{Z_T}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l \\
&\times\omega^{\mu+l-1}t^{\mu+\frac{v}{2}+l-2}\exp\left\{-t\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]\right\}dtd\omega.
\end{aligned}$$

*Using* (2.20) , *it follows*

$$\begin{aligned}
F_{\Omega}(Z_T) &= \int_0^{Z_T}\sum_{l=0}^{\infty}\frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l \\
&\times\omega^{\mu+l-1}\Gamma\left(\mu+\frac{v}{2}+l-1\right)\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]^{-\left(\mu+\frac{v}{2}+l-1\right)}d\omega \\
&= \int_0^{Z_T}\sum_{l=0}^{\infty}\frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m\Gamma\left(\mu+\frac{v}{2}+l-1\right)} \\
&\times\omega^{\mu+l-1}\left[\mu(1+\kappa)\omega+\frac{v}{2}\right]^{-\left(\mu+\frac{v}{2}+l-1\right)}d\omega \\
&= \int_0^{Z_T}\sum_{l=0}^{\infty}\frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m\Gamma\left(\mu+\frac{v}{2}+l-1\right)} \\
&\times\omega^{\mu+l-1}\left\{\frac{v}{2}\left[\frac{2\mu(1+\kappa)\omega}{v}+1\right]\right\}^{-\left(\mu+\frac{v}{2}+l-1\right)}d\omega \\
&= \int_0^{Z_T}\sum_{l=0}^{\infty}\frac{v^{\frac{v}{2}} m^m \mu^{\mu}(1+\kappa)^{\mu}\left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l\left(\frac{v}{2}\right)^{-\left(\mu+\frac{v}{2}+l-1\right)}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m\Gamma\left(\mu+\frac{v}{2}+l-1\right)} \\
&\times\omega^{\mu+l-1}\left[\frac{2\mu(1+\kappa)\omega}{v}+1\right]^{-\left(\mu+\frac{v}{2}+l-1\right)}d\omega.
\end{aligned}$$

*Using* (2.24) *, the outage probability is*

$$
\begin{aligned}
F_{\Omega}(Z_T) &= \sum_{l=0}^{\infty} \frac{v^{\frac{v}{2}} m^m \mu^{\mu} (1+\kappa)^{\mu} \left(\frac{\mu^2\kappa(1+\kappa)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)^l \left(\frac{v}{2}\right)^{-\left(\mu+\frac{v}{2}+l-1\right)}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m\Gamma\left(\mu+\frac{v}{2}+l-1\right)} \\
&\quad \times \frac{Z_T^{\mu+l}}{\mu+l}\, {}_2F_1\left(\mu+\frac{v}{2}+l-1, \mu+l; 1+\mu+l; -\frac{2\mu(1+\kappa)}{v}Z_T\right) \\
&= \sum_{l=0}^{\infty} \frac{\left(\frac{v}{2}\right)^{-\mu-l+1} m^m \mu^{\mu+2l} (1+\kappa)^{\mu+l} \kappa^l Z_T^{\mu+l}}{\theta^m\Gamma\left(\frac{v}{2}\right)\Gamma(\mu)\Gamma\left(\mu+\frac{v}{2}+l-1\right)\left(\mu\kappa+\frac{m}{\theta}\right)^{m+l}(\mu+l)} \\
&\quad \times {}_2F_1\left(\mu+\frac{v}{2}+l-1, \mu+l; 1+\mu+l; -\frac{2\mu(1+\kappa)Z_T}{v}\right) \\
&= \sum_{l=0}^{\infty} \frac{\left(\frac{v}{2}\right)^{-\mu-l+1} m^m \mu^{\mu+2l} (1+\kappa)^{\mu+l} \kappa^l Z_T^{\mu+l}\left(1-\frac{2\mu(1+\kappa)Z_T}{v}\right)^{-\mu-\frac{v}{2}-l+1}}{\theta^m\Gamma\left(\frac{v}{2}\right)\Gamma(\mu)\Gamma\left(\mu+\frac{v}{2}+l-1\right)\left(\mu\kappa+\frac{m}{\theta}\right)^{m+l}(\mu+l)} \\
&\quad \times {}_2F_1\left(\mu+\frac{v}{2}+l-1, \mu+l; 1+\mu+l; \frac{2\mu(1+\kappa)Z_T}{2\mu(1+\kappa)Z_T+v}\right). \qquad (5.15)
\end{aligned}
$$

