Dynamics of Fractional Chaotic Systems with Chebyshev Spectral Approximation Method

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Abstract

The dynamical behavior of chaotic processes with a noninteger-order operator is considered in this work. A lot of scientific reports have justified that modeling of physical scenarios via non-integer order derivatives is more reliable and accurate than integer-order cases. Motivated by this fact, the standard time derivatives in the model equations are formulated with the novel Caputo fractional-order operator. The choice of using the Caputo derivative among several existing fractional derivatives has to do with the fact that it gives way for both the initial conditions and boundary conditions to be incorporated in the development of the chaotic model. Numerical approximation of fractional derivatives has been the major challenge of many scholars in different areas of engineering and applied sciences. Hence, we developed a numerical approximation technique, which is based on the Chebyshev spectral method for solving the integer-order and non-integer-order chaotic systems which are largely found in physics, finance, biology, engineering, and other areas of applied sciences. The proposed numerical method used here is easy to implement on a digital computer, and capable of solving higher-order problems without reduction to the system of lower-order ordinary differential equations with limited computational costs. Experimental results are presented for different instances of fractional-order parameters.

Keywords Chaotic dynamics · Chebyshev spectral method · Fractional differential equation · Spatiotemporal oscillations · Stability analysis

Mathematics Subject Classification 34A34 · 35A05 · 35K57 · 65L05 · 65M06 · 93C10

Introduction

Nowadays, the study of nonlinear dynamics of chaotic models has generated a lot of attention due to their usefulness when describing the evolution of more complex phenomena. Chaos theory is successfully applied to formulate many physical phenomena in areas of biology, robotics electrical circuits, lasers, oscillators, memristors, finance, chemical reactions, neural networks, ecology, weather systems, image and sound encryptions, cryptosystems, secure communication devices [3, 11, 18, 24, 42, 50, 53], among several others.

Recently, the idea of fractional calculus which is known to be an extension meaning from classical to fractional differentiation and integration has been known for over three hundred (300) years [44]. Its applications to engineering, physics, and biology remain the most active area of research in recent years. Researchers have shown that many mathematical problems involving interdisciplinary research and other physical scenarios are adequately formulated by using the concept of fractional derivatives. Many nonlinear dynamics are known to give rise to fractional-order phenomena, like electrode-electrolyte polarization, viscoelastic systems, quantitative finance, dielectric polarization, electromagnetic waves, groundwater systems, Geo-hydrology, and dynamics of complex evolution, see [5–7, 22, 25, 37, 44]. In addition, fractional order idea has been used in no small measure to model a number of non-Markovian processes or spatial or temporal cases. For instance, the reaction-diffusion advection models which include, the Fisher and Fokker-Planck equations [37, 38, 52], the Gray-Scott, Burggers and Ginzburg-Landau models [33, 43], and many other real-life dynamics that are classified in [17, 21, 38–40, 44].

In this paper, the general chaotic fractional equations is considered in the form

$$\begin{aligned} &{}^{C}\mathscr{D}_t^{\beta} g(t) := f(g(t), t), \\ &g^{(s)}(t_0) = g_0^{(s)}, \quad s = 0, 1, \ldots, m-1, \end{aligned} \tag{1}$$

where ${}^{C}\mathscr{D}_t^{\beta}$ denote the Caputo fractional derivative of order β for function $f(g,t) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, with dimension $d \geq 1$. The chaotic dynamics with fractional-order models has began to generate a lot of research attentions in recent years due to the fact that such models are less complicated to implement on digital computer [45]. Chaotic systems are highly sensitive to the variation of the initial conditions or perturbation of the key parameters which make them be required for many applications, there is a need to talk about the coupling of two or more dissipative-chaotic models which is being referred to as synchronization. Synchronization of the chaotic model has been used in various fields of engineering, ecology, biology, finance, and physical systems [30, 41].

The aim of this paper is to model some existing chaotic phenomena arising from science, engineering, and finance using the concept of fractional calculus [27, 29, 34–36]. Since the analytical solutions of such nonlinear fractional dynamical systems are almost nonexistent, therefore we require to seek an appropriate and reliable numerical method to study the chaotic systems. In this paper, we introduce the Chebyshev-spectral method. Spectral methods are known to have an advantage over other classical approaches like finite difference approximation and finite element family of schemes [8, 9, 14, 33, 43]. Some of the proposed methods are accurate but have high computational costs. Thus, spectral algorithms have the edge of being fast-converging methods, because the inherent truncation error decays so fast as the global smooth solution allows [31]. The definite integrals in spectral methods are computed via the quadrature method, see [15, 16, 47–49] for details on spectral methods of approximation.

The rest of this work is categorized into sections as follows. Some useful definitions and preliminaries of fractional calculus in relation to the Caputo fractional operator are given in Sect. 2. Formulation of an approximate method using the Chebyshev spectral methods is given in Sect. 3. In Sect. 4, some chaotic dynamics arising from engineering, finance, and biology are introduced and analyzed. We present the numerical results which revealed the behavior of each dynamics with response to fractional order parameter. Finally, we conclude with the last section.

Some Useful Properties Regarding the Caputo Fractional Operator

There are different types of definitions given to fractional derivatives in the literature, the most notable ones are the Riemann-Liouville (R-L), and the Caputo fractional derivatives which have been used to model a lot of applications in area of engineering and applied sciences. The Riesz fractional operator is a linear representation of both left- and right Riemann–Liouville fractional derivatives. A close relationship/connection exists between the Caputo and Riemann–Liouville fractional operators [44]. The R-L operator can be written as the Caputo derivative under certain regularity and assumptions of the function [44]. In both fractional ordinary- and partial-differential equations, the fractional-in-time derivatives are often represented by the Caputo derivatives. The reason is due to the fact that the R-L technique requires initial conditions that contain its limit values at the origin when $t = 0$, which till date gives no physical meaning interpretation. On the other hand, for the fractional-in-time Caputo derivative, the initial condition takes the same form as that for classical order case, such as the initial values of standard derivatives of functions at the origin of time $t = 0$. Readers are referred to [44] for details. A brief account of some definitions of the Caputo derivative and its general properties are highlighted here.

Let $\beta > 0, t > a, \beta, a, t \in \mathbb{R}$. The Caputo derivative of order α is defined by

$$ {}_a^C\mathscr{D}_t^{\beta} g(t) = \frac{1}{\Gamma(p-\beta)} \int_a^t \frac{g^p(\tau)}{(t-\tau)^{\beta+1-p}} d\tau, \quad p-1 < \beta < p \in \mathbb{N}. $$

As a remark, the Caputo derivative of order $0 < \beta < 1$ for $g = g(t)$ is defined by

$$ {}_a^C\mathscr{D}_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{1}{(t-\tau)^{\beta}} \frac{dg(\tau)}{d\tau} d\tau. $$

The Caputo operator based on the above definition has the following properties.

(i) (Linearity [10, 12]). Let $u(t), v(t) : [a, b] \in \mathbb{R}$ so that ${}_a^C\mathscr{D}_t^{\beta} u(t)$ and ${}_a^C\mathscr{D}_t^{\beta} v(t)$ exist, and assume $\xi_1, \xi_2 \in \mathbb{R}$. Then ${}_a^C\mathscr{D}_t^{\beta} (\xi_1 u(t) + \xi_2(t))$ exists almost every everywhere, and

$$ {}_a^C\mathscr{D}_t^{\beta} (\xi_1 u(t) + \xi_2(t)) = \xi_1 {}_a^C\mathscr{D}_t^{\beta} u(t) + \xi_2 {}_a^C\mathscr{D}_t^{\beta} v(t). $$

(ii) (The derivative of a constant in terms of the Caputo operator [44]). That is, $u(t) = c$ in the sense of Caputo is zero. That is,

$$ {}_a^C\mathscr{D}_t^{\beta} u(t) = 0. $$

Based on the general idea of fractional differential equations, g^* being a constant which stands for an equilibrium state of the Caputo fractional dynamic model (1), provided $f(g^*, t) = 0$.

By following Matignon stability results [28], an equilibrium state g^* of the Caputo fractional chaotic dynamical system (1) is locally-asymptotically-stable if all the eigenvalues of arising from the Jacobian or community matrix of (1), which one evaluates at g^*, satisfies the condition

$$|\arg(\lambda)| > \beta\pi/2.$$

It should also be noted that when $\beta \in (0, 1)$, system (1) has the same steady state as the standard case

$$\frac{dg(t)}{dt} = f(g(t), t). \tag{2}$$

In what follows, we give a known results as reported in [13] based on the Lyapunov direct technique for the Caputo fractional-order equations.

Theorem 2.1 (Asymptotic stability result [13]). *Let g^* be an equilibrium point for chaotic fractional-order differential equation described in (1) and $\Omega \subset \mathbb{R}^m$ be the domain that contains g^*. Also, assume $M : [0, \infty) \times \Omega \to \mathbb{R}]$ be continuously differentiable function, such that*

$$F_1(g) \le M(g(t), t) \le F_2(g),$$

and

$${}_a^C\mathscr{D} M(g(t), t) \le -F_3(g),$$

for all $\beta \in (0, 1)$ and $\forall\, g \in \Omega$, where $F_1(g)$, $F_2(g)$ and $F_3(g)$ are positive definite functions on Ω. Then the point g^* of (1) is uniformly-asymptotically-stable.

