SOLVABILITY AND SUPERSOLVABILITY CRITERIA RELATED TO CHARACTER CODEGREES

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Abstract. Let G be a finite group and $\mathrm{Irr}(G)$ be the set of irreducible characters of G. The number $\mathrm{cod}(\chi) = |G{:}\ker\chi|/\chi(1)$ is called the character codegree of $\chi \in \mathrm{Irr}(G)$. In this article, we show that if G has at most two composite character codegrees, then G is solvable. We also obtain sufficient numerical conditions on the character codegrees and character degrees for the solvability and supersolvability of a group.

1. Introduction

The study of character degrees of irreducible characters of finite groups has received much attention as they proved to have strong influence on the structure of finite groups. Dual to a character degree is the so-called codegree of a character. We shall recall its definition: Let G be a finite group and p be a prime. Let $\mathrm{Irr}(G)$ denote the set of complex irreducible characters of G. The codegree of $\chi \in \mathrm{Irr}(G)$ is defined as $\mathrm{cod}(\chi) := |G{:}\ker\chi|/\chi(1)$. A different definition of a character codegree was used in [6, 7] and this was modified to the above definition in [20] to avoid ambiguity. There has been recent interest on how the structure of a group is affected by the character degrees and codegrees of irreducible characters of a finite group. Gagola and Lewis [10] obtained a characterisation of nilpotent groups in terms of character degrees and codegrees. In particular, they proved that G is nilpotent if and only if $\chi(1)$ divides $\mathrm{cod}(\chi)$ for all $\chi \in \mathrm{Irr}(G)$. In [4], Berkovich showed that G is a p-group if and only if $\mathrm{cod}(\chi) = p^{\alpha_\chi}\chi(1)$, for each non-linear irreducible character χ of G, where α_χ is a positive integer depending on χ. Qian [19] obtained a criterion in terms of character degrees and codegrees of irreducible characters for a finite group to be p-closed and together with Liang [15], they studied finite groups G such that $(\chi(1), \mathrm{cod}(\chi)) = 1$ for each irreducible character χ of G.

A natural question that can be asked is this: *What conditions can be imposed on the codegrees of irreducible characters of G for G to be solvable?*

It is known that a finite non-abelian simple group does not have a character codegree which is a prime power (see [6, Theorem 1]). Recently, the following result was proved giving a sufficient condition for a group to be solvable:

Theorem 1.1. [2, Theorem 4.6] *Let G be a finite group with exactly one composite character codegree. Then G is solvable.*

Our first objective in this article is to extend Theorem 1.1:

Theorem A. *Let G be a finite group with at most two composite character codegrees. Then G is solvable.*

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It is known that $\operatorname{cod}(\chi) > \chi(1)$ for any non-principal irreducible character χ of any finite group (see for example [1, Lemma 2.6]). On the other hand, the result below shows that a finite group is solvable if an inequality involving character degrees and codegrees of non-linear irreducible characters is satisfied:

Theorem 1.2. [25] *Let G be a finite group. If $\operatorname{cod}(\chi) \leqslant p_\chi \chi(1)$ for all non-linear $\chi \in \operatorname{Irr}(G)$, where p_χ is the largest prime divisor of $|G/\ker\chi|$, then G is solvable.*

Our second objective is to improve Theorem 1.2:

Theorem B. *Let G be a finite group. If $\operatorname{cod}(\chi) < p_\chi(\chi(1)+1)$ for all non-linear $\chi \in \operatorname{Irr}(G)$, where p_χ is the largest prime divisor of $|G/\ker\chi|$, then G is solvable.*

Our approach in proving Theorem B is similar to that of Theorem 1.2 with slight changes. We note that these bounds in both Theorems A and B are best possible since for $G = \mathrm{A}_5$, G has three composite codegrees and there is an irreducible character χ of degree 3 such that $\operatorname{cod}(\chi) = p_\chi(\chi(1)+1)$, where $p_\chi = 5$. In [25], the authors did not give any examples of solvable groups with the property in their theorem (Theorem 1.2), so we shall do that for our result. Note that A_4 and S_4 are examples of solvable groups with the property in Theorem B. We also have that the family $\{\mathrm{D}_{2p}\}$, of dihedral groups of order $2p$ for some odd prime has groups with the property in Theorem B. Obviously the condition in Theorem B cannot characterise solvability. In particular, solvable groups G of the form $\mathrm{C}_p \times \mathrm{A}_4$, $\mathrm{C}_p \times \mathrm{S}_4$ or D_{2pq}, where p and q are odd primes are such that there exists $\chi \in \operatorname{Irr}(G)$ such that $\operatorname{cod}(\chi) \geqslant p_\chi(\chi(1)+1)$. In this case C_n is a cyclic group of order n.

The other objective is to provide a sufficient condition for a group to be supersolvable.

Theorem C. *Let G be a finite group. If $\operatorname{cod}(\chi) < \chi(1) + \frac{3}{\chi(1)}$ for all non-linear $\chi \in \operatorname{Irr}(G)$, then G is supersolvable.*

Examples satisfying the hypothesis in Theorem C are S_3, Q_8 and D_8. In Example 3.1 we give an infinite family of supersolvable groups with the property in Theorem C. Note that the bound cannot be improved since $\operatorname{cod}(\chi) = \chi(1) + \frac{3}{\chi(1)}$ for some $\chi \in \operatorname{Irr}(G)$ when $G = \mathrm{A}_4$.

2. Preliminaries

In this section we shall prove some results needed in proofs of our main results.

Lemma 2.1. *Let G be a non-trivial finite group and $\chi \in \operatorname{Irr}(G)$. Then*

$$\chi(1)^2 + \chi(1) \leqslant |G|.$$

Proof. For $\chi \in \operatorname{Irr}(G)$, we note that $\chi(1)^2 = |G : \ker\chi|$ if and only if $\operatorname{Irr}(G/\ker(\chi)) = \{\chi\}$ and this holds if and only if $\chi = 1_G$. Since $|G| \geqslant 2$, we have the desired conclusion if $\chi(1)^2 = |G : \ker\chi|$. Hence we may assume that $\chi(1)^2 < |G : \ker\chi|$ and so $\chi(1)^2 + k = |G|$ for some positive integer k. It follows that $\chi(1)$ divides k and the result follows. □

In the next set of results, we will prove some properties of characters of non-solvable groups. We first show that a simple group has an irreducible character with two good properties.

Lemma 2.2. *Let N be a non-abelian simple group. Then there exists $\chi \in \operatorname{Irr}(N)$ such that χ is extendible to $\operatorname{Aut}(N)$ and $\chi(1)(\chi(1)+1) \leqslant |N|$.*

Proof. Using Lemma 2.1 it is sufficient to showing that every simple group has an irreducible character extendible to $\mathrm{Aut}(N)$. By [18, Lemma 4.2], the result follows. □

For sporadic simple groups and almost all alternating groups more can be said:

Lemma 2.3. *Let N be a sporadic group or an alternating group, A_n, $n \geqslant 5$ with $n \neq 6$. Then there exists some non-linear $\chi \in \mathrm{Irr}(N)$ extendible to $\mathrm{Aut}(N)$ such that $|N| \geqslant p_m \cdot k \cdot (k+1)$, where p_m is the largest prime divisor of $|N|$ and $k = \chi(1)$.*

Proof. Suppose that N is a sporadic simple group. Using Table 1 in [25] and the Atlas [8], it can be shown that the result follows.

