Improved asymptotically optimal error correcting codes for avoidance crosstalk type-IV on-chip data buses

Muhammad Ajmal[1], Masood Ur Rehman[2,*], B G Rodrigues[3]

[1] Department of Mathematics, Nanjing Univesity of Aeronautics and Astronautics, Nanjing, Jiangsu 211100, P. R. China,
Email: majmal@nuaa.edu.cn

[2]Department of Mathematics, Faculty of Science, Selcuk University, Konya 42005, Turkey
Email: masoodqau27@gmail.com

[3]Department of Mathematics and Applied Mathematics, University of Pretoria, Hatfield 0028, South Africa
Emails: bernardo.rodrigues@up.ac.za

Abstract

The first memory-less transition bus encoding technique for low power dissipation, crosstalk avoidance, and error correction simultaneously was presented by Chee et al. [Optimal low-power coding for error correction and crosstalk avoidance in on-chip data buses. *Des. Codes Cryptogr.*, 77 (2-4) (2015), 479–491]. They construct optimal or asymptotically optimal constant weight codes that eliminate each kind of crosstalk. In this article, we construct the improved asymptotically optimal $(n, 4, 3)$-IV code for all even orders $n \geq 14$ by using a combinatorial design approach. Furthermore, we show that an optimal weighted three code avoiding type-III crosstalk is also an optimal code avoiding the crosstalk of type-{III, IV}, for each odd order $n \geq 3$.

Keywords: Crosstalk; Balanced sampling plans avoiding adjacent unit (BSA); BSA*; Incomplete group divisible design (IGDD); Leave graph

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1 Introduction

Coupled switched capacitance causes crosstalk in ultra deep submicron/nonometer VLSI fabrication, which leads to power dissipation, delay faults, and logical malfunctions. The problem of removing or reducing crosstalk is considered a major signal integrity challenge for long on-chip buses implemented in UDSM CMOS technology [14].

The worst crosstalk couplings have been divided into four types [7, 14] and briefly explained in [5]. We restate such types in Table 1. The coupled switched capacitance resulting from type-I, -II, -III, and -IV crosstalks is in the ratio of 1 : 2 : 3 : 4. Therefore, avoiding crosstalks of higher types is of particular interest. However, type-I crosstalk cannot be avoided in any useful communication channel, but must be limited, since type-I crosstalk gives rise to power dissipation. The correction of active errors and limiting power dissipation are also critical issues in the design of bus-encoding despite the crosstalk avoidance.

Research presenting the coding schemes to encode data buses for error correction can be found in [2, 8], for low power dissipation [4, 9, 20, 25, 28] and for crosstalk avoidance [7, 11, 22, 24, 29, 30] and for any two of such three criteria [6, 13, 14, 17, 18, 19, 21]. The first memory-less transition bus-encoding technique that covers all three previously described criteria simultaneously; see [1, 5] for example.

Type-I	**Type-II**	**Type-III**	**Type-IV**
$0 \longleftrightarrow 1$	$001 \longleftrightarrow 110$ $011 \longleftrightarrow 100$	$001 \longleftrightarrow 010$ $010 \longleftrightarrow 100$ $011 \longleftrightarrow 101$ $101 \longleftrightarrow 110$	$010 \longleftrightarrow 101$
Single wire undergoes transition. Adjacent wires maintain previous states	Center wire in opposite transition to an adjacent wire. The other wire in same transition as center wire	Center wire in opposite transition to an adjacent wire. The other wire maintains previous state	All three adjacent wires undergo opposite transitions

Table 1: Types of worst crosstalk couplings [5].

For positive integers $x < y$, denote $[x, y] = \{x, x+1, \ldots, y\}$. Further abbreviate $[1, y]$ to $[y]$. Now, first we recall some basic concepts of classical error correcting codes then define our required codes. Let set $\mathcal{C}$ be a subset of $\{0,1\}^n$. Then we say $\mathcal{C}$ is a binary code of length n, and its elements are *codewords*. The *support* of a vector $u = (u_x)$, $x \in [n]$ of code $\mathcal{C}$ defined to be $supp(u) = \{x \in [n] : u_x \neq 0\}$. The weight of codeword u is the number of elements in $supp(u)$. For any two codewords $u, v \in \mathcal{C}$, the *Hamming distance* between u and v is defined to be $d(u, v) = |\{i : u_i \neq v_i\}|$, where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. The minimum distance of code $\mathcal{C}$, denoted $d(\mathcal{C})$ or simply write d, is the smallest positive integer such that $d(u, v) \geq d$ for all $u, v \in \mathcal{C}$ and $u \neq v$. A code with distance d has capability to correcting any appearance of e or fewer symbol errors, where $e \leq (d-1)/2$. Let (n, d, w)-code be a constant weight code with length n, distance d, and weight w. An (n, d, w)-code with $d \geq 3$ is said to be a *code avoids crosstalk* of type-II (or III, IV) if crosstalk couplings of type-II (or III, IV) do not exist in any three consecutive coordinates of such code [5]. We denote this code as (n, d, w)-II (or -III, IV). The largest size of an (n, d, w)-II code (or III, IV) is denoted by $A^{II}(n, d, w)$ (or $A^{III}(n, d, w)$, $A^{IV}(n, d, w)$). If such a code achieves the corresponding largest size then we call it optimal. Let $\mathcal{S}$

be a subset of set $\{II, III, IV\}$. The notation $A^{\mathcal{S}}(n,d,w)$ denotes the largest size of a code that is simultaneously an (n,d,w)-S for each $S \in \mathcal{S}$.

Let integers $n \geq k \geq 3$. Consider a pair of sets $(X, \mathcal{B})$ as a *set system*, where $X := \{x_0, x_1, \ldots, x_{n-1}\}$ is a finite set of *points* and $\mathcal{B}$ is a set of subsets of X. The cardinality of set X is n which known as *order* of a set system. Suppose a set $B \in \mathcal{B}$, we call B as a block and the cardinality $\mathcal{B}$ is *size* of a set system. If each block of $\mathcal{B}$ having k number of elements then given set system is a k-*uniform*. A k-uniform set system $(X, \mathcal{B})$ of order n is an (n,k)-*packing* if every two distinct elements of X appear in at most one block of $\mathcal{B}$. The pair (X, E) is a *leave graph* of such packing, where E contains all the pairs which are not appear in any block. The notation $D(n,k)$ denotes the largest size of an (n,k)-packing.

Hedayat, Rao and Stufken were designed a set system namely *balanced sampling plan avoiding adjacent units* (BSA in short) for a survey plan when several adjacent units provide similar information [10]. For more background about BSA, see for example [12, 27, 31, 32, 33] and references which are cited in these. We now recall some relevant concepts to this design. If the elements of X are ordered with circular order $x_o \prec x_1 \prec \cdots \prec x_{n-1} \prec x_0$ then we call it a cycle and denote as $(x_0, x_1, \ldots, x_{n-1})$. The set X with order $x_o \prec x_1 \prec \cdots \prec x_{n-1}$ is a *line* and write as $[x_0, x_1, \ldots, x_{n-1}]$. For any two points x_i and x_j of cycle X, define the distance of x_i and x_j to be $\min\{|j-i|, n-|j-i|\}$. The distance between any two points x_i and x_j over line X is defined as $|j-i|$. An (n,k)-packing $(X, \mathcal{B})$ with cyclic X is a CBSA(n,k,α) (or simply write BSA) if every pair $\{i,j\}$ within distance α appears in leave graph and every other pair with distance greater than α appears in exactly one block of $\mathcal{B}$. Similarly, an (n,k)-packing is a LBSA(n,k,α) when each pair $\{i,j\}$ of line X within distance α appears in its leave graph and all other pairs with distance greater than α appear in exactly one block of $\mathcal{B}$. Whenever the necessary conditions of existence of CBSA and LBSA are not hold then these definitions were generalized into *packing sampling plans avoiding adjacent units* (PSA) [5]. If we use the world *at most* instead of world *exactly* in definitions of CBSA$(n,k;\alpha)$ and LBSA$(n,k;\alpha)$ then these concepts become CPSA$(n,k;\alpha)$ and LPSA$(n,k;\alpha)$, respectively. Every CPSA$(n,k;\alpha)$ is a LPSA$(n,k;\alpha)$ but converse may not true. Let $B^\circ(n,k;\alpha)$ be the largest size of any CPSA$(n,k;\alpha)$, a CPSA$(n,k;\alpha)$ is optimal if it achieve the size $B^\circ(n,k;\alpha)$. Similarly, the notation $B(n,k;\alpha)$ be use for largest size of any LPSA$(n,k;\alpha)$, an optimal LPSA$(n,k;\alpha)$ have size $B(n,k;\alpha)$.

In [5], Chee et al. first established the connection between codes avoids crosstalks and PSA then determined $A^S(n,2w,w)$ where $S = II$ (or III, IV) and $A^{II}(n,4,3)$ for each order n. For any $\mathcal{S} \subset \{II, III, IV\}$, they give lower bounds of $A^{\mathcal{S}}(n, 2w-2, w)$. In particular, the authors provided the following result.

Lemma 1. ([5, Lemma 3.4]) *For each $n \not\equiv 0, 1 \pmod 6$ and $n \geq 14$,*

1. $A^{II,IV}(n,4,3) \geq B(n,3;2)$,
2. $A^{IV}(n,4,3) \geq B(n,3;2) + \lfloor \frac{n-1}{2} \rfloor$,

For the upper bound of an $(n,4,3)$-{II, IV} code, they proved $A^{II,IV}(n,4,3) \leq U(n,3;2)$, for each $n \geq 13$, where

$$U(n,3;2) := \left\lfloor \frac{2\lfloor\frac{n-3}{2}\rfloor + 2\lfloor\frac{n-4}{2}\rfloor + (n-4)\lfloor\frac{n-5}{2}\rfloor}{3} \right\rfloor.$$

In 2021, Ajmal and Zhang [1] provided the complete solution of Lemma 1 and consequently showed that $A^{II,IV}(n,4,3) = U(n,3;2)$ for each $n \geq 14$.

From previously established results, we observe that only lower bounds for all crosstalks are provided except for crosstalks of type-II and {II, IV}, when the weight of the codes is three or more. Therefore, such binary codes deserve further investigation. For upper bounds of $(n,4,3)$-III and $(n,4,3)$-IV codes, we present Conjectures 2 and 3, respectively.

Conjecture 2. *Let $n \geq 3$ be an odd integer, then $A^{III}(n,4,3) \leq A(n)$, where*

$$A(n) = \begin{cases} \frac{n^2+6n-3}{24}, & if\ n \equiv 3 \pmod{12}, \\ \frac{n^2+8n-17}{24}, & if\ n \equiv 5 \pmod{12}, \\ \frac{n^2+8n-33}{24}, & if\ n \equiv 1,9 \pmod{12}, \\ \frac{n^2+6n-19}{24}, & if\ n \equiv 7,11 \pmod{12}. \end{cases}$$

Conjecture 3. *Let $n \geq 14$ be an integer, then $A^{IV}(n,4,3) \leq B(n)$, where*

$$B(n) = \begin{cases} \frac{n^2-3n+6}{6}, & if\ n \equiv 0 \pmod{6}, \\ \frac{n^2-2n+1}{6}, & if\ n \equiv 1 \pmod{6}, \\ \frac{n^2-3n+8}{6}, & if\ n \equiv 2,4 \pmod{6}, \\ \frac{n^2-2n-3}{6}, & if\ n \equiv 3,5 \pmod{6}. \end{cases}$$

If an $(n,4,3)$-III (or IV) code achieves the size in Conjecture 2 (or 3), respectively, then we say that it is optimal.

In this article, we construct improved asymptotically optimal $(n,4,3)$-IV code for all even $n \geq 14$. Moreover we show that $A^{III}(n,4,3)$ is also $A^{III,IV}(n,4,3)$ for all $n \geq 3$ and $n \equiv 1 \pmod{2}$. The rest part of this article is organized as follows. In Section 2, we state some relevant concepts and results. We construct an auxiliary design $\mathrm{BSA}^*(n,\{2,3\};2,\ell)$ for all necessary parameters n and ℓ in Section 3. By using a recursive construction, we give the optimal codes avoiding crosstalk type-IV in Section 4. Finally, we give concluding remarks in Section 5.

2 Preliminaries

All graphs used in this paper are finite, undirected, and without loops. Consider a pair (X,E) as a *graph*, where set X is a *vertex set* and set $E \subseteq \binom{|X|}{2}$ is a *edge set*. Two edges are independent if they do not have any common vertex. The collection of such independent edges is called a *matching*. If all independent edges cover all the vertices of a graph, then call it a *perfect matching (or parallel class)*. The *difference* of edge $\{u,v\} \in E$ such that $u < v$ of a graph (X,E) with $|X| = n$, is defined to be $v-u,\ n-(v-u)$, whichever is smaller. For any set $D \subseteq [\lfloor \frac{n}{2} \rfloor]$, define $G(D,n)$ to be a graph with vertex $X = \{0,1,\ldots,n-1\}$ and edge set consisting of all edges having a difference in D. For $d \in D$, a graph $G(\{d\},n)$ is a subgraph of $G(D,n)$. For undefined terminology related to graph theory, the reader is encouraged to consult [3].

For integers a and b, $\gcd(a,b)$ is the greatest common divisor of a and b. The following corollary is taken from [15] which we use in the next lemma.

Corollary 4. [15] *A graph $G(\{d\}, n)$ consists of $h = \gcd(d, n)$ components, and each component is*

1. *a cycle $C_{n/h}$ if $d \neq n/2$, and*
2. *K_2 if $d = n/2$.*

The following result is a modification of Stern and Lenz lemma [26], which use in next section.

Lemma 5. *Given a vertex set $X = \{0, 1, \ldots, 2g-1\}$ and set $D \subseteq [g-1]$. For each $d \in D \cup \{g\}$, if a $G(\{d\}, 2g)$ consists of $h = \gcd(d, 2g)$ components such that $2g/h$ is even, then a graph $G(D \cup \{g\}, 2g)$ can be decomposed into $2|D|+1$ parallel classes of X.*

Proof. From Corollary 4, for each $d \in D$, there exists h cycles of length $2g/h$ in $G(\{d\}, 2g)$. Since $2g/h$ is even, then alternate edges of these cycles form two parallel classes, and for $d = g$, there are g K_2 of $G(\{d\}, 2g)$, which forms one parallel class. Hence, there are in total $2|D|+1$ parallel classes of X. □

In [33], the authors gave the necessary and sufficient conditions of BSA for $\alpha \in \{2, 3\}$. In particular, they prove the result which is stated below, and we use later in the recursive construction.

Lemma 6. [33] *There exists a circular BSA$(n, 3; 2)$ if and only if $n \equiv 3, 5 \pmod 6$ and $n \geq 15$.*

The following concept will be used in our recursive construction as a *master* design.

