Exploring the decay parameter for the exponentially weighted moving average volatility methodology

By

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Abstract

Faculty of Natural and Agricultural Sciences

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Master of Science (MSc) in Financial Engineering

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Volatility estimation is a crucial task for financial institutions, as it affects various aspects of their operations, such as risk management, capital allocation, investment strategy and derivative valuation. However, the traditional method of using equally weighted moving averages to estimate volatility can be inaccurate and incorrectly used, especially in volatile market conditions. It yields financial losses for financial institutions in that the volatility estimates do not correctly reflect financial markets in real time. In this dissertation, we implement the exponentially weighted moving average model instead, which assigns more weight to recent data than older data. We explore how the choice of the decay factor λ influences the performance of the exponentially weighted moving average model in different market scenarios. The optimal value of λ varies depending on the market volatility. We therefore demonstrate that the model can provide more reliable and timely volatility estimates than the equally weighted moving average model. These are useful for different applications in financial, such as Value at Risk or Expected Shortfall.

Declaration of Authorship

I, Sharmaine Fanuel Promise SIBANDA, declare that the thesis, which I hereby submit for the degree of MSc in Financial Engineering at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:

Date:

"*When you stop learning, you stop growing.*"

Kenneth H. Blanchard

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List of Abbreviations

ARCH	**A**uto **R**egressive **C**onditional **H**eteroscedasticity
EGARCH	**E**xponential **G**eneralised **A**uto **R**egressive **C**onditional **H**eteroscedasticity
EWMA	**E**xponential **W**eighted **M**oving **A**verage
GARCH	**G**eneralised **A**uto **R**egressive **C**onditional **H**eteroscedasticity
IGARCH	**I**ntegrated **G**eneralised **A**uto **R**egressive **C**onditional **H**eteroscedasticity
OP	**O**ptimisation **P**roblem
RMSE	**R**oot **M**ean **S**quared **E**rror
SMA	**S**imple **M**oving **A**verages
VaR	**V**alue **a**t **R**isk

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Chapter 1

Introduction

1.1 Introduction

The likelihood of a loss on a position in any financial market, such as the bond market, stock market, commodities market, derivatives market, and others, is referred to as risk in the field of finance. The net risk of maintaining both long and short positions in the market is known as market risk (Metrics, 1996). Since risk exposure cannot be totally avoided in the world of finance, it is crucial to research and investigate various techniques for estimating the risk associated with a trading position. We can decide whether or not this is a position worth taking by quantifying it. In the economy, there are two different types of risks: systematic risk and non-systematic risk. The portion of returns that is connected with the returns of the investment portfolio is referred to as systematic risk. This risk cannot be mitigated by diversification. Non-systematic risk is the proportion of returns that are unrelated to the portfolio's performance. This risk is reduced by diversifying the portfolio (Turvey, Driver, and Baker, 1988). Market risk is the risk connected with price fluctuations in the market. It is possible for a position's value to change due to changes in market prices (Turvey, Driver, and Baker, 1988).

Risk factors

Accounting for anticipated changes in the relevant components is equally crucial when measuring risk. Risk considerations serve as the foundation for financial securities pricing functions.

Commodity risk factor

Spot and future commodity prices determine exposure to commodity prices. Spot prices are used for fast commodities' purchase and delivery. Commodity contracts, such as commodity futures, options, and options on commodity futures, are determined by future prices (Turvey, Driver, and Baker, 1988).

Equity Risk factor

Equities risk indicators are expressed as prices, therefore exposure to them can be described as a time series of prices (these are to be obtained from Yahoo Finance). That is, the underlying determines the equity risk factor. Consider stock in business X; the value of the position in company X is determined by changes in stock prices or its sensitivity to changes in an index. The underlying determines equity contracts such as equity futures and options (Turvey, Driver, and Baker, 1988).

Currency Risk factor

Foreign exchange spot prices create currency risk. Foreign exchange spot rates are the pricing of foreign currency unit amounts (Turvey, Driver, and Baker, 1988).

1.1.1 The importance of measuring market risk

Risk must be identified and correctly measured to be effectively managed. Because financial risk cannot be totally avoided, the best thing to do is to gain control over it. This includes risk exposure and risk management in the financial sector, for example, in the banking industry. The interest in market risk has grown dramatically during the last three decades. This is because financial markets evolved during this period. If capital markets were flawless, investors would be able to achieve the desired level of diversification and there would be no need to manage financial risk. Take, for example, the Modigliani-Miller theorem. Financial risk management would be worthless in an ideal theoretical system with no information inconsistencies, taxes, transaction expenses, or bankruptcy costs, and no market friction.

Banks substitute illiquid instruments with liquid ones through securitisation. This global practice has been rapidly expanding, with both exchange-traded and over-the-counter derivatives becoming important components of markets (Metrics, 1996). These shifts have occurred concurrently with changes in managerial practices as well as technological advancements in data processing. There has been a shift away from management based on accrual accounting and toward risk management based on position marking-to-market (Metrics, 1996).

Accrual accounting is a method of accounting where revenues and expenses are recognised when they are incurred, regardless of when cash is received or paid. This method provides a more long-term view of a company's financial performance, as it takes into account both current and future obligations. However, with the development of securitisation and the growth of global securities markets, there has been a shift towards risk management based on the marking-to-market of positions. This method of accounting involves valuing financial instruments at their current market prices rather than their historical cost. This approach provides more frequent and accurate reporting of investment gains and losses, which has led many firms to manage their earnings on a daily basis.

The shift towards marking-to-market is largely driven by the increased liquidity and pricing availability in financial markets, as well as advancements in data processing technology. With more information available in real time, firms can more accurately assess the risks associated with their positions and adjust their strategies accordingly. Marking-to-market also allows for a more transparent view of a company's financial performance, as it provides a clear picture of the gains and losses associated with a particular position. This level of transparency is especially important for investors, who can more accurately assess the risks associated with a particular investment. In the past, concentrating just on returns resulted in insufficient performance analysis. Return management does not provide an indicator of the cost of risk, which is the volatility of the returns. Banks, financial firms, and corporations are now using integrated market risk indicators (Metrics, 1996).

To appreciate the significance of market risk management, one must analyse market risk regulatory capital requirements before and following the 2008 financial crisis. VaR is the potential loss on a financial position or portfolio over a specific time horizon that is likely to be surpassed, with a given level of confidence. For example, if the VaR estimate for a portfolio is $1 million with a 99% confidence level over a one-day time horizon, this means that there is a 1% chance that the portfolio could lose more than $1 million over the next day.

In the early 1900s, several larger banks began utilising it to assess market risk (O'Brien and Szerszen, 2014). The financial crisis exposed the fact that VaR provides insufficient notice of possible loss, as well as the high frequency and magnitude of losses during a crisis period. O'Brien and Szerszen look into the VaR during the pre-crisis and crisis periods. They conclude that throughout the pre-crisis period, banks' VaR remained constant with minimal VaR exceptions. VaR exceptions occur when actual losses on a financial position or portfolio exceed the VaR estimate. In other words, a VaR exception occurs when the actual losses are greater than what was expected based on the VaR estimate. For example, if a portfolio has a VaR estimate of $1 million with a 95% confidence level, this means that there is a 5% chance that losses could exceed $1 million. If actual losses on the portfolio are $1.5 million, then this would be considered a VaR exception, since the losses were greater than the VaR estimate.

In the context of the study by O'Brien and Szerszen, "minimal VaR exceptions" would mean that banks experienced relatively few instances where actual losses exceeded their

VaR estimates during the pre-crisis period. This suggests that banks' VaR models were performing reasonably well at the time and that they were effectively managing their risk exposure. However, during the crisis, bank VaR exceptions were significantly higher (O'Brien and Szerszen, 2014). With insufficient procedures for monitoring market risk, banks and financial institutions run the risk of incurring massive financial losses. This can have a huge negative impact on the economy as a whole. This highlights the need to accurately evaluate market risk.

Volatility is a popular way to assess market risk. A stock's volatility can be defined as the standard deviation of the return generated by the stock price over a given time frame (Hull et al., 2013). These are some of the elements that influence the volatility of the portfolio:

- The variances and covariances of a portfolio's risk factors.
- The sensitivity of the portfolio's assets to risk variables.

Volatility is a frequent risk indicator, and while there are numerous techniques for measuring it, some have proven to be superior to others. Traditional historical volatility metrics weight each observation equally, regardless of whether it is recent or not. As a result, superfluous "ghost effects" of past events might be seen in volatility jumps when nothing in the real market has occurred to cause these increases. These false mirrors of the real markets cause significant financial losses. While some approaches to forecasting volatility do not produce optimal results, others have been improved to produce more reflective outcomes. One such method forecasts the variances and covariances (volatilities and correlations) of the multivariate normal distribution using an exponentially weighted moving average model (EWMA) (Metrics, 1996). In contrast to the typical volatility forecasting approach, which uses an equally weighted moving average. The simple moving average model (SMA) is the name given to this strategy (Metrics, 1996).

When employing an exponential moving average of historical observations, the most recent observation is given the greatest weight in the volatility. As a result, volatility reacts faster to market leaps and declines exponentially as the weight of the jump observation falls. To estimate volatility, the EWMA requires the decay factor. The decay factor denoted by lambda, (λ), is a parameter that defines the relative weights applied to data (returns) as well as the amount of data collected to evaluate volatility and forecast correlations. An accurate estimate of λ is thus important to evaluating correct volatilities and, by extension, accurate market risk.

1.1.2 Volatility Models

There have been numerous methods developed over the years to estimate standard deviations and correlations. These volatility models include extreme value approaches (Parkinson, 1980), GARCH (Bollerslev, 1986), and stochastic volatility (Ghysels, Harvey, and Renault, 1996). Time series returns typically exhibit time-dependent volatility, which is why GARCH models are the most commonly used (Metrics, 1996). The GARCH model is based on the ARCH model, which was first defined by Engle (Engle, 1982). To mention a few, the ARCH model was followed by the Generalised ARCH, Integrated GARCH, Exponential GARCH, and Switching Regime ARCH. Several tests of the GARCH models of foreign exchange and stock markets have demonstrated that these approaches can produce better estimates of volatility than the SMA.

The ARCH class of models is formally defined by

$$\epsilon_t = z_t \sigma_t$$

where z_t is an i.i.d mean-zero, unit variance stochastic process, and σ_t represents the time-t latent volatility; i.e., $E(\epsilon_t^2|\sigma_t) = \sigma_t^2$. σ must be measurable based on the information set available at time $t-1$ (Andersen, Bollerslev, and Hadi, 2014).

Volatility is a time-varying series. This is where the ARCH model comes in; it depicts the change in variance in a time-dependent series. This is for raising or decreasing volatility. The GARCH is an ARCH extension in that it permits the conditional variance, σ_t^2, to be dependent on preceding conditional variances. As a result, the ARCH models outperform the SMA models in terms of volatility estimation.

Chapter 2

Literature Study

2.1 Risk Measurement

Risk exposure analysis for banks is critical for both financial institutions and non-financial organisations. One of the key sources of risk is market risk; the others are credit risk, operational, liquidity and climate risk.

Volatility is the most widely used risk indicator in financial markets. Even though several ways of determining volatility have been employed in practice over the years, there is still a need to evaluate the dependability of the various methods. Each approach to calculating volatility has its own set of advantages and disadvantages. Inaccurate volatility calculation methodologies can cause large financial losses for financial institutions. There is a need to seek systems that provide real-time volatility forecasts that are accurate representations of financial markets. The results are then employed in VaR calculations, capital allocation models, investment management, and derivative product pricing and hedging (Alexander, 1998). Volatility for any group of assets is determined by the variances and covariances of the relevant risk variables, as well as the sensitivity of individual assets to these risk factors. Extreme value techniques (Parkinson, 1980), nonlinear modelling such as GARCH (Generalised Auto-Regressive Conditional Heteroscedasticity) (Bollerslev, 1986), and stochastic volatility are examples of previously utilised methods (Harvey et al., 1994).

Volatility can be defined as the standard deviation of returns, but because returns rise with the time horizon over which they are assessed, the time horizon over which volatility is determined must be standardised while everything else remains constant (Dowd, 2007). The formula for annualised volatility can be written as:

$$\text{Annualised volatility} = \sigma_d * \sqrt{n}$$

where σ_d is the daily volatility of returns and n is the number of trading days. The volatility is given by the square root of the variance of the returns and the number of trading days is typically taken as 252.

2.1.1 Risk Measurement before VaR

Derivative Risk Measures

Derivatives are financial instruments that derive their value from an underlying asset such as stocks, bonds, commodities, or currencies. The value of a derivative is based on the price movements of the underlying asset. Options are a type of derivative that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (called the strike price) on or before a specified date (called the expiration date). There are two types of options: call options and put options.

A call option gives the holder the right to buy the underlying asset at the strike price, while a put option gives the holder the right to sell the underlying asset at the strike price. The holder of an option pays a premium to the seller of the option for the right to buy or sell the underlying asset. Options are commonly used by investors and traders to manage risk and to speculate on the price movements of the underlying asset. For example, an investor who owns a stock may purchase a put option to protect against a potential price decline, while a trader may purchase a call option if they believe the price of the underlying asset will increase.

Options can be complex instruments and involve various factors such as volatility, time decay, and interest rates. As such, it is important for individuals interested in trading options to have a solid understanding of their mechanics and to carefully consider the risks involved before making any trades.

The Greek parameters of derivative positions can also be used to estimate their risk.

- Delta (δ): This parameter represents the change in the derivative's price in relation to a slight change in the underlying price.
- Gamma (γ): This parameter represents the change in the delta of a derivative in relation to a slight change in the underlying price. This is the second derivative of the derivative's price in relation to the underlying price.
- Rho (ρ): This parameter represents the change in the price of the derivative in relation to the interest rate.
- Theta (θ): This parameter represents the change in the derivative's price over time.
- Vega (v): This parameter represents the change in the derivative's price as a function of volatility.

Gap Analysis

Gap analysis is a method used to determine a financial institution's interest-rate risk exposure. It involves selecting a horizon period, such as one year, and identifying the assets and liabilities in the portfolio that will re-price during that period. Re-price refers to the adjustment of interest rates on financial assets or liabilities that are tied to market rates. These adjustments may occur at predetermined intervals, such as when a loan's interest rate adjusts after a fixed period, or when a bond's interest rate resets at regular intervals. By identifying which assets and liabilities are subject to re-pricing, financial institutions can estimate their exposure to changes in interest rates over a given horizon period. The amounts associated with these rate-sensitive assets and liabilities are used to calculate the gap, which is the difference between the two. The change in net interest income that results from a change in interest rates is then estimated using the gap and the interest rate change, as represented by the equation:

$$\triangle NII = (GAP)\triangle r$$

where $\triangle$ NII is the change in net interest income and $\triangle r$ is the change in interest rates. GAP represents the difference between the rate-sensitive assets and rate-sensitive liabilities of a financial institution that will re-price within a given horizon period. The gap is calculated by subtracting the total amount of rate-sensitive liabilities from the total amount of rate-sensitive assets. For example, if a bank has \$1 million in rate-sensitive assets that will re-price within a one-year horizon period and $\$800,000$ in rate-sensitive liabilities that will re-price within the same period, then the gap is $\$200,000$ (\$1 million $-\$800,000)$.

The gap is an important measure because it provides an estimate of the financial institution's exposure to changes in interest rates over the horizon period. By estimating the change in net interest income resulting from a change in interest rates and comparing it to the gap, a financial institution can assess whether its interest-rate risk exposure is within acceptable limits. While this is a simple approach, it has its downsides (Dowd, 2007):

- Gap analysis assumes that the balance sheet is static and that the interest rate environment will remain constant. In reality, changes in the market environment can significantly impact the interest-rate risk exposure of a financial institution.

- Gap analysis is a static measure that only provides information on the interest-rate sensitivity of a financial institution's balance sheet at a particular point in time. It does not take into account changes in the balance sheet or interest rate environment over time.

- Gap analysis relies on several simplifying assumptions, such as the assumption that interest rates move in parallel across the yield curve, which may not hold in reality. These assumptions can lead to inaccurate estimates of interest-rate risk exposure.

Duration Analysis

Duration analysis is a method used to measure the sensitivity of a financial instrument's price or value to changes in interest rates. It is a widely used approach in fixed-income portfolio management and risk analysis.

Duration measures the weighted average time to receive the present value of the cash flows generated by an investment. It is expressed in years and reflects the bond's sensitivity to changes in interest rates. The higher the duration, the more sensitive the bond's price is to changes in interest rates. Duration analysis involves estimating the duration of a portfolio or single security and then using that estimate to determine how changes in interest rates will impact the value of the portfolio or security. The calculation of duration involves the present value of future cash flows, the period over which the cash flows are expected to be received, and the current price of the security. (Dowd, 2007):

$$D = \frac{\sum_{i=1}^{n} i * PVCF_i}{\sum_{i=1}^{n} PVCF_i}$$

where $PVCF_i$ is the present value of the cash flow for the period i discounted at the fitting period yield. The duration metric is used to gain an idea of how sensitive a bond's price is to changes in yield:

$$\%\triangle BP \approx -\frac{D\triangle y}{(1+y)}$$

where $\%\triangle BP$ is the percentage change in bond price, y is the yield and $\triangle y$ the change in yield. Duration measures are easy to calculate, and the duration of a bond portfolio is a simple weighted average of the durations of the individual bonds in the portfolio. This approach is better compared to gap analysis because it looks at changes in asset and liability values as opposed to just changes in net income (Dowd, 2007). Duration methods, on the other hand, have limitations in that they ignore risks other than interest risk, and even with improvements to improve accuracy, they remain inaccurate in comparison to other methods. The method has become obsolete since its main advantage (easy duration calculations) is no longer significant in the face of more sophisticated approaches that can now be run using programming software (Dowd, 2007).

Portfolio Theory

Portfolio theory assumes that investors select portfolios based on their expected return and standard deviation. The standard deviation of the portfolio return is used to calculate the portfolio's risk. While keeping other variables constant, an investor seeks a portfolio with a high expected value and a low standard deviation. The goal is to choose the portfolio with the best-projected return for any given standard deviation. A portfolio of this type is said to be efficient. The investor selects a subset of optimal investment portfolios and discards the rest. The investor's risk tolerance influences the decision. It is worth noting that the risk of any particular asset is determined not by the standard deviation of its return, but by the extent to which that asset contributes to overall portfolio risk (Dowd, 2007). The entire portfolio's value is determined by the correlation or co-variance of return for each item in the portfolio with the returns of other assets.

Scenario Analysis

This is also referred to as the "What if" analysis. Different scenarios are set and we investigate what can be gained or lost under each of them. Multiple paths for each variable, such as stock price, interest rate, and exchange rate, are considered. The cash flows and accounting values of assets and liabilities are then hypothesised as to how they might develop under each scenario. The results are then used to form an opinion about the situation. One problem with this strategy is that it is strongly dependent on the ability to select the "right" scenarios to include in the collection. The method's accuracy is determined by the analyst's expertise (or lack thereof). The scenarios must be plausible and should not contain contradictory or implausible assumptions. The interdependence of the factors must be thoroughly considered (Dowd, 2007). Another problem is that the approach does not tell us anything about the possibility of various situations. Therefore, we must use our best judgement to determine the significance of each scenario.

2.1.2 Value at Risk

VaR indicates the greatest predicted loss for a certain time frame with a specified level of confidence for banks, financial institutions, and investors. It is a statistical measure used to estimate the potential loss in value of an investment or portfolio of investments over a specified time horizon at a given confidence level. VaR is widely used in risk management to quantify the potential downside risk of an investment or portfolio. Financial organisations and investors typically use it as a risk indicator. VaR is typically calculated using historical data or statistical models, and it represents the maximum amount of loss that could occur with a certain level of confidence over a specified period. For example, a 1-day 95% VaR of $1 million means that there is a 5% chance that the portfolio will lose more than $1 million over the next day.

In the late 1970s and early 1980s, large financial organisations began to develop internal systems to measure and aggregate risks across their institutions. This appeared to be a difficult task to do. In the absence of such systems, good trades or investments were passed over because they exceeded arbitrary limits, risks were taken with insufficient awareness of their overall effects on the firm; reducing risk in one area rarely allowed greater risk-taking in another; and capital allocation was poor. In essence, there were no risk management systems set up. As this complex issue prevailed, there was a unified agreement that a sense of the probability of losses at a firm-wide level. This was the rise of value at risk. Value at risk allowed firms to get a more acceptable sense of their overall risk. JP Morgan RiskMetrics was one of the systems created. (Dowd, 2007).

This began when Dennis Weatherstone, the chairman of JP Morgan, asked his employees to provide a daily one-page report highlighting risk and potential losses for the following 24 hours across the bank's entire trading portfolio. This report was dubbed the "4:15 report" since it was due to Weatherstone each day at 4:15, following the close of business. This was accomplished by creating a method for measuring risks across various trading positions across the entire organisation. These were then aggregated into a single risk measure. This risk indicator was known as "value at risk," or the highest expected loss during the next trading day. The conventional portfolio theory was used to develop value at risk. It took advantage of the standard deviations and correlations between the returns of the many traded instruments. On the surface, this approach appeared to be simple, but putting it into action required a significant amount of effort. This involved selecting measurement conversions, building data sets, generating statistical assumptions and agreeing on them, devising techniques for predicting volatility and correlations, and establishing computing systems to do calculations. (Dowd, 2007). By 1990, the system's fundamental components were in place: data systems, risk measurement methods, and basic mechanics.

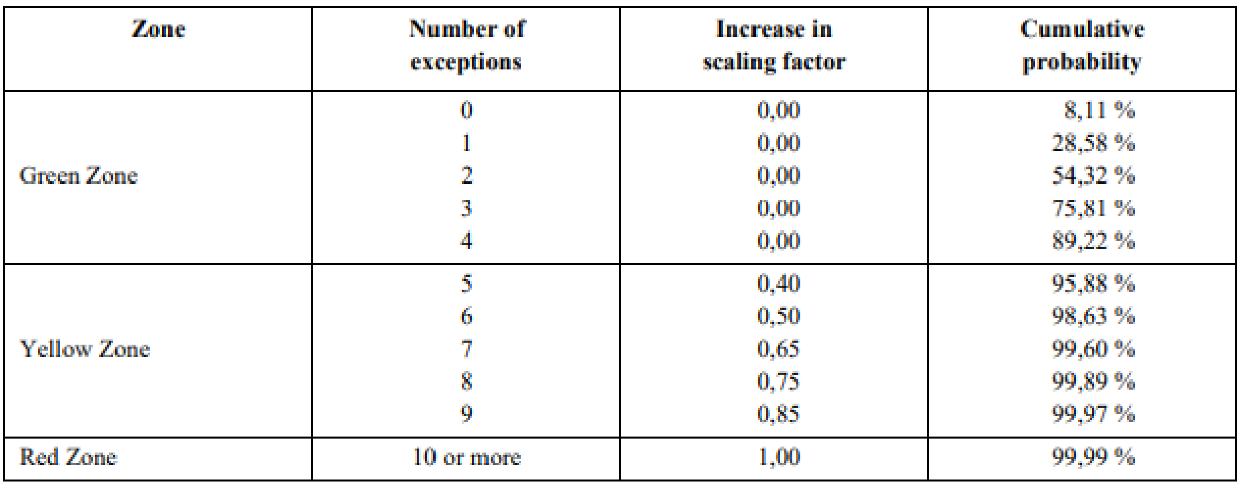
The system's advantage was that it made senior management more sensitive to risk-return trade-offs, resulting in a more efficient distribution of risk across the firm's trades. By 1993, JP Morgan's system had attracted a large number of prospective clients who wanted to buy or lease it for their purposes. Other companies were also focusing on building their risk measurement systems. Although the theoretical concepts were similar, the final systems differed mostly due to assumptions made, data used, parameter estimation processes, and other factors (Dowd, 2007). It is also critical to understand that there is no single way of calculating VaR. The historical simulation approach is one of the VaR calculating approaches. VaR is estimated using histograms of prior profit and loss data. Other techniques include Monte Carlo simulation and parametric methods.

In October 1994, JP Morgan freely made the RiskMetrics model (Morgan, 1994) available on the internet, along with the data used. This aided the proliferation of VaR systems and encouraged other software companies to employ the RiskMetrics approach. Security firms, investment banks, commercial banks, pension funds, other financial institutions, and non-financial institutions have all adopted the RiskMetrics approach. With this, the concept of VaR gained traction in the mid-1990s, and it quickly became a dominating measure of financial risk. VaR systems have also been modified to address credit risks, liquidity hazards, and operational risks. (Dowd, 2007).

VaR is defined as follows by Linsmeier and Pearson:

> "Losses greater than the value at risk are suffered only with a very small probability. Subject to the simplifying assumptions used in its calculations, VaR sums up all the risks in a portfolio into a single value suitable for use in boardrooms, reporting to regulators or disclosure in an annual report. Once one crosses the hurdle of using a statistical measure, the concept of value at risk is straightforward to understand. It is simply a way to describe the magnitude of the likely losses on the portfolio" (Linsmeier and Pearson, 1996).

In this section, T is introduced as the number of trading days or the sample period length. This will corroborate the variables in the formulation of the Research Methodology in Chapter 4. A sequence of historical trading days, $\{t_i\}_{i=0}^{T}$, and the corresponding realised trading losses, $\{L_i\}_{i=1}^{T}$, can be used to assess the accuracy of a VaR forecast calculation using the VaR Coverage Test (Costanzino and Curran, 2018). This involves counting the number of VaR breaches, which is the Traffic Light approach to back-testing VaR initially proposed by the Basel Committee for Banking Supervision in 1996 (Banking Supervision and Banking Supervisors, 1996). The Traffic Light System (Committee et al., 1996) applies the following scaling factors, k, for the number of VaR exceptions:



Zone	Number of exceptions	Increase in scaling factor	Cumulative probability
Green Zone	0	0,00	8,11 %
	1	0,00	28,58 %
	2	0,00	54,32 %
	3	0,00	75,81 %
	4	0,00	89,22 %
Yellow Zone	5	0,40	95,88 %
	6	0,50	98,63 %
	7	0,65	99,60 %
	8	0,75	99,89 %
	9	0,85	99,97 %
Red Zone	10 or more	1,00	99,99 %

Figure 2.1: Basel Traffic Light System

For each trading day $i = 1, ..., T$, $VaR_i(\alpha)$ represents the forecast VaR at level α, defined by

$$VaR(\alpha) := inf\{z \in R : \alpha \leq F_L(z)\}$$

where F_L is the cumulative distribution of the random loss variable L. The VaR breach indicator for each trading day i is defined as

$$X_{VaR}^{(i)}(\alpha) := \begin{cases} 0 & \text{if } L_i > VaR_i(\alpha) \\ 1 & \text{if } L_i \leq VaR_i(\alpha) \end{cases}$$

which keeps track of whether a breach occurred for trading day i. The total number of breaches throughout all T trading days is defined by

$$X_{VaR}^{T}(\alpha) := \sum_{t=1}^{T} 1_{\{L_i \leq VaR_i(\alpha)\}}$$

Under the null hypothesis that the VaR model is correct, $E[X_{VaR}^{T}(\alpha)] = T\alpha$.