Again, in the following lemma we present the result of quadratic Lyapunov functions for some fractional-order systems by Aguila-Camacho et al. [2, 51].

Lemma 2.2 *Let $g(t) \in \mathbb{R}$ be continuously differentiable. Then, for $t \ge t_0$*

$$\frac{1}{2}\, {}_{t_0}^C\mathscr{D}_t^\beta g^2(t) \le {}_{t_0}^C\mathscr{D}_t^\beta g^{(t)},$$

for all $\beta \in (0, 1)$. This shows that

$${}_{t_0}^C\mathscr{D}_t^\beta \left[g(t) - g^* - g^* \ln \frac{g(t)}{g^*} \right] \le \left(1 - \frac{g^*}{g(t)}\right) {}_{t_0}^C\mathscr{D}_t^\beta g(t), \quad g^* \in \mathbb{R}, \quad \forall \beta \in (0, 1).$$

Next, we give a remark by considering the quadratic Lyapunov function

$$M(g_1, g_2, \ldots, g_m) = \sum_{j=1}^{m} \frac{\xi_j}{2} \left(g_j(t) - g_j^*\right)^2, \quad j = 1, 2, \ldots, m,$$

where $\xi_j > 0$. By using above lemma, we obtain

$$ {}_{t_0}^{C}\mathscr{D}_t^{\beta} M(g_1, g_2, \ldots, g_m) \leq \sum_{j=1}^{m} \xi \, {}_{t_0}^{C}\mathscr{D}_t^{\beta} \left(g_j(t) - g_j^* \right), \quad g_i(t),\ g_i^* \in \mathbb{R}^+, \quad \beta \in (0, 1). $$

The Chebyshev polynomial defined on $[-1, 1]$ is determined by [1, 20, 23, 46]

$$ T_{q+1}(x) = 2xT_q(x) - T_{q-1}(x), T_0(x) = 1, T_1(x) = x, \text{for } q = 1, 2, \ldots. $$

The analytic form of $T_q(x)$ is given by

$$ T_q(x) = q \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} (-1)^k 2^{q-2k-1} \frac{(q-k-1)!}{k!(q-2k)!} x^{q-2k}, \ q = 2, 3, \ldots, \tag{3} $$

where $\lfloor \frac{q}{2} \rfloor$ is the integer case of $q/2$. The orthogonality property is defined as

$$ \int_{-1}^{1} \frac{T_k(x)T_s(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & \text{for } k = s = 0, \\ \frac{\pi}{2} & \text{for } k = s \neq 0, \\ 0 & \text{for } k \neq s, \end{cases} \tag{4} $$

In order to apply the Chebyshev polynomial, we introduce $x = 2t - 1$ and follow [20] to define the shifted Chebyshev polynomial in the form

$$ T^*(t) = T_q(2t-1) = T_{2q}(\sqrt{t}) $$

which has the analytic form

$$ T_q^*(t) = q \sum_{r=0}^{q} (-1)^{q-r} \frac{(q+r-1)!}{(2r)!(q-r)!} t^r, \quad q = 2, 3, \ldots. \tag{5} $$

Approximate Method Based on Chebyshev Polynomials

It should be recalled from elementary mathematics that the Chebyshev polynomial (denoted as $T_m(x)$) of the first kind over $x \in [-1, 1]$ is expressed as

$$ T_m(x) = \cos m\theta, \quad x = \cos\theta, $$

which implies that $T_m(x) = \cos(m \cos^{-1} x)$. The corresponding Chebyshev polynomial of second kind is expressed by the relation, say $U_m(x) = \sin(m \cos^{-1} x)$. By using the shift mapping $\omega : x \to \omega(x) = \frac{2x}{b-a} - \frac{b+a}{b-a}$, one can extend the definition of the Chebyshev polynomial to any interval of interest $[a, b]$, where a, b are constants. From the knowledge of trigonometric identities, we have the following useful relation

$$ \cos(m\theta) + \cos(m-2)\theta = 2\cos\theta\cos(m-1)\theta, \tag{6} $$

so that

$$ T_0(x) = 1, T_1(x) = x, \quad T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x), \ m = 2, 3, \ldots \tag{7} $$

which further translates to matrix

$$
\begin{pmatrix}
1 & & & & \\
-2x & 1 & & & \\
1 & -2x & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & -2x & 1
\end{pmatrix}
\begin{pmatrix}
T_0(x) \\ T_1(x) \\ T_2(x) \\ \vdots \\ T_m(x)
\end{pmatrix}
=
\begin{pmatrix}
1 \\ -x \\ 0 \\ \vdots \\ 0
\end{pmatrix}. \tag{8}
$$

and the zeros of T_m are given in terms of the points

$$x_j = -\cos\frac{(j-\frac{1}{2})}{m}, \quad j = 1, 2, 3, \ldots, m.$$

The set of Chebyshev points of the first kind, which are also referred to as the collocation points is denoted by $\{x_j\}_j$. So, for any x, the set $\{T_0(x), T_1(x), \ldots\}$ is an orthogonal basis in accordance to the inner-product given by

$$\langle u, v\rangle = \int_{-1}^{1} \frac{u(x)v(x)}{\sqrt{1-x^2}}dx,$$

valid for any u and v defined in interval $[-1, 1]$. This shows that for any given polynomial with degree $m > 0$, $\exists$ a set of α_j for $j = 1, 2, \ldots, m$ such that

$$P_m(x) = \sum_{j=0}^{m} \alpha_j T_j.$$

For the fact that a set of Chebyshev polynomials is complete and polynomials are dense in $C([-1, 1])$, we provide the following result.

Theorem 3.1 *Let g be Lipschitz and continuous in $[-1, 1]$. Then g has a unique representation*

$$g(x) = \frac{\alpha_0}{2} + \sum_{j=1}^{\infty} \alpha_j T_j(x),$$

where $T_j(x)$ are the usual Chebyshev polynomials, and

$$\alpha_j = \frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} g(x)T_j(x)dx, \quad j = 0, 1, 2, \ldots,$$

which converges absolutely and uniformly.

A Chebyshev approximation of a function g of order $m > 0$ on continuous interval $[-1, 1]$ is given by

$$
\begin{aligned}
g_m(x) &= \sum_{j=0}^{m} g_j T_j(x) \\
&= \bar{\alpha} \cdot T(x)
\end{aligned} \tag{9}
$$

where $\bar{\alpha} = [\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m]$ denotes the coefficients vector associated with g_m. For brevity, we usually write $g(x)$ to represent g_m to stand for the Chebyshev approximation of g of order m at x.

On collocation point, one can write

$$\bar{f}(x) = T(x) \cdot \bar{\alpha}, \tag{10}$$

$$(f(x_0), f(x_1), \ldots, f(x_m)) = \left[\sum_{j=0}^{m} \alpha_j T_j(x_0), \sum_{j=0}^{m} \alpha_j T_j(x_1), \cdots, \sum_{j=0}^{m} \alpha_j T_j(x_m) \right], \tag{11}$$

where

$$T = \begin{pmatrix} T_0(x_0) & T_1(x_0) & \ldots & T_m(x_0) \\ T_0(x_1) & T_1(x_1) & \ldots & T_m(x_1) \\ T_0(x_2) & T_1(x_2) & \ldots & T_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_m) & T_1(x_m) & \ldots & T_m(x_m) \end{pmatrix}.$$

It should be noted that g is represented by a vector $\bar{f}$ on the grids $\bar{x} = [x_0, x_1, x_2, \ldots, x_m]$,that is $\bar{f} = [g(x_0), g(x_1), g(x_2), \ldots, g(x_m)]$, which implies that $\bar{f}$ is the physical state of g. Since $\bar{f} = T\bar{\alpha}$, it means that $\bar{\alpha} = T^{-1}\bar{f}$. The nature of matrix T is often sparse, we apply the Fast Fourier transform in matlab to obtain T^{-1}.