Let $(X, \mathcal{A})$ be a 3-uniform set system. Suppose $\mathcal{G}$ is a partition of X into $G_1, G_2, \ldots, G_u$, (called *groups* or *groops*) such that each $|G_i| = g$, and H is a subset of X (called *hole*) such that $|G_i \cap H| = h$ for each G_i. The quadruple $(X, H, \mathcal{G}, \mathcal{A})$ is an *incomplete group divisible design* (IGDD) of type $(g, h)^u$ with index one, denote 3-IGDD, when each pair of distinct elements x and y of X appears in exactly one block if $\{x, y\} \nsubseteq G_i$ for each $i \in [u]$ and $\{x, y\} \nsubseteq H$, otherwise it appears in no block.

Miao and Zhu established the existence of an 3-IGDD with index one in [16]. We quote that result as below.

Lemma 7. [16] *An 3-IGDD of type $(g, h)^u$ with index one exists if and only if the following properties hold:*

1. *$g \geq 2h$ and $u \geq 3$,*
2. *$g(u-1) \equiv 0 \pmod 2$,*
3. *$(g-h)(u-1) \equiv 0 \pmod 2$,*
4. *$u(u-1)(g^2-h^2) \equiv 0 \pmod 6$.*

The next two lemmas are from [32] by using Langford sequences. Further detail of Langford sequences, see [23].

Lemma 8. [32] *Let d be an odd integer if $m \equiv 0, 1 \pmod 4$, or an even integer if $m \equiv 0, 3 \pmod 4$ such that $m \geq 2d - 1$. Then $[d, d + 3m - 1]$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $i \in [m]$, such that $a_i + b_i = c_i$.*

Lemma 9. [32] *Let d be an odd integer if $m \equiv 2, 3 \pmod 4$, or an even integer if $m \equiv 1, 2 \pmod 4$ such that $m(m - 2d + 1) + 2 \geq 0$. Then $[d, d + 3m] \setminus \{d + 3m - 1\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $i \in [m]$, such that $a_i + b_i = c_i$.*

We close this section with following result which will be used later.

Lemma 10. *Let n be a positive odd integer. Then there is only one pair of codewords having crosstalk type-IV that can be contained in an $(n, 4, 3)$-III code. Furthermore, an optimal $(n, 4, 3)$-III code does not contain such pair of codewords.*

Proof. We start by defining a pair of codewords avoiding the crosstalk type-III but having the crosstalk type-IV. As the entries of five consecutive coordinates $[i-2, i+3]$ for some $i \in [2, n-4]$ are given below and 0 is placed at every other coordinate that are omit here for convenience.

$$\begin{array}{cccccc} i-2 & i-1 & i & i+1 & i+2 & i+3 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

or

$$\begin{array}{cccccc} i-2 & i-1 & i & i+1 & i+2 & i+3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

By observation, the pair above is the only possible pair of codewords in an $(n, 4, 3)$-III code.

Now, we show the second statement of this lemma. By computer search, there exists an optimal $(n, 4, 3)$-III code with $3 \leq n \leq 19$ and $n \equiv 1 \pmod 2$, which does not contain a pair of codewords defined already. For $n \geq 21$, by contrary we suppose an optimal $(n, 4, 3)$-III code contains the two codewords defined above. Then by definition of corsstalks of type-III and type-IV, all entries of columns $i-1$ to $i+2$ are zeros except one, which is presented in one of those two codewords. By shorting any two coordinates between $i-2$ and $i+4$, we have $A(n) \leq A(n-2)+2$. But from Conjecture 2, we have a contradiction. Hence, any optimal $(n, 4, 3)$-III code does not contain such type of two codewords. □

3 Constructions of BSA$^*(n, \{2, 3\}; 2, \ell)$

In [33], Zhang and Chang introduced an auxiliary design that is BSA$^*(n, \{2, 3\}; \alpha, \ell)$. We first recall this design and then give a direct construction of BSA$^*(n, \{2, 3\}; 2, \ell)$ designs for different values of the parameters n and ℓ which are needed in the next section.

Definition 3.1. *A BSA$^*(n, \{2, 3\}; \alpha, \ell)$ is a set system $(X, \mathcal{B})$, where X is a cycle and the size of each block of $\mathcal{B}$ is $k = 2$ or 3, if it satisfies the following properties:*

(i) For points $x, y \in X$, no pair $\{x, y\}$ within distance α appears in any block while any pair $\{x, y\}$ with distance greater than α appears in exactly one block.

(ii) *All blocks with size two can be partitioned into exactly ℓ parallel classes, where a parallel class means that every point of X occurs precisely once in the class.*

The necessary conditions for existence of a BSA$^*(n, \{2,3\}; \alpha, \ell)$ were given in [1], we summarize them in the following lemma.

Lemma 11. *Suppose there exists a BSA$^*(n, \{2,3\}; \alpha, \ell)$. Then the following hold:*

1. $n - 2\alpha - 1 \geq \ell$,
2. $n - \ell - 2\alpha - 1 \equiv 0 \pmod 6$, *and*
3. $n \equiv 0 \pmod 2$ *if* $\ell > 0$.

Note that, if a BSA$^*(n, \{2,3\}; \alpha, \ell)$ have no block of size two then it becomes a simple BSA$(n, 3; \alpha)$ and the notation BSA$^*(n, 2; \alpha, \ell)$ means that the design is a 2-uniform set system only. Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the cyclic additive group of order n. Assume B is a subset of $\mathbb{Z}_n$, let $\Delta B := \{a - b : \text{ where } a, b \in B \text{ and } a \neq b\} \subset \mathbb{Z}_n$. In [1], the authors proved following result.

Lemma 12. [1] *Let n be even and ℓ be odd such that $n - 2\alpha - 1 \geq \ell$ and $n - \ell - 2\alpha - 1 \equiv 0 \pmod 6$. Let R be a set of $\frac{\ell-1}{2}$ positive odd integers less than $n/2$. Suppose there exist 3-subsets $B_i \subset \mathbb{Z}_n$, $i \in [(n - \ell - 2\alpha - 1)/6]$, such that the multiset union*

$$\bigcup_{i \in [(n-\ell-2\alpha-1)/6]} \Delta B_i = \mathbb{Z}_n \setminus (\{0, \pm 1, \ldots, \pm\alpha, n/2\} \cup \pm R),$$

then there exists a BSA$^*(n, \{2,3\}; \alpha, \ell)$.

Given an integer m dividing n, we call the action that $B + im$ where $0 \leq i \leq m-1$ as *developing the base block B* by $m\mathbb{Z}_n$. Here, $B + a := \{b + a : \text{ where } b \in B\}$ for $a \in \mathbb{Z}_n$, the designs BSA* which we will construct over $X = \mathbb{Z}_n$ under modulo n. For some specific parameters, these designs are constructed by a computer search.

Lemma 13. *A BSA$^*(n, 2; 2, \ell)$ exists for each $(n, \ell) \in \{(16, 11), (20, 15), (22, 17)\}$.*

Proof. See Appendix A. □

Lemma 14. *A BSA$^*(n, \{2,3\}; 2, \ell)$ exists for each $(n, \ell) \in \{(16, 5), (28, 11), (28, 17)\}$.*

Proof. For each pair (n, ℓ), all blocks of size three are obtained by developing base blocks under $\mathbb{Z}_n$ and ℓ parallel classes are obtained by decomposing a graph $G(D \cup \{n/2, n\})$ from Lemma 5. We list the base blocks and the set D for each pair (n, ℓ) below.

(n, ℓ)	Base blocks	D
$(16, 5)$	$\{0, 3, 7\}$.	$\{5, 6\}$
$(28, 11)$	$\{0, 4, 12\}$, $\{0, 6, 13\}$.	$\{3, 5, 9, 10, 11\}$
$(28, 17)$	$\{0, 4, 12\}$.	$\{3, 5, 6, 7, 9, 10, 11, 13\}$

□

Lemma 15. *A $BSA^*(n,\{2,3\};2,\ell)$ exists for each $(n,\ell) \in \{(18,3),\ (18,9),(18,11)\}$.*

Proof. For each pair (n,ℓ), a BSA$^*(n,\{2,3\};2,\ell)$ is presented in Appendix B. □

Lemma 16. *A $BSA^*(n,\{2,3\};2,\ell)$ exists for each $(n,\ell) \in \{(20,9),(22,11),(26,9),(26,15)\}$.*

Proof. For each pair (n,ℓ), a BSA$^*(n,\{2,3\};2,\ell)$ is constructed in Appendix C. □

Lemma 17. *A $BSA^*(n,\{2,3\};2,\ell)$ exists for each $(n,\ell) \in \{(24,9),(32,9),(36,11),(40,5),(42,5),$ $(42,17),(64,17)\}$.*

Proof. For each pair (n,ℓ), a BSA$^*(n,\{2,3\};2,\ell)$ is constructed in Appendix D. □

Lemma 18. *A $BSA^*(n,\{2,3\};2,\ell)$ exists for each $(n,\ell) \in \{(22,5),(28,5),(34,5),(44,15),(52,11),$ $(58,5),(58,11),(62,3),(68,15),(70,11),(74,9),(82,5),(82,17),(86,3),(86,15),(94,11),(98,9),$ $(106,\ 17),(110,15)\}$.*

Proof. For each pair (n,ℓ), a BSA$^*(n,\{2,3\};2,\ell)$ is constructed in Appendix E . □

Lemma 19. ([33, Lemma 2.1]) *There exists a $BSA^*(n,\{2,3\};2,\ell)$ in each of the following cases:*

1. *$n = 24r+2$ where $r \geq 2$, $\ell = 15$,*
2. *$n = 24r+8$ where $r \geq 1$, $\ell = 15$,*
3. *$n = 24r+14$ where $r \geq 3$, $\ell = 33$,*
4. *$n = 24r+20$ where $r \geq 5$, $\ell = 69$.*

Lemma 20. *There exists a $BSA^*(24r+2,\{2,3\};2,3)$ where $r \geq 1$.*

Proof. By Lemma 9, the interval $[3,12r] \setminus R$ can be partitioned into $4r-1$ triples $\{a_i,b_i,c_i\}$ such that $a_i+b_i=c_i$ for all $i \in [4r-1]$ and $R=\{12r-1\}$. Transform all base blocks into $\{0,a_i,c_i\}$ then applying Lemma 12, we obtain the desired result. □

Lemma 21. *There exists a $BSA^*(n,\{2,3\};2,\ell)$ in each of the following cases:*

1. *$n = 24r-2$ where $r \geq 5$, $\ell = 53$,*
2. *$n = 24r+2$ where $r \geq 3$, $\ell = 27$,*
3. *$n = 24r+2$ where $r \geq 4$, $\ell = 39$,*
4. *$n = 24r+4$ where $r \geq 2$, $\ell = 17$,*
5. *$n = 24r+8$ where $r \geq 1$, $\ell = 3$,*
6. *$n = 24r+10$ where $r \geq 1$, $\ell = 11$,*
7. *$n = 24r+10$ where $r \geq 2$, $\ell = 23$,*
8. *$n = 24r+10$ where $r \geq 3$, $\ell = 35$,*

9. $n = 24r + 10$ *where* $r \geq 4$, $\ell = 47$,

10. $n = 24r + 14$ *where* $r \geq 1$, $\ell = 9$,

11. $n = 24r + 14$ *where* $r \geq 2$, $\ell = 21$,

12. $n = 24r + 14$ *where* $r \geq 4$, $\ell = 45$,

13. $n = 24r + 20$ *where* $r \geq 1$, $\ell = 9$,

14. $n = 24r + 22$ *where* $r \geq 1$, $\ell = 5$,

15. $n = 24r + 22$ *where* $r \geq 1$, $\ell = 17$,

16. $n = 24r + 22$ *where* $r \geq 2$, $\ell = 29$,

17. $n = 24r + 22$ *where* $r \geq 3$, $\ell = 41$.

Proof. In each of the above cases, the interval $[3, n/2 - 1] \setminus R$ with some $|R| = \frac{\ell-1}{2}$, is partitioned into triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$, for $i \in [(n - \ell - 5)/6]$. Transform all base blocks into $\{0, a_i, c_i\}$, then applying Lemma 12 with corresponding R, we obtained the desired result for each case. We list the sets $\{a_i, b_i, c_i\}$ and R for all cases below. In some of the following cases, we take the set O to consist of all odd integers of $[3, n/2 - 1]$.

(1) $n = 24r - 2$, $r \geq 5$, and $\ell = 53$.

For $r = 5$, let $R = O \setminus \{5, 49\}$. The triples are $\{4, 6, 10\}$, $\{5, 44, 49\}$, $\{8, 30, 38\}$, $\{12, 34, 46\}$, $\{14, 40, 54\}$, $\{16, 42, 58\}$, $\{18, 32, 50\}$, $\{20, 36, 56\}$, $\{22, 26, 48\}$, $\{24, 28, 52\}$.

For $r = 6$, let $R = O \setminus \{5, 7, 9, 11, 55, 59, 63, 69\}$. The triples are $\{4, 6, 10\}$, $\{5, 50, 55\}$, $\{7, 52, 59\}$, $\{8, 12, 20\}$, $\{9, 54, 63\}$, $\{11, 58, 69\}$, $\{14, 32, 46\}$, $\{16, 40, 56\}$, $\{18, 42, 60\}$, $\{22, 48, 70\}$, $\{24, 44, 68\}$, $\{26, 36, 62\}$, $\{28, 38, 66\}$, $\{30, 34, 64\}$.

For $r \geq 7$, let $R = \{3, 5, 7, 9, 4r - 11, 4r - 9, 4r - 7, 4r - 3, 4r - 1, 4r + 1, 4r + 3, 4r + 5, 6r - 1, 6r + 1, 6r + 3, 8r - 5, 8r - 3, 8r - 1, 8r + 1, 8r + 3, 8r + 5, 10r - 3, 10r - 1, 10r + 1, 10r + 3, 12r - 3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r - 5 - h$,	$11 + 2h$,	$6r + 6 + h$,	$[0, r - 7]$;
$5r + 1 - h$,	$2r - 1 + 2h$,	$7r + h$,	$[0, r - 6]$;
$10r - 4 - h$,	$8 + 2h$,	$10r + 4 + h$,	$[0, r - 5]$;
$9r - h$,	$2r + 2h$,	$11r + h$,	$[0, r - 6]$.

Other triples are $\{4, 8r, 8r + 4\}$, $\{6, 4r - 10, 4r - 4\}$, $\{4r - 8, 4r + 6, 8r - 2\}$, $\{4r - 6, 6r + 4, 10r - 2\}$, $\{4r - 5, 6r + 5, 10r\}$, $\{4r - 2, 4r + 4, 8r + 2\}$, $\{4r, 8r - 4, 12r - 4\}$, $\{4r + 2, 6r, 10r + 2\}$, $\{6r - 4, 6r + 2, 12r - 2\}$, $\{6r - 3, 6r - 2, 12r - 5\}$.