#### 5.3.3.2 Moments

From (5.12) the $j^{th}$ moment of the normalized power, $\Omega$, is

$$
\begin{aligned}
E\left(\Omega^j\right) &= \int_0^{\infty} \omega^j f_{\Omega}(\omega)\, d\omega \\
&= \int_0^{\infty} \omega^j \int_0^{\infty} W(t) \frac{m^m\mu^{\mu}(1+\kappa)^{\mu}(t\omega)^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m} \exp\{-\mu(1+\kappa)t\omega\} \\
&\quad \times {}_1F_1\left(m\ ;\ \mu\ ;\ \frac{\mu^2\kappa(1+\kappa)(t\omega)}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right) dt\, d\omega \\
&= \int_0^{\infty} W(t) \frac{m^m\mu^{\mu}(1+\kappa)^{\mu}t^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m} \int_0^{\infty} \omega^{\mu+j-1} \\
&\quad \times \exp\{-\mu(1+\kappa)t\omega\}\, {}_1F_1\left(m; \mu; \frac{\mu^2\kappa(1+\kappa)t\omega}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right) d\omega
\end{aligned}
$$

Using (2.23), the $j^{th}$ moment is

$$\begin{aligned} E\left(\Omega^j\right) &= \int_0^\infty W(t)\frac{m^m\mu^\mu(1+\kappa)^\mu t^{\mu-1}}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m}\frac{\Gamma(\mu+j)}{(t\mu(1+\kappa))^{\mu+j}} \\ &\quad\times {}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\left(\frac{\mu^2\kappa(1+\kappa)t}{\left(\mu\kappa+\frac{m}{\theta}\right)}\right)}{(t\mu(1+\kappa))}\right)dt \\ &= \int_0^\infty W(t)\frac{m^m\Gamma(\mu+j)}{(t)^{j+1}\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j} \\ &\quad\times {}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)dt. \end{aligned} \tag{5.16}$$

**Remark 5.3** *Substituting the weight functions* (2.35) *and* (2.36) *into* (5.16) , *respectively, the following expressions are derived.*

*(i) For the normal distribution from* (5.16) *the* $j^{th}$ *moment is* (*see* (5.6))

$$E\left(\Omega^j\right)=\frac{m^m\Gamma(\mu+j)_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)}{\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j}.$$

*(ii) For the t-distribution from* (5.16) *the* $j^{th}$ *moment is*

$$\begin{aligned} E\left(\Omega^j\right) &= \int_0^\infty \frac{v\left(\frac{vt}{2}\right)^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)}{2\Gamma\left(\frac{v}{2}\right)}\frac{m^m\Gamma(\mu+j)}{(t)^{j+1}\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j} \\ &\quad\times{}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)dt \\ &= \int_0^\infty \frac{v^{\frac{v}{2}}m^m\Gamma(\mu+j)\ {}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j} \\ &\quad\times t^{\frac{v}{2}-j-2}\exp\left(-\frac{vt}{2}\right)dt. \end{aligned}$$

*Using* (2.19) ,

$$\begin{aligned} E\left(\Omega^j\right) &= \frac{v^{\frac{v}{2}}m^m\Gamma(\mu+j)\ {}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j}\frac{\Gamma\left(\frac{v}{2}-j-1\right)}{\left(\frac{v}{2}\right)^{\frac{v}{2}-j-1}} \\ &= \frac{v^{j+1}m^m\Gamma(\mu+j)\Gamma\left(\frac{v}{2}-j-1\right)\ {}_2F_1\left(m\ ,\ \mu+j\ ;\ \mu\ ;\frac{\mu\kappa}{\mu\kappa+\frac{m}{\theta}}\right)}{2^{j+1}\Gamma\left(\frac{v}{2}\right)\theta^m\Gamma(\mu)\left(\mu\kappa+\frac{m}{\theta}\right)^m(\mu(1+\kappa))^j}, \end{aligned} \tag{5.17}$$

*where* $\operatorname{Re}\left(\frac{v}{2}-j-1\right)>0$.

## 5.4 Performance Measures

In this section graphical displays of the composite $\kappa-\mu$ type model and some performance metrics will be shown.

