CODEGREES AND ELEMENT ORDERS OF ALMOST SIMPLE GROUPS

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Abstract. G. Qian proposed a conjecture which states that if an element of a finite group G has order a, then there exists an irreducible character of codegree b of G such that a divides b. He showed that the conjecture holds for solvable groups. In this note, we settle the conjecture for almost simple groups.

1. Introduction

Let G be a finite group. For an irreducible character χ of G, define its codegree by

$$\operatorname{cod}(\chi) = \frac{|G : \ker \chi|}{\chi(1)}.$$

Many authors have studied the set of character codegrees of a finite group and how it influences the structure of the group. One of the most interesting investigations has been the connection between character codegrees and element orders of a finite group. Let g be an element of G and denote by $\operatorname{ord}(g)$, the order of g. It was shown by G. Qian in [16, Theorem 1.1] that if a solvable group G has an element g, then there exists $\chi \in \operatorname{Irr}(G)$ such that p divides $\operatorname{cod}(\chi)$ for every prime divisor p of $\operatorname{ord}(g)$. This was generalized by I. M. Isaacs [14, Theorem] to all finite groups and the proof does not rely on the classification of finite simple groups. In [17], Qian proposed the following conjecture which, if true, will be stronger than the result of Isaacs:

Conjecture 1.1. [17, Conjecture A] *For every element g of a finite group G, there is some $\chi \in \operatorname{Irr}(G)$ such that $\operatorname{ord}(g)$ divides $\operatorname{cod}(\chi)$.*

It was shown in [17] that the conjecture holds for finite solvable groups. Giannelli [11] has recently shown that the conjecture holds for alternating and symmetric groups. In this note, we prove the following:

Theorem A. *Conjecture 1.1 holds for finite almost simple groups.*

Our proof relies on the classification of finite simple groups. For most of the cases of groups of Lie type, the appropriate characters are unipotent characters. The classification of orders of elements of almost simple groups is incomplete (see [7] for some results in this direction). The proof of our main result does not need this classification.

2. Preliminary results

In this section we list a couple of useful observations. We also show that the conjecture holds for an almost simple group whose socle is a sporadic simple group or an alternating group.

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Lemma 2.1. *Let N be a normal subgroup of a finite group G and let $x \in G$. Then* $\mathrm{ord}(x) = rs$ *where* $s = \mathrm{ord}(g)$ *for some* $g \in N$ *and* $r \mid |G/N|$.

Proof. If $x \in N$, then the result follows easily. If $x \in G \setminus N$, then we have that $(xN)^r \in N$ for some integer r and hence the result follows. □

Lemma 2.2. [16, Proposition 2.3] *Let p be a prime. For every p-element g of a finite group G, there is some $\chi \in \mathrm{Irr}(G)$ such that* $\mathrm{ord}(g)$ *divides* $\mathrm{cod}(\chi)$.

Note that all non-trivial irreducible characters χ of a simple group G are faithful and hence $\mathrm{cod}(\chi) = \dfrac{|G|}{\chi(1)}$.

Theorem 2.3. *Conjecture 1.1 holds for almost simple groups whose socle is either a sporadic simple group, ${}^2\mathrm{F}_4(2)'$ or an alternating group.*

Proof. Checking the character tables in the Atlas [9], we have that the result holds for sporadic simple groups, ${}^2\mathrm{F}_4(2)'$ and their respective almost simple groups. Suppose that the socle is an alternating group. Then the result follows from [11, Corollary B] with the exception of A_6, which we check its respective character tables in the Atlas [9]. This concludes our proof. □

Lemma 2.4. *Let p be a prime. Let G be a finite simple group of Lie type of characteristic p and $g \in G$. If g is p-regular, then the Steinberg character χ of G is such that* $\mathrm{ord}(g) \mid \mathrm{cod}(\chi)$.

Proof. This follows easily. □

Hence for finite groups of Lie type of characteristic p, it is sufficient to consider p-singular elements which are not p-elements by Lemmas 2.2 and 2.4.

3. Classical groups

Let $(n,m) = \gcd(n,m)$ for some positive integers m,n. Let $[n,m] = \mathrm{lcm}(n,m)$, $\omega_p(G)$, the set of orders of p-singular elements of G. By $\epsilon \in \{\pm 1\}$, we also mean $\epsilon \in \{+,-\}$ where appropriate. Table 3.1 below has the appropriate character degrees of some classical groups of Lie type that we shall use in our arguments. These character degrees can be found in [8, Section 13.8].

