Numerical Simulations of Cryptocurrency Asset Flow Fractional Differential Equations

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Abstract: The cryptocurrency market has grown exponentially since its inception in 2009. Asset price movements in this emerging market have been the subject of several research studies aimed at explaining their patterns. This article proposes a robust fractional time spectral method for studying a three-dimensional fractional differential equation which describes the flow and stability of cryptocurrency assets. The method relies on the fractional spectral integration matrix operator. We demonstrate the efficiency of the numerical method with comparison to existing methods.

Keywords: Cryptocurrency, Chebyshev polynomial, Fractional integral, Spectral Method

1 Introduction

In the recent years, the concept of cryptocurrency has become more and more understood by a large group of people. By definition, a cryptocurrency is essentially a type of digital asset used as currency within the meaning of the Austrian School of Economics, that is to say, that the currency emerges from a competition of means of exchange [1,2]. The value of cryptocurrency is primarily driven by supply and demand. It develops ideally in a free economy where the intervention of an entity or a government is not possible in the issuance of these new units. Unlike fiat currencies which, for their part, follow the Keynesian school of economics, i.e. a sovereign government, their organization can have a positive effect on an economy, especially when the latter begin to slow down or take a blow [3,4].

The technology that forms the basis of cryptocurrency is called the blockchain. It is a distributed public ledger that records all the transactions since inception in a safe, secure, verifiable and non-editable manner. Newly issued units of a cryptocurrency is realized through a process called mining. Miners verify transactions in a block by solving of complex hard mathematical problems which requires an enormous amount of computing power. This concept is known as the proof of work. During this process, the most competitive miner receives mining rewards and transactions fees. This is how new coins are issued [5]. Bitcoin uses the proof of work verification process to ensure the integrity of the system. There exist other types of verification processes such proof of stake, delegated proof of stake, proof of authority, etc. [6,7].

A lot of research has been carried out to gain insights into the dynamics of the bitcoin price. However, most of the work focuses on time series modelling [8]. Bitcoin prices and derivatives are barely studied from a modelling point of view. Some papers dealing with option pricing are available but an extensive empirical analysis is missing. A dynamical approach instead is performed by Caginalp [9] to analyse stability of cryptocurrency markets. The model describes a system of nonlinear ordinary differential equations.

However, due to high variability in the dataset, total differentiation can in some instance leave some challenges in the degree of the accuracy of the model and by using fractional differentiation, the accuracy on pricing models may be improved. We extend the study of the model proposed by Caginalp [9] to fractional differential equations (FDE).

Fractional derivatives are unfortunately not unique. The classical ones are the Riemann-Liouville fractional

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derivatives and the Caputo fractional derivative [10]. One challenge with classical fractional calculus is to handle nonlinear phenomena [11,12]. Various fractional derivative operators have been introduced. Recently among them, the Atangana-Baleanu (AB) fractional derivative. Fractional derivatives have been tested with success in many fields including chaotic behavior and epidemiology [13,14,15]. Since our cryptocurrency model also is valid on a short period of time, singularities are not observed. Hence a Caputo fractional operator is preferred in order to facilitate the introduction of fractional integration. A fractional dynamical approach for cryptocurrency in the Caputo sense is therefore proposed. Attempts to solve FDEs has been an ongoing active topic of research. There are several analytical methods such as Adomian decomposition methods, homotopy- perturbation methods, variational iteration method and homotopy analysis methods [16,17,18,19]. In general, most FDEs do not have exact analytical solutions, so approximations and numerical techniques must be used. Most works developed in numerical methods for solving FDEs have focused on lower (or classical) methods which include the class of finite difference and finite element methods. These methods have shown a slow convergence.

Higher order (or spectral) methods, however, have the advantage of being fast converging methods. Though only sparely explored in the context of FDEs, high order methods have the potential to reduce computational cost by allowing the use of fewer points, while achieving the same accuracy as that of lower order methods [20]. In this paper we intend to solve the fractional dynamical system governing the price process of cryptocurrency by means of a spectral method, following the footsteps of [21,22] with an extension to the three dimensional problem. More precisely the Caputo fractional operator is used for handling fractional differentiation.

This paper is organised as follows. Section 2 presents some basic concepts of fractional integrals and fractional derivatives. In Section 3, we introduce the fractional spectral integral method that will be used, then we apply this to the cryptocurrency problem in Section 4. In the same section we present results and conduct an error analysis. The last section is devoted to the conclusion.

2 Basic definitions and notation on fractional calculus

2.1 Fractional Integral

In order to define fractional integral and differential operators, we need to first introduce the following Euler's Gamma function.

Definition 21 *The function $\Gamma : (0,+\infty) \to \mathbb{R}$ defined by:*

$$\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-t}, \tag{1}$$

is called the Euler's Gamma function (or Euler's integral of second kind), in particular we have $\Gamma(n) = (n-1)!$ for any positive integer n.

Assuming that a function $f(x)$ is well defined where $x > 0$, we can form the definite integral from 0 to x. Let call this

$$(Jf)(x) = \int_0^x f(t)dt. \tag{2}$$

Repeating the process gives

$$(J^2 f)(x) = \int_0^x (Jf)(t)dt = \int_0^x \left(\int_0^t f(s)ds \right) dt, \tag{3}$$

and this can be extended arbitrarily.
The Cauchy formula for repeated integration, namely

$$(J^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt, \tag{4}$$

leads to a straightforward way to the generalisation for n being a real number. However, instead of the factorial, let us insert the Gamma function defined in (1) into (4). In this way, we get a natural candidate for the definition of fractional integral operator.

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt. \tag{5}$$

This is in fact a well-defined operator. It can be shown that J operator is both commutative and additive, that is,

$$J^\beta J^\alpha f = J^\alpha J^\beta f$$
$$= J^{\alpha+\beta} f = \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1} f(t)dt. \tag{6}$$

A more difficult question is how to define a fractional derivative operator D.