Suppose that N is an alternating group A_n, $n \geqslant 5$. Consider $n = 5$. Then N has $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 3$ and $|N| = 60 \geqslant 5 \cdot 3 \cdot 4 = p_m \cdot \chi(1) \cdot (\chi(1)+1)$.

If $n > 6$, then A_n has an irreducible character of degree $n-1$ that is extendible to S_n. Now $p_m \leqslant n$ and $\frac{(n-2)!}{2} > n$. Then $|\mathrm{A}_n| = n \cdot (n-1) \cdot \frac{(n-2)!}{2} > n \cdot (n-1) \cdot n \geqslant p_m \cdot \chi(1) \cdot (\chi(1)+1)$. □

The alternating group A_6 is indeed an exception since there is no irreducible character of A_6 that extends $\mathrm{Aut}(\mathrm{A}_6)$ and which satisfy the numerical condition in Lemma 2.3. However, since $\mathrm{A}_6 \cong \mathrm{A}_1(9)$, we obtain a result on A_6 in Lemma 2.4(i) below.

For most of the simple groups of Lie type, instead of an extendible character with the right properties, we show that the corresponding result of [25, Lemma 2.4] holds. We use this new result to prove Theorem B.

Let $\Phi_r(q)$ denote the r-th cyclotomic polynomial evaulated at q, or Φ_r for short. Note that $\Phi_r(q)$ is a polynomial with integer coefficients, so Φ_r is an integer.

Lemma 2.4. *Let N be a finite simple group of Lie type or the Tits group ${}^2\mathrm{F}_4(2)'$. Let p_m be the largest prime divisor of $|N|$.*

(a) *There exists some non-linear $\theta \in \mathrm{Irr}(N)$ such that $|N| \geqslant p_m \cdot \theta(1) \cdot (\theta(1)+1)$.*
(b) *If N is not one of the following groups: $\mathrm{A}_1(q)$, $\mathrm{A}_2(q)$, ${}^2\mathrm{A}_2(q^2)$ with $(3, q+1) = 3$, ${}^2\mathrm{B}_2(q^2)$, or ${}^2\mathrm{G}_2(q^2)$, then there exists some non-linear $\theta \in \mathrm{Irr}(N)$ such that*
$$|N| \geqslant p_m^2 \cdot \theta(1) \cdot (\theta(1)+1).$$

The respective bounds for p_m and values of $\theta(1)$ are in Tables 2.1 and 2.2.

Proof. We shall first prove (a) for the following groups: $\mathrm{A}_1(q)$, $\mathrm{A}_2(q)$, ${}^2\mathrm{A}_2(q^2)$ with $(3, q+1) = 3$, ${}^2\mathrm{B}_2(q^2)$ and ${}^2\mathrm{G}_2(q^2)$. Let p be a prime such that q is a power of p.

Suppose that $N \cong \mathrm{A}_1(q)$. If $p = 2$, then $\mathrm{cd}(N) = \{1, q-1, q, q+1\}$. Let $\theta \in \mathrm{Irr}(N)$ be such that $\theta(1) = q-1$. Since $p_m \leqslant q+1$, it follows that $|N| = q(q-1)(q+1) = (q+1)(q-1)q \geqslant p_m \cdot \theta(1) \cdot (\theta(1)+1)$.

Suppose that $p > 2$. Using Lemma 2.3, we may assume that $q \geqslant 7$. Now $\mathrm{cd}(N) = \{1, q-1, q, q+1, \frac{q+\epsilon}{2}\}$, where $\epsilon = (-1)^{\frac{q-1}{2}}$. Consider $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = \dfrac{q+\epsilon}{2}$. Then since $p_m \leqslant q+1$ and $q(q-1) > \frac{(q+1)(q+3)}{2}$, we have that
$$|N| = \frac{q(q^2-1)}{2} > (q+1)\frac{(q+1)}{2}\frac{(q+3)}{2} \geqslant p_m \cdot \theta(1) \cdot (\theta(1)+1).$$

Suppose that $N \cong \mathrm{A}_2(q)$, $q > 2$ since $\mathrm{A}_2(2) \cong \mathrm{A}_1(7)$. Consider $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = q(q+1)$. Then since $p_m \leqslant q^2+q+1$,
$$\frac{|N|}{p_m \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{q^3(q^2-1)(q^3-1)}{\gcd(3, q-1)(q^2+q+1)^2(q(q+1))} \geqslant \frac{q^2(q-1)}{q^2+q+1} > 1.$$

Suppose that $N \cong {}^2\mathrm{A}_2(q^2)$, $\gcd(3, q+1) = 3$ and q^2 is a power of p. Note that $|{}^2\mathrm{A}_2(q^2)| = \frac{1}{3}q^3(q^3+1)(q^2-1)$. Consider $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = q(q-1)$. Since $p_m \leqslant q^2-q+1$, we have that

$$\frac{|N|}{p_m \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{q^3(q^2-1)(q^3+1)}{3(q^2-q+1)^2(q(q-1))} \geqslant \frac{q^2(q+1)^2}{3(q^2-q+1)} > 1.$$

Suppose that $N \cong {}^2\mathrm{B}_2(q^2)$, $q^2 = 2^{2m+1}$. Then $|{}^2\mathrm{B}_2(q^2)| = q^4(q^4+1)(q^2-1)$. Consider a unipotent character $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = \frac{1}{\sqrt{2}}q(q^2-1)$.

Since $p_m \leqslant q^2 + \sqrt{2}q + 1$, we have that

$$\frac{|N|}{p_m \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{|N|}{2p_m \cdot \theta(1)^2} \geqslant \frac{q^2(q^4+1))}{(q^2+\sqrt{2}q+1)(q^2-1)} > 1.$$

Suppose that $N \cong {}^2\mathrm{G}_2(q^2)$, $q^2 = 3^{2m+1}$, $q^2 \neq 3$ since ${}^2\mathrm{G}_2(3)' \cong A_1(8)$. Consider a unipotent character $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = \frac{1}{\sqrt{3}}q(q^4-1)$. Since $p_m \leqslant q^4 - q^2 + 1$, we have that

$$\frac{|N|}{p_m \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{|N|}{2p_m \cdot \theta(1)^2} \geqslant \frac{q^6(q^4-q^2+1)(q^4-1)}{\frac{2}{3}(q^4-q^2+1)q^2(q^4-1)^2} = \frac{3q^4}{2(q^4-1)} > 1.$$

Let $N \cong {}^2\mathrm{F}_4(2)'$. Then

$$|N| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13 > 13^2 \cdot (26)(27) = p_m^2 \cdot \theta(1) \cdot (\theta(1)+1).$$

For the rest of the groups of Lie type, we shall prove that (b) holds. This is sufficient to prove that (a) also holds since $|N| \geqslant p_m^2 \cdot \theta(1) \cdot (\theta(1)+1) \geqslant p_m \cdot \theta(1) \cdot (\theta(1)+1)$.