3.1. **Linear and unitary groups.** The spectra of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ are well known and we list the orders of p-singular elements below:

Lemma 3.1. *Let $n \geqslant 2$ and let q be a power of a prime p, $\epsilon \in \{\pm 1\}$. Let $G \cong \mathrm{PSL}_n^\epsilon(q)$ and $d = (n, q-1)$. Then $\omega_p(G)$ consists of all divisors of the following numbers:*

(i) *$p^t(q^{n_1} - \epsilon^{n_1})/d$ for $t, n_1 > 0$ such that $p^{t-1} + 1 + n_1 = n$,*
(ii) *$p^t[q^{n_1} - \epsilon^{n_1}, \dots, q^{n_s} - \epsilon^{n_s}]$, where $s \geqslant 2$, $t, n_i > 0$ such that $p^{t-1} + 1 + n_1 + \cdots + n_s = n$,*
(iii) *p^t if $p^{t-1} + 1 = n$ for $t > 0$.*

Proof. This follows from [2, Corollary 3]. □

Table 3.1 Character degrees of classical groups of Lie type

G	Labels	Degrees
$\mathrm{A}_n(q)$, $n \geqslant 3$	$(1, n)$	$\frac{q(q^n-1)}{q-1}$
${}^2\mathrm{A}_n(q^2)$, $n \geqslant 3$	$(1, n)$	$\frac{q(q^n-(-1)^n)}{q+1}$
$\mathrm{B}_n(q)$, $\mathrm{C}_n(q)$, $n \geqslant 2$	$\begin{pmatrix} 1 & n \\ 0 & \end{pmatrix}$	$\frac{q(q^n+1)(q^{n-1}-1)}{2(q-1)}$
	$\begin{pmatrix} 0 & n \\ 1 & \end{pmatrix}$	$\frac{q(q^n-1)(q^{n-1}+1)}{2(q-1)}$
$\mathrm{D}_n(q)$, $n \geqslant 4$	$\begin{pmatrix} n-1 \\ 1 \end{pmatrix}$	$\frac{q(q^{n-2}+1)(q^n-1)}{q^2-1}$
	$\begin{pmatrix} 1 & 2 & n \\ 0 & 1 & 2 \end{pmatrix}$	$\frac{q^6(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)(q^4-1)}$
${}^2\mathrm{D}_n(q^2)$, $n \geqslant 4$	$\begin{pmatrix} 1 & n-1 \\ & - \end{pmatrix}$	$\frac{q(q^{n-2}-1)(q^n+1)}{q^2-1}$
	$\begin{pmatrix} 0 & 1 & 2 & n \\ & 1 & 2 & \end{pmatrix}$	$\frac{q^6(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)(q^4-1)}$

Theorem 3.2. *Conjecture 1.1 holds when $G \cong \mathrm{PSL}_n^\epsilon(q)$.*

Proof. Let $G \cong \mathrm{PSL}_2(q)$ and let $g \in G$ be a p-singular element. It means that $\mathrm{ord}(g) = p$. Using Lemma 2.2, our result follows.

Let $G \cong \mathrm{PSL}_3(q)$ and let $g \in G$ be a p-singular element. Then either $\mathrm{ord}(g) = p$ or $\mathrm{ord}(g) = p(q-1)$ (and $\mathrm{ord}(g) = p^2$ if $p = 2$). In all these cases the appropriate character of G is the unipotent character of degree $q(q^2+q+1)$.

Let $G \cong \mathrm{PSU}_3(q)$ and let $g \in G$ be a p-singular element. Then either $\mathrm{ord}(g) = p$ or $\mathrm{ord}(g) = p(q+1)$ (and $\mathrm{ord}(g) = p^2$ if $p = 2$). In all these cases the appropriate character of G is the unipotent character of degree $q(q^2-q+1)$.

Suppose that $G \cong \mathrm{PSL}_n^\epsilon(q)$, $n \geqslant 4$ and let $g \in G$ be a p-singular element. The largest possible p-power order of $\mathrm{ord}(g)$ is in the case Lemma 3.1(iii) when $p^{t-1} = n-1$. Since the p-part of $|G|$ is $q^{\frac{(n)(n-1)}{2}}$, we may consider $q^{\frac{n(n-1)}{2}}/p^t$. We may assume that $p = q$. Since $t \leq p^{t-1} = n-1$, we have that $\frac{n(n-1)}{2} - n + 1 = (n^2-3n+2)/2 \geq 3$ for $n \geq 4$. This means that our character degree may have a p-part up to q^3.

Let $G \cong \mathrm{PSL}_n^+(q)$. Note that the largest possible degree of its p'-part polynomial of $\mathrm{ord}(g)$ is when $t = 1$ in Lemma 3.1(i), that is when, $\mathrm{ord}(g) = p(q^{n-2}-1)$. In particular, no element of G has order divisible by $\frac{q^{n-1}-1}{q-1}$. Hence the unipotent character of degree $\frac{q(q^{n-1}-1)}{q-1}$ is such that $\mathrm{ord}(g) | \mathrm{cod}(\chi)$ as required.