2.2 Fractional Derivative

The Riemann-Liouville fractional order derivative of f is defined as the m^{th} derivative of the fractional integral of order $m-q$. That is:

Definition 22 *Let $f(t)$ be an integrable function on $[a,T]$. For all $a < t < T$ the Riemann-Liouville fractional derivative of order $q > 0$ of f is given by:*

$${}_a^{RL}D_t^q f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} f(\tau)d\tau \right],$$
$$m = [q]+1. \tag{7}$$

And for the case $q = k \in \mathbb{N}$ then $m = k+1$, we recover the normal differentiation formula

$${}_a^{RL}D_t^k f(t) = \frac{1}{\Gamma(1)} \frac{d^{k+1}}{dt^{k+1}} \int_a^t f(\tau)d\tau = \frac{d^k}{dt^k} f(t).$$

The classical integer derivatives becomes like singularities among the Riemann-Liouville fractional derivatives. They turn out to be the only fractional derivatives that do not depend on the lower bound a. Note that if f is a monomial i.e. $f(t) = t^r$ then,

$$ {}_{a}^{RL}D^{\alpha}t^{r} = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}t^{r-\alpha}, \quad \alpha, t > 0, \quad r > -1. \quad (8)$$

Thus, for a constant function f we have the remarkable fact that its fractional derivative will not be zero as with any normal integer differentiation. In fact from (8) and taking $r = 0$ we have,

$$D^{\alpha}1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, \quad t > 0. \quad (9)$$

Similarly to the Riemann-Liouville fractional derivative, let introduce another fractional derivative, the Caputo derivative. Which is defined as the fractional integral of the m^{th}-derivative, That is:

Definition 23*Let q be a positive number, $f \in C^m[0,t]$ and $0 \leq \tau \leq t$. Then the Caputo fractional derivative of $f(t)$ is defined as*

$$ {}_{a}^{C}D_{t}^{q}f(t) = \frac{1}{\Gamma(m-q)}\int_{a}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}}d\tau, \quad m-1 \leq q < m \quad (10)$$

and for the case $k \in \mathbb{N}$ then $m = k+1$ thus we recover the usual derivative

$$ {}_{a}^{C}D_{t}^{k}f(t) = \frac{1}{\Gamma(1)}\frac{d^{k+1}}{dt^{k+1}}\int_{a}^{t}f(\tau)d\tau = \frac{d^k}{dt^k}f(t).$$

With the Caputo derivative we recover the fact that the derivative of a constant function is indeed zero, however we have pay the price that f has to be m-differentiable.
The following relations allow us to see the equivalence between the Riemann-Liouville and the Caputo fractional derivatives:

$$ {}_{a}^{RL}D^{\alpha}f(t) = {}_{a}^{C}D^{\alpha}f(t) + \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0^+), \quad (11)$$

Consequently,

$$ {}_{a}^{RL}D^{\alpha}f(t) := D^m J^{m-\alpha}f(t) \neq J^{m-\alpha}D^m f(t) := {}_{a}^{C}D^{\alpha}f(t), \quad (12)$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$.

3 Chebyshev approximation

3.1 Definition and usual properties

Let $\Lambda = [-1,1]$ and $T_k(x)$ be the standard Chebyshev polynomial of degree k. Denote by $w(x) = \frac{1}{(1-x^2)}$ the Chebyshev weight function. Clearly, $T_0(x) = 1$, $T_1(x) = x$ and

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \qquad k = 2,3\ldots \quad (13)$$

The set of $\{T_k(x)\}$ is a complete $L_w^2(\Lambda)$-orthogonal system, namely,

$$\int_{\Lambda}T_k(x)T_j(x)w(x)dx = \frac{\pi}{2}c_{kj}\delta_{kj}(x)dx, \quad (14)$$

where δ_{kj} is the Kronecker symbol, $c_0 = 2$ and $c_k = 1$ for $k \geq 1$.
The points $\{x_k\}$ defined as

$$x_k = -\cos\frac{(k-\frac{1}{2})\pi}{n}, \qquad k = 1,2,\ldots,n. \quad (15)$$

are called the collocation points
Thus, any continuous function u on the interval $[-1,1]$ can be approximated by the following Chebyshev expansion:

$$u_n(x) = \sum_{k=0}^{n}c_kT_k(x) \quad (16)$$
$$= \underline{c}\cdot T(x), \quad (17)$$

for some coefficients c_k and $\underline{c} = (c_0, c_1, \ldots, c_n)$ is the vector of coefficients associated with the approximation u_n.
Denote by $\underline{v}(x) = (u(x_0), u(x_1), ..., u(x_n))$ the physical representation, then there exists a conversion matrix T (see [22]) such that

$$\underline{v} = T\underline{c} \qquad \text{i.e.} \qquad \underline{c} = T^{-1}\underline{v}. \quad (18)$$

3.2 Fractional Integration and matrix operator

We recall the definition of the fractional integral in (5)

$$J^q f(x) = \frac{1}{\Gamma(q)}\int_0^x(x-t)^{q-1}f(t)dt. \quad (19)$$

Let us consider a function $f : [0,b] \to \mathbb{R}$, with the Chebyshev approximation:

$$f(x) = \sum_{k=0}^{n-1}c_kT_k(\alpha x+\beta), \quad \alpha = \frac{2}{b}, \quad \beta = -1. \quad (20)$$