(2) $n = 24r + 2$, $r \geq 3$, and $\ell = 27$.

For $r = 3$, let $R = O \setminus \{9, 27, 33, 35\}$. The triples are $\{4, 18, 22\}$, $\{6, 27, 33\}$, $\{8, 28, 36\}$, $\{9, 26, 35\}$, $\{10, 24, 34\}$, $\{12, 20, 32\}$, $\{14, 16, 30\}$.

For $r \geq 4$, let $R = \{3, 4r - 5, 4r - 3, 4r - 1, 4r + 1, 6r - 1, 6r + 1, 8r + 1, 8r + 3, 10r - 1, 10r + 1, 10r + 3, 12r - 1\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-2-h$,	$5+2h$,	$6r+3+h$,	$[0,r-4]$;
$5r+1-h$,	$2r-1+2h$,	$7r+h$,	$[0,r-3]$;
$10r-2-h$,	$6+2h$,	$10r+4+h$,	$[0,r-4]$;
$9r+1-h$,	$2r+2h$,	$11r+1+h$,	$[0,r-3]$.

Other triples are $\{4,8r-2,8r+2\}$, $\{4r-4,4r+3,8r-1\}$, $\{4r-2,6r+2,10r\}$, $\{4r,8r,12r\}$, $\{4r+2,6r,10r+2\}$.

(3) $n=24r+2$, $r\geq 4$, and $\ell=39$.

For $r=4$, let $R=O\setminus\{5,7,41,45\}$. The triples are $\{4,6,10\}$, $\{5,36,41\}$, $\{7,38,45\}$, $\{8,24,32\}$, $\{12,28,40\}$, $\{14,34,48\}$, $\{16,30,46\}$, $\{18,26,44\}$, $\{20,22,42\}$.

For $r=5$, let $R=O\setminus\{5,7,9,11,13,35,39,53,55,59\}$. The triples are $\{4,6,10\}$, $\{5,35,40\}$, $\{7,46,53\}$, $\{8,12,20\}$, $\{9,50,59\}$, $\{11,44,55\}$, $\{13,39,52\}$, $\{14,28,42\}$, $\{16,32,48\}$, $\{18,36,54\}$, $\{22,38,60\}$, $\{24,34,58\}$, $\{26,30,56\}$.

For $r\geq 6$, let $R=\{3,5,7,4r-5,4r-1,4r+1,6r-3,6r-1,6r+1,6r+3,8r-3,8r-1,8r+1,8r+3,8r+5,10r-1,10r+1,10r+3,12r-3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-4-h$,	$9+2h$,	$6r+5+h$,	$[0,r-6]$;
$5r+1-h$,	$2r-1+2h$,	$7r+h$,	$[0,r-4]$;
$10r-2-h$,	$6+2h$,	$10r+4+h$,	$[0,r-4]$;
$9r+1-h$,	$2r+2h$,	$11r+1+h$,	$[0,r-6]$.

Other triples are $\{4,4r-10,4r-6\}$, $\{4r-3,4r+3,8r\}$, $\{4r-2,4r+4,8r+2\}$, $\{4r+2,6r,10r+2\}$, $\{4r-4,6r+4,10r\}$, $\{4r-7,8r+6,12r-1\}$, $\{4r,8r-2,12r-2\}$, $\{4r-8,8r+4,12r-4\}$, $\{6r-2,6r+2,12r\}$.

(4) $n=24r+4$, $r\geq 2$, and $\ell=17$.

For $r=2$, let $R=O\setminus\{5,7,15,25\}$. The triples are $\{4,10,14\}$, $\{5,20,25\}$, $\{6,12,18\}$, $\{7,15,22\}$, $\{8,16,24\}$.

For $r=3$, let $R=\{9,13,15,21,25,27,29,31\}$. The triples are $\{3,23,26\}$, $\{4,6,10,\}$, $\{5,32,37\}$, $\{7,28,35\}$, $\{8,16,24,\}$, $\{11,22,33\}$, $\{12,18,30,\}$, $\{14,20,34,\}$, $\{17,19,36\}$.

For $r\geq 4$, let $R=\{3,5,4r+1,6r-3,6r+1,8r-1,12r-1,12r+1\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-4-h$,	$7+2h$,	$6r+3+h$,	$[0,r-4]$;
$5r-1-h$,	$2r+2+2h$,	$7r+1+h$,	$[0,r-3]$;
$10r-3-h$,	$4+2h$,	$10r+1+h$,	$[0,r-3]$;
$9r-1-h$,	$2r+1+2h$,	$11r+h$,	$[0,r-2]$.

Other triples are $\{6r-2,6r+2,12r\}$, $\{4r-1,7r,11r-1\}$, $\{4r-2,6r,10r-2\}$, $\{2r,8r,10r\}$, $\{4r,6r-1,10r-1\}$.

(5) $n=24r+8$, $r\geq 1$, and $\ell=3$.

For $r=1$, let $R=\{13\}$. The triples are $\{3,9,12\}$, $\{4,7,11\}$, $\{5,10,15\}$, $\{6,8,14\}$.

For $r \geq 2$, let $R = \{6r+3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+1-h$,	$4+2h$,	$6r+5+h$,	$[0, r-2]$;
$5r+2-h$,	$2r+2+2h$,	$7r+4+h$,	$[0, r-2]$;
$10r+2-h$,	$3+2h$,	$10r+5+h$,	$[0, r-2]$;
$9r+3-h$,	$2r+1+2h$,	$11r+4+h$,	$[0, r-2]$.

Other triples are $\{4r-1, 6r+4, 10r+3\}$, $\{4r, 8r+3, 12r+3\}$, $\{4r+1, 4r+3, 8r+4\}$, $\{4r+2, 6r+2, 10r+4\}$.

(6) $n = 24r+10$, $r \geq 1$, and $\ell = 11$.

For $r = 1$, let $R = \{5, 9, 11, 13, 15\}$. The triples are $\{3, 7, 10\}$, $\{4, 12, 16\}$ and $\{6, 8, 14\}$.

For $r \geq 2$, let $R = \{4r+1, 4r+3, 6r+3, 8r+3, 12r+3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+1-h$,	$4+2h$,	$6r+5+h$,	$[0, r-2]$;
$5r+2-h$,	$2r+2+2h$,	$7r+4+h$,	$[0, r-2]$;
$10r+2-h$,	$3+2h$,	$10r+5+h$,	$[0, r-2]$;
$9r+3-h$,	$2r+1+2h$,	$11r+4+h$,	$[0, r-2]$.

Other triples are $\{4r-1, 6r+4, 10r+3\}$, $\{4r, 8r+4, 12r+4\}$, $\{4r+2, 6r+2, 10r+4\}$.

(7) $n = 24r+10$, $r \geq 2$, and $\ell = 23$.

For $r = 2$, let $R = O \setminus \{11, 19\}$. The triples are $\{4, 20, 24\}$, $\{6, 16, 22\}$, $\{8, 11, 19\}$, $\{10, 18, 28\}$, $\{12, 14, 26\}$.

For $r = 3$, let $R = \{5, 7, 11, 13, 15, 17, 19, 21, 23, 27, 33\}$. The triples are $\{3, 25, 28\}$, $\{4, 35, 39\}$, $\{6, 31, 37\}$, $\{8, 24, 32\}$, $\{9, 20, 29\}$, $\{10, 30, 40\}$, $\{12, 26, 38\}$, $\{14, 22, 36\}$, $\{16, 18, 34\}$.

For $r \geq 4$, let $R = \{3, 5, 4r+3, 6r+1, 6r+3, 8r+1, 10r+1, 10r+3, 10r+5, 12r+1, 12r+3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-h$,	$6+2h$,	$6r+6+h$,	$[0, r-3]$;
$5r+2-h$,	$2r+2+2h$,	$7r+4+h$,	$[0, r-4]$;
$10r-1-h$,	$7+2h$,	$10r+6+h$,	$[0, r-4]$;
$9r+2-h$,	$2r+1+2h$,	$11r+3+h$,	$[0, r-3]$.

Other triples are $\{4, 4r+1, 4r+5\}$, $\{4r-4, 6r+4, 10r\}$, $\{4r-3, 6r+5, 10r+2\}$, $\{4r-2, 4r+4, 8r+2\}$, $\{4r-1, 8r+3, 12r+2\}$, $\{4r, 8r+4, 12r+4\}$, $\{4r+2, 6r+2, 10r+4\}$.

(8) $n = 24r+10$, $r \geq 3$, and $t = 35$.

For $r = 3$, let $R = O \setminus \{5, 39\}$. The triples are $\{4, 14, 18\}$, $\{5, 34, 39\}$, $\{6, 22, 28\}$, $\{8, 24, 32\}$, $\{10, 30, 40\}$, $\{12, 26, 38\}$, $\{16, 20, 36\}$.

For $r = 4$, let $R = O \setminus \{17, 23, 37, 39, 45, 47, 49, 51\}$. The triples are $\{4, 38, 42\}$, $\{6, 39, 45\}$, $\{8, 22, 30\}$, $\{10, 37, 47\}$, $\{12, 40, 52\}$, $\{14, 36, 50\}$, $\{16, 32, 48\}$, $\{17, 34, 51\}$, $\{18, 28, 46\}$, $\{20, 24, 44\}$, $\{23, 26, 49\}$.

For $r \geq 5$, let $R = \{3, 5, 4r-5, 4r+1, 4r+3, 6r-1, 6r+1, 6r+5, 6r+7, 8r+1, 8r+3, 8r+5, 10r+1, 10r+3, 10r+5, 12r+1, 12r+3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-2-h,$	$10+2h,$	$6r+8+h,$	$[0, r-5];$
$5r+2-h,$	$2r+2+2h,$	$7r+4+h,$	$[0, r-4];$
$10r-1-h,$	$7+2h,$	$10r+6+h,$	$[0, r-4];$
$9r+2-h,$	$2r+1+2h,$	$11r+3+h,$	$[0, r-4].$

Other triples are $\{4, 12r, 12r+4\}, \{6, 6r, 6r+6\}, \{8, 4r-4, 4r+4\}, \{4r-3, 6r+3, 10r\}, \{4r-2, 6r+4, 10r+2\}, \{4r-1, 4r+5, 8r+4\}, \{4r, 8r+2, 12r+2\}, \{4r+2, 6r+2, 10r+4\}$.

(9) $n = 24r+10$, $r \geq 4$, and $\ell = 47$.

For $r = 4$, let $R = O \setminus \{3, 37\}$. The triples are $\{3, 37, 40\}$, $\{4, 6, 10\}$, $\{8, 30, 38\}$, $\{12, 32, 44\}$, $\{14, 34, 48\}$, $\{16, 36, 52\}$, $\{18, 24, 42\}$, $\{20, 26, 46\}$, $\{22, 28, 50\}$.

For $r = 5$, let $R = O \setminus \{7, 9, 11, 13, 51, 59, 61, 63\}$. The triples $\{4, 6, 10\}$, $\{7, 44, 51\}$, $\{8, 12, 20\}$, $\{9, 50, 59\}$, $\{11, 52, 63\}$, $\{13, 48, 61\}$, $\{14, 32, 46\}$, $\{16, 38, 54\}$, $\{18, 40, 58\}$, $\{22, 42, 64\}$, $\{24, 36, 60\}$, $\{26, 30, 56\}$, $\{28, 34, 62\}$.

For $r \geq 6$, let $R = \{5, 7, 9, 4r-5, 4r-3, 4r-1, 4r+1, 4r+5, 4r+7, 6r+1, 6r+3, 6r+7, 8r-1, 8r+1, 8r+3, 8r+5, 8r+7, 10r+1, 10r+5, 10r+7, 10r+9, 12r+1, 12r+3\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-3-h,$	$12+2h,$	$6r+9+h,$	$[0, r-6];$
$5r+2-h,$	$2r+2+2h,$	$7r+4+h,$	$[0, r-6];$
$10r-3-h,$	$13+2h,$	$10r+10+h,$	$[0, r-6];$
$9r+2-h,$	$2r+3+2h,$	$11r+5+h,$	$[0, r-6].$

Other triples are $\{3, 8, 11\}$, $\{4, 8r, 8r+4\}$, $\{6, 10r-2, 10r+4\}$, $\{10, 4r-8, 4r+2\}$, $\{4r-7, 6r+6, 10r-1\}$, $\{4r-6, 6r+8, 10r+2\}$, $\{4r-4, 6r+4, 10r\}$, $\{4r-2, 8r+6, 12r+4\}$, $\{4r, 8r+2, 12r+2\}$, $\{4r+3, 6r+5, 10r+8\}$, $\{4r+4, 6r-1, 10r+3\}$, $\{4r+6, 6r, 10r+6\}$, $\{6r-2, 6r+2, 12r\}$.

(10) $n = 24r+14$, $r \geq 1$, and $\ell = 9$.

For $r = 1$, let $R = \{7, 11, 13, 17\}$. The triples are $\{3, 15, 18\}$, $\{4, 8, 12\}$, $\{5, 9, 14\}$, $\{6, 10, 16\}$.

For $r \geq 2$, let $R = \{4r+1, 8r+5, 10r+5, 12r+5\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+2-h,$	$3+2h,$	$6r+5+h,$	$[0, r-1];$
$5r+2-h,$	$2r+3+2h,$	$7r+5+h,$	$[0, r-2];$
$10r+3-h,$	$4+2h,$	$10r+7+h,$	$[0, r-2];$
$9r+4-h,$	$2r+2+2h,$	$11r+6+h,$	$[0, r-2].$

Other triples are $\{4r, 6r+4, 10r+4\}$, $\{4r+2, 8r+4, 12r+6\}$, $\{4r+3, 6r+3, 10r+6\}$.

(11) $n = 24r+14$, $r \geq 2$, and $\ell = 21$.

For $r = 2$, let $R = O \setminus \{3, 23, 27, 29\}$. The triples are $\{3, 24, 27\}$, $\{4, 16, 20\}$, $\{6, 23, 29\}$, $\{8, 22, 30\}$, $\{10, 18, 28\}$, $\{12, 14, 26\}$.

For $r \geq 3$, let $R = \{3, 5, 4r-1, 4r+3, 4r+5, 6r+5, 8r+5, 10r+3, 10r+5, 12r+5\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+1-h$,	$6+2h$,	$6r+7+h$,	$[0, r-3]$;
$5r+3-h$,	$2r+2+2h$,	$7r+5+h$,	$[0, r-3]$;
$10r+1-h$,	$7+2h$,	$10r+8+h$,	$[0, r-3]$;
$9r+3-h$,	$2r+3+2h$,	$11r+6+h$,	$[0, r-3]$.