Next, we define $g(t)$ in terms of the shifted-chebyshev polynomials in space [0, 1] as

$$g(t) = \sum_{j=0}^{\infty} \alpha_j T_j^*(t), \tag{12}$$

where α_j is the coefficient written as

$$\alpha_j = \frac{2}{\pi \hbar_j} \int_0^1 \frac{1}{\sqrt{t - t^2}} g(t) T_j^*(t) dt, \;\; \hbar_0 = 2, \;\; \hbar_j = 1, \;\; j = 1, 2, \ldots. \tag{13}$$

In practice, it is habitual to consider just $(x+1)$ terms of the shifted-Chebyshev polynomials [32]. In such a way that

$$g_x(t) = \sum_{j=0}^{x} \alpha_j T_j^*(t). \tag{14}$$

The approximation of the Caputo operator with order $0 < \beta < 1$ via the Chebyshev polynomials takes the form

$$\mathcal{D}_t^{\beta}(g_x(t)) = \sum_{j=\lceil \beta \rceil}^{x} \sum_{r=\lceil \beta \rceil}^{j} \alpha_j f_{j,r}^{(\beta)} t^{r-\beta}, \tag{15}$$

where

$$f_{j,r}^{(\beta)} = (-1)^{j-r} \frac{2^{2r} j (j+r-1)! \Gamma(r+1)}{(j-r)!(2r)! \Gamma(r+1-\beta)}.$$

The expression for the Caputo derivative via the shifted Chebyshev polynomials is written as

$$ {}^{C}\mathcal{D}_0^{\beta}[T_j^*(t)] = \sum_{r=\lceil\beta\rceil}^{j} \sum_{s=0}^{r-\lceil\beta\rceil} \chi_{j,r,s} T_s^*(t), \tag{16} $$

where

$$ \chi_{j,r,s} = \frac{(-1)^{j-r} 2j(j+r-1)!\Gamma(r-\beta+\frac{1}{2})}{\hbar_s \Gamma(r+\frac{1}{2})(j-r)!\Gamma(r-\beta-s+1)\Gamma(r+s-\beta+1)}, \text{ for } s = 0, 1, 2, \ldots. $$

The error $|E_r(x)|$ is computed as $|\mathcal{D}^\beta g(t) - \mathcal{D}^\beta g_x(t)|$ when $\mathcal{D}^\beta g(t)$ is approximated by $\mathcal{D}^\beta g_x(t)$, is bound by

$$ |E_r(x)| \leq \left| \sum_{j=x+1}^{\infty} \alpha_j \left(\sum_{r=\lceil\beta\rceil}^{j} \sum_{s=0}^{r-\lceil\beta\rceil} \chi_{j,r,s} \right) \right|. $$

readers are referred to [46] for details.

Since we are considering a three species chaotic dynamics with components x_1, x_2, x_3, we require to give a solution guideline for the Caputo fractional-order chaotic models by approximating these components as

$$ x_1^q(t) = \sum_{j=0}^{q} x_j T_j^*(t), \quad x_2^q(t) = \sum_{j=0}^{q} y_j T_j^*(t), \quad x_3^q(t) = \sum_{j=0}^{q} z_j T_j^*(t). \tag{17} $$

From (21) and (15) we have

$$ \begin{aligned} \sum_{j=\lceil\beta\rceil}^{q} \sum_{r=\lceil\beta\rceil}^{j} x_j f_{i,r}^{(\beta)} t^{r-\beta} &= a_1 \left(\sum_{j=0}^{q} x_j T_j^*(t) - \sum_{j=0}^{q} y_j T_j^*(t) \right), \\ \sum_{j=\lceil\beta\rceil}^{q} \sum_{r=\lceil\beta\rceil}^{j} y_j f_{i,r}^{(\beta)} t^{r-\beta} &= -4a_1 \sum_{j=0}^{q} y_j T_j^*(t) + \left(\sum_{j=0}^{q} x_j T_j^*(t) \right) \left(\sum_{j=0}^{q} z_j T_j^*(t) \right) \\ &\quad + a_2 \left(\sum_{j=0}^{q} z_j T_j^*(t) \right)^3 \\ \sum_{j=\lceil\beta\rceil}^{q} \sum_{r=\lceil\beta\rceil}^{j} z_j f_{i,r}^{(\beta)} t^{r-\beta} &= -a_1 a_4 \sum_{j=0}^{q} z_j T_j^*(t) + \left(\sum_{j=0}^{q} x_j T_j^*(t) \right)^3 \left(\sum_{j=0}^{q} y_j T_j^*(t) \right) \\ &\quad + a_3 \left(\sum_{j=0}^{q} z_j T_j^*(t) \right)^2. \end{aligned} \tag{18} $$

By collocating (18) at $(q+1-\lceil\beta\rceil)$ points $t_\eta(\eta=0,1,2,\dots,m+1-\lceil\beta\rceil)$ we obtain

$$\sum_{j=\lceil\beta\rceil}^{q}\sum_{r=\lceil\beta\rceil}^{j} x_j f_{i,r}^{(\beta)} t_\eta^{r-\beta} = a_1\left(\sum_{j=0}^{q} x_j T_j^*(t_\eta) - \sum_{j=0}^{q} y_j T_j^*(t_\eta)\right),$$

$$\sum_{j=\lceil\beta\rceil}^{q}\sum_{r=\lceil\beta\rceil}^{j} y_j f_{i,r}^{(\beta)} t_\eta^{r-\beta} = -4a_1\sum_{j=0}^{q} y_j T_j^*(t_\eta) + \left(\sum_{j=0}^{q} x_j T_j^*(t_\eta)\right)\left(\sum_{j=0}^{q} z_j T_j^*(t_\eta)\right)$$
$$+a_2\left(\sum_{j=0}^{n} z_j T_j^*(t_\eta)\right)^3$$

$$\sum_{j=\lceil\beta\rceil}^{q}\sum_{r=\lceil\beta\rceil}^{j} z_j f_{i,r}^{(\beta)} t_\eta^{r-\beta} = -a_1 a_4\sum_{j=0}^{q} z_j T_j^*(t_\eta) + \left(\sum_{j=0}^{q} x_j T_j^*(t_\eta)\right)^3\left(\sum_{j=0}^{q} y_j T_j^*(t_\eta)\right)$$
$$+a_3\left(\sum_{j=0}^{q} z_j T_j^*(t_\eta)\right)^2. \tag{19}$$

To get accurate collocation points. We apply the roots of the shifted-Chebyshev polynomials $T^*_{q+1-\lceil\beta\rceil}(t)$. By putting (17) into initial conditions

$$x_1(t=0)=x_1^0,\ x_2(t=0)=x_2^0,\ \text{and}\ x_3(t=0)=x_3^0,$$

one gets

$$\sum_{j=0}^{q}(-1)^j x_j = x_1^0,\ \sum_{j=0}^{q}(-1)^j y_j = x_2^0,\ \sum_{j=0}^{q}(-1)^j z_j = x_3^0. \tag{20}$$

Obviously, the above (19) and (20) lead to algebraic equations which can be advanced with any time-solver. In this paper, we utilize the Matlab ode45 code for the time integration.

Fractional-Order Chaotic Systems

Three notable examples of chaotic fractional models which are largely encounter in engineering, science and finance, and which are of current and recurring interests are presented here. In each of the cases, we require to examine the linear-stability analysis, give the dynamics of Lyapunov exponents of the chaotic system, and finally carry out the simulation experiments for some instances of fractional order.

Chaotic Problem in Engineering

The dynamic of fractional order chaotic model which has a lot of applications in engineering has been considered by many authors [3, 4] in the form of circuit synchronizations. In the present case, we consider a three component chaotic model described by the Caputo fractional derivative is given as

$$
\begin{aligned}
{}^{C}\mathscr{D}_t^{\beta} x_1(t) &= f(x_1, x_2, x_3) = \gamma_1(x_1(t) - x_2(t)), \\
{}^{C}\mathscr{D}_t^{\beta} x_2(t) &= g(x_1, x_2, x_3) = -4\gamma_1 x_2(t) + x_1(t)x_3(t) + \gamma_2 x_1^3(t), \\
{}^{C}\mathscr{D}_t^{\beta} x_3(t) &= h(x_1, x_2, x_3) = -\gamma_1\gamma_4 x_3(t) + x_1^3(t)x_2(t) + \gamma_3 x_3^2(t),
\end{aligned} \tag{21}
$$

where x_1, x_2 and x_3 are the variables, $\gamma_i > 0$ for $i = 1(1)4$ are parameters.

Before examining system (21) for linear stability analysis, we require to show that the Caputo fractional order dynamics satisfy the existence and uniqueness of solutions. This can be achieved by adopting the fundamental theorem of calculus to (21) as

$$
\begin{aligned}
x_1(t) - x_1(0) &= \frac{1}{\Gamma(\beta)} \int_0^t (\gamma_1(x_1(\tau) - x_2(\tau)))(t-\tau)^{\beta-1} d\tau, \\
x_2(t) - x_2(0) &= \frac{1}{\Gamma(\beta)} \int_0^t \left(-4\gamma_1 x_2(\tau) + x_1(\tau)x_3(\tau) + \gamma_2 x_1^3(\tau)\right)(t-\tau)^{\beta-1} d\tau, \\
x_3(t) - x_3(0) &= \frac{1}{\Gamma(\beta)} \int_0^t \left(-\gamma_1\gamma_4 x_3(\tau) + x_1^3(\tau)x_2(\tau) + \gamma_3 x_3^2(\tau)\right)(t-\tau)^{\beta-1} d\tau.
\end{aligned} \tag{22}
$$

Let $\mathscr{C}_{a,b}$ be compact set, so that

$$\mathscr{C}_{a,b} = \mathscr{A}_a(t_0) \times \mathscr{B}_b(\xi)$$

where

$$\mathscr{C} = \min\{x_1(0), x_2(0), x_3(0)\}$$

and

$$\mathscr{A}_a(t_0) = [t_0 - a, t_0 + a], \quad \mathscr{B}_0(\mathscr{C}) = [\mathscr{C} - b, \mathscr{C} + b].$$