***1**. Lemma 2.4(b) holds for $N \cong \mathrm{A}_n(q)$, $n \geqslant 3$.*

Let $N \cong \mathrm{A}_3(q)$, where $q \geqslant 3$ since $A_3(2) \cong \mathrm{A}_8$. Then

$$\frac{|N|}{p_m^2 \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{q^4(q^2-1)^2(q-1)}{\gcd(4, q-1)(q^2+q+1)(q^4+q^2+1)} > 1,$$

since $q^4(q-1) > q^4+q+1$ and $(q^2-1)(q+1) > q^2+q+1$. From here going forward we will take advantage of the fact that

$$\frac{|N|}{p_m^2 \cdot \theta(1) \cdot (\theta(1)+1)} \geqslant \frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2}.$$

Let $N \cong \mathrm{A}_4(q)$. Then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^{10}(q^2-1)(q^3-1)(q^4-1)(q^5-1)}{2 \cdot (5, q-1)(q^5-1)^2 q^2(q+1)^2(q^2+1)^2} > \frac{q^8(q-1)^2}{2 \cdot (q^5-1)} > 1.$$

Let $N \cong \mathrm{A}_n(q)$ with $n \geqslant 5$. Then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^{\frac{n(n+1)}{2}}(q^2-1)(q^3-1)\cdots(q^{n+1}-1)(q-1)^2(q^2-1)^2}{2 \cdot (n+1, q-1)(q^{n+1}-1)^2 q^6(q^{n-1}-1)^2(q^n-1)^2} > \frac{(q^3-1)(q-1)^3}{2 \cdot (n+1, q-1)q^3} > 1$$

if $q \geqslant 3$. If $q = 2$, then since $n \geqslant 5$,

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{9 \cdot 2^{\frac{1}{2}(n^2-5n)-6}(2^2-1)(2^3-1)}{2} > 1.$$

2. Lemma 2.4(b) holds for $N \cong \mathrm{B}_n(q)$, $n \geqslant 2$ and $N \cong \mathrm{C}_n(q)$, $n \geqslant 3$.

Let $n = 2$. If $2 \leqslant q \leqslant 4$, then the result follows by direct calculation. Suppose that $q \geqslant 5$. Then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^4(q^2-1)(q^4-1)}{2 \cdot (2, q-1)(q^2+1)^2(\frac{1}{4}q^2(q-1)^4)} \geqslant \frac{q^2}{q^2-1} \cdot \frac{(q+1)^2}{q^2+1} > 1.$$

Let $N \cong \mathrm{B}_3(q)$. If $2 \leqslant q \leqslant 3$, then the result follows from character tables in the Atlas [8]. Suppose that $N \cong \mathrm{B}_3(q)$, $q \geqslant 4$ or $N \cong \mathrm{B}_n(q)$, $n \geqslant 4$. If q is odd, then it was shown in [25, Lemma 2.4] that

$$\frac{|N|}{p_m^2 \cdot \theta(1)^2} \geqslant \frac{|N|}{(q^n+1)^2 \cdot \theta(1)^2} > 1.$$

We note that $p_m \leqslant \frac{q^n+1}{2}$. So

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{|N|}{2 \cdot (\frac{q^n+1}{2})^2 \cdot \theta(1)^2} = \frac{2|N|}{(q^n+1)^2 \cdot \theta(1)^2} > 2.$$

If q is even, then $\mathrm{B}_n(q) \cong \mathrm{C}_n(q)$. It is sufficient to consider $\mathrm{C}_n(q)$ to conclude our proof. For $\mathrm{C}_3(2)$ and $\mathrm{C}_3(3)$, the character tables in the Atlas [8] shows that the result is true.

If $N \cong \mathrm{C}_3(q)$, $q \geqslant 4$, then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^9(q^2-1)(q^4-1)(q^6-1)(q^2-1)^4}{2 \cdot (q^3+1)^2(q^2-1)^2 q^6(q^6-1)^2} \geqslant \frac{q^3(q-1)^2(q^4-1)}{2 \cdot (q^2-q+1)^3(q^2+q+1)} \geqslant 1$$

since $(q^4-1)(q-1)^2 > (q^2-q+1)^3$ and $q^3 > 2 \cdot (q^2+q+1)$.

If $N \cong \mathrm{C}_n(q)$, with $n \geqslant 4$ and $q \geqslant 4$, then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^{n^2}(q^2-1)\cdots(q^{2(n-2)}-1)(q^{2(n-1)}-1)(q^{2n}-1)(q^2-1)^4}{2 \cdot (q^n+1)^2 q^6 (q^{2(n-2)}-1)^2(q^{2n}-1)^2} \geqslant$$
$$\frac{q^{n^2}(q^2-1)^5}{2 \cdot (q^n+1)(q^{2n}-1)} > \tfrac{1}{2}q^{n^2-4n-6}(q^2-1)^5 > 1.$$

3. Lemma 2.4(b) holds for $N \cong \mathrm{D}_n(q)$, $n \geqslant 4$, $N \cong {}^2\mathrm{A}_n(q^2)$, $n \geqslant 2$ and $N \cong {}^2\mathrm{D}_n(q^2)$, $n \geqslant 4$.

Let $N \cong \mathrm{D}_4(q)$. For $\mathrm{D}_4(2)$ and $\mathrm{D}_4(3)$, the result easily follows from calculations using the character tables in the Atlas [8]. Suppose that $q \geqslant 4$. Since $q^2+q+1 > q^2+1 > q^2-q+1$,

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} \geqslant \frac{q^{12}(q^4-1)(q^2-1)(q^4-1)(q^6-1)}{2 \cdot (4, q^n-1)(q^2-q+1)^2 \frac{1}{4} q^6 (q+1)^8 (q^2-q+1)^2} \geqslant \frac{q^4(q-1)^4}{2 \cdot (q+1)^4} > 1.$$

Let $N \cong \mathrm{D}_5(q)$. Then

$$\frac{|N|}{2 \cdot p_m^2 \cdot \theta(1)^2} > \frac{q^5(q^4+1)}{8 \cdot (q^4+q^2+1)(q-1)^4} > 1.$$

Suppose that $N \cong \mathrm{D}_n(q)$, $n \geqslant 6$. Then

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} > \frac{q^{n(n-1)}(q^n-1)(q^2-1)^5\cdots(q^{2(n-4)}-1)(q^{2(n-3)}-1)(q^{2(n-2)}-1)(q^{2(n-1)}-1)(q^4-1)^2}{2\cdot(4,q^n-1)(q^n-1)^2(q^{n-4}+1)^2q^{12}(q^{2(n-3)}-1)^2(q^{2(n-1)}-1)^2(q^n-1)^2} \geqslant \frac{1}{8}\cdot q^{n(n-1)-5n+10}\cdot(q^2-1)^4(q^4-1)\cdot\frac{(q^2-1)(q^{n-4}-1)}{q^{n-4}+1} > 1.$$

Let $N\cong{}^2\mathrm{A}_2(q^2)$, with $(3,q-1)=1$. Then

$$\frac{|N|}{p_m^2\cdot\theta(1)\cdot(\theta(1)+1)} \geqslant \frac{q^3(q^2-1)(q^3+1)}{(q^2-q+1)^3(q(q-1))} \geqslant \frac{q^2(q+1)^2}{(q^2-q+1)^2} > 1.$$

Suppose that $N\cong{}^2\mathrm{A}_3(q^2)$. Then

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} \geqslant \frac{q^6(q-1)^2(q+1)^2(q^3+1)(q^2+1)}{2\cdot(4,q+1)q^4(q^2+1)^4} \geqslant \frac{q^2(q-1)^2(q+1)^2(q^3+1)}{2\cdot(4,q+1)(q^2+1)^3} > 1.$$