The k-factor in Basel's traffic light system is a parameter that determines the boundaries of different zones for back-testing VaR models (Banking Supervision and Banking Supervisors, 1996). The traffic light system assigns a green, yellow, or red zone to a VaR model based on the number of exceptions (losses exceeding VaR - exceptions) observed over a given period. The null hypothesis of the Traffic Light approach is that the VaR model is correct, which implies that the number of breaches should follow a binomial distribution with parameters T (the number of trading days) and the VaR coverage rate (the proportion of trading days for which the VaR forecast is exceeded). Under this null hypothesis, the expected number of breaches is given by T times the VaR coverage rate. To test the null hypothesis, the Traffic Light approach uses a three-level colour scheme to indicate the level of confidence in the VaR model

- The green zone indicates that the model is acceptable: if the number of breaches is less than or equal to the 95% VaR confidence interval around the expected number of breaches, then the VaR model is considered to be accurate at the 95% confidence level, and the green light is shown.

- The yellow zone indicates that the model should be reviewed: if the number of breaches falls outside the 95% VaR confidence interval but is within the 99% VaR confidence interval around the expected number of breaches, then the VaR model is considered to be accurate at the 99% confidence level, and the yellow light is shown.

- The red zone indicates that the model should be rejected: if the number of breaches is outside the 99% VaR confidence interval around the expected number of breaches, then the VaR model is considered to be inaccurate at the 99% confidence level, and the red light is shown.

VaR has both downsides and upsides. In this section, we highlight some of these.

Advantages of Value at Risk

- VaR is a widespread and consistent risk measurement. It can be used in various positions, risk factors, and portfolio types. It enables inter-portfolio analysis because it compares the risks of different portfolios.
- VaR aggregates the risks of individual positions into a single measure of portfolio risk. It enables intra-portfolio analysis since it considers how different risk factors affecting the portfolio interact.
- Unlike traditional approaches like Greek measures, which only assess one risk factor at a time, VaR provides a holistic risk measure by taking into account all of the driving risk factors.
- Unlike traditional methodologies such as the Greeks, which address the question "what if," VaR is probabilistic and speaks to the possibility of a loss.
- VaR gives a single and simple measure of risk, ’lost money.’

Disadvantages of Value at Risk

- VaR estimates are notoriously inaccurate. VaR estimates can differ depending on the VaR model used (Beder, 1995). This is due to the various underlying assumptions made.
- VaR has a detrimental impact on both upside and downside variance.
- The mean-variance uses quadratic utility functions, which are not growing functions and so are not practical (Joshi and Paterson, 2013).
- It is overly simplistic for general distributions other than the normal distribution.
- VaR is insufficient in capturing the potential for extreme losses in the tail of loss distributions.

Uses of Value at Risk

- VaR is used to set an institution's total risk target.
- VaR can inform hedging, trading, investment, and portfolio management decisions.
- VaR can be used to decide on capital requirements. The higher the VaR and the greater the capital requirement, the riskier the activity (Dowd, 2007).
- Other types of risk, such as credit risk and operational risk, can be measured using item risk measurement systems based on VaR.
- VaR is included in reports, disclosures, and annual reports.

2.1.3 Expected Shortfall

According to Rosenberg and Schuermann (Rosenberg and Schuermann, 2006), VaR is insufficient in capturing the potential for extreme losses in the tail of loss distributions. Therefore, a conditional measure is needed to assess the severity of losses once a VaR threshold has been exceeded. Expected Shortfall (ES) addresses this issue by taking into account the probability-weighted losses beyond the VaR threshold, which accounts for both the severity of losses and their likelihood, as noted by Nadarajah, Zhang, and Chan (Nadarajah, Zhang, and Chan, 2014).

VaR is frequently criticised for relying on historical data and therefore having limited predictive power for uncertain futures. However, this same criticism also applies to ES. To address this issue, the Basel Committee on Banking Supervision (BCBS, 2013) has mandated the use of ES for determining both the capital requirements based on internal models and the risk weights for the revised standardised approach.

Under the proposed Basel framework, the VaR confidence level will be lowered from 99% to 97.5%, and ES will measure the probability-weighted losses beyond this threshold. As illustrated in Figure 2, this new confidence level provides a risk level similar to that of the existing 99% VaR threshold, with only a 0.5% difference for the normal distribution. With a larger number of observations in the 2.5% tail, compared to the previous 1% tail, the move to ES is expected to result in more stable model output and less sensitivity to extreme outlier observations. Banks may choose to adopt fatter-tailed distributions, as indicated in Figure 2.2 (c), which shows the difference between normal and t-distributed assumptions.

Figure 2.2 displays three different scenarios for portfolio volatility of 1%: (a) the normal distribution VaR at the 99% confidence level, as well as the ES counterpart, (b) the same as (a), but with the 97.5% confidence interval, and (c) the VaR and ES for both normal and t-distributed assumptions, with excess kurtosis of 3 and degrees of freedom parameter of 6.

The expected shortfall at a particular quantile, q, is represented by ES_q, which is defined as the probability-weighted average of values in the tail of the distribution to the left of q, such that

$$ES_q = E(L|L < VaR_q)$$

.

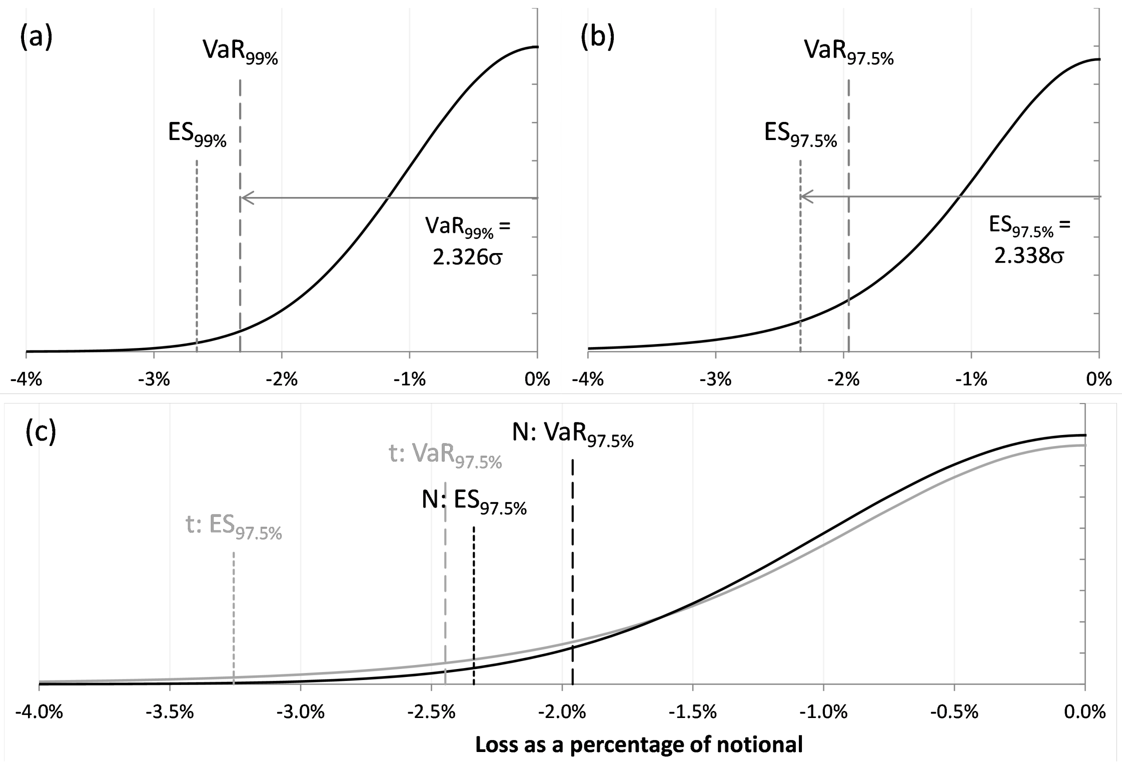


Figure 2.2: Expected Shortfall

In the case of a normal distribution, the formula for ES_q is expressed as

$$ES_q = \frac{f(VaR_q)}{q}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

that is, the probability density function of the normal distribution is represented by $f(x)$, where σ denotes the volatility, and it is assumed that mean, $\mu = 0$. To calculate ES_q for any volatility, σ, and at any significance level, q, the function below must be integrated:

$$\begin{aligned} ES_q &= \int_{-\infty}^{q} x f(x) dx \\ &= \int_{-\infty}^{q} \frac{x}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \end{aligned} \tag{2.1}$$

Let $u = \exp(-\frac{x^2}{2\sigma^2})$ then $d\chi = -\frac{x}{\sigma^2}\exp(-\frac{x^2}{2\sigma^2})$ so $-\sigma^2 d\chi = -x\exp(-\frac{x^2}{2\sigma^2})d\chi$.

Substituting $ES_q = -\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{q} d\chi = -\frac{\sigma}{\sqrt{2\pi}} \exp -\frac{x^2}{2\sigma^2}\Big|_q^{-\infty} = -\frac{\sigma}{\sqrt{2\pi}} \exp -\frac{q^2}{2\sigma^2}$.

Consider the t-distribution which has fat tails and is commonly considered a better representation of VaR:

$$ES_q = \int_{-\infty}^{q} tf(t)dt$$

where $f(t)$ is the probability density function of the t-distribution (for $\mu = 0$ and standard deviation, σ):

$$f(t) = \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})\sigma}(1+\frac{t^2}{\sigma^2\nu})^{(\frac{\nu+1}{2})}$$

where ν counts the degrees of freedom:

$$k = \frac{6}{\nu-4} + 3$$

where is the kurtosis of the data (Rozga and Arneric, 2009). For ν even

$$\frac{\Gamma(\nu+1}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} = \frac{(\nu-1)(\nu-3)\ldots5\cdot3}{2\sqrt{\nu}(\nu-2)(\nu-4)\ldots4\cdot2}$$

and for ν odd

$$\frac{\Gamma(\nu+1}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} = \frac{(\nu-1)(\nu-3)\ldots4\cdot2}{2\sqrt{\nu}(\nu-2)(\nu-4)\ldots5\cdot3}$$

To calculate ES_q for any volatility, σ, any number of degrees of freedom, ν, and any significant level, , the integral below must be determined:

$$ES_q = \int_{-\infty}^{q} t \cdot \frac{\Gamma(\nu+1)}{\sigma\sqrt{\nu\pi}\cdot\Gamma(\frac{\nu}{2})} \cdot \left(1+\frac{t^2}{\sigma^2\nu}\right)^{-(\frac{\nu+1}{2})} dt$$

Let $\Theta = \frac{\Gamma(\nu+1)}{\sqrt{\nu\pi}\cdot\Gamma(\nu/2)}$ and $\chi = 1 + \frac{t^2}{\sigma^2\nu}$ then $d\chi = \frac{2t}{\sigma^2\nu}dt$ so $\frac{\sigma^2\nu}{2}, d\chi = tdt$

Substituting

$$
\begin{aligned}
ES_q &= \Theta\frac{\sigma\nu}{2}\int_{-\infty}^{q}\chi^{-((\nu+1)/2)}d\chi \\
&= \frac{\Theta\sigma\nu}{1-\nu}\left[\chi^{((1-\nu)/2)}\right]_{-\infty}^{q} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu/2)}\cdot\frac{\sigma}{1-\nu}\sqrt{\frac{\nu}{\pi}}\cdot\left(1+q^2/(\sigma^2\nu)\right)^{\frac{1-\nu}{2}}
\end{aligned}
$$

For ES tail event measurements, advanced simulation and sampling techniques are required; as a result, banks will find it much harder to back-test ES, which takes into account both loss size and likelihood, as opposed to VaR, which only takes into account loss likelihood (Yamai, Yoshiba, et al., 2002). In VaR, violations are observable variables, making it easier to apply formal statistical techniques to assess whether the distribution of the violations complies with a known underlying model. This is done by comparing model predictions with observed results. This is not true for ES model predictions, as these may only be compared to model outcomes. Despite the wide range of back-testing ES procedures available, these are vastly inferior to the VaR equivalents (Nadarajah, Zhang, and Chan, 2014).

2.2 Volatility Forecast Models

The variability of volatility over different periods is known as time-varying volatility. This is critical for pricing derivatives, calculating risk measures, and hedging portfolio risk, among other things. As a result, scholars and practitioners alike are interested in and studying conditional variance. Because conditional variance is unobserved, evaluating and comparing different models has become challenging. Unobserved variance is sometimes substituted with squared returns, but this results in poor out-of-sample performance (Hansen and Lunde, 2005). This was resolved by Anderson and Bollerslev (Andersen and Bollerslev, 1998). They concluded that volatility models are useful and responded, "Yes, typical volatility models do produce accurate estimates." Andersen and Bollerslev based their evaluation on an estimated measure of volatility using intra-day returns rather than noisy measurements of daily volatility, the squared intra-day returns. These achieved good out-of-sample volatility model performance. Hansen and Lunde (Hansen and Lunde, 2005) explored the relative effectiveness of various volatility models in terms of their ability to forecast realised volatility. They employed White's superior predicting ability tests to do this (White, 2000) and Hansen (Hansen et al., 2001). These are also known as "data snooping tests."

A moving average is essentially an arithmetic average of consecutive data points from a time series collected across a rolling window. (Alexander, 1998). This looks to have been a useful tool in financial forecasting over the years. If portfolio volatility is consistent over time, the historical volatility estimator (Figlewski, 1994) is an efficient method. The estimator is defined as

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \bar{r})^2}. \tag{2.2}$$

The exponentially weighted moving average (EWMA) model should be employed when there are fast high and low volatility swings. This weighs recent returns more heavily. The equation is given as follows

$$\sigma = \sqrt{(1-\lambda)\sum_{t=1}^{T}\lambda^{t-1}(r_t-\bar{r})^2}. \tag{2.3}$$

For the equations above, the portfolio return in month t is given by r_t, T is the sample period length, $\bar{r}$ is the mean return across the sample period, and λ $(0 \leq \lambda \leq 1)$ is the decay factor.

The SMA model is a volatility forecasting strategy based on moving averages with constant, equal weights (Metrics, 1996). The pitfalls that come with this method result in "ghost effects." To avoid this, J.P Morgan and Reuters (Metrics, 1996) introduced the use of exponentially weighted moving averages. To employ this method, the decay factor, λ must be determined, depending on whether the market is quiet or turbulent. This parameter controls the relative weights assigned to the observations as well as the effective amount of data used in estimating volatility. According to Moosa and Bollen (Bollen, 2015), when forecasting risk measures over short time horizons, the EWMA volatility estimator outperforms the Auto-Regressive Conditional Heteroscedasticity (ARCH) volatility models.

2.2.1 Timeline for Volatility Forecast Models

- 1982 – RF Engle: ARCH models
- 1986 – T Bollerslev: GARCH models, Taylor: IGARCH models
- 1990 – GW Schwert: IGARCH
- 1996 – JP Morgan and Reuters: RiskMetrics™–Technical Document, EWMA Model
- 1998 – Andersen and Bollerslev: Yes, standard volatility models do provide accurate forecasts
- 2000 – H White: Data Snooping
- 2001 – PR Hansen: A Test for Superior Predictive Ability

2.3 Simple Moving Average

The Simple Moving Average weights all the historical data equally. The unbiased estimator of variance is given by

$$\sigma_t^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_{t-i} - \bar{r})^2. \tag{2.4}$$

When using daily data the average return, $\bar{r}$, will be significantly low so we can treat it as though it were zero (Dowd, 2007). This does not affect the estimates significantly and it will usually reduce the standard errors.

Downsides of the Simple Moving Average Model

- "Ghost effects": Since this model assigns the same weight to all historical data, old events have the same impact as recent occurrences. That is, even if an extreme event occurs relatively early, it will still have an impact on the volatility estimate for later periods, even if the event is old and the markets have returned to normal. This introduces "ghost effects," in which estimates are artificially high or low for particular periods, even though this is not the case in real markets.

- Constant volatility: Since the underlying volatility is assumed to be constant, all volatility variations occur in the sampling error. A short-term volatility estimate will generate a more volatile volatility estimate than a long-term volatility estimate.

2.4 GARCH Models

A time series method called Auto-Regressive Conditional Heteroskedasticity (ARCH) predicts volatility as a function of prior returns. The generalised ARCH (GARCH) model of volatility is based on past returns and volatility Metrics, 1996. This is the fundamental difference between the ARCH and GARCH models. According to GARCH, the value of volatility today is determined not just by yesterday's return but also by yesterday's volatility. ARCH models generate bursting, which causes volatility to rise and decrease instantly. GARCH models handle this by taking into account previous volatility values. Volatility is determined by q previous volatility and p past returns, according to the GARCH(p,q) model. In general, the GARCH models predict volatility based on one or more previous volatility and return periods. It varies from the EWMA model in that it determines volatility in a more generalised manner (Dowd, 2007). The GARCH(p,q) model is given as

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \cdots + \alpha_p r_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 > 0$$

where $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \geq 0$. The constraints on the parameter values ensure that the conditional variance is always positive. Different GARCH models use a different number of past terms. The number of parameters is determined by the minimum that fits the data acceptably (Dowd, 2007).

GARCH(1,1) is the most popular GARCH model. It is given as

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{2.5}$$

where $\omega \geq 0, \alpha, \beta \geq 0, \alpha + \beta < 1$. This model usually fits the data significantly well. This can be shown by estimating the GARCH(1,1) parameters on statistical software such as SAS. The α and β give information about the movement of volatility. A high value of α means that the volatility has pronounced peaks and troughs, that is, it reacts to market movements quickly. A high value of β means that volatility takes a long time to change. Usually the values of α and β are less than 0.25 and over 0.7, respectively (Dowd, 2007).

The EWMA model is a special case of the GARCH(1,1) where $\omega = 0$, $\alpha = 1 - \lambda$ and $\beta = \lambda$, resulting in

$$\sigma_{1,t}^2 = (1-\lambda) r_{1,t-1}^2 + \lambda \sigma_{1,t-1}^2. \tag{2.6}$$

The main distinction between the EWMA and the GARCH model is the addition of a positive ω element in the GARCH model. This is the GARCH(1,1) model's mean reverting component. That is, if volatility becomes too high, it tends to fall back to long-run variance over time, and if volatility becomes too low, it tends to climb back to long-run variance over time. The long-run variance is the value to which the variance tends to revert over time (Dowd, 2007). It is given by

$$V_{LR} = \frac{\omega}{1-\alpha-\beta}. \tag{2.7}$$

Factoring (2.7) into (2.5) we get

$$\begin{aligned} \sigma_t^2 &= (1-\alpha-\beta)V_{LR} + \alpha r_{t-1}^2 + \beta \sigma_{t-l}^2 \\ \sigma_t^2 - V_{LR} &= \alpha(r_{t-1}^2 - V_{LR}) + \beta(\sigma_{t-1}^2 - V_{LR}) \end{aligned} \tag{2.8}$$

Now, assuming that the forecast is for k periods, (2.8) becomes:

$$\sigma_{t+k}^2 - V_{LR} = \alpha(r_{t+k-1}^2 - V_{LR}) + \beta(\sigma_{t+k-1}^2 - V_{LR}) \tag{2.9}$$

Given that the expected value $E[r_{t+k-1}^2] = \sigma_{n+k-1}^2$, it follows that

$$E[\sigma_{t+k}^2 - V_{LR}] = (\alpha+\beta)(\sigma_{n+k-1}^2 - V_{LR}). \tag{2.10}$$

Therefore the forecast for the k^{th} period ahead is given by

$$E(\sigma_{t+k}^2) = V_{LR} + (\alpha+\beta)^k(\sigma_t^2 - V_{LR}) \tag{2.11}$$

Since $\alpha + \beta < 1$, the second term in (2.11) falls away as k increases, therefore the variance forecast converges to the long-run variance. This is the mean reversion of GARCH forecasts as they revert back to the long-run variance V_{LR}. The forecasts are then used to obtain estimates of the volatility term structure using geometric returns. The return and variance at time t over the next n periods are:

$_{i=1} \sum_{j=1}^{n} corr_t(r_{t+i}, r_{t+j})$ (2.12)

The correlation terms are usually small relative to the volatilities and co-variances, therefore

$$var_t(r_{t,n}) \approx \sum_{i=1}^{n} var_t(r_{t+i}). \tag{2.13}$$

2.4.1 Estimating GARCH Models

In the book, Measuring market risk (Dowd, 2007), these are the steps provided for the estimation of GARCH models:

1. Apply a simple filter, such as an autoregressive moving average (ARMA) model, to the returns. This is done to eliminate any serial correlations in the data.

2. The residuals are squared and conditional heteroscedasticity is assessed (changing variance). There are several conventional tests for this, including Pierce tests and Ljung-Box tests.

3. To choose a certain GARCH specification, look at the partial autocorrelation coefficients or other data. These include the selection of a specific noise process, either normal or t process.

4. Use a maximum likelihood technique to estimate the model's parameters.

5. Check the GARCH model's sufficiency by ensuring that the standardised innovations are i.i.d. and follow the anticipated noise distribution.

The GARCH model has several advantages over the simple moving average (SMA) method for forecasting volatility. Firstly, GARCH is a more sophisticated model that takes into account the complex dynamics of financial markets, including the tendency for volatility to cluster and for large changes in volatility to occur. In contrast, SMA is a simple method that only considers the average of past observations, without taking into account any underlying patterns or dynamics.

Secondly, GARCH provides a more accurate forecast of volatility by incorporating both past and current information about volatility. This is because the GARCH model includes both autoregressive and moving average components that allow for the model to adapt to changing market conditions. In contrast, SMA only considers past data, so it may not be able to capture changes in volatility that occur in real-time.

2.5 Exponentially Weighted Moving Average

The Exponentially Weighted Moving Average Model is a refinement of the SMA. This model places more emphasis on recent data and less emphasis on historical data. The estimate is given as

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i r_{t-i}^2, \sum_{i=1}^{n} \alpha_i = 1 \tag{2.14}$$

where the weights α_i decline as i increases, the weights sum up to 1. In the EMWA, the weights decrease exponentially with time. That is, the influence of an observation decreases with time because the EMWA model depends on the λ parameter, $\frac{\alpha_{i+1}}{\alpha_i} = \lambda$, where the constant λ falls between 0 and 1. The volatility forecasting equation is given by

$$\sigma_t^2 \approx (1-\lambda) \sum_{i=1}^{n} \lambda^{i-1} r_{t-i}^2 \tag{2.15}$$

The approximation is valid if the n is sufficiently large. The derivation of the volatility estimate is presented in the Research Method chapter:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2 \tag{2.16}$$

From this, we see that the volatility estimate for time t, σ_t, obtained at $t-1$ is calculated using the volatility estimate, σ_{t-1} from the previous time point and the return r_{t-1} from the previous time point. This is a straightforward updating rule that allows us to adjust the volatility from time to time based on the most recent return (Dowd, 2007). A high value of λ indicates a slower weight loss, whereas a low value indicates a rapid weight loss. The value is calculated using the available data, but the RiskMetrics-Technical Document (Metrics, 1996) specifies a value of 0.94 for daily return data and 0.97 for monthly return data. Advocating a single number implies that it will remain constant over time. However, this has been proven to be false.

Chapter 3

Research Data

3.1 Selection of Stocks

In this chapter, the data for the investigation are discussed. The data consist of daily prices of 10 equities each from the US, UK, and RSA, representing two developed and one emerging market. The equities are selected from various industries to diversify the portfolio risk. The data are obtained from Yahoo Finance (Finance, n.d.) from January 1, 2012, to March 31, 2022. The portfolio is constructed using equal weighting, which means that each asset has the same proportion in the portfolio. This is a simple asset allocation method that does not require any optimisation or estimation of expected returns or covariances. Portfolio optimisation using Modern Portfolio Theory is not the focus of this study, but rather the impact of different risk measures on portfolio performance. The standard deviation volatility is calculated using one year of historical data and the k-factor must be estimated using data from a year of VaR.

3.1.1 Johannesburg Stock Exchange Stocks

The following industries are represented by the stocks chosen for the South African market:

- Pharmaceutical: Adcock Ingram Holdings Limited was established in 1890 in Krugersdorp by EJ Adcock as a pharmacy. It produces, sells, and distributes medical supplies both domestically and abroad. The South African city of Midrand serves as its headquarters. It was worth $R9.38B$ as of October 2023.

- Mining, precious metals, and other metals: Ernest Oppenheimer, a businessman, formed Anglo American Platinum Limited in 1917. Before May 2011, it was known as Anglo Platinum Ltd. The business produces and supplies platinum group metals, base metals, and precious metals both domestically and abroad. Johannesburg, South Africa, serves as its corporate headquarters. It was valued at $R188.36B$ in October 2023.

- Accommodations: Hans Enderle founded City Lodge Hotels Limited in 1985 with financial support from the Mine Pension Funds. The company's main office is in Bryanston, South Africa. The corporation manages hotels both domestically and abroad. It was worth $R2.62B$ as of October 2023.

- Financial Services: Discovery Limited was founded in 1992 and is based in Sandton, South Africa. It offers private medical insurance products, commercial short-term risk insurance products, as well as health, life, car, and home insurance, as well as insurance for vehicles, buildings, the contents of homes, and movable possessions. Additionally, it offers managed care services, financial solutions, investment products, and retail banking services like deposits, loans, and advances, as well as Vitality, a program that promotes healthy living. It was worth $R92.21B$ in October 2023

- Pharmaceutical Retailers: Before changing its name in June 2009, Clicks Group Limited was known as New Clicks Holdings Limited. Its headquarters are in Cape Town, South Africa, where Jack Goldin founded it. It was worth $R61.99B$ in October 2023.

- Thermal coal: Before changing its name in November 2006, Exxaro Resources Limited was known as Kumba Resources Limited. Mxolisi Mgoyo founded it in 2000, and it is headquartered in Centurion, South Africa. The business creates semi-sifted coking coal, thermal coal, and metallurgical coal. Its estimated value as of October 2023 was $R63.43B$.

- Real Estate Investment Trust: Investec Property Fund Limited was established in 2008. Investec Property Proprietary Limited is in charge of managing it. The fund consists of R27 billion in direct and indirect South African real estate investments. It was worth $R14.9B$ in October 2023.

- Medical Care Facilities: Dr Jackie Shevel started Netcare Limited. It was founded in 1996, and Sandton, South Africa, serves as its corporate home. Hospitals are run by the corporation in South Africa. It was worth $R19.39B$ in October 2023.

- Energy and chemicals company: Sasol Limited was established in 1950 in Sasolburg, South Africa, to produce speciality chemicals. The South African Government Employees Pension Fund, Allan Gray, and Industrial Development Corporation of South Africa Limited are a few of the company's key shareholders. It engages in mining, gas, fuels, and chemical businesses. Johannesburg, South Africa, serves as its corporate headquarters. It was valued at $R166.03B$ in October 2023.

- Retail: Shoprite Holdings Limited is an investment holding company for the retail sale of food. Its segments include Furniture, Other Operating, Supermarkets RSA, and Supermarkets Non-RSA. Christo Wiese founded it in 1979, and its main office is in Brackenfell, South Africa. It has a $R142.71B$ market value in October 2023.

3.1.2 London Stock Exchange Stocks

The following industries are represented by the stocks chosen on the British market:

- Financial services: Henry Allan Engelhardt founded Admiral Group plc in 1993 as an insurance company. The business offers automobile insurance solutions both domestically and abroad. It has a October 2023 value of £$7.38B$.

