*Discussiones Mathematicae*
*Graph Theory 42 (2022) 1281–1312*
*https://doi.org/10.7151/dmgt.2346*

# NESTED LOCALLY HAMILTONIAN GRAPHS AND THE OBERLY-SUMNER CONJECTURE

Johan P. de Wet[1,2] and Marietjie Frick[1]

[1] *University of Pretoria, Private Bag X20*
*Hatfield 0028, South Africa*

[2] *DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)*

**e-mail:** johan.dewet@up.ac.za
marietjie.frick@up.ac.za

**Abstract**

A graph $G$ is *locally* $\mathcal{P}$, abbreviated $L\mathcal{P}$, if for every vertex $v$ in $G$ the open neighbourhood $N(v)$ of $v$ is non-empty and induces a graph with property $\mathcal{P}$. Specifically, a graph $G$ without isolated vertices is *locally connected* ($LC$) if $N(v)$ induces a connected graph for each $v \in V(G)$, and *locally hamiltonian* ($LH$) if $N(v)$ induces a hamiltonian graph for each $v \in V(G)$. A graph $G$ is *locally locally* $\mathcal{P}$ (abbreviated $L^2\mathcal{P}$) if $N(v)$ is non-empty and induces a locally $\mathcal{P}$ graph for every $v \in V(G)$. This concept is generalized to an arbitrary degree of nesting. For any $k \geq 0$ we call a graph *locally* $k$*-nested-hamiltonian* if it is $L^mC$ for $m = 0, 1, \ldots, k$ and $L^kH$ (with $L^0C$ and $L^0H$ meaning *connected* and *hamiltonian*, respectively). The class of locally $k$-nested-hamiltonian graphs contains important subclasses. For example, Skupień had already observed in 1963 that the class of connected $LH$ graphs (which is the class of locally 1-nested-hamiltonian graphs) contains all triangulations of closed surfaces. We show that for any $k \geq 1$ the class of locally $k$-nested-hamiltonian graphs contains all simple-clique $(k+2)$-trees. In 1979 Oberly and Sumner proved that every connected $K_{1,3}$-free graph that is locally connected is hamiltonian. They conjectured that for $k \geq 1$, every connected $K_{1,k+3}$-free graph that is locally $(k+1)$-connected is hamiltonian. We show that locally $k$-nested-hamiltonian graphs are locally $(k+1)$-connected and consider the weaker conjecture that every $K_{1,k+3}$-free graph that is locally $k$-nested-hamiltonian is hamiltonian. We show that if our conjecture is true, it would be "best possible" in the sense that for every $k \geq 1$ there exist $K_{1,k+4}$-free locally $k$-nested-hamiltonian graphs that are non-hamiltonian. We also attempt to determine the minimum order of non-hamiltonian locally $k$-nested-hamiltonian graphs and investigate the complexity of the Hamilton Cycle Problem for locally $k$-nested-hamiltonian graphs with restricted maximum degree.

**Keywords:** locally traceable, locally hamiltonian, Hamilton Cycle Problem, locally $k$-nested-hamiltonian, Oberly-Sumner Conjecture.

**2010 Mathematics Subject Classification:** 05C60.

## 1. Introduction and Background

For a given graph property $\mathcal{P}$, we say a graph $G$ is *locally* $\mathcal{P}$ if for each $v \in V(G)$ the open neighbourhood $N(v)$ of $v$ is nonempty and the graph induced by $N(v)$ has the property $\mathcal{P}$. A graph $G$ is *hamiltonian* if it has a Hamilton cycle (a cycle that visits every vertex). Our interest in local properties that imply hamiltonicity was sparked by the well-known theorem of Oberly and Sumner, stated below.

**Theorem 1.1** [11]. *If $G$ is a $K_{1,3}$-free, connected, locally connected graph of order at least* 3, *then $G$ is hamiltonian.*

Throughout the paper $k$ will denote a positive integer. A graph $G$ is $k$-*connected* if $G$ has at least $k+1$ vertices and for every subset $S$ of $V(G)$ consisting of fewer than $k$ vertices, the graph $G - S$ is connected. Oberly and Sumner conjectured an extension of Theorem 1.1.

**Conjecture 1.2** [11]. *If $G$ is a $K_{1,k+3}$-free, connected, locally $(k+1)$-connected graph, then $G$ is hamiltonian.*

The Oberly-Sumner Conjecture has not been settled for any $k \geq 1$. In fact, it is not even known whether there exists an integer $t$ such that every $K_{1,4}$-free locally $t$-connected graph is hamiltonian. However, it is easy to prove the weaker result that every graph satisfying the conditions of Conjecture 1.2 is 1-tough. (A connected graph $G$ is 1-*tough* if for every subset $S$ of $V(G)$ the number of components of $G - S$ is less than or equal to $|S|$.) The proof uses the following result of Chartrand and Pippert.

**Theorem 1.3** [2]. *If $G$ is a connected, locally $k$-connected graph, then $G$ is $(k+1)$-connected.*

**Theorem 1.4.** *If $G$ is a $K_{1,k+3}$-free, connected, locally $(k+1)$-connected graph, then $G$ is* 1-*tough.*

***Proof.*** Let $S$ be any vertex cut of $G$. Since $G$ is locally $(k+1)$-connected, it follows from Theorem 1.3 that $G$ is $(k+2)$-connected. So each component of $G-S$ has at least $k+2$ neighbours in $S$. On the other hand, since $G$ is $K_{k+3}$-free, each vertex in $S$ has neighbours in at most $k+2$ different components of $G-S$. This implies that $G-S$ has at most $|S|$ components. Therefore $G$ is 1-tough. ■

Theorem 1.4 can also be derived as a corollary to a theorem by Chen *et al.* [3]. Our proof is included here because it gives a direct insight into why the graphs are 1-tough. It is easily seen that 1-toughness is a necessary condition for hamiltonicity.

We are interested in replacing the local connectivity condition in the Oberly-Sumner Conjecture with a stronger local condition that might guarantee hamiltonicity. For example, local hamiltonicity is stronger than local 2-connectivity, so it follows that the following conjecture is weaker than the case $k = 1$ of the Oberly-Sumner Conjecture.

**Conjecture 1.5.** *If $G$ is a $K_{1,4}$-free, connected, locally hamiltonian graph, then $G$ is hamiltonian.*

In this paper we shall consider a conjecture that extends Conjecture 1.5 and is weaker than Conjecture 1.2 for each $k \geq 1$. First we need some definitions.

If $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $\langle S\rangle$. We use the abbreviation $LC$ and $LH$ for *locally connected* and *locally hamiltonian*, respectively. Thus a graph $G$ is $LC$ (respectively, $LH$) if $\langle N(v)\rangle$ is a connected (respectively, hamiltonian) graph for each $v \in V(G)$.

We define a graph $G$ to be *locally locally* $\mathcal{P}$ (abbreviated $L^2\mathcal{P}$) if $N(v) \neq \emptyset$ and $\langle N(v)\rangle$ is locally $\mathcal{P}$ for every $v \in V(G)$. We extend this concept inductively to arrive at the following definition.

**Definition 1.6.** A graph is $L^0\mathcal{P}$ if it has the property $\mathcal{P}$. For any integer $k \geq 1$, a graph $G$ is $L^k\mathcal{P}$ if $N(v) \neq \emptyset$ and $\langle N(v)\rangle$ is $L^{k-1}\mathcal{P}$ for every $v \in V(G)$.

We note that an $L^kH$ graph is also $L^kC$ but not necessarily $L^mC$ for $0 \leq m \leq k-1$. For example, Figure 1 depicts a $K_{1,3}$-free $L^3H$ graph that is $L^mC$ for $m = 0, 2, 3$ but not for $m = 1$, and is obviously not hamiltonian. This observation motivated us to study locally $k$-nested-hamiltonian graphs, which we define as follows.

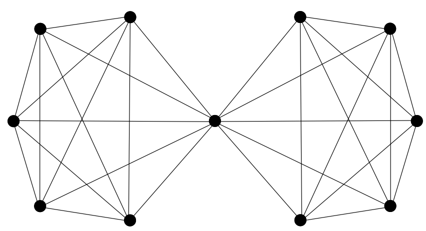

Figure 1. An $L^3H$ graph that is not $LC$.

**Definition 1.7.** A graph $G$ is *locally $k$-nested-hamiltonian* if $G$ is $L^m\mathcal{C}$ for $m = 0, 1, \ldots, k-1$ and $L^kH$.

Requiring $G$ to be $L^m C$ for $m = 0, 1, \ldots, k-1$ in Definition 1.7 is analogous to restricting our investigation to connected graphs when studying the hamiltonicity of $LH$ graphs.

For ease of notation, we give the following definition.

**Definition 1.8.** A graph $G$ is $L^{\leq k}\mathcal{P}$ if $G$ is $L^m\mathcal{P}$ for $m = 1, \ldots, k$.

Thus, since an $L^k H$ graph is also $L^k C$, a graph $G$ is locally $k$-nested-hamiltonian if and only if $G$ is connected, $L^{\leq k}C$ and $L^k H$.

We shall show in Section 2 that every locally $k$-nested-hamiltonian graph is locally $(k+1)$-connected. Thus the following conjecture, which extends Conjecture 1.5, is indeed weaker than Conjecture 1.2.

**Conjecture 1.9.** *If $G$ is a $K_{1,k+3}$-free graph that is locally $k$-nested-hamiltonian, then $G$ is hamiltonian.*

We shall show in Section 3 that if Conjecture 1.9 is true, it would be "best possible" in the sense that for each $k \geq 1$ there exists a $K_{1,k+4}$-free locally $k$-nested-hamiltonian graph that is non-hamiltonian.

Pareek and Skupień [13] proved that the graph of order 11 depicted in Figure 2 is the smallest connected non-hamiltonian $LH$ graph. (This graph, known as the *Goldner-Harary graph*, was shown by Goldner and Harary [9] to be the smallest non-hamiltonian maximal planar graph.) De Wet [4, 6] showed that there are four non-hamiltonian, connected $LH$ graphs of order 11 and they all have maximum degree 8.

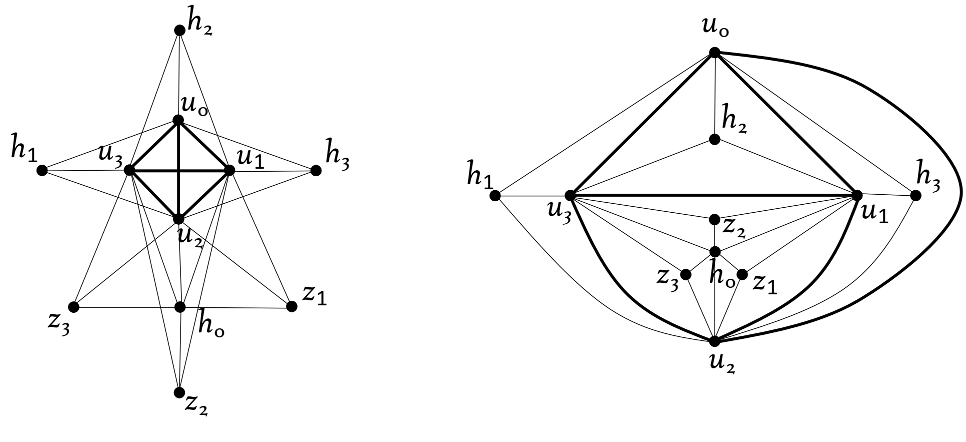


Figure 2. Two depictions of the Goldner-Harary graph.

In Section 3 we prove that the minimum order of a non-hamiltonian locally 2-nested-hamiltonian graph is 13. By generalizing the graph in Figure 2 we obtain for each $k \geq 1$ a locally $k$-nested-hamiltonian graph of order $9 + 2k$ that is non-$L^m H$ for $m = 0, 1, \ldots, k-1$. On the other hand, a generalization of the 11-vertex $LH$ graph in Figure 7(a) yields for each $k \geq 1$ a non-hamiltonian connected graph that is $L^{\leq k}H$. We show that if Conjecture 1.9 is true, the minimum order of a non-hamiltonian connected $L^{\leq k}H$ graph would be $9 + 2k$, as is the case for $k = 1, 2$.

Pareek [12] claimed that every non-hamiltonian connected $LH$ graph has maximum degree at least 8, but there are flaws in his "proof" that we have not been able to rectify, as discussed in [4, 6]. However, if an $LH$ graph contains an induced $K_{1,4}$ centred at a vertex $x$, then $|N(x)| \geq 8$, since $\langle N(x) \rangle$ is hamiltonian but contains 4 mutually independent vertices. This proves that any $LH$ graph with maximum degree less than 8 is $K_{1,4}$-free. Thus, if Conjecture 1.5 is true, it would immediately prove Pareek's (as yet unproved) claim.

A graph $G$ is *fully cycle extendable* if every vertex in $G$ lies in a 3-cycle and for every non-hamiltonian cycle $C$ there is a cycle $C^*$ that contains all the vertices of $C$ plus a single new vertex. It is shown in [16] that every connected $LH$ graph with maximum degree at most 6 is fully cycle extendable (and hence hamiltonian). However,we showed in [5] that the Hamilton Cycle Problem for $LH$ graphs with maximum degree 9 is NP-complete. We show in Section 4 that every locally 2-nested-hamiltonian graph with maximum degree at most 7 is fully cycle extendable, while the Hamilton Cycle Problem for locally 2-nested-hamiltonian graphs with maximum degree 12 is NP-complete.

