# Fragmentation-coagulation equations with growth

by


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# Declaration

I, Poka David Wetsi, declare that the thesis, which I hereby submit for the degree Doctor of Philosophy at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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# Abstract

The theory of fragmentation-coagulation equations began around 1916 with a series of papers by Smoluchowski on pure coagulation and since then continued to incorporate other processes into the model. The intention was to study the evolution of objects undergoing breakdown and/or merging. The scientific goals are to determine the conditions under which solutions exist, are unique and identify them accordingly.

In this study, we considered the continuous fragmentation-coagulation equation with transport (decay or growth), subject to homogenous/McKendrick-von Foerster boundary condition in the latter case. The theory of semigroups of linear operators and, in particular, the Miyadera-Desch perturbation theorem are used to show the existence of semigroup solutions for the linear transport-fragmentation equation. We proved that the established semigroups have the moment improving property. The latter result plays a crucial role in the analysis of the complete transport-fragmentation-coagulation equation which is treated as a Lipschitz perturbation of the former linear problem. Under mild restrictions on the model coefficients, the existence of positive local classical solutions is established. Further, under additional conditions, their global in time existence is proved. Finally, a systematic technique is developed for obtaining closed-form solutions to continuous transport-fragmentation equations with homogenous boundary conditions and power-law coefficients. New solutions for the constant and linear decay/growth coefficients are presented. Furthermore, it is shown that the technique extends to some cases of the growth-fragmentation equation with the McKendrick-von Foerster boundary condition.

# Contents

Chapter I

# Introduction

## 1 Overview

Fragmentation and coagulation processes describe a wide range of natural and man-made events in animate and inanimate matter. Fragmentation means the breakdown of matter into smaller particles, whereas coagulation, its reciprocal, is the merging of particles to form new, larger clusters. Specific instances are evident in polymerisation and degradation, planetesimal formation, evolution of the marine world e.t.c., see [18, 40, 41, 55, 56] and references therein. Therefore, a better understanding of these processes is crucial.

The study of coagulation dates back to the early $20^{th}$ century and was done in two papers [51, 52] by Smoluchowski, who modeled the kinetics of colloidal formation. He considered coagulation equations with discrete particle sizes. The model was formulated as an infinite set of ordinary differential equations. Subsequently, Becker and Döring [21] extended the model to encompass discrete fragmentation.

In some applications, considering the discrete particle sizes turns out to be disadvantageous; as a result, a continuous form of the model is necessary. Müller [46] proposed a continuous coagulation equation given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)dy - \int_0^\infty k(x,y)u(x,t)u(y,t)dy, \qquad \text{(I.1.1)}$$

with

$$u(x,0) = u_0(x).$$

In this case, $x \geq 0$ and $t \geq 0$ denote the particle size and time, respectively. The density of particles $x$ at time $t$ is denoted by $u(x,t)$, while $k(x,y)$ is a coagulation kernel, which gives the rate at which particles of size $x$ merge with particles of size $y$ and it is assumed to be symmetric, that is, $k(x,y) = k(y,x)$. Factor $\frac{1}{2}$ is used to avoid double counting.

The extension to cover the fragmentation is due to Melzak [44], and it can be written as

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= -a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)dy \\ &+ \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)dy - \int_0^\infty k(x,y)u(x,t)u(y,t)dy.\end{aligned} \tag{I.1.2}$$

The first term on the right side of (I.1.2) represents the rate at which particles of mass/size $x$ break to form smaller particles and the second is the rate at which we get particles of size/mass $x$. The overall fragmentation rate is represented by $a(x)$ and the fragmentation kernel sometimes called the daughter distribution function, is represented by $b(x,y)$, describes the distribution of $x-$mass/size particles produced by a fragmentation of $y-$mass/size particle, while the remaining terms are defined as before.

Fragmentation and coagulation often occur in combination with other processes such as growth, decay or diffusion, and to account for, a term with partial derivative with respect to the size variable was introduced. This broadens the application domain of the model by incorporating e.g., the evolution of the species that can grow or die.

In this thesis, we deal with the mathematical analysis of a continuous fragmentation-coagulation equation with transport, which is given by the integro-partial differential equation [16]:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \pm\partial_x(r(x)u(x)) - a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)dy \\ &+ \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)dy - \int_0^\infty k(x,y)u(x,t)u(y,t)dy,\end{aligned} \tag{I.1.3}$$

subject to initial and, in some instances, boundary conditions. In (I.1.3), the function $r(x)$ is the transport coefficient that represents either the growth $(-)$ or the decay $(+)$. Two techniques have received attention in the wellposedness analysis of the models: the theory of semigroups and the weak compactness method. In the thesis, we employ the theory of semigroups of linear operators for the wellposedness of the linear part of (I.1.3) and its extension to semilinear problems for the analysis of the complete equation.

In addition to the qualitative analysis of (I.1.3), we also provide explicit solutions for some cases of the transport-fragmentation equation given by the integro-partial differential equation [24, 35, 36]:

$$\frac{\partial u(x,t)}{\partial t} = \pm\partial_x(r(x)u(x)) - a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)dy. \tag{I.1.4}$$

To obtain the explicit solutions of (I.1.4), the method of characteristics and the operator theory method will be employed.

Models (I.1.1), (I.1.2), (I.1.3) and (I.1.4) have been extensively studied from the beginning of the $20^{th}$ century to the present, see [2, 3, 16, 18, 24, 25, 28, 35, 36, 37] and references cited therein. We provide a brief survey of the existing results for continuous fragmentation and coagulation equations.

# 2 Literature review

## 2.1 Fragmentation-coagulation equation

Equation (I.1.2) with $k = 0$ is called pure fragmentation and, as a linear equation, it is easier to solve; hence, we begin our presentation with it. A systematic mathematical treatment of pure fragmentation equation was performed, both quantitatively and qualitatively, in many papers. One of the earliest studies of the equation was conducted in [44] and the main result was the identification of the conditions that guarantee the existence and uniqueness of nonnegative solutions.

Contribution to the analysis of pure fragmentation was further made by McLaughlin et al. [42] using the semigroup approach to fragmentation models and establishing the existence and uniqueness of nonnegative mass-conserving solutions. A systematic treatment using the same method was presented in [10] to determine the criteria for the existence or absence of shattering. Shattering is a phase transition phenomenon that results in the creation of a dust composed of particles of zero size. It was observed that shattering does not occur if the fragmentation rate is bounded for small particles. Also, nonlocal fragmentation models were investigated via the semigroup theory in [19] in the non-shattering regime. In [9], the author demonstrated that a pure fragmentation equation, despite its simplicity, can exhibit multiple solutions emanating from given a initial data. Nevertheless, based on the Hille and Phillips [34] results, the uniqueness for a large class of solutions was established.

Two papers [55, 56], provided closed-form solutions of pure fragmentation models relevant to droplet breakup, depolymerisation, e.t.c. Different types of the fragmentation kernels were considered. In particular, for the constant kernel, the explicit expression was found, which is a simplified version of the solution also obtained in [49]. Solutions for the linear and power rates of the fragmentation kernel can be expressed in terms of the confluent hypergeometric function.

Next, we consider the complete fragmentation-coagulation equation (I.1.2). It is semilinear and two methods have been used for its study: the theory of semigroups and the weak compactness method. The latter method was first successfully applied to the discrete fragmentation-coagulation equation by McLeod [43], Carr, Bell and Penrose [8] and Ball and Carr [6, 7]. This approach was extended to continuous models by Stewart [53], Dubovskiĭ and Stewart [25], Laurençot [39, 40], Giri, Kumar and Warnecke [31], and Giri and Warnecke [32]. The disadvantages of this approach are that the fragmentation term should be dominated by coagulation and that it provides only the existence of a weak solution, while other relevant properties, such as uniqueness and mass conservation, must be established separately.

Since the last decade of the $20^{th}$ century, we have witnessed an increasing number of applications of the semigroup theory to the fragmentation-coagulation equation [11, 15, 16, 25, 38, 42]. The main idea of this method is to recast the model as an abstract Cauchy problem and then establish the existence and uniqueness of solutions using the relevant results from semigroup theory. This method handles (I.1.2) as a Lipschitz perturbation of the linear fragmentation equation and hence, the fragmentation process is assumed to dominate coagulation. The treatment of (I.1.2) in this manner can be found in [14, 13, 17, 42].

## 2.2 Fragmentation-coagulation equation with transport

Often, fragmentation and coagulation occur along with other processes, such as growth and decay, resulting in equation (I.1.3). The decay of particles due to chemical reactions or dissolution is important in chemical engineering and other fields. The investigation of this phenomenon can be traced to the paper [3]. In particular, some explicit solutions were found in [24, 35, 36]. On the other hand, particles' growth is important in biological applications, see [37], where the author modeled the growth and formation of marine algal flocs and showed the conditions under which coagulation is essential for controlling algal concentrations and settling.