Suppose that $N\cong{}^2\mathrm{A}_n(q^2)$, $n\geqslant 4$. Then

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} \geqslant \frac{q^{\frac{n(n+1)}{2}}(q^2-1)\cdots(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)(q^{n+1}-(-1)^{n+1})(q+1)^2(q^2-1)^2}{2\cdot(n+1,q+1)(q^{n+1}-(-1)^{n+1})^2q^6(q^{n-1}-(-1)^{n-1})^2(q^n-(-1)^n)^2} \geqslant q^{\frac{1}{2}(n^2+n)-3n-4}(q^2-1)^3(q-1)^2 > 1.$$

Let $N\cong{}^2\mathrm{D}_4(q^2)$. Since $q^3-q-1\geqslant 5$,

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} \geqslant \frac{q^{12}(q^4+1)(q^2-1)(q^4-1)(q^6-1)(q+1)^2}{2\cdot(4,q^4+1)(q^4+1)^4\frac{1}{4}q^6(q^3+1)^2} = \frac{(q^4-1)(q^4+q^3-q-1)}{(q^4+1)^2}\cdot\frac{2q^4}{q^4+1}\cdot\frac{q^2(q-1)(q+1)^2}{(4,q^n+1)(q^3+1)} > 1.$$

Let $N\cong{}^2\mathrm{D}_5(q^2)$. If $q=2$, then the result follows from the character table in Atlas [8]. Suppose $q\geqslant 3$. Then

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} \geqslant \frac{q^{20}(q^5+1)(q^2-1)(q^4-1)(q^6-1)(q^8-1)\cdot 4\cdot(q^2+1)^2(q-1)^4}{2\cdot(4,q^4+1)(q^5+1)^4q^6(q^2-1)^2(q^3+1)^2(q^4-1)^2} = \frac{q^{15}(q-1)^3}{(q^5+1)^3}\cdot\frac{2(q^3-1)(q^4+1)(q^2+1)^2}{(4,q^5+1)q(q+1)(q^3+1)} > 1.$$

Let $N\cong{}^2\mathrm{D}_n(q^2)$, $n\geqslant 6$. Then

$$\frac{|N|}{2\cdot p_m^2\cdot\theta(1)^2} \geqslant \frac{4q^{n(n-1)}(q^n+1)(q^2-1)\cdots(q^{2(n-3)}-1)(q^{2(n-2)}-1)(q^{2(n-1)}-1)(q^2+1)^2(q-1)^4}{2\cdot(4,q^n+1)(q^{n+1}+1)^4q^6(q^{n-3}-1)^2(q^{n-2}+1)^2(q^{n-1}+1)^2} \geqslant \frac{2\cdot(q^{n-2}-1)}{(q^{n-2}+1)}\cdot\frac{(q^{n+1})^3(q^2-1)(q^2+1)^2}{(4,q^n+1)(q^n+1)^3}\cdot q^{n^2-4n-9} > 1.$$

4. Lemma 2.4(b) holds for exceptional simple groups with the exception of ${}^2\mathrm{B}_2(q^2)$ *and* ${}^2\mathrm{G}_2(q^2)$.

This easily follows using the values in Table 2.2. ☐

Table 2.1 Character degrees and p_m of classical groups of Lie type

N	$p_m \leqslant$	Labels	Degrees
$\mathrm{A}_2(q)$	q^2+q+1		$\theta(1)=q(q+1)$
$\mathrm{A}_3(q)$	q^2+q+1	$(1,3)$	$\theta(1)=q(q^2+q+1)$
		St	q^6
		$(2,2)$	$q^2(q^2+1)$
$\mathrm{A}_4(q)$	q^5-1	$(1,4)$	$\theta(1)=q(q+1)(q^2+1)$
		St	q^{10}
		$(2,3)$	$q^2(q^4+q^3+q^2+q+1)$
$\mathrm{A}_n(q)$, $n\geqslant 5$	$q^{n+1}-1$	$(1,1,n-1)$	$\theta(1)=\frac{q^3(q^{n-1}-1)(q^n-1)}{(q-1)(q^2-1)}$
		St	$q^{\frac{n(n+1)}{2}}$
		$(1,n)$	$\frac{q(q^n-1)}{q-1}$
$\mathrm{B}_2(q)\cong\mathrm{C}_2(q)$	q^2+1	$\begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix}$	$\theta(1)=\frac{1}{2}q(q-1)^2$
		St	q^4
		$\begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}$	$\frac{q(q+1)^2}{2}$
$\mathrm{B}_n(q)$, $n\geqslant 3$	q^n+1		$\theta(1)=\frac{q^4(q^{n-2}-1)(q^{n-1}-1)q^{n-1}+1)(q^n+1)}{2(q^2-1)^2}$
		St	q^{n^2}
		$\begin{pmatrix} 1 & n \\ 0 & \end{pmatrix}$	$\frac{q(q^n+1)(q^{n-1}-1)}{2(q-1)}$
		$\begin{pmatrix} 0 & n \\ 1 & \end{pmatrix}$	$\frac{q(q^n-1)(q^{n-1}+1)}{2(q-1)}$
$\mathrm{C}_n(q)$, $n\geqslant 3$	q^n+1		$\theta(1)=\frac{q^3(q^{2(n-2)}-1)(q^{2n}-1)}{(q^2-1)^2}$
		St	q^{n^2}
		$\begin{pmatrix} 1 & n \\ 0 & \end{pmatrix}$	$\frac{q(q^n+1)(q^{n-1}-1)}{2(q-1)}$
		$\begin{pmatrix} 0 & n \\ 1 & \end{pmatrix}$	$\frac{q(q^n-1)(q^{n-1}+1)}{2(q-1)}$
$\mathrm{D}_4(q)$, $q>3$	q^2+q+1		$\theta(1)=\frac{1}{2}q^3(q+1)^4(q^2-q+1)$
		St	q^{12}
		$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	$q(q^2+1)^2$
		$\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$	$\frac{1}{2}q^3(q^2+1)^2(q^2+q+1)$

