A Burden of Wonder

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1 Introduction

I was recently preparing for a lecture on recursion for a first year programming course. The examples include computing the first n Fibonacci numbers, generating the permutations of a string and testing whether a string is a palindrome. For some reason I was inspired to find out a little bit more about the Fibonacci numbers in order to provide the class with a few interesting – for me, at least – historical and mathematical trivia. I was astonished to find a magnificently huge amount of material on the matter.

The work presented here is clearly not my own. It does however provide a sense of the marvellously magical world in which we live.

Since the theme of this volume is "From Operations Research to Software Engineering and Beyond", I had hoped to find a thread starting at statistics, through some interesting numbers, and ending at computer science. Specifically had I hoped to start this piece with the Wishart distribution[1]. I was, alas, not able to find a suitable link between this distribution and the rest of the topics in the paper. In true academic style, finding this link is left as future work.

2 Fibonacci

The Fibonacci sequence is attributed to Leonardo of Pisa, known as Fibonacci. He considered the growth of an idealised rabbit population. The number of rabbits in the population in month n is equal to F_n (which is defined below). However, in [1, p. 197] Livio mentions that the Fibonacci sequence first appeared in poetry—long before Fibonacci studied his rabbits. One of the types of metre in Sanskrit poetry is called mātrā-vittas. Work has been published in which metres are constructed by the sum of two previous metres, resulting in the Fibonacci sequence.

The n^{th} Fibonacci number F_n is defined as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases}$$

The first few values in the sequence is then $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

Since 60 is a significant number in the present context let us consider $F_{60} = 1548008755920$. I could not immediately find anything special about the number. It is easy to show that it is not a prime since F_n prime implies that n is prime[2]. The first prime F_n after F_{60} is $F_{83} = 99194853094755497$. Perhaps 83 is a number of significance.

Staying with the "60" theme, let us consider the unit – or right-most – digits in the Fibonacci sequence.

$$\underline{0}, \underline{1}, \underline{1}, \underline{2}, \underline{3}, \underline{5}, \underline{8}, 1\underline{3}, 2\underline{1}, 3\underline{4}, 5\underline{5}, 8\underline{9}, 14\underline{4}, 23\underline{3}, 37\underline{7}, 61\underline{0}, \ldots$$

A new sequence can be constructed from these unit digits, yielding:

$$0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, \ldots$$

The sequence of the last digits UD can be represented by a short sequence – say $U = u_1, u_2, \ldots, u_m$ – that repeats through the Fibonacci sequence. Thus $UD = U, U, U, \ldots$ The length of the cycle, and thus the value of m, is 60 [2].

Let us consider an extension of the Fibonacci sequence to strings. One can define the so-called Fibonacci word by using strings instead of numbers

1. This is the probability distribution of the maximum-likelihood estimator of the covariance matrix of a multivariate normal distribution.

2. Except for $F_4 = 3$

and concatenation instead of addition.

$$S_n = \begin{cases} 1 & \text{if } n = 0, \\ 10 & \text{if } n = 1, \\ S_{n-1}S_{n-2} & \text{if } n > 1. \end{cases}$$

The Fibonacci word is then S_∞. The table below shows the first few steps for generating the Fibonacci word. It also shows the number of zeros and ones, respectively, that occur in each S_i

TABLE 1
Generating the Fibonacci word.

S_0	1	0	1
S_1	10	1	1
S_2	101	1	2
S_3	10110	2	3
S_4	10110101	3	5
S_5	1011010110110	5	8
S_6	101101011011010110101	8	13

Let us consider the Fibonacci word and apply to it a transformation [1, p.213]. Start from the left and whenever you encounter a 1, mark three symbols and whenever you reach a 0, mark two symbols. The groups of symbols should not overlap. The first part of the word is shown below.

$$\widehat{101}\,\widehat{10}\,\widehat{101}\,\widehat{101}\,\widehat{10}\,\widehat{101}\ldots$$

Since the first symbol is a 1, three symbols are marked. The second symbols is a 0 so two symbols are marked. Note that the two symbols that are marked are the symbols following the previously marked symbols. The third symbol is a 1 and thus three symbols are marked.