The mathematical analysis of the linear part of the transport-fragmentation-coagulation equation (I.1.3), that is, (I.1.4), was performed using sub-stochastic semigroup theory. There is also another approach to the problem, aimed mainly at finding the long-term asymptotics of the solutions, see [22, 45] and references cited therein.

In this thesis, we focus on extending the results of [16], the cases of decay-fragmentation-coagulation and growth-fragmentation-coagulation with McKendrick-von Foerster boundary condition. In particular, we generalize the moment improving property of [22], which is an important tool in proving the wellposedness of (I.1.3). Second, there is a growing need for exact solutions of the transport-fragmentation problems. These solutions are essential for many applications and for understanding of dynamic features of the model (I.1.3) that are not easily detectable by theoretical analysis. We extend the work of [36, 35] to include constant and linear growth rates under the power-law coefficients assumption. In addition, we consolidate fragments of specific former results and provide a unified setting for the problem.

# 3 Structure

## 3.1 Outline

In this thesis, we study continuous fragmentation-coagulation equations with decay or growth subject to the McKendrick-von Foerster boundary condition (which includes the previously considered homogenous condition) using the theory of semigroup of linear operators. We also provide explicit solutions for the decay/growth fragmentation equation under the homogenous and the McKendrick-von Foerster boundary condition.

In Chapter II, we describe the theories and tools used in the thesis. In particular, we provide a survey of relevant aspects of the semigroup theory, including generation results and perturbation theory. Then, Chapter III is dedicated to establishing the wellposedness of the transport-fragmentation-coagulation problem using the semigroup theory of linear operators. Dealing with the nonlinear coagulation requires an extension of the moment improving property of [22] to cover the decay-fragmentation and the growth-fragmentation equation with general McKendrick-von Foerster boundary condition. We combine this property with a fixed-point argument to establish local classical wellposedness of the model (I.1.3). We conclude the chapter with a result on its global in time solvability of (I.1.3). Chapter IV concentrates on the closed-form solution for the decay/growth-fragmentation equations with homogenous boundary conditions. Here, we assume power-law coefficients. These results are new and extend those of [54, 35, 36]. Furthermore, we show that our method of obtaining solutions applies to some cases of the growth-fragmentation equation with the McKendrick-von Foerster boundary condition. Finally, we conclude with a discussion of our findings and of future work.

## 3.2 Novelty

Achievements of the thesis consist in:

- extending the moment improving result in [22];
- extending the solvability results of transport-fragmentation-coagulation equation to more general settings, including the decay and the growth with the McKendrick-von Foerster boundary conditions;
- finding explicit solutions to the continuous decay/growth fragmentation equation with homogenous boundary conditions; developing a systematic approach for obtaining solutions for constant and linear decay/growth coefficients and extending the method to some spe-

cial cases of growth-fragmentation equation with the McKendrick-von Foerster boundary condition.

Parts of Chapter III and Chapter IV have been submitted to a journal for publication and a part of Chapter IV has been published [20].

Chapter II

# Preliminaries

In this chapter, we lay a foundation for the results and theories which we shall develop in this thesis. We are interested in solving partial differential equations (PDEs) that are central to modern mathematics and are not very easy to handle, since there is no robust way of solving all types. While we would like to solve problems in an explicit way, most often it is impossible. Hence, it is essential to study problems in a qualitative way, that is, to find whether the solutions exist, are unique and how they evolve. This thought leads us to an elegant theory of semigroup of linear operators, which we shall use in the thesis. Further, to illustrate the theoretical findings, we find exact solutions in some special cases of equations considered in this thesis. For this, we employ different methods, like the method of characteristics, power series e.t.c. We begin with notations, definitions and useful results.

## 1 General notation

We start by setting up the stage, introducing the basic notations necessary to study the fragmentation-coagulation equation with transport by the theory of semigroups of linear operators.

To avoid confusion, $\partial_x$ and $\frac{d}{dx}$ denote the partial derivative and the ordinary derivative with respect to $x$, respectively.

Further, we set $\mathbb{R}_+ := (0, \infty)$. We use $\Re x$ and $\Im x$ for the real and imaginary part of a complex number $x$, respectively.

Our analysis will be conducted in Banach spaces, hence, we begin with general definitions in these spaces.

**Definition II.1.1.** *[29] A Banach space $X$ is a complete normed linear space.*

Let $X$ be a Banach space with norm $\|\cdot\|$. Examples of Banach spaces are given by the Lebesgue spaces of integrable $X-$valued function $L_p(\Omega,\mu;X)$, $1 \le p \le \infty$, where $(\Omega,\mu)$ is a measure space. Elements of $L_p(\Omega,\mu;X)$ are equivalence classes of $\mu-$Bochner measurable functions $f:\Omega \to X$, satisfying

$$\begin{aligned} \|f\|_{L_p(\Omega,\mu;X)} &:= \left[\int_\Omega \|f\|^p d\mu\right]^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L_\infty(\Omega,\mu;X)} &:= \operatorname*{esssup}_{\mu} \|f\| = \inf\{a \in \mathbb{R} : \mu\{\|f\| > a\} = 0\} < \infty. \end{aligned} \tag{II.1.1}$$

In Chapter III, we either use measure $d\mu = dx$ or its weighted versions, $d\mu = (1+x^m)dx$ and $d\mu = x^m dx$, $m \ge 0$, in $\Omega = \mathbb{R}_+ = (0,\infty)$. In the first case, we abbreviate $L_m(\Omega,\mu;X)$ to $L_m(\Omega)$, while in the second case, we replace $L_1(\mathbb{R}_+,(1+x^m)dx)$ and $L_1(\mathbb{R}_+,x^m dx)$ with $X_m$ and $X_{[m]}$, respectively.

**Definition II.1.2.** *[29] Let $X$ and $Y$ be Banach spaces. We say that operator $T : D(T) \subset X \to Y$ is linear if $T(x+y) = T(x)+T(y)$ for all $x,y \in D(T)$ and $T(\lambda x) = \lambda T(x)$ for all $x \in D(T)$ and $\lambda \in \mathbb{R}$. The sets $D(T) \subset X$ and $Rg(T) := TD(T) \subset Y$ are called the domain and the range of $T$, respectively.*

**Definition II.1.3.** *We say that the linear map $T : D(T) \subset X \to Y$ is bounded if $D(T) = X$ and*

$$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y < \infty.$$

The Banach space of all bounded linear maps from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$. If a linear map $T : D(T) \subset X \to Y$ fails to be bounded, we call it unbounded. In this situation, we shall write the operator as the pair $(T, D(T))$. In the case when $D(T) = X$ and $(T, D(T))$ is bounded or there is no ambiguity in the definition of its domain $D(T)$, we write shortly $T$. The class of unbounded linear maps contains a vast number of operators that arise in modern mathematics and its applications. For example, it contains all differential operators.

**Definition II.1.4.** *[12] Let $X$ and $Y$ be Banach spaces. We say the linear map $T : D(T) \subset X \to Y$ is closed if and only if for every sequence $\{x_n\}_{n\ge 0} \subset D(T)$, such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Tx_n = y$, it follows that $x \in D(T)$ and $Tx = y$.*

**Definition II.1.5.** *[12] An operator $(T, D(T))$ is said to be invertible if there is a bounded linear operator $T^{-1} \in \mathcal{L}(Y, X)$ such that $T^{-1}Tx = x$ for all $x \in D(T)$ and $T^{-1}y \in D(T)$ with $TT^{-1}y = y$ for any $y \in Y$.*

# 2 Semigroup theory

Often in mathematics and its applications equations of the type

$$u'(t) = Tu(t), \quad u(0) = u_0 \tag{II.2.1}$$

arise. If $T \in \mathcal{L}(X, X)$ for some Banach space $X$, then for $u_0 \in X$, the solution of (II.2.1) reads

$$u(t) = e^{tT}u_0, \quad \text{where} \quad e^{tT} = \sum_{n=0}^{\infty} \frac{t^n T^n}{n!}.$$

The family of exponentials $(e^{tT})_{t \geq 0} \subset \mathcal{L}(X, X)$ satisfy

(i) $e^{tT} \in C(\mathbb{R}_+, \mathcal{L}(X, X))$,

(ii) $e^{(s+t)T} = e^{sT}e^{tT}$ for all $s, t \geq 0$,

(iii) $e^{0T} = I$,

where $I$ is the identity operator in $X$. The main goal of the semigroup theory is to find conditions that allows for a generalisation of the exponential function to a class of unbounded linear operators so that the properties (i) – (iii) hold.