- Asset management: Aberdeen Diversified Income and Growth Trust is a fund that is jointly held by Aberdeen Fund Managers Limited and BlackRock Investment Management (UK) Limited. The fund makes investments in securities from diverse companies operating in various industries. It was established in January 1898 and valued at £$230.77M$ in October 2023.

  A fund called Scottish Mortgage Investment Trust plc was established in 1909. Baillie

Gifford Co. launched it. The fund is managed by them, and their global headquarters are in the UK. It was worth £12.777M as of October 2023.

Investment firm 3i Infrastructure plc specialises in making investments in infrastructure. In January 2016, it was established. It was worth £3.7B in pounds as of October 2023.

In 1945, 3i Group plc was established. It is a private equity firm with a focus on management leveraged buyouts and buy-ins, middle markets, infrastructure, mature companies, growth capital, and middle markets. Additionally, it offers debt management and financing. Its main office is in London, United Kingdom. It was valued at £18.558M in October 2023.

- Biotechnology company: Abcam plc was founded in 1998. It is a biology-based business that specialises in finding, creating, and disseminating reagents and equipment for scientific study, diagnostics, and medication discovery. Its corporate headquarters are in Cambridge, Great Britain. It had a value of £986.1M in the financial year 2022. The company 4D Pharma plc was founded in 2014. The business creates live biotherapeutic items in the UK. Before that, it went by the name Schosweveen 18 Limited. Its main office is in Leeds, Great Britain. It was worth £2.6B in financial year 2022.

- Gambling: Avi, Aaron, Shay, and Ron Ben-Yitzhak created 888 Holdings plc in 1997. The business offers products and services for online gaming and betting. Gibraltar is home to its headquarters. It was worth £3.0B in financial year 2022.

- Agencies for advertising: 4imprint Group plc was founded in 1921. In North America, the UK, and Ireland, the business sells promotional products directly to consumers. Its main office is in London, United Kingdom. Its worth in the financial year 2022 was £240.3M.

- Application of software: 1Spatial plc was founded in 2005. In the United Kingdom, Ireland, the United States, and Australia, the company creates, markets, and supports IT products while also offering consulting and support services. Its corporate headquarters are in Cambridge, Great Britain. It was valued at £38.5M in financial year 2023.

- Publishing: Pearson PLC was founded in 1844. It creates educational materials, assessments, and services in the United Kingdom and around the world. The content is intended for Assessment and Qualifications, English Language Learning, Higher Education, Virtual Learning, and Workforce Skills. Its headquarters are in London, United Kingdom. It was worth £1.507M in December 2019.

3.1.3 New York Stock Exchange Stocks

The stocks chosen for the market in the United States of America are from the following industries:

- Telecommunications Services: AT&T Inc is a global telecommunications, media, and technology services provider. It was founded in 1983 and was previously known as SBC Communications Inc until its name was changed in 2005. Its headquarters are in Dallas, Texas. It was valued at \$402.9B in the financial year 2022.
- Diagnostics and research: Danaher Corporation was founded in 1969. The company creates, manufactures, and sells professional, medical, industrial, and commercial products and services all over the world. Its businesses include biology, diagnostics, and environmental and applied solutions. Its headquarters are in the District of Columbia, Washington. It was valued at \$160.39B in October 2023.
- Mining, precious metals, and other metals: Freeport-McMoRan Inc was founded in 1987. The company's operations include mineral property mining in North America, South America, and Indonesia. The company's headquarters are in Phoenix, Arizona. It was valued at \$58B in October 2023.
- Automobile Manufacturers: Elon Musk founded Tesla Inc in 2003. In the United States, China, and internationally, the company designs, develops, manufactures, leases, and sells electric vehicles as well as energy generation and storage systems. It was previously known as Tesla Motors, Inc until its name was changed in February

2017. The company's headquarters are in Austin, Texas. It was valued at \$870B in October 2023.

- Utilities: NextEra Energy Inc was founded in 1925. In North America, the company generates, transmits, distributes, and sells electricity to retail and wholesale customers. The company's headquarters are in Juno Beach, Florida. It was valued at \$153.57B in October 2023.

- Entertainment: Netflix Inc was founded in 1997. The company offers streaming services for TV shows, documentaries, feature films, and mobile games in a variety of genres and languages to its members. The company's headquarters are in Los Gatos, California. It was valued at \$119.74B in October 2023.

- Beverages: The Coca-Cola Company was established in 1886. This is a beverage company that manufactures, markets, and sells various non-alcoholic beverages all over the world. The company's headquarters are in Atlanta, Georgia. It was valued at \$87.6B in October 2023.

- Airlines: United Airlines Holdings Inc, founded in 1968, is an airline company. It operates in North America, Asia, Europe, Africa, the Pacific, and the Middle East. The company's headquarters are in Chicago, Illinois. It was valued at \$13.25B in October 2023.

- Discount Store: In 1945, Walmart Inc was founded. The company's global operations include retail and wholesale. It has stores in supercenters, supermarkets, hypermarkets, warehouse clubs, and cash and carry. The company's headquarters are in Bentonville, Arkansas. It was valued at \$550B in October 2023.

- Information Technology Services: Xerox Holdings Corporation was founded in 1906. The corporation is a technology firm that designs, develops, and sells document management systems and solutions in the United States, Europe, Canada, and around the world. The company's headquarters are in Norwalk, Connecticut. It was valued at \$2.38B in October 2023.

Chapter 4

Research Methodology

4.1 RiskMetrics Forecasting Methodology

This chapter introduces the RiskMetrics forecasting methodology, which employs the exponentially weighted moving average model (EWMA) to predict the variances and covariances of the multivariate normal distribution (Metrics, 1996). We have reproduced the derivation and implementation of the RiskMetrics forecast methodology, which is essential to understand the approach used in the dissertation. In comparison to the simple moving average model, this method produces more accurate results. It accomplishes this by focusing more on the most recent data. This chapter introduces the RiskMetrics forecasting methodology, which employs the exponentially weighted moving average model (EWMA) to predict the variances and covariances of the multivariate normal distribution (Metrics, 1996). We have reproduced the derivation and implementation of the RiskMetrics forecast methodology, which is essential to understand the approach used in the dissertation. In comparison to the simple moving average model, this method produces more accurate results. It accomplishes this by focusing more on the most recent data. This chapter introduces the RiskMetrics forecasting methodology, which employs the exponentially weighted moving average model (EWMA) to predict the variances and covariances of the multivariate normal distribution (Metrics, 1996). We have reproduced the derivation and implementation of the RiskMetrics forecast methodology, which is essential to understand the approach used in the dissertation. In comparison to the simple moving average (SMA) model, this method produces more accurate results. It accomplishes this by focusing more on the most recent data.

When adopting the EWMA model for estimating, volatility reacts faster to market surges since new data have a higher weight than older data. When the spike observation diminishes after the market surge, the volatility reduces exponentially. This removes any ghost effects from the findings.

To forecast volatility, the EWMA model takes use of previous data. In essence, the method's fundamental premise is that future patterns will be similar to historical ones. We observe and study the following attributes, as we would with any time series model:

- The term "trend" refers to the long-term movement of a time series, indicating whether it is increasing, decreasing, or remaining stable over time. A trend can be either linear or non-linear, and it is often used to identify the underlying direction of a time series.
- Seasonality recognises the pattern that repeats itself at fixed intervals within a given period, such as a year, a month, or a day. This pattern is often influenced by seasonal factors such as weather, holidays, or cultural events.
- Cycles are fluctuations in a time series that occur at irregular intervals and are not necessarily related to seasonal factors. These fluctuations can be caused by economic cycles, technological changes, or other factors that affect the underlying dynamics of the time series. Cycles can be either short-term or long-term, and they are often modelled using a cyclical component, which captures the cyclic fluctuations of the time series.

4.1.1 Volatility Estimation and Forecasting

Recall the volatility estimators of the equally weighted moving average model and the exponentially weighted moving average for a set of T returns as shown in Chapter 2

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \bar{r})^2} \tag{4.1}$$

and

$$\sigma = \sqrt{(1-\lambda)\sum_{t=1}^{T}\lambda^{t-1}(r_t - \bar{r})^2}, \tag{4.2}$$

respectively, where r_t is the portfolio return in month t, T is the sample period length, $\bar{r}$ is the mean return across the sample period, and λ $(0 \leq \lambda \leq 1)$ is the decay factor. The exponentially weighted moving average model is dependent on the decay factor. The relative weights given to the observations (returns) and the effective quantity of data utilised in calculating volatility are determined by λ. The decay factor is best determined for the volatilities (variations) and correlations (covariances) that are compatible with their respective covariance matrices. RiskMetrics works with 480 data points. There is an ideal decay factor that minimises the variance forecast's root mean square error (RMSE). RiskMetrics (Metrics, 1996), for example, uses RMSE as the forecast error measuring criteria. The best λ value must be based on the most current data and evolves. The decay factor for a 1-day prediction is 0.94, according to the RiskMetrics (Metrics, 1996).

1-day RiskMetrics Volatility Forecast

Recall Equation (2.14) from Chapter 2. This is a Geometric sum to infinity and allows us to approximate the volatility forecasts as

$$\sigma_t^2 \approx (1-\lambda)\sum_{i=1}^{n}\lambda^{i-1}r_{t-i}^2 \tag{4.3}$$

The approximation is valid assuming that $n \longrightarrow \infty$. The exponentially weighted estimator is utilised recursively. For the derivation of the recursive form, we assume the following assumption:

- Assume a limitless quantity of data are accessible.
- Assume that the sample mean is equal to zero, $r = 0$.

For the 1-day RiskMetrics volatility estimate, we take the data available at time t (one day earlier) to calculate the variance forecast for the period $t+1$ as follows

$$\sigma_{1,t+1|t}^2 = \lambda\sigma_{1,t|t-1}^2 + (1-\lambda)r_{1,t}^2 \tag{4.4}$$

Therefore, the formula for the 1-day RiskMetrics volatility estimate is

$$\sigma_{1,t+1|t} = \sqrt{\lambda\sigma^2_{1,t|t-1} + (1-\lambda)r^2_{1,t}} \tag{4.5}$$

The subscript $t+1|t$ is read as "the time $t+1$ forecast given information up to and including time t." This reading format follows for all the subscripts. This explicitly shows that the variance (volatility) is time-dependent (4.4) and is derived as follows

$$\begin{aligned}
\sigma^2_{1,t+1|t} &= (1-\lambda)\sum_{i=0}^{\infty}\lambda^i r^2_{1,t-i} \\
&= (1-\lambda)\{r^2_{1,t} + \lambda(r^2_{1,t-1} + \lambda r^2_{1,t-2} + \ldots)\} \\
&= (1-\lambda)\{r^2_{1,t} + \lambda r^2_{1,t-1} + \lambda^2 r^2_{1,t-2} + \ldots\} \\
&= (1-\lambda)r^2_{1,t} + \lambda(1-\lambda)(r^2_{1,t-1} + \lambda r^2_{1,t-2} + r^2_{1,t-3}) \\
&= \lambda\sigma^2_{1,t|t-1} + (1-\lambda)r^2_{1,t}
\end{aligned} \tag{4.6}$$

The covariance estimators for a set of T returns of both the equally weighted moving average model and the exponentially weighted moving average are given as follows

$$\sigma^2_{12} = \frac{1}{T}\sum_{t=1}^{T}(r_{1t} - \bar{r}_1)(r_{1t} - \bar{r}_2) \tag{4.7}$$

and

$$\sigma^2_{12} = (1-\lambda)\sum_{t=1}^{T}\lambda^{j-1}(r_{1t} - \bar{r}_1)(r_{1t} - \bar{r}_2) \tag{4.8}$$

respectively, where r_t is the portfolio return in month t, T is the length of the sample period, $\bar{r}$ is the mean return over the sample period and λ $(0 \leq \lambda \leq 1)$ is the decay factor.

The recursive form of the 1-day RiskMetrics covariance forecast is given as

$$\sigma^2_{12,t+1|t} = \lambda\sigma^2_{12,t|t-1} + (1-\lambda)r_{1t}.r_{2t} \tag{4.9}$$

(4.9) is derived as follows.

$$
\begin{aligned}
\sigma^2_{12,t+1|t} &= (1-\lambda)\sum_{i=0}^{\infty}\lambda^i r^{1,t-i}\cdot r^{2,t-i} \\
&= (1-\lambda)\{r_{1,t}\cdot r_{2,t} + \lambda r_{1,t-1}\cdot r_{2,t-1} + \lambda^2 r_{1,t-2}\cdot r_{2,t-2} + \ldots\} \\
&= (1-\lambda) r_{1,t}\cdot r_{2,t} + \lambda(1-\lambda)(r_{1,t-1}\cdot r_{2,t-1} + \lambda r_{1,t-2}\cdot r_{2,t-2} + \lambda^2 r_{1,t-3}\cdot r_{2,t-3}) \\
&= \lambda\sigma^2_{12,t|t-1} + (1-\lambda) r_{1,t-1}\cdot r_{2,t-1}
\end{aligned}
$$

Consequently, the 1-day RiskMetrics prediction of correlation is given as

$$
\rho_{12,t+1|t} = \frac{\sigma^2_{12,t+1|t}}{\sigma_{1,t+1|t}\cdot\sigma_{2,t+1|t}} \tag{4.10}
$$

4.2 Optimal Decay Factor

The volatility and correlation estimates from RiskMetrics need to be produced using the ideal decay factor, λ. We describe how to select the ideal decay factor in this section. We use an example with two return series, $r_{1,t}$ and $r_{2,t}$, to illustrate our point. The covariance matrices linked to these results are provided by

$$\sum = \begin{bmatrix} \sigma_1^2(\lambda_1) & \sigma_{12}^2(\lambda_3) \\ \sigma_{21}^2(\lambda_3) & \sigma_2^2(\lambda_2) \end{bmatrix}$$

Each variance and covariance is expressed explicitly as a function of its decay factor therefore the covariance matrix $\sum$ is a function of three decay factors, λ_1, λ_2 and λ_3. For $\sum$ to be a covariance matrix, the following conditions must be met:

- The variances σ_1^2 and σ_2^2 are non-negative.
- The variances σ_{12}^2 and σ_{21}^2 must be equal.
- The correlation between $r_{1,t}$ and $r_{2,t}$ has the range $-1 \leq \rho \leq 1$ where $\rho = \frac{\sigma_{12}^2}{\sigma_1 \sigma_2}$.

The whole covariance matrix is subject to a single optimum decay factor application by RiskMetrics, which means that separate decay factors are utilised for the daily volatility and correlation matrix and the monthly volatility matrices. To calculate a decay factor, 480 variance data point projections were used (Metrics, 1996). Although it is theoretically feasible to select ideal decay factors that are compatible with each covariance matrix, doing so, in reality, is far more difficult (Metrics, 1996).

4.2.1 Forecast Error Measure: Root Mean Squared Error Criterion

In this part, we go over the criteria RiskMetrics uses to pick the decay factor that will produce the best forecast accuracy.

Definitions:

- The forecast of time $t+1$ of the variance of the return r_{t+1} at time period t is given by

$$E_t[r^2_{1,t+1}] = \sigma^2_{1,t+1|t} \tag{4.11}$$

  A forecast made for time $t+j$, $j \geq 1$ is given by

$$E_t[r^2_{1,t+j}] = \sigma^2_{1,t+j|t} \tag{4.12}$$

  This is the expected value of the squared return one time period earlier.

- The forecast of the time $t+1$ of the covariance between two series $r_{1,t+1}$ and $r_{2,t+1}$ at time period t is given by

$$E_t[r_{1,t+1} \cdot r_{2,t+1}] = \sigma^2_{12,t+1|t} \tag{4.13}$$

  A forecast made at time $t+j$, $j \geq 1$ is given by

$$E_t[r_{1,t+j} \cdot r_{2,t+j}] = \sigma^2_{12,t+j|t} \tag{4.14}$$

Since the variance forecast error is

$$\epsilon_{t+1|t} = r^2_{t+1} - \sigma^2_{t+1|t}$$

the expected value of the forecast error is zero, from (4.9):

$$E_t[\epsilon_{t+1|t}] = E_t[r^2_{t+1}] - \sigma^2_{t+1|t} = 0.$$

To choose the optimal decay factor, the average squared errors must be minimised. Applying this to daily forecasts of variance, we get the daily root mean squared predictions error as

$$RMSE_v = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_{t+1}^2 - \hat{\sigma}_{t+1|t}^2(\lambda))^2} \tag{4.15}$$

where r_{t+1} is the portfolio return in month $t+1$, T is the length of the sample period, $\hat{\sigma}^2$ is the unbiased estimator of σ^2 and λ $(0 \leq \lambda \leq 1)$ is the decay factor. The expression of the variance's prediction value as a function of λ is given. To choose the best decay factor in practice, we look for the lowest RMSE across a range of λ (Metrics, 1996) values. This is an optimisation problem, linear programming to be exact. This is achieved on MS Excel VBA using the solver.

Similarly, for the accuracy of covariance forecasts, we have that the covariance forecast error is

$$\epsilon_{12,t+1|t} = r_{1,t+1}r_{2,t+1} - \sigma_{12,t+1|t}^2$$

such that, by (4.12) we have

$$E_t[\epsilon_{12,t+1|t}] = E_t[r_{1,t+1}r_{2,t+1}] - \sigma_{12,t+1|t}^2 = 0.$$

Then applying this to daily forecasts of covariance, we get the daily root mean squared predictions error as

$$RMSE_c = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_{1,t+1}r_{2,t+1} - \hat{\sigma}_{12,t+1|t}^2(\lambda))^2}$$

where r_{t+1} is the portfolio return in month $t+1$, T is the length of the sample period, $\hat{\sigma}^2$ is the unbiased estimator of σ^2 and λ $(0 \leq \lambda \leq 1)$ is the decay factor. It is crucial to keep in mind that although the criteria listed above are solely statistical, they might not be the greatest option for risk management because other variables are taken into account when choosing the best projection (Metrics, 1996). To guarantee that the variance and co-variance predictions are helpful for risk managers who do not update their systems daily, for example, the ideal decay factor must provide for enough stability on such estimates

(West, Edison, and Cho, 1993).

Mathematical Optimisation Problem:

The objective of the optimisation problem is to find the value of that minimises the $RMSE_v$, subject to the constraint that λ must be between 0 and 1. This is a constrained optimisation problem, and various optimisation techniques can be used to find the optimal value of λ that satisfies the constraint and minimises the objective function. The optimisation problem is formulated as follows

$$(OP)\begin{cases} minRMSE_v = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_{t+1}^2 - \hat{\sigma}_{t+1|t}^2(\lambda))^2} \\ s.t. \\ 0 \leq \lambda \leq 1 \end{cases}$$

where:

- $RMSE_v$ is the root mean square error with respect to the unknown decision variable λ,
- r_{t+1}^2 is the portfolio return in month $t+1$,
- T is the length of the sample period,
- $\hat{\sigma}_{t+1|t}^2$ is the unbiased estimator of σ^2 for the period $t+1$, given the value of λ,
- and λ is the decision variable representing the decay factor that affects the calculation of the estimator $\hat{\sigma}_{t+1|t}^2$.

4.2.2 RiskMetrics: The Optimal Decay Factor

In the application of the EWMA method, $T = 250$. The forecast error measuring criteria is selected to be the $RMSE$. Basel recommends the size of the historical data to be a minimum of one year and a maximum of 3 years.

For the daily data sets, one optimal decay factor is computed from the 480 data points. Let $\hat{\lambda}_i$ the i^{th} optimal decay factor where $1 \leq i \leq N$ with N denoting the number of time series in the RiskMetrics database. Let τ_i be the i^{th} RMSE associated with $\hat{\lambda}_i$, that is, τ_i is the minimum RMSE for the i^{th} time series. The decay factor is derived as follows (Metrics, 1996). This is the linear programming problem:

1. Find $\prod$, the sum of all N minimal RMSE's, τ_i's:

$$\prod = \sum_{i=1}^{N} \tau_i \tag{4.16}$$

2. Define the relative error measure:

$$\theta_i = \frac{\tau_i}{\sum_{i=1}^{N} \tau_i} \tag{4.17}$$

3. Define the weight ϕ_i:

$$\phi_i = \frac{\theta_i^{-1}}{\sum_{i=1}^{N} \theta_i^{-1}} \tag{4.18}$$

4. The optimal decay factor λ is defined as

$$\lambda = \sum_{i=1}^{N} \phi_i \hat{\lambda}_i \tag{4.19}$$

   The optimal decay factor applied by RiskMetrics is a weighted average of individual optimal decay factors where the weights are a measure of individual forecast accuracy.

4.2.3 Interpretation of Optimal Decay Factor

Consider the two terms in (4.4).

- Term 1: $\lambda\sigma^2_{1,t|t-1}$, the dictates the persistence in volatility. Regardless of what happens in the market, if the volatility was high yesterday it will be high today. The closer λ is to 1, the more persistent the volatility will follow the market shock (Alexander, 2008). The closer λ is to 1, the more the EWMA is like the SMA.

- Term 2: $(1-\lambda)r^2_{1,t}$, this dictates the intensity of the reaction of volatility to market events. The smaller λ is, the more the volatility reacts to the data observed in the previous day's return (Alexander, 2008).

Consider the term $\sum_{i=1}^{T}\lambda^{i-1} \cong \frac{1}{(1-\lambda)}$ in (4.3). This equivalence is true for $T \longrightarrow \infty$. To compare with the SMA, a better representation of the EWMA is (Metrics, 1996):

$$\frac{\lambda^{t-1}}{\sum_{i=1}^{T}\lambda^{i-1}} \tag{4.20}$$

as opposed to $(1-\lambda)\lambda^{t-1}$.

From this, it is clear to see that (4.20) is $\frac{1}{T}$ for $\lambda = 1$. That is, the EWMA is a SMA for $\lambda = 1$.

4.2.4 EWMA Implementation

For the implementation of this research project, the volatility used to calculate VaR is estimated using two different methods: simple moving averages (SMA) and exponentially weighted moving averages (EWMA). After estimating the volatility using both methods, the number of exceptions obtained in the VaR calculation is checked to determine which method has fewer exceptions. By comparing the results of both methods, the research aims to identify which volatility estimation method is more effective for calculating VaR.

Hypothesis

The null hypothesis for the research is: There is no significant difference in the number of exceptions obtained in the VaR calculation using simple moving averages (SMA) and

exponentially weighted moving averages (EWMA).
The alternative hypothesis is: The number of exceptions obtained in the VaR calculation using EWMA is lower than the number obtained using SMA.

The data used for the implementation are described in Chapter 3. We implement the EWMA in three different economies, two developed countries and one developing country:

- South Africa (SA), United Kingdom (UK), and the United States of America (USA).
- 10 stocks each from the Johannesburg Stock Exchange (JSE), London Stock Exchange (LSE) and New York Stock Exchange (NYSE).
- Three diversified portfolios.
- The data are collected at a daily frequency from January 1, 2012, to March 31, 2021, from Yahoo Finance. Note that this includes the Global COVID-19 pandemic period.
- Build up a time series of λs using one year of historical data and rolling the sample period forward three months each time to determine a new measurement.

Chapter 5

Results and Analysis

5.1 Exponentially Weighted Moving Average and Simple Moving Average

While all the stocks in the respective portfolios are used in the model and portfolio results are shown where relevant, specific stock results are used to demonstrate certain points.

5.1.1 UK Stocks

The SMA method applies equal weights to all the returns, despite how old or recent it is. Unlike the SMA method, the EWMA model depends on the λ parameter known as the decay factor. This determines the relative weights that are applied to the returns and the amount of data used in estimating volatility (Metrics, 1996).

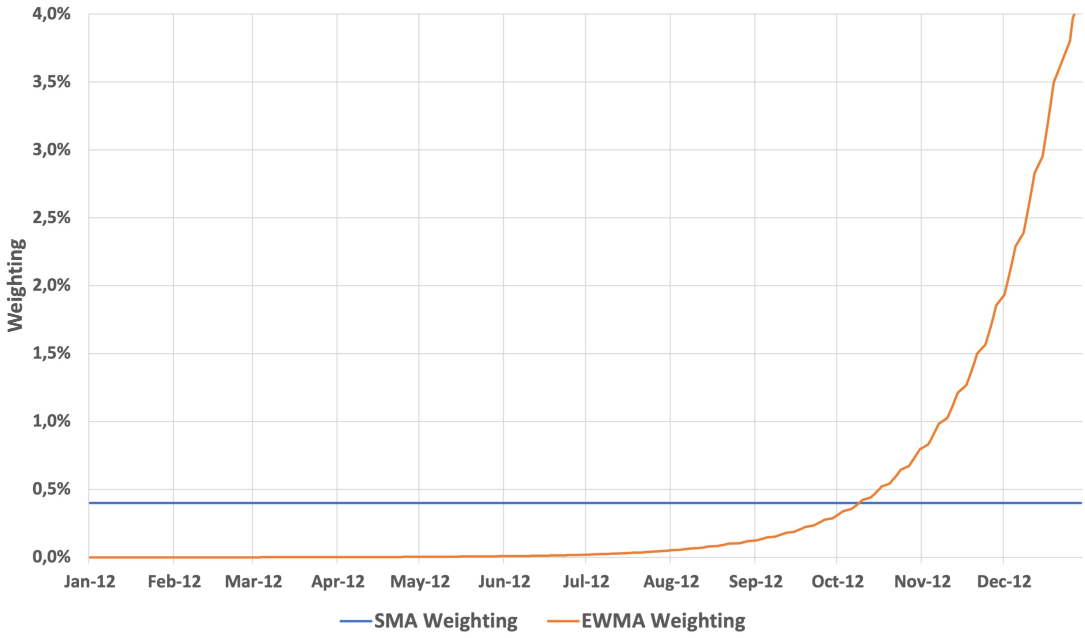


Figure 5.1: A comparison between the EWMA and SMA weighting: January 2012 - December 2012, UK Stocks.

Figure 5.1 shows the difference in how the data is weighted by the EWMA and the SMA. Consider the investment portfolio consisting of stocks in different industries (developed economy), e.g. asset management, insurance and biotechnology. Figure 5.2 shows a time series of λs built using one year of historical data and rolling the sample period forward three months each time to get a new measurement. For the UK economy, throughout the 2012 - 2021 time period, the decay factor varies considerably: $\lambda_P \in [0.79, 0.99]$. Consider turbulent markets: October 2015 - September 2016, $\lambda_P \in [0.79, 0.92]$ from $\lambda_P = 0.96$ during the previous time point and April 2019 - December 2020, $\lambda_P \in [0.85, 0.87]$ from $\lambda_P = 0.97$ during the previous period. The BREXIT vote and COVID-19 pandemic happened during these turbulent markets in October 2015 - September 2016 and March 2020 - December 2020, respectively. The decay factor considers the state of the economy. λ is at its lowest so the EWMA weighting is more on the most recent data and therefore gives a better reflection of the volatility of the market.