A *k-tree* is a graph that can be constructed by starting with a $K_{k+1}$ and then recursively performing the following operation. Choose a $k$-clique in the graph, add a new vertex and add an edge between the new vertex and each vertex in the chosen $k$-clique. If no $k$-clique is chosen more than once during the construction, the resulting $k$-tree is called a *simple-clique k-tree*. The Goldner-Harary graph happens to be a maximal planar graph that is a simple-clique 3-tree. In Section 5 we investigate the connection between $LH$ graphs, 3-trees and maximal planar graphs, and we show that for every integer $k \geq 2$, the simple-clique $k$-trees constitute a subclass of the class of $L^{k-2}H$ graphs.

For standard concepts we use the notation and terminology of [1]. In particular, $n(G)$ denotes the *order* of $G$ (i.e., the number of vertices in $G$). The *degree* of a vertex $v$ is denoted by $d_G(v)$, or $d(v)$ if $G$ is understood, and the minimum degree and maximum degree of $G$ are denote by $\delta(G)$ and $\Delta(G)$, respectively. The maximum number of independent vertices in $G$ is denoted by $\alpha(G)$.

## 2. Basic Properties and Constructions of $L^kC$ Graphs and $L^kH$ Graphs

The proposition below provides a characterization of $L^k\mathcal{P}$ graphs for $k \geq 1$, which may be used as a convenient alternative definition for these graphs.

**Proposition 2.1.** *For $k \geq 1$, a graph $G$ is $L^k\mathcal{P}$ if and only if each of the following holds.*

(1) *If $1 \leq m \leq k$, and $X$ is an $m$-clique in $G$, then $X$ is contained in an $(m+1)$-clique in $G$.*

(2) *If $X$ is a $k$-clique in $G$, then the neighbourhood intersection $\bigcap_{x \in V(X)} N(x)$ induces a graph with property $\mathcal{P}$.*

***Proof.*** The proof is by induction on $k$.

First, suppose $G$ is $L^k\mathcal{P}$. If $k = 1$, then it follows immediately from the definition of an $L\mathcal{P}$ graph that (1) and (2) hold.

Now let $k \geq 2$ and let $X$ be an $m$-clique in $G$. If $m = 1$, then $X$ is contained in a 2-clique by Definition 1.6. If $m \geq 2$, let $x \in V(X)$. Then $X - x$ is an $(m-1)$-clique in $\langle N(x) \rangle$. But $\langle N(x) \rangle$ is an $L^{k-1}\mathcal{P}$ graph by Definition 1.6. So by our induction hypothesis, $X - x$ is contained in an $m$-clique $Y$ in $\langle N(x) \rangle$. But then $\langle V(Y) \cup \{x\} \rangle$ is an $(m+1)$-clique in $G$ that contains $X$. Thus $G$ satisfies (1).

Now let $X$ be a $k$-clique in $G$ with $V(X) = \{x_1, \ldots, x_k\}$. Then $\{x_1, \ldots, x_{k-1}\}$ induces a $(k-1)$-clique in $\langle N(x_k) \rangle$. But $\langle N(x_k) \rangle$ is $L^{k-1}\mathcal{P}$ by Definition 1.6. So our induction hypothesis implies that the subgraph of $\langle N(x_k) \rangle$ induced by $\bigcap_{i=1}^{k-1} N_{N(x_k)}(x_i)$ has the property $\mathcal{P}$. But $\bigcap_{i=1}^{k-1} N_{N(x_k)}(x_i) = \bigcap_{i=1}^{k} N(x_i)$ (since $N_{N(x_k)}(x_i) = N(x_i) \cap N(x_k)$ for $i = 1, \ldots, k-1$). So $G$ also satisfies (2).

Now suppose (1) and (2) hold. If $k = 1$, then (1) implies that $N(v)$ is nonempty for every $v \in V(G)$, and (2) implies that $\langle N(v) \rangle$ induces a graph with property $\mathcal{P}$ for every $v \in V(G)$. So then $G$ is $L\mathcal{P}$.

Now let $k \geq 2$ and consider any $v \in V(G)$. If $1 \leq m \leq k-1$ and $X$ is an $m$-clique in $\langle N(v) \rangle$, then $V(X) \cup \{v\}$ induces an $(m+1)$-clique in $G$. So (1) implies that $X$ lies in an $(m+2)$-clique $Y$ in $G$. But then $Y - v$ is an $(m+1)$-clique in $\langle N(v) \rangle$ that contains $X$. Thus the graph $\langle N(v) \rangle$ satisfies (1) with $k$ replaced by $k-1$. Now let $X$ be a $(k-1)$-clique in $\langle N(v) \rangle$ with $V(X) = \{x_1, \ldots, x_{k-1}\}$. Then $\{x_1, \ldots, x_{k-1}, v\}$ induces a $k$-clique in $G$. So (2) implies that $\bigcap_{i=1}^{k-1} N(x_i) \cap N(v)$, induces a graph with property $\mathcal{P}$, i.e., the subgraph of $\langle N(v) \rangle$ induced by $\bigcap_{i=1}^{k-1} N_{N(v)}(x_i)$ has the property $\mathcal{P}$. Hence $\langle N(v) \rangle$ also satisfies (2) with $k$ replaced by $k-1$. Hence, by our induction hypothesis, $\langle N(v) \rangle$ is an $L^{k-1}\mathcal{P}$ graph. So by Definition 1.6, $G$ is an $L^k\mathcal{P}$ graph. ■

If $v$ is any vertex in an $LH$ graph, then $\langle N[v] \rangle$ contains a wheel of order $d(v) + 1$, centered at $v$, as spanning subgraph. The following result is therefore useful when dealing with $LH$ graphs.

**Lemma 2.2.** *Suppose a graph $G$ contains a wheel $W$ with centre $v$ and rim $C$. Let $C$ be the cycle $v_0 v_1 \cdots v_t v_0$ $(t \geq 2)$ and let $V(G) - V(W) = \{x_1, \ldots, x_r\}$. If $d_C(x_i) \geq 4$ for each $i \in \{1, \ldots, r\}$, then the following hold.*

(*a*) *If* $r = 1$*, then* $G$ *is hamiltonian.*

(*b*) *If* $r = 2$ *and there are two consecutive vertices* $v_i, v_{i+1}$ *on* $C$ *that each has a neighbour in* $\{x_1, x_2\}$*, then* $G$ *is hamiltonian.*

(*c*) *If* $r = 3$ *and there are two pairs of consecutive vertices* $\{v_i, v_{i+1}\}$ *and* $\{v_j, v_{j+1}\}$ $(i \neq j)$ *on* $C$ *such that* $\{v_i, v_{i+1}\} \subseteq N(x_1)$ *and* $\{v_j, v_{j+1}\} \subseteq N(x_2)$*, then* $G$ *is hamiltonian.*

***Proof.*** Hamilton cycles that illustrate the proof of the result in each of the three cases are shown in Figure 3. Since $d_C(x_2) \geq 4$, we may assume in Case (b) that $v_i, v_{i+1}, v_k, v_l$ are four distinct vertices where $\{v_i, v_{i+1}, v_k, v_l\} \subseteq (N(x_1) \cup N(x_2)) \cap V(C)$. Figure 3(b)(i) represents the case in which $v_i$ and $v_{i+1}$ are neighbours of the same vertex in $V(G) - V(W)$. Figure 3(b)(ii) represents the case in which $v_i$ and $v_{i+1}$ do not share a neighbour in $V(G) - V(W)$. In Case (c), we may for example have $v_k = v_i$, but then we can choose $v_l$ to be neither $v_{i+1}$ nor $v_{j+1}$ and we still have a valid Hamilton cycle. The details are straightforward and are left to the reader. ■

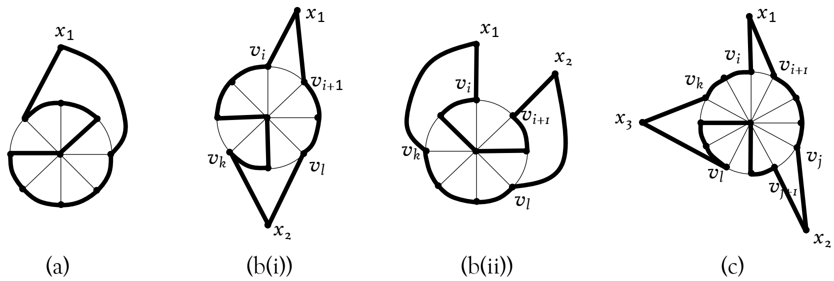


Figure 3. Hamilton cycles illustrating the proof of Lemma 2.2.

**Corollary 2.3.** *If* $G$ *is a connected non-hamiltonian LH graph, then* $n(G) \geq \Delta(G) + 3$.

Our next result is implied by Proposition 2.1.

**Lemma 2.4.** *If* $G$ *is an* $L^kH$ *graph, then every vertex of* $G$ *lies in a* $(k+2)$*-clique and* $\delta(G) \geq k + 2$.

***Proof.*** Let $v \in V(G)$. By applying Proposition 2.1(1) repeatedly, we see that $v$ lies in a $k$-clique $X$. By Proposition 2.1(2), the neighbourhood intersection $\bigcap_{x \in V(X)} N(x)$ is nonempty and induces a hamiltonian graph. But a hamiltonian graph has at least three vertices and contains $K_2$'s. So it follows that $|N(v)| \geq k + 2$ and $v$ lies in a $K_{k+2}$. ■

**Corollary 2.5.** *The* $L^kH$ *graph of smallest order is* $K_{k+3}$.

**Corollary 2.6.** *If $G$ is an $L^kH$ graph and $d(v) = k+2$ for some $v \in V(G)$, then $\langle N(v) \rangle$ is a $(k+2)$-clique in $G$.*

***Proof.*** From Definition 1.6 we see that $\langle N(v) \rangle$ is an $L^{k-1}H$ graph of order $k+2$, and by Corollary 2.5, $\langle N(v) \rangle$ is isomorphic to $K_{k+2}$. ■

**Lemma 2.7.** *If $v$ is a vertex in an $L^kH$ graph $G$ such that $\langle N(v) \rangle$ is contained in a clique of order at least $k+3$ in $G$, then $G - v$ is also $L^kH$.*

***Proof.*** Only the neighbourhoods of vertices adjacent to $v$ are affected by the removal of $v$ from $G$. So, by Proposition 2.1(2) in order to show that $G - v$ is $L^kH$, we need only show that the neighbourhood intersection of $k$-cliques in $\langle N(v) \rangle$ are hamiltonian. Let $X$ be a $k$-clique in $\langle N(v) \rangle$. Then the graph $\big\langle \bigcap_{x \in V(X)} N(x) \big\rangle$ contains the vertex $v$ plus at least three vertices in $G - v$, and therefore it has a Hamilton cycle $C$ that contains a subpath $u_1 v u_2$, with $u_1, u_2 \in N(v)$. Replacing the path $u_1 v u_2$ in $C$ with the edge $u_1 u_2$ yields a Hamilton cycle of $\big\langle \bigcap_{x \in V(X)} N_{G-v}(x) \big\rangle$. Hence $G - v$ is $L^kH$. ■

Theorem 1.3 implies the following.

**Theorem 2.8.** *If $G$ is a locally $k$-nested-hamiltonian graph, then $G$ is locally $(k+1)$-connected and $(k+2)$-connected.*

***Proof.*** It is easily seen that a connected $LH$ graph is locally 2-connected and thus, by Theorem 1.3, 3-connected. So the result holds for $k = 1$. Now let $k \geq 2$, and let $v \in V(G)$. Then, by Definitions 1.6 and 1.7, $\langle N(v) \rangle$ is locally $(k-1)$-nested-hamiltonian. So by the induction hypothesis, $\langle N(v) \rangle$ is $(k+1)$-connected. Hence $G$ is locally $(k+1)$-connected and therefore, by Theorem 1.3, $G$ is $(k+2)$-connected. ■

De Wet [4, 6] developed a method, called *triangle identification*, for obtaining $LH$ graphs with certain properties by combining suitable $LH$ graphs. A triangle $Y$ in an $LH$ graph $G$ is called *suitable for triangle identification*, or simply a *suitable triangle*, if for every $y \in V(Y)$ there is a Hamilton cycle in $\langle N(y) \rangle$ that contains the edge between the two vertices in $Y - y$. The following results are proved in [4, 6].

**Theorem 2.9** [6]**.** *For $i = 1, 2$ let $G_i$ be an $LH$ graph that contains a suitable triangle $Y_i$. Let $G$ be the graph obtained from $G_1$ and $G_2$ by identifying the triangle $Y_1$ with the triangle $Y_2$. Then the following hold.*

(1) *$G$ is $LH$.*

(2) *If $G$ is hamiltonian, then both $G_1$ and $G_2$ are hamiltonian.*

In order to generalize Theorem 2.9 for $L^kH$ graphs, we generalize the concept of a suitable triangle as follows.

**Definition 2.10.** A $(k+2)$-clique $Y$ in an $L^kH$ graph $G$ is called *suitable for $K_{k+2}$-identification*, or simply a *suitable* $(k+2)$*-clique*, if for each $k$-clique $X$ in $Y$, the graph induced by $\bigcap_{v \in V(X)} N(v)$ has a Hamilton cycle that contains the edge between the two vertices in $V(Y) - V(X)$. The procedure of combining two $L^2H$ graphs by means of $K_4$-identification is illustrated in Figure 4.

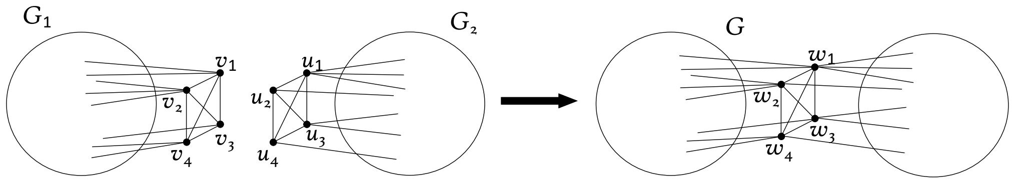


Figure 4. The $K_4$-identification procedure.