## 2.1 Semigroups and generators

**Definition II.2.1.** *[47] A strongly continuous semigroup ($C_0$-semigroup) is a one parameter family $(S(t))_{t \geq 0}$ of bounded linear operators in Banach space $X$ satisfying*

*(i)* $S(0) = I$,

*(ii)* $S(t+s) = S(t)S(s), \qquad s,t \geq 0,$

*(iii)* $\lim_{t\to 0^+} S(t)x = x, \quad \forall x \in X.$

**Proposition II.2.1.** *[5, 47] Let $(S(t))_{t\geq 0}$ be a $C_0$-semigroup on a Banach space $X$. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\| S(t) \| \leq Me^{\omega t}, \quad \text{for all } t \geq 0. \tag{II.2.2}$$

*Proof.* See [47]. □

**Definition II.2.2.** *[12] The type or uniform growth bound $\omega_0(S)$, of $(S(t))_{t\geq 0}$ is defined as*

$$w_0(S) = \inf\{\omega; \text{there is } M \text{ such that (II.2.2) holds}\}.$$

A semigroup $(S(t))_{t\geq 0}$ is called a contraction semigroup if $M = 1$ and $\omega = 0$, that is, if $\|S(t)\| \leq 1$, $t \geq 0$.

Now, we connect the semigroup $(S(t))_{t\geq 0}$ with a linear operator $(T, D(T))$. This leads us to the definition of the infinitesimal generator.

**Definition II.2.3.** *[47] A linear operator $(T, D(T))$ is called the infinitesimal generator of $(S(t))_{t\geq 0}$ if*

$$Tx = \lim_{h\to 0^+} \frac{S(h)x - x}{h}, \tag{II.2.3}$$

*with $D(T)$ defined as the set of all $x \in X$ for which the limit exists.*

We shall denote by $(S_T(t))_{t\geq 0}$ the semigroup generated by $T$.

**Theorem II.2.2.** *[12] Let $(S_T(t))_{t\geq 0}$ be a $C_0$-semigroup. Then*

*(i) the domain of $T$ is dense in $X$;*

*(ii) the operator $T$ is closed.*

*Proof.* See [12]. □

A semigroup is defined uniquely by its generator.

**Theorem II.2.3.** *[47] Let $(S_T(t))_{t\geq 0}$ and $(S_B(t))_{t\geq 0}$ be $C_0$-semigroups of bounded linear operators with infinitesimal generators $(T, D(T))$ and $(B, D(B))$. If $(T, D(T)) = (B, D(B))$, then $S_T(t) = S_B(t)$, $t \geq 0$.*

*Proof.* See [47]. □

**Definition II.2.4.** *We say that $(T, D(T)) \in \mathcal{G}(M, \omega)$ if it generates $(S_T(t))_{t\geq 0}$ satisfying* (II.2.2).

We further mention the following properties of semigroups and their generators.

**Theorem II.2.4.** *[47] Let $(T, D(T))$ be the infinitesimal generator of a $C_0$-semigroup $(S_T(t))_{t\geq 0}$. Then, the following statements are true.*

*i) For $x \in X$,*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} S_T(r)x dr = S_T(t)x. \tag{II.2.4}$$

*ii) For $x \in X, \int_0^t S_T(r)xdr \in D(T)$ and*

$$T\bigg( \int_0^t S_T(r)xdr \bigg) = S_T(t)x - x. \tag{II.2.5}$$

*iii) For $x \in D(T), S_T(t)x \in D(T)$ and*

$$\frac{dS_T(t)x}{dt} = TS_T(t)x = S_T(t)Tx. \tag{II.2.6}$$

*iv) For $x \in D(T)$,*

$$T(t)x - T(s)x = \int_s^t S_T(r)Txdr = \int_s^t TS_T(r)xdr. \tag{II.2.7}$$

*Proof.* See [47]. □

## 2.2 Resolvent of the generator

In order to decide if a closed operator $(T, D(T))$ is the generator of a semigroup $(S_T(t))_{t\geq 0}$, we use its resolvent. First, we define the resolvent set.

**Definition II.2.5.** *Let $(T, D(T))$ be a linear operator in $X$. The set*

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T : D(T) \subset X \mapsto X \textit{ is invertible with bounded inverse}\}, \qquad \text{(II.2.8)}$$

*is called the resolvent set of $(T, D(T))$. Its complement $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of $T$.*

We also mention two important notions related to the spectrum of linear operators:

**Definition II.2.6.** *[27]*

1. *For a linear operator $T$ in a Banach space $X$,*

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}, \qquad \text{(II.2.9)}$$

   *is called the spectral radius of $T$.*

2. *For a linear operator $T$ in a Banach space $X$,*

$$s(T) := \sup\{\Re(\lambda) : \lambda \in \sigma(T)\}, \qquad \text{(II.2.10)}$$

   *is called the spectral bound of $T$.*

**Definition II.2.7.** *[27] Let $(T, D(T))$ be a linear operator on a Banach space $X$. For $\lambda \in \rho(T)$ the resolvent operator $R(\lambda, T) : X \to X$ is defined by*

$$R(\lambda, T) = (\lambda I - T)^{-1}.$$

We note that if $\big(S_T(t)\big)_{t\geq 0}$ is a semigroup in $X$ and $(T, D(T))$ is its generator, then the resolvent operator is given explicitly by

$$R(\lambda, T) := \int_0^\infty e^{-\lambda t} S_T(t) dt, \quad \lambda > s(T), \tag{II.2.11}$$

see [12, Theorem 3.34]. In particular, from (II.2.2) and (II.2.11) it follows that the resolvent set $\rho(T)$ of the generator $(T, D(T))$ contains the ray $\{\lambda > s(T)\} \subset \mathbb{R}$.

## 2.3 Generation theorems

As we did see earlier, every $C_0$-semigroup $(S_T(t))_{t\geq 0}$ is associated with a closed, densely defined linear operator $(T, D(T))$, whose resolvent set contains the ray $\{\lambda > s(T)\} \subset \mathbb{R}$. Our problem is: "Does any linear operator satisfying the above conditions always generate a $C_0$-semigroup?" The answer to the fundamental question is contained in the generation theorems. The first classical result is due to Hille and Yosida.

**Theorem II.2.5** (Hille-Yosida)**.** *[47] A linear (unbounded) operator $(T, D(T))$ is the infinitesimal generator of a $C_0$-semigroup of contractions $(S_T(t))_{t\geq 0}$ if and only if*

*(i) $T$ is closed and $\overline{D(T)} = X$;*

*(ii) the resolvent set $\rho(T)$ of $(T, D(T))$ contains $\mathbb{R}_+$ and for every $\lambda > 0$, $\|R(\lambda, T)\| \leq \frac{1}{\lambda}$.*

We can extend this result to quasi-contractive semigroups, i.e., semigroups satisfying (II.2.2) with $M = 1$ and some $\omega \in \mathbb{R}$. In this case, instead of (ii), we have

$$\| R(\lambda, T) \| \leq \frac{1}{\lambda - \omega}, \quad \lambda > \omega. \tag{II.2.12}$$

For the general case, we have the following:

**Theorem II.2.6** (Feller, Miyadera, Phillips)**.** *[27] Let $(T, D(T))$ be a linear operator on a Banach space $X$ and let $\omega \in \mathbb{R}, M \geq 1$ be constants. Then the following properties are equivalent.*

*i)* $(T, D(T))$ *generates a strongly continuous semigroup* $(S_T(t))_{t\geq 0}$ *satisfying*

$$\| S_T(t) \| \leq M e^{\omega t}, \quad \textit{for } t \geq 0.$$

*ii)* $(T, D(T))$ *is closed, densely defined, and for every* $\lambda > \omega$ *one has* $\lambda \in \rho(T)$ *and*

$$\| \left[(\lambda - \omega) R(\lambda, T)\right]^n \| \leq M, \quad \textit{for all } n \in \mathbb{N}.$$

*iii)* $(T, D(T))$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$, *with* $\Re\lambda > \omega$ *one has* $\lambda \in \rho(T)$ *and*

$$\| R(\lambda, T)^n \| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad \textit{for all } n \in \mathbb{N}.$$

## 2.4 Positivity of semigroups

In many evolution problems, the positivity of solutions is an essential feature. For instance, the unknown function we are looking for may define the size of a population, the number of particles e.t.c. This requires that the solutions should be positive throughout the evolution. Dealing with the positivity in the setting of Banach spaces requires prior definitions which we state below.

**Definition II.2.8.** *[12] Let* $X$ *be an arbitrary set. A partial order on* $X$ *is a binary relation, denoted by '*$\geq$*', which is reflexive, antisymmetric and transitive, that is*

1. $x \geq x$ *for any* $x \in X$*;*
2. $x \geq y$ *and* $y \geq x$ *implies* $x = y$ *for any* $x, y \in X$*;*
3. $x \geq y$ *and* $y \geq z$ *implies* $x \geq z$ *for any* $x, y, z \in X$*.*

An upper bound of a set $M \subset X$ is an element $x \in X$ satisfying $x \geq y$, for all $y \in M$, while its lower bound is an element $x \in X$ such that $x \leq y$ for all $y \in M$.