Note that $c = (c_0, c_1, \ldots, c_n)$ is its spectral representation. The fractional integral of order q of the function f at any collocation point x_k is:

$$\begin{aligned} J^q f(x_j) &= \frac{1}{\Gamma(q)}\int_0^{x_j}(x_j-t)^{q-1}\sum_{k=0}^{n-1}c_kT_k(\alpha t-1)dt \\ &= \sum_{k=0}^{n-1}c_k\int_0^{x_j}(x_j-t)^{q-1}T_k(\alpha t-1)dt \\ &= \sum_{k=0}^{n-1}c_kI_k(x_j), \end{aligned}$$

where

$$I_k(x_j) = \int_0^{x_j} (x_j - t)^{q-1} T_k(\alpha t - 1) dt. \qquad (21)$$

Thus the physical representation of the fractional integral of f on the entire interval $[0,b]$ is:

$$\underline{v}(x) = J^q f(x) = (J^q f(x_0), J^q f(x_1), ..., J^q f(x_n))$$

$$\left(\sum_{k=0}^{n} \tilde{c}_k T_k(x_0), ..., \sum_{k=0}^{n} \tilde{c}_k T_k(x_n)\right) =$$

$$\left(\sum_{k=0}^{n} c_k I_k(x_0), ..., \sum_{k=0}^{n} c_k I_k(x_n)\right).$$

The above relation implies the existence of a matrix I such that

$$T\,\tilde{c} = I\,c$$
$$\tilde{c} = T^{-1} I\,c$$

where $\tilde{c}$ is the spectral representation of the fractional integral of f, the matrix I is defined as follows

$$I = (I_{kj}), \quad I_{kj} = I_k(x_j), \quad i,j = 1,2,...,n \qquad (22)$$

$I_k(x_j)$ being defined as in (21). Consequently the physical representation of the fractional integral operator is $I \cdot T^{-1}$ and the spectral representation of the fractional integral operator is $T^{-1}I$. It remains therefore to compute the matrix I. To this extend we have the following lemma, see also [23,24].

Lemma 31*Let f be a continuous function defined on [0,b] and vanishing at 0, and define $I_k(x)$ again as in (21). Then*

$$I_0(x) = \frac{x^{1-q}}{1-q}/2,$$

$$I_1(x) = \frac{\alpha x^{2-q}}{(2-q)(1-q)} - \frac{x^{1-q}}{k(1-q)},$$

$$I_2(x) = \frac{4\alpha^2 x^{3-q}}{(3-q)(2-q)(1-q)} - \frac{4\alpha x^{2-q}}{(2-q)(1-q)} + \frac{x^{1-q}}{1-q},$$

and

$$\left(1 + \tfrac{1-q}{k}\right) \cdot I_k(x) =$$

$$2(\alpha x - 1) \cdot I_{k-1}(x) + \left(\tfrac{1-q}{k-2} - 1\right) \cdot I_{k-2}(x) - \tfrac{2(-1)^k}{k(k-2)} x^{1-q}.$$

See the appendix for the proof.

Consequently let $0 < q_0 < q_1 < ... < q_m$, and consider a general multiple order fractional differential equation $\mathscr{A}u = f$ of order q_m with constant coefficients. Suppose the fractional differential operator can be written as $\mathscr{A} = L + \mathscr{N}$ where L and $\mathscr{N}$ are respectively the linear part and the nonlinear part, then the equation can be written as

$$Lu(t) + \mathscr{N}u(t) = f(t) \qquad (23)$$

$$\sum_{k=0}^{m} D^{q_k} u(t) = -\mathscr{N}u(t) + f(t). \qquad (24)$$

Taking the fractional integral of order q_m to (24) and applying relation (12) we get:

$$\sum_{k=0}^{m} J^{q_m - q_k} u(t) = J^{q_m} \left[-\mathscr{N}u(t) + f(t)\right]. \qquad (25)$$

The above equation can be represented in the frequency space as:

$$\sum_{k=0}^{m} \mathbf{J}^{q_m - q_k} \underline{c} = -\mathbf{n} + \tilde{\mathbf{f}}$$

$$\mathbf{A}\,\underline{c} = \mathbf{f} \qquad (26)$$

implying $\qquad \underline{c} = \mathbf{A}^{-1}(\mathbf{f})$

for some $\mathbf{A} = \sum_{k=0}^{m} \mathbf{J}^{q_m - q_k}$; where $\mathbf{n}, \tilde{\mathbf{f}}$ are the spectral representation of $J^{q_m}\mathscr{N}u$ and $J^{q_m}f$ respectively, and $\mathbf{f} = -\mathbf{n} + \tilde{\mathbf{f}}$.

4 Application and numerical results

In this section, we apply our fractional spectral integral method (FSIM) to problems in cryptocurrency world. We also test the convergence of our proposed method against the existing fde12 method [25,26]. Whenever the analytical solution is not found, we choose our FSIM with relative tolerance 10^{-14} (that is 2000 collocation points) as the benchmark solution. The error E is the maximal error given by

$$||E|| = ||Sol_{Benchmark} - Sol_{Numerical}||_\infty. \qquad (27)$$

All the numerical simulations are performed on an Intel core I5, 8th Generation processor.

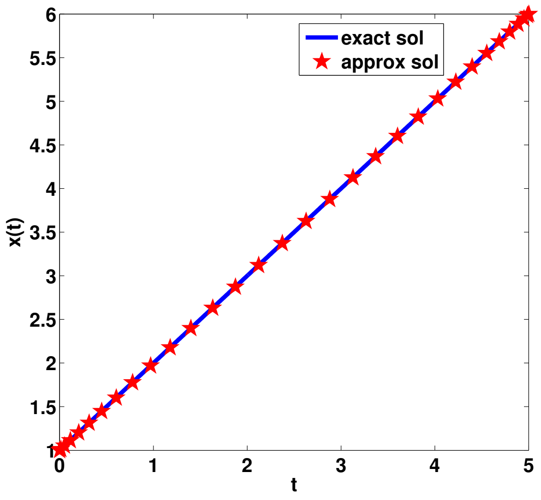
4.1 Benchmark problem

Here we consider the following system of fractional differential equations [27]