Our next result is a straightforward generalization of Theorem 2.9.

**Theorem 2.11.** *For $i = 1, 2$ suppose $G_i$ is a locally $k$-nested-hamiltonian graph that contains a suitable $(k+2)$-clique $Y_i$. Let $G$ be the graph obtained from $G_1$ and $G_2$ by identifying the $(k+2)$-clique $Y_1$ with the $(k+2)$-clique $Y_2$. Then the following hold.*

(1) *$G$ is locally $k$-nested-hamiltonian.*

(2) *If $G$ is hamiltonian, then both $G_1$ and $G_2$ are hamiltonian.*

***Proof.*** (1) We denote by $Y$ the $(k+2)$-clique obtained by identifying $Y_1$ and $Y_2$. We regard $G_1$ and $G_2$ as subgraphs of $G$ that intersect in $Y = Y_1 = Y_2$. Let $X$ be an $m$-clique in $G$, $1 \leq m \leq k$ and let

$$N = \bigcap_{x \in V(X)} N_G(x) \text{ and } N_i = \bigcap_{x \in V(X)} N_{G_i}(x), \ i = 1, 2.$$

Then $N = N_1 \cup N_2$.

We shall use Proposition 2.1 to show that $G$ is $L^mC$ for $m = 1, \ldots, k-1$ as well as $L^kH$. We consider two cases.

*Case* 1. *$X$ is not contained in $Y$.* In this case we may assume without loss of generality that $X$ has a vertex in $V(G_1) - Y$. Then, since there are no edges between $G_1 - V(Y)$ and $G_2 - V(Y)$, it follows that $X$ lies completely in $G_1$, and $N = N_1$. Since $G_1$ is $L^mC$, it follows from Proposition 2.1(1) that $X$ lies in an $(m+1)$-clique in $G_1$, and from Proposition 2.1(2) that $\langle N \rangle$ is a connected graph. Moreover, if $m = k$, then $\langle N \rangle$ is also hamiltonian, since $G_1$ is $L^kH$.

*Case* 2. *$X$ is contained in $Y$.* In this case $X$ obviously lies in an $(m+1)$-clique (since $Y$ is a $(k+2)$-clique). Moreover, $X$ is an $m$-clique in $G_1$ as well as in $G_2$. So for $i = 1, 2$, it follows from Proposition 2.1(2) that $\langle N_i \rangle$ is a connected graph (since $G_i$ is $L^m H$), which is also hamiltonian if $m = k$ (since $G_i$ is $L^k H$). We note that both $\langle N_1 \rangle$ and $\langle N_2 \rangle$ contain the clique $Y - V(X)$, and hence $\langle N \rangle$ is connected. If $m = k$, then $X$ is a $k$-clique in $Y_i$ for $i = 1, 2$. Since $Y_i$ is a suitable $(k + 2)$-clique in $G_i$, $i = 1, 2$, each of $\langle N_1 \rangle$ and $\langle N_2 \rangle$ has a Hamilton cycle containing the edge between the two vertices in $Y - V(X)$, and hence $\langle N \rangle$ is hamiltonian.

It now follows from Proposition 2.1 that $G$ is $L^m C$ for $m = 1, \ldots, k-1$ and $L^k H$, which implies that $G$ is locally $k$-nested hamiltonian.

(2) Suppose $v_0 v_1 \cdots v_n v_0$ is a Hamilton cycle of $G$. If $v_i v_{i+1} \cdots v_j$ is a path of length at least 2 on $C$ that has its two end-vertices in $Y$ (where $Y$ is defined as in the proof of part (1)) and its internal vertices in $G - V(G_2)$, we replace that path with the edge $v_i v_j$. We do this for every such path on $C$. The result is a Hamilton cycle of $G_2$. A similar argument shows that $G_1$ also has a Hamilton cycle. ∎

**Definition 2.12.** The procedure described in the statement of Theorem 2.11 will be referred to as $K_{k+2}$-*identification*.

**Remark 2.13.** If $m < k$, then $K_{k+2}$-identification of two locally $m$-nested-hamiltonian graphs does not necessarily result in an $L^m H$ graph. For example, we shall see in the next section that the graph $S_2$ in Figure 5 is obtained.

By a slight modification of the proof of Theorem 2.11 we can also prove the following.

**Theorem 2.14.** *Suppose $G$ is a locally $k$-nested-hamiltonian graph and $G$ contains two disjoint suitable $(k+2)$-cliques $X_1$ and $X_2$ such that $N(X_1) \cap N(X_2) = \emptyset$. Then the graph obtained from $G$ by identifying $X_1$ with $X_2$ is also locally $k$-nested-hamiltonian.*

A suitable $(k + 2)$-clique in a locally $k$-nested-hamiltonian graph $G_1$ may be used only once in $K_{k+2}$-identification. That is to say, if $G$ was obtained by identifying suitable $(k + 2)$-cliques of two locally $k$-nested-hamiltonian graphs $G_1$ and $G_2$ to a single $(k + 2)$-clique $Y$, then $Y$ is not a suitable $(k + 2)$-clique of $G$. This is because, if $X$ is a $k$-clique in $Y$ and $V(Y) - V(X) = \{y_1, y_2\}$, then any Hamilton cycle in $\bigcap_{x \in V(X)} N_G(x)$ contains vertices in both $G_1$ and $G_2$, and therefore does not contain the edge $y_1 y_2$.

However, the following result implies that a vertex of degree $(k + 2)$ in a locally $k$-nested-hamiltonian graph $G_0$ may be used $(k + 1)$ times in successive $K_{k+2}$-identification, each time as a member of a different $(k + 2)$-clique in its closed neighbourhood (which is a $(k + 3)$-clique by Lemma 2.4). Moreover, if $G_0 \cong K_{k+3}$, then all of the $k + 2$ distinct $(k + 2)$-cliques in $G_0$ may be used in successive $K_{k+2}$-identifications.

**Lemma 2.15.** *Let $G_0$ be a locally $k$-nested-hamiltonian graph that has a vertex $u_0$ of degree $k+2$ and let $N(u_0) = \{u_1, \dots, u_{k+2}\}$. For $i = 0, 1, \dots, k+2$, let $Y_i$ be the $(k+2)$-clique in the $(k+3)$-clique $\langle N[u_0]\rangle$ that does not contain the vertex $u_i$. Then the following hold.*

(a) *For $i = 1, \dots, k+2$, let $G_i$ be the graph obtained from $G_{i-1}$ by identifying $Y_i$ with a suitable $(k+2)$-clique in a locally $k$-nested-hamiltonian graph $H_i$. Then $Y_i$ is a suitable $(k+2)$-clique in $G_{i-1}$, and $G_i$ is a locally $k$-nested-hamiltonian graph for $i = 1, \dots, k+2$.*

(b) *In the special case where $G_0 \cong K_{k+3}$, the $(k+2)$-clique $Y_0$ is a suitable $(k+2)$-clique in the graph $G_{k+2}$ defined in* (a).

***Proof.*** (a) Let $U = N_{G_0}[u_0] = \{u_0, \dots, u_{k+2}\}$ and let $W = V(G_0) - U$. Then $Y_i = U_i - \{u_i\}$.

We first show that $Y_1$ is a suitable $k$-clique in the locally $k$-nested-hamiltonian graph $G_0$. Let $X_1$ be any $k$-clique in $Y_1$. Then there are two vertices $u_l, u_m \in V(Y_1)$ such that $0 \le l < m \le k+2$ and $V(X_1) = U - \{u_1, u_l, u_m\}$. Now let

$$N_1 = \bigcap_{x \in V(X_1)} N_{(G_0)}(x).$$

Since $G_0$ is locally $k$-nested-hamiltonian, it follows from Proposition 2.1 that $\langle N_1\rangle_{G_0}$ has a Hamilton cycle $C$. If $l = 0$, then $N_1 \subseteq \{u_1, u_l, v_m\} \cup W$. But $u_0$ has no neighbour in $W$. So then $u_1$ and $u_m$ are the only two neighbours of $u_l$ in $N_1$ and hence $C$ contains the edge $u_l u_m$. On the other hand, if $l \ne 0$, then $u_0 \in V(X)$ and hence $N_1 = \{u_1, u_l, u_m\}$. So in this case $C$ is a 3-cycle containing the edge $u_l u_m$. Thus, by Definition 2.10, $Y_1$ is a suitable $(k+2)$-clique in the graph $G_0$ and hence, by Theorem 2.11, $G_1$ is a locally $k$-nested-hamiltonian graph.

Now let $r \in \{1, 2, \dots, k+2\}$ and suppose we have shown that $G_{r-1}$ is a locally $k$-nested-hamiltonian graph and that $Y_r$ is a suitable $(k+2)$-clique in $G_{r-1}$. We note that

$$V(G_r) = U \cup W \cup \left(\bigcup_{i=1}^{r} V(H_i)\right)$$

and

$$N_{G_r}(u_i) \subseteq \begin{cases} V(G_r) - W - \{u_i\} & \text{if} \quad i = 0, \\ V(G_r) - V(H_i) - \{u_i\} & \text{if} \quad 1 \le i \le r, \\ V(G_r) - \{u_i\} & \text{if} \quad r+1 \le i \le k+2. \end{cases}$$

By Theorem 2.11 and our induction hypothesis, $G_r$ is a locally $k$-nested-hamiltonian graph. In order to show that $Y_{r+1}$ is a suitable $(k+2)$-clique in $G_r$, let $X_{r+1}$ be a $k$-clique in $Y_{r+1}$. Then $V(Y_{r+1}) - V(X) = \{u_l, u_m\}$ for some pair $l, m$ such that $0 \le l < m \le k+2$. Now let

$$N_{r+1} = \bigcap_{x \in V(X_{r+1})} N_{G_r}(x).$$

Then $\langle N_{r+1}\rangle_{G_r}$ has a Hamilton cycle $C$, by Proposition 2.1. We consider three cases.

*Case* 1. $m > l \geq r+1$. In this case $N_{r+1}$ contains no vertex in $W$ (since $u_0 \in V(X)$) and no vertex in $H_i$ for $i = 1, \ldots, r$ (since $\{u_1, \ldots, u_r\} \subseteq V(X)$). Thus $N_{r+1} = \{u_{r+1}, u_l, u_m\}$. So in this case $C$ is a 3-cycle containing the edge $u_l u_m$.

*Case* 2. $l \leq r$ *and* $m \geq r+1$. If $l = 0$, then $N_{r+1} \subseteq \{u_{r+1}, u_l, u_m\} \cup W$, and if $l \neq 0$, then $N_{r+1} \subseteq \{u_{r+1}, u_l, u_m\} \cup V(H_l)$. In either case, $u_m$ and $u_{r+1}$ are the only neighbours of $u_l$ in $N_{r+1}$. So $u_l u_m \in E(C)$.

*Case* 3. $l < m \leq r$. First, suppose $l \neq 0$. Then $N_{r+1} \subseteq \{u_{r+1}, u_l, u_m\} \cup V(H_l) \cup V(H_m)$. If $u_l u_m \notin E(C)$, then $u_{r+1}$ is a neighbour of both $u_l$ and $u_m$ on $C$ and the other neighbour of $u_l$ on $C$ is in $V(H_m)$, while the other neighbour of $u_m$ on $C$ is in $H_l$. But there are no edges between $H_l$ and $H_m$, and therefore $C$ cannot be a Hamilton cycle of $N_r+1$. This contradiction proves that $u_l u_m \in E(C)$. Next, suppose $l = 0$. Then $N_{r+1} \subseteq \{u_{r+1}, u_l, u_m\} \cup W \cup V(H_m)$, and by a similar proof as for the case $l \neq 0$, we can show that $v_l v_m \in E(G)$.

Thus $Y_{r+1}$ is a suitable $(k+2)$-clique in $G_r$, by Definition 2.10.

We have shown by induction that $Y_{r+1}$ is a suitable $(k+2)$-clique in $G_r$ for $r = 0, 1, \ldots, k+1$.

(b) Now suppose $G_0 \cong K_{k+3}$ and let $X_0$ be a $k$-clique in $Y_0$. Then, since we have shown in (a) that $G_{k+2}$ is locally $k$-nested-hamiltonian, the subgraph of $G_{k+2}$ induced by $\bigcap_{x \in V(X_0)} N_{G+2}(x)$ has a Hamilton cycle $C$. But $\bigcap_{x \in V(X_0)} N_{G+2}(x) = \{u_0, u_l, u_m\} \cup V(H_l) \cup V(H_m)$, where $u_l, u_m$ are the two vertices in $Y_0 - V(X_0)$. As in Case 3 above, we can prove that $u_l u_m \in E(C)$, and hence $Y_0$ is a suitable $(k+2)$-cycle in $G_{k+2}$. ■

## 3. Non-Hamiltonian and Nontraceable Locally $k$-Nested-Hamiltonian Graphs of Small Order

As mentioned in Section 1, Pareek and Skupień proved the following.

**Theorem 3.1** [13]**.** *The minimum order of a non-hamiltonian connected LH graph is* 11.

A graph is *traceable* if it has a Hamilton path (a path that visits every vertex). De Wet *et al.* proved the following.

**Theorem 3.2** [4, 7]**.** *The minimum order of a nontraceable connected LH graph is* 14.

In this section we construct non-hamiltonian and nontraceable locally $k$-nested-hamiltonian graphs of small order and determine the minimum order of non-hamiltonian and nontraceable locally 2-nested-hamiltonian graphs. We shall need the following result of de Wet.