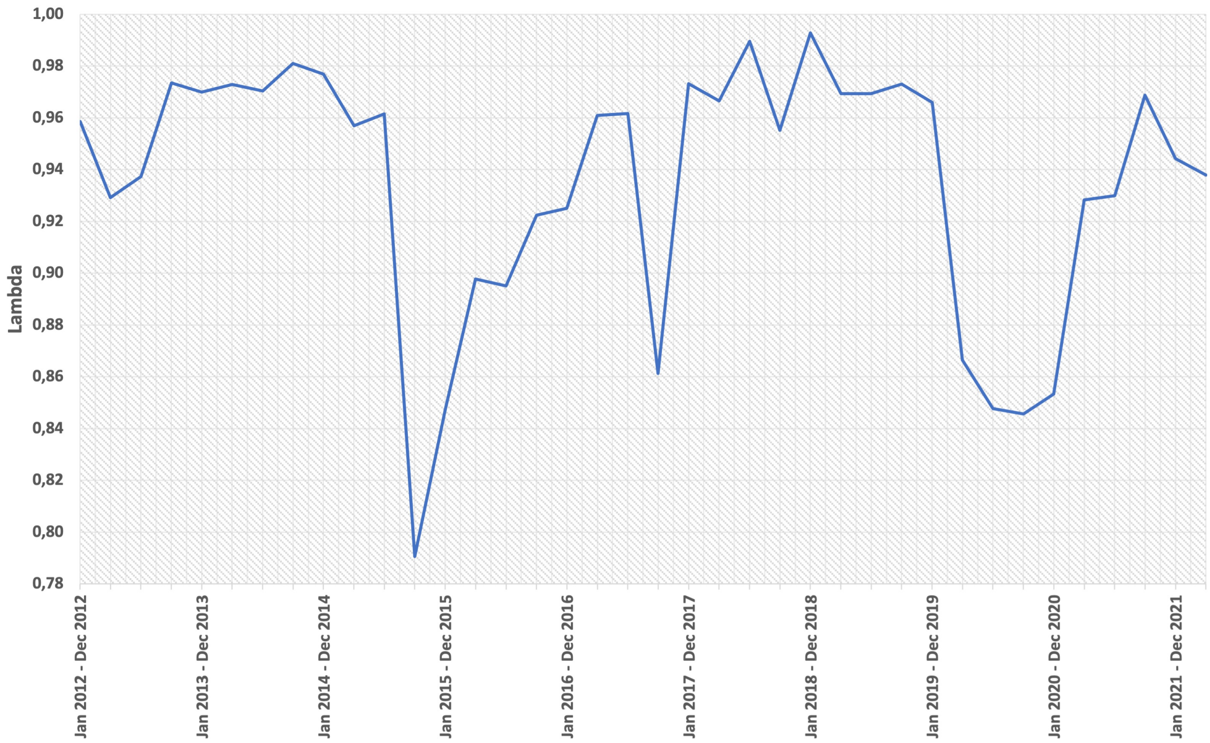


Figure 5.2: UK λ series measured over quarters from 2012 to 2021.

Figure 5.3 depicts a disadvantage of the SMA. It requires a time window for the data before before the volatility can be determined, whereas the EWMA volatility does not. To make the two volatility metrics similar, a cutoff is applied such that the SMA period is used instead, as seen in 5.4.

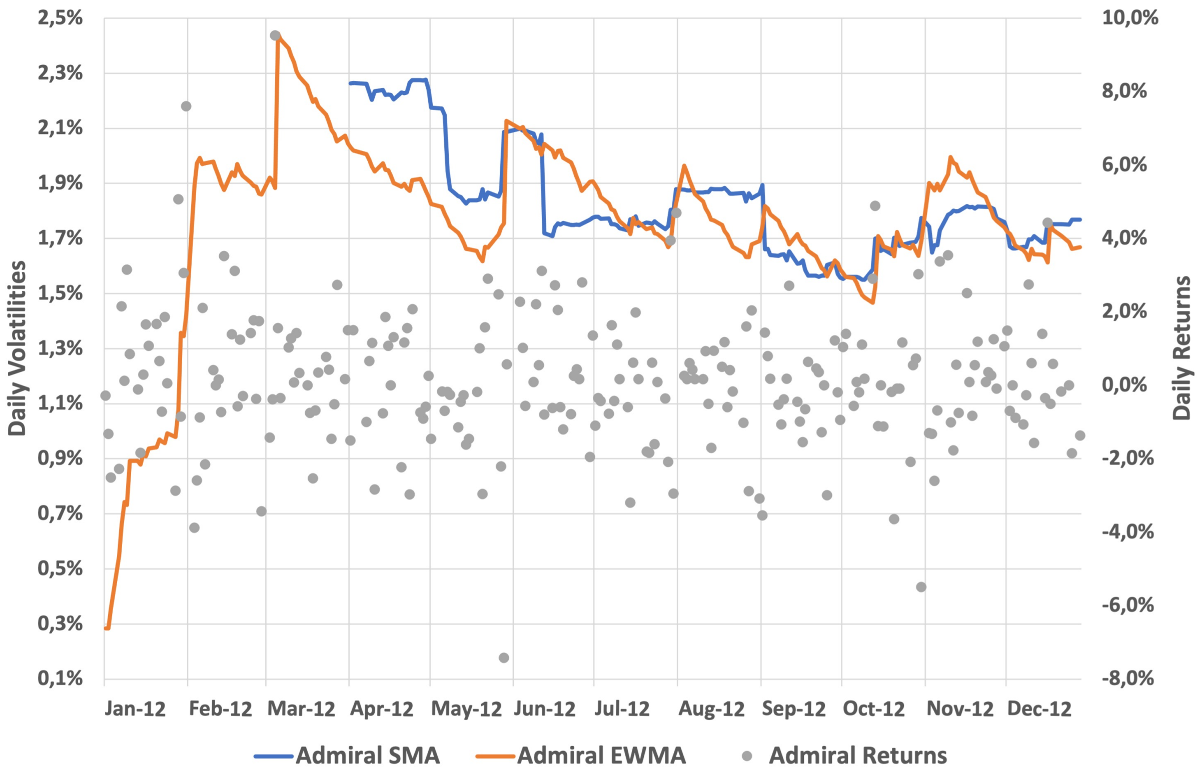


Figure 5.3: A comparison between the EWMA and SMA volatility: January 2012 - December 2012, Admiral Group plc.
$\lambda_{Admiral} = 0.97$.

Figure 5.4 illustrates a major difference between the EWMA and the SMA reaction time using the Admiral Returns in 2012. In May the return dropped significantly, from -2.2% to -7.4%. The volatility estimate obtained from the EWMA model rapidly indicated this with an increase from 1.7% to 2.1%. In mid-June, there is a sudden drop in the volatility estimate obtained by the SMA model from 2.1% to 1.7%. This is not a good representation of the stock at this time point because there was no sudden or rapid change in the Admiral return during this time frame. In August the return has a rise to 4.7% followed by a drop to 0.4% in mid-August where it was normalising. The EWMA reflects these shocks promptly with a gradual decrease from 1.9% to 1.7% while the SMA took longer to show the returns normalising.

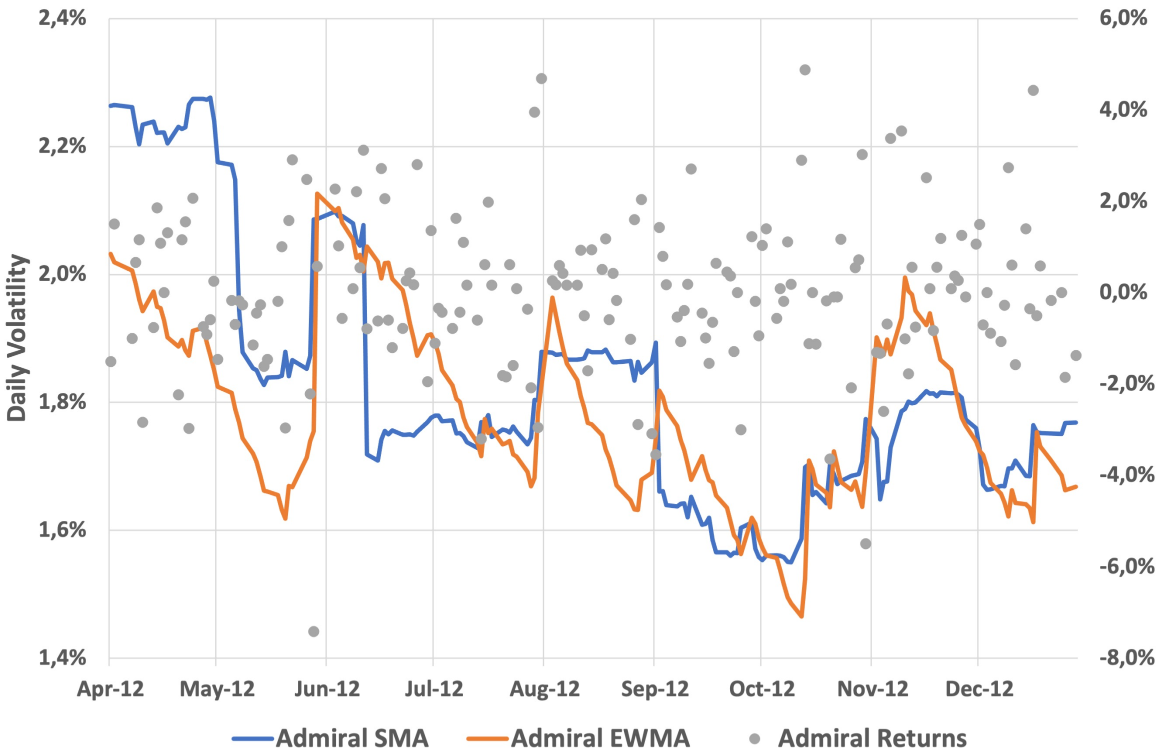


Figure 5.4: A comparison between the EWMA and SMA volatility: April 2012 - December 2012, Admiral Group plc.
$\lambda_{Admiral} = 0.97$.

Figures 5.5 and 5.6 display, respectively, the λs for the stocks for April 2012 to March 2013 and returns for 4imprint from July 2012 to March 2013. The return is generally unstable over this time, with both positive and negative outliers. The SMA typically does not represent the spikes and drops of the return properly, but the EWMA more accurately captures the volatility of the returns.

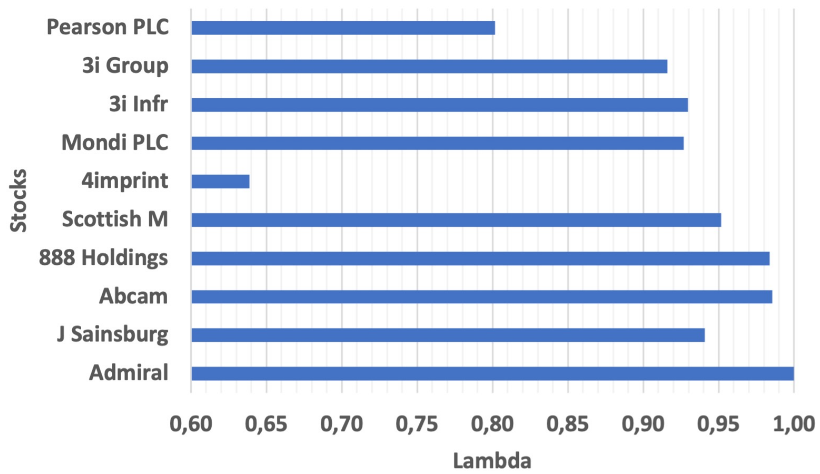


Figure 5.5: Decay parameters, λ for UK stocks: April 2012 - March 2013. $\lambda_{Portfolio} = 0.93$.

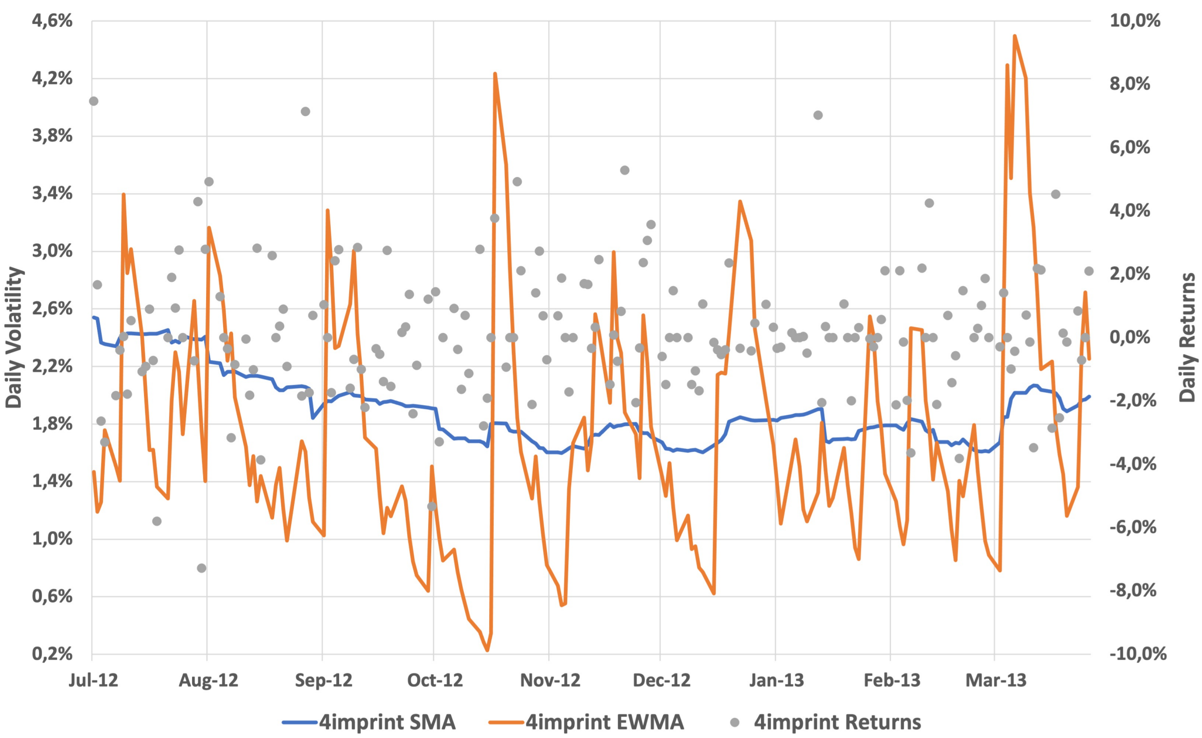


Figure 5.6: A comparison between EWMA and SMA volatility: July 2012 - March 2013, 4imprint Group plc. $\lambda_{4imprint} = 0.64$.

Consider Figure 5.7. In late January, the SMA shows a sudden drop in volatility from 2.08% to 1.3%, however there is not drastic change daily returns to constitute the drop. $\lambda_{4imprint} = 0.99$ is close to one as a result we see that the two moving averages follow a relatively similar trend as demonstrated in Chapter 4 (4.20).

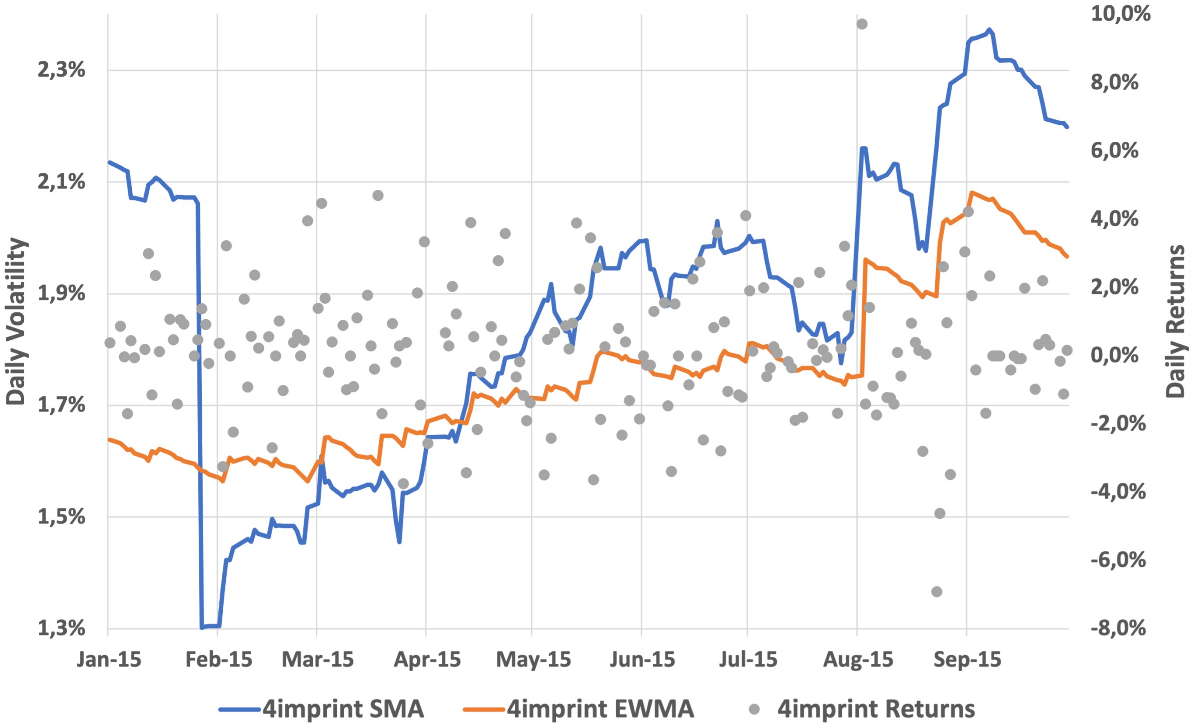


Figure 5.7: A comparison between EWMA and SMA volatility: January 2015 - September 2015, 4imprint Group plc.
$\lambda_{4imprint} = 0.99$

The volatility of the Covid-19 period is shown in Figure 5.8. When the returns fluctuated rapidly in March, the EWMA reflected in the volatility much faster than the SMA. The SMA also takes longer to normalise after May as the returns gradually normalise. This is the ghost effect of the rapid fluctuation of the returns in March.

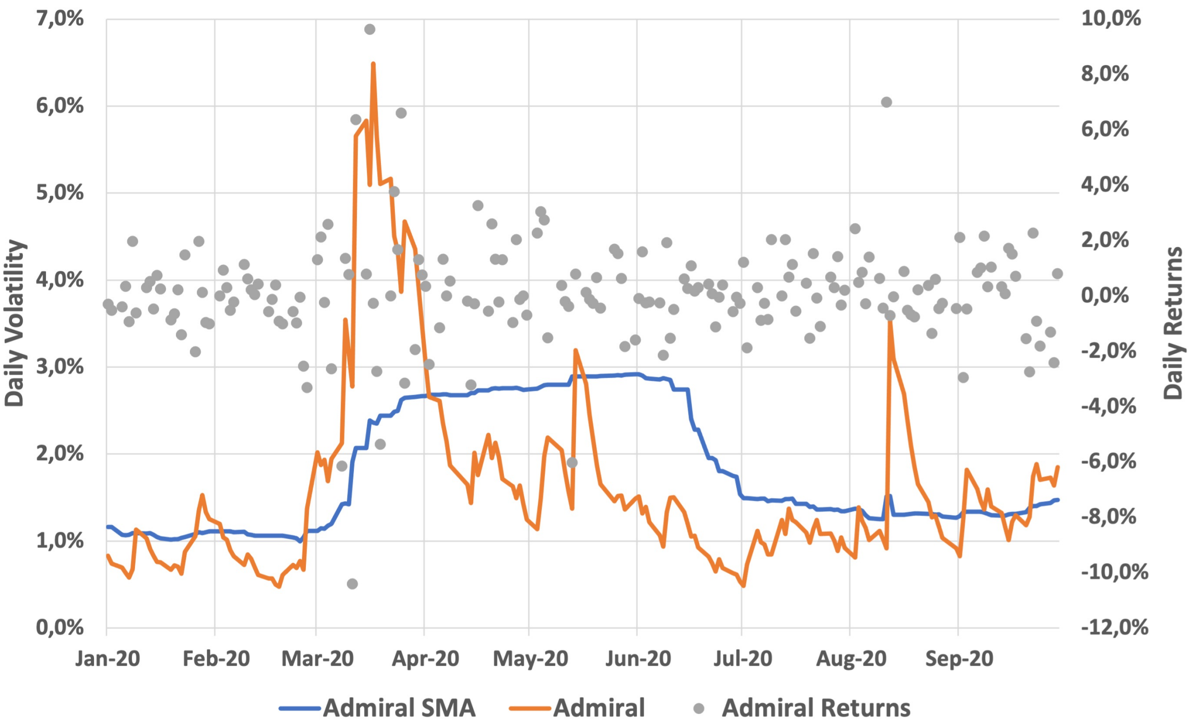


Figure 5.8: A comparison between EWMA and SMA volatility: January 2020 - September 2020, 4imprint Group plc.
$\lambda_{4imprint} = 0.73$

5.1.2 RSA Stocks

Consider the investment portfolio consisting of stocks in different industries (developing economy), e.g. pharmaceutical manufacturing, lodging and mining. Figure 5.9 shows a time series of λs built using one year of historical data and rolling the sample period forward three months each time to get a new measurement. For the RSA economy, throughout the 2012 - 2021 time period, the decay factor varies considerably: $\lambda_P \in [0.84, 0.99]$. Consider turbulent markets: January 2016 - December 2017, $\lambda_P \in [0.90, 0.98]$ from $\lambda_P = 0.96$ during the previous time point and January 2019 - December 2021, $\lambda_P \in [0.84, 0.97]$ from $\lambda_P = 0.96$ during the previous period. In particular, during the final quarter of 2016, production in the mining and manufacturing sectors fell during the period of the turbulent market from January 2016 to December 2017. The pandemic occurred between March 2020 and December 2021, during the turbulent market.

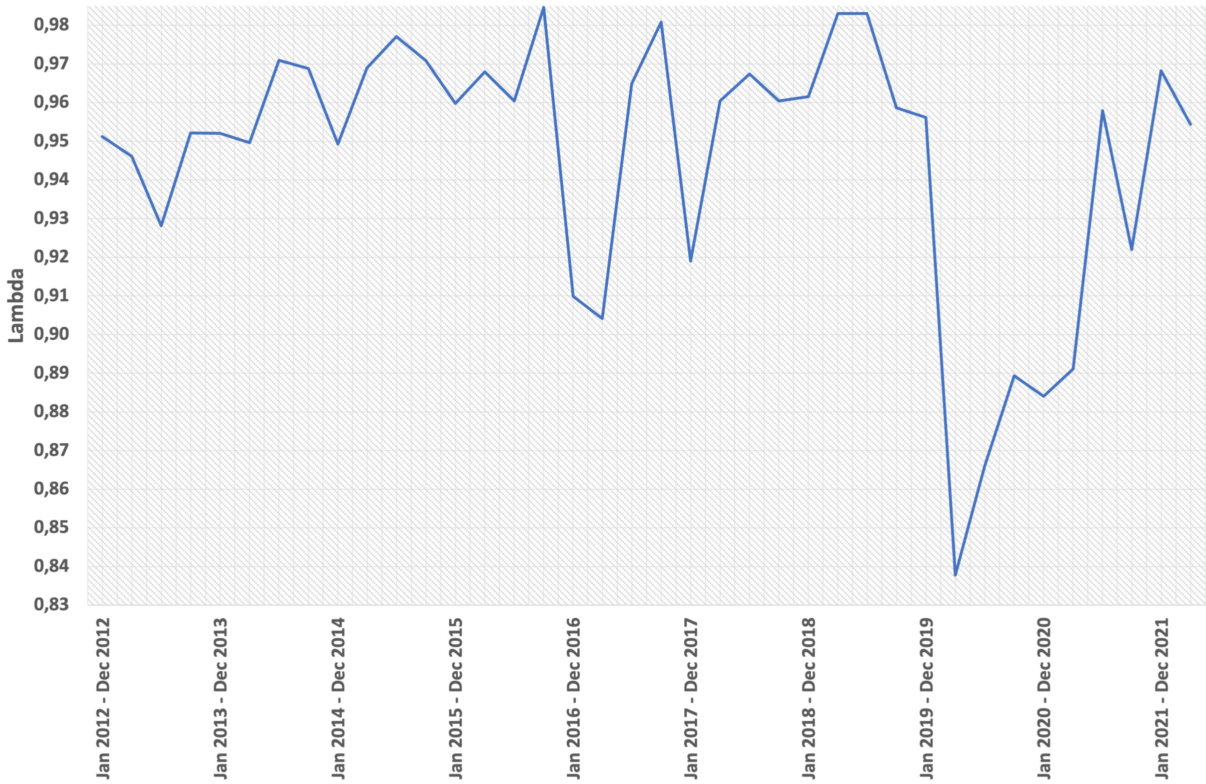


Figure 5.9: RSA λ series measured over quarters from 2012 to 2021.

Figure 5.10 shows λ for the different stocks from January 2012 to December 2012. Consider the lodging industry as an example to observe the impact of the economy on stocks, $\lambda_{City} = 0.96$. From Figure 5.11, we see the result of the value of λ_{City} being close to 1. That is, the EWMA approaches the SMA or greater values of λ and is the SMA for $\lambda = 1$. It is clear to see that the EWMA and the SMA follow a relatively similar path but both have significantly different values. For instance, in mid-May, the EWMA and SMA volatilities are 1.9% and 1.7% respectively. The return had fluctuated from -1.7% to 1.6%.

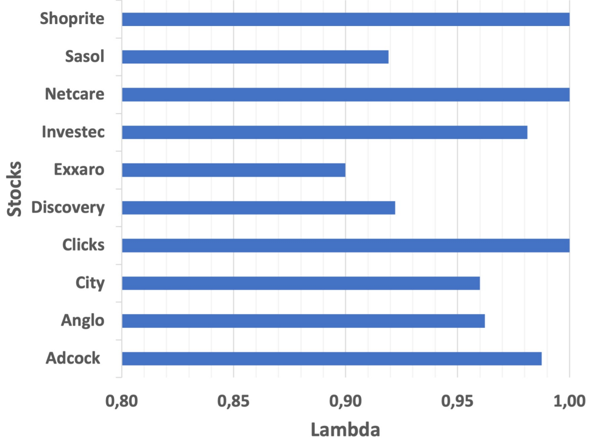


Figure 5.10: Decay parameters, λ for RSA stocks: January 2012 - December 2012. $\lambda_{Portfolio} = 0.95$

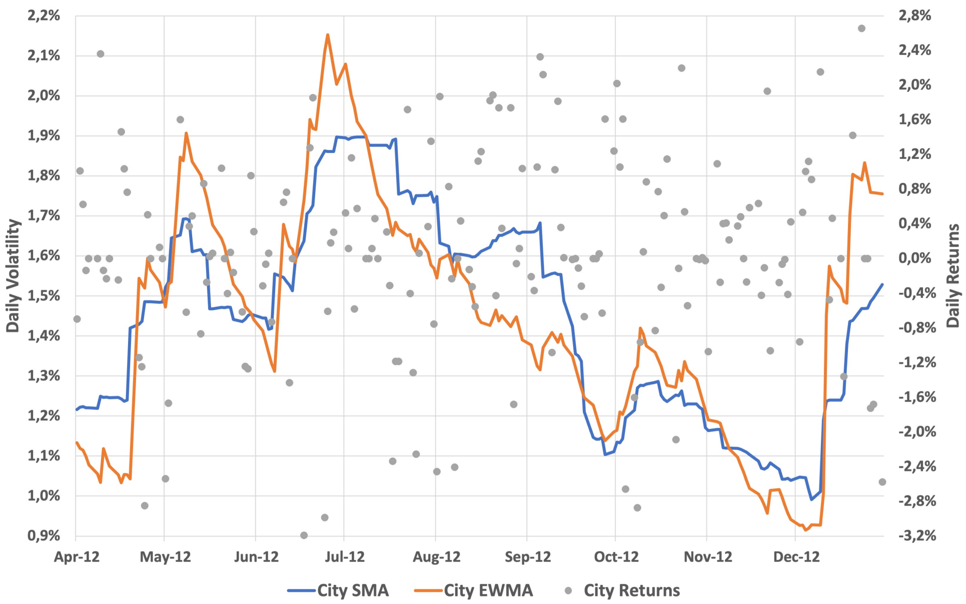


Figure 5.11: A comparison between EWMA and SMA volatility: January 2012 - December 2012, City Lodge Hotels. $\lambda_{City} = 0.96$

Consider λs for the period from July 2012 to June 2013. $\lambda_{Anglo} = 0.89$. is shown in Figure 5.12. A very low value, causes the EWMA to deviate completely from the SMA. Figure 5.13 shows that the returns are more volatile during January, May, and June. In mid-January, returns range between 6.1% and 4.9%. While the EWMA reflects this volatility, the SMA has changed much less. Throughout the month, the EWMA volatility swings between 1.04% and 2.54%, while the SMA swings between 1.5% and 1.56%. The conclusions drawn from Figures 5.10 to 5.13 emphasise the significance of weighting data exponentially rather than equally. Observations that result in significantly low values for λ will be completely unrepresented if the SMA is used.