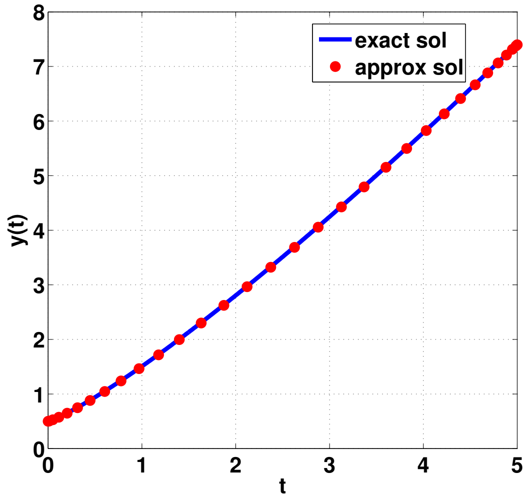
$$\begin{cases} D^{q_1} x(t) = \sqrt{t} + \sqrt[6]{(y(t) - 0.5)(z(t) - 0.3)} \\ D^{q_2} y(t) = \Gamma(2.2) \\ D^{q_3} z(t) = \frac{\Gamma(2.8)}{\Gamma(2.2)} \end{cases} \qquad (28)$$

with $q_1 = 0.5$ $q_2 = 0.2$ and $q_3 = 0.6$ together with initial condition: $x(0) = 1, y(0) = 0.5$ and $z(0) = 0.3$, for which the exact solution is found to be $x(t) = t + 1, y(t) = t^{1.2}, z(t) = t^{1.8} + 0.3$.

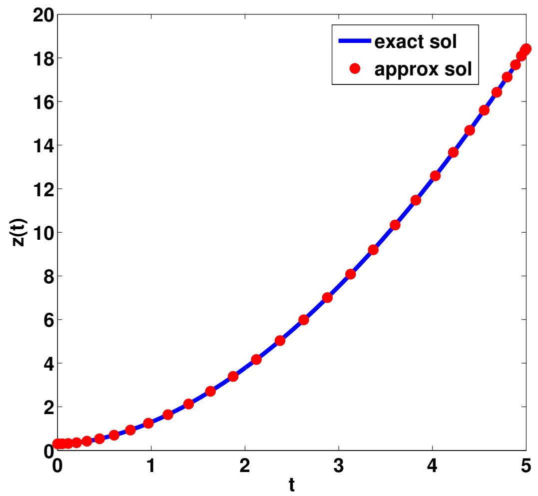
Applying our method and comparing with the exact solution we get the following plot for the solutions x, y, z



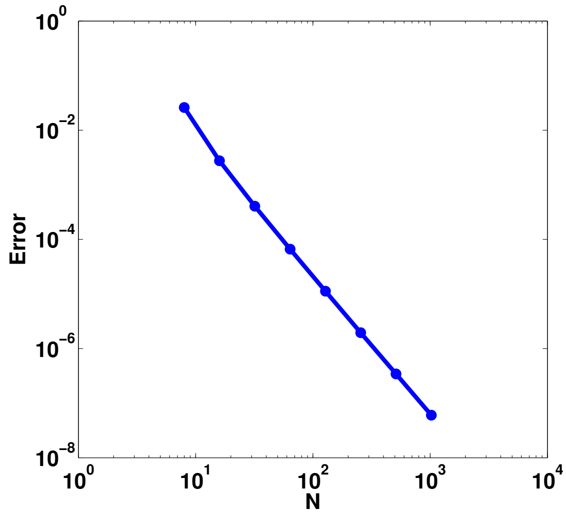
$x(t)$



$y(t)$



$z(t)$



Convergence

Fig. 1: plot of the variables x,y,z for $T = 5$ and $N = 32$ collocation points and convergence

in Fig. 1 obtained for $T = 5$ and $N = 32$ collocation points. One clearly see that all 32 solution points from numerical method lie on the exact solution. This shows that the numerical solution from (FSIM) and the exact solution are in good agreement.

We also run a comparison with another already existing numerical method, here fde12 which is based on a Taylor expansion approximation.

As we vary the number of collocation points on the FSIM, we record again in Fig. 1, the evolution of the error dynamics on the variable z. The log-log graph shows an exponential decay of the error which is expected from a spectral method. in fact, for N ranging from $n = 2^5$ to 2^{10} ie from 16 to 1024 collocation points, the error decays rapidly from 0.0027 to 6.08e-8. This is another confirmation of the high precision of spectral methods. The same result holds also for the variables x and y.

4.2 Cryptocurrency model

The behaviour of the cryptocurrency price dynamics in the market is based on some key factors:

- $P(t)$: The market price of cryptocurrency.
- $L(t)$: The Liquidity price at time t.
- $\zeta_1(t)$: The trend-based component of investor preference at time t.

Caginalp [9] proposed a dynamical system based on asset flow differential equations to describe the behaviour of those three variables in the market as :

$$\begin{cases} \tau_0 \frac{dP}{dt} &= (1+2\zeta_1)L - P \\ c_0 \frac{dL}{dt} &= 1 - L + q(1+2\zeta_1)L - qP \\ c_1 \frac{d\zeta_1}{dt} &= q_1(1+2\zeta_1)\frac{L}{P} - q_1 - \zeta_1 \end{cases} \tag{29}$$

The system admits only one equilibrium point obtained for $L = P$ and $\zeta_1 = 0$.
It is known that integer order derivatives may fail to take into consideration the history of the system and fails to address some technical issues describing this system. For this reason, let us substitute the integer derivatives by Caputo fractional derivatives into the system. We get:

$$\begin{cases} \tau_0^c D_t^{\alpha_1} P &= (1+2\zeta_1)L - P \\ c_0^c D_t^{\alpha_2} L &= 1 - L + q(1+2\zeta_1)L - qP \\ c_1^c D_t^{\alpha_3} \zeta_1 &= q_1(1+2\zeta_1)\frac{L}{P} - q_1 - \zeta_1. \end{cases} \tag{30}$$