From (21) we know that

$$
\begin{aligned}
f(x_1, x_2, x_3, t) &= \gamma_1(x_1(t) - x_2(t)), \\
g(x_1, x_2, x_3, t) &= -4\gamma_1 x_2(t) + x_1(t)x_3(t) + \gamma_2 x_1^3(t), \\
h(x_1, x_2, x_3, t) &= -\gamma_1\gamma_4 x_3(t) + x_1^3(t)x_2(t) + \gamma_3 x_3^2(t).
\end{aligned} \tag{23}
$$

We let

$$\mathcal{M} = \max_{\mathscr{C}_{a,b}} \left\{ \sup_{\mathscr{C}_{a,b}} \|f_1\|, \ \sup_{\mathscr{C}_{a,b}} \|f_2\|, \ \sup_{\mathscr{C}_{a,b}} \|f_3\| \right\}.$$

and apply the infinite norm, to have

$$\|E\|_\infty = \sup_{t \in \mathscr{A}_a} ||E(t)|.$$

Next, we create a function,

$$\Gamma : \mathscr{C}_{a,b} \to \mathscr{C}_{a,b}$$

in such a way that

$$\Gamma\mathcal{L}(t) = \mathcal{L}_0 + \frac{1}{\Gamma(\beta)} \int_0^t \mathcal{F}(x_1, x_2, x_3, t)(t-\tau)^{\beta-1} d\tau \tag{24}$$

where

$$\mathcal{L}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

and

$$F(u, v, w, t) = \begin{pmatrix} f(x_1, x_2, x_3, t) \\ g(x_1, x_2, x_3, t) \\ h(x_1, x_2, x_3, t) \end{pmatrix}$$

To show that our operator is well-defined, we evaluate the condition

$$\|\Gamma \mathcal{L}(t) - \mathcal{L}_0\|_\infty < \begin{pmatrix} b \\ b \\ b \end{pmatrix},$$

where

$$\|\Gamma_1 x_1(t) - x_1(0)\|_\infty < b,$$
$$\|\Gamma_2 x_2(t) - x_2(0)\|_\infty < b,$$
$$\|\Gamma_3 x_3(t) - x_3(0)\|_\infty < b.$$

By examining first the variable x_1, we get

$$\begin{aligned} \|\Gamma_1 x_1(t) - x_1(0)\|_\infty &= \left\| \frac{1}{\Gamma(\beta)} \int_0^t f(x_1, x_2, x_3, \tau)(t-\tau)^{\beta-1} d\tau \right\|_\infty \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t \|f(x_1, x_2, x_3, \tau)\|_\infty (t-\tau)^{\beta-1} d\tau \\ &\leq \frac{\mathcal{M}}{\Gamma(\beta)} \int_0^t (t-\tau) d\tau \\ &\leq \frac{\mathcal{M} a^\beta}{\Gamma(\beta+1)} < b \end{aligned} \tag{25}$$

where

$$a < \left(\frac{b\Gamma(\beta+1)}{\mathcal{M}} \right)^{1/\beta}, \tag{26}$$

with similar expression for variables x_2 and x_3.

Therefor, we say

$$\|\Gamma \mathcal{L}(t) - \mathcal{L}_0\|_\infty \leq \frac{\mathcal{M} a^\beta}{\Gamma(\beta+1)},$$

Γ is well-defined provided the condition (26) holds.

Secondly, we require to verify that the Caputo fractional system (21) satisfy a Lipshitz condition, that is,

$$\|\Gamma\mathcal{L}_1 - \Gamma\mathcal{L}_2\|_\infty < \phi\,\|\mathcal{L}_1 - \mathcal{L}_2\|$$

$$\begin{aligned}
\left\|\Gamma_1 x_1^a - \Gamma_1 x_1^b\right\|_\infty &= \left\|\frac{1}{\Gamma(\beta)}\int_0^t f_1(x_1^a, x_2, x_3, \tau)(t-\tau)^{\beta-1}d\tau \right.\\
&\quad \left. -\frac{1}{\Gamma(\beta)}\int_0^t f_1(x_1^b, x_2, x_3, \tau)(t-\tau)^{\beta-1}d\tau\right\|_\infty \qquad (27)\\
&= \frac{1}{\Gamma(\beta)}\left\|\int_0^t \left(f(x_1^a, x_2, x_3, \tau) - f(x_1^b, x_2, x_3, \tau)\right)(t-\tau)^{\beta-1}d\tau\right\|_\infty\\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t \left\|\left(f(x_1^a, x_1, x_2, \tau) - f(x_1^b, x_2, x_3, \tau)\right)\right\|_\infty (t-\tau)^{\beta-1}d\tau\\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t \left\|\left(\epsilon x - \epsilon x_1^a - \epsilon x + \epsilon x_1^b\right)\right\|_\infty (t-\xi)^{\beta-1}d\tau\\
&\leq \frac{|\epsilon|}{\Gamma(\beta)}\int_0^t \left\|x_1^a - x_1^b\right\|_\infty (t-\tau)^{\beta-1}d\tau\\
&\leq \frac{|\epsilon|}{\Gamma(\beta)}\left\|x_1^a - x_1^b\right\|_\infty \cdot \frac{a^\beta}{\beta}\\
&\leq \frac{|\epsilon|\,\|x_1^a - x_1^b\|_\infty \cdot a^\beta}{\Gamma(\beta+1)} \leq \Phi_1\left\|x_1^a - x_1^b\right\|_\infty \qquad (28)
\end{aligned}$$

where

$$\Phi_1 = \frac{|\epsilon| a^\beta}{\Gamma(\beta+1)}.$$

By following a similar process for the remaining components, we have

$$\begin{aligned}
\left\|\Gamma_2 x_2^a - \Gamma_2 x_2^b\right\|_\infty &= \left\|\frac{1}{\Gamma(\beta)}\int_0^t g(x_1, x_2^a, x_3, \tau)(t-\tau)^{\beta-1}d\tau \right.\\
&\quad \left. -\frac{1}{\Gamma(\beta)}\int_0^t g(x_1, x_2^b, x_3, \tau)(t-\tau)^{\beta-1}d\tau\right\|_\infty\\
&\leq \Phi_2\|x_2^a - x_2^b\|_\infty \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\Gamma_3 x_3^a - \Gamma_3 x_3^b\right\|_\infty &= \left\|\frac{1}{\Gamma(\beta)}\int_0^t h(x_1, x_2, x_3^a, \tau)(t-\tau)^{\beta-1}d\tau \right.\\
&\quad \left. -\frac{1}{\Gamma(\beta)}\int_0^t h(x_1, x_2, x_3^b, \tau)(t-\tau)^{\beta-1}d\tau\right\|_\infty\\
&\leq \frac{\left(|\tau|\|x_3^b - x_3^a\|\right)a^\beta}{\Gamma(\beta+1)}\\
&\leq \Phi_3\|x_3^b - x_3^a\| \qquad (30)
\end{aligned}$$

where

$$\Phi_2 = \frac{(|\kappa| + |q| \|x_1(t)\|_\infty)\, a^\beta}{\Gamma(\beta+1)}, \quad \Phi_3 = \frac{|p| a^\beta}{\Gamma(\beta+1)}$$

So, Γ is a contraction provided

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} < \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, \tag{31}$$

for

$$a < \left(\frac{\Gamma(\beta+1)}{b} \right)^{1/\beta}, \quad a < \left(\frac{\Gamma(\beta+1)}{|p|} \right)^{1/\beta}, \quad a < \left(\frac{\Gamma(\beta+1)}{|\kappa| + |q| \|x_1(t)\|_\infty} \right)^{1/\beta}.$$

So to obtain a contraction

$$a < \min \left\{ a < \left(\frac{\Gamma(\beta+1)}{b} \right)^{1/\beta}, \quad a < \left(\frac{\Gamma(\beta+1)}{|p|} \right)^{1/\beta}, \quad a < \left(\frac{\Gamma(\beta+1)}{|\kappa| + |q| \|x_1(t)\|_\infty} \right)^{1/\beta} \right\}.$$

With this condition, Γ has a unique solution.