Suppose that $G \cong \mathrm{PSL}_n^-(q)$. The largest possible degree of its p'-part polynomial of $\mathrm{ord}(g)$ is when $\mathrm{ord}(g) = p(q^{n-2}-(-1)^{n-2})$. Then no element of G has order divisible by $\frac{q^{n-1}-(-1)^{n-1}}{q-1}$. Hence the unipotent character of degree $\frac{q(q^{n-1}-(-1)^{n-1})}{q-1}$ is such that $\mathrm{ord}(g) | \mathrm{cod}(\chi)$. □

3.2. **Symplectic and Orthogonal groups.** The next set of results lists the orders of p-singular elements of symplectic and orthogonal groups.

Lemma 3.3. *Let $G \cong \mathrm{PSp}_{2n}(q)$, where $n \geqslant 2$ and let q be a power of an odd prime p. Then $\omega_p(G)$ consists of all divisors of the following numbers:*

(i) $p^t[q^{n_1}+\epsilon_1, q^{n_2}+\epsilon_2, \ldots, q^{n_s}+\epsilon_s]$, *where* $s \geqslant 1$, $\epsilon_j \in \{\pm 1\}$ *and* $t, n_i > 0$ *with* $p^{t-1}+1+2n_1+2n_2+\cdots+2n_s = 2n$,

(ii) p^t *if* $p^{t-1}+1 = 2n$ *for some* $t > 1$.

Proof. This follows from [3, Corollary 2]. □

Lemma 3.4. *Let q be a power of 2 and let $G \cong \mathrm{Sp}_{2n}(q) \cong \Omega_{2n+1}(q)$, where $n \geqslant 2$. Then $\omega_2(G)$ consists of all divisors of the following numbers:*

(i) $2[q^{n_1}+\epsilon_1, q^{n_2}+\epsilon_2, \ldots, q^{n_s}+\epsilon_s]$ *for all* $s \geqslant 1$, $\epsilon_j \in \{\pm 1\}$ *and* $n_i > 0$ *with* $n_1+n_2+\cdots+n_s = n-1$,

(ii) $2^t[q^{n_1}+\epsilon_1, q^{n_2}+\epsilon_2, \ldots, q^{n_s}+\epsilon_s]$, *where* $s \geqslant 1$, $t \geqslant 2$, $\epsilon_j \in \{\pm 1\}$ *and* $n_i > 0$ *with* $2^{t-2}+1+n_1+n_2+\cdots+n_s = n$,

(iii) 2^t *if* $2^{t-2}+1 = n$ *for some* $t > 1$.

Proof. This follows from [3, Corollary 3]. □

Lemma 3.5. *Let q be a power of an odd prime p and let $G \cong \Omega_{2n+1}(q)$, where $n \geqslant 3$. Then $\omega_p(G)$ consists of all divisors of the following numbers:*

(i) $p^t(q^{n_1} \pm 1)/2$ *for all* t *and* n_1 *with* $p^{t-1}+1+2n_1 = 2n$,

(ii) $p^t[q^{n_1}+\epsilon_1, q^{n_2}+\epsilon_2, \ldots, q^{n_s}+\epsilon_s]$ *for all* $s \geqslant 2$, $\epsilon_i \in \{\pm 1\}$ *and* $n_i > 0$ *with* $p^{t-1}+1+2n_1+2n_2+\cdots+2n_s = 2n$,

(iii) p^t *if* $n = p^{t-1}+1$ *for some* $t \geqslant 1$.

Proof. This follows from [3, Corollary 6]. □

Theorem 3.6. *Let $\mathcal{L} = \{\mathrm{PSp}_{2n}(q), n \geqslant 2\} \cup \{\Omega_{2n+1}(q), n \geqslant 3\}$. Then Conjecture 1.1 holds when $G \in \mathcal{L}$.*

Proof. Suppose that $G \cong \mathrm{PSp}_4(q)$ and let g be a p-singular element. Then $\mathrm{ord}(g)$ is given in Lemmas 3.4 and 3.3. In particular, $\mathrm{ord}(g)$ is a divisor of $p(q \pm 1)$ (and p^2 when $p = 2$ or $p = 3$). By considering $|G|$, we have that the unipotent character of degree $\frac{q(q^2+1)}{2}$ will give us our result.