Table 2.1 continued

N	$p_m \leqslant$	Labels	Degrees
$\mathrm{D}_5(q)$	q^5-1		$\theta(1)=(q-1)^2(q^4-1)(q^5-1)(q^6-1)$
		St	q^{20}
		$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\frac{q(q^3+1)(q^5-1)}{q^2-1}$
		$\begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$	$\frac{q^3(q^5-1)(q^2+1)^2(q^3+1)}{2(q^2-1)^2}$
$\mathrm{D}_n(q)$, $n \geqslant 6$	q^n-1		$\theta(1)=\frac{q^6(q^{n-4}+1)(q^{2(n-3)}-1)(q^{2(n-1)}-1)(q^n-1)}{(q^2-1)^2(q^4-1)}$
		St	$q^{n(n-1)}$
		$\begin{pmatrix} n-1 \\ 1 \end{pmatrix}$	$\frac{q(q^{n-2}+1)(q^n-1)}{q^2-1}$
		$\begin{pmatrix} 0 & 2 & n \\ 0 & 1 & 2 \end{pmatrix}$	$\frac{q^6(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)(q^4-1)}$
${}^2\mathrm{A}_2(q^2)$	q^2-q+1		$\theta(1)=q(q-1)$
		St	q^3
${}^2\mathrm{A}_3(q^2)$	q^2+1	$(2,2)$	$q^2(q^2+1)$
		St	q^6
		$(1,n)$	$q(q^2-q+1)$
${}^2\mathrm{A}_n(q^2)$, $n \geqslant 4$	$q^{n+1}-(-1)^{n+1}$	$(1,1,n-1)$	$\theta(1)=\frac{q^3(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)}{(q+1)(q^2-1)}$
		St	$q^{\frac{n(n+1)}{2}}$
		$(1,n)$	$\frac{q(q^n-(-1)^n)}{q+1}$
${}^2\mathrm{D}_4(q^2)$	q^4+1		$\theta(1)=\frac{1}{2}q^3(q^4+1)(q^2-q+1)$
		St	q^{12}
		$\begin{pmatrix} 1 & 3 \\ & - \end{pmatrix}$	$q(q^4+1)$
		$\begin{pmatrix} 0 & 2 & 3 \\ & 1 & \end{pmatrix}$	$\frac{q^3(q^4+1)(q^2-q+1)(q^2+1)}{2(q+1)}$
${}^2\mathrm{D}_n(q^2)$, $n \geqslant 5$	q^n+1	$\begin{pmatrix} 0 & 2 & n-1 \\ & 1 & \end{pmatrix}$	$\theta(1)=\frac{q^3(q^{n-3}-1)(q^{n-2}+1)(q^{n-1}-1)(q^n+1)}{2(q^2+1)(q-1)^2}$
		St	q^{12}
		$\begin{pmatrix} 1 & n-1 \\ & - \end{pmatrix}$	$\frac{q(q^{n-2}-1)(q^n+1)}{q^2-1}$

We note that that the character degree of ${}^2\mathrm{D}_4(q^2)$ is $\frac{1}{2}q^3(q^4+1)(q^2-q+1)$ and not $\frac{1}{2}q^3(q+1)^4(q^2-q+1)$ as in [25, Table 4].

Table 2.2 Character degrees and p_m of exceptional groups of Lie type

N	$p_m \leqslant$	Labels	Degrees
$\mathrm{G}_2(q)$, $q>2$	q^2+q+1	$\mathrm{G}_2[1]$	$\theta(1)=\frac{1}{6}q\Phi_1^2\Phi_6$
		$\mu_{1,6}$	q^6
		$\mu_{2,2}$	$\frac{1}{2}q\Phi_2^2\Phi_6$
$\mathrm{F}_4(q)$	q^4+1	$\mu_{9,2}$	$\theta(1)=q^2\Phi_3^2\Phi_6^2\Phi_{12}$
		$\mu_{1,24}$	q^{24}
		$\mu_{9,10}$	$q^{10}\Phi_3^2\Phi_6^2\Phi_{12}$
$\mathrm{E}_6(q)$	q^8+q^4+1	$\mu_{6,1}$	$\theta(1)=q\Phi_8\Phi_9$
		$\mu_{1,36}$	q^{36}
		$\mu_{20,2}$	$q^2\Phi_4\Phi_5\Phi_8\Phi_{12}$
$\mathrm{E}_7(q)$	q^7+1	$\mu_{7,1}$	$\theta(1)=q\Phi_7\Phi_{12}\Phi_{14}$
		$\mu_{1,63}$	q^{63}
		$\mu_{21,3}$	$q^3\Phi_7\Phi_9\Phi_{14}\Phi_{18}$
$\mathrm{E}_8(q)$	$q^{10}+q^5+1$	$\mu_{8,1}$	$\theta(1)=q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
		$\mu_{1,120}$	q^{120}
		$\mu_{8,91}$	$q^{91}\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
${}^3\mathrm{D}_4(q^3)$	q^8+q^4+1	$\mu'_{1,3}$	$\theta(1)=q\Phi_{12}$
		$\mu_{1,6}$	q^{12}
		$\mu_{2,1}$	$\frac{1}{2}q^3\Phi_2^2\Phi_6^2$
${}^2\mathrm{E}_6(q^2)$	q^8+q^4+1	$\mu'_{2,4}$	$\theta(1)=q\Phi_8\Phi_{18}$
		$\mu_{1,24}$	q^{36}
		$\mu_{4,1}$	$q^2\Phi_4\Phi_8\Phi_{10}\Phi_{12}$
${}^2\mathrm{F}_4(q^2)$, $q^2>2$	q^8-q^4+1	${}^2\mathrm{B}_2[a],1$	$\theta(1)=\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_6$
		ε	q^{24}
		ε'	$q^2\Phi_{12}\Phi_{24}$
${}^2\mathrm{B}_2(q^2)$	$q^2+\sqrt{2}q+1$		$\theta(1)=\frac{1}{\sqrt{2}}q\Phi_1\Phi_2$
${}^2\mathrm{G}_2(q^2)$, $q^2\neq 3$	q^4-q^2+1	cusp	$\theta(1)=\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$
		ε	q^6
		cusp	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi'_{12}$

Theorem 2.5. *Let N be a non-abelian simple group other than $\mathrm{A}_1(q)$. Then there exist non-trivial characters $\chi,\varphi\in\mathrm{Irr}(N)$ such that both χ and φ extend to $\mathrm{Aut}(N)$ and $\chi(1)\neq\varphi(1)$.*

Proof. This follows from [16, Proposition 3.3]. □

In the next theorem we prove that almost simple groups have at least three composite character codegrees:

Theorem 2.6. *Let G be an almost simple group with socle N. Then G has at least three composite character codegrees*

Proof. Suppose that G is simple. Then $|\mathrm{cd}(G)| \geqslant 4$ by [14, Theorem 12.15]. Note that G has at least three non-linear irreducible characters whose degrees are pairwise distinct. Note that these characters are faithful. Hence for each non-trivial $\chi \in \mathrm{Irr}(G)$, $\mathrm{cod}(\chi) = \frac{|G|}{\chi(1)}$ is composite by [2, Theorem 4.1]. Hence we may assume that $N < G \leqslant \mathrm{Aut}(N)$.

If N is a sporadic simple group, then from the Atlas [8] we have our result. We may assume that N is an alternating group, A_n, $n \geqslant 5$. If $5 \leqslant n \leqslant 14$, then using the Atlas [8] and GAP [11], we can verify that G has at least three composite character codegrees. If $n \geqslant 15$, then by [23, Corollary 5], the first four minimal character degrees of N are extendible to G and hence our result follows.

Let N be a simple group of Lie type. Then by Tables 2.1 and 2.2, with some exceptions, N has at least three unipotent characters which are extendible to $\mathrm{Aut}(N)$. By [3, Lemma 5], all these characters are extendible to G. We are only left to consider $\mathrm{A}_1(q), \mathrm{A}_2(q)$, ${}^2\mathrm{A}_2(q^2)$, ${}^2\mathrm{B}_2(q^2)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{F}_4(2)'$ and ${}^2\mathrm{G}_2(3)$.

Suppose that $N \cong \mathrm{A}_1(q)$, $q \geqslant 5$. The character degrees of G are known by [24, Theorem A]. In particular, G has a character of degree q and has at least two more irreducible characters of degrees which are multiples of character degrees of N. Since N is the only minimal normal subgroup of G, we have that G has three non-linear irreducible faithful characters of distinct degrees. Thus G has three composite character codegrees.