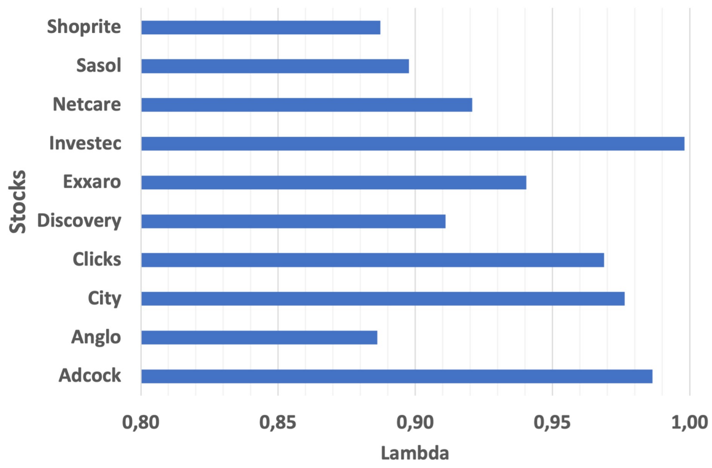


Figure 5.12: Decay parameters, λ for RSA stocks: July 2012 - June 2013.
$\lambda_{Portfolio} = 0.93$

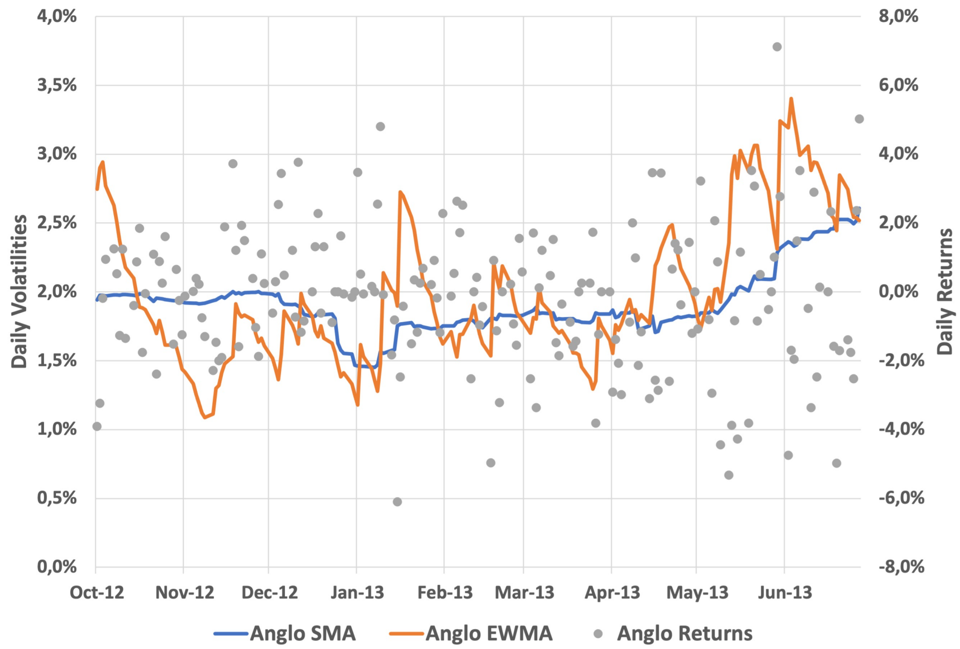


Figure 5.13: A comparison between EWMA and SMA volatility: October 2012 - June 2013, Anglo-American.
$\lambda_{Anglo} = 0.89$

Figures 5.14 to 5.15 indicate "ghost effects" resulting from the SMA during the respective time frames. A "ghost effect" is when the volatility estimates are incorrectly high (or low) for some periods after the event has passed and the returns have stabilised. In May and August 2014, the SMA is slow to stabilise as the returns become less volatile. The SMA is also slow to react to the volatility of the returns in November and December.

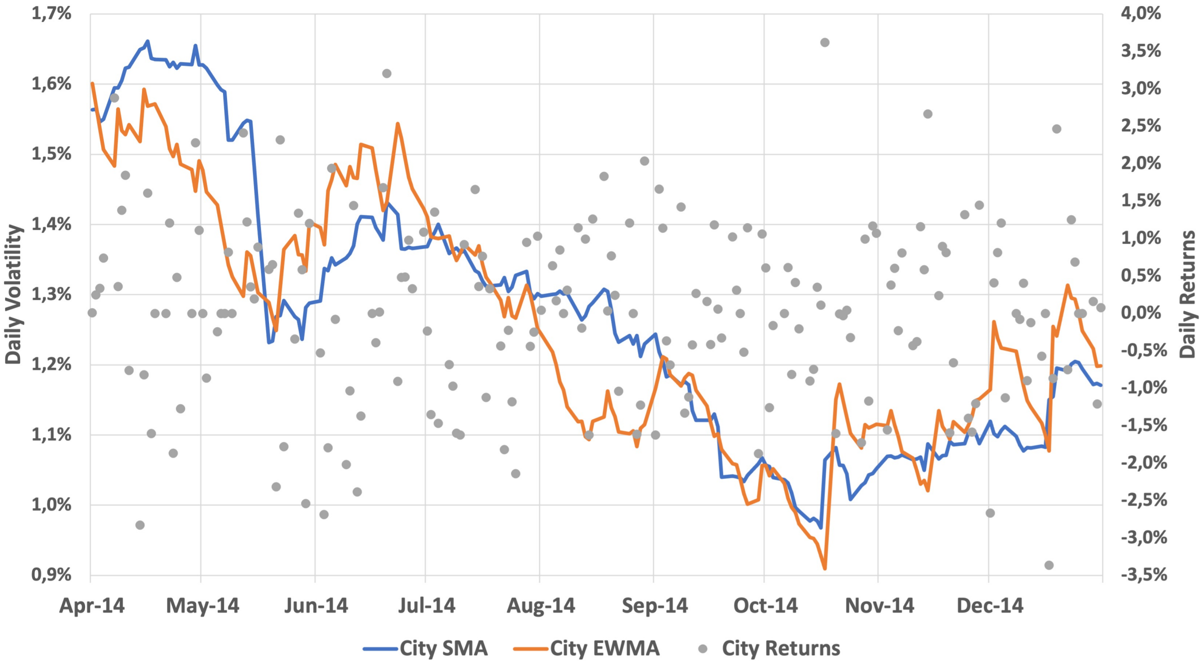


Figure 5.14: A comparison between EWMA and SMA volatility: April 2014 - December 2014, City Lodge Hotels.
$\lambda_{City} = 0.96$

The return falls from 3.6% to -2.6% just before June 2015, then begins to stabilise during the month. As a result, the EWMA gradually falls to stability at 1.4%, while the SMA remains unchanged at 1.62%.

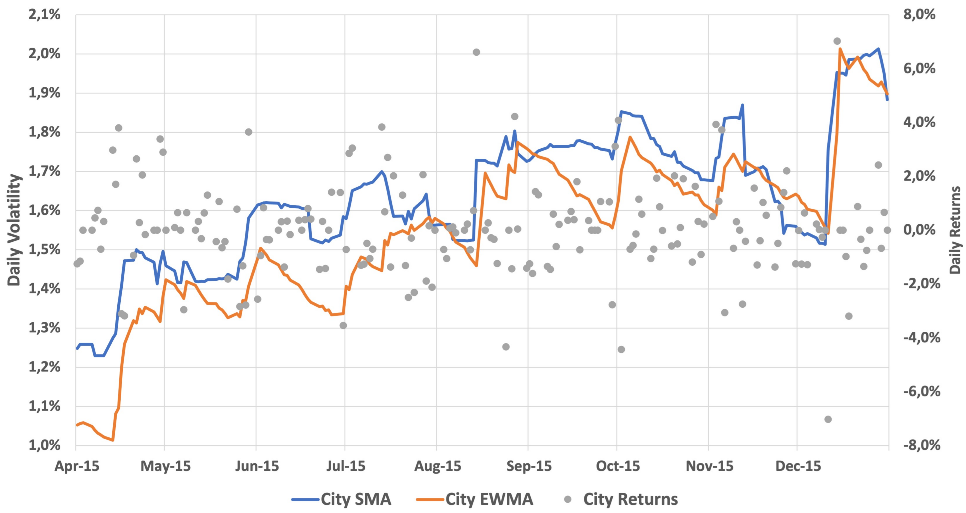


Figure 5.15: A comparison between EWMA and SMA volatility: April 2015 - December 2015, City Lodge Hotels.
$\lambda_{City} = 0.98$

Consider Figure 5.16. In 2016, January and February had erratic results, fluctuating rapidly between -8.3% and 6.1% before they begin to stabilise in March. The EWMA records this volatility in a more timely way than the SMA, which is slow to react to the volatility. The EWMA peaks at 3.3% before beginning to stabilise in March, whereas the SMA gradually oscillates between 2.6% and 2.7% from mid-February to mid-March before beginning to fall in an attempt to stabilise. From March to the middle of May, the SMA effectively registers a "ghost effect".

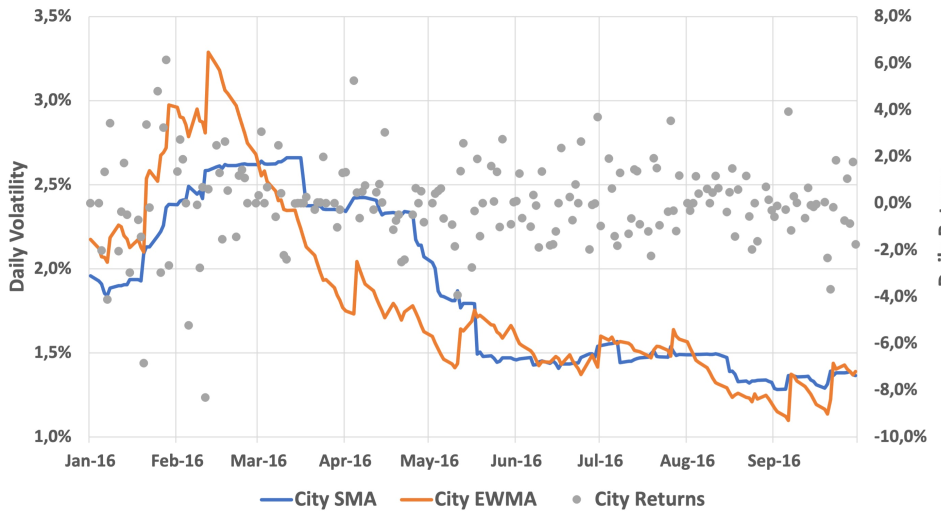


Figure 5.16: A comparison between EWMA and SMA volatility: January 2016 - September 2016, City Lodge Hotels.
$\lambda_{City} = 0.95$

The SMA overestimates the volatility in Figures 5.17 and 5.18. The returns fluctuate between −3.9% and 3.8% in February 2019. In contrast to the SMA volatility surge from 1.5% to 1.78%, the EWMA volatility jump is from 1.45% to 1.62%. The returns stabilise in May. While the SMA volatility abruptly decreases from 1.78% to 1.44%, the EWMA volatility decreases steadily from 1.61% to 1.52%.

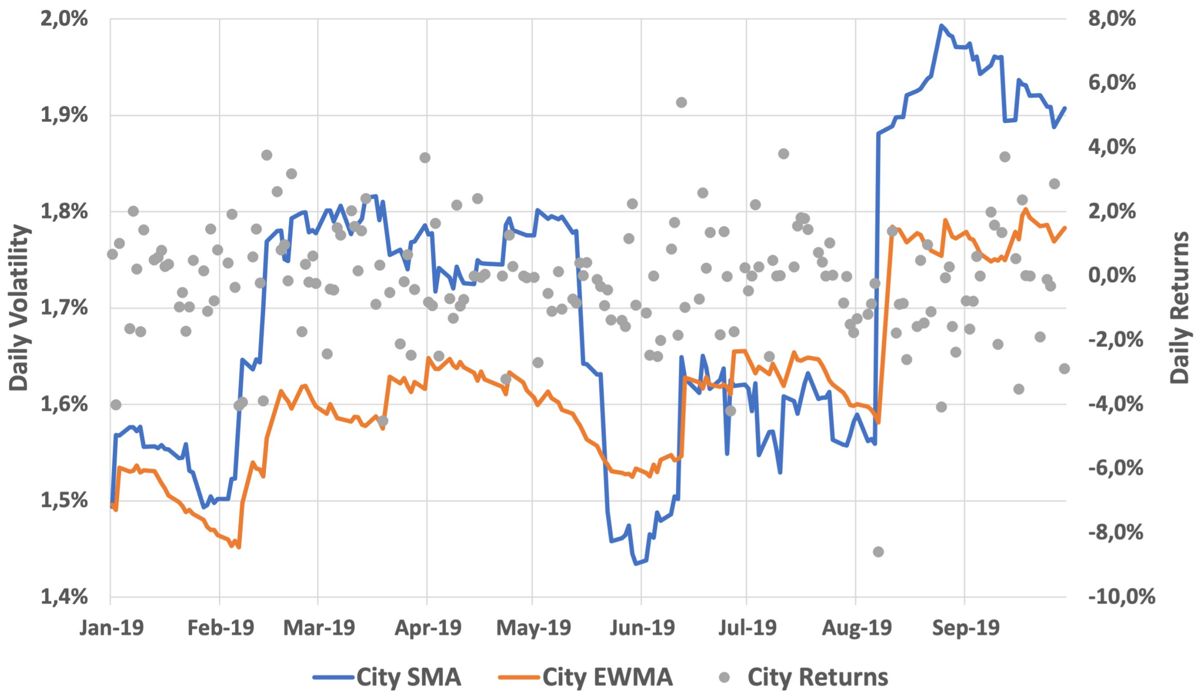


Figure 5.17: A comparison between EWMA and SMA volatility: January 2019 - September 2019, City Lodge Hotels.
$\lambda_{City} = 0.99$

Consider Figure 5.18. From October through March 2019–2020, returns are largely stable, with some volatility in early November, mid–December, and February. Between December 2019 and February 2020, the SMA's volatility spikes from 1.5% to 8% and stays high until late April 2020. The returns are erratic starting in March 2020, with a low of −35.9% and a high of 18.2%. The EWMA volatility rises between 2% and 18.5% at this time.

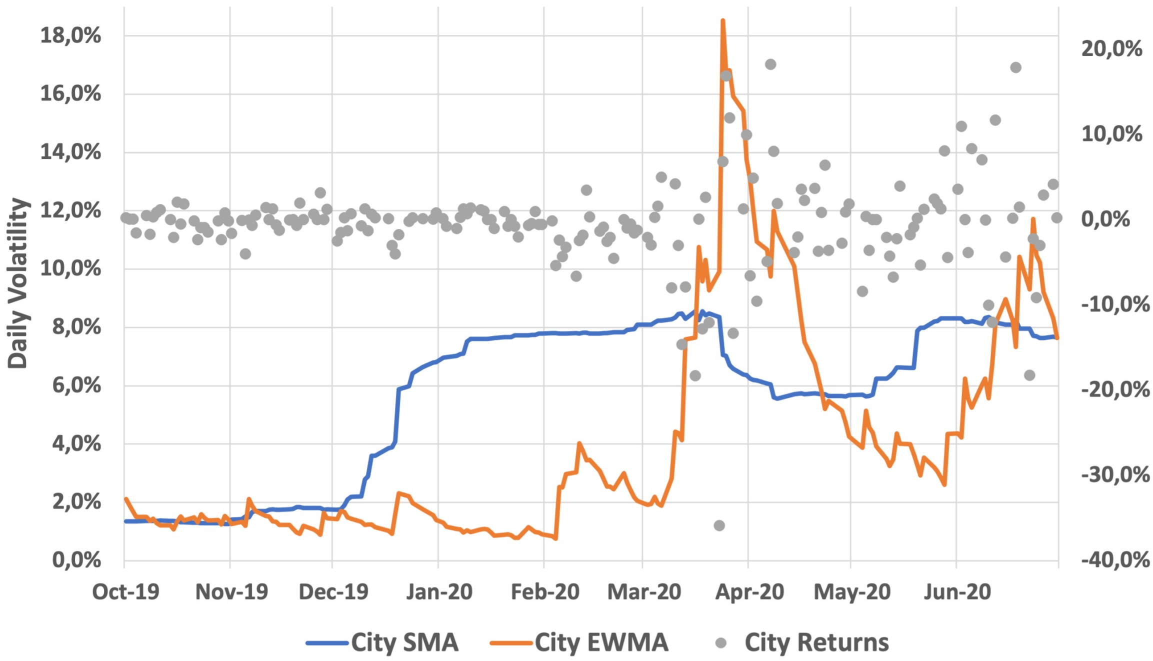


Figure 5.18: A comparison between EWMA and SMA volatility: October 2019 - June 2020, City Lodge Hotels.
$\lambda_{City} = 0.79$

The findings in Figure 5.19 are highlighted and expanded in Figure 5.18. In cases where the returns are erratic, the SMA volatility is either constant or less pronounced than expected. The returns are erratic in June 2020, peaking at 17.9% and decreasing to -18.2%. While the SMA volatility progressively declines between 8.3% and 7.5%, the EWMA volatility spikes from 4.2%to 10.8%. Early in August, when the returns reach their top of 20%, the EWMA volatility also reaches its peak of 8.9%. The return falls to -34.2% in the middle of August, and the EWMA volatility reaches a maximum of 14%. The SMA volatility, in contrast to the EWMA, progressively declines from 7.4% to 6.9% from early to mid-August before dropping abruptly to 5.5%.

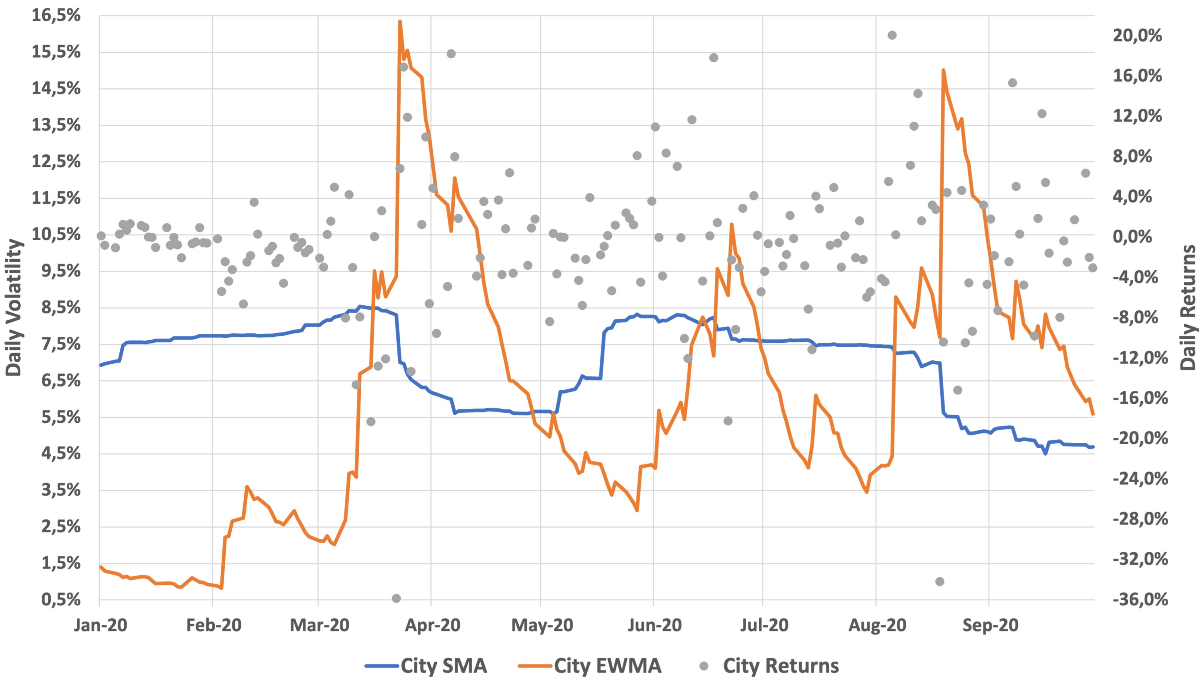


Figure 5.19: A comparison between EWMA and SMA volatility: January 2020 - September 2020, City Lodge Hotels.
$\lambda_{City} = 0.85$

5.1.3 USA Stocks

Consider the investment portfolio consisting of stocks in different industries (developed economy), e.g. entertainment, and utilities. Figure 5.20 shows a time series of λs built using one year of historical data and rolling the sample period forward three months each time to get a new measurement. For the US economy, throughout the 2012 - 2021 time period, the decay factor varies considerably: $\lambda_P \in [0.77, 0.99]$. Consider turbulent market: April 2019 - April 2021, $\lambda_P = [0.80, 0.84]$ from $\lambda_P = 0.97$ during the previous period. The 2019 Global pandemic was during this time frame.

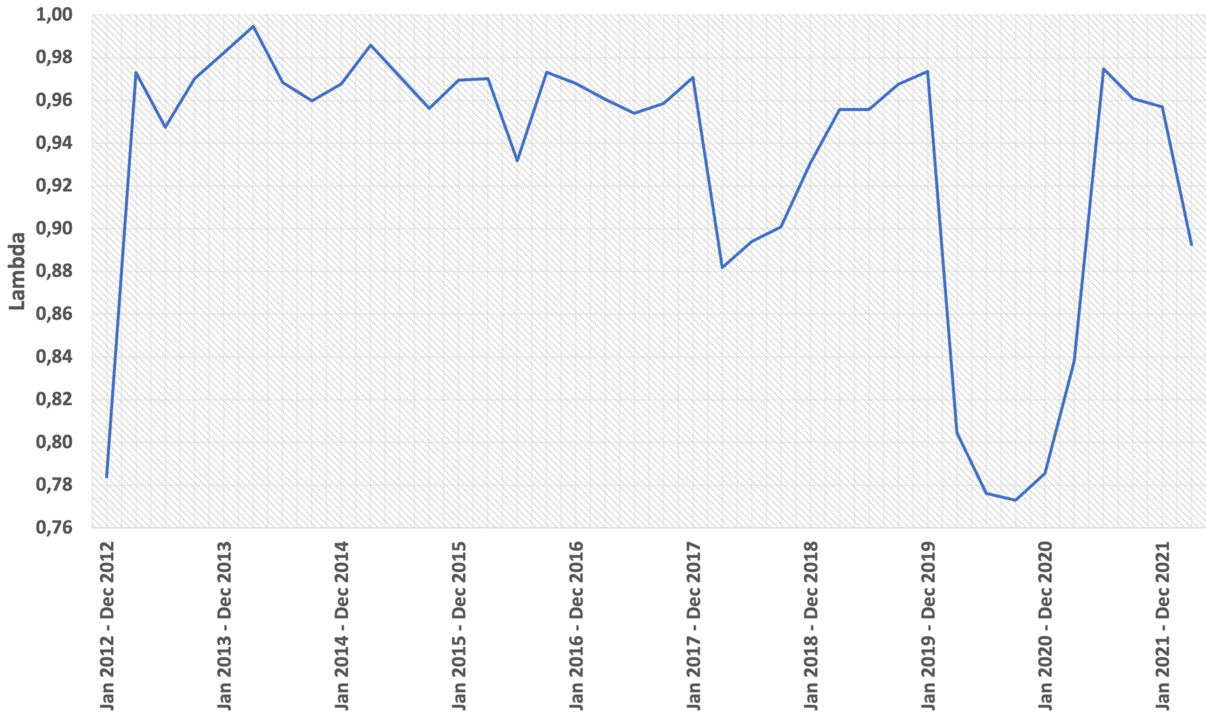


Figure 5.20: USA λ series measured over quarters from 2012 to 2021.

Figure 5.21 illustrates the AT&T Inc returns' volatile nature from April to December 2012. The return peaked in April, late July, and mid-November at 3.6%, 2.7%, and 2%, respectively. In addition, it fell to its lowest levels at -2.1%, -2.4%, and 3.4%, respectively, in the middle of July, the middle of September, and the beginning of November. The EWMA volatility consistently responds to these peaks and troughs throughout the period, reaching its highest recorded value of 2% in late April. The SMA, on the other hand, exhibits volatility across the whole time frame bounded by $[0.6\%, 1\%]$.

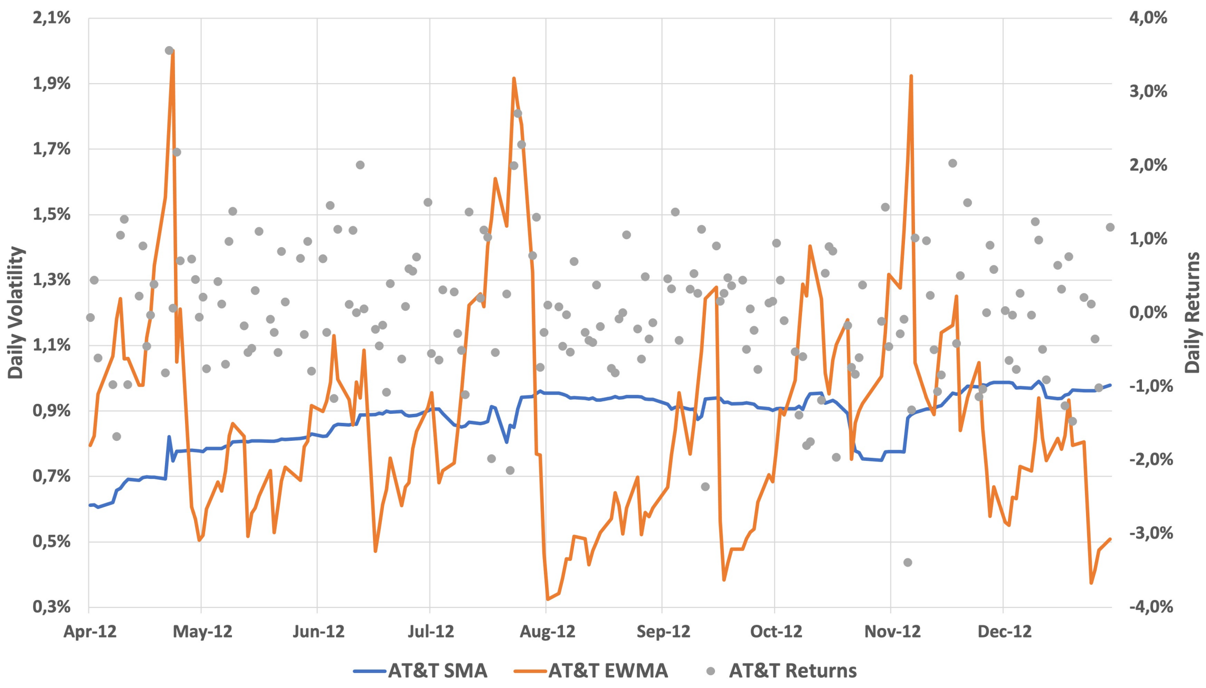


Figure 5.21: A comparison between EWMA and SMA volatility: April 2012 - December 2012, AT&T Inc.
$\lambda_{AT\&T} = 0.74$

Figure 5.22 depicts the erratic nature of the Tesla Inc returns from July 2012 to March 2013. The EWMA follow a relatively similar pathway. The return reached its highest points at 6.8%, 8.5%, and 5.9% in mid-September, early November, and late February, respectively. In addition, it reached its lowest points in late September and late February, when it was -10.3% and -9.2%, respectively. Throughout the period, the EWMA volatility consistently reacts to these peaks and troughs, reaching its highest recorded value of 3.7% in late September and its lowest value of 1.95% in mid-February. Similarly, the SMA records a reactive volatility pathway with its highest value at 3.75% in August and the lowest value at 1.7% in mid-February.