To examine the linear stability of (21), we let ${}^{C}\mathscr{D}_t^\beta x_i = 0, i = 1, 2, 3$ and the local kinetics $f(x_1, x_2, x_3) = 0$, $g(x_1, x_2, x_3) = 0$ and $h(x_1, x_2, x_3) = 0$. So that,

$$\begin{aligned} &\gamma_1(x_1(t) - x_2(t)) = 0, \\ &-4\gamma_1 x_2(t) + x_1(t) x_3(t) + \gamma_2 x_1^3(t) = 0, \\ &-\gamma_1\gamma_4 x_3(t) + x_1^3(t) x_2(t) + \gamma_3 x_3^2(t) = 0. \end{aligned} \tag{32}$$

With this in place, we can easily verify that model (21) has four equilibrium states, denoted her as $E^0 = (0, 0, 0)$ which correspond to extinction of chemical or biological species. The second state $E^1 = (0, 0, \frac{\gamma_1\gamma_4}{\gamma_3})$ has both x_1 and x_2 being washed out with just component x_3 exists. The last two equilibrium is nontrivial, its more feasible since all the three species exist. These points are

$$E_a^* = \left(\sqrt{\frac{4\gamma_1 - x_3^*}{\gamma_2}}, \quad \sqrt{\frac{4\gamma_1 - x_3^*}{\gamma_2}}, \quad \frac{\gamma_1(\gamma_4\gamma_2^2 + 8) + \gamma_1\gamma_2\sqrt{\gamma_4^2\gamma_2 2 + 16\gamma_4 - 64\gamma_3}}{2\gamma_3\gamma_2^2 + 2} \right)$$

and

$$E_b^* = \left(-\sqrt{\frac{4\gamma_1 - x_3^*}{\gamma_2}}, \quad -\sqrt{\frac{4\gamma_1 - x_3^*}{\gamma_2}}, \quad \frac{\gamma_1(\gamma_4\gamma_2^2 + 8) + \gamma_1\gamma_2\sqrt{\gamma_4^2\gamma_2 2 + 16\gamma_4 - 64\gamma_3}}{2\gamma_3\gamma_2^2 + 2} \right)$$

where

$$x_3^* = \frac{\gamma_1(\gamma_4\gamma_2^2 + 8) + \gamma_1\gamma_2\sqrt{\gamma_4^2\gamma_2 2 + 16\gamma_4 - 64\gamma_3}}{2(\gamma_3\gamma_2^2 + 1)}.$$

The corresponding Jacobian matrix of (21) is

$$A_{(x_1,x_2,x_3)} = \begin{pmatrix} \gamma_1 & -\gamma_1 & 0 \\ 3\gamma_2 x_1^2 + x_3 & -4\gamma_1 & x_1 \\ 3x_1^2 x_2 & x_1^3 & 2\gamma_3 x_3 - \gamma_1\gamma_4 \end{pmatrix}. \tag{33}$$

With $\det(\lambda I - A(0,0,0)) = 0$, we get the characteristic equation

$$\lambda^3 + \zeta_1\lambda^2 + \zeta_2\lambda - \zeta_3, \tag{34}$$

where $\zeta_i, i = 1,2,3 = (1.80, 1.61, 35)$, respectively. On simplification, we have to be $\lambda_1 = 1.80, \lambda_2 = -2.7$ and $\lambda_3 = -7.2$. Obviously, one of the three real roots is nonnegative, having opposite sign, then the equilibrium state E^0 is saddle and unstable. Applying similar process, the corresponding eigenvalues for point E^1 are $\lambda_1 = 6.77, \lambda_2 = -12.17$ and $\lambda_3 = 2.70$, respectively. This implies that E^1 is saddle and unstable. For nontrivial states E_a^* and E_b^* , we have the corresponding eigenvalues as $(\lambda_1 = 0.83 - 3.83j, \lambda_2 = 0.83 + 3.83j, \lambda_3 = -10.75)$ and $(\lambda_1 = 0.83 - 3.83j, \lambda_2 = 0.83 + 3.83j, \lambda_3 = -10.75)$, respectively. The presence of opposite signs in the eigenvalues shows that both interior equilibrium states are unstable. At $t = 1000$, we compute the Lyapunov exponent dynamics to obtain $(-2.697921, -2.699263, -8.101339)$. Since the sum of these exponents is negative, the possibility of obtaining chaotic attractors is evident.

When $\gamma_1 = 1.8, \gamma_2 = 0.12, \gamma_3 = -0.07$ and $\gamma_4 = 1.5$ with initial conditions $(x_1, x_2, x_3) = (2.6, 1.8, 2.5)$, we obtain chaotic attractor for system (21). The simulation results reported in Figs. 1 and 2 correspond to $\beta = 0.54$ and $\beta = 0.99$, respectively. All simulations run for time $t = 1000$.

Chaotic Problem in Finance

The chaotic finance model is a three parameter dynamics which is govern by nonlinear fractional differential equations [26, 31, 42]

$$\begin{aligned} {}^C\mathscr{D}_t^\beta x_1(t) &= f(x_1,x_2,x_3) = (x_2(t) - \phi)x_1(t), \\ {}^C\mathscr{D}_t^\beta x_2(t) &= g(x_1,x_2,x_3) = 1 - \varphi x_2(t) - x_1^2(t) \\ {}^C\mathscr{D}_t^\beta x_3(t) &= h(x_1,x_2,x_3) = -x_1(t) - \psi x_3(t) \end{aligned} \tag{35}$$

where ${}^C\mathscr{D}_t^\beta$ is the Caputo fractional derivative of order β satisfying $0 < \beta \leq 1$, variables $x_1(t), x_2(t)$ and $x_3(t)$ represents the interest rate, investment demand and price index of stock,respectively. The parameter ϕ stands for the savings, φ is the per-investment cost, and ψ denotes the elasticity demand. Following the procedure for obtaining linear stability analysis, model (35) has three steady states

$$E^0 = (0, \frac{1}{\varphi}, 0),$$

$$E^1 = \left(\sqrt{1-\varphi\phi-\frac{\varphi}{\psi}},\quad \phi+\frac{1}{\psi},\quad -\frac{1}{\psi}\sqrt{1-\varphi\phi-\frac{\varphi}{\psi}}\right),$$
$$E^2 = \left(-\sqrt{1-\varphi\phi-\frac{\varphi}{\psi}},\quad \phi+\frac{1}{\psi},\quad \frac{1}{\psi}\sqrt{1-\varphi\phi-\frac{\varphi}{\psi}}\right).$$

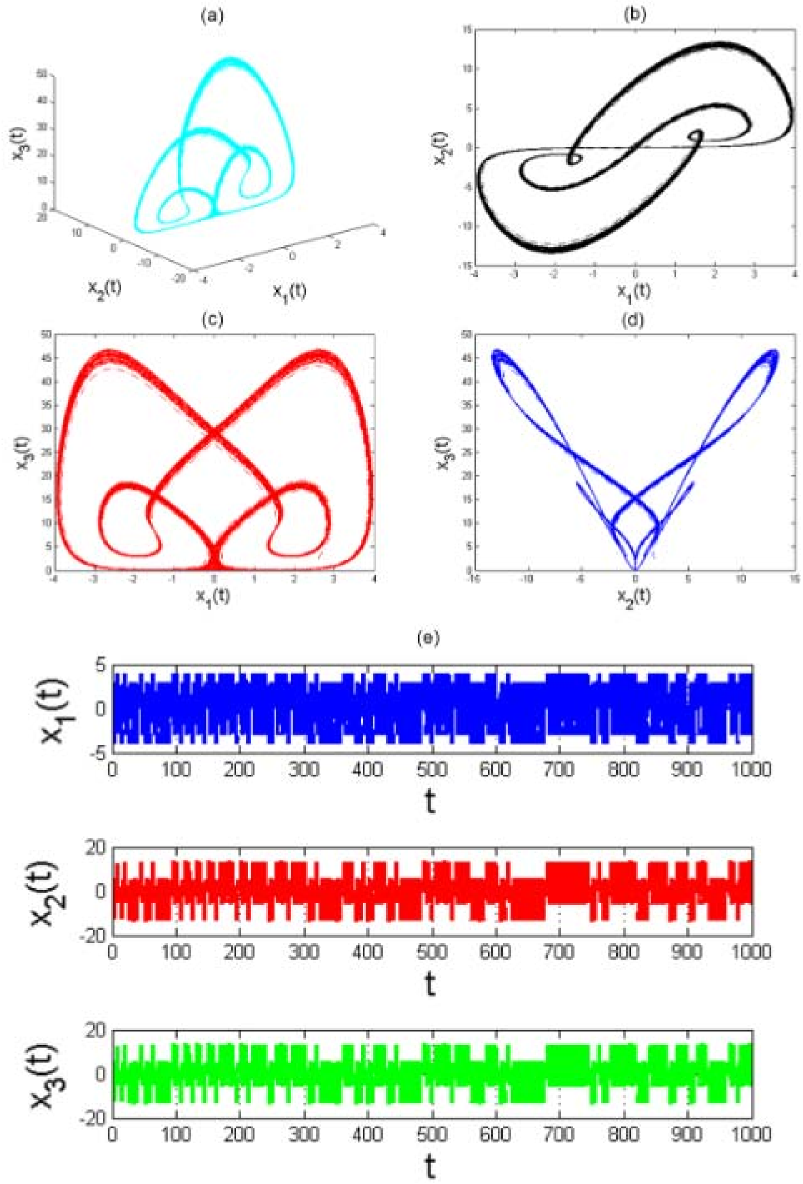

Fig. 1 Chaotic attractors and time-series plot for fractional model (21) with $\beta = 0.54$

The associated Jacobian is

$$A_{(x_1,x_2,x_3)} = \begin{pmatrix} x_3-\phi & x_1 & 1 \\ -2x_1 & -\varphi & 0 \\ -1 & 0 & -\psi \end{pmatrix}. \tag{36}$$