Suppose that $N \cong \mathrm{A}_2(q)$, $q \geqslant 2$. Observe that $N \cong \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and this has been considered above. If $q = 3$, then by the Atlas [8], G has more than three composite character codegrees. If $q = 4$, then by the Atlas [8], $\mathrm{Out}(N) \cong \mathrm{D}_{12}$. Using the Atlas [8] and GAP [11], G has more than three character composite codegrees. We may assume $q \geqslant 5$. Note that $|\mathrm{Out}(N)| = 2df$ with $q = p^f$ and $d = \gcd(3, q-1)$. Also note that N has two irreducible characters of degrees q^3 and $q(q+1)$ which are extendible to G. We want to find a third irreducible character of G. By [22, Table 2], N has an irreducible character φ of degree $(q+1)(q^2+q+1)$. Let χ be an irreducible constituent of φ^G. Then $\chi(1) = a\varphi(1)$, where a divides $2df$. Clearly, this is distinct from the first two degrees. Since all these characters are faithful, we have that G has at least three composite character codegrees.

Suppose that $N \cong {}^2\mathrm{A}_2(q^2)$, $q \geqslant 3$. By the Atlas [8] and GAP [11], we may assume that $q \geqslant 5$. Then N has irreducible characters of degrees q^3 and $q(q-1)$ extendible to G. Using [22, Table 2], we have that N has an irreducible character of degree $(q-1)(q^2-q+1)$. Arguing as above, G has an irreducible character of degree $b(q-1)(q^2-q+1)$, where b divides $2df$. Hence the result follows.

Suppose that $N \cong {}^2\mathrm{B}_2(q^2)$, $q = 2^{2n+1}$, $n > 1$. By [5, Section 13.9], N has unipotent characters of degrees $\frac{q}{\sqrt{1}}\Phi_1\Phi_2$ and q^4 which are extendible to $\mathrm{Aut}(N)$ using [17, Theorems 2.4 and 2.5]. Hence G has two composite character codegrees since these characters are faithful. We want to show that G has another composite character codegree. Consider $\varphi \in \mathrm{Irr}(N)$ such that $\varphi(1) = \Phi_8$. Then its inertia group I is such that $N \leqslant I \leqslant G \leqslant \mathrm{Aut}(N)$ and so G has an irreducible character of degree $|G{:}I|\Phi_8$. This is different from the two above and this character is also faithful. Hence the result follows. For $N \cong {}^2\mathrm{B}_2(8)$, we use the Atlas [8] to verify our result.

If N isomorphic to one of $G_2(2)'$, ${}^2F_4(2)'$ and ${}^2G_2(3)$, then the result follows by checking the Atlas [8]. □

We shall need the following result by W. Gaschutz to prove Theorem C.

Theorem 2.7. [12] *Let G be a finite group and S be the socle of G. Then G has a faithful irreducible character if and only if S is generated by the G-conjugates of a single element of S.*

3. Proof of Main Results

In this section we prove Theorems A, B and C.

Proof of Theorem A. Suppose the theorem is not true and let G be a minimal counterexample. Using Theorem 2.6, we have that G is not simple. We first claim that G has a unique non-abelian minimal normal subgroup. If G has minimal normal subgroups N_1 and N_2, then G/N_i is solvable by the minimality of $|G|$, where $i = 1, 2$. So $G/(N_1 \cap N_2) \cong G$ is solvable since the class of finite solvable groups is a formation, a contradiction.

Let N be the unique minimal normal subgroup of G which is non-abelian. Then $N = T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T$, T is a non-abelian simple group for $i = 1, 2, \ldots, k$. Since $\mathbf{C}_G(N) = 1$, we have that $G \leq \mathrm{Aut}(N) \cong \mathrm{Aut}(T) \wr \mathrm{S}_k := \Gamma$. Let $B = \mathrm{Aut}(T)^k \cap G$. We have that G/B is a permutation group of $\Omega = \{1, 2, \ldots, k\}$. We want to show that $k = 1$. We may assume that $k \geqslant 2$.

Assume first that T is a simple group other than $A_1(q)$, $q \geqslant 5$. By Theorem 2.5, T has non-linear irreducible characters φ and ψ which are extendible to $\mathrm{Aut}(T)$ and $\varphi(1) \neq \psi(1)$. Consider the characters

$$\phi_1 = \varphi \times \varphi \times \cdots \times \varphi,\ \phi_2 = \psi \times \psi \times \cdots \times \psi \text{ and } \phi_3 = 1_T \times \varphi \times \cdots \times \varphi.$$

By [3, Lemma 5], ϕ_1 and ϕ_2 are irreducible characters of N that extend to G and so $\phi_1(1)^k$ and $\phi_2(1)^k$ are character degrees of G. For ϕ_3, we have that ϕ_3 extends to its inertia group $I = I_\Gamma(\phi_3)$ in Γ by [13, Lemma 25.5]. Since $I \cap G$ is the inertia group of ϕ_3 in G, we have that

$$|G{:}I \cap G|\phi_3(1) \in \mathrm{cd}(G)$$

by Clifford's theorem [14, Theorem 6.11]. Note that kernels of the three characters of G are trivial. Hence G has three composite character codegrees, a contradiction.

We may assume that $T \cong A_1(q)$, $q \geqslant 5$ and $k \geqslant 3$. Let $\varphi, \psi \in \mathrm{Irr}(T)$ be such that $\varphi(1) = q$ and $\psi(1) = q-1$. Note that φ is extendible to $\mathrm{Aut}(T)$. Consider the following irreducible characters of N:

$$\phi_1 = \varphi \times \varphi \times \cdots \times \varphi,\ \phi_2 = \psi \times 1_T \times \cdots \times 1_T \text{ and } \phi_3 = 1_T \times \psi \times \cdots \times \psi.$$

Clearly, $\phi_1(1) = \varphi(1)^k \in \mathrm{cd}(G)$. Now both ϕ_2 and ϕ_3 extend to $I = I_\Gamma(\phi_2) = I_\Gamma(\phi_3)$. It follows by Clifford's theorem [14, Theorem 6.11] that the character degrees

$$|G{:}I \cap G|\psi(1),\ |G{:}I \cap G|\psi(1)^{m-1} \in \mathrm{cd}(G).$$

Since $|G{:}I \cap G|\psi(1) \neq |G{:}I \cap G|\psi(1)^{m-1}$, G has three distinct character degrees. But the kernels of the respective characters are trivial and so G has three composite character codegrees, a contradiction.

We may assume that $T \cong A_1(q)$, $q \geqslant 5$ and $k = 2$. Note that $|G/B| = 2$. Let us consider

$$\phi_1 = \varphi \times \varphi,\ \phi_2 = \varphi \times 1_T \text{ and } \phi_3 = \varphi \times \psi,$$

where $\varphi, \psi \in \mathrm{Irr}(T)$ such that $\varphi(1) = q$ and $\psi(1) = q-1$. Then $q^2, 2q, |G{:}I \cap G|q(q-1) \in \mathrm{cd}(G)$, where $I = I_\Gamma(\phi_3)$ is the inertia group in Γ. Obviously these character degrees are pairwise distinct. Note that the kernels of the respective characters are trivial. Thus G has three composite character codegrees, a contradiction.