**Definition II.2.9.** *i) The supremum of a set* $M \subset X$ *is its least upper bound and is denoted by* $\sup M$*.*

*ii) The infimum of a set* $M \subset X$ *is its greatest upper bound and is denoted by* $\inf M$*.*

The supremum or infimum of a set is unique if it exists.

**Definition II.2.10.** *[12] The set $X$ is a lattice if for every finite collection $M$ of elements of $X$ there exists its supremum and infimum.*

**Definition II.2.11.** *[12] A vector space $X$, equipped with partial order which is compatible with its vector structure in the sense that*

1. $x \geq y$ *implies* $x + z \geq y + z$ *for all* $x, y, z \in X$*;*
2. $x \geq y$ *implies* $\alpha x \geq \alpha y$ *for any* $x, y \in X$ *and* $\alpha \geq 0$,

*is called a partially ordered vector space. If an ordered vector space $X$ is also a lattice, then it is called a vector lattice or a Riesz space.*

For an element $x$ of a Riesz space $X$, we can define the positive and the negative part of $x$ as $x_+ = \sup\{x, 0\}$ and $x_- = \sup\{0, -x\}$. From our definitions, it follows that $x = x_+ - x_-$. Further, we let $|x| = x_+ + x_-$. The quantity $|x|$ is known as the modulus of $x$.

**Definition II.2.12.** *[12] A norm on a vector lattice $X$ is called a lattice norm if*

$$|x| \leq |y| \qquad \textit{implies} \quad \|x\| \leq \|y\|. \tag{II.2.13}$$

**Definition II.2.13.** *The set $X_+ = \{x \in X : x \geq 0\}$ is referred to as the positive cone of $X$.*

At this point we define a Banach lattice as a Riesz space under the lattice norm. Further, we define a positive operator.

**Definition II.2.14.** *[12] A linear operator $(T, D(T))$ from a Banach lattice $X$ into a Banach lattice $Y$ is called positive and is denoted as $T \geq 0$, if $Tx \geq 0$ for any $0 \leq x \in D(T)$.*

**Definition II.2.15.** *An operator $(T, D(T))$ in a Banach lattice $X$ is resolvent positive if there exists $r$ such that $(r, \infty) \subset \rho(T)$ and $R(\lambda, T) \geq 0$ for all $\lambda > r$.*

This leads us to the following theorem.

**Theorem II.2.7.** *[4, 27] A strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach lattice $X$ is positive if and only if its generator $(T, D(T))$ is resolvent positive.*

*Proof.* See [27] □

## 3 Abstract Cauchy Problem (ACP)

The theory of semigroups is successful due to the fact that many evolution problems can be written in the form of an Abstract Cauchy Problems (ACP) that we state immediately below.

**Definition II.3.1.** *[27] Let $X$ be a Banach space, $(T, D(T))$ be a linear operator and $f \in L_1(\mathbb{R}_+; X)$. The initial value problem of the form*

$$u_t(t) = Tu(t) + f(t), \quad \text{for} \quad t > 0, \quad u(0) = u_0 \in X, \tag{II.3.1}$$

*is termed an abstract inhomogeneous Cauchy problem and if $f = 0$, it is termed an abstract homogeneous Cauchy problem on $X$, associated with $(T, D(T))$ and initial value $u_0$.*

Solutions of (II.3.1) can be understood in a number of ways. We discuss two of them, namely mild and classical solutions.

**Definition II.3.2.** *[27] A function $u : \mathbb{R}_+ \to X$ is a classical solution of (II.3.1) on $[0, \infty)$ if $u$ is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$, $u(t) \in D(T)$ for $t \geq 0$ and (II.3.1) is satisfied.*

**Theorem II.3.1.** *[27] Let $T : D(T) \subset X \to X$ generate a $C_0$-semigroup $(S_T(t))_{t\geq 0}$ on $X$, then for all $u_0 \in D(T)$ and $f = 0$, (II.3.1) has a unique classical solution given by $u(t) = S_T(t)u_0$, $t \geq 0$.*

*Proof.* See [27]. □

**Definition II.3.3.** *[47] Let $(T, D(T))$ be an infinitesimal generator of a $C_0$-semigroup $(S_T(t))_{t\geq 0}$ in $X$. Let $u_0 \in X$ and $f \in L_1((0,\infty), X)$. The function $u \in C([0,\infty), X)$ given by*

$$u(t) = S_T(t)u_0 + \int_0^t S_T(t-s)f(s)\mathrm{d}s$$

*is called a mild solution of the ACP (II.3.1) on $[0,\infty)$.*

It is important to note that mild solutions of (II.3.1) exist for every initial value $u_0 \in X$. In contrast, the solutions are classical if $u_0 \in D(T)$, $u \in C^1((0,\infty), X)$ and (II.3.1) is satisfied in the sense of $X$ for all $t > 0$.

## 4 Perturbation theory

It is worth pointing out that if $(T, D(T))$ generates a $C_0$-semigroup in $X$, then the existence, uniqueness, and continuous dependence of solutions on the data $u_0$, follows immediately. However, problems arise if more than one operator is involved, making it difficult to apply the theory of semigroups directly. In such a case, we use the perturbation theory. The perturbation theory problem can be stated roughly as: suppose that $(T, D(T))$ is a linear operator in Banach space $X$ that generates a $C_0$-semigroup $\left(S_T(t)\right)_{t\geq 0}$. Let $B$ be another operator acting in $X$. The question is: under what conditions does the sum $T + B$ generate a semigroup? We note that the sum of operators is defined by

$$(T + B)x := Tx + Bx,$$

with

$$D(T + B) := D(T) \cap D(B).$$

For a bounded $B$, the relevant result is given in the theorem below.

**Theorem II.4.1.** *[12] Let $(T, D(T)) \in \mathcal{G}(M,\omega)$. If $B \in \mathcal{L}(X)$, then $(K, D(T)) := (T + B, D(T)) \in \mathcal{G}(M, \omega + M\|B\|)$. Moreover, the semigroup $\left(S_{T+B}(t)\right)_{t\geq 0}$ generated by $T + B$*

*satisfies either of the Duhamel equations:*

$$S_{T+B}(t)x = S_T(t)x + \int_0^t S_T(t-s)BS_{T+B}(s)xds, \quad t \geq 0, \quad x \in X, \tag{II.4.1a}$$

*and*

$$S_{T+B}(t)x = S_T(t)x + \int_0^t S_{T+B}(t-s)BS_T(s)xds, \quad t \geq 0, \quad x \in X, \tag{II.4.1b}$$

*where the integrals are defined in the strong operator topology. The semigroup* $\big(S_{T+B}(t)\big)_{t\geq 0}$ *is also given by the Dyson-Phillips series obtained by iterating* (II.4.1)*:*

$$S_{T+B}(t) = \sum_{n=0}^{\infty} S_n(t),$$

*where* $S_0(t) = S_T(t)$ *and*

$$S_{n+1}(t)x = \int_0^t S_T(t-s)BS_n(s)xds, \quad t \geq 0, \quad x \in X.$$

*The series converges in the operator norm of* $\mathcal{L}(X)$ *and uniformly for* $t$ *in bounded intervals.*

*Proof.* See [12]. □

We note that the Dyson-Phillips expansion show that if $\big(S_T(t)\big)_{t\geq 0}$ is positive and $B \geq 0$ then $\big(S_{T+B}(t)\big)_{t\geq 0}$ is positive.

The situation is very delicate for unbounded perturbations and in this thesis we make use of the Miyadera-Desch perturbation theorem.

**Theorem II.4.2** (Miyadera-Desch)**.** *[12, Lemma 5.12] If* $T$ *is the generator of a positive* $C_0$*-semigroup in* $X = L_1(\Omega, \mu)$ *and* $B \in \mathcal{L}(D(T), X)$ *is a positive operator such that for some* $\lambda > s(T)$*, we have* $\| B(\lambda I - T)^{-1} \| < 1$*, then* $(T + B, D(T))$ *generates a positive semigroup.*

*Proof.* See [12]. □

## 5 Auxiliary results

In this section, we state some important results that arise often in the calculations.