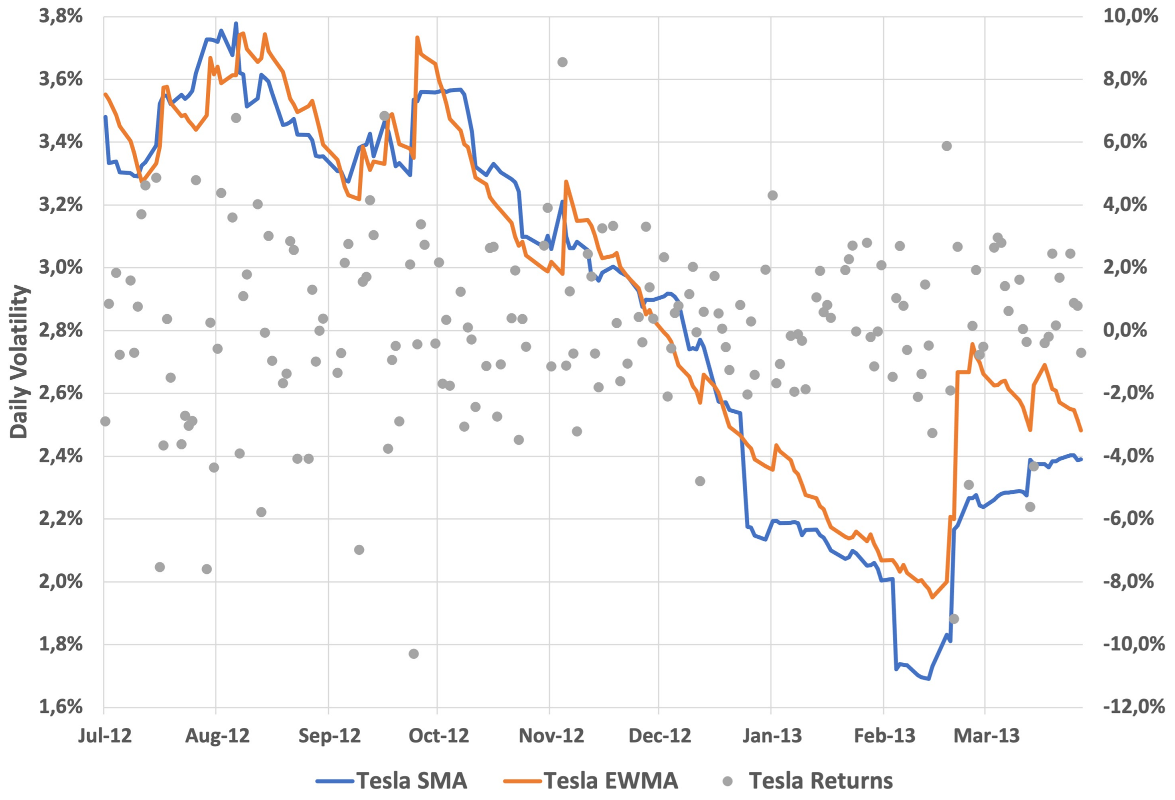


Figure 5.22: A comparison between EWMA and SMA volatility: July 2012 - March 2013, Tesla Inc.
$\lambda_{Tesla} = 0.97$

Consider Figure 5.23, which exhibits Coca-Cola Company's performance from October 2012 to June 2013. The returns are erratic, peaking at 3.7% and 5.5% respectively in January and mid-April. The returns fell to -2.8% and -3.2% in mid-February and late June, respectively. The EWMA volatility peaks in January between 0.81% and 0.95% before progressively falling over the month to 0.85%. The SMA volatility, on the other hand, progressively varies in the range $[0.9\%, 0.93\%]$. The EWMA peaks at 1.24% in mid-April and then progressively declines until 1.05% in late May. Before starting to decline to 1.06%, the SMA volatility oscillates within the range $[1.18\%, 1.21\%]$ for about a month.

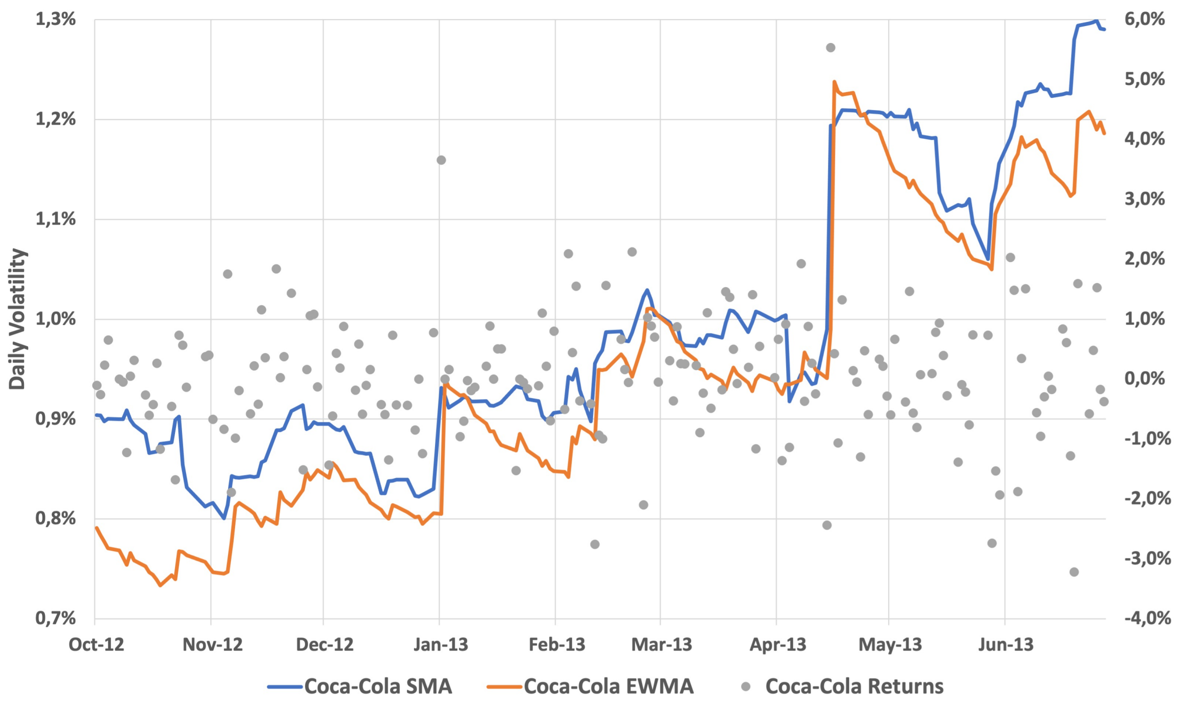


Figure 5.23: A comparison between EWMA and SMA volatility: October 2012 - June 2013, Coca-Cola Company.
$\lambda_{Coke} = 0.98$

Figure 5.24 demonstrates the steadiness of Netflix Inc's returns between January and September 2013. With occasional peaks at 35.2% and 21.9% in late January and April, the returns are largely consistent. The EWMA volatility peaks from 2.9% to 4.7% in late January and progressively declines to 4% in late April before increasing to 4.5% in late January. Early in January, before the return peak, the SMA volatility peaked between $[5.4\%, 5.8\%]$. Between February and May, the returns stabilise, while the SMA volatility oscillates sharply within the range $[4.8\%, 5.4\%]$.

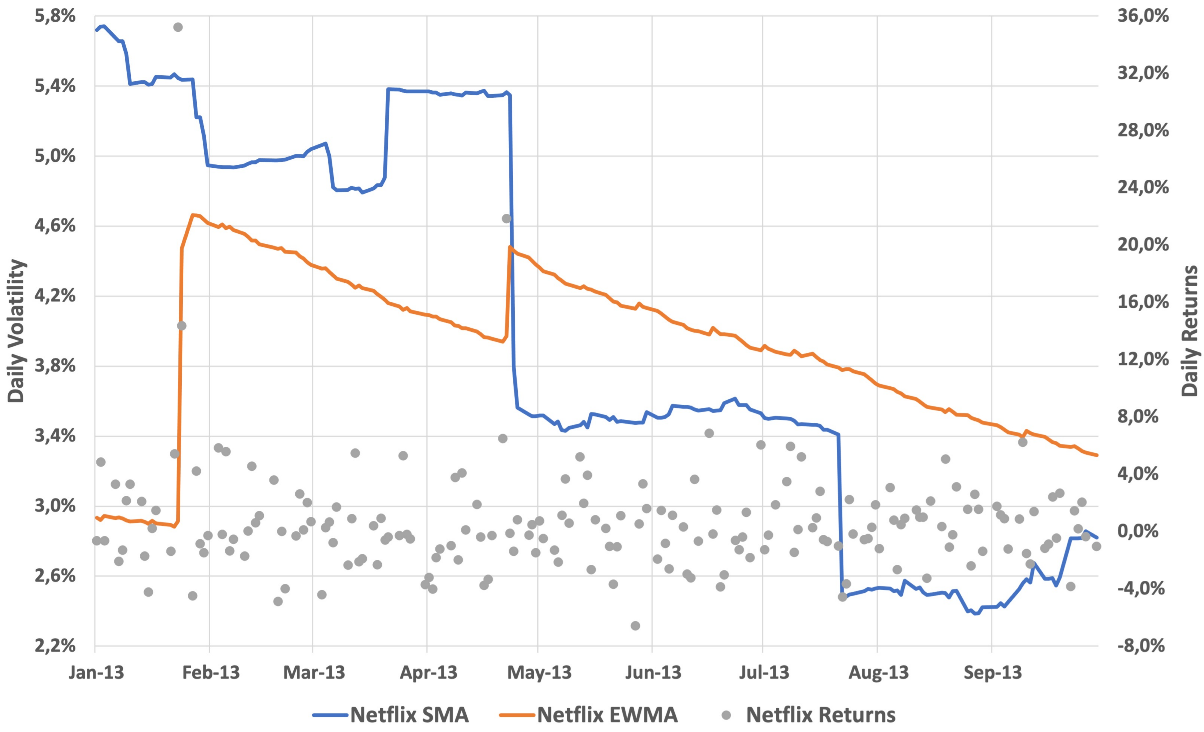


Figure 5.24: A comparison between EWMA and SMA volatility: January 2013 - September 2013, Netflix Inc.
$\lambda_{Netflix} = 0.99$

Consider the United Airlines Holdings Inc returns from Figure 5.25. In general, the returns fluctuate, with certain peaks and valleys scattered throughout. In the middle of July, late in November, and in February, respectively, there are maxima of 12%, 8%, and 5.3%. The EWMA peaks at 3.3%, 2.8%, and 2.9%, respectively, during these times, while the SMA volatility peaks just below each of these levels. From mid-August to October, "ghost effects" are recorded by the SMA volatility.

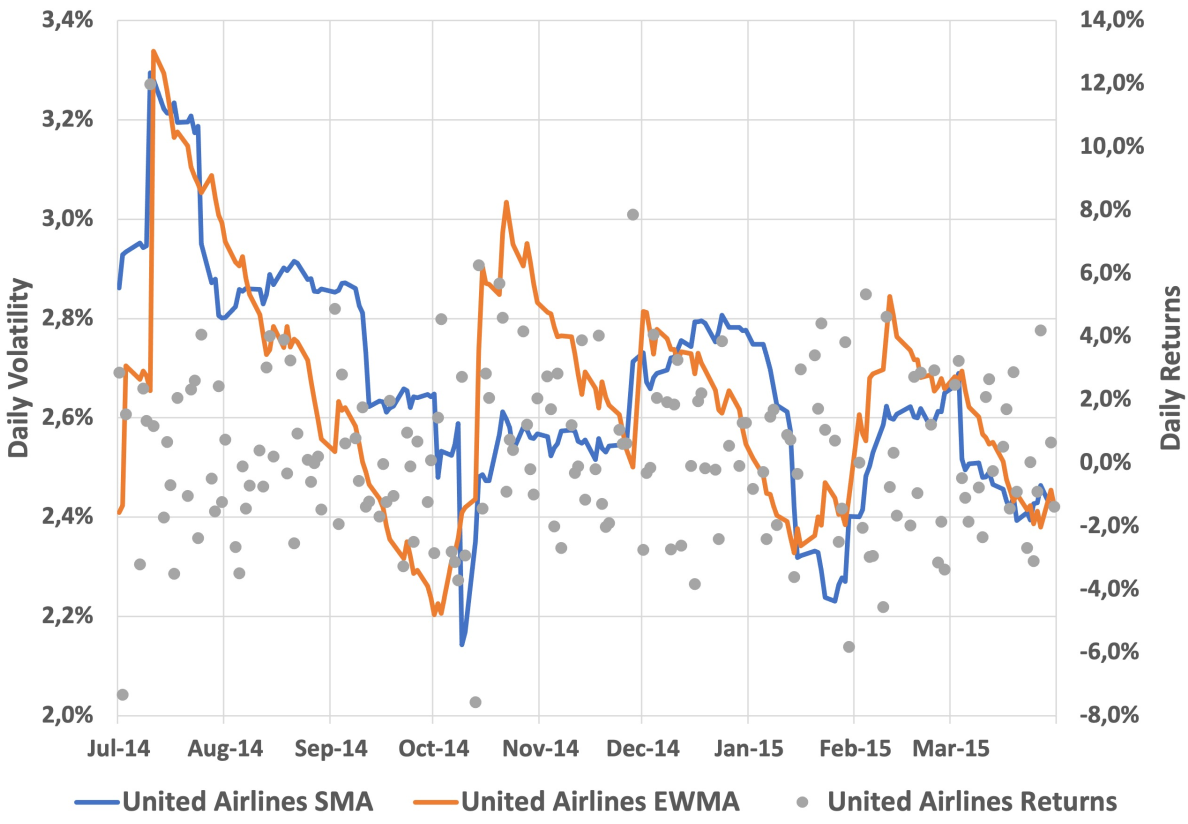


Figure 5.25: A comparison between EWMA and SMA volatility: July 2014 - March 2015, United Airlines Holdings Inc.
$\lambda_{Airlines} = 0.97$

According to Figure 5.26, the returns for United Airlines Holdings Inc were somewhat erratic from April to December 2019. Throughout the period, there are many points where the returns peak and dip. For instance, the return peaks at 4.7% in mid-April, while the EWMA volatility rises to 2.89% and the SMA volatility stays at 1.9%. When returns stabilise, the SMA does not accurately represent this, creating "ghost effects" throughout the period.

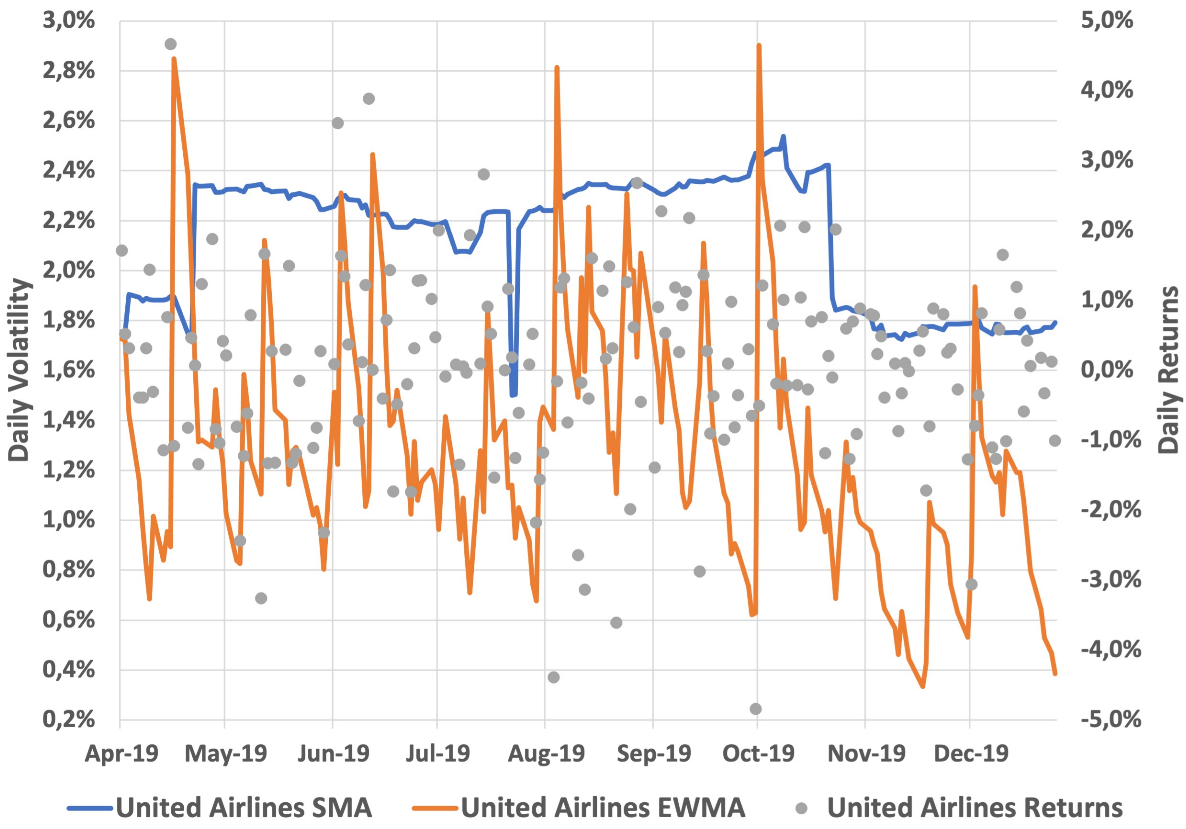


Figure 5.26: A comparison between EWMA and SMA volatility: April 2019 - December 2019, United Airlines Holdings Inc.
$\lambda_{Airlines} = 0.65$

Figure 5.27 shows United Airlines Holdings Inc's spotty returns during the pandemic. The peak returns are 19%, 17%, 14%, and 17%, respectively, in mid-May, June, July, and November. In June, the returns fall to -17%. In reaction to these peaks and falls, the EWMA volatility increases. For instance, the EWMA volatility is recorded at 12.8% in June, with a return peak of 17% and a drop of -17%. The SMA volatility, however, is listed at 10%.

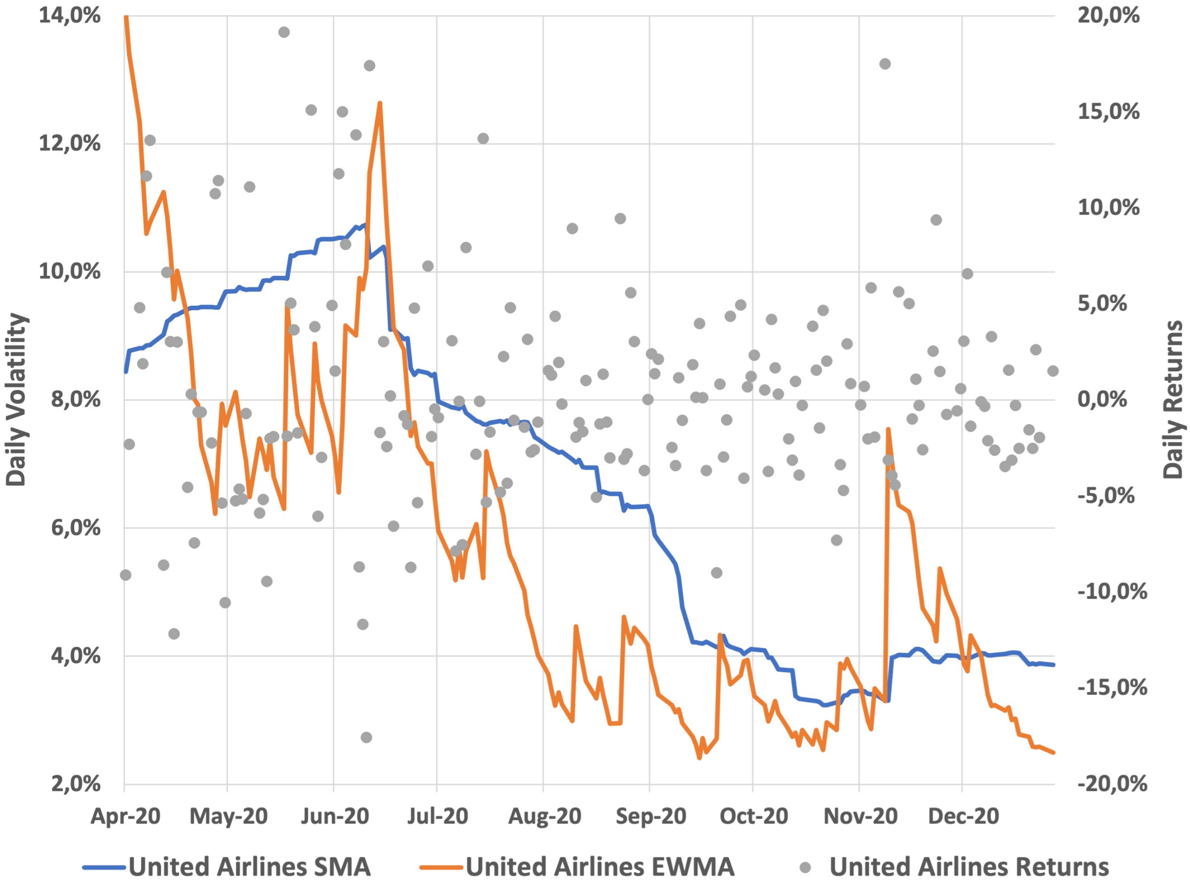


Figure 5.27: A comparison between EWMA and SMA volatility: April 2020 - December 2020, United Airlines Holdings Inc.
$\lambda_{Airlines} = 0.78$

5.2 EWMA, GARCH(1,1) and SMA

The EWMA is a special case of the GARCH(1,1), where $\omega = 0$, $\alpha = 1 - \lambda$ and $\beta = \lambda$, as demonstrated in Chapter 2, (2.6). The GARCH(1,1) has the mean-reverting whereas the EWMA does not, since $\omega = 0$ for the EWMA and non-zero for GARCH(1,1). In other words, if volatility increases too much, it tends to fall back to the long-run variance over time, and if volatility decreases too much, it tends to climb back to the long-run variance over time. The value that the variance will typically return to over the long run is known as the long-run variance.

The returns, EWMA, GARCH, and SMA volatilities for Scottish Mortgage Investment Trust plc from January to September 2015 are displayed in Figure 5.28. In general, returns are unpredictable throughout the year and especially so in August. 4.1% is its highest, and -4.9% is its lowest point. The EWMA and SMA record volatility peaks at 3.09% and 2.2%, respectively, while the GARCH(1,1) records volatility of 2.2%. The EWMA is more reactive than the GARCH(1,1) despite following a similar trend throughout, and this is due to the GARCH's mean-reversion (1,1) to the long-run variance.

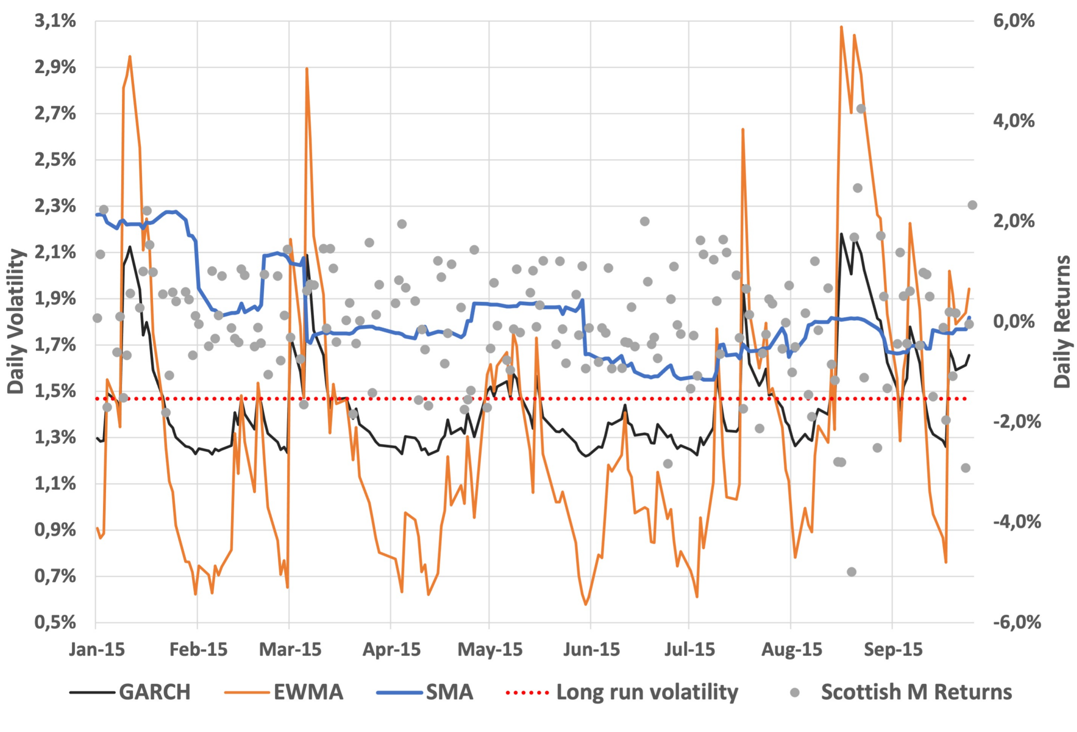


Figure 5.28: A comparison between GARCH(1,1), EWMA and SMA: January 2015 - September 2015, Scottish Mortgage Investment Trust plc. $\lambda_{Scottish} = 0.68$ and $V_{LR} = 1.47\%$

Figure 5.29 displays the EWMA and the GARCH(1,1) moving in the pattern with different values for volatility. From April to December 2020, the returns are unstable with the highest value at 11.1% and the lowest value at -13.3% in April. The EWMA, GARCH(1,1) and SMA record their respective volatility peaks to be 10%, 6.5% and 2.2%.