**Theorem 3.3** [4]. *If $G$ is a non-hamiltonian connected LH graph of order* 12*, then* $\Delta(G) = 9$.

The smallest non-hamiltonian connected $LH$ graph $S_1$, which is shown in Figure 2 and again in Figure 5(a), may be obtained by the following construction.

Let $G$ be a $K_4$ with vertices labelled $u_0, u_1, u_2, u_3$. For $i = 0, 1, 2, 3$, denote by $Y_i$ the triangle in $G$ that does not contain the vertex $u_i$, then take a new vertex $h_i$ and add an edge between $h_i$ and every vertex in $Y_i$. This is equivalent to identifying $Y_i$ with a triangle in a graph $H_i \simeq K_4$ and denoting by $h_i$ the vertex in $H_i$ that was not involved in the identification. In the resulting graph $G^*$ the vertex $h_0$ is of degree 3. The vertices $z_1$, $z_2$, $z_3$ in Figure 5 can be seen as the result of three further such triangle identifications, using the triangles in $\langle N_{G^*}[h_0]\rangle$ that contain $h_0$. Lemma 2.15 confirms that the resulting graph $S_1$ is $LH$, and the fact that $S_1 - \{h_0, u_0, u_1, u_2, u_3\}$ has 6 components confirms that $S_1$ is non-hamiltonian.

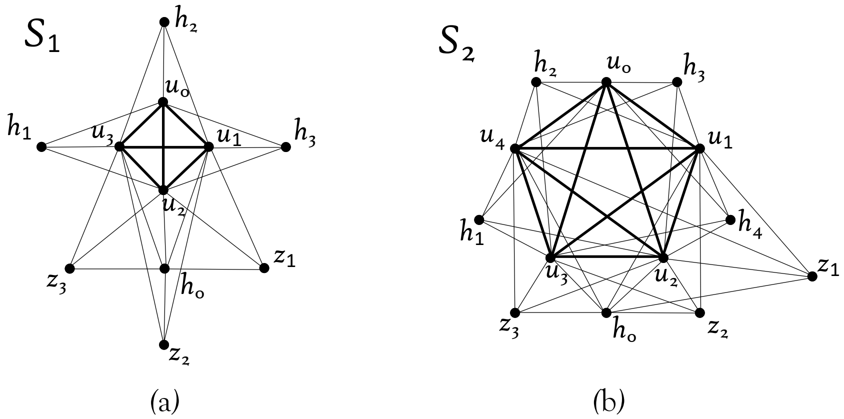


Figure 5. (a) A non-hamiltonian connected $LH$ graph of order 11. (b) A non-hamiltonian locally 2-connected graph of order 13.

The graph $S_2$ in Figure 5(b) is obtained by emulating the construction of $S_1$, but using $K_5$'s and $K_4$-identification instead of $K_4$'s and triangle identification. It follows from Lemma 2.15 that $S_2$ is locally 2-nested-hamiltonian. Since $S_2 - \{h_0, u_0, u_1, u_2, u_3, u_4\}$ has 7 components, $S_2$ is non-hamiltonian. We note that $N_{S_2}(u_2) = V(S_2) - \{u_2, h_2\}$ and that every 4-clique in $S_2$ reduces to a 3-clique in $\langle N(u_2)\rangle$. So it is clear from the construction of $S_1$ and $S_2$ that $\langle N(u_2)\rangle \cong S_1$. Thus $\langle N(u_2)\rangle$ is non-hamiltonian, which implies that $S_2$ is not $LH$.

There also exist non-hamiltonian connected $L^2H$ graphs of order 13 that are $LH$. Such a graph is depicted in Figure 7(b).

In the $K_5$ induced by the vertex set $\{h_0, u_1, u_2, u_3, u_4\}$ in $S_2$ there are four 4-cliques that contain the vertex $h_0$, but in the construction of $S_2$, only three of those 4-cliques were used in $K_4$-identification. By Lemma 2.15(a), the fourth such 4-clique may also be used, and doing so results in the locally 2-nested hamiltonian graph $S_2'$ shown in Figure 6. Since $S_2 - \{h_0, u_0, u_1, u_2, u_3, u_4\}$ has 8 components, $S_2'$ is nontraceable.

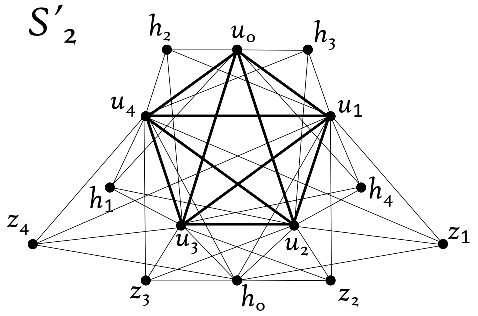


Figure 6. A nontraceable locally 2-nested-hamiltonian graph of order 14.

We are now ready to prove the main results of this section.

**Theorem 3.4.** *The minimum order of a non-hamiltonian locally* 2*-nested-hamiltonian graph is* 13.

***Proof.*** Let $G$ be a non-hamiltonian locally 2-nested-hamiltonian graph of minimum order. The graph $S_2$ in Figure 5(b) illustrates that $n(G) \leq 13$. Now suppose $n(G) \leq 12$.

First, suppose $G$ is not $LH$. Then there is a vertex $v \in V(G)$ such that $\langle N(v)\rangle$ is non-hamiltonian. Thus, by Theorem 3.1, $|N(v)| = 11$ and $n(G) = 12$. Since $G$ is $L^2H$, $\langle N(v)\rangle$ is $LH$. Now since $\langle N(v)\rangle$ is $LH$, it follows from Theorem 3.2 that $\langle N(v)\rangle$ is traceable. But then $G$ is hamiltonian, contrary to our assumption.

We therefore assume that $G$ is $LH$. Then $\Delta(G) \geq 7$ by Theorem 4.1, $n(G) - \Delta(G) \geq 3$, by Corollary 2.3, and it follows from Theorem 3.1 that $n(G) \in \{11, 2\}$.

Now let $w$ be a vertex of degree $\Delta(G)$ in $G$. Then $\langle N(w)\rangle$ has a Hamilton cycle $C = v_0v_1 \cdots v_t$ where $t = \Delta(G) - 1$. Let $X = G - N(w) - \{w\}$ and let $V(X) = \{x_1, \ldots, x_r\}$. Then $r = n(G) - \Delta(G) - 1$.

If $n(G) = 11$, then either $\Delta(G) = 7$ and $r = 3$, or $\Delta(G) = 8$ and $r = 2$. If $n(G) = 12$, then by Theorem 3.3, $\Delta(G) = 9$ and $r = 2$.

Thus there are three cases to consider. Each case has subcases depending on *comp*$(X)$, the number of components of $X$. Since $G$ is non-hamiltonian, *comp*$(X) \geq 2$. In each case where *comp*$(X) = 2$ and $r = 3$, we assume that

$E(X) = \{x_1x_2\}$. Then, since Lemma 2.4 implies that $\delta(G) \geq 4$, it follows that $x_i$ has at least three neighbours on $C$ for $i = 1, 2$, and $x_3$ has at least four neighbours on $C$.

*Case* 1. *$n(G) = 11$ and $\Delta = 7$ (so $r = 3$).* If *comp*$(X) = 2$, then by Lemma 2.4, $d_C(x_i) \geq 3$ for $i = 1, 2$ and $d_C(x_3) \geq 4$. Therefore by the pigeonhole principle $N(x_3)$ contains two consecutive vertices $v_i, v_{i+1}$ of $C$ and there are two distinct vertices $v_l, v_k$ in $V(C) - \{v_i, v_{i+1}\}$ such that $v_l \in N(x_1)$ and $v_k \in N(x_2)$ (since $G$ is 4-connected by Theorem 2.8). Thus $G$ has a Hamilton cycle $v_ix_3v_{i+1}v_{i+2}\cdots v_lx_1x_2v_kv_{k-1}\cdots v_{l+1}wv_{k+1}v_{k+2}\cdots v_i$ if $l < k$. A similar Hamilton cycle can be found if $k < l$.

If *comp*$(X) = 3$, then by Lemma 2.4, $d_C(x_i) \geq 4$ for $i = 1, 2, 3$ and hence by the pigeonhole principle each set $N(x_i)$ contains a pair of consecutive vertices of $C$. It therefore follows from Lemma 2.2(c) that there is a pair of consecutive vertices $v_j, v_{j+1}$ of $C$ that is contained in $N(x_1) \cap N(x_2) \cap N(x_3)$. Then $\{w, x_1, x_2, x_3\}$ is an independent set, and hence $\alpha(\langle N(v_j)\rangle) \geq 4$. Since $|N(v_j)| \leq \Delta(G) = 7$, it follows that $\langle N(v)\rangle$ is non-hamiltonian, contradicting our assumption that $G$ is $LH$.

*Case* 2. *$n(G) = 11$ and $\Delta(G) = 8$ (so $r = 2$).* By Lemma 2.4 $\delta(G) \geq 4$. Therefore by Lemma 2.2(b), we may assume without loss of generality that $N(x_1) = \{v_1, v_3, v_5, v_7\}$ and that $N(x_2) = N(x_1)$. Since $G$ is $L^2H$, it follows from Corollary 2.6 that $\langle\{v_1, v_3, v_5, v_7\}\rangle \cong K_4$. Therefore $\{v_{i-1}, v_{i+1}, w, x_1, x_2\} \subset N(v_i)$ so that $d(v_i) = 8$ for $i = 1, 3, 5, 7$. Since $\Delta(G) = 8$, $v_2$ is not adjacent to either of $v_5$ or $v_7$. If $v_2$ is adjacent to $v_0$, then $v_1x_1v_7v_6v_5x_2v_3v_4wv_2v_0v_1$ is a Hamilton cycle in $G$. Hence $N(v_1) \cap N(v_2) = \{w, v_3\}$ and $|N(v_1) \cap N(v_2)| = 2$ contradicting that $\langle N(v_1) \cap N(v_2)\rangle$ is hamiltonian as is required by Definition 1.6.

*Case* 3. *$n(G) = 12$.* In this case, $\Delta(G) = 9$ by Theorem 3.3, and therefore $r = 2$.

By Lemma 2.2(b) we may assume without loss of generality that $N(x_1) = \{v_1, v_3, v_5, v_7\}$ and it follows that $N(x_2) = N(x_1)$. Since $G$ is $L^2H$, $\langle N(x_1)\rangle$ is $LH$ and since $d(x_1) = 4$, we get $\langle\{v_1, v_3, v_5, v_7\}\rangle \cong K_4$. We now show that, with the exception of $v_8v_0$ there are no edges in $G$ between vertices in $\{v_0, v_2, v_4, v_6, v_8\}$, since otherwise $G$ would be hamiltonian.

If $v_2v_6 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6v_2wv_8v_0v_1$ is a Hamilton cycle in $G$. If $v_2v_4 \in E(G)$, then $v_1x_1v_3v_4v_2wv_6v_5x_2v_7v_8v_0v_1$ is a Hamilton cycle in $G$.

Hence $v_4, v_6 \notin N(v_2)$, and by symmetry $v_4 \notin N(v_6)$. So $\{v_2, v_4, v_6\}$ is an independent set in $G$.

If $v_0v_2 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6wv_8v_0v_2v_1$ is a Hamilton cycle in $G$. If $v_0v_4 \in E(G)$, then $v_1v_2v_3x_2v_5v_6wv_4v_0v_8v_7x_1v_1$ is a Hamilton cycle in $G$. If $v_0v_6 \in E(G)$, then $v_1v_2wv_6v_0v_8v_7x_2v_5v_4v_3x_1v_1$ is a Hamilton cycle in $G$.

Hence $v_0$ does not have a neighbour in $\{v_2, v_4, v_6\}$ and by symmetry, neither does $v_8$.

Since $\delta(G) \geq 4$, it follows that each of $v_2, v_4, v_6$ has three neighbours in the set $\{v_1, v_3, v_5, v_7\}$ and each of $v_0, v_8$ has two neighbours in this set. From the pigeonhole principle it follows that at least one of $v_1, v_3, v_5, v_7$ has degree at least 10, contradicting our assumption that $\Delta(G) = 9$.

Thus we have proved that $n(G) \geq 13$. ■

In view of Theorem 3.2, our next result is somewhat surprising.

**Theorem 3.5.** *The minimum order of a nontraceable locally* 2*-nested-hamiltonian graph is* 14.

***Proof.*** Let $G$ be a nontraceable locally 2-nested-hamiltonian graph of minimum order. The graph in Figure 6 illustrates that $n(G) \leq 14$.

Now suppose $n(G) \leq 13$. Then $G$ is not $LH$, by Theorem 3.2. Thus there is a vertex $v \in V(G)$ such that $\langle N(v) \rangle$ is $LH$ but non-hamiltonian. So $|N(v)| \geq 11$ by Theorem 3.1. But $|N(v)| \leq 12$, and therefore $\langle N(v) \rangle$ is traceable by Theorem 3.2. Thus $\langle N[v] \rangle$ is a hamiltonian subgraph of $G$ with 12 vertices. Since $G$ is nontraceable, this implies that $n(G) \geq 14$. ■

We now turn our attention to non-hamiltonian and nontraceable locally $k$-nested-hamiltonian graphs of small order, for higher values of $k$. We first construct such graphs that are non-$L^m H$ for each $m \in \{0, 1, \ldots, k-1\}$.