**Proposition II.5.1.** *Let $x > 0$ and $m > 1$. Then*

$$x^{m-1} + x^m \leq \begin{cases} 1 + x^m, & \text{for } x < 1 \\ 2x^m, & \text{for } x \geq 1, \end{cases} \leq 2(1 + x^m).$$

*Proof.* For $0 < x < 1$, $x^{m-1} < 1$ and for $x \geq 1$, $x^{m-1} < x^m$. Combining this facts, the assertion follows. □

**Proposition II.5.2.** *[16, Equation (3.5)] Let $0 \leq p \leq \eta$. Then*

$$(1 + x^\eta)(1 + x^p) \leq 4(1 + x^{\eta+p}),$$

*for $x > 0$.*

**Theorem II.5.3.** *[18, Lemma 7.4.2] Let $m > 1$. Then*

$$(x + y)^m - x^m - y^m \leq C_m(xy^{m-1} + yx^{m-1}), \quad (x, y) \in \mathbb{R}_+^2, \tag{II.5.1}$$

*with $C_m := 2^{m-1} - 1$, for $m \in (1, 2] \cup [3, \infty)$ and $C_m := m$, for $m \in (2, 3)$.*

*Proof.* See [18]. □

**Proposition II.5.4.** *Let $m \geq l$. Then*

$$\frac{1 + x^l}{1 + x^m} \leq K_{m,l} \leq 2, \tag{II.5.2}$$

*with $K_{m,l} = \dfrac{1 + (\frac{l}{m})^{\frac{l}{m-l}}}{1 + (\frac{l}{m})^{\frac{m}{m-l}}}$ for $x \in (0, 1)$ and $K_{m,l} = 1$ for $x \in [1, \infty)$.*

*Proof.* For $m \geq l$, we let $f(x) = \dfrac{1 + x^l}{1 + x^m}$. Then $f(0) = f(1) = 1$ and in the interval $(0, 1)$, we have $x^m < x^l$. Function $f(x)$ is increasing until it attains the maximum at $c = \left(\frac{l}{m}\right)^{\frac{1}{m-l}} < 1$ and then decreases. We note that $1 < f(c) < 2$. In the interval $[1, \infty)$, $x^l < x^m$ and the function is decreasing and $f(x) \leq 1$ which concludes the proof. □

**Lemma II.5.5** (Gronwall-Henry inequality)**.** *[18, Lemma 7.5.1] Let* $u \in L_{\infty,loc}((0,T]) \cap L_1(0,T)$, $0 < T < \infty$, *be a non-negative function satisfying*

$$u(t) \leq \frac{c}{t^\gamma} + c\int_0^t u(s)(t-s)^{-\alpha} ds, \quad t \in (0,T], \tag{II.5.3}$$

*where* $\gamma < 1, \quad 0 < \alpha < 1$ *and* $c > 0$. *Then there is a constant* $C = cC(\gamma, \alpha, T)$ *such that*

$$u(t) \leq \frac{C}{t^\gamma}, \quad t \in (0,T]. \tag{II.5.4}$$

*Proof.* See [18]. □

**Theorem II.5.6** (Young's inequality)**.** *[50] For* $a, b \geq 0$ *and* $p, q \geq 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *one has*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{II.5.5}$$

*Proof.* See [50]. □

**Theorem II.5.7** (Hölder's inequality)**.** *[50] If* $p, q \geq 1$ *be such that* $\frac{1}{p}+\frac{1}{q} = 1$, *and if* $f \in L_p(\Omega, \mu)$ *and* $g \in L_q(\Omega, \mu)$, *then* $fg \in L_1(\Omega, \mu)$ *and*

$$\| fg \|_1 \leq \| f \|_p \| g \|_q . \tag{II.5.6}$$

*Proof.* See [50]. □

Chapter III

# Analysis of the model

## 1 Introduction

The fragmentation-coagulation-transport equations appear in applications, describing various physical and biological phenomena [13, 18, 35]. In this thesis, we provide a unified treatment of both decay and growth cases, extending the latter to cover the McKendrick-von Foerster boundary conditions. Our aim is to prove its well-posedness using an appropriate semigroup theory presented in Chapter II. We start by stating the problem and defining the terms involved and Banach spaces of choice.

### 1.1 The model

The continuous fragmentation-coagulation-transport equation reads

$$\begin{aligned}\partial_t u(x,t) &= \pm\partial_x(r(x)u(x,t)) - a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)dy \\ &\quad - u(x,t)\int_0^\infty k(x,y)u(y,t)dy + \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)dy, \\ u(x,0) &= u_0(x), \quad x \geq 0,\end{aligned} \tag{III.1.1}$$

where $-$ and $+$ denote the growth and the decay scenarios, respectively. In the growth case, the initial condition is complemented by the McKendrick-von Foerster boundary condition

$$\lim_{x\to 0^+} r(x)u(x,t) = \int_0^\infty \beta(x)u(x)dx. \tag{III.1.2}$$

In the model (III.1.1), $u(x,t)$ is the density of particles of mass/size $x \in (0,\infty)$ and the overall fragmentation rate of particles of mass/size $x$ is $a(x)$. The fragmentation kernel $b(x, y)$ describes mass distribution of $x-$mass/size particles spawned by the fragmentation of a mass/size $y$ particle. Transport coefficient, which defines the rate of growth/decay of particle of mass/size $x$ is represented by $r(x)$. The term $k(x, y)$ is the coagulation kernel which is symmetric and represents the rate at which particles of mass/size $x$ coalesce with particles of mass/size $y$. Coefficient $\beta(x)$ gives the rate at which the daughter cells enter the population.

In our analysis, we rewrite (III.1.1) in the ACP form

$$\partial_t u(x,t) = T^{\pm}u(x,t) + Fu(x,t) + C[u,u](x,t), \qquad (x,t) \in \mathbb{R}^2_+, \tag{III.1.3}$$

where the transport operator $T^{\pm}$, the fragmentation operator $F$ and the coagulation operator $C$ are respectively defined by

$$\begin{aligned} T^{\pm}u(x) &= \pm\partial_x\big(r(x)u(x)\big), \\ Fu(x) &= Au(x) + B_m u(x) = -a(x)u(x) + \int_x^{\infty} a(y)b(x,y)u(y)dy, \\ C[u,u](x) &= \frac{1}{2}\int_0^x k(x-y)u(x-y)u(y)dy - u(x)\int_0^{\infty} k(x,y)u(y)dy. \end{aligned} \tag{III.1.4}$$

The choice of the space is motivated by the interpretation of the two integral norms for non-negative $u$. In $X_0 := L_1(\mathbb{R}_+, dx)$, for $u \geq 0$, the norm

$$\| u \|_0 := \int_0^{\infty} u(x)dx,$$

yields the total number of particles in the ensemble, while the $X_1 := L_1(\mathbb{R}_+, xdx)$ norm

$$\| u \|_1 := \int_0^{\infty} u(x)xdx, \tag{III.1.5}$$

gives the total mass of the particles in the system. Further, to introduce some control on the large particles and thus improve the properties of the solutions, we shall study (III.1.1) in the higher moment spaces $X_m := L_1(\mathbb{R}_+, (1+x^m)dx)$, for $m \geq 1$. The norm in $X_m$ is defined by

$$\| u \|_m := \int_0^\infty |u(x)|(1+x^m)dx, \quad m \geq 1. \tag{III.1.6}$$

## 1.2 Assumptions

The appearance of the term involving the derivative with respect to the state variable makes the solvability of (III.1.1) not a straightforward procedure and requires specific assumptions on the coefficients, which we adopt from [12, 18], with slight modification.

The transport coefficient $r$ is assumed to be positive and continuous in $(0, \infty)$. The fragmentation rate $a$ is assumed to satisfy

$$0 \leq a \in L_{\infty,loc}([0,\infty)).$$

The assumptions imply that $1/r, a/r \in L_{1,loc}((0,\infty))$. For future use, we introduce

$$R(x) := \int_\zeta^x \frac{ds}{r(s)}, \quad Q(x) := \int_\zeta^x \frac{a(s)}{r(s)}ds, \tag{III.1.7}$$

for some $\zeta > 0$. We note that $R(\zeta) = Q(\zeta) = 0$ by definition of definite integrals. An immediate consequence of (III.1.7) is that $R$ is strictly increasing and $Q$ is non-decreasing on $(0, \infty)$. We have the following limits,

$$\begin{aligned} \lim_{x\to 0} R(x) = m_R, \quad \lim_{x\to\infty} R(x) = M_R, \\ \lim_{x\to 0} Q(x) = m_Q, \quad \lim_{x\to\infty} Q(x) = M_Q. \end{aligned} \tag{III.1.8}$$

In general, $m_R$ and $m_Q$ can either be finite or $-\infty$ and $M_R$ and $M_Q$ can be finite or $+\infty$.