Figure 5.30 displays Investec's volatile returns, notably in July, September, and December. For instance, August's erratic returns are depicted by volatilities measured by EWMA, GARCH(1,1), and SMA which were 9.5%, 4.8%, and 1.6%, respectively.

Consider the returns in Figure 5.31. The Figure displays Xerox's unstable financial performance. for April 2020 through December 2020. the returns are quite unpredictable between April and June and November. The highest returns are 11.9% and 11.5% in November and May, respectively. In April, May, and June, it declines to -9.5% and -10.3%, respectively. EWMA and Garch(1,1) reported maxima in the volatility of 8.9% and 4.8% in May, respectively. The SMA's volatility during this time was 1.8%. As of November, Volatility maxima were reported by EWMA and Garch(1,1) at 8.26% and 4.4%, respectively. During The SMA had 1.8% volatility throughout this time. The SMA, in this erratic market, from April 2020 - December 2020, toggles between $[1.6\%, 2.3\%]$.

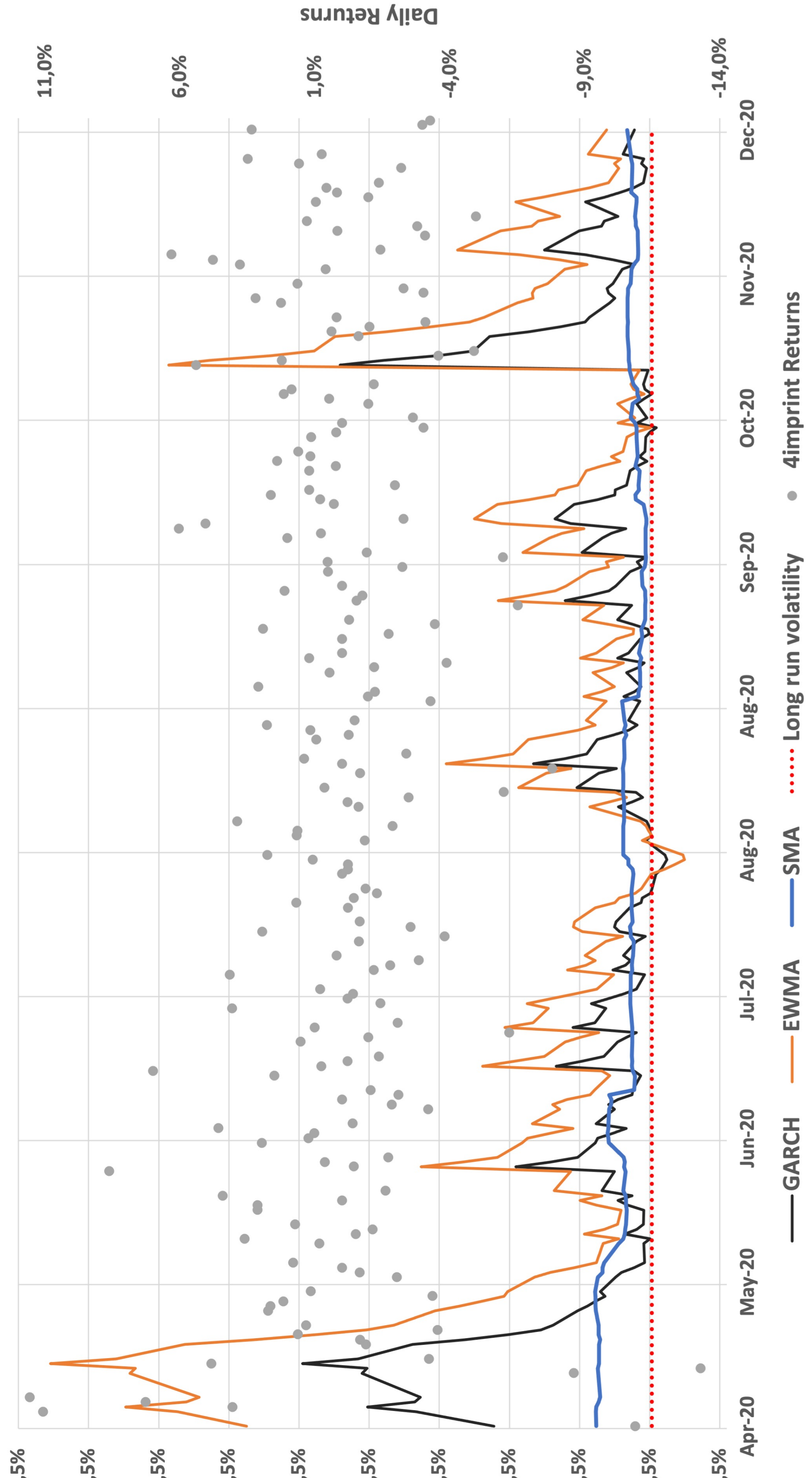


Figure 5.29: A comparison between GARCH(1,1), EWMA and SMA: April 2020 - December 2020, 4imprint Group plc. $\lambda_{4imprint} = 0.77$ and $V_{LR} = 1.47\%$

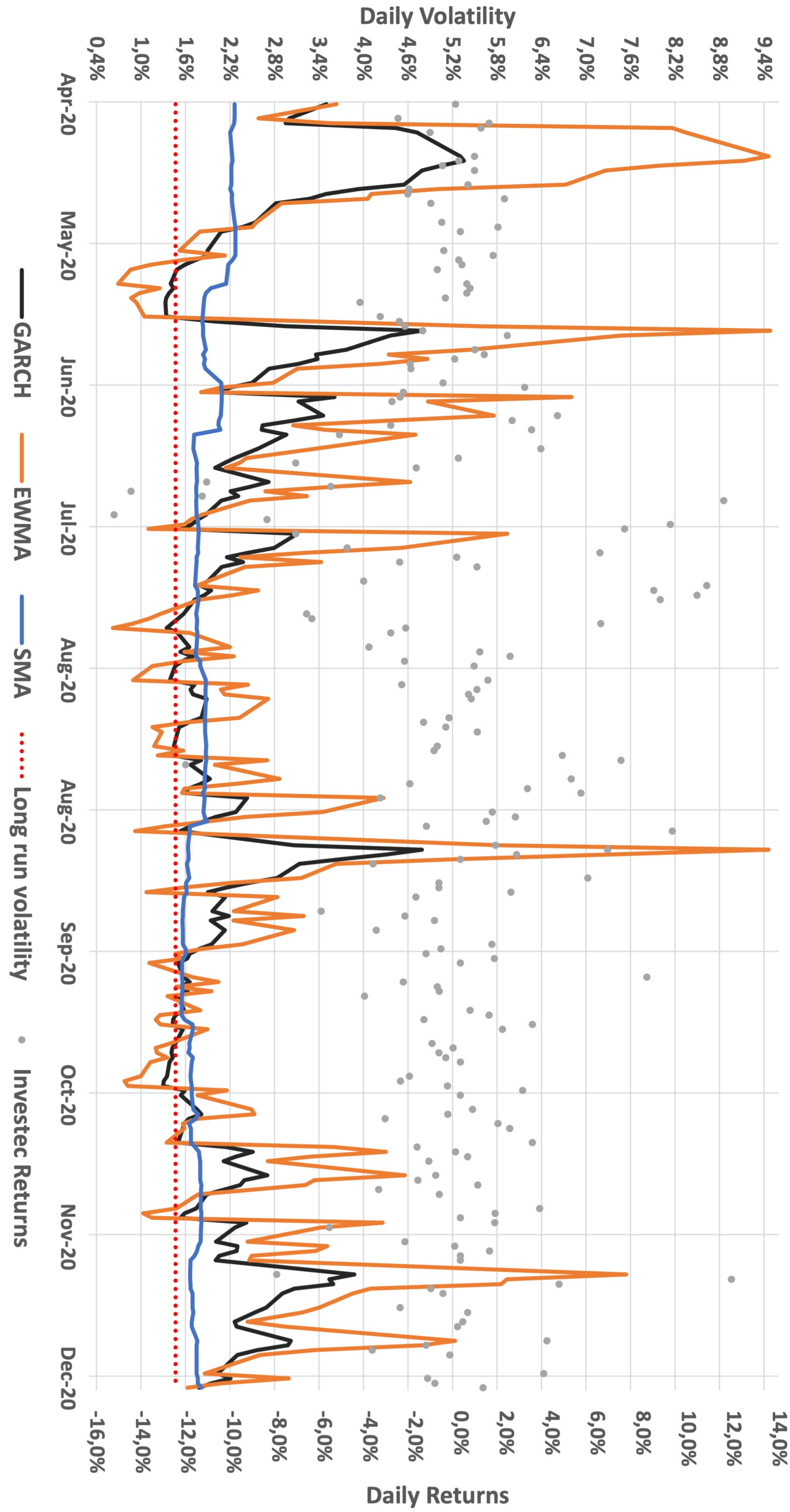


Figure 5.30: A comparison between GARCH(1,1), EWMA and SMA: April 2020 - December 2020, Investec Property Fund ltd. $\lambda_{Investec} = 0.49$ and $V_{LR} = 1.47\%$

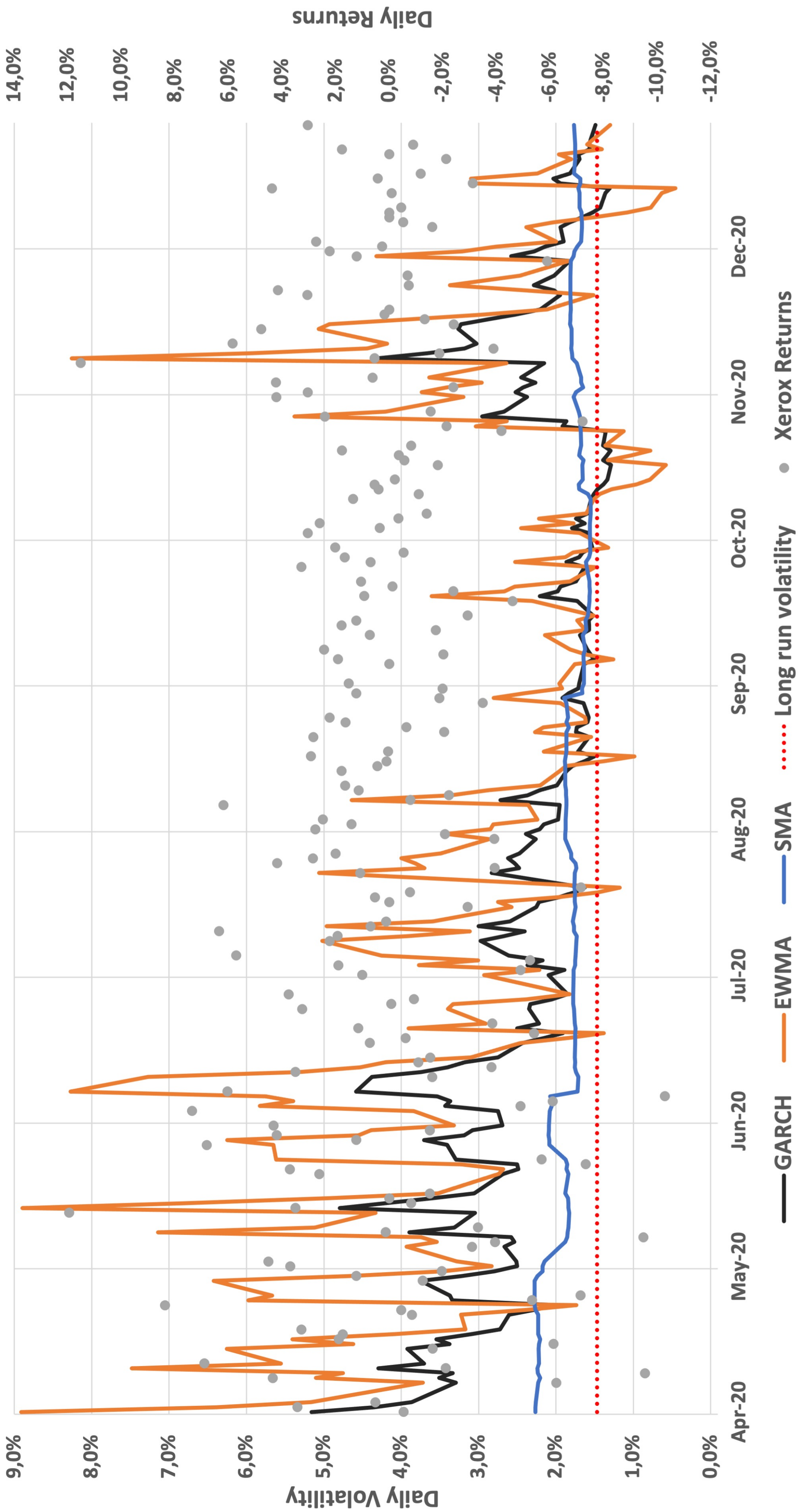


Figure 5.31: A comparison between GARCH(1,1), EWMA and SMA: April 2020 - December 2020, Xerox Holdings Corporation. $\lambda_{Xerox} = 0.51$ and $V_{LR} = 1.47\%$

5.3 Value at Risk

Consider Figure 5.32, the differences in VaR exceptions observed between the two methods are explained by the increased volatility in the market during the period considered. The EWMA method was better able to capture this volatility, resulting in more VaR exceptions in the orange zone. The SMA method was lower to react to the increased volatility, resulting in fewer VaR exceptions in the green zone. However, as the market volatility decreased towards the end of the period considered, the differences between the two methods became less pronounced, with both methods observing fewer VaR exceptions in the green zone.

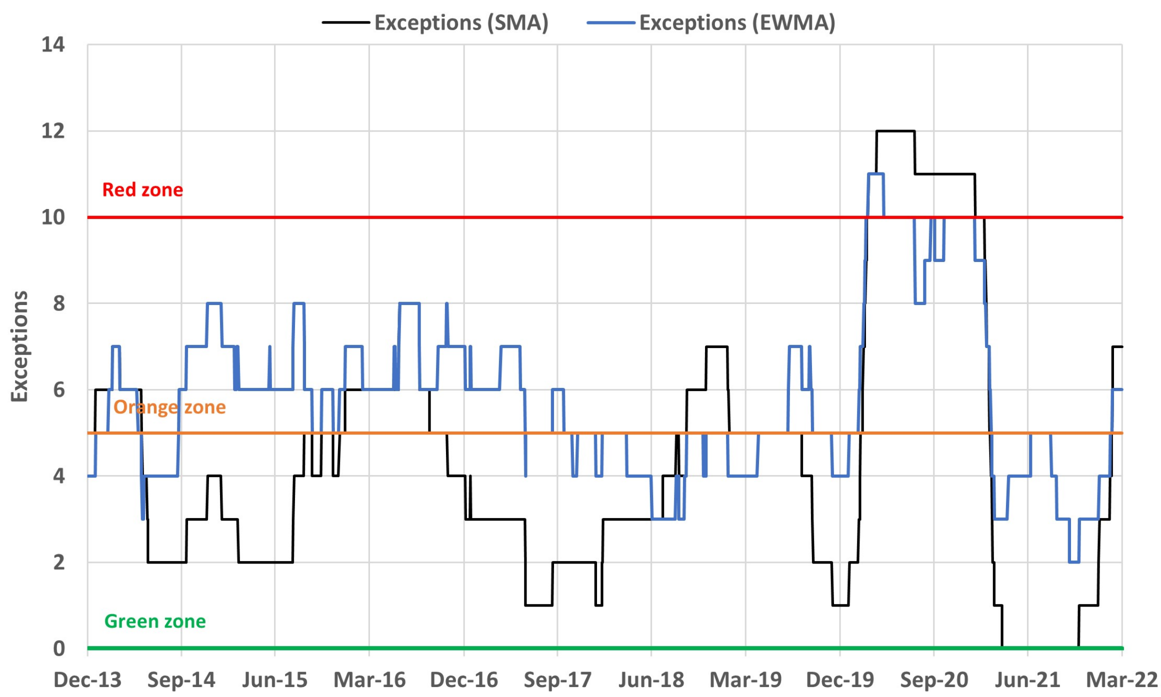


Figure 5.32: VaR exceptions: UK market

Several events occurred during the period considered that could have impacted the economy and, consequently, affected the VaR of the portfolio. One of the most notable events was Brexit. The negotiations between the UK and the EU, along with the uncertainty regarding the future relationship between the two entities, led to increased volatility in the stock market, which affected VaR.

In addition to Brexit, the trade tensions between the US and China, began in 2018 and continued to escalate throughout the period considered. These tensions resulted in increased market volatility and uncertainty. COVID-19 and the subsequent restrictions had significant impacts on the global economy, with many businesses forced to close or reduce their operations, leading to increased unemployment and reduced economic activity. The result was an increase in market volatility and thereby affecting the portfolio VaR.

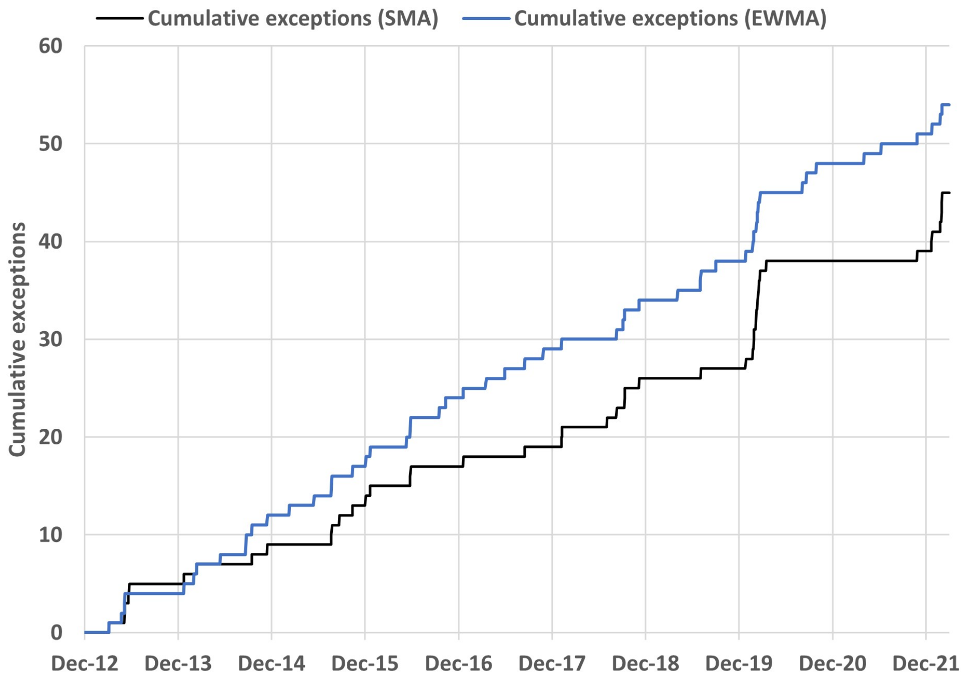


Figure 5.33: Cumulative Exceptions: UK market

As with the rest of the world, the pandemic had a profound impact on the South African stock market, with many companies experiencing a decline in revenue and profits. Figure 5.34 shows the VaR exceptions throughout the period of consideration. A significant event that could have impacted the South African economy is the political turmoil that occurred during this period. In 2017, South Africa experienced political instability as the then President faced corruption allegations. Zuma was eventually forced to resign in February 2018, which led to an improvement in market sentiment. However, political instability remained a concern in South Africa during this period.

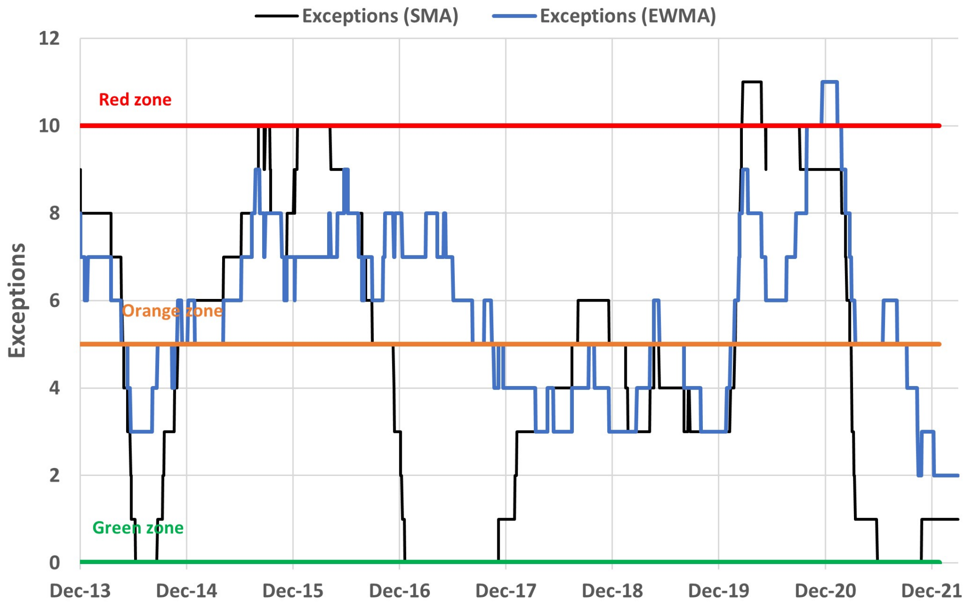


Figure 5.34: VaR exceptions: RSA market

In terms of stock movements, there were several notable movements in the South African stock market during this period. For example, Sasol, one of the stocks in the portfolio, experienced significant volatility due to a decline in oil prices and the impact of the COVID-19 pandemic on demand for oil.

Consider 5.35 which shows the cumulative VaR exceptions for SMA and EWMA during 2013-2020. The observed higher cumulative exceptions for the EWMA method suggest that the EWMA method captured more of the market's volatility compared to the SMA method. Additionally, the gradual tiering of the EWMA cumulative exceptions in the periods of 2013, 2015, 2016, and 2017 suggests that the method was able to capture the gradual changes in volatility during those periods. On the other hand, the flat cumulative exceptions observed using the SMA method in 2013, 2014, 2016, 2017, 2020, and 2021 suggest that the SMA method was not as sensitive to changes in volatility during these periods. The big jump in cumulative exceptions observed for both methods in 2020 is attributed to the impact of the COVID-19 pandemic on the global economy, which led to heightened market volatility. This event had far-reaching impacts on the South African economy, including disruptions in supply chains, a decline in global demand for commodities, and a significant reduction in foreign investment. The cumulative exceptions for the

SMA show a slightly higher jump before remaining flat until the end of 2021, while the EWMA cumulative exceptions show a gradual increase up until the end of 2021.

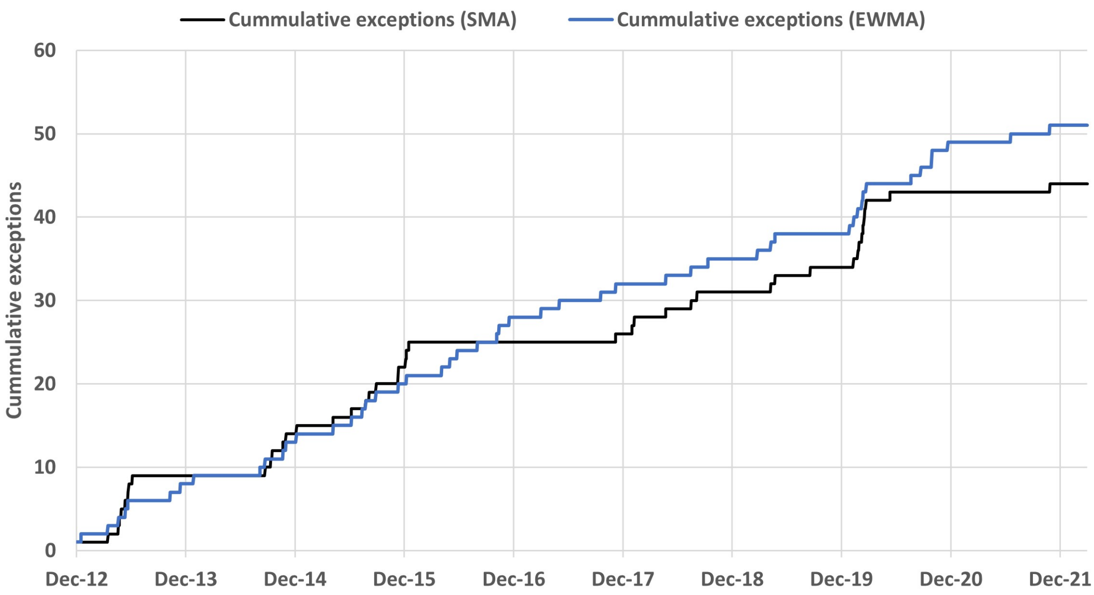


Figure 5.35: Cumulative VaR Exceptions: RSA market

Consider Figure 5.36 showing the VaR exceptions in the USA market. From 2013 to 2016, the EWMA VaR exceptions were observed to be predominantly in the orange zone while the SMA VaR exceptions were in the green zone in 2014 and 2015 and in the orange zone in 2016. The green SMA VaR exceptions in 2014 and 2015 could have been influenced by the economy's recovery from the 2008 financial crisis. On the other hand, the orange EWMA VaR exceptions during the same period indicate that the financial markets were still volatile as influenced by the Federal Reserve starting a policy of gradually raising interest rates in 2015, which could have impacted the performance of the stocks in the portfolio. Higher interest rates can make it more expensive for companies to borrow money, which can lead to decreased profits and a decline in stock prices. This decline in stock prices could have led to increased VaR exceptions, particularly in the SMA method which is more sensitive to short-term fluctuations in stock prices.

Similarly, the EWMA VaR exceptions being fairly distributed in the green and orange zones in 2017-2018 could reflect a period of relative stability in the market, but the sudden rapid jump of the SMA to the red zone in early 2019 might suggest a sudden surge in market volatility. It's worth noting that this period coincided with the US-China trade war, which could have played a role in the market instability. Additionally, the increased uncertainty and volatility in financial markets resulting from the pandemic could have also impacted the performance of the portfolio. In 2020, the EWMA and the SMA VaR exceptions rose to the middle of the red zone after which the EWMA gradually starts to fall while the SMA VaR exceptions remain in the mid-red zone.