Other triples are $\{4, 10r+2, 10r+6\}$, $\{4r-2, 6r+6, 10r+4\}$, $\{4r, 8r+4, 12r+4\}$, $\{4r+1, 4r+2, 8r+3\}$, $\{4r+4, 6r+3, 10r+7\}$, $\{6r+2, 6r+4, 12r+6\}$.

(12) $n = 24r+14$, $r \geq 4$, and $\ell = 45$.

For $r = 4$, let $R = O \setminus \{17, 33, 39, 41\}$. The triples are $\{4, 40, 44\}$, $\{6, 33, 39\}$, $\{8, 28, 36\}$, $\{10, 22, 32\}$, $\{12, 42, 54\}$, $\{14, 38, 52\}$, $\{16, 34, 50\}$, $\{17, 24, 41\}$, $\{18, 30, 48\}$, $\{20, 26, 46\}$.

For $r = 5$, let $R = O \setminus \{5, 7, 9, 11, 13, 43, 51, 55, 59, 63\}$. The triples are $\{4, 6, 10\}$, $\{5, 46, 51\}$, $\{7, 48, 55\}$, $\{8, 12, 20\}$, $\{9, 50, 59\}$, $\{11, 52, 63\}$, $\{13, 43, 56\}$, $\{14, 26, 40\}$, $\{16, 38, 54\}$, $\{18, 42, 60\}$, $\{22, 44, 66\}$, $\{24, 34, 58\}$, $\{28, 36, 64\}$, $\{30, 32, 62\}$.

For $r \geq 6$, let $R = \{3, 5, 7, 9, 11, 4r-5, 4r-3, 4r+3, 6r+1, 6r+5, 6r+9, 8r+3, 8r+5, 8r+7, 10r-1, 10r+1, 10r+3, 10r+5, 10r+7, 10r+9, 12r+3, 12r+5\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-2-h$,	$12+2h$,	$6r+10+h$,	$[0, r-6]$;
$5r+3-h$,	$2r+2+2h$,	$7r+5+h$,	$[0, r-5]$;
$10r-2-h$,	$13+2h$,	$10r+11+h$,	$[0, r-6]$;
$9r+3-h$,	$2r+3+2h$,	$11r+6+h$,	$[0, r-5]$.

Other triples are $\{4, 6, 10\}$, $\{4r-4, 4r+6, 8r+2\}$, $\{4r-1, 4r+7, 8r+6\}$, $\{4r, 4r+4, 8r+4\}$, $\{4r-6, 6r+6, 10r\}$, $\{4r-2, 6r+8, 10r+6\}$, $\{4r+2, 6r+2, 10r+4\}$, $\{4r+5, 6r+3, 10r+8\}$, $\{8, 10r+2, 10r+10\}$, $\{4r+1, 8r+1, 12r+2\}$, $\{6r, 6r+4, 12r+4\}$, $\{6r-1, 6r+7, 12r+6\}$.

(13) $n = 24r+20$, $r \geq 1$, and $\ell = 9$.

For $r = 1$, let $R = \{9, 13, 17, 19\}$. The triples are $\{3, 8, 11\}$, $\{4, 16, 20\}$, $\{5, 10, 15\}$, $\{6, 12, 18\}$, $\{7, 14, 21\}$.

For $r \geq 2$, let $R = \{3, 10r+5, 12r+7, 12r+9\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+1-h$,	$5+2h$,	$6r+6+h$,	$[0, r-2]$;
$5r+2-h$,	$2r+3+2h$,	$7r+5+h$,	$[0, r-2]$;
$10r+3-h$,	$4+2h$,	$10r+7+h$,	$[0, r-2]$;
$9r+4-h$,	$2r+2+2h$,	$11r+6+h$,	$[0, r-2]$.

Other triples are $\{4r, 6r+4, 10r+4\}$, $\{4r+1, 6r+5, 10r+6\}$, $\{4r+2, 8r+4, 12r+6\}$, $\{4r+3, 8r+5, 12r+8\}$, $\{6r+2, 6r+3, 12r+5\}$.

(14) $n = 24r+22$, $r \geq 2$, and $\ell = 5$.

For $r \geq 2$, let $R = \{6r+5, 12r+9\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+3-h$,	$4+2h$,	$6r+7+h$,	$[0, r-2]$;
$5r+4-h$,	$2r+2+2h$,	$7r+6+h$,	$[0, r-1]$;
$10r+6-h$,	$3+2h$,	$10r+9+h$,	$[0, r-1]$;
$9r+6-h$,	$2r+3+2h$,	$11r+9+h$,	$[0, r-1]$.

Other triples are $\{4r+2, 6r+6, 10r+8\}, \{4r+3, 6r+4, 10r+7\}, \{4r+4, 8r+6, 12r+10\}$.

(15) $n = 24r+22$, $r \geq 1$, and $\ell = 17$.

For $r = 1$, let $R = O \setminus \{3, 15\}$. The triples are $\{3, 15, 18\}, \{4, 16, 20\}, \{6, 8, 14\}$, $\{10, 12, 22\}$.

For $r \geq 2$, let $R = \{3, 4r+1, 4r+5, 6r+5, 8r+5, 10r+7, 10r+9, 12r+9\}$. Then the triples as follows:

a_i	b_i	c_i	$h \in$
$6r+3-h$,	$4+2h$,	$6r+7+h$,	$[0, r-2]$;
$5r+4-h$,	$2r+2+2h$,	$7r+6+h$,	$[0, r-2]$;
$10r+5-h$,	$5+2h$,	$10r+10+h$,	$[0, r-2]$;
$9r+6-h$,	$2r+3+2h$,	$11r+9+h$,	$[0, r-2]$.

Other triples are $\{4r, 6r+6, 10r+6\}$, $\{4r+2, 8r+6, 12r+8\}$, $\{4r+3, 8r+7, 12r+10\}$, $\{4r+4, 6r+4, 10r+8\}$.

(16) $n = 24r+22$, $r \geq 2$, and $\ell = 29$.

For $r = 2$, let $R = O \setminus \{3, 29\}$. The triples are $\{3, 29, 32\}$, $\{4, 14, 18\}$, $\{6, 20, 26\}$, $\{8, 22, 30\}$, $\{10, 24, 34\}$, $\{12, 16, 28\}$.

For $r \geq 3$, let $R = \{3, 5, 4r-1, 4r+1, 4r+5, 6r+3, 6r+7, 8r+5, 8r+7, 10r+5, 10r+7, 10r+9, 12r+7, 12r+9\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r+2-h$,	$6+2h$,	$6r+8+h$,	$[0, r-3]$;
$5r+4-h$,	$2r+2+2h$,	$7r+6+h$,	$[0, r-3]$;
$10r+4-h$,	$7+2h$,	$10r+11+h$,	$[0, r-3]$;
$9r+6-h$,	$2r+3+2h$,	$11r+9+h$,	$[0, r-3]$.

Other triples are $\{4, 10r+6, 10r+10\}$, $\{4r-2, 4r+6, 8r+4\}$, $\{4r, 8r+8, 12r+8\}$, $\{4r+2, 4r+4, 8r+6\}$, $\{4r+3, 6r+5, 10r+8\}$, $\{6r+4, 6r+6, 12r+10\}$.

(17) $n = 24r+22$, $r \geq 3$, and $\ell = 41$.

For $r = 3$, let $R = O \setminus \{3, 37\}$. The triples are $\{3, 34, 37\}$, $\{4, 16, 20\}$, $\{6, 22, 28\}$, $\{8, 30, 38\}$, $\{10, 36, 46\}$, $\{12, 32, 44\}$, $\{14, 26, 40\}$, $\{18, 24, 42\}$.

For $r = 4$, let $R = O \setminus \{5, 7, 9, 11, 45, 49, 53, 57\}$. The triples are $\{4, 6, 10\}$, $\{5, 40, 45\}$, $\{7, 42, 49\}$, $\{8, 12, 20\}$, $\{9, 44, 53\}$, $\{11, 46, 57\}$, $\{14, 34, 48\}$, $\{16, 36, 52\}$, $\{18, 38, 56\}$, $\{22, 28, 50\}$, $\{24, 30, 54\}$, $\{26, 32, 58\}$.

For $r \geq 5$, let $R = \{3, 5, 4r-9, 4r-7, 4r-3, 4r-1, 4r+1, 6r-1, 6r+1, 6r+3, 8r-1, 8r+$

$1, 10r+1, 10r+3, 10r+5, 10r+7, 12r+3, 12r+5, 12r+7, 12r+9\}$. Then the triples are as follows:

a_i	b_i	c_i	$h \in$
$6r-2-h$,	$7+2h$,	$6r+5+h$,	$[0, r-5]$;
$5r+2-h$,	$2r-1+2h$,	$7r+1+h$,	$[0, r-5]$;
$10r-h$,	$8+2h$,	$10r+8+h$,	$[0, r-4]$;
$9r+3-h$,	$2r+2+2h$,	$11r+5+h$,	$[0, r-4]$.

Other triples are $\{4, 10r+2, 10r+6\}$, $\{6, 8r-3, 8r+3\}$, $\{4r-5, 4r+5, 8r\}$, $\{4r-4, 4r+2, 8r-2\}$, $\{4r-2, 8r+6, 12r+4\}$, $\{4r, 6r+4, 10r+4\}$, $\{4r+3, 8r+5, 12r+8\}$, $\{4r+4, 8r+2, 12r+6\}$, $\{4r+6, 8r+4, 12r+10\}$, $\{6r, 6r+2, 12r+2\}$. $\square$

Lemma 22. *A BSA*$^*(n, \{2,3\}; 2, \ell)$ *exists for each pair* $(n, \ell) \in \{(24r+4, 5), (24r+16, 11)\}$ *where* $r \geq 1$ *and* $r \geq 2$*, respectively.*

Proof. For each pair (n, ℓ), the triples and the set R are obtained by direct application given in Lemmas 20 and 21 (case 10), respectively.

For pairs $(n, \ell) = (24r+4, 5)$ where $r \geq 1$, insert a new odd element $12r+1$ into set R of Lemma 20.

For pairs $(n, \ell) = (24r+16, 11)$ where $r \geq 2$, inset a new odd element $12r+7$ into set R of Lemma 21 (case 10).

Finally apply Lemma 12, a BSA$^*(n, \{2,3\}; 2, \ell)$ exists for each pair (n, ℓ). $\square$

4 $(n, 4, 3)$-IV codes

In this section, we construct an $(n, 4, 3)$-IV code for all $n \geq 14$ with size $B(n)$, where the values of $B(n)$ are given in Conjecture 3.

First, we show the following recursive construction, which plays an important role in the construction of $(n, 4, 3)$-IV codes for even orders n.

Construction 23. *Let g and ℓ be the positive integers such that* $(g, \ell) \equiv (0,1), (2,3), (4,5) \pmod 6$*, $g > \ell$, and $g + \ell \geq 21$. Suppose that there exist the following:*

1. *a 3-IGDD of type $(g, 5)^u$ with index one,*
2. *a BSA$(5u, 3; 2)$,*
3. *a BSA$^*(g, \{2,3\}; 2, \ell)$, and*
4. *an $(g+\ell-1, 4, 3)$-IV code with size $B(g+\ell-1)$ such that three edges $\{\frac{g+\ell-3}{2}, \frac{g+\ell-1}{2}\}, \{\frac{g+\ell-5}{2}, \frac{g+\ell-1}{2}\}$ and $\{\frac{g+\ell-3}{2}, \frac{g+\ell+1}{2}\}$ must be contained in its leave graph,*

then there exists an $(gu+\ell-1, 4, 3)$-IV code with size $B(gu+\ell-1)$.

Proof. There exists a 3-IGDD of $(g,5)^u$ with index one, $(X,H,\mathcal{G},\mathcal{A})$, where $X=[u]\times\mathbb{Z}_g$ is a point set, $\mathcal{G}=\{G_i:1\le i\le u\}$ with $G_i=\{i\}\times\mathbb{Z}_g$ is a group set , and $H=[u]\times\{0,1,g-3,g-2,g-1\}$ is a hole set. For our convenience, each point (i,j) of X is denoted by i_j. Adjoin a set $Y=\{\infty_1,\infty_2,\dots,\infty_{\ell-1}\}$ of $\ell-1$ new points with X, and let $X'=X\cup Y$. We will construct an $(gu+\ell-1,4,3)$-IV code with size $B(n)$ on line X' depicted in Fig. 1 as follows:

For each $i\in[u]\backslash\lceil\frac{u}{2}\rceil$, on the cyclic group $G_i=(i_0,i_1,\dots,i_{g-1})$, we construct BSA$^*(g,\{2,3\};2,\ell)$ which is denoted by $(G_i,\mathcal{B}_i)$. By definition, each $\mathcal{B}_i=\mathcal{B}_i'\cup(\cup_{j\in[\ell]}\mathcal{P}_{i,j})$, where $\mathcal{B}_i'$ is a block set with size 3 and each $\mathcal{P}_{i,j}$, $j\in[\ell]$ is a parallel class of G_i with block size 2. For each $j\in[\ell-1]$, let $\mathcal{P}_{i,j}'=\{\{\infty_j\}\cup P:P\in\mathcal{P}_{i,j}\}$ and $\mathcal{P}_i'=\cup_{j\in[\ell-1]}\mathcal{P}_{i,j}'$. Finally, let $\mathcal{B}=\cup_{i\in[u]\backslash\lceil\frac{u}{2}\rceil}(\mathcal{B}_i'\cup\mathcal{P}_i')$. Here, we leave a parallel class $\mathcal{P}_{i,\ell}$ for each $i\in[u]\setminus\lceil\frac{u}{2}\rceil$, which will be included in the desired leave graph.

On the cyclic hole $H=(1_0,1_1,1_{g-3},1_{g-2},1_{g-1},2_0,2_1,2_{g-3},2_{g-2},2_{g-1},\dots,u_0,u_1,u_{g-3},u_{g-2},u_{g-1})$, we construct a BSA$(5u,3;2)$ and its block set $\mathcal{C}$.