The fragmentation kernel is assumed to be a measurable function of two variables satisfying

$$b \geq 0, \quad b(x,y) = 0, \quad \text{for } x > y. \tag{III.1.9}$$

For $m \geq 0$, we define the quantities as in [18],

$$n_m(y) = \int_0^y x^m b(x,y)dx, \tag{III.1.10}$$

$$N_m(y) = y^m - n_m(y). \tag{III.1.11}$$

In particular, $n_0(y)$ is the mean number of daughter aggregates spawned by the fragmentation of a mass/size $y$ aggregate. If the fragmentation process is conservative, i.e., if $N_1(y) = 0$, we have

$$\begin{aligned} &N_m(y) > 0, \quad m > 1, \\ &N_m(y) < 0, \quad 0 \leq m < 1. \end{aligned} \tag{III.1.12}$$

We further assume that there is $l \geq 0$ and $b_0 \in \mathbb{R}_+$, such that

$$n_0(x) \leq b_0(1 + x^l), \quad x \in (0, \infty) \tag{III.1.13}$$

and there exists $m_0 > 1$ such that

$$\lim_{y \to \infty} \inf \frac{N_{m_0}}{y^{m_0}} > 0. \tag{III.1.14}$$

It is noted in [11] that if (III.1.14) holds for some $m_0 > 1$, then it holds for all $m > 1$. Then (III.1.14) implies that for any $m > 1$ there is a $c_m < 1$ and $y_m > 0$, such that

$$n_m(y) \leq c_m y^m, \quad y \geq y_m. \tag{III.1.15}$$

The analysis commences with the decay scenario and is followed by the growth case.

# 2 The decay-fragmentation equation

In this section, we consider the continuous fragmentation equation with decay, defined by

$$\partial_t u(x,t) = T^+ u(x,t) + Fu(x,t), \qquad (x,t) \in \mathbb{R}_+^2, \tag{III.2.1}$$

with

$$u(x,0) = u_0(x), \qquad x \geq 0. \tag{III.2.2}$$

## 2.1 The transport semigroup

We start by assuming that $B = 0$. Then problem (III.2.1) reduces to

$$\begin{aligned} \partial_t u(x,t) &= \partial_x(r(x)u(x,t)) - a(x)u(x,t), \\ u(x,0) &= u_0(x), \quad x \geq 0. \end{aligned} \tag{III.2.3}$$

We reformulate (III.2.3) as an abstract Cauchy problem posed in $X_m$

$$\begin{aligned} \frac{du}{dt} &= T^+u - Au =: Z_m^+ u, \quad u \in D(Z_m^+) \subset D_{Z_m}, \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{III.2.4}$$

where $Z_m^+ u = \frac{d(ru)}{dx} - au$. The natural domain of $Z_m^+$ is

$$D_{Z_m} = \left\{ u \in X_m : \frac{d(ru)}{dx}, au \in X_m \right\}. \tag{III.2.5}$$

Our principal aim is to find a $D(Z_m^+)$ so that $(Z_m^+, D(Z_m^+))$ generates a semigroup in $X_m$.

As the first step in this direction, for a fixed value of $m \geq 0$, we find the resolvent of $Z_m^+$, which is formally given as the solution of the ordinary differential equation

$$\lambda u(x) + a(x)u(x) - \frac{d(r(x)u(x))}{dx} = g(x), \quad g \in X_m. \tag{III.2.6}$$

For $\lambda > 0$, the formal solution reads

$$u(x) = \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(x)-Q(x)} g(y)dy + C\frac{e^{\lambda R(x)+Q(x)}}{r(x)}, \tag{III.2.7}$$

where $C$ is a constant. As in [18], an investigation for the possible eigenfunctions of (III.2.6) shows that $v_\lambda = \frac{e^{\lambda R(x)+Q(x)}}{r(x)}$ is a formal eigenfunction for the homogenous part of (III.2.6) while the other term solves (III.2.6) and vanishes at infinity for functions $g \in X_m$. The norm of $v_\lambda$ in $X_m$ is given by

$$\| v_\lambda \|_m = \int_0^\infty \frac{e^{\lambda R(x)+Q(x)}}{r(x)}(1+x^m)dx. \tag{III.2.8}$$

We point out that if $\| v_\lambda \|_m < \infty$ for some choice of $a$ and $r$, the solution in (III.2.7) is in $X_m$, for $C \in \mathbb{R}$, see [18]. In this situation, $(\lambda I - Z_m^+, D_{Z_m})$ is not invertible, hence the operator $(Z_m^+, D_{Z_m})$ cannot generate a semigroup. To cater for this non-trivial case, we use [18, Lemma 5.2.1]. That is, we let

$$D(Z_m^+) = \begin{cases} D_{Z_m}, & \text{if } \| v_1 \|_m = \infty; \\ D_{Z_m} \text{ and } \lim_{x\to\infty} \frac{u(x)}{v_1(x)} = 0, & \text{if } \| v_1 \|_m < \infty, \end{cases}$$

where $v_1(x) = \frac{e^{R(x)+Q(x)}}{r(x)}$.

**Lemma III.2.1.** *[18, Lemma 5.2.1] Under the adopted assumptions, for any $m \geq 1$ and $\lambda > 0$, the family of operators, defined by*

$$[R(\lambda)g](x) = \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(y)-Q(y)} g(y)dy, \quad g \in X_m, \tag{III.2.9}$$

*is the resolvent of $(Z_m^+, D(Z_m^+))$, i.e., $R(\lambda, Z_m^+) = R(\lambda), \lambda > 0$. Furthermore,*

$$\| R(\lambda, Z_m^+) \|_m \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

*Proof.* To begin, we show that $R(\lambda)X_m \subset D(Z_m^+)$. Let $g \in X_m$ and $v = [R(\lambda)g]$, then for

$\lambda > 0$ we have

$$\begin{aligned}
\| R(\lambda)g \|_m &= \int_0^\infty (1+x^m)\frac{e^{\lambda R(x)+Q(x)}}{r(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}g(y)dydx \\
&= \int_0^\infty g(y)e^{-\lambda R(y)-Q(y)}\left(\int_0^y (1+x^m)\frac{e^{\lambda R(x)+Q(x)}}{r(x)}dx\right)dy \\
&\leq \int_0^\infty g(y)e^{-\lambda R(y)}\left(\int_0^y (1+x^m)\frac{e^{\lambda R(x)}}{r(x)}dx\right)dy \\
&\leq \int_0^\infty g(y)e^{-\lambda R(y)}\left(\frac{1}{\lambda}\int_0^y (1+x^m)\frac{d[e^{\lambda R(x)}]}{dx}dx\right)dy \\
&\leq \frac{1}{\lambda}\int_0^\infty g(y)e^{-\lambda R(y)}[(1+y^m)e^{\lambda R(y)} - \lim_{x\to 0^+}(1+x^m)e^{\lambda R(x)}]dy \\
&\leq \frac{1}{\lambda} \| g \|_m,
\end{aligned} \tag{III.2.10}$$

where we used the Fubini-Tonelli theorem to change the order of integration and $e^{Q(y)} = \sup_{s\in(0,y)} e^{Q(s)}$ since it is non-decreasing. Hence, $R(\lambda)g$ is a bounded operator and $\| R(\lambda)g \|_m \leq \frac{1}{\lambda} \| g \|_m$. Further,

$$\begin{aligned}
\| aR(\lambda)g \|_m &= \int_0^\infty a(x)(1+x^m)\frac{e^{\lambda R(x)+Q(x)}}{r(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}g(y)dydx \\
&= \int_0^\infty g(y)e^{-\lambda R(y)-Q(y)}\left(\int_0^y a(x)(1+x^m)\frac{e^{\lambda R(x)+Q(x)}}{r(x)}dx\right)dy \\
&\leq \int_0^\infty g(y)e^{-\lambda R(y)-Q(y)}\left(\int_0^y (\lambda+a(x))(1+x^m)\frac{e^{\lambda R(x)+Q(x)}}{r(x)}dx\right)dy \\
&= \int_0^\infty g(y)e^{-\lambda R(y)-Q(y)}\left(\int_0^y (1+x^m)\frac{d[e^{\lambda R(x)+Q(x)}]dx}{dx}\right)dy \\
&\leq \int_0^\infty g(y)e^{-\lambda R(y)-Q(y)}\left[(1+x^m)e^{\lambda R(x)+Q(x)}\right]_0^y dy \\
&\leq \| g \|_m,
\end{aligned} \tag{III.2.11}$$

where again we changed the order of integration using the Fubini-Tonelli theorem. We notice that $\partial_x(rv) = (\lambda + a)v - g$, therefore

$$\| \partial_x(rv) \|_m = \| (\lambda+a)v - g \|_m \leq \| g \|_m + \| g \|_m = 2 \| g \|_m, \tag{III.2.12}$$

on the account of (III.2.6) and (III.2.11). Using (III.2.10)–(III.2.12), direct calculations give

$$\lambda R(\lambda)g + a(x)R(\lambda)g - \partial_x(r(x)R(\lambda)g) = g, \tag{III.2.13}$$

for almost all $x > 0$. If $\| v_1 \|_m = \infty$, we have immediately $R(\lambda)X_m \subset D(Z_m^+)$, while for $\| v_1 \|_m < \infty$,

$$\frac{R(\lambda)g(x)}{v_1(x)} = \int_x^\infty e^{-\lambda R(y) - Q(y)} g(y)dy \tag{III.2.14}$$

approaches zero as $x \to \infty$ and $R(\lambda)X_m \subset D(Z_m^+)$. Thus, $R(\lambda)g \in D(Z_m^+)$, while (III.2.13) shows that $R(\lambda)$ is the right inverse of $\lambda I - Z_m^+$.