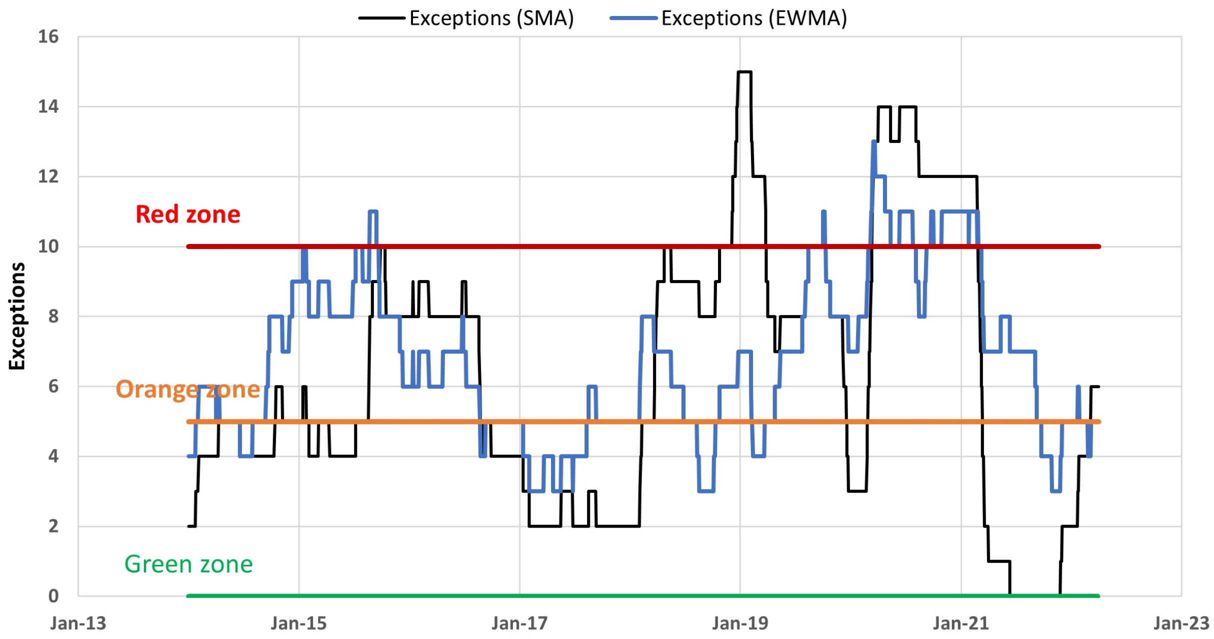


Figure 5.36: VaR exceptions: USA market

When plotting the cumulative exceptions for this market, Figure 5.37, the EWMA is observed to predominantly have higher cumulative exceptions throughout the period. The tiering on the EWMA cumulative exceptions is more gradual than that of the SMA. In particular, the SMA observes flat cumulative exceptions in 2020 and 2021 while the EWMA observes gradual tiering of the cumulative exceptions in these periods. Both methods observed a big jump in cumulative exceptions in 2020. However, the SMA has a significantly bigger jump before it stays flat until the end of 2021, while the EWMA cumulative exceptions gradually increase until the end of 2021.

From 2013 to 2016, the EWMA VaR exceptions were observed to be predominantly in the orange zone, while the SMA VaR exceptions were in the green zone in 2014 and 2015 and in the orange zone in 2016. From 2017 to 2018, both methods observed VaR exceptions that were fairly distributed in both the green zone and the orange zone. However, the SMA VaR exceptions were in the lower region of the green zone and the high region of the orange zone, which demonstrates a sudden rapid jump of the SMA as it goes on to the red zone in early 2019. At the end of 2019, SMA VaR exceptions were observed to fall from the mid-orange zone to the green zone while EWMA fell from the floor of the red zone to the middle of the orange zone. This reduction could be attributed to the easing of trade tensions between the USA and China during this period. However, in 2020, both methods observed a significant rise in VaR exceptions due to the COVID-19 pandemic, which led to unprecedented levels of market volatility.

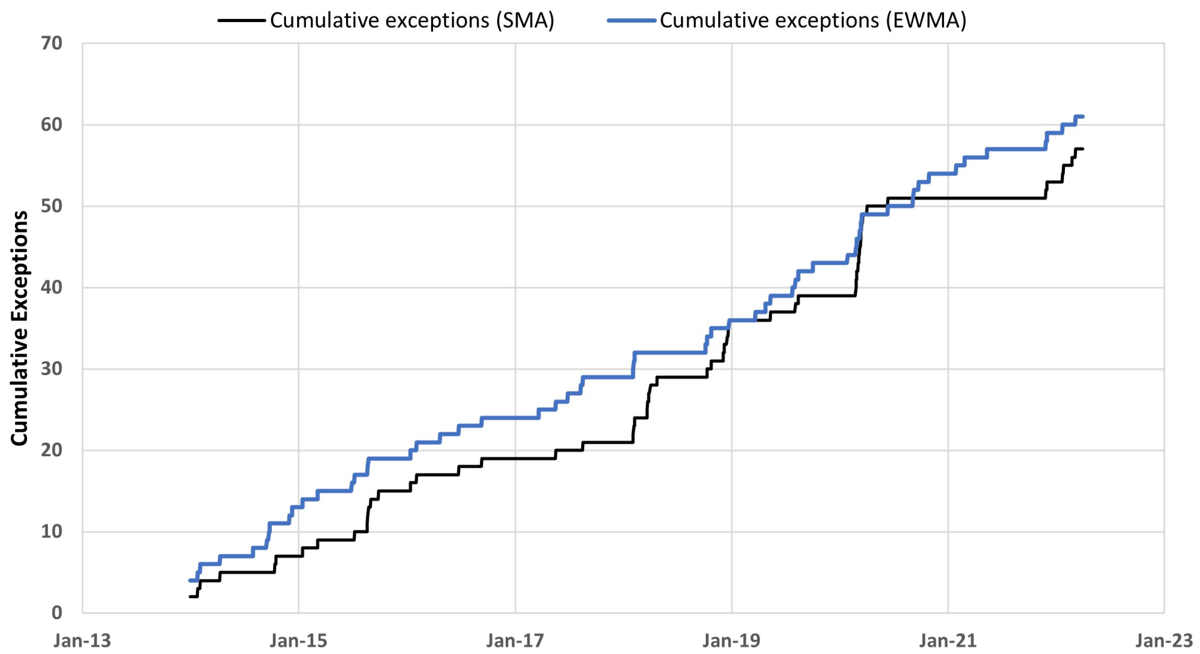


Figure 5.37: Cumulative VaR Exceptions: USA market

Figure 5.39 shows the VaR exceptions for the single stock Admiral, calculated from the three volatility methods SMA, EWMA and GARCH(1,1). The results indicate that GARCH(1,1) VaR exceptions were less conservative compared to SMA and EWMA, thereby observing more exceptions in volatile times and periods of gradually increasing volatility.

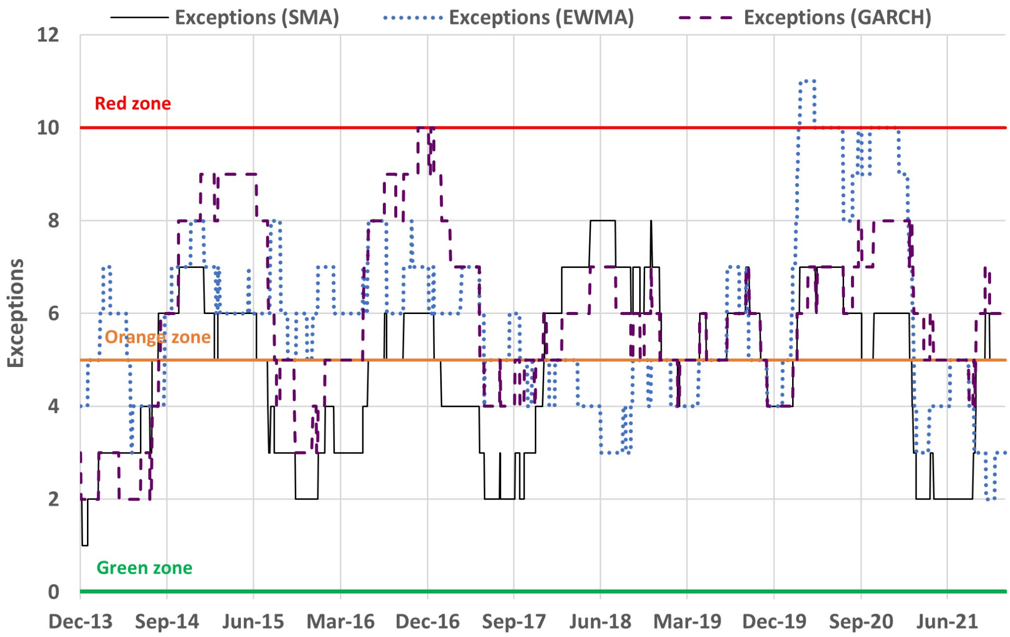


Figure 5.38: VaR exceptions: UK market - Admiral

Furthermore, the cumulative exceptions for Admiral stock show that the GARCH(1,1) and EWMA methods exhibit higher cumulative exceptions compared to the SMA method. The tiering on the EWMA and GARCH(1,1) cumulative exceptions is more gradual than that of the SMA as shown in Figure 5.39.

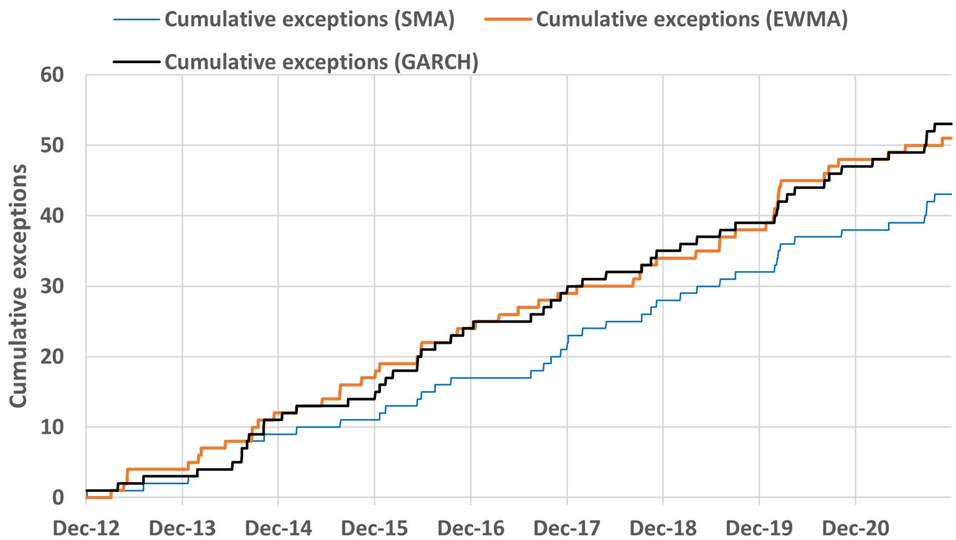


Figure 5.39: Cumulative VaR Exceptions: UK Market - Admiral

Chapter 6

Conclusion

This study focused on investigating the decay parameter of the EWMA model and comparing it to other techniques of determining volatility. (Metrics, 1996) addresses the use of standard deviation forecasts in VaR estimates. This is because the volatility of financial returns is predictable. As a result, estimating volatility accurately is critical. This paper's findings imply that EWMA represents stock price volatility better than SMA. This is corroborated by data collected, particularly during volatile market moments. Where SMA appears to be unresponsive, EWMA appears to be more responsive to the volatility of stock price returns. The model's reliability is consistent in both developing and developed markets, as well as in both quiet and violent markets. The SMA yields noticeable "ghost effects," and EWMA is used to eliminate this shortcoming, as evidenced by the paper's results.

While the EWMA is more adaptable and can accommodate shifting volatilities and covariances over time, it has some limitations that are related to computational issues. In this study, the number of distinct λ values for various periods was determined. This differs from the single $\lambda = 0.94$ used by RiskMetrics in their approach to estimating the variance-covariance matrix of daily returns. There are many data to deal with, and they must return a positive definite estimate of the variance-covariance matrix. As a result, computational challenges in the form of long run-time emerge. This opens up future research options on how to execute the model in a time-efficient manner.

The use of a constant λ value reduces the accuracy of the model. This fact is highlighted by the results of sequential periods with various λ values. An incorrect value of λ

completely skews the assessment of volatility. Such inaccurate results would subsequently have an impact on the accuracy of the VaR calculation, thus undermining an institution's financial stability. The model considers the specific market conditions in those periods rather than having an umbrella view over all periods with different market conditions when optimising the $RMSE$ for distinct periods independently.

GARCH models are typically thought to be superior to EWMA models. While this is true, GARCH models face higher computing challenges because there are more parameters to calculate. This suggests another research avenue to pursue to improve computational practicality and run-time efficiency. Using GARCH(1,1) as a prototype, this research demonstrates that EWMA outperforms SMA. With a few small exceptions, the results reveal that the EWMA and the GARCH(1,1) follow the same route. The distinction between the two is the latter's mean reversion feature. While GARCH models are generally better, it is also crucial to remember that there is no guarantee that a stock price will always revert to its average. The stock price is unlikely to recover to its average in extreme conditions, such as the risk of liquidation.

For each of the three economies, a λ time series was constructed. To get a new measurement, the sample period was shifted forward three months each time, employing historical data dating back one year. The time series revealed that λ's value changed from 2012 to 2021. The λ value was much lower in volatile markets than in quieter environments. This is especially true for stocks that were significantly impacted by market volatility.

In contrast to the EWMA volatility, the SMA volatility requires a time window for the data to be determined. This is a drawback of the SMA that necessitated the need to recalibrate the data for illustration purposes. The SMA period was substituted with a cutoff to make the two volatility metrics comparable. The optimisation was employed to find the decay factor with the lowest RMSE across a range of λ values to select the optimum decay factor in practice. The best value of λ is determined in this paper by using mathematical optimisation to reduce the RMSE.

The examples in Chapter 5 support the interpretation of λ discussed in Chapter 4. Lower values produce dramatically distinct EWMA and SMA courses, with the latter recording less volatility in some circumstances. This lack of responsiveness reflects the SMA's insensitivity to more recent data rather than equal weighting. The former's exponential weighting allows for a faster reaction to changes in returns. The predicted volatility allows for more precise VaR calculations.

This paper analysed the performance of SMA, EWMA, and GARCH(1,1) - in the context of Basel's traffic light system for VaR exceptions. The analysis was conducted for three different economies (South Africa, the United States, and the United Kingdom) using stock market data. The GARCH(1,1) and EWMA methods predominantly observed higher cumulative exceptions throughout the period, while the tiering on the EWMA cumulative exceptions is more gradual than that of SMA. The findings reveal that GARCH(1,1) consistently produces less conservative VaR estimates than SMA and EWMA. This means that GARCH(1,1) is more likely to identify exceptions in volatile times and during periods when volatility gradually becomes steady. In contrast, SMA and EWMA tend to produce more conservative VaR estimates, resulting in fewer exceptions.

The observed differences in performance between the three methods can be attributed to how they calculate volatility. SMA is a simple method that calculates volatility as the average of past observations, giving equal weight to all observations. In contrast, EWMA gives more weight to recent observations and less weight to older observations. Finally, GARCH(1,1) is a more complex method that models volatility as a function of past observations and the residuals of the model. The balance in the trade-off between complexity and accuracy lead to EWMA being preferred. The choice of volatility calculation method can significantly impact the VaR estimates and the number of exceptions identified. Hence, careful consideration must be given to the method used in any given application, taking into account the specific characteristics of the data and the requirements of the analysis.

Appendix A

Statistical Models of Financial Time Series

A.1 Definitions

(Franke, Härdle, and Hafner, 2004)

Stochastic Process

A stochastic process X_t, $t \in \mathbb{Z}$, is a family of random variables, defined in a probability space $(\Omega, \mathcal{F}, P)$.

X_t is a random variable at time t, with a specific density function. For a unique $\omega \in \Omega$, $X(\omega) = \{X_t(\omega), t \in \mathbb{Z}\}$ is a realisation or a path of the process.

CDF of a Stochastic Process

The joint cumulative distribution (cdf) of a stochastic process X_t is defined as

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = P(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n).$$

The underlying stochastic process is uniquely determined if the joint distribution function $F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$ is known $\forall\, t_1, \ldots, t_n \in \mathbb{Z}$.

Conditional CDF

The conditional CDF of a stochastic process X_t $\forall\, t_1, \ldots, t_n \in \mathbb{Z}$ with $t_1 < t_2 < \ldots < t_n$ is defined as

$$F_{t_n|t_{n-1},\dots,t_1}(x_n|x_{n-1},\dots,x_1) = P(X_{t_n} \leq x_n|X_{t_{n-1}} = x_{n-1},\dots,X_{t_1} = x_1).$$

Mean Function

The mean function μ_t of a stochastic process X_t is defined as

$$\mu_t = \mathbf{E}[X_t] = \int_{\mathbf{R}} x dF_t(x)$$

Auto-covariance Function

The auto-covariance function of a stochastic process X is defined as

$$\gamma(t,\tau) = \mathbf{E}[(X_t - \mu_t)(X_t - \tau - \mu_t - \tau)] = \int_{\mathbf{R}^2} (x_1 - x_2)(x_2 - \mu_t - \tau) dF_t, t - \tau(x_1, x_2)$$

for $\tau \in \mathbf{Z}$.

Stationary

A stochastic process X is covariance stationary if

1. $\mu_t = \mu$,
2. $\gamma(t, \tau) = \gamma_\tau$

A stochastic process X_t is strictly stationary if $\forall\ t_1, ..., t_n$ and $\forall\ n, s \in \mathbf{Z}$

$$F_{t_1,...,t_n}(x_1, ..., x_n) = F_{t_1+s,...,t_n+s}(x_1, ..., x_n).$$

Auto-correlation function

The auto-correlation function ρ of a covariance stationary stochastic process is defined as

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0}$$

where $\rho \in [-1, 1]$.

White Noise

The stochastic process X_t is white noise if

1. $\mu_t = 0$
2. $\gamma_\tau = \begin{cases} \sigma^2 & , x = 0 \\ 0 & , x = 0 \end{cases}$

   where σ^2 is the variance.

Random Walk

The stochastic process X_t follows a random walk if it can be represented as

$$X_t = c + X_t - 1 = c + \epsilon_t$$

where c is a constant and ϵ_t is a white noise.

If c is non-zero, the variables $Z_t = X_t - X_t - 1 = c + \epsilon_t$ have a non-zero mean. This is known as a random walk with drift.

Markov Process

A stochastic process has the Markov property if $\forall\, t \in \mathbf{Z}$ and $k \geq 1$

$$F_{t|t-1,...,t-k}(x_t|x_{t-1},...,x_{t-k}) = F_{t|t-1}(x_t|x_{t-1}).$$

That is, the conditional distribution of a Markov process at time t is determined by the condition of the process at time $t-1$. An example of a Markov process is the random walk with independent variables.

Martingale

The stochastic process X_t is a martingale if

$$E[X_t|X_{t-1} = x_{t-1},...,X_{t-k} = x_{t-k}] = x_{t-1}$$

$\forall\, k > 0$.

An example of a Martingale is the random walk without a drift.

Log Return

The log return r_t is defined as

$$r_t = ln(\frac{P_t}{P_{t-1}}) = ln(1 + R_t).$$

The log return is defined for the case of continuous compounding.

Linear Process

If the process X_t has the form

$$X_t = \mu + \sum_{i=-\infty}^{\infty} a_i \epsilon_{t-i}$$

with white noise ϵ_t and absolute summability of the filter $(a_i) : \sum_{i=-\infty}^{\infty} |a_i| < \infty$, then it is a linear process.

Moving Average Processes

The moving average process of order q, $MA(q)$, is defined as

$$X_t = \beta_1 \epsilon_{t-1} + ... + \beta_q \epsilon_{t-q} + \epsilon_t$$

with white noise ϵ_t.

Auto-regressive Process

The linear autoregressive process order p, (AR(p)), is defined as

$$X_t = v + \alpha_1 X_{t-1} + ... + \alpha_p X_{t-p} + \epsilon_t.$$

ARMA Models

The $ARMA(p,q)$ model is defined as

$$X_t = v + \alpha_1 X_{t-1} + ... + \alpha_p X_{t-p} + \beta_1 \epsilon_{t-1} + ... + \beta_q \epsilon_{t-q} + \epsilon_t.$$

A.2 ARCH and GARCH Models

(Franke, Härdle, and Hafner, 2004)

Definitions

(Franke, Härdle, and Hafner, 2004)

ARCH(q)

The process ϵ_t, $t \in \mathbf{Z}$ is $ARCH(q)$, if $E[\epsilon_t | F_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + ... + \alpha_q \epsilon_{t-q}^2$$

with $\omega > 0$, $\alpha_1, ..., \alpha_q \geq 0$ and

- $Var(\epsilon_t | F_{t-1}) = \sigma_t^2$ and $Z_t = \frac{\epsilon_t}{\sigma_t}$ is *i.i.d.* (strong ARCH)
- $Var(\epsilon_t | F_{t-1}) = \sigma_t^2$ (semi-strong ARCH)
- $P(\epsilon_t^2 | 1, \epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_{t-1}^2, \epsilon_{t-2}^2, ...) = \sigma_t^2$ (weak ARCH),

where P is the best linear projection.

GARCH(p,q)

The process ϵ_t, $t \in \mathbf{Z}$ is $GARCH(p,q)$, if $E[\epsilon_t|F_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \epsilon_{t-j}^2$$

and

- $Var(\epsilon_t|F_{t-1}) = \sigma_t^2$ and $Z_t = \frac{\epsilon_t}{\sigma_t}$ is $i.i.d.$ (strong GARCH)
- $Var(\epsilon_t|F_{t-1}) = \sigma_t^2$ (semi-strong GARCH)
- $P(\epsilon_t^2|1, \epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_{t-1}^2, \epsilon_{t-2}^2, ...) = \sigma_t^2$ (weak GARCH).

Appendix B

Basic Properties of Covariance and Correlation Matrices

(Freund and Walpole, 1986)

A covariance matrix is a square, symmetric $m \times m$ matrix of variances and co-variances of a set of m returns:

$$A = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \dots & \dots & \sigma_{2m} \\ \sigma_{31} & \sigma_3^2 & \sigma_3^2 & \dots & \sigma_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{m1} & \dots & \dots & \dots & \sigma_m^2 \end{bmatrix}.$$

Since correlation is given by

$$\rho_{ij} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \in [-1, 1]$$

and

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \dots & \dots & \sigma_{2m} \\ \sigma_{31} & \sigma_3^2 & \sigma_3^2 & \dots & \sigma_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{m1} & \dots & \dots & \dots & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \dots & \dots & \rho_{1m}\sigma_1\sigma_m \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \dots & \dots & \rho_{2m}\sigma_2\sigma_m \\ \rho_{31}\sigma_3\sigma_1 & \rho_{32}\sigma_3\sigma_2 & \sigma_3^2 & \dots & \rho_{3m}\sigma_3\sigma_m \\ \dots & \dots & \dots & \dots & \dots \\ \rho_{m1}\sigma_m\sigma_1 & \dots & \dots & \dots & \sigma_m^2 \end{bmatrix},$$

the covariance matrix can be represented as

$$A = DCD$$

where D is a diagonal matrix containing the standard deviations of the returns and C is the correlation matrix of the returns.

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \dots & \dots & \sigma_{2m} \\ \sigma_{31} & \sigma_3^2 & \sigma_3^2 & \dots & \sigma_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{m1} & \dots & \dots & \dots & \sigma_m^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 & 0 & \dots & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \sigma_m \end{bmatrix} \times \begin{bmatrix} 1 & \rho_{12} & \dots & \dots & \rho_{1n} \\ \rho_{12} & 1 & \dots & \dots & \rho_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \rho_{1m} & \rho_{2m} & \dots & \dots & 1 \end{bmatrix} \times \begin{bmatrix} \sigma_1 & 0 & \dots & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \sigma_m \end{bmatrix}$$

The covariance matrix is mathematically convenient for expressing the asset volatilities and their correlations. A is positive semi-definite. This is true if and only if C is positive semi-definite. D will always be positive definite.

Definition

Positive Semi-definite

A square symmetric matrix $C \in R^{m \times m}$ is positive semi-definite if

$$\vec{v}^T C \vec{v} \geq 0, \forall \vec{v} \in \mathbb{R}^m \tag{B.1}$$

and positive definite if the inequality holds with equality only for vectors $\vec{v} = 0$.

Corollary

A symmetric matrix is positive semi-definite $\iff$ all eigenvalues are non-negative.

Appendix C

Basics of Mathematical Optimisation

Concept

Minimise or maximise a given objective function of several decision variables that satisfy constraints.

(Yang, 2008)

Optimisation Problem

Given $f(\vec{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a set $S \subset \mathbb{R}^n$, find $\vec{x}^* \in \mathbb{R}^n$ that solves

$$(OP) \begin{cases} min_{\vec{x}} f(\vec{x}) \\ s.t. \\ \vec{x} \in S \end{cases}$$

where S is a feasible region.

$$S = \{\vec{x} : g_i(\vec{x}) \leq 0, i \in I \text{ and } h_j(\vec{x}) = 0, j \in \epsilon\}$$

where ϵ and I are index sets for equality and inequality constraints:

$$(OP) \begin{cases} min_{\vec{x}} f(\vec{x}) \\ s.t. \\ g_i(\vec{x}) \leq 0; i \in I \\ h_j(\vec{x}) = 0; j \in \epsilon \end{cases}$$

Appendix D

Unbiased Estimator

In statistics, one of the qualities of an estimator is unbiasedness.

Definition

The statistic $\hat{\theta}$ is an unbiased estimator of $\theta \iff E[\hat{\theta}] = \theta$ (Voinov, n.d.).

Now let $\hat{\theta} = \hat{\sigma^2}$ and $\theta = \sigma^2$. Consider estimator used in the SMA

$$\hat{\sigma_t^2} = \frac{1}{n-1}\sum_{i=1}^{n}(r_{t-i} - \bar{r})^2. \tag{D.1}$$

where r_t is the portfolio return in month t, T is the sample period length, $\bar{r} = \frac{\sum_{i=1}^{n} r_i}{n}$ is the mean return across the sample period. Show that $\hat{\sigma^2}$ is an unbiased estimator of σ^2. Note: consider this outside the time series space, so discard the time parameter t for this proof. It holds true for the time series as well.

Proof

Let $r_1, r_2, ..., r_n$ to be i.i.d random variables, each with the expected value μ and variance σ^2. For the whole population, $\sigma^2 = E[(r_i - \mu)^2]$.

For a sample from this population, we want a statistic s.t $E[\hat{\sigma^2}] = \sigma^2$.

$$
\begin{aligned}
E[\hat{\sigma^2}] &= E[\frac{\sum_{i=1}^{n}(r_i - \bar{r})^2}{n-1}] \\
&= \frac{1}{n-1}E[\sum_{i=1}^{n}(r_i - \bar{r})^2] \\
&= \frac{1}{n-1}E[\sum_{i=1}^{n}(r_i - \mu)^2 - (\bar{r} - \mu))^2] \\
&= \frac{1}{n-1}E[\sum_{i=1}^{n}(r_i - \mu)^2 - 2\sum_{i=1}^{n}(r_i - \mu)(\bar{r} - \mu) + \sum_{i=1}^{n}(\bar{r} - \mu))^2] \\
&= \frac{1}{n-1}[\sum_{i=1}^{n} E(r_i - \mu)^2 - nE(\bar{r} - \mu))^2]
\end{aligned}
$$

Then substituting $\sigma^2 = E(r_i - \mu)^2$ and $Var(\bar{r}) = E(\bar{r} - \mu)^2 = \frac{\sigma^2}{n}$ by the Central limit theorem:

$$
\begin{aligned}
E[\hat{\sigma^2}] &= \frac{1}{n-1}(\sum_{i=1}^{n}\sigma^2 - n\frac{\sigma^2}{n}) \\
&= \frac{1}{n-1}(n\sigma^2 - \sigma^2) \\
&= \frac{1}{n-1}(\sigma^2(n-1)) \\
&= \sigma^2
\end{aligned}
$$

Therefore since $E(\hat{\sigma^2}) = \sigma^2$, $\hat{\sigma^2}$ is an unbiased estimator of σ^2.

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