For the middle group G_i where $i=\lceil\frac{u}{2}\rceil$, we construct an $(g+\ell-1,4,3)$-IV code with size $B(g+\ell-1)$ on line $G_i\cup Y$ with order $[\infty_{\frac{\ell-1}{2}},\dots,\infty_1,i_{\frac{g}{2}-1},\dots,i_1,i_0,i_{g-1},i_{g-2},\dots,i_{\frac{g}{2}},\infty_{\ell-1},\dots,\infty_{\frac{\ell+1}{2}}]$ and block set $\mathcal{D}$. Denote its leave graph by $(G_{\lceil\frac{u}{2}\rceil}\cup T,\mathcal{P}_{\lceil\frac{u}{2}\rceil})$, the edge set $\mathcal{P}_{\lceil\frac{u}{2}\rceil}=\mathcal{U}\cup\mathcal{V}$ such that $\mathcal{U}\cap\mathcal{V}=\phi$, where set $\mathcal{U}=\{\{\lceil\frac{u}{2}\rceil_0,\lceil\frac{u}{2}\rceil_{g-1}\},\{\lceil\frac{u}{2}\rceil_0,\lceil\frac{u}{2}\rceil_{g-2}\},\{\lceil\frac{u}{2}\rceil_1,\lceil\frac{u}{2}\rceil_{g-1}\}\}$, $|\mathcal{V}|=x$ and integer $x=((g+\ell-1)(g+\ell-2)-2B(g+\ell-1)-6)/2$.

Define a set $\mathcal{E}=\sum_{i=1}^{\lfloor\frac{u}{2}\rfloor}(\mathcal{E}_i^1\cup\mathcal{E}_i^2)\cup\sum_{i=\lceil\frac{u}{2}\rceil+1}^{u}(\mathcal{E}_i^3\cup\mathcal{E}_i^4)$, where $\mathcal{E}_i^1=\{\{i_{2j},i_{2j+1},i_{2j+2}\}:0\le j\le g/2-2\}$, $\mathcal{E}_i^2=\{\{i_{g-2},i_{g-1},(i+1)_0\}\}$, $\mathcal{E}_i^3=\{\{(i-1)_{g-1},i_0,i_1\}\}$, and $\mathcal{E}_i^4=\{\{i_{2j+1},i_{2j+2},i_{2j+3}\}:0\le j\le g/2-2\}$.

Define a set $\mathcal{W}=\sum_{i=1}^{\lfloor\frac{u}{2}\rfloor}(\mathcal{W}_i^1\cup\mathcal{W}_i^2)\cup\sum_{i=\lceil\frac{u}{2}\rceil+1}^{u}(\mathcal{W}_i^3\cup\mathcal{W}_i^4)$, where $\mathcal{W}_i^1=\{\{i_{2j+1},i_{2j+3}\}:0\le j\le g/2-2\}$, $\mathcal{W}_i^2=\{\{i_{g-1},(i+1)_1\}\}$, $\mathcal{W}_i^3=\{\{(i-1)_{g-2},i_0\}\}$, and $\mathcal{W}_i^4=\{\{i_{2j},i_{2j+2}\}:0\le j\le g/2-2\}$.

Let the block set be $\mathcal{M}=\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}$. We will prove that the pair $(X',\mathcal{M})$ is an $(gu+\ell-1,4,3)$-IV code with size $B(gu+\ell-1)$ over line X', and the set $\mathcal{P}:=(\cup_{i\in[u]\backslash\lceil\frac{u}{2}\rceil}\mathcal{P}_{i,\ell})\cup\mathcal{V}\cup\mathcal{W}\cup\mathcal{Y}$, where $\mathcal{Y}=\{\{1_0,u_{g-1}\},\{1_0,u_{g-2}\},\{1_1,u_{g-1}\}\}$, consists on all edges of its leave graph.

First, we prove that every pair $\{x,y\}$ on X' appears in at least one block of $\mathcal{M}\cup\mathcal{P}$. There arise the following four cases.

1. For $x\in G_i$, $y\in G_j$ such that $i\ne j$.

   If a pair $\{x,y\}$ is not contained in H, then a pair $\{x,y\}$ appears in one block of $\mathcal{A}$. If a pair $\{x,y\}$ is contained in H, then a pair $\{x,y\}$ with distance greater than two appears in one block of $\mathcal{C}$, while each pair within distance two appears either in $\mathcal{E}_i^2\subset\mathcal{E}$, $\mathcal{W}_i^2\subset\mathcal{W}\subset\mathcal{P}$, for some $i\in[1,\lfloor\frac{u}{2}\rfloor]$, $\mathcal{E}_i^3\subset\mathcal{E}$, $\mathcal{W}_i^3\subset\mathcal{W}\subset\mathcal{P}$, for some $i\in[\lceil\frac{u}{2}\rceil,u]$, or $\mathcal{Y}\subset\mathcal{P}$.

2. For $x,y\in G_i$, where $i\in[u]\setminus\lceil\frac{u}{2}\rceil$.

   If a pair $\{x,y\}$ is at distance greater than two on G_i, then it appears in one block of $\mathcal{B}_i$, that is, it appears either in a block of $\mathcal{B}_i'$ or a block in $\mathcal{P}_{i,j}$ for some $j\in[\ell]$. If $j\in[\ell-1]$, then there is a block of $\mathcal{P}_{i,j}'\subset\mathcal{B}$ containing $\{x,y\}$. If $j=\ell$, then there is a block of $\mathcal{P}_{i,\ell}\subset\mathcal{P}$ containing $\{x,y\}$.

   If a pair $\{x,y\}$ is within distance two on G_i, then each pair $\{x,y\}$ appears either in a block of $\mathcal{E}_i^1\subset\mathcal{E}$, $\mathcal{W}_i^1\subset\mathcal{W}\subset\mathcal{P}$, for some $i\in[1,\lfloor\frac{u}{2}\rfloor]$, $\mathcal{E}_i^4\subset\mathcal{E}$, or $\mathcal{W}_i^4\subset\mathcal{W}\subset\mathcal{P}$, for some $i\in[\lceil\frac{u}{2}\rceil,u]$

except three pairs $\{i_0, i_{g-1}\}$, $\{i_0, i_{g-2}\}$, and $\{i_1, i_{g-1}\}$. On cycle H, such three exceptional pairs have distance greater than two, therefore, they appear in some block of $\mathcal{C}$.

3. For $x, y \in G_{\lceil \frac{u}{2} \rceil} \cup Y$.

   Each pair $\{x, y\}$ appears in one block of $\mathcal{D}$ or in $\mathcal{P}_{\lceil \frac{u}{2} \rceil} \setminus \mathcal{U}$ except for three pairs of $\mathcal{U}$. As in the above case, such three exceptional pairs of $\mathcal{U}$ appear in some block of $\mathcal{C}$ because they have distance greater than two on the cycle H.

4. For $x \in G_i$, $i \in [u] \setminus \lceil \frac{u}{2} \rceil$ and $y = \infty_j$, $j \in [\ell - 1]$.

   We know that $\mathcal{P}_{i,j}$ is a parallel class of G_i, there exists a pair containing x, denoted by $\{x, z\}$. Then $\{x, z, \infty_j\}$ is a block of $\mathcal{P}'_{i,j} \subset \mathcal{B}$ containing $\{x, y\}$.

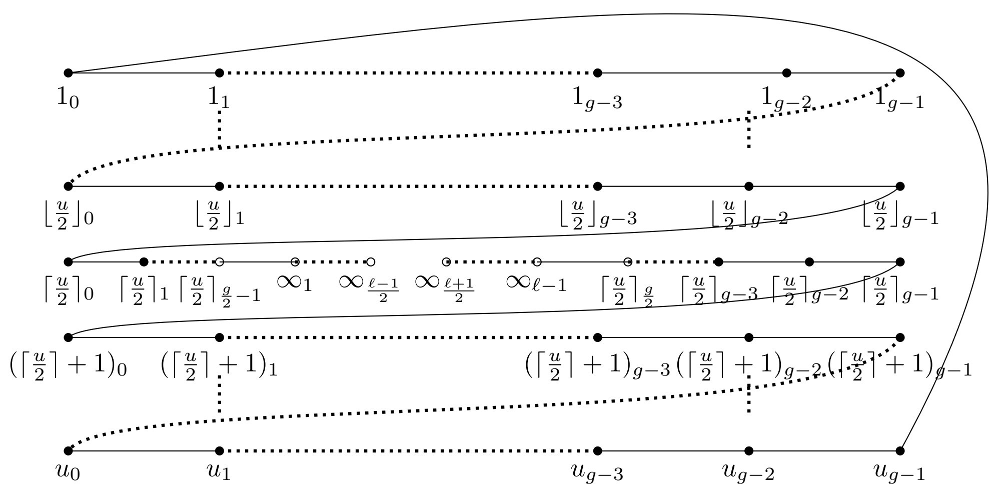

Figure 1: The set hole points are indicated by $\bullet$ and all others by $\circ$

Since there are $\frac{(gu+\ell-1)(gu+\ell-2)}{2}$ distinct pairs on X', if we prove that $\mathcal{M} \cup \mathcal{P}$ have the same number of pairs, then we will have shown that each such pair appears exactly once in $\mathcal{M} \cup \mathcal{P}$. That is, $\mathcal{P}$ is the leave graph of an $(gu + \ell - 1, 4, 3)$-IV code with block set $\mathcal{M}$. We show this by counting the number of blocks in $\mathcal{M}$ and $\mathcal{P}$.

We know that

$$|\mathcal{A}| = \frac{u(u-1)(g^2 - 5^2)}{6},$$

$$|\mathcal{B}'_i| = \frac{g(g - \ell - 5)}{6}| \text{ and } |\mathcal{P}'_i| = \frac{g(\ell - 1)}{2} \text{ for } i \in [u] \setminus \lceil \frac{u}{2} \rceil,$$

$$|\mathcal{C}| = \frac{25u(u-1)}{6},$$

$$|\mathcal{D}| = \frac{(g + \ell - 1)(g + \ell - 2) - 2(x + 3)}{6},$$

and

$$|\mathcal{E}| = \frac{g(u-1)}{2}$$

then

$$|\mathcal{M}| = \frac{(gu+\ell-1)(gu+\ell-2) - 2g(u-1) - 2(x+3)}{6}$$

and

$$|\mathcal{P}| = g(u-1) + x + 3.$$

Therefore, the total number of pairs they have in common is

$$3|\mathcal{M}| + |\mathcal{P}| = \frac{(gu+\ell-1)(gu+\ell-2)}{2}.$$

Thus, every pair of the set X' appears exactly once in $\mathcal{M} \cup \mathcal{P}$. Hence, the Hamming distance is at least four.

Second, we need to show that the blocks of the set $\mathcal{M}$ do not create the crosstalk of type-IV.

Let x, y, z be the any three consecutive coordinates on X'. Then, by the definition of crosstalk type-IV, we need to verify that the following occur:

1. If there exists a pair $\{x, z\}$ in a block and another coordinate not equal to y, then the coordinate y is not allowed to appear in any other blocks.
2. If y has distance more than two with other coordinates which are not equal to x and z in the same block, then the pair $\{x, z\}$ is not allowed in any other blocks.
3. A block of the form $\{x, y, z\}$ is permitted

Every coordinate of the set $X' \setminus G_{\lceil \frac{u}{2} \rceil}$ appears in at least one block of $\mathcal{M} \setminus \mathcal{D}$. So, for each pair $\{x, z\} \subset X' \setminus G_{\lceil \frac{u}{2} \rceil}$, we need to check the situations (2) and (3) in blocks of $\mathcal{M} \setminus \mathcal{D}$. A pair $\{x, z\}$ does not appear in any block of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ because each pair has distance greater than two in a block of these sets. A pair $\{x, z\}$ has appeared in blocks of $\mathcal{E}$ because its blocks are in $\{x, y, z\}$ form. While, the blocks of $\mathcal{D}$ over point set $G_{\lceil \frac{u}{2} \rceil}$ already avoided the crosstalk of type-IV. Hence, all blocks of $\mathcal{M}$ avoided the crosstalk type-IV. This completes the proof. $\square$

It is easy to check that the lower bound $B(n, 3; 2) + \lfloor \frac{n-1}{2} \rfloor$ for an $(n, 4, 3)$-IV code in Lemma 1 coincides with $B(n)$ and $B(n) - 1$ for each odd order $n \geq 15$ and even order $n \geq 14$, respectively. The values of $B(n, 3; 2)$ when $n \geq 14$ are already determined in [1, Theorem 5.1]. Therefore, we have the following result.

Lemma 24. *There exists an optimal* $(n, 4, 3)$*-IV code when* $n \equiv 1 \pmod 2$ *and* $n \geq 15$.

Now, we handle the case when $n \equiv 0 \pmod 2$ and $n \geq 14$. For some small orders, optimal sizes of an $(n, 4, 3)$-IV codes are listed in Table 2 and Table 3. All these codes are found by computer search and are available upon request. Note that the three pairs $\{n/2 - 2, n/2\}$, $\{n/2 - 1, n/2\}$, and $\{n/2 - 1, n/2 + 1\}$ appeared in their leave graph for each order n mentioned in Table 3. The codes of these small orders will be used in our recursive construction as initial terms.

n	14	16	22	50
$B(n)$	27	36	71	393

Table 2: Optimal sizes of an $(n, 4, 3)$-IV code.

n	18	20	24	26	28	30	32	34	36	38	40	42	44	46	48
$B(n)$	46	58	85	101	118	136	156	177	199	223	248	274	302	331	361

Table 3: Optimal sizes of an $(n, 4, 3)$-IV code for some small orders n.

Lemma 25. *There exists an optimal $(n, 4, 3)$-IV code when $n \equiv 0 \pmod 6$ and $n \geq 18$.*

Proof. For each $n \in [18, 48]$ and $n \equiv 0 \pmod 6$, there exists an $(n, 4, 3)$-IV code by Table 3. When $n \geq 54$ and $n \equiv 0 \pmod 6$, the proof is similar to that of [1, Lemma 4.8]. We just apply Construction 23 instead of [1, Construction 4.7] in the proof of [1, Lemma 4.8] and obtain the desired result. □

Lemma 26. *There exists an optimal $(n, 4, 3)$-IV code for each $n \in \{14, 16, 22, 50\}$.*

Proof. This follows by using Table 2. □

Lemma 27. *There exists an optimal $(n, 4, 3)$-IV code when $n \equiv 2, 4 \pmod 6$ and $n \geq 14$ except when $n \in \{14, 16, 22, 50\}$.*

Proof. For each $n \in \{20, 26, 28, 32, 34, 38, 40, 44, 46\}$, an optimal $(n, 4, 3)$-IV code exists from Table 3.