Taking fractional integrals appropriately for each equation in the system and using (12) we get

$$\begin{cases} \tau_0 P = J^{\alpha_1}(1+2\zeta_1)L - J^{\alpha_1} P \\ c_0 L = J^{\alpha_2}(1 - L + q(1+2\zeta_1)L - qP) \\ c_1 \zeta_1 = J^{\alpha_3}(q_1(1+2\zeta_1)\frac{L}{P} - q_1 - \zeta_1). \end{cases} \tag{31}$$

Equation (31) is a 3 dimensional system of nonlinear ordinary fractional integral equations. It can be written as:

$$Au + Nu = f. \tag{32}$$

Using our Fractional Spectral Integral Method described in Section 3, we transport the equation in the frequency space and it becomes

$$\mathbf{A}\underline{c} = \mathbf{f} \tag{33}$$

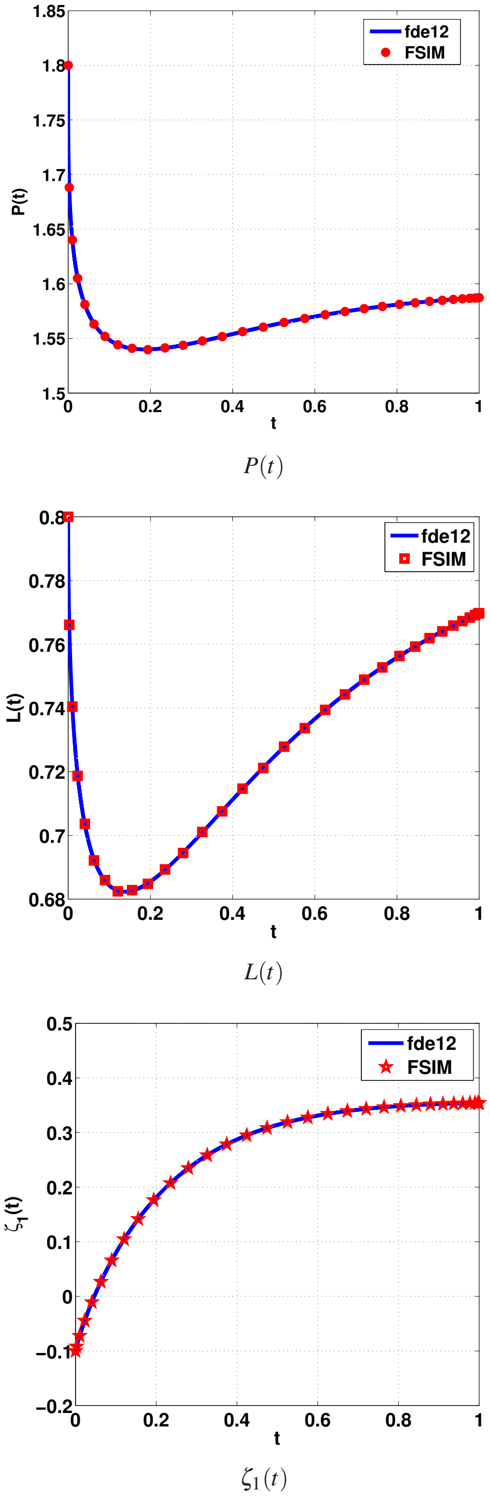
where $\underline{c}$ and $\mathbf{f}$ are spectral representations of the unknown solution vector $u = (P, L, \zeta_1)$ and the nonlinear part $f - Nu$, respectively. In addition, the matrix $\mathbf{A}$ is of the form:

$$\mathbf{A} = \begin{bmatrix} \tau_0 \mathscr{I} + J^{\alpha_1} & -J^{\alpha_1} & 0 \\ qJ^{\alpha_2} & c_0 \mathscr{I} + (1-q)J^{\alpha_2} & 0 \\ 0 & 0 & c_1 \mathscr{I} + J^{\alpha_3} \end{bmatrix}$$

where $\mathscr{I}$ is the identity matrix and J is the integral matrix as defined in (22). The nonlinear part will be written as:

$$N = \begin{bmatrix} 2J^{\alpha_1}\zeta_1 L \\ J^{\alpha_2}(1 + 2q\zeta_1 L) \\ J^{\alpha_3}\left(q_1(1+2\zeta_1)\frac{L}{P} - q_1\right) \end{bmatrix}.$$

We run the algorithm for the following set of parameters $\tau_0 = 1.8,\ c_0 = 1,\ q_0 = 0.75,\ q_1 = -2.5,\ c_1 = 1$; considering an initial solution to be $P(0) = 1.8,\ L(0) = 0.8,\ \zeta_1(0) = -0.1$, and $\alpha_1 = 0.5, \alpha_2 = 0.7, \alpha_3 = 0.9$. We compare the results with the solution from fde12 method. The solutions are plotted in Fig. 2 for the variables $x(t), y(t), z(t)$, for $n = 32$ collocation points and $T = 1$. Again as with Fig. 1, all