Suppose that $G \cong \mathrm{PSp}_{2n}(q)$, $n \geqslant 3$ and let g be a p-singular element. Let us consider the largest possible p-power order of g. Then $\mathrm{ord}(g)$ is given in Lemmas 3.4 and 3.3. Hence this happens in the cases in Lemma 3.3(ii) and Lemma 3.4(iii) when $p^{t-1} = 2n-1$ for odd p and $2^{t-2} = n-1$. Let us consider q^{n^2}/p^t and $q^{n^2}/2^t$. We may assume that $q = p$. Since $t < p^{t-1} = 2n-1$ and $t < 2^t < n-1$ we have that $n^2 - t > n^2 - 2n + 1 \geqslant 4$ and $n^2 - t > n^2 - n + 1 \geqslant 7$. Hence the p-part of our character degree maybe up to q^4.

We now consider the p'-part of $\mathrm{ord}(g)$. The highest possible degree of the p'-part polynomial of $\mathrm{ord}(g)$ is when $t = 1$ and $n_s = n_1$ for odd p, and $n_s = n_1$ for $p = 2$, that is, $q^{n-1} \pm 1$. In particular, no value of $\mathrm{ord}(g)$ is divisible by $q^n - 1$ or $q^n + 1$. Hence appropriate characters are unipotent characters of degree $\frac{q(q^n-1)(q^{n-1}+1)}{2(q-1)}$ and $\frac{q(q^n+1)(q^{n-1}-1)}{2(q-1)}$, respectively.

Suppose that $G \cong \Omega_{2n+1}(q)$, where $n \geqslant 3$ and q is a power of an odd prime. Let g be a p-singular element. Then $\mathrm{ord}(g)$ is given in Lemma 3.5. The largest possible p-part of $\mathrm{ord}(g)$ is when $n_s = n_1$ in case (i), that is, when $p^{t-1} = 2n-3$. Let us consider q^{n^2}/p^t and assume that $q = p$. Since $t \leqslant p^{t-1} = 2n-3$, $n^2 - t \geqslant n^2 - 2n + 3 \geqslant 3$ for $n \geqslant 3$. Hence the p-part of our character degree maybe up to q^6.

Let us consider the p'-part of $\mathrm{ord}(g)$. The highest possible degree of the p'-part polynomial of $\mathrm{ord}(g)$ is when $t = 1$, that is, when $(q^{n-1} \pm 1)/2$. Arguing as above, we have our result. □

Lemma 3.7. *Let q be a power of 2 and let $G \cong \Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$. Then $\omega_2(G)$ consists of all divisors of the following numbers:*

(i) $2^t[q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 1$, $r \geqslant 1$ *and* $n_i > 0$ *with* $2^{t-2} + 2 + n_1 + n_2 + \cdots + n_s = n$,

(ii) $2[q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 1$, $r \geqslant 1$ *and* $n_i > 0$ *with* $2 + n_1 + n_2 + \cdots + n_s = n$,

(iii) $2[q \pm 1, q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 1$ *and* $n_i > 0$ *with* $2 + n_1 + n_2 + \cdots + n_s = n$ *and* r *is even if* $\epsilon = +$ *and odd if* $\epsilon = -$,

(iv) $4[q-1, q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}+1, q^{n_{r+2}}+1, \dots, q^{n_s}+1]$ *for all* $s \geqslant 1$ *and* $n_i > 0$ *with* $3 + n_1 + n_2 + \cdots + n_s = n$ *and* r *is even if* $\epsilon = +$ *and odd if* $\epsilon = -$,

(v) $4[q+1, q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 1$ *and* $n_i > 0$ *with* $3 + n_1 + n_2 + \cdots + n_s = n$ *and* r *is even if* $\epsilon = +$ *and odd if* $\epsilon = -$,

(vi) 2^t *if* $n = 2^{t-2} + 2$ *for some* $t \geqslant 2$.

Proof. This follows from [3, Corollary 4]. □

Lemma 3.8. *Let q be a power of an odd prime p and let $G \cong \mathrm{P}\Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$, $\epsilon \in \{\pm 1\}$ and $(4, q^n - \epsilon) = 4$. For $t \geqslant 1$, let $n(t) = (t^{n-1} + 3)/2$. Then $\omega_p(G)$ consists of all divisors of the following numbers:*

(i) $p^t(q^{n-n(t)} \pm 1)/2$, *for* t *with* $n(t) < t$,

(ii) $p^t[q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 2$, $r \geqslant 1$ *and* $n_i > 0$ *with* $n(t) + n_1 + n_2 + \cdots + n_s = n$

(iii) $p[q \pm 1, q^{n_1}+1, q^{n_2}+1, \dots, q^{n_r}+1, q^{n_{r+1}}-1, q^{n_{r+2}}-1, \dots, q^{n_s}-1]$ *for all* $s \geqslant 2$ *and* $n_i > 0$ *with* $2 + n_1 + n_2 + \cdots + n_s = n$ *and* r *is even if* $\epsilon = +$ *and odd if* $\epsilon = -$,

(iv) $p[q \pm 1, (q^{n-2} - \epsilon)/2]$,

(v) p^t *if* $n = n(t)$ *for some* t.