Hence N is simple and so G is almost simple. Our result then follows from Theorem 2.6. □

Proof of Theorem B. Suppose the theorem is not true and let G be a minimal counterexample. By Lemmas 2.3 and 2.4, G is not a simple group. Clearly, G has a unique minimal normal subgroup N and N is non-abelian. Since $\mathbf{C}_G(N) = 1$, $G \leq \mathrm{Aut}(N)$.

Our next claim is that N is simple. Let $N = T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T$, T is a non-abelian simple group for $i = 1, 2, \ldots, k$ and $k \geqslant 2$. By Lemmas 2.3 and 2.4, there exists a non-linear $\theta \in \mathrm{Irr}(T)$ such that $|T| \geqslant p_m \cdot \theta(1) \cdot (\theta(1) + 1)$. Let $\phi = \theta \times \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$. We have that $|N| \geqslant p_m^k \theta(1)^k (\theta(1)+1)^k > p_m^k \phi(1)(\phi(1)+1)$. Let χ be an irreducible constituent of ϕ^G. By Clifford's theory, $\chi_N = e\sum_{j=1}^t \phi_j$, where $\phi_j = \phi$ and $\frac{\chi(1)}{\phi(1)} = et \mid |G{:}N|$ and so $et \leqslant |G{:}N|$. By Lemma 2.2, there exists $\psi \in \mathrm{Irr}(G)$ such that $\psi_N = \varphi \in \mathrm{Irr}(N)$ and $\varphi(1)(\varphi(1) + 1) \leqslant N$. Since N is the unique minimal normal subgroup of G and $N \nsubseteq \ker(\chi)$, $N \nsubseteq \ker(\psi)$, we see that $\ker(\chi) = \ker(\psi) = 1$, and so $p_\chi = p_\psi$, where p_χ and p_ψ are largest prime divisors of $|G/\ker(\chi)| = |G|$ and $|G/\ker(\psi)| = |G|$, respectively. If $p_\chi \leqslant |G{:}N|$, we conclude that

$$|G| = |G{:}N||N| \geqslant p_\chi \varphi(1)(\varphi(1) + 1) = p_\psi \cdot \psi(1) \cdot (\psi(1) + 1).$$

Hence $\mathrm{cod}(\psi) \geqslant p_\psi(\psi(1) + 1)$, contradicting our hypothesis. Thus $et \leqslant |G{:}N| < p_\chi$. Since $k \geqslant 2$, $|N| > p_m^2 \phi(1)(\phi(1) + 1)$ and so

$$|G| = |G{:}N||N| > etp_m^2 \phi(1)(\phi(1) + 1) > p_m e^2 t^2 \phi(1)(\phi(1) + 1) > p_\chi \chi(1)(\chi(1) + 1),$$

contradicting our hypothesis, which says $\mathrm{cod}(\psi) < p_\psi(\psi(1) + 1)$. Therefore $k = 1$ and G is an almost simple group with $N < G$.

We now show that N is isomorphic to $\mathrm{A}_1(q)$, $\mathrm{A}_2(q)$, ${}^2\mathrm{A}_2(q^2)$ with $(3, q + 1) = 3$, ${}^2\mathrm{B}_2(q^2)$ or ${}^2\mathrm{G}_2(q^2)$.

Suppose that $N \cong \mathrm{A}_n$, where $n \geqslant 5$. Let $n = 5$. Then N has an irreducible character θ of degree 4 which is extendible to G. Let $\chi_N = \theta$ for some $\chi \in \mathrm{Irr}(G)$.

$$|G| = 120 > 5 \cdot 4 \cdot 5 = p_m \theta(1)(\theta(1) + 1).$$

Let $n = 6$. Then N has an irreducible character θ of degree 9 that is extendible to G. So

$$|G| \geqslant 2 \cdot 360 > 5 \cdot 9 \cdot 10 = p_\chi \theta(1)(\theta(1) + 1).$$

Suppose that N is isomorphic to a sporadic simple group or A_n, $n \geqslant 7$. It follows that

$$|G| \geqslant 2|N| > p_m \theta(1)(\theta(1) + 1) = p_m \chi(1)(\chi(1) + 1).$$

for some $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$ using Lemma 2.3.

Let N be the Tits group ${}^2\mathrm{F}_4(2)'$ or a simple group of Lie type with the following exceptions: $\mathrm{A}_1(q)$, $\mathrm{A}_2(q)$, ${}^2\mathrm{A}_2(q^2)$ with $(3, q+1) = 3$, ${}^2\mathrm{B}_2(q^2)$, or ${}^2\mathrm{G}_2(q^2)$. Using Lemma 2.4(b), there exists some non-linear $\theta \in \mathrm{Irr}(N)$ such that $|N| \geqslant p_m^2 \theta(1)(\theta(1) + 1)$. Suppose that $\chi_N = e_1 \sum_{j=1}^{t_1} \theta_j$ with $\theta_j = \theta$ for some $\chi \in \mathrm{Irr}(G)$. It follows that $\chi(1) = e_1 t_1 \theta(1)$ and $e_1 t_1 \leqslant |G{:}N| < p_\chi$. It follows that

$$|G| = |G{:}N||N| > e_1t_1p_m^2\theta(1)(\theta(1)+1) > p_\chi e_1^2t_1^2\theta(1)(\theta(1)+1) > p_\chi\chi(1)(\chi(1)+1).$$

This is obviously a contradiction. Hence our claim is true. We are left with the cases mentioned above.

Suppose that $N \cong \mathrm{A}_1(q)$, $q = p^r \geqslant 7$ for some prime p, with $q \neq 9$(since $\mathrm{A}_6 \cong \mathrm{A}_1(9)$). Note that $p_m \leqslant q+1$. Let $q > p$. This means that $(2, q-1)p_m \leqslant q+1$. Since $|\mathrm{A}_1(q)| = \frac{1}{(2,q-1)}q(q+1)(q-1)$,

$$|G| \geqslant 2|N| = \frac{2q(q+1)(q-1)}{(2,q-1)} \geqslant p_m \cdot 2q(q-1) > p_m\chi(1)(\chi(1)+1),$$

where $\chi \in \mathrm{Irr}(G)$ is such that $\chi_N = \theta$, where θ is the Steinberg character θ of N of degree q.

We may assume that $q = p$ and p is an odd prime. Using [24, Lemma 4.5], N has an irreducible character θ of degree $p-1$ which is extendible to G. Hence

$$|G| = 2|N| = (p+1)(p-1)p \geqslant p_m\theta(1)(\theta(1)+1),$$

a contradiction.

Suppose that $N \cong \mathrm{A}_2(q)$, $q = p^r$ for some prime p. Using the character tables in the Atlas [8], we may assume that $q \geqslant 11$. Let $|G{:}N| \leqslant 8$. There exists $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) = q(q+1)$. Let $\chi \in \mathrm{Irr}(G)$ be an irreducible constituent of θ^G. Since $\chi(1) \leqslant |G{:}N|\theta(1)$ and $p_m \leqslant q^2+q+1$, we have

$$\frac{|G|}{p_m\chi(1)(\chi(1)+1)} \geqslant \frac{q^3(q^3-1)(q^2-1)}{(3,q-1)\cdot 8 \cdot (q^2+q+1)^2q(q+1)} = \frac{3(q-1)}{24}\cdot\frac{\frac{q^2(q-1)}{3}}{q^2+q+1},$$

since $q \geqslant 11$ and $\frac{q-1}{3}q^2 > 3q^2 > q^2+q+1$.