We further show that $R(\lambda)$ is the left inverse of $\lambda I - Z_m^+$ in $D(Z_m^+)$. Consider $R(\lambda)(\lambda I - Z_m^+)g$, with $g \in D(Z_m^+)$. We have

$$\begin{aligned}
\left[R(\lambda)(\lambda I - Z_m^+)g\right](x) &= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \int_x^\infty e^{-\lambda R(x) - Q(x)} (\lambda I - Z_{0,m}^+)g(y)dy \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \int_x^\infty e^{-\lambda R(x) - Q(x)} (\lambda g(y) - [r(y)g(y)]_y + a(y)g(y))dy \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \int_x^\infty -\frac{d}{dy}[e^{-\lambda R(x) - Q(x)} r(y)g(y)]dy \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} [-e^{-\lambda R(x) - Q(x)} r(y)g(y)]_x^\infty \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \left[e^{-\lambda R(x) - Q(x)} r(x)g(x) - \lim_{y\to\infty} \frac{g(y)}{v_\lambda(y)}\right] \\
&= g(x) - v_\lambda(x) \lim_{y\to\infty} \frac{g(y)}{v_1(y)}.
\end{aligned} \tag{III.2.15}$$

Since $g \in D(Z_m^+)$, the limit in the last line vanishes and it follows that $R(\lambda)(\lambda I - Z_m^+)g = g$. Hence, $R(\lambda, Z_m^+) = R(\lambda)$ is the resolvent of $Z_m^+$. □

We note that $(\lambda I - Z_m^+)^{-1}$ is bounded by (III.2.10) and, since every bounded operator is closed, $(\lambda I - Z_m^+)$ is closed and therefore $Z_m^+$ is also closed.

**Theorem III.2.2.** *[18, Theorem 5.2.4] The operator $(Z_m^+, D(Z_m^+))$ generates a strongly continuous positive semigroup of contractions, say $(G_{Z_m^+}(t))_{t\geq 0}$, on $X_m$ for any given $m \geq 1$.*

*Proof.* The proof follows from Lemma III.2.9, the positivity of $R(\lambda, Z_m^+)$ and Theorem II.2.5. □

## 2.2 The decay-fragmentation semigroup

We have established the existence of a strongly continuous positive semigroup of contractions $(G_{Z_m^+}(t))_{t\geq 0}$ for the associated problem (III.2.3). Now we consider the complete linear part of (III.2.1) , i.e., (III.2.1) with $C = 0$:

$$\partial_t u(x,t) = T^+u(x) + Fu(x) = Z_m^+u(x) + B_m u(x), \quad u(0) = u_0, \quad t \geq 0, \tag{III.2.16}$$

where $Z_m^+u(x)$ is defined in (III.2.4) and $B_m u(x) = \int_x^\infty a(y)b(x,y)u(y)dy$.

**Theorem III.2.3.** *Let $m > \max\{1, l\}$ and (III.1.2), (III.1.9) and (III.2.7) hold. Then $B_m$ is a Miyadera perturbation of $Z_m^+$, and hence $(K_m^+, D(Z_m^+)) := (Z_m^+ + B_m, D(Z_m^+))$ generates a positive semigroup, $(S_{K_m^+}(t))_{t\geq 0}$ in $X_m$.*

*Proof.* We shall use Theorem II.4.2 to establish that the operator $(Z_m^+ + B_m, D(Z_m^+))$ generates a positive semigroup in $X_m$. The theorem requires that $(Z_m^+, D(Z_m^+))$ generates a positive semigroup, $B_m$ is a positive and $Z_m^+$- bounded operator that satisfies the condition

$$\| B_m R(\lambda, Z_m^+) \|_m < 1. \tag{III.2.17}$$

The operator $B_m$ is an integral operator with a positive kernel and therefore is positive. Further,

$$\begin{aligned} \| B_m R(\lambda, Z_m^+)g \|_m &= \int_0^\infty (1+x^m) \int_x^\infty a(y)b(x,y) \left[R(\lambda, Z_m^+)g\right](y)dydx \\ &= \int_0^\infty a(y) \left[R(\lambda, Z_m^+)g\right](y) \int_0^y (1+x^m)b(x,y)dxdy \\ &= \int_0^\infty a(y) \left[R(\lambda, Z_m^+)g\right](y)(n_0(y) + n_m(y))dy, \end{aligned} \tag{III.2.18}$$

where we used the Fubini-Tonelli theorem to change the order of integration and formulae (III.1.10) and (III.1.13) to write

$$\int_0^y (1+x^m)b(x,y)dx = n_0(y) + n_m(y). \tag{III.2.19}$$

Now, since the fragmentation rate is unbounded, we split the last integral in (III.2.18) into two,

$$\int_0^s a(y)\big[R(\lambda, Z_m^+)g\big](y)(n_0(y)+n_m(y))dy \\ + \int_s^\infty a(y)\left[R(\lambda, Z_m^+)g\right](y)(n_0(y)+n_m(y))dy = I_1 + I_2, \tag{III.2.20}$$

for some $s > y_m$, where $y_m$ is defined in (III.1.15). Consider the first integral. We have

$$\begin{aligned} I_1 &= \int_0^s a(y)\left[R(\lambda, Z_m^+)g\right](y)(n_0(y)+n_m(y))dy \\ &\le a_s \int_0^s \left[R(\lambda, Z_m^+)g\right](y)(b_0(1+y^l)+y^m)dy \\ &\le a_s D_{m,l} \int_0^s \left[R(\lambda, Z_m^+)g\right](y)(1+y^m)dy \\ &\le \frac{a_s D_{m,l}}{\lambda} \parallel g \parallel_m, \end{aligned} \tag{III.2.21}$$

where $a_s = \operatorname{ess\,sup}_{y\in(0,s)} a(y)$ and

$$D_{m,l} = \sup_{y\in(0,s)} b_0 \frac{1+y^l}{1+y^m} + \frac{y^m}{1+y^m} \le 2b_0 + 1, \tag{III.2.22}$$

on the account of (III.1.13), (III.1.15) and Proposition II.5.4. Since $l < m$,

$$b_0 \frac{1+y^l}{1+y^m} \to 0, \quad y \to \infty. \tag{III.2.23}$$

Therefore, there is $s$ such that $b_0 \frac{1+y^l}{1+y^m} < 1 - c_m$, for $0 < y_m < s < y$. Then it follows that

$$b_0 \frac{1+y^l}{1+y^m} + c_m < 1, \quad \text{for } y > s. \tag{III.2.24}$$

Hence, if we let $0 < \beta = \operatorname{ess\,sup}_{y>s} b_0 \frac{1+y^l}{1+y^m}$, then $\beta + c_m < 1$. As a result, the second integral in (III.2.20) can be estimated as

$$\begin{aligned} I_2 &\le (\beta + c_m) \int_s^\infty a(y)\left[R(\lambda, Z_m^+)g\right](y)(1+y^m)dy \\ &\le (\beta + c_m) \parallel g \parallel_m, \end{aligned} \tag{III.2.25}$$

where we used (III.2.11). Hence,

$$\| B_m R(\lambda, Z_m^+) g \|_m \leq \left( \frac{a_s D_{m,l}}{\lambda} + (\beta + c_m) \right) \| g \|_m . \tag{III.2.26}$$

Since $\frac{a_s D_{m,l}}{\lambda} \to 0$ as $\lambda \to \infty$ and $(\beta + c_m) < 1$, there exist $\lambda_0$ such that for $\lambda > \lambda_0$

$$\left( \frac{a_s D_{m,l}}{\lambda} + (\beta + c_m) \right) < 1. \tag{III.2.27}$$

Hence, $B_m$ is a Miyadera perturbation of $Z_m^+$. By Theorem II.4.2, we conclude that the sum $(Z_m^+ + B_m, D(Z_m^+)) =: (K_m^+, D(Z_m^+))$ generates a positive semigroup in $X_m$, say $(S_{K_m^+}(t))_{t \geq 0}$. □

# 3 The growth-fragmentation equation

In this section, we study the fragmentation equation with growth. As in the previous section, we express (III.1.1) as an abstract Cauchy problem

$$\partial_t u(x,t) = T^- u(x,t) + F u(x,t), \qquad (x,t) \in \mathbb{R}_+^2, \tag{III.3.1}$$

posed in the Banach space $X_m, m \geq 1$, where $T^-$ and $F$ are defined in (III.1.4). It is worth to mention that in the growth settings there are two possible cases, that depend on the behaviour of the function $r(x)$ near $x = 0$, that is,

$$\int_{0^+} \frac{ds}{r(s)} = \infty \tag{III.3.2}$$

and

$$\int_{0^+} \frac{ds}{r(s)} < \infty, \tag{III.3.3}$$

where $\int_{0^+}$ is the integral in a positive neighbourhood of $0$. When (III.3.2) holds, the characteristics do not reach $x = 0$, hence there is no need for boundary conditions, while in the case (III.3.3) the characteristics reach $x = 0$ and boundary conditions are necessary. It should be mentioned that $R(x)$ and $Q(x)$ are defined by integrals with $\zeta = 0$.