**Theorem 3.6.** *For each $k \geq 1$ there exists a locally $k$-nested-hamiltonian graph of order $9 + 2k$ that is non-$L^m H$ for $m = 0, 1, \ldots, k-1$, and for $k \geq 2$ there exists a nontraceable locally $k$-nested-hamiltonian graph of order $10 + 2k$ that is non-$L^m H$ for $m = 0, 1, \ldots, k-1$.*

***Proof.*** We already know that $S_1$ is a locally hamiltonian graph of order 11 that is non-$L^0 H$ and that $S_2$ is a locally 2-nested-hamiltonian graph of order 13 that is neither $L^0 H$ nor $LH$. We also know that $S_2'$ is a nontraceable locally $k$-nested-hamiltonian graph of order 14. We now generalize these constructions for $k \geq 3$.

Let $G$ be a $K_{k+3}$ with vertex set $\{u_0, u_1, \ldots, u_{k+2}\}$. For $i = 0, 1, \ldots, k+2$ let $Y_i$ be the $(k+2)$-clique in $Y$ that does not contain the vertex $y_i$, then take a new vertex $h_i$ and add an edge between $h_i$ and every vertex in $Y_i$. This is equivalent to identifying $Y_i$ with a $K_{k+2}$ in a graph $H_i \cong K_{k+3}$ and denoting by $h_i$ the vertex in $H_i$ that was not used in the identification, for $i = 0, 1, \ldots, k+2$. Call the resulting graph $G^*$. Since $G \cong K_{k+3}$, Lemma 2.15 implies that all the $(k+2)$-cliques in $G$ may be used in $(k+2)$-identification. So $G^*$ is a locally $k$-nested-hamiltonian graph of order $2(k+3)$.

Since $d_{G^*}(h_0) = k+2$, Lemma 2.15(a) implies that $h_0$ may now be used $k+2$ times in $(k+2)$-identifications, each time as a member of a different $(k+2)$-clique in $\langle N_{G^*}[h_0]\rangle$. To create a non-hamiltonian graph we only need to use three of the $(k+2)$-cliques that contain $h_0$, but to create a nontraceable graph we need to use four, as we did in the construction of $S_2$ and $S_2'$. To create $S_k$, we choose three $(k+2)$-cliques $Q_1, Q_2, Q_3$ in $G^*$ that each contain the vertices $h_0$ and $u_2$, then we take three new vertices $z_1, z_2, z_3$ and add an edge between $z_i$ and every vertex in $Q_i$ for $i = 1, 2, 3$. To create $S_k'$ we choose a fourth $(k+2)$-clique in $G^*$ and join each of its vertices to a new vertex $z_4$. By Lemma 2.15(a), both $S_k$ and $S_k'$ are locally $k$-nested-hamiltonian. We note that $n(S_k) = 9 + 2k$ and $n(S_k) = 10 + 2k$.

Let $W = V(G) \cup \{h_0\}$. Then $|W| = k + 4$ and $S_k - W$ has $k + 5$ components, while $S_k' - W$ has $k + 6$ components. So $S_k$ is non-hamiltonian and $S_k'$ is nontraceable.

Next, we prove by induction that $S_k$ is not $L^m H$ for $m = 0, 1, \ldots, k-1$. We already know that this holds for $k = 1, 2$. Now let $k \geq 3$. Then $N(u_2) = V(S_k) - \{u_2, h_2\}$. So it is clear from the construction of $S_{k-1}$ that $\langle N(u_2)\rangle \cong S_{k-1}$. But by our induction hypothesis, $S_{k-1}$, and hence $\langle N(u_2)\rangle$, is not $L^m H$ for $m = 0, 1, \ldots, k-2$. Thus, by Definition 1.6, $S_k$ is not $L^m H$ for $m = 1, \ldots, k-1$. We have already shown that $S_k$ is not $L^0 H$. ■

Next we construct a connected non-hamiltonian $L^{\leq k} H$ graph of order $9 + 2k$ for each $k \geq 2$ by generalizing the graph in Figure 7(a).

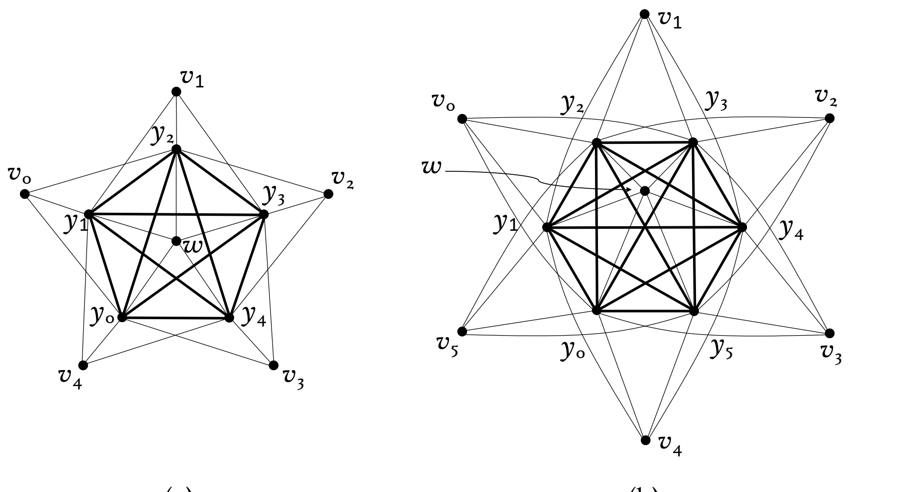


Figure 7. (a) A non-hamiltonian $LH$ graph of order 11. (b) A non-hamiltonian $L^{\leq 2} H$ graph of order 13.

**Theorem 3.7.** *For each $k \geq 1$ there exists a connected non-hamiltonian $L^{\leq k} H$ graph of order $9 + 2k$.*

***Proof.*** A connected non-hamiltonian $L^{\leq k}H$ graph $G_k$ of order $9+2k$ can be constructed in the following way. Let $Y$ be a $K_{k+4}$ with $V(Y)=\{y_0,y_1,\ldots,y_{k+3}\}$ and add a vertex $w$ that is adjacent to all vertices in $V(Y)$. Then add $k+4$ vertices $v_i$, $i=0,1,\ldots,k+3$, and let $N(v_i)=\{y_i,y_{i+1},\ldots,y_{i+k+1}\}$, where subscripts are taken modulo $k+4$. The graphs $G_1$ and $G_2$ are shown in Figure 7, where the edges belonging to $Y$ are represented by heavy lines.

It is easily seen that the graph $G_1$ is $LH$. Now let $k\geq 2$ and suppose we have shown that $G_{k-1}$ is $L^{\leq k-1}H$. We note that $N_{G_k}(y_0)=V(G_k)-\{y_0,v_1,v_2\}$, and therefore $\langle N_{G_k}(y_0)\rangle\cong G_{k-1}-v_1$. It is easily seen that $G_{k-1}-v_1$ is hamiltonian, and it follows from Lemma 2.7 that $G_{k-1}-v_1$ is $L^{\leq k-1}H$. Thus $\langle N_{G_k}(y_0)\rangle$ is $L^m(H)$ for $m=0,1,\ldots,k-1$. Since $\langle N_{G_k}(y_i)\rangle\cong N_{G_k}(y_0)$, this proves that for $i=0,1,\ldots,k+4$, the graph $\langle N_{G_k}(y_i)\rangle$ is $L^m(H)$ for $m=0,1,\ldots,k-1$. Furthermore, $\langle N_{G_k}(w)\rangle\cong K_{k+4}$ and $\langle N_{G_k}(v_i)\rangle\cong K_{k+2}$ for $i=0,1,\ldots,k+4$. So it follows that the neighbourhood of every vertex of $G_k$ induces a graph that is $L^mH$ for $m=0,1,\ldots,k-1$. Hence, by Definition 1.6, $G_k$ is $L^mH$ for $m=1,\ldots,k$, i.e., $G$ is $L^{\leq k}H$. To see that $G_k$ is non-hamiltonian, note that $V(Y)$ is a vertex cut, $|V(Y)|<|V(G)|/2$ and $V(G)-V(Y)$ is an independent set of vertices. ∎

Now suppose $G$ is a connected non-hamiltonian $L^{\leq k}H$ graph that contains an induced $K_{1,k+3}$, with $v$ as its central vertex. Then $\alpha(\langle N(v)\rangle)\geq k+3$, and therefore $|N(v)|\geq 2k+6$ since $G$ is $LH$. Hence, by Corollary 2.3, $n(G)\geq 9+2k$. Thus, if Conjecture 1.9 is true, Theorem 3.6 would imply that the minimum order of a connected non-hamiltonian $L^{\leq k}H$ graph is $9+2k$. By Theorems 3.1, 3.4 and 3.6, this is indeed the case for $k=1,2$.

It should be pointed out that the graphs constructed in the proof of Theorem 3.7 were first constructed in [5], where they were described as $LH$ graphs that are $k+2$-connected. The fact that they are also $L^mH$ for every $m=2,3,\ldots,k$ was not addressed there. In the light of Conjecture 1.9 it is interesting to note that these graphs are locally $(k+1)$-connected and contain an induced $K_{1,k+3}$. We shall now show that they do not contain an induced $K_{1,4}$.

**Corollary 3.8.** *For any $k\geq 1$ there exists a $K_{1,k+4}$-free connected non-hamiltonian $L^{\leq k}H$ graph of order $9+2k$.*

***Proof.*** Consider the graph $G_k$ that is $L^{\leq k}H$ constructed in the proof of Theorem 3.7. We use the same nomenclature as in the proof of Theorem 3.7. The vertex in a $K_{1,q}$ star that has degree greater than 1 is referred to as the *centre* vertex of the star. Since the neighbourhoods of the vertices $u,v_1,v_2,\ldots,v_{k+4}$ all induce complete graphs, it is clear that none of these vertices can be the centre vertex of an induced $K_{1,k+4}$. Since $\langle N(w_i)\rangle\cong\langle N(w_j)\rangle$ for $\{i,j\}\subseteq\{0,1,\ldots,k+3\}$, we need only consider the subgraph induced by $N(w_{k+3})=$

$\{w_0, w_1, \ldots, w_{k+2}, u, v_2, v_3, \ldots, v_{k+3}\}$. Since $\langle\{w_0, w_1, \ldots, w_{k+2}\}\rangle$ induces a complete graph, say $W_{k+3}$, and $w_i \in N(u)$, $i = 0, 1, \ldots, k+3$, and $v_i$, $i = 0, 1, \ldots, k+3$, only has neighbours in $V(W)$, it follows that $\alpha(\langle N(w_{k+3})\rangle) = k + 3$. ■

Thus Conjecture 1.9, if true, would be a best possible result.

Similar constructions for connected nontraceable graphs that are $L^{\leq k}H$ do not yield graphs of order $10 + 2k$, as is the case for nontraceable $L^kH$ graphs that are $L^{\leq k-1}C$, but rather graphs of order $12 + 2k$. This is because it is not possible to add another vertex of degree $k + 2$ to the non-hamiltonian graph in such a way that the resulting graph is still $L^{\leq k}H$. Figure 8 gives an example of a nontraceable graph that is $L^{\leq 2}H$ of order 16. It is not known at this stage whether it is possible to improve on this result. It is speculated that this is due to these graphs being $LH$, since for connected $LH$ graphs, the smallest non-hamiltonian graph has order 11 $(= 9 + 2k)$, but the smallest nontraceable graph has order 14 $(= 12 + 2k)$.

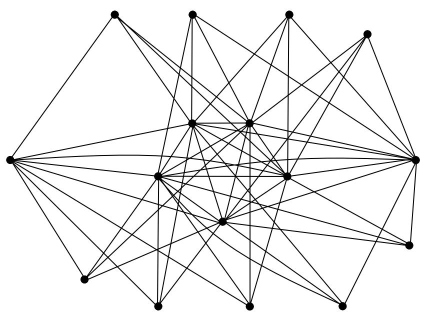

Figure 8. A nontraceable $L^{\leq 2}H$ graph of order 16.

**Observation 3.9.** *If $G$ is any non-hamiltonian connected $LH$ graph, then according to Corollary* 2.3, $\Delta(G) \leq n - 3$. *However, if $G$ is a non-hamiltonian, locally* 2*-nested-hamiltonian graph, then $\Delta(G)$ can be as large as $n - 1$.*

The graph in Figure 9 is an example of a non-hamiltonian locally 2-nested-hamiltonian graph of order 15 for which the maximum degree is 14. To see that 15 is the smallest order for which this is possible, note that if $G$ is $L^2H$ with $\Delta(G) = n - 1$, there exists a vertex $v \in V(G)$ such that $d(v) = n - 1$ and $\langle N(v)\rangle$ is $LH$ and nontraceable, otherwise $G$ is hamiltonian. Therefore $|N(v)| \geq 14$ and $n(G) \geq 15$.

In the next section we consider locally $k$-nested-hamiltonian graphs with small maximum degree.

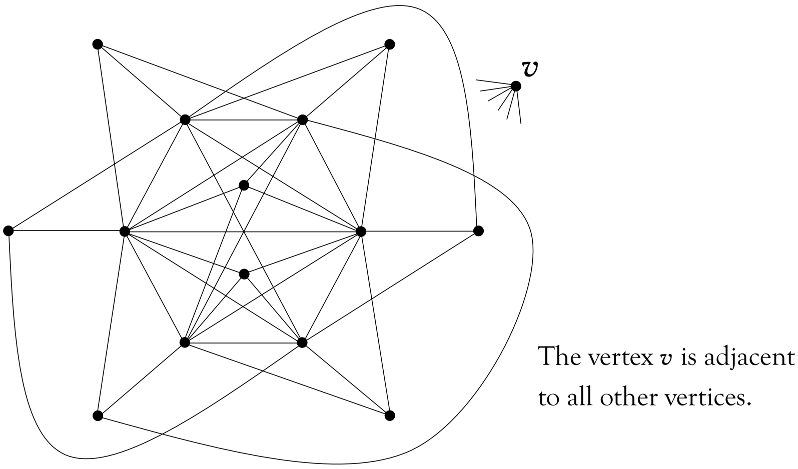


Figure 9. A non-hamiltonian locally 2-nested-hamiltonian graph of order 15 with maximum degree 14.