At point E^0 we have

$$A_{E^0} = \begin{pmatrix} \frac{1}{\varphi} - \phi & 0 & 1 \\ 0 & -\varphi & 0 \\ -1 & 0 & -\psi \end{pmatrix}, \tag{37}$$

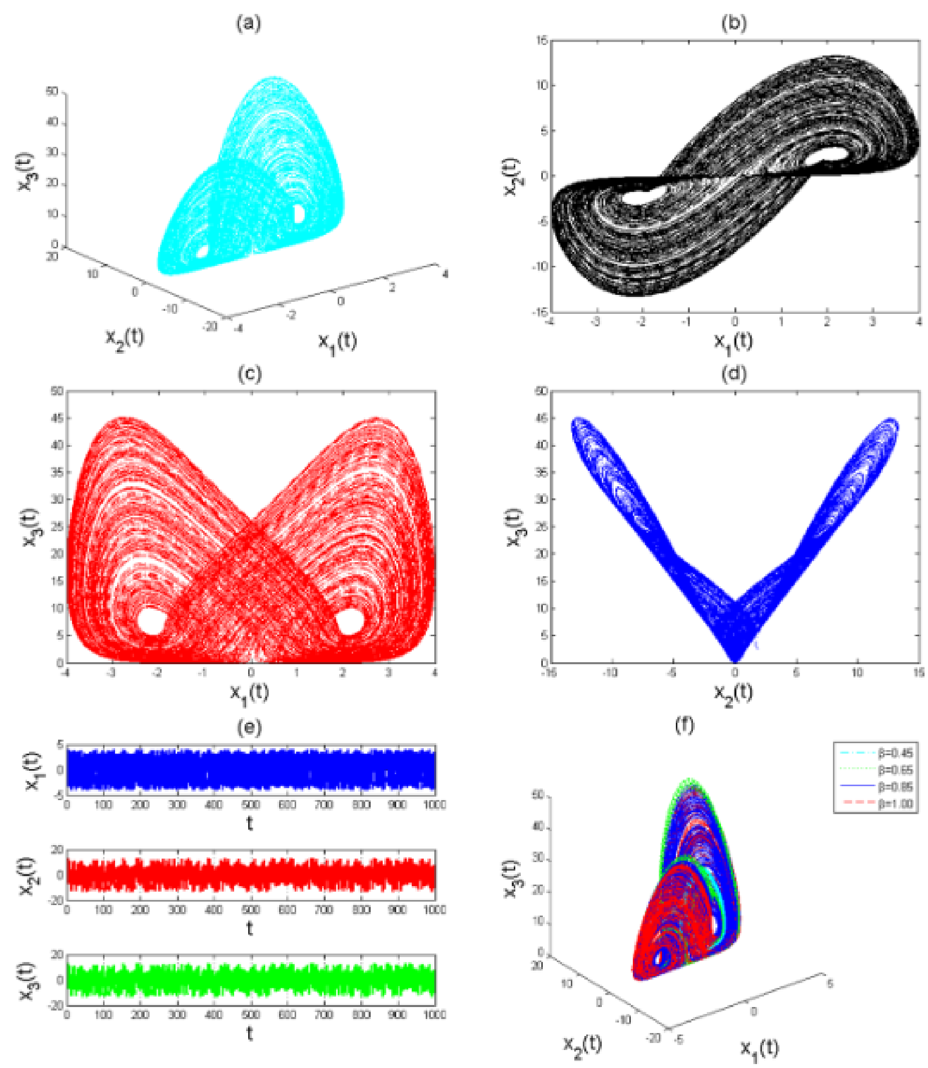

Fig. 2 Chaotic attractors and time-series plot for fractional model (21) with $\beta = 0.99$

and the corresponding characteristic equation becomes

$$p(\lambda) = \lambda^3 - \left(\frac{1}{\varphi} - \phi - \varphi - \psi\right)\lambda^2 - \left(\frac{\psi}{\varphi} - \phi\varphi - \phi\psi - \varphi\psi\right)\lambda - (\psi - \varphi - \phi\varphi\psi). \tag{38}$$

By following the Routh–Hurwitz stability criterion, for a polynomial of degree three, the following inequalities must be satisfied:

$$\begin{aligned} &-\left(\frac{1}{\varphi} - \phi - \varphi - \psi\right) > 0, \\ &-(\psi - \varphi - \phi\varphi\psi) > 0, \\ &\left(\frac{1}{\varphi} - \phi - \varphi - \psi\right)\left(\frac{\psi}{\varphi} - \phi\varphi - \phi\psi - \varphi\psi\right) + (\psi - \varphi - \phi\varphi\psi) > 0. \end{aligned} \tag{39}$$

Obviously, if $\lambda = -\varphi$ in (38), we have negative real part for $\varphi > 0$. If for instance, we select $\varphi = 0.1$, $\psi = 1$ arbitrarily and keep ϕ as control parameter, the conditions in (39) becomes $\phi > 8.9$, $\phi > 9$ and $(\phi - 8.9)\left(\frac{11\phi-99}{10}\right) - \frac{\phi-9}{10} > 0$, respectively. This implies that, for $\phi > 9$ all the eigenvalues are negative and real. Hence, we say that the equilibrium point E^0 is asymptotically stable.

At equilibrium state E^1, the Jacobian is defined by

$$A_{E^1} = \begin{pmatrix} \frac{1}{\psi} & \sqrt{1 - \phi\varphi - \frac{\varphi}{\psi}} & 1 \\ -2\sqrt{1 - \phi\varphi - \frac{\varphi}{\psi}} & -\varphi & 0 \\ -1 & 0 & -\psi \end{pmatrix}$$

which has characteristic polynomial

$$p(\lambda) = \lambda^3 + \left(\varphi + \psi - \frac{1}{\psi}\right)\lambda^2 + \left(1 + \varphi\psi - 2\phi\varphi - \frac{3\varphi}{\psi}\right)\lambda + 2(\psi - \varphi - \phi\varphi\psi). \tag{40}$$

In accordance, the eigenvalues have the negative real parts if the following conditions hold:

$$\begin{aligned} &\varphi + \psi - \frac{1}{\psi} > 0, \\ &2(\psi - \varphi - \phi\varphi\psi) > 0, \\ &\left(\varphi + \psi - \frac{1}{\psi}\right)\left(1 + \varphi\psi - 2\phi\varphi - \frac{3\varphi}{\psi}\right) - 2(\psi - \varphi - \phi\varphi\psi) > 0. \end{aligned} \tag{41}$$

Arbitrarily, we let $\varphi = 0.1$ and $\psi = 1$, with ϕ as free parameter, the first two conditions in (41) indicate that $\phi < 9, \phi > 9$, which implies that E^1 is unstable regardless the value of ϕ. The same analysis can be followed to examine the stability of the equilibrium point E^2. The dynamics of Lyapunov exponents for $\phi = 0.9, \varphi = 0.2$, $\psi = 1$, and initials $x_1(0) = x_2(0) = x_3(0) = 0.1$ for different instances of time is computed as:

```
t=200.0000   0.064181   0.002846  -0.692949 t=655.0000   0.080138
-0.000850  -0.696358 t=900.0000   0.076418  -0.002402  -0.694733
t=1000.0000  0.079907  -0.000838  -0.697682
```

It is obvious that the sum of Lyapunov exponents at any value of t is negative, this means that system (35) is dissipative, and hyper-chaotic for two of the exponents having the same plus(+) or minus(−) signs. The behaviour of the system is given in Fig. 3 for $t = 200$ and $t = 1000$.

When $\phi = 0.9, \varphi = 0.2$ and $\psi = 1$, Caputo fractional finance system (35) has a chaotic attractor. In the simulation experiments, we only vary the fractional derivative order β and the control parameter ϕ, the remaining model parameters are fixed. Simulation results in Figs. 4 and 5 correspond to parameter pairs $(\beta, \phi) = (0.92, 0.90)$ and $(\beta, \phi) = (0.78, 1.00)$, respectively, with initial conditions $x_1(0) = x_2(0) = x_3(0) = 0.1$.