We may assume that $|G{:}N| \geqslant 9$ and let $\theta \in \mathrm{Irr}(N)$ be the Steinberg character. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$. Now $|G| \geqslant 9|N| \geqslant 3q^3(q^3-1)(q^2-1)$ and

$$\frac{|G|}{p_m\chi(1)(\chi(1)+1)} \geqslant \frac{3q^3(q^3-1)(q^2-1)}{(q^2+q+1)q^3(q^3+1)} > \frac{3(q-1)^2}{q^2-q+1} > \frac{3(q-1)^2}{(q+1)^2} > 1.$$

Suppose that $N \cong {}^2\mathrm{A}_2(q^2)$, where $(3, q+1) = 3$, $q = p^r$ for some prime p. Using the character tables in the Atlas [8], we may assume that $q \geqslant 11$. Let $\theta \in \mathrm{Irr}(N)$ be the Steinberg character with $\chi_N = \theta$ for some $\chi \in \mathrm{Irr}(G)$. Then

$$\frac{|G|}{p_m\chi(1)(\chi(1)+1)} \geqslant \frac{3q^3(q^3+1)(q^2-1)}{(3,q+1)(q^2-q+1)q^3(q^3+1)} \geqslant \frac{3(q^2-1)}{(3,q+1)(q^2-q+1)} > 1.$$

We may assume that $|G{:}N| = 2$. Let $\theta \in \mathrm{Irr}(N)$ be such that $\theta(1) = q(q-1)$ and χ be an irreducible constituent of θ^G. Then $\chi(1) \leqslant 2\theta(1) = 2q(q-1)$. Thus

$$\frac{|G|}{2p_m\chi(1)^2} \geqslant \frac{2|N|}{2\cdot 3\cdot 4p_m\theta(1)^2} = \frac{q(q+1)^2}{12(q-1)} > 1,$$

since $q \geqslant 11$.

Suppose that $N \cong {}^2\mathrm{B}_2(q^2)$ (or $N \cong {}^2\mathrm{G}_2(q^2)$). Since $\mathrm{Out}(N)$ is cyclic, let $\theta \in \mathrm{Irr}(N)$ be such that

$$\theta(1) = \tfrac{1}{\sqrt{2}}q(q^2-1) \text{ (respectively, } \theta(1) = \tfrac{1}{\sqrt{3}}q(q^4-1)).$$

Since $(\theta(1), \mathrm{Out}(N)) = 1$ and θ is invariant in G, θ is extendible to G by [14, Corollary 11.22]. Suppose that $\chi_N = \theta$ for some $\chi \in \mathrm{Irr}(G)$. Then

$$\frac{|G|}{2p_m\chi(1)^2} \geqslant \frac{2q^4(q^4+1)(q^2-1)}{(q^2+\sqrt{2}q+1)q^2(q^2-1)^2} > \frac{2q^2(q^2-1)}{q^2+\sqrt{2}q+1} > 1$$

(and respectively,

$$\frac{|G|}{2p_m\chi(1)^2} \geqslant \frac{2q^6(q^6+1)(q^2-1)}{2(q^4-q^2+1)\frac{1}{3}q^2(q^4-1)^2} > \frac{3q^4}{(q^2+1)^2} > 1).$$

This final contradiction concludes our proof. □

Proof of Theorem C. Suppose G is non-trivial. We prove the result by induction on $|G|$. If N is a non-trivial minimal normal subgroup of G, then G/N satisfies the hypothesis and so G/N is supersolvable. If there exists another minimal normal subgroup M of G, then $G/(N\cap M) \cong G$ is also supersolvable. We may assume that N is the unique minimal normal subgroup of G.

Suppose that $\ker\chi = 1$ for some χ. Then by hypothesis, $|G/\ker\chi| = |G| \leqslant \chi(1)^2+2$. This means that $|G| = \chi(1)^2+2$. Since G is non-abelian and χ is the only one non-linear irreducible character. By [21], G is an extra-special 2-group or a Frobenius group of order $m(m-1)$ with an abelian kernel of order m where m is a power of a prime. In the first case G is supersolvable and in the latter case $G \cong \mathrm{S}_3$ since $|G/G'| = 2$. Hence G is supersolvable.

We may assume that $\ker\chi \neq 1$ (for all non-linear $\chi \in \mathrm{Irr}(G)$). Note that $|G\colon\ker\chi| \leqslant \chi(1)^2+2 \leqslant |G|$ by Lemma 2.1. If $|G\colon\ker\chi| = \chi(1)^2$, then $\chi(1)$ vanishes on $G\setminus\ker\chi$ using [14, Lemma 2.30], a contradiction. Suppose that $|G\colon\ker\chi| = \chi(1)^2+1$. Since $\chi(1) \mid \chi(1)^2+1$, we have another contradiction. We may assume that $|G\colon\ker\chi| = \chi(1)^2+2$. It follows that $\chi(1) = 2$. Hence $\mathrm{cd}(G) = \{1,2\}$. By [14, Theorem 12.5], either G has an abelian normal subgroup of index 2 or G is a direct product of a 2-group and an abelian group. Now G cannot be the latter case because it will have at least two minimal normal subgroups. Note that G has a normal abelian 2-complement by Thompson's theorem [14, Corollary 12.2]. Since N is a unique minimal normal subgroup of G, the abelian 2-complement is a p-group for some p. Moreover $|G\colon\ker\chi| \leqslant 6$, that is, $G/\ker\chi$ is isomorphic to S_3. Hence $p = 3$.

Since $\ker\chi \neq 1$, N cannot be generated by the G-conjugates of any single element of N by Theorem 2.7. Consider $x \in N$ an element of order 3 and y an involution. Consider $\langle x, y\rangle = N_1\langle y\rangle$, where N_1 is a normal 3-subgroup of G. Since N is unique a minimal subgroup of G, $N_1 = N$. This means that N_1 is generated by conjugates of y, a contradiction. That concludes our proof. □

Example 3.1. Consider $G_n = [\mathrm{C}_3 \times \mathrm{C}_3 \times \cdots \times \mathrm{C}_3] \rtimes \mathrm{C}_2$, the semidirect product of the direct product of n copies of C_3 and C_2, where C_2 acts on $[\mathrm{C}_3 \times \mathrm{C}_3 \times \cdots \times \mathrm{C}_3]$ without non-trivial fixed points. For $n \geqslant 2$, G_n has no faithful irreducible characters using Theorem 2.7. Note that $\mathrm{cd}(G_n) = \{1,2\}$. We prove that $|G_n\colon\ker\chi| = 6 = \chi(1)^2+2$ for all non-linear irreducible characters of G_n by induction on $|G_n|$. For G_1, G_2, G_3 and G_4, the result is true since these are the following groups respectively, S_3, SmallGroup(18,4), SmallGroup(54,14), SmallGroup(162,54) using GAP [11] notation. If $\chi \in \mathrm{Irr}(G_n)$ in non-linear, then $\ker\chi \neq 1$ and so χ is a faithful character of $G/\ker\chi \cong G_{n-k}$ for some positive integer k. Since G_1 is the only one with a faithful non-linear character, $G/\ker\chi \cong \mathrm{S}_3$ and the result follows.

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