When $n \in \{52, 56, 58, 62, 64, 68, 70\}$, write $n = 3s + \ell - 1$ where pair $(s, \ell) \in \{(16, 5), (18, 3), (16, 11), (18, 9), (18, 11), (20, 9), (22, 5)\}$. For each s, there exists a 3-IGDD of type $(s, 5)^3$ from Lemma 7. For each pair (s, ℓ), a BSA$^*(s, \{2, 3\}; 2, \ell)$ comes from Lemmas 13, 15, 16, 18 and there exists an optimal $(s+\ell-1, 4, 3)$-IV code by above. A BSA$(15, 3; 2)$ exists by Lemma 6. By applying Construction 23, there exists an optimal $(n, 4, 3)$-IV code for each $n \in \{52, 56, 58, 62, 64, 68, 70\}$. When $n \geq 74$ and $n \equiv 2, 4 \pmod 6$, write $n = 3s + \ell - 1$ where values of parameters s and ℓ are listed in Table 4 except some orders of n. Now, we use induction on r. For $r = 1$, $n \in [74, 142]$ and $n \equiv 2, 4 \pmod 6$, write each $n = su + \ell - 1$. When $u = 3$, then pair $(s, \ell) \in \{(20, 15), (22, 11), (26, 3), (22, 17), (26, 9), (28, 5), (26, 15), (28, 11), (32, 3), (28, 17), (32, 9), (34, 5), (32, 15), (34, 11), (36, 11), (38, 9), (40, 5), (42, 5), (42, 17), (44, 9)\}$ and when $u = 7$ then pair $(s, \ell) \in \{(16, 5), (18, 3), (18, 9), (18, 11)\}$. For each s, a 3-IGDD of type $(s, 5)^u$, when $u = 3$ or 7 exists by Lemma 7. For each pair (s, ℓ), a BSA$^*(s, \{2, 3\}; 2, \ell)$ exists by Lemmas 13-22 and an optimal $(s + \ell - 1, 4, 3)$-IV code exists by above. For $u = 3$ or 7, there exists a BSA$(5u, 3; 2)$ by Lemma 6. Then applying Construction 23, an optimal $(n, 4, 3)$-IV code exists for each $n \in [74, 142]$ and $n \equiv 2, 4, \pmod 6$.

For exceptional cases when $r = 2, 3, 4, 5$ listed in Table 4, we have $n \in \{146, 148, 158, 166, 176, 178, 188, 190, 200, 208, 218, 220, 230, 250, 260, 262, 272, 292, 302, 334, 344, 416\}$. For each n, write $n = us+\ell-1$ where $u = 3$ or 7. When $u = 3$, pair $(s, t) \in \{(44, 15), (52, 11), (58, 5), (62, 3), (64, 17),$

n	$= 3s + (\ell - 1)$	s	ℓ	
$72r+2$	$= 3(24(r-1)+14)+32$	$24(r-1)+14$	33	$r \geq 1$, $r \neq 1,2,3$
$72r+4$	$= 3(24(r-1)+10)+46$	$24(r-1)+10$	47	$r \geq 1$, $r \neq 1,2,3,4$
$72r+8$	$= 3(24r+2)+2$	$24r+2$	3	$r \geq 1$
$72r+10$	$= 3(24(r-1)+22)+16$	$24(r-1)+22$	17	$r \geq 1$, $r \neq 1$
$72r+14$	$= 3(24(r-1)+14)+44$	$24(r-1)+14$	45	$r \geq 1$, $r \neq 1,2,3,4$
$72r+16$	$= 3(24r+4)+4$	$24r+4$	5	$r \geq 1$
$72r+20$	$= 3(24r+2)+14$	$24r+2$	15	$r \geq 1$, $r \neq 1$
$72r+22$	$= 3(24(r-1)+22)+28$	$24(r-1)+22$	29	$r \geq 1$, $r \neq 1,2$
$72r+26$	$= 3(24r+8)+2$	$24r+8$	3	$r \geq 1$
$72r+28$	$= 3(24r+4)+16$	$24r+4$	17	$r \geq 1$, $r \neq 1$
$72r+32$	$= 3(24r+2)+26$	$24r+2$	27	$r \geq 1$, $r \neq 1,2$
$72r+34$	$= 3(24(r-1)+22)+40$	$24(r-1)+22$	41	$r \geq 1$, $r \neq 1,2,3$
$72r+38$	$= 3(24r+8)+14$	$24r+8$	15	$r \geq 1$
$72r+40$	$= 3(24r+10)+10$	$24r+10$	11	$r \geq 1$
$72r+44$	$= 3(24r+2)+38$	$24r+2$	39	$r \geq 1$, $r \neq 1,2,3$
$72r+46$	$= 3(24r-2)+52$	$24r-2$	53	$r \geq 1$, $r \neq 1,2,3,4$
$72r+50$	$= 3(24r+14)+8$	$24r+14$	9	$r \geq 1$
$72r+52$	$= 3(24r+10)+22$	$24r+10$	23	$r \geq 1$, $r \neq 1$
$72r+56$	$= 3(24(r-1)+20)+68$	$24(r-1)+20$	69	$r \geq 1$, $r \neq 1,2,3,4,5$
$72r+58$	$= 3(24r+16)+10$	$24r+16$	11	$r \geq 1$, $r \neq 1$
$72r+62$	$= 3(24r+14)+20$	$24r+14$	21	$r \geq 1$, $r \neq 1$
$72r+64$	$= 3(24r+10)+34$	$24r+10$	35	$r \geq 1$, $r \neq 1,2$
$72r+68$	$= 3(24r+20)+8$	$24r+20$	9	$r \geq 1$
$72r+70$	$= 3(24r+22)+4$	$24r+22$	5	$r \geq 1$, $r \neq 1$

Table 4: Parameters for proof of Lemma 27.

$(68,15),(70,11),(74,9),(82,5),(86,3),(82,17),(86,15),(94,11),(98,9),(106,17),(110,15)\}$ and when $u=7$, pair $(s,\ell) \in \{(20,9),(22,5),(24,9),(26,9),(28,5),(58,11)\}$. For each s, having $u=3$ or 7, there exists a 3-IGDD of type $(s,5)^u$ by Lemma 7. For each pair (s,ℓ), a BSA$^*(s,\{2,3\};2,\ell)$ comes from Lemmas 16,17,18 and an optimal $(s+\ell-1,4,3)$ code just proved. For $u=3,7$, there exists a BSA$(5u,3;2)$ by Lemma 6. Then applying Construction 23, an optimal $(n,4,3)$-IV code exists for each n written above.

Assume that there exists an optimal $(n,4,3)$-IV code for all orders n, except $n \notin \{14,16,22,50\}$, listed in Table 4 with $1 \leq r \leq p$. Now, we prove the case $r=p+1$ except the 22 cases proved above. For each s, there exists a 3-IGDD of type $(s,5)^3$ from Lemma 7. For each pair (s,ℓ) corresponding to each class that mentioned in Table 4, there exits a BSA$^*(s,\{2,3\};2,\ell)$ by Lemmas 19-22 and an optimal $(s+\ell-1,4,3)$-IV code exists by assumption. From Lemma 6, there exists a BSA$(15,3;2)$. Then applying Construction 23, an optimal $(n,4,3)$-IV code exists for all classes when $r=p+1$.

Finally, an optimal $(n,4,3)$-IV code exists whenever $n \geq 14$ and $n \equiv 2,4 \pmod 6$, except if

$n \in \{14, 16, 22, 50\}$. □

5 Concluding remarks and open problems

Motivated by binary codes that cover the three features like, low-power consumption, error-correction, and avoiding crosstalks, simultaneously on-chip data buses, we have studied such binary codes as well. In [5], the authors presented the lower bounds of these codes, which avoid the crosstalk of type-III (or IV). In this paper, we present Conjecture 2 (or 3) for the upper bound of $(n, 4, 3)$-III (or IV) codes, respectively. By computer search, these conjectures are true for small orders, but their theoretical proofs are very difficult. However, our main contribution to this paper is summarized in the following two results.

Theorem 28. *There exists an optimal $(n, 4, 3)$-IV code for each integer $n \geq 14$.*

Proof. By combining the Lemmas 24-27, we have proof of this result. □

Theorem 29. *For $n \geq 3$ and $n \equiv 1 \pmod 2$, every optimal $(n, 4, 3)$-III code is also an optimal $(n, 4, 3)$-$\{III, IV\}$ code.*

Proof. The proof of this result follows directly from Lemma 10. □

There are still intriguing open problems in the underlying direction. For instance,

Problem 30. *Prove Conjectures 2 and 3.*

Problem 31. *Give a tight upper bound for the size of an $(n, 4, 3)$-III code when n is even and construct optimal codes.*

Finally, we close this section with the following remarks.

Remark 32. *If Conjecture 3 is true, then the optimal size of an $(n, 4, 3)$-IV code for all $n \geq 14$ is determined.*

Remark 33. *Conjecture 2 is confirmed for odd n in the range $3 \leq n \leq 19$ by [5]. If Conjecture 2 holds for all odd $n \geq 21$ and solution of Problem 31 exists, then an optimal size of an $(n, 4, 3)$-III code for all $n \geq 3$ can be determined.*

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Data Availability

The figures, tables and other data used to support this study are included within the article.

Conflict of Interest

The authors declare that there is no conflicts of interests regarding the publication of this paper.

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Appendices

Appendix A. There exists a BSA$^*(n, 2; 2, \ell)$ for each pair $(n, \ell) \in \{(16, 11), (20, 15), (22, 17)\}$ as below. Following each column is considered as one parallel class.

$(n, \ell) = (16, 11)$

{0, 3},	{0, 4},	{0, 5},	{0, 6},	{0, 7},	{0, 8},	{0, 9},	{0, 10},	{0, 11},	{0, 12},	{0, 13},
{1, 4},	{1, 5},	{1, 6},	{1, 7},	{1, 8},	{1, 9},	{1, 10},	{1, 11},	{1, 12},	{1, 13},	{1, 14},
{2, 5},	{2, 6},	{2, 7},	{2, 8},	{2, 10},	{2, 11},	{2, 12},	{2, 9},	{2, 13},	{2, 14},	{2, 15},
{6, 9},	{3, 7},	{3, 11},	{3, 12},	{3, 13},	{3, 10},	{3, 6},	{3, 14},	{3, 15},	{3, 9},	{3, 8},
{7, 10},	{8, 12},	{4, 12},	{4, 11},	{4, 14},	{4, 13},	{4, 15},	{4, 8},	{4, 9},	{4, 7},	{4, 10},
{8, 13},	{9, 13},	{8, 14},	{5, 13},	{5, 11},	{5, 12},	{5, 14},	{5, 15},	{5, 8},	{5, 10},	{5, 9},
{11, 14},	{10, 14},	{9, 15},	{9, 14},	{6, 15},	{6, 14},	{7, 13},	{6, 13},	{6, 10},	{6, 11},	{6, 12},
{12, 15}.	{11, 15}.	{10, 13}.	{10, 15}.	{9, 12}.	{7, 15}.	{8, 11}.	{7, 12}.	{7, 14}.	{8, 15}.	{7, 11}.

$(n, \ell) = (20, 15)$

{0, 3},	{0, 4},	{0, 5},	{0, 6},	{0, 7},	{0, 8},	{0, 9},	{0, 10},
{1, 4},	{1, 5},	{1, 6},	{1, 7},	{1, 8},	{1, 9},	{1, 10},	{1, 11},
{2, 5},	{2, 6},	{2, 7},	{2, 8},	{2, 9},	{2, 10},	{2, 12},	{2, 13},
{6, 9},	{3, 7},	{3, 8},	{3, 9},	{3, 6},	{3, 11},	{3, 13},	{3, 12},
{7, 10},	{8, 12},	{4, 9},	{4, 14},	{4, 15},	{4, 16},	{4, 17},	{4, 18},
{8, 11},	{9, 13},	{10, 13},	{5, 15},	{5, 14},	{5, 17},	{5, 16},	{5, 19},
{12, 15},	{10, 16},	{11, 16},	{10, 17},	{10, 18},	{6, 14},	{6, 15},	{6, 16},
{13, 17},	{11, 17},	{12, 18},	{11, 18},	{11, 19},	{7, 15},	{7, 18},	{7, 14},
{14, 18},	{14, 19},	{14, 17},	{12, 16},	{12, 17},	{12, 19},	{8, 19},	{8, 15},
{16, 19}.	{15, 18}.	{15, 19}.	{13, 19}.	{13, 16}.	{13, 18}.	{11, 14}.	{9, 17}.

{0, 11},	{0, 12},	{0, 13},	{0, 14},	{0, 15},	{0, 16},	{0, 17},
{1, 12},	{1, 13},	{1, 14},	{1, 15},	{1, 16},	{1, 17},	{1, 18},
{2, 14},	{2, 11},	{2, 15},	{2, 16},	{2, 17},	{2, 18},	{2, 19},
{3, 10},	{3, 14},	{3, 16},	{3, 17},	{3, 18},	{3, 19},	{3, 15},
{4, 13},	{4, 19},	{4, 7},	{4, 10},	{4, 12},	{4, 8},	{4, 11},
{5, 18},	{5, 8},	{5, 11},	{5, 12},	{5, 9},	{5, 10},	{5, 13},
{6, 17},	{6, 18},	{6, 12},	{6, 11},	{6, 19},	{6, 13},	{6, 10},
{7, 19},	{7, 17},	{8, 17},	{7, 13},	{7, 11},	{7, 12},	{7, 16},
{8, 16},	{9, 16},	{9, 18},	{8, 18},	{8, 13},	{9, 14},	{8, 14},
{9, 15}.	{10, 15}.	{10, 19}.	{9, 19}.	{10, 14}.	{11, 15}.	{9, 12}.

$(n, \ell) = (22, 17)$

{0, 3},	{0, 4},	{0, 5},	{0, 6},	{0, 7},	{0, 8},	{0, 9},	{0, 10},	{0, 11},
{1, 4},	{1, 5},	{1, 6},	{1, 7},	{1, 8},	{1, 9},	{1, 10},	{1, 11},	{1, 12},
{2, 5},	{2, 6},	{2, 7},	{2, 8},	{2, 9},	{2, 10},	{2, 11},	{2, 12},	{2, 13},
{6, 9},	{3, 7},	{3, 8},	{3, 9},	{3, 6},	{3, 11},	{3, 14},	{3, 15},	{3, 16},
{7, 10},	{8, 12},	{4, 9},	{4, 10},	{4, 11},	{4, 15},	{4, 16},	{4, 14},	{4, 17},
{8, 11},	{9, 13},	{10, 13},	{5, 11},	{5, 16},	{5, 18},	{5, 15},	{5, 19},	{5, 14},
{12, 15},	{10, 14},	{11, 17},	{12, 17},	{10, 17},	{6, 16},	{6, 17},	{6, 18},	{6, 15},
{13, 16},	{11, 18},	{12, 18},	{13, 19},	{12, 19},	{7, 19},	{7, 20},	{7, 16},	{7, 18},
{14, 19},	{15, 19},	{14, 20},	{14, 18},	{13, 20},	{12, 20},	{8, 19},	{8, 20},	{8, 21},
{17, 20},	{16, 20},	{15, 21},	{15, 20},	{14, 21},	{13, 21},	{12, 21},	{9, 21},	{9, 19},
{18, 21}.	{17, 21}.	{16, 19}.	{16, 21}.	{15, 18}.	{14, 17}.	{13, 18}.	{13, 17}.	{10, 20}.