The pdf for the normalized power (5.12) is shown Figure 5.4.1 for $\kappa = 2$, $m = 2$ and $\Omega = 1$, Figure 5.4.2 for $\mu = 0.5$, $m = 2$ and $\Omega = 1$, and Figure 5.4.3 for $\kappa = 2$, $\mu = 0.5$ and $\Omega = 1$ for both the special cases (normal (5.3), on left, and t-distribution, where $v = 3$ degrees of freedom was considered, (5.13), on right, respectively).

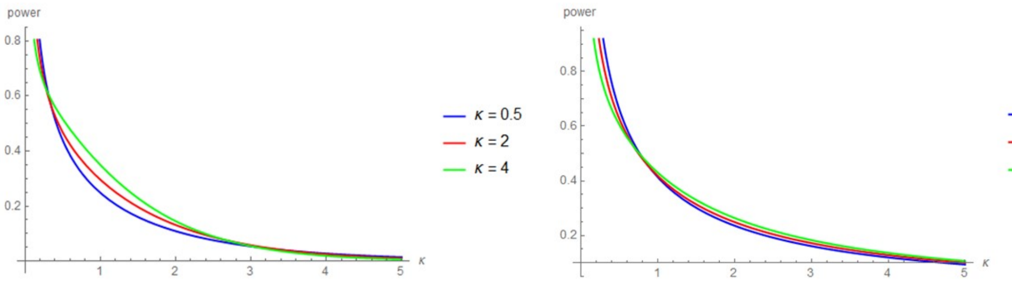



Figure 5.4.1

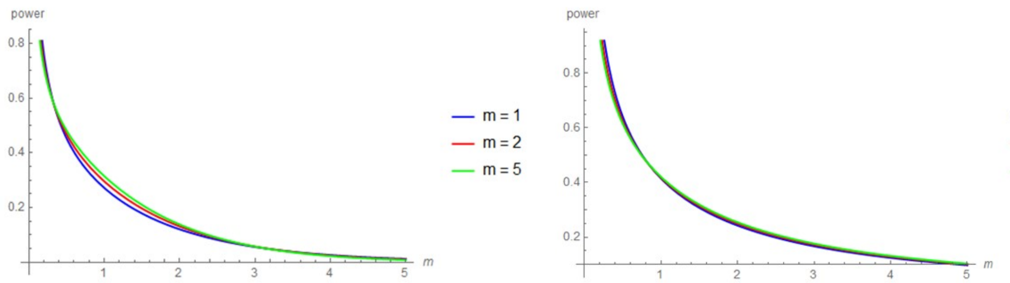



Figure 5.4.2

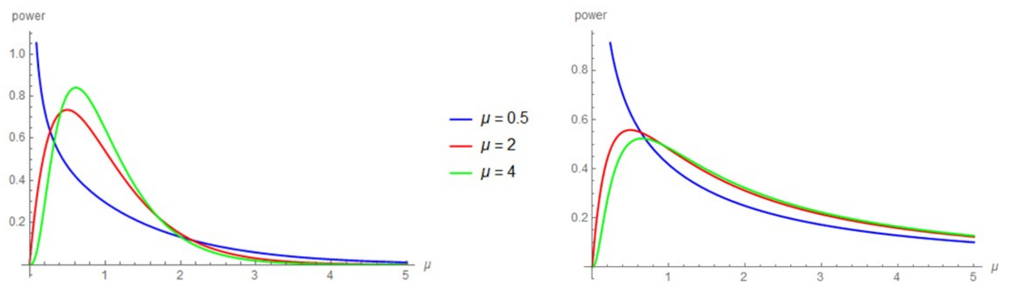



Figure 5.4.3

The outage probability (5.14) is illustrated in Figure 5.4.4 for $\kappa = 2$, $m = 2$ and $\Omega = 1$, Figure 5.4.5 for $\mu = 1$, $m = 2$ and $\Omega = 1$, and Figure 5.4.6 for $\kappa = 2$, $\mu = 2$ and

$\Omega = 1$, respectively. Both the special cases normal (5.5), and t-distribution, where $v = 3$ degrees of freedom is considered, (5.15). For a fixed $\mu$, the outage probability for the t-distribution outperforms the normal distribution as shown in Figure 5.4.4. For a fixed $\kappa$, the outage probability for the t-distribution only outperforms the normal distribution when $\mu$ (the number of clusters) is large as shown in Figure 5.4.5. Thus, as the number of clusters increase the assumption of the t-distribution as the underlying model is more appropriate.