## 4. Hamiltonicity of Locally $k$-Nested-Hamiltonian Graphs with Restricted Maximum Degree

The following result is proved in [16].

**Theorem 4.1** [16]**.** *If $G$ is a connected LH graph with $\Delta(G) \leq 6$, then $G$ is fully cycle extendable.*

As mentioned in Section 1, it is known that there exist non-hamiltonian connected $LH$ graphs with maximum degree 8, but it remains an open question whether all connected $LH$ graphs with maximum degree 7 are hamiltonian. We suspect that the Hamilton Cycle Problem for $LH$ graphs with maximum degree at most 8 is solvable in polynomial time. In [5] we proved the following.

**Theorem 4.2.** *The Hamilton Cycle Problem for LH graphs with maximum degree 9 is NP-complete.*

We now consider the hamiltonicity of locally 2-nested-hamiltonian graphs. First we prove the following.

**Theorem 4.3.** *If $G$ is a locally 2-nested-hamiltonian graph with maximum degree at most 7, then $G$ is fully cycle extendable.*

***Proof.*** Suppose $G$ contains a nonextendable non-hamiltonian cycle $C = v_0v_1 \cdots v_{t-1}v_0$. Then, since $G$ is connected, some vertex on $C$, say $v_0$, has a neighbour $x$ in $V(G) - V(C)$. We consider two cases.

*Case* 1. $v_{t-1}v_1 \notin E(G)$. In this case $\{v_{t-1}, v_1, x\}$ is an independent set in $\langle N(v_0)\rangle$. Since $\langle N(v_0)\rangle$ is $LH$, it follows from Lemma 2.4 that $\delta(\langle N(v_0)\rangle) \geq 3$. So each vertex in $\{v_{t-1}, v_1, x\}$ has at least three neighbours in $N(v_0) - \{v_{t-1}, v_1, x\}$. But $|N(v_0)| \leq 7$, and therefore $N(v_0) - \{v_{t-1}, v_1, x\}$ has at most four vertices. So it contains a vertex $v_j$ on $C$ that is a common neighbour of $v_1$, $v_{t-1}$ and $x$. (Since $C$ is a nonextendable cycle, any neighbour of $v_1$ in $N(v_0)$ necessarily lies on $C$.) Since $G$ is $L^2H$, it follows that $\langle N(v_0) \cap N(v_i)\rangle$ is hamiltonian, and therefore $x$ has at least two neighbours in $N(v_0) \cap N(v_i)$. But since $C$ is nonextendable, neither $v_{i-1}$ nor $v_{i+1}$ is adjacent to $x$. So $N(v_i)$ contains at least two vertices other than $v_{i-1}, v_{i+1}, v_{t-1}, v_{t+1}, x, v_0$. Since $d(v) \leq 7$, this implies that either $v_{i-1} = v_1$ or $v_{i+1} = v_{t-1}$, and therefore $i \in \{2, t-2\}$.

By symmetry, we may assume that $i = 2$. Then there are two vertices $v_j, v_k$ on $C$ with $4 \leq j \leq t-2$ such that

$$N(v_2) \cap N(v_0) = \{x, v_1, v_3, v_{t-1}, v_j, v_k\},$$

and $v_j$ and $v_k$ are the only neighbours of $x$ in $N(v_0) \cap N(v_2)$. But $v_3 \notin N(v_1)$, since otherwise $v_1v_3v_4 \cdots v_{t-1}v_0xv_2v_1$ would extend the cycle $C$. So $v_j$ and $v_k$ are also the only neighbours of $v_1$ in $N(v_0) \cap N(v_2)$. But then $\langle N(v_0) \cap N(v_2)\rangle$ is not hamiltonian, since the union of the neighbourhoods of any two distinct vertices on a 6-cycle is at least 3. By symmetry, a similar contradiction is obtained if $i = t-2$.

*Case* 2. $v_{t-1}v_1 \in E(G)$. In this case $x$ has at least three neighbours in the set $N(v_0) - \{v_1, x, v_{t-1}\}$ and $v_1$ has at least two neighbours in that set. So $v_1$ and $x$ have a common neighbour $v_i$ in $N(v_0)$, with $2 \leq i \leq t-2$. If $v_{i-1}v_{i+1} \in E(G)$, then the cycle $v_{i-1}v_{i+1}v_{i+2} \cdots v_{t-1}v_0xv_iv_1 \cdots v_{i+1}$ is an extension of $C$. Hence $v_{i-1}v_{i+1} \notin E(G)$, but then we have Case 1. ∎

It is routine to confirm that the graph in Figure 7(b) is a non-hamiltonian connected locally 2-nested-hamiltonian graph with $\Delta = 10$ (it is also $LH$). We do not know whether non-hamiltonian connected locally 2-nested-hamiltonian graphs with maximum degree 8 or 9 exist.

The proof of our next theorem relies on the well-known result of Garey, Johnson and Tarjan [8] that the Hamilton Cycle Problem for planar cubic graphs is NP-complete.

**Theorem 4.4.** *The Hamilton Cycle Problem for locally* 2*-nested-hamiltonian graphs with maximum degree* 12 *is NP-complete.*

***Proof.*** The proof is based on transforming a case of the Hamilton Cycle Problem for cubic graphs to locally 2-nested-hamiltonian graphs. We start with a cubic graph $G'$ and construct a locally 2-nested-hamiltonian graph $G$ that is hamiltonian if and only if $G'$ is hamiltonian. This is sufficient to establish the result.

Each vertex in $G'$ is represented by a copy of $K_5$ in $G$, and will be referred to as a *node* in $G$.

Each edge in $G'$ is represented by a more complex structure, that is based on the graph $H$ in Figure 10, which is the graph in Figure 5(b) *the vertices of which have been relabeled* for convenience. We use $K_4$-identification to combine $H$ with two copies of graph $D$ in Figure 10 in the following way: using the first copy of $D$ we identify $u_j$ and $x_j$, $j = 1, 2, 3, 4$, and using the second copy of $D$ we identify $v_j$ and $x_j$, $j = 1, 2, 3, 4$. This creates the graph $B_i$ shown in Figure 11.

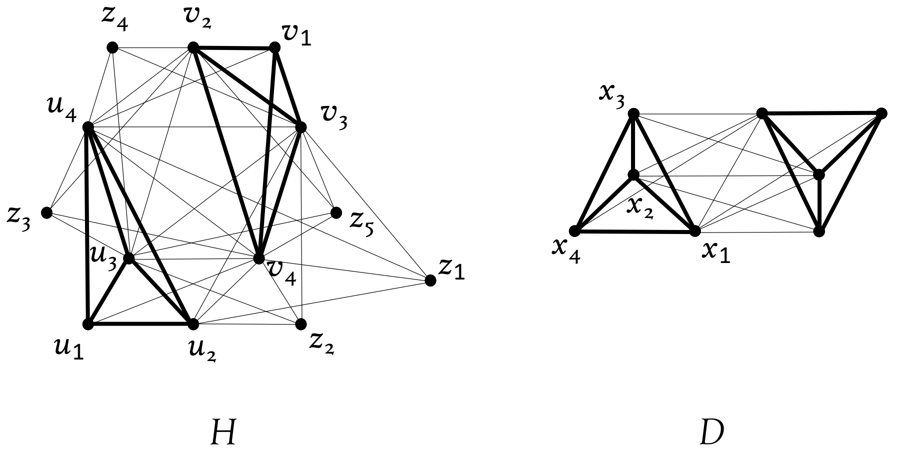


Figure 10. The graphs $H$ and $D$ used in the proof of Theorem 4.4.

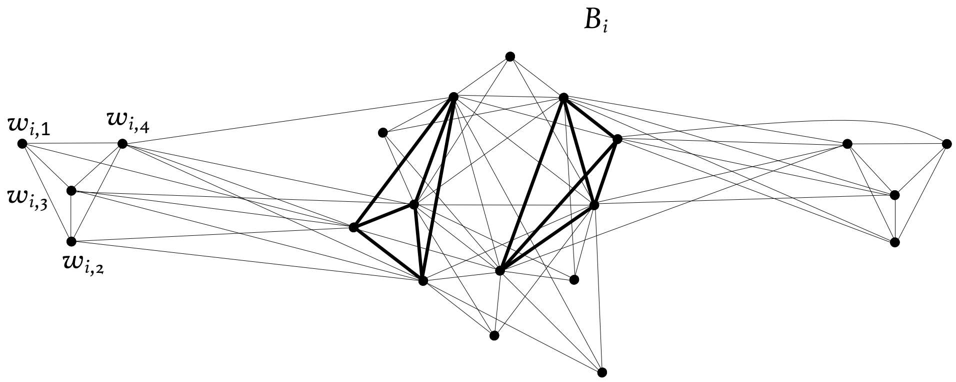


Figure 11. The graph $B_i$ used in the proof of Theorem 4.4.

The edges in $G'$ are represented by copies of $B_i$ in $G$, and will be referred to as *borders*. The borders are connected to the nodes by means of $K_4$-identification.

Let the vertices in a node in $G$ be $y_1, y_2, y_3, y_4, y_5$ and let the vertices in $B_i$

be labeled as shown in Figure 11. Since each vertex in $G'$ has degree three, each node in $G$ is attached to three copies of $B_i$. We identify the vertices as shown in Table 1 (after each vertex identification, the resulting vertex retains the $y$-label). We use the graphs $B_1$, $B_2$ and $B_3$ for illustrative purposes. See Figure 12 (the heavy lines in $G$ represent edges belonging to the nodes).

| Vertex in node | Vertex in $B_i$ |
|---|---|
| $y_1$ | $w_{1,2}$ |
| $y_2$ | $w_{1,1}$ |
| $y_4$ | $w_{1,4}$ |
| $y_5$ | $w_{1,3}$ |
| $y_1$ | $w_{2,3}$ |
| $y_2$ | $w_{2,2}$ |
| $y_3$ | $w_{2,1}$ |
| $y_5$ | $w_{2,4}$ |
| $y_1$ | $w_{3,1}$ |
| $y_2$ | $w_{3,2}$ |
| $y_3$ | $w_{3,3}$ |
| $y_4$ | $w_{3,4}$ |

Table 1. Vertices identified in the proof of Theorem 4.4.

Checking the degrees of the vertices that have been identified shows that $\Delta(G) = 12$ and by Theorems 2.11 and 2.14 and Lemma 2.15, $G$ is connected, $LC$ and $L^2H$.

Figure 13 shows how a Hamilton cycle in $G'$ can be translated to a Hamilton cycle in $G$ (the heavy lines represent edges that are in the Hamilton cycles). To see that if $G$ is hamiltonian, then $G'$ is also hamiltonian, consider the graph $H$ in Figure 10 that forms the core of the connection between two nodes in $G$. Note that $u_2, u_3, u_4, v_2, v_3, v_4$ are the only neighbours of the five vertices labeled $z$ in Figure 10. Therefore any Hamilton cycle in $G$ must contain a subpath of order 11 that contains only the vertices in $\{u_2, u_3, u_4, v_2, v_3, v_4, z_1, z_2, z_3, z_4, z_5\}$, in some order. Thus if there is a border between two nodes $Z_i$ and $Z_j$, then every Hamilton cycle in $G$ has at most one path from node $Z_i$ to node $Z_j$ that passes through the border between them. Since each node has three borders incident to it, the result follows. ■

The proof of Theorem 4.4 relies on the fact that the graph $H$ in Figure 10 is locally 2-nested-hamiltonian and non-hamiltonian, has order 13, contains 7 independent vertices of degree 4 each and is traceable between any two vertices of degree 4. In Section 3 we constructed, for each $k \geq 2$, a non-hamiltonian locally $k$-nested-hamiltonian graph $G_k$ of order $9 + 2k$ that has $k + 5$ vertices of degree

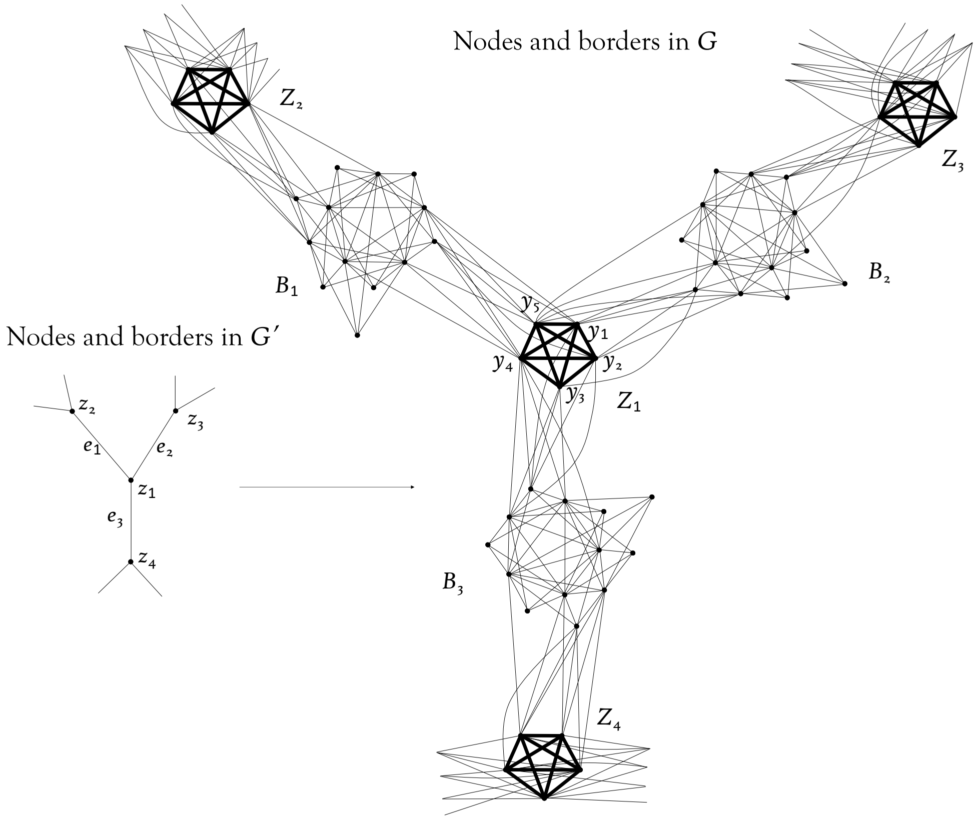


Figure 12. Converting the graph $G'$ to the graph $G$ in Theorem 4.4.