From every group of marked symbols, discard the right-most symbol. This yields the following string.

$$10\,1\,10\,10\,1\,10\ldots$$

It should be clear that this is again the original string. This means that the Fibonacci word is self-similar. For more on this theme see [3] where a related sequence – the rabbit sequence – is defined.

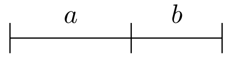
One last important property to mention concerns the relative number of ones and zeros in the Fibonacci word. Consider Table 1 once more. The number of ones in S_n equals F_{n+1} and the number of zeros in S_n equals F_n. The ratio of ones to zeros in S_∞ is therefore

$$\tau = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1.6180339887\ldots$$

This special number is the theme of the next section.

3 Golden Ratio

Consider the following illustration.



The line segment is split into two parts with the length of the left a and the length of the right b. If the point at which the line is split is chosen in such a manner that

$$\frac{a+b}{a} = \frac{a}{b} = \tau,$$

the ratio τ is called the golden ratio. A bit of algebra yields

$$\tau = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

It should be noted that this number is intimately connected with the Fibonacci sequence as can be seen from the final result in the previous section. The golden ratio also allows us to find a closed form for F_n [4], [1].

$$F_n = \frac{\tau^n - (1-\tau)^n}{\sqrt{5}}$$

The golden ratio can also be expressed as a continued fraction.

$$1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}$$

To square the golden ratio, one merely has to add one to it. Further, to find the reciprocal of the golden ration, one subtracts one from it.

$$\tau^2 = \tau + 1$$

$$\tau^{-1} = \tau - 1$$

For a thorough discussion on the golden ratio and see Livio's book delightful book [1]. In it he recounts the history of the number and describes how the number keeps occurring in nature, art, and beauty. He also dispels a few myths regarding the number.

4 Palindromes

A palindrome is a string, or sequence of symbols, that is constant under reversal. Palindromes are not constrained to strings or text, but may be found elsewhere, for example in music. Bach's Musical Offering contains a number of retrograde canons which are musical palindromes. Hofstadter [5] refers to these as Crab Canons.

Consider again the generation of the Fibonacci word above in Table 1. If one removes the two rightmost symbols from an S_i the string that remains is a palindrome. A formal treatment of palindromes in the Fibonacci word can be found in [6].

The language that consists of palindromic words over some alphabet is an example of a language that is not regular. It is therefore not possible to represent the language using regular expressions or a finite automaton.

A proof for this non-reglarity is presented below. The proof is from [7] and employs the pumping lemma. Let us only consider the palindromes over the alphabet $\{0, 1\}$. Call this language L_{pal}. Assume that L_{pal} is regular. Consider the palindrome $w = 0^n 10^n \in L_{pal}$. Since we assume that L_{pal} is regular we can split w in such a way that $w = xyz$, where y consists of at least one 0 from the first group of zeros in w. From the pumping lemma xz should still be in L_{pal}. However, xz is not a palindrome since it has at least one zero fewer on the left. This implies that L_{pal} is not regular.

The Sator Square [8] is a word square containing the Latin words SATOR AREPO TENET OPERA ROTAS. The earliest known appearance of the square was found in the ruins of Herculaneum.

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

The words may be translated as follows [9]. Sator – sower, planter; Arepo – an invented proper name; Tenet – he holds; Opera – works, efforts; Rotas – wheels. These can then be used to produce two possible translations. 'The sower Arepo holds the wheels with effort' and 'The sower Arepo leads with his hand the plough'. According to [9], C. W. Ceram read the square in alternating directions producing the following translation: 'The Great Sower holds in his hand all works; all works the Great Sower holds in his hand.' It should be noted that the origin and meaning of the Sator Square is still an open problem [8], [10].

It is also possible to rearrange the letters in the Sator Square to form the same phrase verticaly and horisontally, leaving out two A's and two O's. This is left as an exercise for the reader.

References

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