Proof. This follows from [3, Corollary 9]. □

Theorem 3.9. *Conjecture 1.1 holds when $G \cong P\Omega^{\epsilon}_{2n}(q)$, $n \geqslant 4$.*

Proof. Let $G \cong P\Omega^{\epsilon}_{2n}(q)$, $n \geqslant 4$ and $g \in G$ be a p-singular element. Then the possible values of $\mathrm{ord}(g)$ are listed in Lemmas 3.7 and 3.8. If $p = 2$, then the highest 2-power order is in case (vi) of Lemma 3.7, that is, when $2^{t-2} = n - 2$ and if p is odd, then the highest p-power order is in case (v) of Lemma 3.8, that is, when $t^{n-1} = 2n - 3$. Let us consider $q^{n(n-1)}/2^t$ and $q^{n(n-1)}/p^t$ and assume that $p = q$. Since $t \leqslant 2^{t-1} \leqslant n - 2$ and $t \leqslant p^{n-1} = 2n - 3$, we have that $n(n-1) - t \geqslant 10$ and $n(n-1) - t \geqslant 7$. Hence the p-part of our character degree maybe up to q^7.

We now consider the p'-part of $\mathrm{ord}(g)$. The highest possible degree of the p'-part polynomial of $\mathrm{ord}(g)$ is when $n_s = n_1$ in case (ii) of Lemma 3.7 for $p = 2$, that is, $q^{n-2} \pm 1$ and in case (iv) of Lemma 3.8 for odd p, that is, $[q \pm 1, (q^{n-2} \pm 1)]$. In particular, no $\mathrm{ord}(g)$ is divisible by $q^n - 1$, $q^{2(n-1)} - 1$ or $q^n + 1$.

If $G \cong P\Omega_{2n}^{+}(q)$, then the appropriate character is either the unipotent character of degree $\frac{q(q^{n-2}+1)(q^n-1)}{q^2-1}$ or that of degree $\frac{q^6(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)(q^4-1)}$.

If $G \cong P\Omega_{2n}^{-}(q)$, then the appropriate character is either the unipotent character of degree $\frac{q(q^{n-2}-1)(q^n+1)}{q^2-1}$ or that of degree $\frac{q^6(q^{2(n-2)}-1)(q^{2(n-1)}-1)}{(q^2-1)(q^4-1)}$. □

4. Exceptional groups

4.1. **Exceptional groups of small Lie rank.** We shall consider exceptional groups of small Lie rank. Table 4.2 has the appropriate character degrees of some exceptional groups of Lie type that we shall use in our arguments. These character degrees can be found in [8, Section 13.9].

We shall collect information on the orders of some of the groups. Let $\omega(G)$ denote the set of orders of elements of G and let $\mu(G)$ denote the subset of $\omega(G)$ of maximal elements of $\omega(G)$ under divisibility.

Lemma 4.1. [19, Lemma 1.4] *Let $G \cong \mathrm{G}_2(q), q \geqslant 3$ and let q be a power of a prime p. Then*

(i) $\mu(G) \subseteq \{8, 12, 2(q \pm 1), q^2-1, q^2 \pm q + 1\} \subseteq \omega(G)$ *for* $p = 2$*;*
(ii) $\mu(G) = \{p^2, p(q \pm 1), q^2-1, q^2 \pm q+1\}$ *for* $p = 3, 5$*;*
(iii) $\mu(G) = \{p(q \pm 1), q^2-1, q^2 \pm q+1\}$ *for* $p > 5$.

Lemma 4.2. *Let $G \cong {}^3\mathrm{D}_4(q)$, with q a power of a prime p. Then $\omega_p(G)$ consists of all the appropriate divisors of the following numbers:*

(i) $p(q^3 \pm 1)$*;*
(ii) $4(q^2 \pm q+1)$ *and* 8 *if* $p = 2$*;*
(iii) p^2 *if* $p \in \{3, 5\}$.

Proof. This follows from [13, Theorem 3.2]. □

Let $\mathcal{C} = \{{}^2\mathrm{B}_2(q^2), q^2 = 2^{2f+1} > 2\} \cup \{{}^2\mathrm{G}_2(q^2), q^2 = 3^{2f+1} > 3\} \cup \{{}^2\mathrm{F}_4(q^2), q^2 \geqslant 3\} \cup \{\mathrm{G}_2(q), q \geqslant 3\} \cup \{{}^3\mathrm{D}_4(q), q \geqslant 2\}$.