$k+2$ each, such that $G_k$ is traceable between any two vertices of degree $k+2$. We conclude that NP-completeness theorems for locally $k$-nested-hamiltonian graphs with restricted maximum degree are possible for all $k \geq 3$. The smallest value of the maximum degree that these constructions yield depends on the choice of neighbours for the vertices of degree $k+2$ in the graphs of order $9+2k$. As $k$ increases, there is increasing flexibility in the choice of neighbours for the vertices of degree $k+2$. Detailed calculations show that for $k = 3, 4, 5, 6, 7, 8$ the Hamilton Cycle Problem for locally $k$-nested-hamiltonian graphs with maximum degree $3k+6$ is NP-complete. Since the constructions follow a regular pattern, we expect that this is the case for all $k \geq 1$.

When investigating the possible NP-completeness of the Hamilton Cycle Problem for graphs that are $L^{\leq k}H$, we do not have the advantage of a theorem equivalent to Theorem 2.11 (see Remark 2.13). This means that any construction has to be checked in detail to confirm that the resulting graph is $L^{\leq k}H$. We begin with $k = 2$.

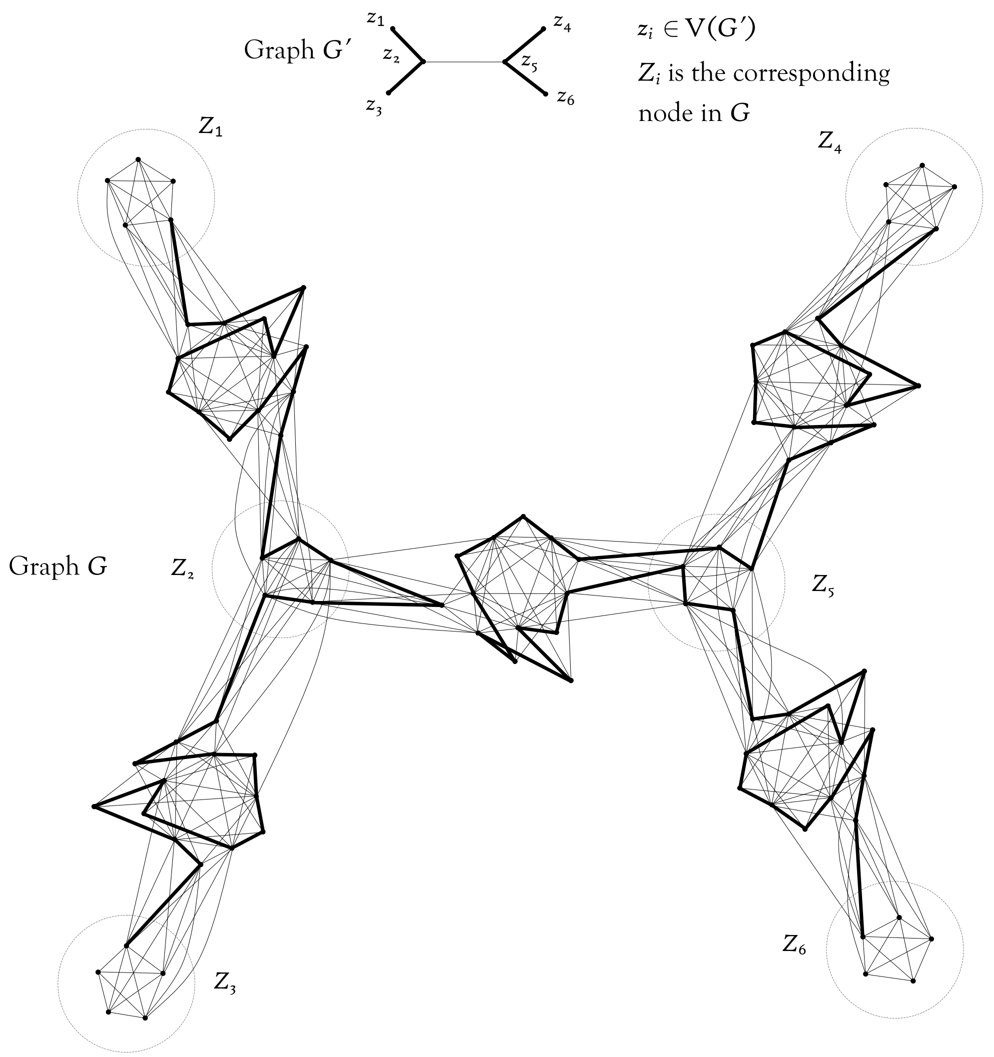


Figure 13. Translating a Hamilton cycle from $G'$ to $G$ in Theorem 4.4.

**Theorem 4.5.** *The Hamilton Cycle Problem for $L^{\leq 2}H$ graphs with maximum degree* 13 *is NP-complete.*

***Proof.*** As in the proof of Theorem 4.4, the proof is based on transforming a case of the Hamilton Cycle Problem for cubic graphs to locally 2-nested-hamiltonian graphs. Starting with any cubic graph $G'$, we construct a $L^{\leq 2}H$ graph $G$ that is hamiltonian if and only if $G'$ is hamiltonian. In this case, the graph $H$ is the graph shown in Figure 7(b).

We combine $H$ with two copies of the graph $D$ to create the graph shown in Figure 14. When using $K_4$-identification to connect borders to nodes to construct the graph $G$, we take care to limit the degree of vertices in the nodes to 10, as shown in Figure 15. Since the smallest connected non-hamiltonian $LH$ graph has order 11, this ensures that in $G$, for any vertex $v$ that lies in a node, $\langle N(v)\rangle$ is a hamiltonian graph. We still have to confirm that for any vertex $u$ that is in a border and adjacent to a node, $\langle N(u)\rangle$ is hamiltonian. This is easily done, since there are only eight such vertices in any border (and only 6 of them have degree at least 11), and by symmetry, only one border has to be checked (see Figures 14 and 15). It follows that $G$ is both $LH$ and $L^2H$.

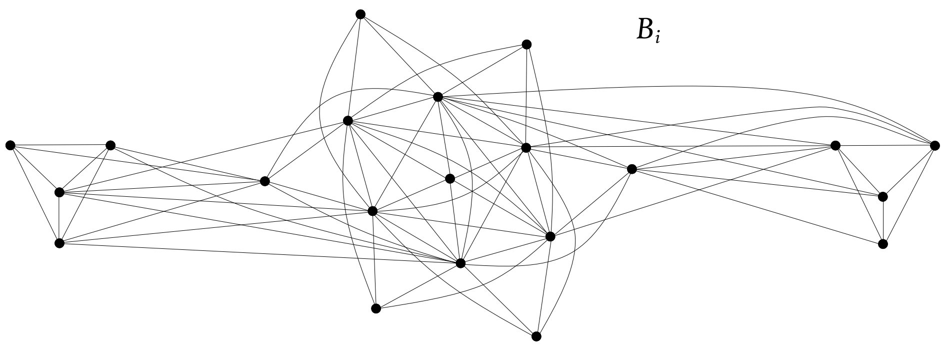


Figure 14. A border used in the construction of the graph $G$ in Theorem 4.5.

An argument similar to the one used in Theorem 4.4 can be used to show that if $G$ is hamiltonian then $G'$ is hamiltonian. To see that $G$ is hamiltonian if $G'$ is hamiltonian, the reader is referred to Figure 15, where the heavy lines represent edges that are in a Hamilton cycle. ■

Detailed calculations for the cases $k = 3$ and $k = 4$ show that the Hamilton Cycle Problem is NP-complete for $L^{\leq k}H$ graphs that have maximum degree 16 for $k = 3$ and maximum degree 19 for $k = 4$. There appears to be a pattern according to which the Hamilton Cycle Problem is NP-complete for $L^{\leq k}H$ graphs that have maximum degree $3k + 7$, for $k \geq 2$. Again there is reason to expect that the relationship will hold for all values of $k \geq 2$, since the pattern of the construction is quite regular. It is an interesting question whether these results would be best possible, particularly since for $k = 1$ we know the Hamilton Cycle Problem is NP-complete for maximum degree $3k + 6$. It should be noted that constructions very similar to the ones used in Theorem 4.5 and the discussion in this paragraph appeared in [5]. However, in [5] we used them to prove the Hamilton Cycle Problem is NP-complete for $LH$ graphs that are $(k+2)$-connected. The fact that these graphs are also $L^{\leq k}H$ was not addressed there.

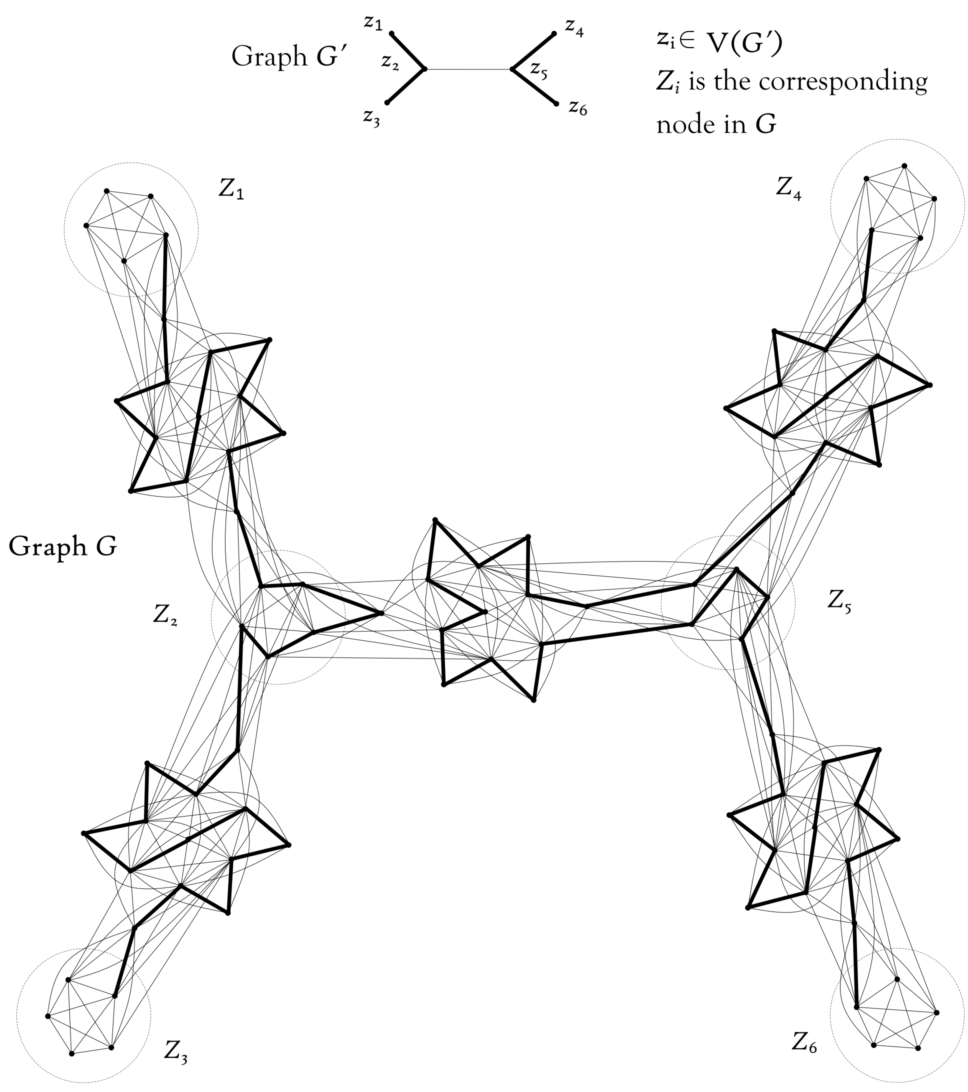


Figure 15. Translating a Hamilton cycle from $G'$ to $G$ in Theorem 4.5.

## 5. The Connection Between $k$-Trees and $L^{k-2}H$ Graphs

We begin this section by stating some basic properties of $k$-trees. The first follows directly from the definition of a $k$-tree.

**Proposition 5.1.** *If $G$ is a $k$-tree, then the neighbourhood of every vertex of degree $k$ in $G$ induces a $k$-clique, and vertices of degree $k$ may be recursively removed until only a $k$-clique remains.*

A graph $G$ of order $n$ has a *perfect elimination ordering* if the vertices in $G$ may be labelled $v_1, v_2, \ldots, v_n$ such that $\langle N[v_i]\rangle$ is a clique in $G - \{v_1, \ldots, v_{i-1}\}$ for $i = 1, \ldots, n-1$. It is well-known that a graph has a perfect elimination ordering if and only if it is a *chordal graph* (a graph in which every cycle of length greater than 3 has a chord). Thus Proposition 5.1 implies the following.