$P(t)$

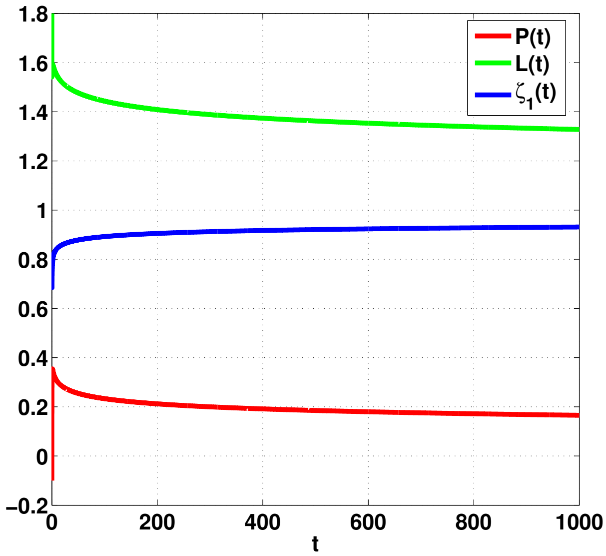
$L(t)$

$\zeta_1(t)$

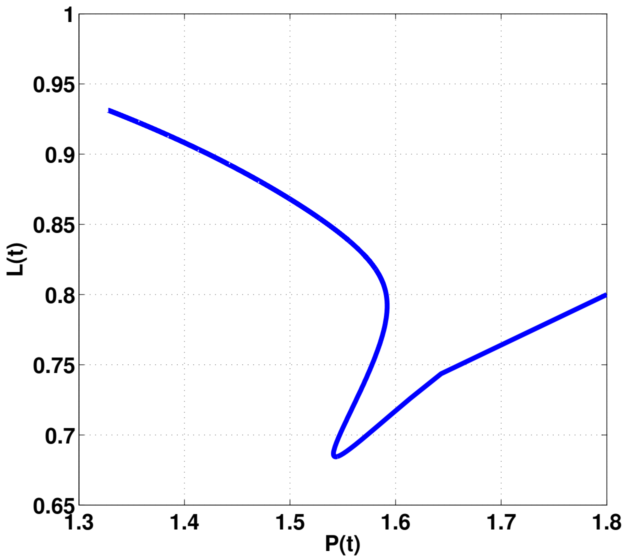
Fig. 2: plot of the variables P, L, ζ_1 for $T = 1, N = 32$ collocation points and $h = 10^{-5}$ for the fde12 method

solution points from FSIM method lie on the solution from fde12.

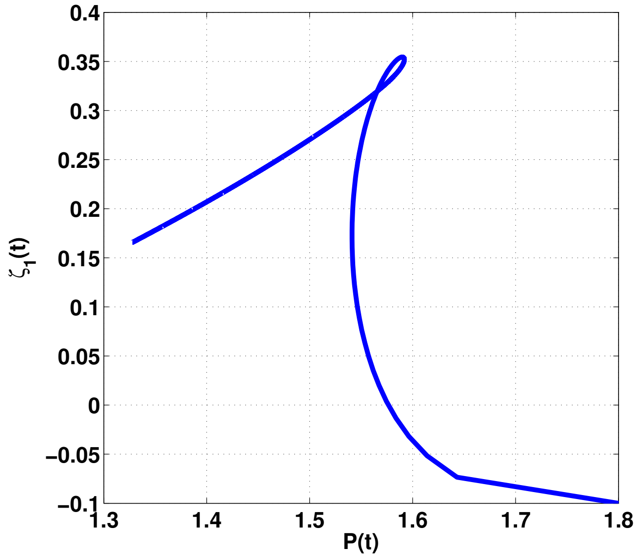
A long run behaviour of the solutions is plotted in Fig. 3 together with a phase plane $PL, P\zeta_1$ and $L\zeta_1$ for $T = 1000$. This is an indication that the method is capable of handling even large time scale evolution problems. Also, these plots confirm the stability analysis announced earlier, that is, there is no chaotic behaviour observed in the cryptocurrency pricing problem; see also [28] for more on the stability analysis.



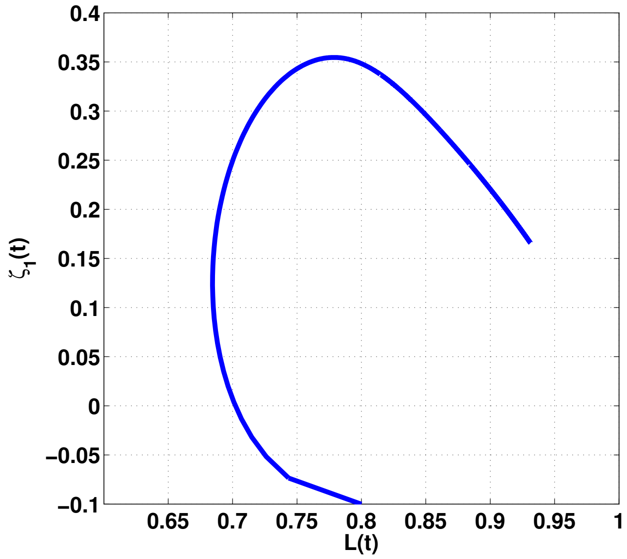
P, L, ζ_1 plot



PL plot



$P\zeta_1$ plot

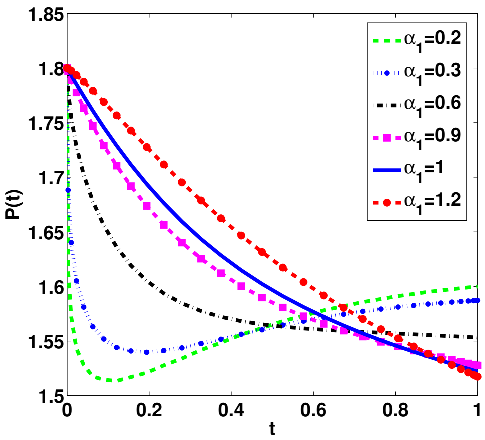


$L\zeta_1$ plot

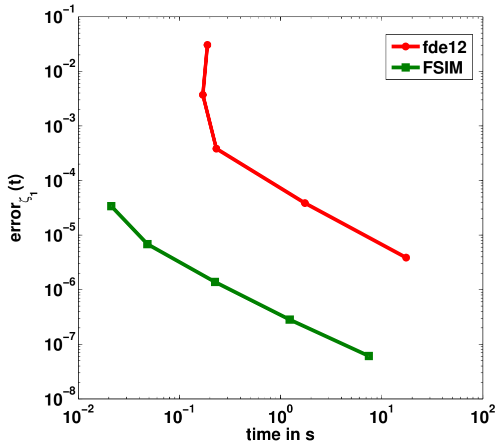
Fig. 3: Phase planes for large $T = 1000$

Looking at the effect of varying the fractional order of differentiation, Fig. 4 shows that as $\alpha \to 1$ the solution of the fractional differential equation converges to the solution of the ordinary differential equation. In addition, the numerical results demonstrate that a decrease in the derivative order is associated with a decrease in the minimum value of P and L and in the maximum value of ζ_1.