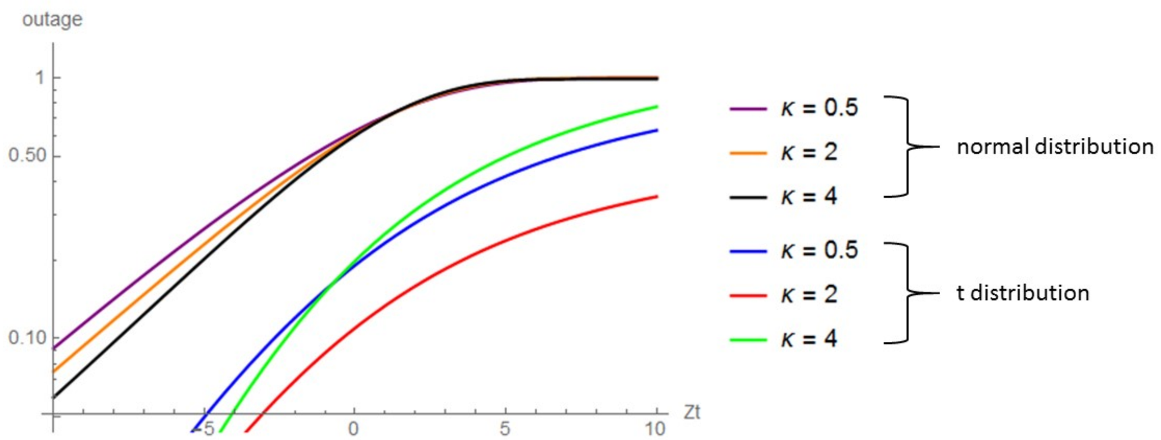



Figure 5.4.4

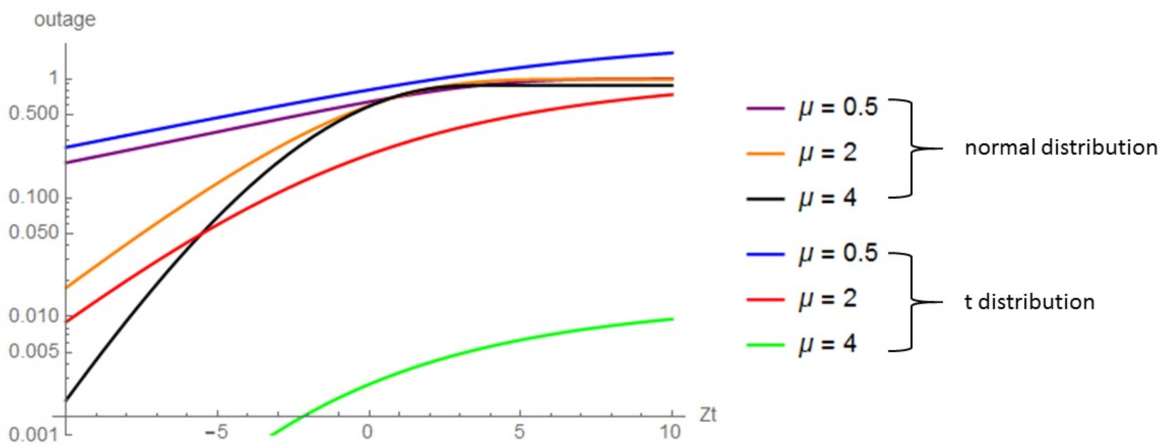



Figure 5.4.5

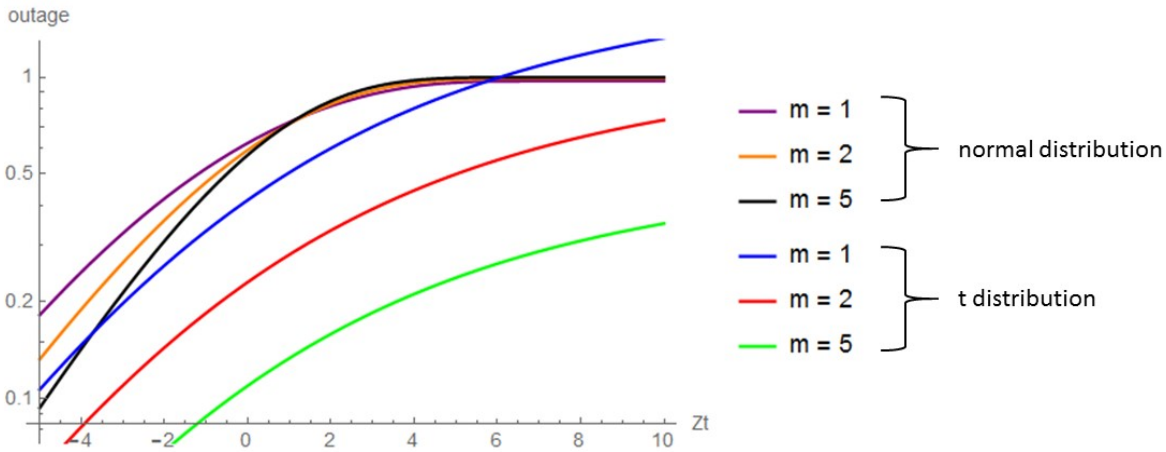

outage
1
0.5
0.2
0.1
-4
-2
0
2
4
6
8
10
Zt
m = 1
m = 2
m = 5
normal distribution
m = 1
m = 2
m = 5
t distribution

Figure 5.4.6

# Chapter 6

# Summary and future work

In this mini-dissertation the univariate and bivariate $\kappa - \mu$ *type* fading models as well as the composite $\kappa - \mu$ *type* fading model were developed; these distributions emanated from the elliptical assumption as the point of departure for the description of the envelope of the physical model. For each model the basic statistical characteristics and performance metrics were derived. By assuming the elliptical model various different distributions are included, but specific the t distribution is focused of this study, since Yacoub (2007) [35] commented about the inadequacy of tail fitting of some distributions to experimental data from the communications system domain.

Graphical illustrations accompanied these characteristics of the different models. The analytical framework that was used (stemming from an integral representation) provides computationally convenient forms of these distributions (Chu, 1973 [6]; Loots et al., 2013 [3]). By considering the t-distribution as a special member of the elliptical class, it was noted that the outage probability was lower in most cases. This has a significant impact on the assumption made by the researcher/practitioner in the communication systems field.

There are many opportunities for future research based on this study, to name a few:

- In a statistical context the univariate $\kappa - \mu$ type fading model can be further explored in terms of fitting this model to a real data case and investigating more characteristics, amongst others the skewness. Similarly for the bivariate $\kappa - \mu$ type model the correlation structure needs investigation. The physical contribution in the communications systems area needs to be investigated, the fit to field measurements in scenarios in this area, together with the study of other metrics such as the ergodic capacity (see Garcia-Corrales et al 2014 [12]).