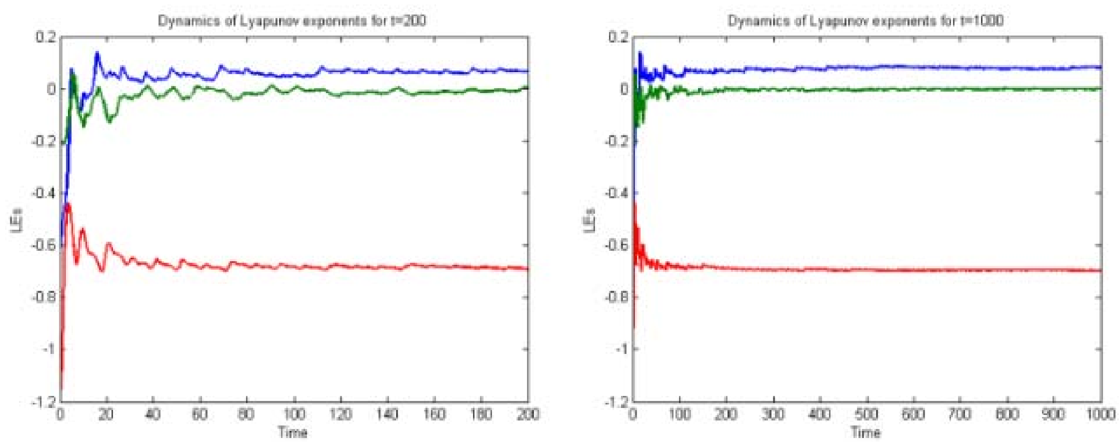

Fig. 3 Lyapunov exponents of system (35) for $t = 200$ and $t = 1000$

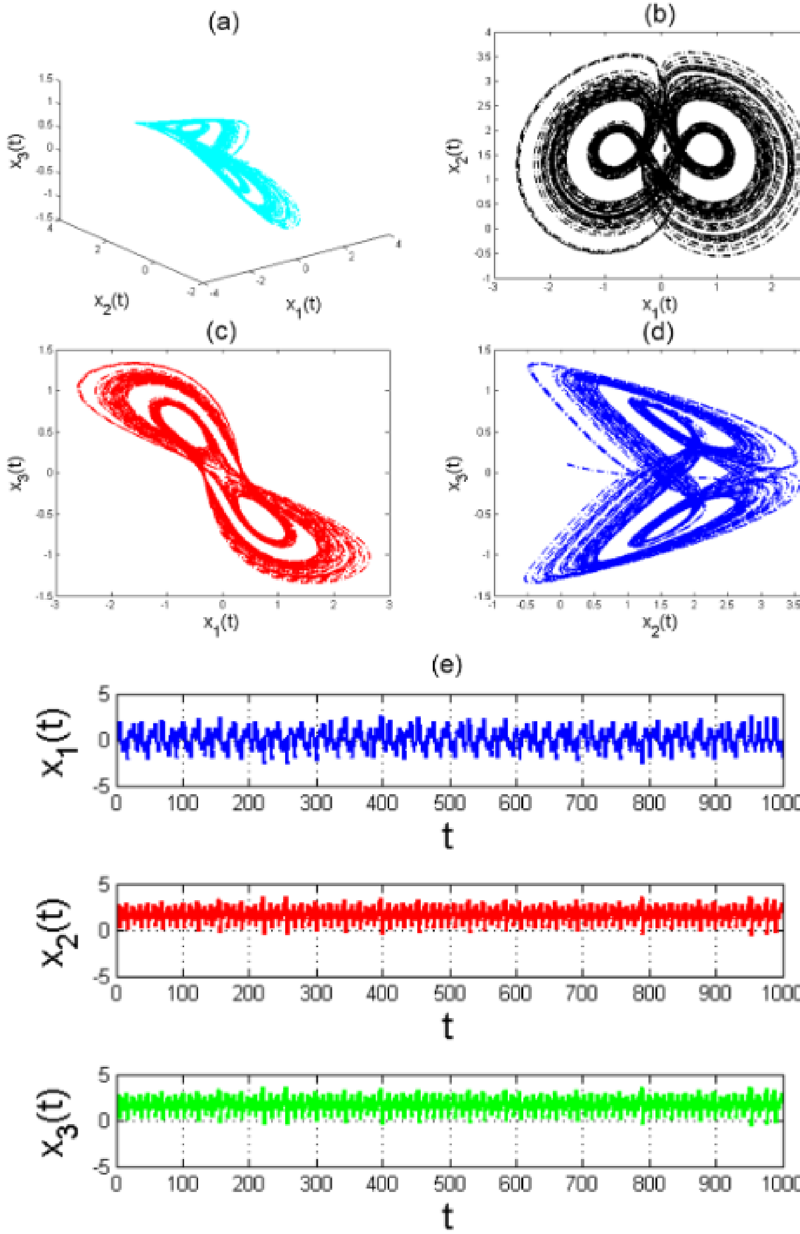

Fig. 4 Chaotic attractors for system (35) with $\beta = 0.92$ and $\phi = 0.90$

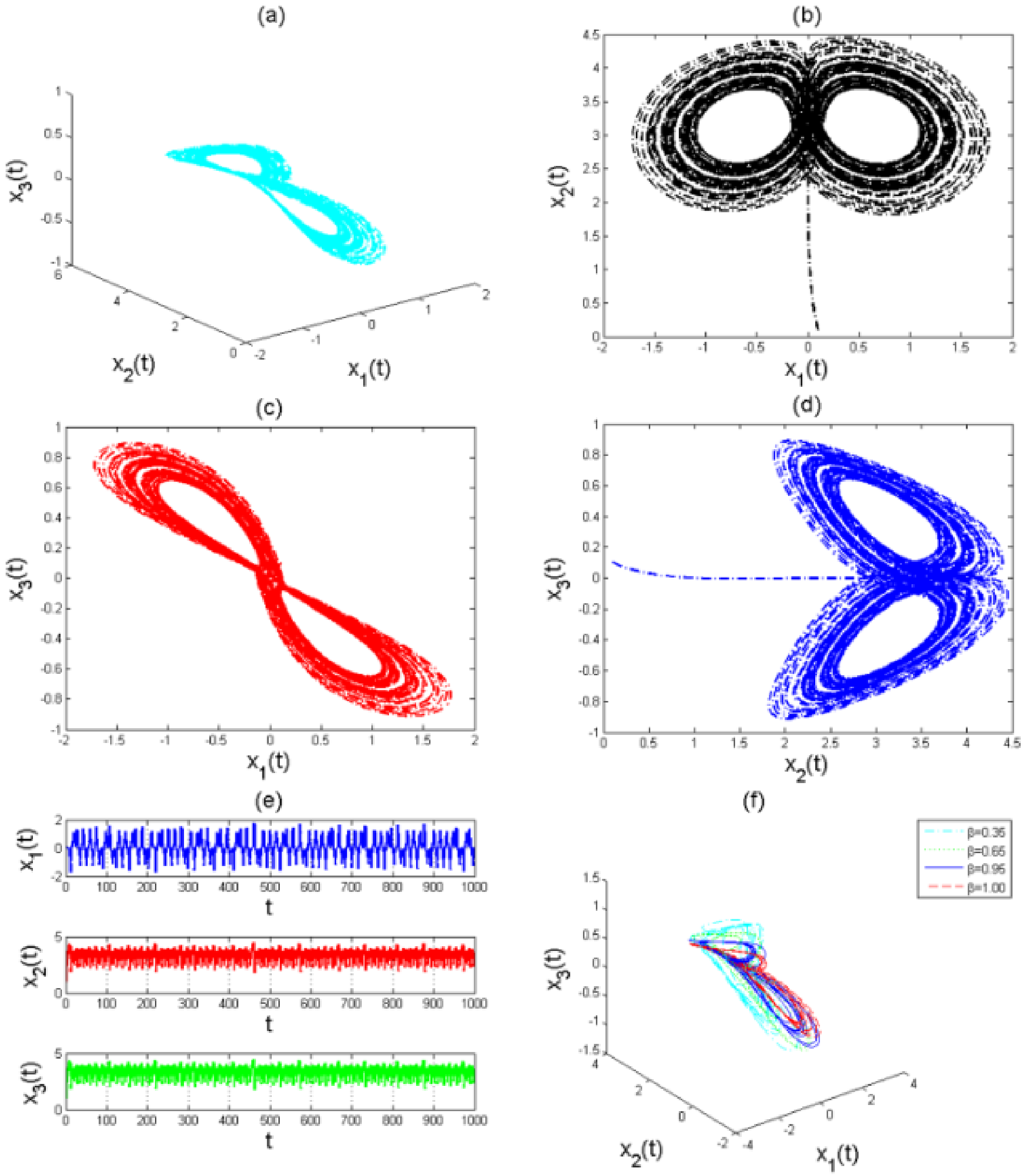

Fig. 5 Chaotic attractors for system (35) with $\beta = 0.78$ and $\phi = 1.00$

Chaotic Problem in Biology

For this example, we consider a three-dimensional chaotic cancer model [19] in the sense of Caputo to obtain the system of equations

$$\begin{aligned} {}^{C}\mathscr{D}_t^{\beta} x_1(t) &= x_1(t)(1 - x_1(t)) - \rho_1 x_1(t)x_2(t) - \rho_2 x_1(t)x_3(t), \\ {}^{C}\mathscr{D}_t^{\beta} x_2(t) &= \tau_1 x_2(1 - x_(2)) - \rho_3 x_1(t)x_2(t), \\ {}^{C}\mathscr{D}_t^{\beta} x_3(t) &= \tau_3 \frac{x_1(t)x_3(t)}{x_1(t) + \sigma_1} - \rho_4 x_1(t)x_3(t) - \sigma_2 x_3(t), \end{aligned} \tag{42}$$

where variable $x_1(t)$, $x_2(t)$ and $x_3(t)$ are used to represent the total population of tumor tissue cells, healthy tissue cells and recovery or those cells that are not affected by tumor at time t. All parameters are assumed positive. Linear stability analysis of the classical case (42) can be found in [19]. With $x_1(t) = 0.1$, $x_2(t) = 0.1$, $x_3(t) = 0.1$ and $\tau_1 = 0.6$, $\tau_2 = 4.5$, $\rho_1 = 1$, $\rho_2 = 1.5$, $\rho_3 = 1.5$, $\rho_4 = 0.9$, $\sigma_1 = 1.0$, $\sigma_2 = 0.5$, we obtain the phase plots of the chaotic cancer attractors as displayed in Fig. 6 for different instances of fractional order β. Also, we let $\rho_1 = 1.5$ and vary β to get the 3-D surface plots and their corresponding time series results in Fig. 7.