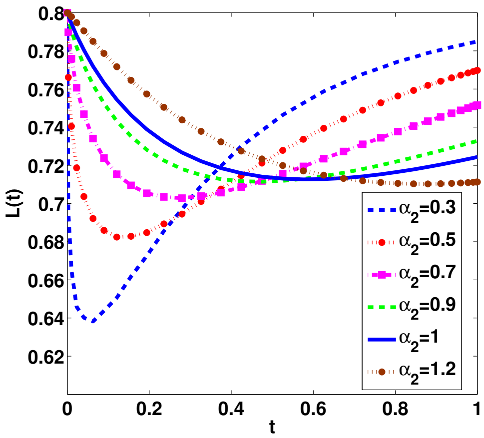
We investigate the convergence and efficiency of the method compared to the fde12 method. The plot confirms



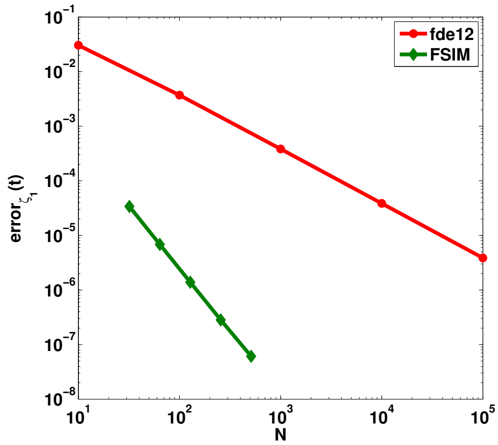
$P(t)$ for $\alpha_2 = 0.5$, $\alpha_3 = 0.9$



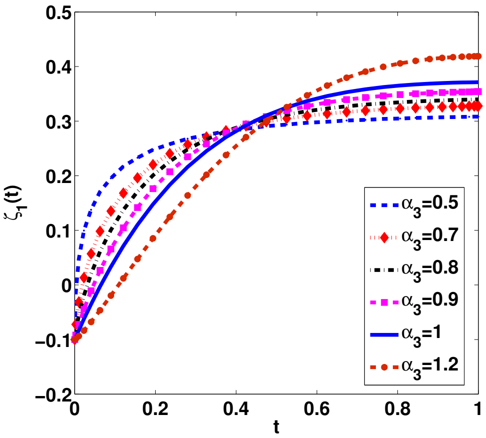
Efficiency



$L(t)$ for $\alpha_1 = 0.3$, $\alpha_3 = 0.9$



Convergence



$\zeta_1(t)$ for $\alpha_2 = 0.3$, $\alpha_2 = 0.5$

Fig. 4: plot of the variables x, y, z for $T = 1$

the fast convergence of the spectral method. Indeed it only takes 32 points and 512 points to already reach accuracy of order 10^{-3} and 10^{-7} respectively while it would necessitate respectively 1000 and 100,000 points to achieve the same order of accuracy for the fde12 method. In terms of efficiency, again Fig. 4 shows that fde12 method takes 1.7s to achieve accuracy order 10^{-4} whereas our spectral method covers this same accuracy within 0.2s. Since fde12 is mostly based on Adams–Bashforth–Moulton and some elementary integration rules, it turns out that such method would struggle to reach high order of accuracy as it would requires more points and consequently the number of unknowns to be determined gets larger. See [29] for more on the stability of fde12 method.

5 Conclusion

In this paper we have presented a spectral integral method to numerically solve systems of FDEs especially in the case of cryptocurrency models where the problem involves multiple fractional orders. The solution obtained using the suggested method shows that this approach can solve the problem effectively. Moreover, only a small number of shifted Chebyshev polynomials is needed to obtain a satisfactory result. The result from error analysis shows that our method maintains its spectral convergence. An efficiency analysis was also conducted against an existing method, here the fde12 and results reinforced again the ability of our spectral method to obtain high accuracy rapidly. For further applications it would be interesting to couple our method with a splitting method in order to handle large time scale FDEs. Also see how the method can be adjusted in the case of problems with kernels that would require Atangana-Baleanu fractional derivatives in the modeling instead of the Caputo derivative.

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Conflict of Interest The authors declare that they have no conflict of interest.

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Appendix

We are interested in computing the following integral

$$I_k(x) = \int_0^x \frac{T_k(\alpha t-1)}{(x-t)^{q-1}}dt \tag{34}$$

We have

$$\int_0^x \frac{1}{(x-t)^{1-q}}dt = \frac{x^q}{2q}, \tag{35}$$

$$\int_0^x \frac{t}{(x-t)^{1-q}}dt = \frac{x^{q+1}}{q(q+1)}, \tag{36}$$

$$\int_0^x \frac{t^2}{(x-t)^{1-q}}dt = \frac{2x^{q+2}}{q(q+1)(q+2)}. \tag{37}$$

Thus,

$$I_0(x) = \int_0^x \frac{1}{(x-t)^{1-q}}dt = \frac{x^q}{2q},$$

$$\begin{aligned} I_1(x) &= \int_0^x \frac{(\alpha t-1)}{(x-t)^{1-q}}dt \\ &= \alpha\int_0^x \frac{t\,dt}{(x-t)^{1-q}} + \int_0^x \frac{dt}{(x-t)^{1-q}} \\ &= \alpha\frac{x^{q+1}}{q(q+1)} - \frac{x^q}{2q}, \end{aligned}$$

$$\begin{aligned} I_2(x) &= \int_0^x \frac{2(\alpha t-1)^2-1}{(x-t)^{1-q}}dt \\ &= 2\alpha^2\int_0^x \frac{t^2dt}{(x-t)^{1-q}} \\ &\quad -4\alpha\int_0^x \frac{t\,dt}{(x-t)^{1-q}} + \int_0^x \frac{1}{(x-t)^{1-q}}dt \\ &= 4\alpha^2\frac{x^{q+2}}{q(q+1)(q+2)} - 4\alpha\frac{x^{q+2}}{q(q+1)} + \frac{x^q}{q}. \end{aligned}$$