- If the overall fading in a channel is the result of multiple scattering components, further research can include the assumption that the received SNR of the cascaded channel is expressed as the product of $\kappa - \mu$ type random variables. (See for the gamma case Shankar, 2011, p 232. [28])
- For the above scenario obtain derive, in a manageable form, asymptotic approximations for products and/or ratios of independent product of $\kappa - \mu$ type random variables which have pdf and cdf easily implementable and computationally appealing. (See Marques and Longeville, 2016 [17]; Nicolas and Florence Tupin, 2016 [20].)
- Extension of other models, for example the $\alpha - \kappa - \mu$ fading model with the elliptical assumption.

- For the dependent components $(X, Y)$ in the bivariate $\kappa - \mu$ type fading model the pdf of ratio of the components is also of interest, if $X$ represents the signal power and $Y$ the interference power.
- Exploring a gamma mixture approach (see Atapattu, et al 2011 [4]) for the univariate $\kappa - \mu$ type fading and the composite $\kappa - \mu$ type fading models, as well as exact and approximate moment generating functions (MGF) expression for these generalized fading models (see Salahat, et al 2017 [26])
- The generalised elliptical assumption (see Arashi and Nadarajah, 2016 [2]) can be assumed as the underlying model for the descriptions in this study.
- Investigate bivariate extensions such as the bivariate $\kappa - \mu$ –lognormal or the bivariate composite $\kappa - \mu$ type model, since Reig et al (2014) [25] described the usefulness of the bivariate Nakagami- lognormal in the assessment of performances in receivers with highly dynamical environments.

In conclusion, with the elliptical model as the point of departure, this study contributed to the distribution field by proposing new $\kappa - \mu$ type fading distributions that may serve as alternatives to the existing models in the literature.

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