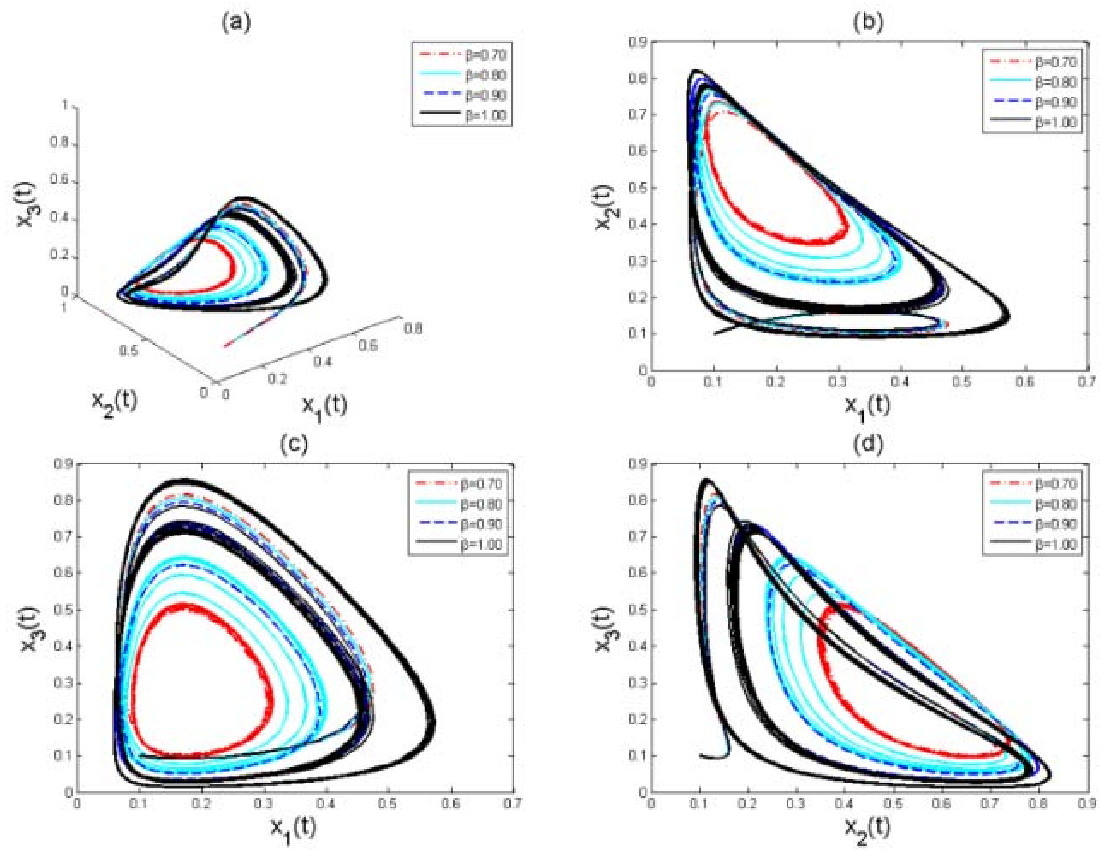

Fig. 6 Chaotic attractors for fractional cancer model (42) with $\beta = 0.61$

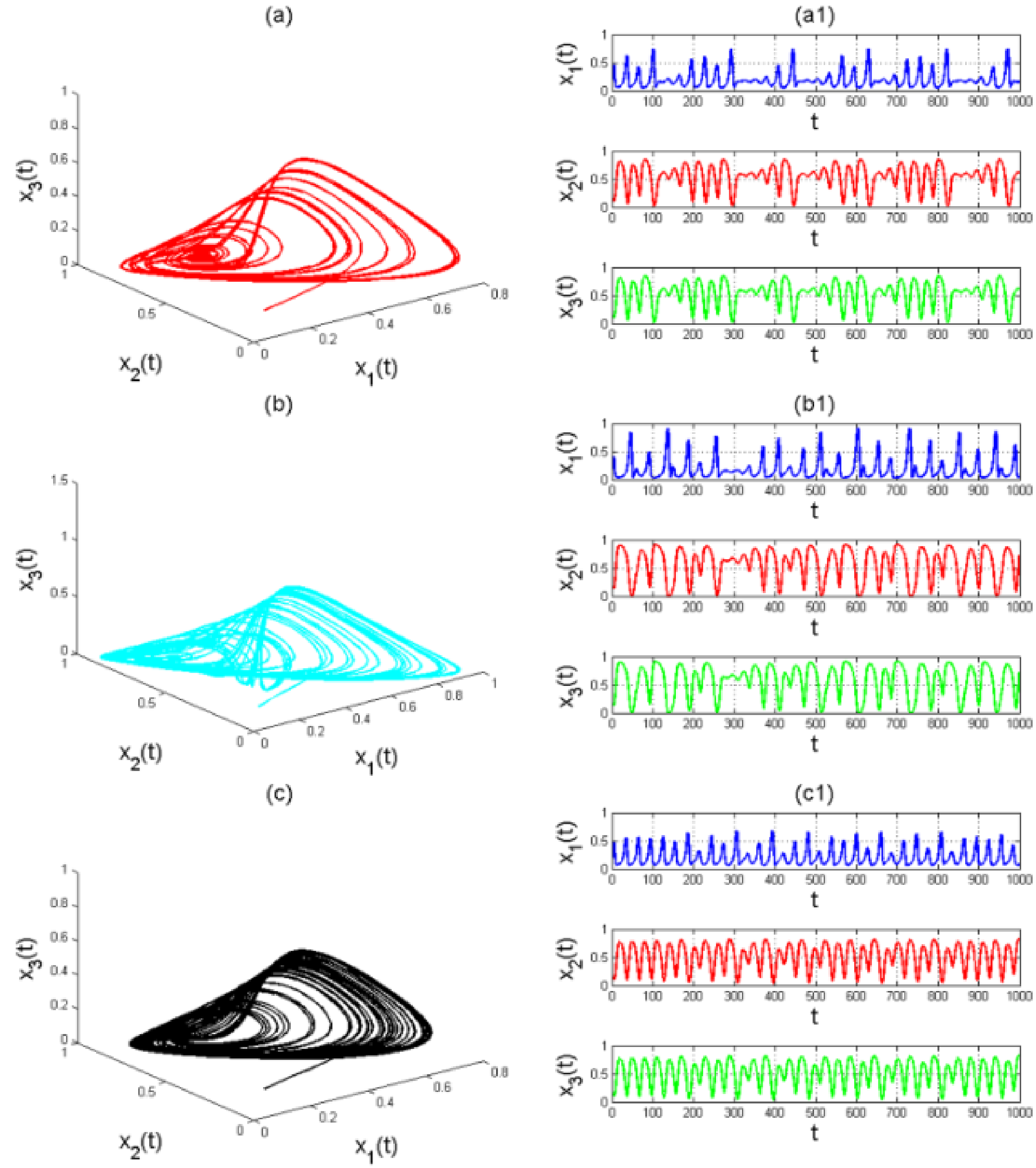

Fig. 7 3-D attractors and corresponding time-series plots showing chaotic evolution of system (42) for $\rho_1 = 1.00$. Rows 1–3 correspond to $\beta = 0.56, 0.73, 0.99$ at $t = 1000$

Conclusion

This paper considers the dynamics of fractional chaotic systems that are of current and recurring interest in application areas of engineering, science, and finance. The standard time derivative is modeled by the Caputo fractional operator of order $\beta \in (0, 1]$. To provide a guide in the appropriate choice of parameters, we examine the dynamics for linear stability analysis and calculate their Lyapunov exponents. We adopt the novel Chebyshev spectral method to numerically approximate the fractional operator, and integrate the resulting system with the help of inbuilt Ode45 in Matlab. The proposed method is easy to implement on a digital computer and capable of solving higher-order problems without reducing to the system of lower-order ordinary differential equations. Some numerical experiments which show chaotic behaviors are given for some instances of fractional order. Based on a comparison of results, it is obvious that chaotic patterns observed for $0 < \beta < 1$ are almost similar to the ones obtained when $\beta = 1$. The idea presented in this paper can be extended to any reaction-diffusion problems with limited computational costs.

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Declarations

Conflict of interest The authors have no competing interests

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