{0, 12},	{0, 13},	{0, 14},	{0, 15},	{0, 16},	{0, 17},	{0, 18},	{0, 19},
{1, 13},	{1, 14},	{1, 15},	{1, 16},	{1, 17},	{1, 18},	{1, 19},	{1, 20},
{2, 14},	{2, 15},	{2, 16},	{2, 17},	{2, 18},	{2, 19},	{2, 20},	{2, 21},
{3, 10},	{3, 12},	{3, 13},	{3, 18},	{3, 19},	{3, 20},	{3, 21},	{3, 17},
{4, 18},	{4, 7},	{4, 8},	{4, 19},	{4, 20},	{4, 21},	{4, 12},	{4, 13},
{5, 17},	{5, 20},	{5, 21},	{5, 8},	{5, 12},	{5, 9},	{5, 13},	{5, 10},
{6, 19},	{6, 21},	{6, 12},	{6, 20},	{6, 10},	{6, 13},	{6, 11},	{6, 14},
{7, 15},	{8, 17},	{7, 17},	{7, 13},	{7, 21},	{7, 11},	{7, 14},	{7, 12},
{8, 16},	{9, 16},	{9, 18},	{9, 12},	{8, 13},	{8, 14},	{8, 15},	{8, 18},
{9, 20},	{10, 18},	{10, 19},	{10, 21},	{9, 14},	{10, 15},	{9, 17},	{9, 15},
{11, 21}.	{11, 19}.	{11, 20}.	{11, 14}.	{11, 15}.	{12, 16}.	{10, 16}.	{11, 16}.

Appendix B. BSA$^*(n, \{2, 3\}; 2, \ell)$s, where $(n, \ell) \in \{(18, 3), (18, 9), (18, 11)\}$. All triples and ℓ parallel classes (in column) are listed below.

$(n, \ell) = (18, 3)$

Triples	{0, 3, 14},	{0, 7, 15},	{1, 5, 11},	{1, 12, 16},	{2, 8, 14},	{3, 6, 10},	{3, 11, 15},	{4, 12, 15},	{5, 13, 17},	{7, 10, 14},
	{0, 4, 10},	{0, 8, 11},	{1, 8, 15},	{2, 5, 16},	{2, 9, 12},	{3, 7, 13},	{4, 8, 16},	{5, 8, 12},	{6, 9, 16},	{7, 11, 17},
	{0, 6, 12},	{1, 4, 7},	{1, 9, 13},	{2, 6, 13},	{2, 10, 17},	{3, 9, 17},	{4, 11, 14},	{5, 9, 15},	{6, 14, 17},	{10, 13, 16}.

Parallel classes	$\{2l, 5+2l\}$	$\{2l+1, 6+2l\}$	$\{l, 9+l\}$	where $l \in [0, 8]$

$(n, \ell) = (18, 9)$

Triples	{0, 3, 6},	{1, 4, 8},	{2, 5, 8},	{3, 11, 15},	{7, 12, 16},	{10, 13, 17},
	{0, 4, 7},	{1, 5, 9},	{2, 10, 14},	{6, 12, 15},	{9, 13, 16},	{11, 14, 17}.

Parallel classes	{0, 5},	{0, 8},	{0, 9},	{0, 10},	{0, 11},	{0, 12},	{0, 13},	{0, 14},	{0, 15},
	{1, 6},	{1, 7},	{1, 10},	{1, 11},	{1, 12},	{1, 13},	{1, 14},	{1, 15},	{1, 16},
	{2, 7},	{2, 11},	{2, 12},	{2, 6},	{2, 9},	{2, 15},	{2, 16},	{2, 17},	{2, 13},
	{3, 8},	{3, 12},	{3, 13},	{3, 14},	{3, 10},	{3, 9},	{3, 7},	{3, 16},	{3, 17},
	{4, 13},	{4, 14},	{4, 11},	{4, 12},	{4, 15},	{4, 16},	{4, 17},	{4, 10},	{4, 9},
	{9, 14},	{5, 13},	{5, 14},	{5, 15},	{5, 16},	{5, 17},	{5, 10},	{5, 12},	{5, 11},
	{10, 15},	{6, 17},	{6, 16},	{7, 13},	{6, 13},	{6, 10},	{6, 11},	{6, 9},	{6, 14},
	{11, 16},	{9, 15},	{7, 15},	{8, 16},	{7, 17},	{7, 14},	{8, 15},	{7, 11},	{7, 10},
	{12, 17}.	{10, 16}.	{8, 17}.	{9, 17}.	{8, 14}.	{8, 11}.	{9, 12}.	{8, 13}.	{8, 12}.

$(n, \ell) = (18, 11)$

Triples	{0, 3, 6},	{1, 4, 7},	{2, 5, 8},	{9, 12, 15},	{10, 13, 16},	{11, 14, 17}.

Parallel classes	{0, 4},	{0, 5},	{0, 7},	{0, 8},	{0, 9},	{0, 10},	{0, 11},	{0, 12},	{0, 13},	{0, 14},	{0, 15},
	{1, 5},	{1, 6},	{1, 8},	{1, 10},	{1, 11},	{1, 9},	{1, 12},	{1, 13},	{1, 14},	{1, 15},	{1, 16},
	{2, 6},	{2, 7},	{2, 9},	{2, 11},	{2, 10},	{2, 12},	{2, 13},	{2, 14},	{2, 15},	{2, 16},	{2, 17},
	{3, 7},	{3, 8},	{3, 12},	{3, 13},	{3, 14},	{3, 11},	{3, 9},	{3, 15},	{3, 16},	{3, 17},	{3, 10},
	{8, 11},	{4, 13},	{4, 14},	{4, 12},	{4, 15},	{4, 16},	{4, 10},	{4, 9},	{4, 17},	{4, 8},	{4, 11},
	{9, 14},	{9, 16},	{5, 13},	{5, 14},	{5, 12},	{5, 15},	{5, 16},	{5, 17},	{5, 11},	{5, 10},	{5, 9},
	{10, 15},	{10, 14},	{6, 15},	{6, 16},	{6, 13},	{6, 14},	{6, 17},	{6, 10},	{6, 9},	{6, 11},	{6, 12},
	{12, 16},	{11, 15},	{10, 17},	{7, 15},	{7, 16},	{7, 17},	{7, 14},	{7, 11},	{7, 10},	{7, 12},	{7, 13},
	{13, 17}.	{12, 17}.	{11, 16}.	{9, 17}.	{8, 17}.	{8, 13}.	{8, 15}.	{8, 16}.	{8, 12}.	{9, 13}.	{8, 14}.

Appendix C. A BSA$^*(n,\{2,3\};2,\ell)$, where $(n,\ell)\in\{(20,9),(22,11),(26,9),(26,15)\}$. For each (n,t), the baseblocks and all t parallel classes (in columns) are listed below. Each baseblocks of size three is developed in $\mathbb{Z}_n$.

$(n,\ell)=(20,9)$: Baseblock {0, 4, 9}.

Parallel classes	{0, 3},	{0, 6},	{0, 7},	{0, 8},	{0, 10},	{0, 12},	{0, 13},	{0, 14},	{0, 17},
	{1, 4},	{1, 7},	{1, 8},	{1, 9},	{1, 11},	{1, 13},	{1, 14},	{1, 15},	{1, 18},
	{2, 5},	{2, 8},	{2, 9},	{2, 12},	{2, 14},	{2, 10},	{2, 15},	{2, 16},	{2, 19},
	{6, 9},	{3, 9},	{3, 6},	{3, 13},	{3, 15},	{3, 11},	{3, 10},	{3, 17},	{3, 16},
	{7, 10},	{4, 14},	{4, 17},	{4, 16},	{4, 12},	{4, 18},	{4, 7},	{4, 11},	{4, 10},
	{8, 11},	{5, 13},	{5, 15},	{5, 17},	{5, 18},	{5, 19},	{5, 12},	{5, 8},	{5, 11},
	{12, 19},	{10, 17},	{10, 16},	{6, 14},	{6, 13},	{6, 16},	{6, 19},	{6, 18},	{6, 12},
	{13, 16},	{11, 18},	{11, 14},	{7, 15},	{7, 17},	{7, 14},	{8, 18},	{7, 19},	{7, 13},
	{14, 17},	{12, 15},	{12, 18},	{10, 18},	{8, 16},	{8, 15},	{9, 16},	{9, 12},	{8, 14},
	{15, 18}.	{16, 19}.	{13, 19}.	{11, 19}.	{9, 19}.	{9, 17}.	{11, 17}.	{10, 13}.	{9, 15}.

$(n,\ell)=(22,11)$: Baseblock {0, 4, 10}.

Parallel classes	{0, 3},	{0, 5},	{0, 7},	{0, 8},	{0, 9},	{0, 11},	{0, 13},	{0, 14},	{0, 15},	{0, 17},	{0, 19},
	{1, 4},	{1, 6},	{1, 8},	{1, 9},	{1, 10},	{1, 12},	{1, 14},	{1, 15},	{1, 16},	{1, 18},	{1, 20},
	{2, 5},	{2, 7},	{2, 9},	{2, 10},	{2, 11},	{2, 13},	{2, 15},	{2, 16},	{2, 19},	{2, 21},	{2, 17},
	{6, 9},	{3, 8},	{3, 6},	{3, 14},	{3, 16},	{3, 17},	{3, 10},	{3, 11},	{3, 18},	{3, 20},	{3, 12},
	{7, 10},	{4, 9},	{4, 11},	{4, 15},	{4, 17},	{4, 18},	{4, 12},	{4, 13},	{4, 7},	{4, 19},	{4, 21},
	{8, 11},	{10, 13},	{5, 10},	{5, 16},	{5, 14},	{5, 19},	{5, 18},	{5, 12},	{5, 20},	{5, 8},	{5, 13},
	{12, 15},	{11, 18},	{12, 19},	{6, 17},	{6, 15},	{6, 14},	{6, 19},	{6, 20},	{6, 21},	{6, 13},	{6, 11},
	{13, 16},	{12, 17},	{13, 20},	{7, 18},	{7, 20},	{7, 15},	{7, 16},	{7, 21},	{8, 13},	{7, 12},	{7, 14},
	{14, 19},	{14, 21},	{14, 17},	{11, 19},	{8, 19},	{8, 16},	{8, 21},	{8, 17},	{9, 12},	{9, 14},	{8, 15},
	{17, 20},	{15, 20},	{15, 18},	{12, 20},	{12, 21},	{9, 20},	{9, 17},	{9, 18},	{10, 17},	{10, 15},	{9, 16},
	{18, 21}.	{16, 19}.	{16, 21}.	{13, 21}.	{13, 18}.	{10, 21}.	{11, 20}.	{10, 19}.	{11, 14}.	{11, 16}.	{10, 18}.

$(n,\ell)=(26,9)$: Baseblocks {0, 4, 10}, {0, 5, 12}.

Parallel classes	{0, 3},	{0, 8},	{0, 9},	{0, 11},	{0, 13},	{0, 15},	{0, 17},	{0, 18},	{0, 23},
	{1, 4},	{1, 9},	{1, 10},	{1, 12},	{1, 14},	{1, 16},	{1, 18},	{1, 24},	{1, 19},
	{2, 5},	{2, 10},	{2, 11},	{2, 13},	{2, 15},	{2, 17},	{2, 19},	{2, 20},	{2, 25},
	{6, 9},	{3, 6},	{3, 12},	{3, 16},	{3, 18},	{3, 11},	{3, 14},	{3, 21},	{3, 20},
	{7, 10},	{4, 7},	{4, 13},	{4, 17},	{4, 19},	{4, 21},	{4, 15},	{4, 22},	{4, 12},
	{8, 11},	{5, 13},	{5, 18},	{5, 20},	{5, 16},	{5, 8},	{5, 22},	{5, 23},	{5, 14},
	{12, 15},	{11, 14},	{6, 19},	{6, 21},	{6, 17},	{6, 24},	{6, 23},	{6, 14},	{6, 15},
	{13, 24},	{12, 23},	{7, 20},	{7, 18},	{7, 22},	{7, 25},	{7, 16},	{7, 15},	{7, 24},
	{14, 17},	{15, 18},	{8, 21},	{8, 19},	{8, 23},	{9, 18},	{8, 25},	{8, 17},	{8, 16},
	{16, 19},	{16, 25},	{14, 22},	{9, 22},	{9, 20},	{10, 19},	{9, 24},	{9, 12},	{9, 17},
	{18, 21},	{17, 20},	{15, 23},	{10, 23},	{10, 21},	{12, 20},	{10, 13},	{10, 25},	{10, 18},
	{20, 23},	{19, 22},	{16, 24},	{14, 25},	{11, 24},	{13, 22},	{11, 20},	{11, 19},	{11, 22},
	{22, 25}.	{21, 24}.	{17, 25}.	{15, 24}.	{12, 25}.	{14, 23}.	{12, 21}.	{13, 16}.	{13, 21}.

$(n,\ell)=(26,15)$: Baseblock {0, 4, 10}.

Parallel classes	{0, 3},	{0, 5},	{0, 7},	{0, 8},	{0, 9},	{0, 11},	{0, 12},	{0, 13},
	{1, 4},	{1, 6},	{1, 8},	{1, 9},	{1, 10},	{1, 12},	{1, 13},	{1, 14},
	{2, 5},	{2, 7},	{2, 9},	{2, 10},	{2, 11},	{2, 13},	{2, 14},	{2, 15},
	{6, 9},	{3, 8},	{3, 6},	{3, 11},	{3, 12},	{3, 10},	{3, 16},	{3, 17},
	{7, 10},	{4, 9},	{4, 11},	{4, 7},	{4, 13},	{4, 15},	{4, 17},	{4, 16},
	{8, 11},	{10, 13},	{5, 10},	{5, 12},	{5, 18},	{5, 22},	{5, 19},	{5, 20},
	{12, 15},	{11, 14},	{12, 19},	{6, 19},	{6, 20},	{6, 18},	{6, 21},	{6, 23},
	{13, 16},	{12, 17},	{13, 20},	{13, 22},	{7, 21},	{7, 24},	{7, 18},	{7, 19},
	{14, 17},	{15, 18},	{14, 21},	{14, 23},	{8, 23},	{8, 21},	{8, 20},	{8, 22},
	{18, 21},	{16, 23},	{15, 22},	{15, 20},	{14, 19},	{9, 23},	{9, 22},	{9, 18},
	{19, 24},	{19, 22},	{16, 24},	{16, 21},	{15, 24},	{14, 25},	{10, 24},	{10, 21},
	{20, 23},	{20, 25},	{17, 25},	{17, 24},	{16, 25},	{16, 19},	{11, 25},	{11, 24},
	{22, 25}.	{21, 24}.	{18, 23}.	{18, 25}.	{17, 22}.	{17, 20}.	{15, 23}.	{12, 25}.
.	{0, 14},	{0, 15},	{0, 17},	{0, 18},	{0, 19},	{0, 21},	{0, 23},	
	{1, 15},	{1, 16},	{1, 18},	{1, 19},	{1, 20},	{1, 22},	{1, 24},	
	{2, 16},	{2, 17},	{2, 19},	{2, 20},	{2, 21},	{2, 23},	{2, 25},	
	{3, 18},	{3, 14},	{3, 15},	{3, 21},	{3, 22},	{3, 24},	{3, 20},	
	{4, 12},	{4, 18},	{4, 21},	{4, 22},	{4, 23},	{4, 25},	{4, 19},	
	{5, 17},	{5, 8},	{5, 23},	{5, 24},	{5, 13},	{5, 14},	{5, 16},	
	{6, 24},	{6, 25},	{6, 11},	{6, 13},	{6, 14},	{6, 17},	{6, 15},	
	{7, 20},	{7, 22},	{7, 16},	{7, 14},	{7, 25},	{7, 15},	{7, 12},	
	{8, 19},	{9, 20},	{8, 13},	{8, 25},	{8, 15},	{8, 16},	{8, 17},	
	{9, 21},	{10, 23},	{9, 24},	{9, 17},	{9, 16},	{9, 12},	{9, 14},	
	{10, 22},	{11, 19},	{10, 25},	{10, 15},	{10, 17},	{10, 19},	{10, 18},	
	{11, 23},	{12, 21},	{12, 20},	{11, 16},	{11, 18},	{11, 20},	{11, 22},	
	{13, 25}.	{13, 24}.	{14, 22}.	{12, 23}.	{12, 24}.	{13, 18}.	{13, 21}.	