**Corollary 5.2.** *Every $k$-tree is a chordal graph.*

From the construction procedure of $k$-trees we also observe the following.

**Observation 5.3.** *Let $X$ be a $k$-clique in a graph that is a $k$-tree. If a vertex is added to the graph with an edge between the new vertex and each vertex in $V(X)$, we call this* using *$X$ in the construction of a larger $k$-tree. If $X$ is used $r$ times $(r \geq 0)$ in the construction of a $k$-tree $G$, then $G - V(X)$ has $r+1$ components, each of which contains one vertex of $\bigcap_{x\in V(X)} N(x)$.*

The next result is due to Rose [14].

**Lemma 5.4** [14]**.** *Let $G$ be a $k$-tree and let $u$ and $v$ be any pair of nonadjacent vertices in $G$. Then there are exactly $k$ internally disjoint $u$-$v$ paths in $G$.*

The smallest non-hamiltonian connected $LH$ graph (depicted in Figure 2) happens to be a maximal planar graph as well as a 3-tree. (Note that we can recursively delete a vertex of degree 3 whose neighbourhood is a $K_3$, until only a $K_4$ remains.) This prompted us to have a closer look at the connection between $LH$ graphs, 3-trees and maximal planar graphs.

It is well known that every maximal planar graph of order $n$ has exactly $3n-6$ edges, and an easy calculation shows that the same is true for 3-trees. Markenzon, Justel and Paciornik [10] found a relationship between maximal planar graphs and simple-clique 3-trees.

**Theorem 5.5** [10]**.** *A graph $G$ of order $n \geq 3$ is a simple-clique 3-tree if and only if it is a chordal maximal planar graph.*

Skupień [15] found a relationship between $LH$ graphs and maximal planar graphs.

**Theorem 5.6** [15]**.** *A connected $LH$ graph $G$ of order $n \geq 3$ is a maximal planar graph if and only if $|E(G)| = 3n-6$.*

For ease of reference, we combine and restate these two theorems as Theorem 5.7.

**Theorem 5.7.** *A connected graph $G$ of order $n \geq 3$ is a simple-clique 3-tree if and only if $G$ is a chordal $LH$ graph with $|E(G)| = 3n-6$.*

Markenzon, Justel and Paciornik [10] also found a relationship between 2-trees and maximal outerplanar graphs.

**Theorem 5.8.** [10] *A graph $G$ of order $n \geq 3$ is a simple-clique 2-tree if and only if $G$ is a maximal outerplanar graph.*

A maximal outerplanar graph of order at least 3 is obviously hamiltonian. So it follows from Theorems 5.7 and 5.8 that for $k = 2, 3$, the class of $L^{k-2}H$ graphs contains all simple-clique $k$-trees.

We note that the non-hamiltonian locally $k$-nested-hamiltonian graphs of order $9 + 2k$ constructed in the proof of Theorem 3.6 are $(k + 2)$-trees. We now prove the main result of this section, which establishes a relationship between simple-clique $k$-trees and $L^{k-2}H$ graphs.

**Theorem 5.9.** *For $k \geq 3$ a $k$-tree is locally $(k - 2)$-nested-hamiltonian if and only if it is a simple-clique $k$-tree.*

***Proof.*** First, suppose $G$ is a $k$-tree that is not a simple clique $k$-tree. Then some $k$-clique $X$ was used more than once in the $k$-tree construction of $G$. By Observation 5.3, there are three independent vertices $u_1, u_2, u_3$ in $\bigcap_{x \in V(X)} N_G(x)$. Now let $Y$ be any $(k-2)$-clique in $X$ and let $\{v_1, v_2\} = V(X) - V(Y)$. By Theorem 5.4, there are exactly $k$ internally disjoint paths between any two vertices in $\{u_1, u_2, u_3\}$. Each such path contains exactly one vertex of $X$. Since $\{v_1, v_2\}$ are the only vertices of $X$ in $\bigcap_{y \in V(Y)} N_G(y)$, any cycle in $\left\langle \bigcap_{y \in V(Y)} N_G(y) \right\rangle$ misses at least one of the vertices in $\{u_1, u_2, u_3\}$. Thus $\left\langle \bigcap_{y \in V(Y)} N_G(y) \right\rangle$ is not hamiltonian and hence $G$ is not $L^{k-2}H$.

Now let $G$ be a simple clique $k$-tree of order $n$. We prove by induction on $n$ that $G$ is locally 2-nested-hamiltonian. If $n = k + 1$, then $G = K_{k+1}$, which is obviously $L^{k-2}H$. Now assume $n \geq k + 2$. Let $z$ be the last vertex added in the $k$-tree construction of $G$. Then $G - z$ is a simple clique $k$-tree of order $n - 1$ and $\langle N_G(z) \rangle$ is a $k$-clique in $G - z$ that has not been used in the $k$-clique construction of $G - z$. Let $N_G(z) = \{v_1, \ldots, v_k\}$. By Observation 5.3, $\left\langle \bigcap_{v \in N(z)} N_{G-z}(v) \right\rangle$ consists of a single vertex, say $v_{k+1}$. By our induction hypothesis, $G - z$ is $L^{k-2}H$. Thus, to prove that $G$ is $L^{k-2}H$, we only need to show that the $k$-clique $\langle N(z) \rangle$ is suitable for $k$-clique identification.

Now consider any $(k - 2)$-clique $Y$ in $\langle N(z) \rangle$. Then $\left\langle \bigcap_{y \in V(Y)} N_{G-z}(y) \right\rangle$ has a Hamilton cycle $C$, since $G - z$ is $L^{k-2}H$. We may assume that $V(Y) = \{v_1, \ldots, v_{k-2}\}$. Then $\{v_{k-1}, v_k, v_{k+1}\} \subseteq \bigcap_{y \in V(Y)} N_{G-z}(y)$ and $v_{k+1}$ is the only common neighbour of $v_{k-1}$ and $v_k$ in $\bigcap_{y \in V(Y)} N_{G-z}(y)$. Suppose $C$ does not contain the edge $v_{k-1}v_k$. Then $\bigcap_{y \in V(Y)} N_{G-z}(y)$ contains a $v_{k-1} - v_k$ path that contains neither the edge $v_{k-1}v_k$ nor the vertex $v_{k+1}$. Let $P$ be a shortest such path. We note that $v_{k-1}$ and $v_k$ do not have a common neighbour on $P$. So $P$ has at least four vertices and, by the minimality of $P$, the cycle $v_k v_{k-1} P v_k$ is chordless, contradicting Corollary 5.2. Hence $C$ contains the edge $v_{k-1}v_k$, and therefore $\langle N(z)\rangle$ is suitable for $k$-clique identification. This proves that $G$ is locally 2-nested-hamiltonian. ■

From Theorems 5.7, 5.8 and 5.9, we conclude the following.

**Corollary 5.10.** *For each integer $k \geq 1$, the class of locally $k$-nested-hamiltonian graphs contains the class of simple-clique $(k+2)$-trees.*

## 6. Conclusion and Open Problems

We considered the conjecture that every $K_{1,k+3}$-free, locally $k$-nested-hamiltonian graph is hamiltonian. Since $k$-nested-hamiltonian graphs are locally $(k+1)$-connected, our conjecture seems somewhat weaker than the Oberly-Sumner Conjecture, which asserts that every $K_{1,k+3}$-free locally $(k+1)$-connected graph is hamiltonian. However, we have not succeeded in settling our conjecture. We therefore investigated two special classes of locally $k$-nested-hamiltonian graphs, namely the connected $L^{\leq k}H$ graphs and the simple-clique $(k+2)$-trees. An affirmative answer to the following question would prove the restriction of our conjecture to the class of $(k+2)$-trees.

- Is every $K_{1,k+1}$-free simple-clique $k$-tree hamiltonian for $k \geq 3$?

Since simple-clique $k$-trees are highly structured and have a perfect elimination order, one would expect that answering the question above should not be too difficult, but it is still an open problem.

From the construction in Theorem 3.6, it can be deduced that $S_k$ is a non-hamiltonian, simple-clique $(k+2)$-tree. We note that it is $K_{1,k+4}$-free (but it contains an induced $K_{1,k+3}$, centred at $u_2$). We have also shown that the graph $G_k$ constructed in Theorem 3.7 is a non-hamiltonian, connected, $K_{1,k+4}$-free $L^{\leq k}H$-graph. This demonstrates that the Oberly-Sumner Conjecture is best possible in a strong sense.

The graph $S_k$ is of order $9+2k$ and has maximum degree $6+2k$. For $k \leq 2$ it has been shown that $9+2k$ is the minimum order of a non-hamiltonian locally $k$-connected graph. The following questions are still unanswered for $k \geq 3$.

- Is every locally $k$-nested-hamiltonian graph of order less than $9+2k$ hamiltonian?
- Is every connected $L^{\leq k}H$ graph of order less than $9+2k$ hamiltonian? (This will indeed be the case if Conjecture 1.9 is true.)

It was shown in [16] that every connected $LH$ graph with maximum degree at most 6 is fully cycle extendable and that the Hamilton Cycle Problem for LH graphs with maximum degree 9 is NP-complete. We know that the graph $S_1$ is a non-hamiltonian, connected $LH$ graph with maximum degree 8. The following two questions which appeared in [5] are still unanswered.

- Does there exist a non-hamiltonian, connected $LH$ graph with maximum degree 7?
- Is the HCP for LH graphs with maximum degree at most 8 solvable in polynomial time?

We conclude with two more open problems.

- Does there exist a non-hamiltonian locally 2-nested-hamiltonian graph with maximum degree 8 or 9? (We have shown that every locally 2-nested-hamiltonian graph with maximum degree at most 7 is fully cycle extendable, and that the graph $S_2$ is a non-hamiltonian, locally 2-nested-hamiltonian graph with maximum degree 10.)
- We have shown that the Hamilton Cycle Problem is NP-complete for locally 2-nested-hamiltonian graphs with maximum degree 12 and for $L^{\leq 2}H$ graphs with maximum degree 13. Are these results best possible?

**Acknowledgements**

The authors would like to thank both anonymous referees for their careful reading of the manuscript and their valuable comments and suggestions.

Thanks to the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) for financial support (reference number BA2017/268). Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

This material is based on work supported by the National Research Foundation of S.A. under Grant number 107668.

## References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, 2008).

[2] G. Chartrand and R. Pippert, *Locally connected graphs*, Čas. Pěst. Mat. **99** (1974) 158–163.

[3] G.T. Chen, A. Saito and S.L. Shan, *The existence of a 2-factor in a graph satisfying the local Chvátal-Erdős condition*, SIAM J. Discrete Math. **27** (2013) 1788–1799. https://doi.org/10.1137/12090037X

[4] J.P. de Wet, *Local Properties of Graphs* (PhD Thesis, University of South Africa, 2016).
http://uir.unisa.ac.za/bitstream/handle/10500/22278/thesis˙de “%20wet˙jp.pdf?sequence=1&isAllowed=y

[5] J.P. de Wet and M. Frick, *The Hamilton cycle problem for locally traceable and locally Hamiltonian graphs*, Discrete Appl. Math. **266** (2019) 291–308. https://doi.org/10.1016/j.dam.2019.02.046

[6] J.P. de Wet, M. Frick and S.A. van Aardt, *Hamiltonicity of locally Hamiltonian and locally traceable graphs*, Discrete Appl. Math. **236** (2018) 137–152. https://doi.org/10.1016/j.dam.2017.10.030

[7] J.P. de Wet and S.A. van Aardt, *Traceability of locally traceable and locally Hamiltonian graphs*, Discrete Math. Theor. Comput. Sci. **17** (2016) 245–262.

[8] M.R. Garey, D.S. Johnson and R.E. Tarjan, *The planar Hamiltonian circuit problem is NP-complete*, SIAM J. Comput. **5** (1976) 704–714. https://doi.org/10.1137/0205049

[9] A. Goldner and F. Harary, *Note on a smallest non-hamiltonian maximal planar graph*, Bull. Malays. Math. Sci. Soc. **6** (1975) 41–42.

[10] L. Markenzon, C.M. Justel and N. Paciornik, *Subclasses of $k$-trees: characterization and recognition*, Discrete Appl. Math. **154** (2006) 818–825. https://doi.org/10.1016/j.dam.2005.05.021

[11] D.J. Oberly and D.P. Sumner, *Every locally connected nontrivial graph with no induced claw is Hamiltonian*, J. Graph Theory **3** (1979) 351–356. https://doi.org/10.1002/jgt.3190030405

[12] C.M. Pareek, *On the maximum degree of locally Hamiltonian non-Hamiltonian graphs*, Util. Math. **23** (1983) 103–120.

[13] C.M. Pareek and Z. Skupień, *On the smallest non-Hamiltonian locally Hamiltonian graph*, J. Univ. Kuwait (Sci.) **10** (1983) 9–17.

[14] D.J. Rose, *On simple characterizations of $k$-trees*, Discrete Math. **7** (1974) 317–322. https://doi.org/10.1016/0012-365X(74)90042-9

[15] Z. Skupień, *Locally Hamiltonian and planar graphs*, Fund. Math. **58** (1966) 193–200. https://doi.org/10.4064/fm-58-2-193-200

[16] S.A. van Aardt, M. Frick, O. Oellermann and J.P. de Wet, *Global cycle properties in locally connected, locally traceable and locally Hamiltonian graphs*, Discrete Appl. Math. **205** (2016) 171–179. https://doi.org/10.1016/j.dam.2015.09.022

Received 8 January 2019
Revised 3 July 2020
Accepted 3 July 2020