Theorem 4.3. *Conjecture 1.1 holds when $G \in \mathcal{C}$.*

Proof. Let $G \cong \mathrm{G}_2(q)$ and let $g \in G$ be a p-singular element. Since $|G|_p = q^6$ and the largest possible p-power order of $\mathrm{ord}(g)$ is p^3 by Lemma 4.1, we have that the p-part of an appropriate character degree is up to q^3. Considering the p'-part of $\mathrm{ord}(g)$ we also have that maximal orders are $p(q \pm 1)$. Hence the unipotent character of degree $\frac{1}{3}q(q^2+q+1)(q^2-q+1)$ is appropriate.

Let $G \cong {}^3\mathrm{D}_4(q)$ and let $g \in G$ be a p-singular element. Considering the possible values of $\mathrm{ord}(g)$ in Lemma 4.2, we have that the unipotent character of degree $q(q^4-q^2+1)$ will give us the desired result.

Let $G \cong {}^2\mathrm{B}_2(q^2)$, where $q^2 = 2^{2m+1}$ and let $g \in G$ be a 2-singular element. The orders of elements of G are well known (see for example [18]) and in particular the only 2-singular elements of G are of orders 2 and 4. Hence the result follows from Lemma 2.2.

Let $G \cong {}^2\mathrm{G}_2(q^2)$, where $q^2 = 3^{2m+1}$ and let $g \in G$ be a 3-singular element. From [1, XI, Theorems 13.2 and 13.4], the only possible values of $\mathrm{ord}(g)$ are $3, 6$ and 9. Hence the unipotent character of degree $\frac{1}{2\sqrt{3}}q(q-1)(q+1)(q^2-\sqrt{3}q+1)$ will be appropriate.

Let $G \cong {}^2\mathrm{F}_4(q^2)$, where $q^2 = 2^{2m+1}$ and let $g \in G$ be a 2-singular element. The orders of elements of G are listed in [10, Lemma 3]. In particular, the only 2-singular elements are of orders 2, 4, 8, 16, all divisors of the following: $2(q^2+1)$, $4(q^2-1)$, $4(q^2+\sqrt{2}q+1)$ and $4(q^2-\sqrt{2}q+1)$. Hence the character of degree $q^2\Phi_{12}\Phi_{24}$ gives our result. This concludes our proof. $\square$

4.2. **Exceptional groups of large Lie rank.** We finally consider exceptional groups of large Lie rank.

Let $\mathcal{D} = \{\mathrm{F}_4(q), \mathrm{E}_6(q), {}^2\mathrm{E}_6(q), \mathrm{E}_7(q), \mathrm{E}_8(q)\}$.

Theorem 4.4. *Let G be a finite simple group of exceptional Lie type. Then Conjecture 1.1 holds.*

Proof. Let $G \cong \mathrm{F}_4(q)$ and let $g \in G$ be p-singular element. The orders of G are found in [13, Theorem 3.1]. In particular, the possible values of $\mathrm{ord}(g)$ are not divisible by Φ_4^2, Φ_8 or by Φ_{12}. The unipotent character of degree $\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$ will be appropriate since $|G|$ is divisible by $\Phi_4^2\Phi_8\Phi_{12}$.

Let $G \cong \mathrm{E}_6(q)$ or ${}^2\mathrm{E}_6(q)$ and let $g \in G$ be a p-singular element. The orders of G are described in [4, Theorem 1]. In particular, no p-singular elements have order divisible by q^4+1 or $q^6+\epsilon q^3+1$. Hence for $G \cong \mathrm{E}_6(q)$, the unipotent character of degree $q(q^4+1)(q^6+q^3+1)$ is appropriate whilst for $G \cong {}^2\mathrm{E}_6(q)$, we have the unipotent character of degree $q(q^4+1)(q^6-q^3+1)$.

Let $G \cong \mathrm{E}_7(q)$ and let $g \in G$ be a p-singular element. The orders of G are described in [5, Theorem 2]. In particular, no p-singular elements have order divisible by Φ_7, Φ_9, Φ_{14} or Φ_{18}. Hence the unipotent character of degree $q^3\Phi_7\Phi_9\Phi_{14}\Phi_{18}$ will give us our desired result.

Let $G \cong \mathrm{E}_8(q)$ and let $g \in G$ be a p-singular element. The orders of elements of G are described in [6, Theorem 1]. In particular, no p-singular elements have order divisible by Φ_{12}, Φ_{20} or Φ_{24}. Moreover, no p-singular elements are divisible by Φ_4^2, Φ_8^2 or Φ_{12}^2 but $|G|$ is divisible by $\Phi_4^3\Phi_8^2\Phi_{12}^2$. Hence the unipotent character of degree $q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$ is what we need. $\square$