Appendix D. For each pair $(n,\ell)\in\{(24,9),(32,9),(36,11),(40,5),(42,5),(42,17),(64,17)\}$, all blocks of size three are available upon request. Here, we only list below the set R corresponding to each pair (n,ℓ).

(n,ℓ)	R	(n,ℓ)	R
$(24,9)$	$\{3,5,9,11\}$	$(42,5)$	$\{5,11\}$
$(32,9)$	$\{5,9,11,13\}$	$(42,17)$	$\{5,7,9,11,13,15,17,19\}$
$(36,11)$	$\{5,7,9,11,13\}$	$(64,17)$	$\{5,7,15,19,23,25,29,31\}$
$(40,5)$	$\{5,11\}$		

Appendix E. For each pair (n,ℓ), the baseblocks, value of m, and the set S of a $\mathrm{BSA}^*(n,\{2,3\};2,\ell)$ are listed in the following table.

(n,ℓ)	**Baseblocks**	m	R
$(22,5)$	$\{0,4,12\}$, $\{0,9,15\}$, $\{1,5,13\}$, $\{0,6,13\}$.	2	$\{3,5\}$
$(28,5)$	$\{0,4,12\}$, $\{0,5,11\}$, $\{0,3,13\}$.	1	$\{7,9\}$.
$(34,5)$	$\{0,4,10\}$, $\{0,9,15\}$, $\{0,13,25\}$, $\{3,11,21\}$, $\{0,7,21\}$, $\{0,11,18\}$, $\{0,19,23\}$, $\{0,8,20\}$	2	$\{3,5\}$.
$(44,15)$	$\{0,4,10\}$, $\{1,11,24\}$, $\{2,17,25\}$, $\{1,27,39\}$, $\{0,20,38\}$, $\{3,33,39\}$, $\{2,18,31\}$, $\{0,8,23\}$, $\{3,27,43\}$, $\{0,30,34\}$, $\{3,13,26\}$, $\{0,12,26\}$, $\{0,16,29\}$, $\{1,17,21\}$, $\{1,13,31\}$, $\{2,26,34\}$.	4	$\{3,5,7,9,11,17,19\}$.
$(52,11)$	$\{0,4,10\}$, $\{0,17,30\}$, $\{0,19,35\}$, $\{1,26,40\}$, $\{0,25,29\}$, $\{2,12,30\}$, $\{2,22,37\}$, $\{3,25,31\}$, $\{0,20,34\}$, $\{3,18,26\}$, $\{1,19,39\}$, $\{3,7,32\}$, $\{3,9,24\}$, $\{1,22,41\}$, $\{3,13,42\}$, $\{3,11,30\}$, $\{2,19,32\}$, $\{3,34,40\}$, $\{0,12,33\}$, $\{0,28,36\}$, $\{3,33,43\}$, $\{1,9,25\}$, $\{1,15,33\}$, $\{2,14,18\}$.	4	$\{3,5,7,9,11\}$.
$(58,5)$	$\{0,4,10\}$, $\{3,7,23\}$, $\{2,40,49\}$, $\{3,10,29\}$, $\{3,47,55\}$, $\{0,17,44\}$, $\{3,21,40\}$, $\{1,10,36\}$, $\{2,17,38\}$, $\{1,25,37\}$, $\{3,13,28\}$, $\{1,34,46\}$, $\{3,26,54\}$, $\{0,8,24\}$, $\{0,18,45\}$, $\{1,18,29\}$.	3	$\{3,5\}$.
$(58,11)$	$\{0,3,25\}$, $\{0,4,11\}$, $\{0,6,15\}$, $\{0,7,27\}$, $\{0,8,26\}$, $\{0,10,24\}$, $\{0,12,28\}$.	1	$\{13,17,19,21,23\}$.
$(62,3)$	$\{0,4,9\}$, $\{0,29,39\}$, $\{0,37,43\}$, $\{1,13,27\}$, $\{3,21,41\}$, $\{0,13,53\}$, $\{2,36,54\}$, $\{0,48,54\}$, $\{0,11,26\}$, $\{0,16,49\}$, $\{1,5,35\}$, $\{3,46,58\}$, $\{2,24,47\}$, $\{1,6,48\}$, $\{0,21,32\}$, $\{0,17,24\}$, $\{0,25,41\}$, $\{0,27,35\}$.	2	$\{3\}$.
$(68,15)$	$\{0,4,10\}$, $\{0,8,56\}$, $\{3,27,50\}$, $\{1,7,15\}$, $\{3,29,56\}$, $\{1,31,48\}$, $\{0,16,46\}$, $\{3,22,48\}$, $\{3,14,39\}$, $\{0,18,24\}$, $\{3,49,67\}$, $\{2,38,50\}$, $\{3,55,61\}$, $\{3,23,45\}$, $\{2,12,25\}$, $\{3,13,30\}$, $\{0,22,50\}$, $\{3,24,52\}$, $\{2,44,55\}$, $\{1,12,55\}$, $\{2,16,54\}$, $\{0,17,36\}$, $\{1,39,51\}$, $\{2,17,29\}$, $\{1,33,49\}$, $\{2,15,66\}$, $\{0,27,55\}$, $\{2,26,45\}$, $\{1,41,45\}$, $\{0,14,25\}$, $\{0,45,53\}$, $\{2,10,57\}$.	4	$\{3,5,7,29,31,33\}$.
$(70,11)$	$\{0,4,10\}$, $\{2,38,59\}$, $\{1,34,52\}$, $\{2,41,49\}$, $\{0,8,51\}$, $\{1,25,46\}$, $\{2,43,60\}$, $\{1,21,39\}$, $\{3,37,43\}$, $\{3,31,57\}$, $\{3,28,60\}$, $\{2,16,46\}$, $\{2,25,29\}$, $\{0,15,29\}$, $\{0,16,33\}$, $\{0,20,42\}$, $\{1,11,23\}$, $\{0,24,55\}$.	2	$\{3,5,7,9,11\}$.
$(74,9)$	$\{0,4,10\}$, $\{2,30,53\}$, $\{3,36,67\}$, $\{2,17,71\}$, $\{3,51,72\}$, $\{3,11,61\}$, $\{3,28,42\}$, $\{0,26,38\}$, $\{3,44,63\}$, $\{0,39,43\}$, $\{2,29,47\}$, $\{1,35,50\}$, $\{3,25,31\}$, $\{0,20,42\}$, $\{3,35,65\}$, $\{0,29,56\}$, $\{0,17,55\}$, $\{0,13,30\}$, $\{0,21,34\}$, $\{0,16,24\}$.	2	$\{3,7,9,11\}$.
$(82,5)$	$\{0,4,10\}$, $\{1,7,62\}$, $\{1,10,48\}$, $\{2,49,66\}$, $\{2,41,53\}$, $\{0,43,57\}$, $\{3,19,40\}$, $\{2,39,52\}$, $\{2,51,69\}$, $\{3,29,53\}$, $\{0,46,53\}$, $\{1,50,61\}$, $\{3,23,33\}$, $\{1,45,49\}$, $\{3,57,76\}$, $\{2,31,77\}$, $\{0,8,34\}$, $\{0,16,71\}$, $\{0,19,59\}$, $\{2,22,62\}$, $\{0,13,70\}$, $\{3,11,70\}$, $\{0,14,31\}$, $\{0,30,54\}$.	2	$\{3,5\}$
$(82,17)$	$\{0,4,10\}$, $\{1,28,47\}$, $\{0,34,56\}$, $\{0,28,36\}$, $\{3,47,73\}$, $\{3,19,37\}$, $\{3,26,55\}$, $\{2,29,66\}$, $\{1,23,73\}$, $\{2,32,55\}$, $\{3,34,65\}$, $\{2,35,41\}$, $\{3,7,42\}$, $\{2,22,72,\}$ $\{0,14,35\}$, $\{3,31,45\}$, $\{2,39,63\}$, $\{0,25,44\}$, $\{0,49,57\}$, $\{2,26,42\}$.	2	$\{3,5,7,9,11,13,15,17\}$.
$(86,3)$	$\{0,4,9\}$, $\{3,44,67\}$, $\{1,31,56\}$, $\{1,38,80\}$, $\{0,18,29\}$, $\{1,7,46\}$, $\{2,14,28\}$, $\{1,8,72\}$, $\{3,21,50\}$, $\{2,10,64\}$, $\{2,32,68\}$, $\{0,35,75\}$, $\{2,53,61\}$, $\{3,53,73\}$, $\{1,6,75\}$, $\{2,27,65\}$, $\{2,29,78\}$, $\{0,52,58\}$, $\{1,45,54\}$, $\{0,48,65\}$, $\{3,13,75\}$, $\{0,16,71\}$, $\{0,13,73\}$, $\{0,19,53\}$, $\{1,5,33\}$, $\{2,48,69\}$.	2	$\{3\}$.
$(86,15)$	$\{0,4,10\}$, $\{1,18,58\}$, $\{1,31,50\}$, $\{2,27,37\}$, $\{2,21,52\}$, $\{2,54,76\}$, $\{2,32,53\}$, $\{3,28,81\}$, $\{1,28,67\}$, $\{0,17,41\}$, $\{0,27,63\}$, $\{1,13,73\}$, $\{0,45,49\}$, $\{3,42,56\}$, $\{2,40,56\}$, $\{0,24,68\}$, $\{2,28,59\}$, $\{0,23,65\}$, $\{3,25,71\}$, $\{1,7,59\}$, $\{1,17,49\}$, $\{2,10,68\}$.	2	$\{3,5,7,9,11,13,15\}$.
$(94,11)$	$\{0,4,10\}$, $\{1,7,20\}$, $\{3,55,80\}$, $\{3,62,70\}$, $\{0,30,46\}$, $\{1,21,38\}$, $\{1,19,70\}$, $\{1,32,56\}$, $\{3,33,69\}$, $\{2,35,78\}$, $\{2,60,73\}$, $\{1,17,80\}$, $\{3,24,56\}$, $\{3,35,89\}$, $\{2,62,82\}$, $\{1,30,51\}$, $\{1,34,72\}$, $\{1,36,73\}$, $\{2,21,69\}$, $\{1,42,91\}$, $\{1,27,83\}$, $\{0,40,82\}$, $\{0,44,66\}$, $\{1,35,50\}$, $\{1,11,25\}$, $\{0,29,68\}$.	2	$\{3,5,7,9,11\}$.
$(98,9)$	$\{0,4,10\}$, $\{1,32,64\}$, $\{1,36,81\}$, $\{0,61,75\}$, $\{1,20,79\}$, $\{0,43,60\}$, $\{1,5,48\}$, $\{1,17,87\}$, $\{3,28,59\}$, $\{3,13,75\}$, $\{1,42,69\}$, $\{2,44,88\}$, $\{1,25,77\}$, $\{3,57,72\}$, $\{3,61,69\}$, $\{2,16,21\}$, $\{3,16,37\}$, $\{1,28,51\}$, $\{2,26,74\}$, $\{0,58,76\}$, $\{0,17,78\}$, $\{0,13,46\}$, $\{1,66,74\}$, $\{0,30,69\}$, $\{0,16,93\}$, $\{1,7,58\}$, $\{0,15,53\}$, $\{2,30,66\}$.	2	$\{3,7,9,11\}$.
$(106,17)$	$\{0,4,10\}$, $\{3,24,43\}$, $\{1,52,82\}$, $\{1,20,62\}$, $\{3,21,52\}$, $\{3,72,88\}$, $\{2,28,50\}$, $\{3,61,105\}$, $\{0,60,88\}$, $\{1,72,84\}$, $\{2,36,83\}$, $\{0,61,67\}$, $\{2,42,75\}$, $\{0,14,82\}$, $\{0,27,77\}$, $\{3,11,78\}$, $\{3,13,68\}$, $\{3,63,85\}$, $\{0,59,71\}$, $\{1,53,79\}$, $\{3,44,93\}$, $\{0,50,79\}$, $\{3,37,79\}$, $\{1,39,71\}$, $\{0,8,70\}$, $\{0,20,52\}$, $\{0,69,83\}$, $\{1,44,87\}$.	2	$\{3,5,7,9,11,13,15,17\}$.
$(110,15)$	$\{0,4,10\}$, $\{0,8,52\}$, $\{0,14,94\}$, $\{0,48,60\}$, $\{0,17,83\}$, $\{0,64,85\}$, $\{1,55,59\}$, $\{3,68,90\}$, $\{0,24,49\}$, $\{2,20,39\}$, $\{0,28,57\}$, $\{1,39,58\}$, $\{0,39,79\}$, $\{0,32,65\}$, $\{2,22,58\}$, $\{0,27,35\}$, $\{0,34,77\}$, $\{3,26,66\}$, $\{2,53,70\}$, $\{0,38,69\}$, $\{0,61,71\}$, $\{0,26,67\}$, $\{1,15,52\}$, $\{1,30,105\}$, $\{1,23,43\}$, $\{1,22,85\}$, $\{3,63,79\}$, $\{1,29,93\}$, $\{1,13,37\}$, $\{1,31,63\}$.	2	$\{3,5,7,9,11,13,15\}$.