Using the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2,3,\dots$, we get the following

$$\begin{aligned} I_k(x) &= \int_0^x \frac{T_k(\alpha t-1)}{(x-t)^q}dt \\ &= \int_0^x \frac{2(\alpha t-1)T_{k-1}(\alpha t-1) - T_{k-2}(\alpha t-1)}{(x-t)^q}dt \\ &= 2\alpha\int_0^x \frac{tT_{k-1}(\alpha t-1)}{(x-t)^{1-q}}dt - 2\int_0^x \frac{T_{k-1}(\alpha t-1)}{(x-t)^{1-q}}dt \\ &\quad - \int_0^x \frac{T_{k-2}(\alpha t-1)}{(x-t)^{1-q}}dt \\ &= 2\alpha\left[t\int_0^x \frac{T_{k-1}(\alpha t-1)}{(x-t)^{1-q}}dt\right]_{t=0}^{t=x} \\ &\quad - 2\alpha\int_0^x \frac{T_{k-1}(\alpha t-1)}{(x-t)^{1-q}}dt - 2I_{k-1}(x) - I_{k-2}(x) \\ &= 2\alpha x I_{k-1}(x) - 2\alpha\int_0^x \frac{T_{k-1}(\alpha t-1)}{(x-t)^{1-q}}dt - 2I_{k-1}(x) \\ &\quad -I_{k-2}(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} \tilde{I}_{k-1}(x) &= \int_0^x (x-t)^q T_{k-1}(\alpha t-1)dt \\ &= \int_0^x (x-t)^q\left[\frac{T_k'(\alpha t-1)}{2k} - \frac{T_{k-2}'(\alpha t-1)}{2(k-2)}\right]dt \\ &= \frac{1}{2k}\int_0^x (x-t)^q T_k'(\alpha t-1)dt \\ &\quad - \frac{1}{2(k-2)}\int_0^x (x-t)^q T_{k-2}'(\alpha t-1)dt \end{aligned}$$

That is,

$$\begin{aligned} &= \frac{1}{2k}\left\{\left[\frac{1}{\alpha}(x-t)^q T_k(\alpha t-1)\right]_{t=0}^{t=x} \right.\\ &\quad \left. + \int_0^x \frac{q}{\alpha}(x-t)^{q-1}T_k(\alpha t-1)dt\right\} \\ &\quad - \frac{1}{2(k-2)}\left\{\left[\frac{1}{\alpha}(x-t)^q T_{k-2}(\alpha t-1)\right]_{t=0}^{t=x} \right.\\ &\quad \left. + \int_0^x \frac{q}{\alpha}(x-t)^{q-1}T_{k-2}(\alpha t-1)dt\right\} \\ &= \frac{1}{2k}\left[-\frac{x^q}{\alpha}T_k(-1)\right] + \frac{q}{2k\alpha}\int_0^x \frac{T_k(\alpha t-1)}{(x-t)^{1-q}}dt \\ &\quad -\frac{1}{2(k-2)}\left[-\frac{x^q}{\alpha}T_{k-2}(-1)\right] \\ &\quad -\frac{q}{2\alpha(k-2)}\int_0^x (x-t)^{q-1}T_{k-2}(\alpha t-1)dt \end{aligned}$$

$$\begin{aligned} \tilde{I}_{k-1}(x) &= \frac{x^q}{2\alpha}\left[\frac{T_{k-2}(-1)}{k-2} - \frac{T_k(-1)}{k}\right] \\ &\quad +\frac{q}{2k\alpha}\int_0^x (x-t)^{q-1}T_k(\alpha t-1)dt \\ &\quad -\frac{q}{2\alpha(k-2)}\int_0^x (x-t)^{q-1}T_k(\alpha t-1)dt \\ &= \left[\frac{(-1)^{k-2}}{k-2} - \frac{(-1)^k}{k}\right]\frac{x^q}{2k\alpha} + \frac{q}{2k\alpha}I_k(x) \\ &\quad -\frac{q}{2\alpha(k-2)}I_{k-2}(x) \\ &= \frac{(-1)^k x^q}{\alpha k(k-2)} + \frac{q}{2k\alpha}I_k(x) - \frac{q}{2\alpha(k-2)}I_{k-2}(x). \end{aligned}$$

Therefore

$$\begin{aligned} I_k(x) &= -2\alpha\left[\frac{(-1)^k x^q}{\alpha k(k-2)} + \frac{qI_k(x)}{2k\alpha} - \frac{qI_{k-2}(x)}{2\alpha(k-2)}\right] \\ &\quad +2(\alpha x-1)I_{k-1}(x) - I_{k-2}(x) \\ &= \frac{-2(-1)^k x^q}{k(k-2)} - \frac{q}{k}I_k(x) + \frac{q}{k-2}I_{k-2}(x) \\ &\quad +2(\alpha x-1)I_{k-1}(x) - I_{k-2}(x) \end{aligned}$$

$$\begin{aligned} \left(1+\frac{q}{k}\right)I_k &= 2(\alpha x-1)I_{k-1}(x) + \left(\frac{q}{k-2}-1\right)I_{k-2}(x) \\ &\quad -\frac{2(-1)^k x^q}{k(k-2)}. \end{aligned}$$

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