Table 4.2 Character degrees of exceptional groups of Lie type

G	Labels	Degrees
$\mathrm{G}_2(q)$, $q>2$	$\phi'_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$
$\mathrm{F}_4(q)$	$\phi''_{2,4}$	$\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$
$\mathrm{E}_6(q)$	$\phi_{6,1}$	$q\Phi_8\Phi_9$
$\mathrm{E}_7(q)$	$\phi_{21,3}$	$q^3\Phi_7\Phi_9\Phi_{14}\Phi_{18}$
$\mathrm{E}_8(q)$	$\phi_{8,1}$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
${}^3\mathrm{D}_4(q^3)$	$\phi'_{1,3}$	$q\Phi_{12}$
${}^2\mathrm{E}_6(q^2)$	$\phi'_{2,4}$	$q\Phi_8\Phi_{18}$
${}^2\mathrm{F}_4(q^2)$, $q^2>2$	ε'	$q^2\Phi_{12}\Phi_{24}$
${}^2\mathrm{G}_2(q^2)$, $q^2\neq 3$	cusp	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2(q^2-\sqrt{3}q+1)$

5. Almost simple groups

We start this section by mentioning a result proved in previous sections. Note that the appropriate characters used were unipotent characters. Hence we have the following result using [15, Theorems 2.4 and 2.5]:

Theorem 5.1. *Let S be a non-abelian simple group of Lie type which is not $\mathrm{PSL}_2(q)$. Then there exists a set of three character degrees Γ such that for every element $x \in G$:*

(i) *there exists $\chi \in \mathrm{Irr}(S)$ with $\chi(1) \in \Gamma$ such that $\mathrm{ord}(x) \mid \mathrm{cod}(\chi)$;*
(ii) *χ is extendible to $\mathrm{Aut}(S)$.*

Let G be an almost simple group with socle S. Consider $S \cong \mathrm{PSL}_2(q)$, where $q = p^f \geqslant 4$ for a prime p. The character degrees of G are in the following result. The outer automorphism group of S is of order $\gcd(2, q-1) \cdot f$ and is generated by a field automorphism φ of order f and a diagonal automorphism δ of order $\gcd(2, q-1)$.

Theorem 5.2. [21, Theorem A] *Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f \geqslant 4$ for a prime p, $A = \mathrm{Aut}(S)$ and let $S \leqslant G \leqslant A$. Set $H = \mathrm{PGL}_2(q)$ if $\delta \in G$ and $H = S$ if $\delta \notin G$, and let $|G{:}H| = 2^a m = d$, m odd. If p is odd, let $\varepsilon = (-1)^{(q-1)/2}$. Then*

$$\mathrm{cd}(G) = \{1, q, (q+\varepsilon)/2\} \cup \{(q-1)2^a i : i \mid m\} \cup \{(q+1)j : j \mid d\},$$

with the following exceptions:

(i) *If p is odd with $G \nleqslant S\langle\varphi\rangle$ or if $p = 2$, then $(q+\varepsilon)/2$ is not a degree of G.*
(ii) *If f is odd, $p = 3$, and $G = S\langle\varphi\rangle$, then $i \neq 1$.*
(iii) *If f is odd, $p = 3$, and $G = A$, then $j \neq 1$.*
(iv) *If f is odd, $p = 2, 3$ or 5, and $G = S\langle\varphi\rangle$, then $j \neq 1$.*
(v) *If $f \equiv 2 \mod 4$, $p = 2$ or 3, and $G = S\langle\varphi\rangle$ or $G = S\langle\delta\varphi\rangle$, then $j \neq 2$.*

Lemma 5.3. *Let $S \cong \mathrm{PSL}_2(q)$ with q a power of $p > 3$. Then there exists $\chi \in \mathrm{Irr}(S)$ such that $(\chi(1), p) = 1$ and χ is extendible to $\mathrm{Aut}(S)$.*

Proof. This follows from [12, Lemma 4.4]. □

Theorem 5.4. *Conjecture 1.1 holds for finite almost simple groups.*

Proof. Let G be an almost simple group with socle S. Using Theorem 2.3, we may assume that S is a non-abelian simple group of Lie type of characteristic p. Let $x \in G$. Using Lemma 2.1, $\mathrm{ord}(x)$ divides $|G : S|s$ where $s = \mathrm{ord}(g)$ for some $g \in S$. If x is p-regular, then the Steinberg character of S gives us our result. Hence going forward, we may assume that x is p-singular.

Let S be a non-abelian simple group of Lie type which is not $\mathrm{PSL}_2(q)$. By Theorem 5.1, there exists $\theta \in \mathrm{Irr}(S)$ such that $\mathrm{ord}(g) \mid \mathrm{cod}(\theta)$ and $\chi_S = \theta$ for some $\chi \in \mathrm{Irr}(G)$. Hence $\mathrm{ord}(x)$ divides $|G : S|s$ which divides $|G : S|\mathrm{cod}(\theta) = \mathrm{cod}(\chi)$.

Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$ and $p > 3$. Note that the only p-singular elements of S are of order p. Then by Lemma 5.3, S has an irreducible character χ extendible to $\mathrm{Aut}(S)$ with $(\chi(1), p) = 1$. Hence this character degree shows that the conjecture holds in this case.

For the remaining cases we may assume that G is as in Theorem 5.2. It is sufficient to consider the exceptions in Theorem 5.2.

Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$ and $p = 3$. We may assume that x is p-singular. If $G = S\langle\varphi\rangle$, then $(q+\varepsilon)/2$ is an appropriate character degree. If $G = S\langle\delta\varphi\rangle$, then G has the character degree $q+1$ which is appropriate. We are left with (i) and (iii). Let us consider (iii). Then f is odd and so $2^a = 1$, which means that $q-1$ is a character degree of G and the result follows. Hence we may assume that $G \nleq S\langle\varphi\rangle$, $G \neq A$ and $G \neq S\langle\delta\varphi\rangle$. In this case $q+1$ is a character degree of G and it gives us the result.

Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$ and $p = 2$. We may assume that x is p-singular. Then $\mathrm{ord}(x)$ divides $q|G/S|$. If f is even, then G has an irreducible character of degree $q+1$ and the result follows. If f is odd, then $|G{:}H| = m$, m odd and so the appropriate character degree is $q-1$. This concludes our proof. □

Using the argument in the proof of Theorem 5.4, the following result can be shown:

Corollary 5.5. *Let N be a minimal normal subgroup of a finite group G. If N is a non-abelian simple group of Lie type which is not* $\mathrm{PSL}_2(q)$*, then Conjecture 1.1 holds.*

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References

[1] N. Blackburn, B. Huppert, Finite groups, III, Springer-Verlag, 1982.
[2] A. A. Buturlakin, Spectra of finite linear and unitary groups, *Algebra Logic* **47** (2008), no. 2, 91–99.
[3] A. A. Buturlakin, Spectra of finite symplectic and orthogonal groups, *Siberian Adv. Math.* **21** (2011), no. 3, 176–210.
[4] A. A. Buturlakin, Spectra of finite simple groups $E_6(q)$ and ${}^2E_6(q)$, *Algebra Logic* **52** (2013), no. 3, 188–202.
[5] A. A. Buturlakin, Spectra of finite simple group $E_7(q)$, *Siberian Math. J.* **57** (2016), no. 5, 769–777.
[6] A. A. Buturlakin, Spectra of groups $E_8(q)$, *Algebra Logic* **57** (2018), no. 1, 1–8.
[7] A. A. Buturlakin, M. A. Grechkoseeva, Orders of inner-diagonal automorphisms of some simple groups of Lie type, *to appear in J. Group Theory* (2022), 26 pages.
[8] R. W. Carter. *Finite Groups of Lie Type : Conjugacy Classes and Complex Characters.* (A Wiley-Interscience Publication, Pure and Applied Mathematics, John Wiley and Sons Inc., New York, 1985).
[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of Finite Groups*, Oxford, Clarendon Press, 1985.
[10] H. W. Deng, W. J. Shi, A characterization of Ree groups ${}^2\mathrm{F}_2(q)$ by element orders, *J. Algebra* **217** (1999) 180–187.
[11] E. Giannelli, Character codegrees and element orders in symmetric and alternating groups, *submitted*, 2022.
https://eugeniomaths.files.wordpress.com/2022/05/giann_codeg2.pdf
[12] E. Giannelli, N. Rizo and A. A. Schaeffer Fry, Groups with few p'-character degrees, *J. Pure Appl. Algebra* **224** (2020) 106338.
[13] M. A. Grechkoseeva, M. A. Zvezdina, On spectra of automorphic extensions of finite simple groups $F_4(q)$ and ${}^3D_4(q)$, *J. Algebra Appl.* **15** (2016) 1650168 (13 pages).
[14] I. M. Isaacs, Element orders and character codegrees, *Arch. Math.* **97** (2011) 499–501.
[15] G. Malle, Extensions of unipotent characters and the inductive Mckay condition, *J. Algebra* **320** (2008) 2963–2980.
[16] G. Qian, A note on element orders and character codegrees, *Arch. Math.* **97** (2011) 99–103.
[17] G. Qian, Element orders and codegrees, *Bull. London Math. Soc.* **53** (2021) 820–824.
[18] M. Suzuki, On a class of doubly transitive groups, *Ann. Math.* **75** (1962) 105–145.
[19] A. V. Vasil'ev, A. M. Starolev, Recognizability of the groups $G_2(q)$ by the spectrum, *Algebra Logic* **52** (2013) 1–14.
[20] H. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121** (1966) 62–89.
[21] D. White, Character degree extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$, *J. Group Theory* **16** (2013) 1–33.

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