In this section, we focus on the second scenario. That is, we assume that the growth is governed by

$$0 < r(x) \leq b_0 + b_1 x \leq r_0(1+x), \quad b_0, b_1 > 0, \quad r_0 = \max\{b_0, b_1\} \tag{III.3.4}$$

and consider the case (III.3.3) coupled with the McKendrick-von Foerster renewal boundary condition

$$\lim_{x \to 0^+} r(x)u(x,t) = \int_0^\infty \beta(y)u(y,t)dt. \tag{III.3.5}$$

Further, we assume that function $\beta$ satisfies

$$0 < \beta \in X_m^*, \tag{III.3.6}$$

where $X_m^*$ is the dual of $X_m$ equipped with the norm

$$\beta_\infty := \| g \|_m^* = \text{ess} \sup_{x \in \mathbb{R}_+} \frac{g(x)}{1+x^m} \tag{III.3.7}$$

and the duality pairing is given by

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)dx, \quad f \in X_m, \quad g \in X_m^*. \tag{III.3.8}$$

It should be noted that for $\beta = 0$ we get the homogeneous boundary condition which was considered in [16]. We observe that if $\beta > 0$, then the boundary condition in (III.3.5) depends on the solution on the entire interval $\mathbb{R}_+$.

## 3.1 The transport semigroup

As in Section III.2, we focus on (III.3.1) with $B = 0$ and $C = 0$,

$$\partial_t u(x,t) = T^- u(x) + Au(x) = -\partial_x(r(x)u(x)) - a(x)u(x) := Z^- u(x), \tag{III.3.9}$$

with the initial condition $u(x,0) = u_0(x)$ and the boundary condition (III.3.5). To begin, we note that the maximal realisation $Z_m^-$ of $Z^-$ in $X_m$ is defined as $Z_m^- := Z^-|_{D(Z_m^-)}$, where

$$D(Z_m^-) := \{u \in X_m : \partial_x(ru), au \in X_m\}. \tag{III.3.10}$$

To incorporate the boundary condition (III.3.5), we restrict $Z_m^-$ to

$$D(Z_{\beta,m}^-) := \left\{u \in D(Z_m^-) : \lim_{x\to 0^+} r(x)u(x) = \int_0^\infty \beta(y)u(y)dy\right\}. \tag{III.3.11}$$

In particular, when $\beta = 0$, the domain of $Z_{0,m}^-$ is given by

$$D(Z_{0,m}^-) := \{u \in D(Z_m^-) : \lim_{x\to 0^+} r(x)u(x) = 0\}, \tag{III.3.12}$$

and corresponds to homogenous Dirichlet boundary conditions considered in [18].

The resolvent equation for $(Z_{\beta,m}^-, D(Z_{\beta,m}^-))$ is given by

$$\lambda u(x) + \partial_x(r(x)u(x)) + a(x)u(x) = f(x), \tag{III.3.13}$$

with the side condition

$$\lim_{x\to 0^+} r(x)u(x) = \int_0^\infty \beta(x)u(x)dx. \tag{III.3.14}$$

To solve (III.3.13), we rewrite it as

$$\partial_x(r(x)u(x)) + \frac{1}{r(x)}[\lambda + a(x)]r(x)u(x) = f(x).$$

Then, setting $v(x) = r(x)u(x)$ and using the integrating factor $e^{\lambda R(x)+Q(x)}$, from the last formula we obtain

$$\partial_x\left(v(x)e^{\lambda R(x)+Q(x)}\right) = f(x)e^{\lambda R(x)+Q(x)}$$

so that

$$v(x) = e^{-\lambda R(x) - Q(x)} \int_0^x f(y) e^{\lambda R(y) + Q(y)} dy + C e^{-\lambda R(x) - Q(x)}.$$

Hence, the general solution of the resolvent equation is given by

$$u(x) = \frac{e^{-\lambda R(x) - Q(x)}}{r(x)} \int_0^x f(y) e^{\lambda R(y) + Q(y)} dy + C \frac{e^{-\lambda R(x) - Q(x)}}{r(x)}, \tag{III.3.15}$$

where $C$ is an arbitrary constant. It is known from [16] that for $\beta = 0$ the resolvent $R(\lambda, Z_{0,m}^{-})$ reads

$$\left[R(\lambda, Z_{0,m}^{-}) f\right](x) = \frac{e^{-\lambda R(x) - Q(x)}}{r(x)} \int_0^x f(y) e^{\lambda R(y) + Q(y)} dy \tag{III.3.16}$$

and satisfies

$$\| R(\lambda, Z_{0,m}^{-}) \|_m \leq \frac{1}{\lambda - w_{r,m}}, \quad w_{r,m} := 2 m r_0. \tag{III.3.17}$$

The analysis of the general case $\beta \neq 0$, relies on the following result:

**Lemma III.3.1.** *[16] Let $m \geq 1$ be fixed and $\lambda > w_{r,m} = 2mr_0$. Then,*

*(a) for any $0 < a < b \leq \infty$,*

$$P_{m,1}(a,b) = \int_a^b \frac{e^{-\lambda R(s)}}{r(s)} (1 + s^m) ds \leq \frac{1}{\lambda - w_{r,m}} e^{-\lambda R(a)} (1 + a^m), \tag{III.3.18}$$

*(b) for any $0 < a < b \leq \infty$,*

$$P_{m,2}(a,b) = \int_a^b \frac{(\lambda + a(s)) e^{-\lambda R(s) - Q(s)}}{r(s)} (1 + s^m) ds \leq \frac{\lambda}{\lambda - w_{r,m}} e^{-\lambda R(a) - Q(a)} (1 + a^m). \tag{III.3.19}$$

For the following, we set

$$d_\lambda(x) := e^{-\lambda R(x) - Q(x)}. \tag{III.3.20}$$

**Lemma III.3.2.** *Let (III.1.2),(III.1.7) and (III.3.4) be satisfied. Then, for any $m \geq 1$ and*

$\lambda > w_{r,m} + \beta_\infty$, $R(\lambda)$ *defined by*

$$\Big[R(\lambda)f\Big](x) = R(\lambda, Z^-_{0,m})f(x) + \frac{d_\lambda(x)}{r(x)} \frac{\langle \beta, R(\lambda, Z^-_{0,m})f\rangle}{1 - \langle \beta, \frac{d_\lambda}{r}\rangle}, \qquad u \in X_m, \tag{III.3.21}$$

*is the resolvent* $R(\lambda, Z^-_{\beta,m})$ *of* $(Z^-_{\beta,m}, D(Z^-_{\beta,m}))$ *that satisfies the estimate*

$$\| R(\lambda, Z^-_{\beta,m}) \|_m \leq \frac{1}{\lambda - w_{r,m} - \beta_\infty}. \tag{III.3.22}$$

*Proof.* Let $f \in X_m$ and $u = R(\lambda)f$. The resolvent, if it exists, must be of the form (III.3.15), hence we need to determine the constant $C$. Multiplying (III.3.15) by $r(x)$, we have initially

$$r(x)u(x) = r(x)R(\lambda, Z^-_{0,m})f(x) + Cd_\lambda(x), \tag{III.3.23}$$

and then, taking the limit as $x \to 0^+$,

$$\lim_{x\to 0^+} r(x)u(x) = \int_0^\infty \beta(x)u(x)dx = \lim_{x\to 0^+} r(x)R(\lambda, Z^-_{0,m})f(x) + C \lim_{x\to 0^+} d_\lambda(x). \tag{III.3.24}$$

Considering the first term on the right hand side, we have

$$\lim_{x\to 0^+} r(x)R(\lambda, Z^-_{0,m})f(x) = 0, \tag{III.3.25}$$

because the integral defining $R(\lambda, Z^-_{0,m})f(x)$ tends to zero as $x \to 0^+$. Recall (III.1.7) and (III.1.8) and notice that $e^{-\lambda R(x)}$ and $e^{-Q(x)}$ are positive and decreasing with $R(0) = Q(0) = 0$. Using (III.3.3) leads to

$$\lim_{x\to 0^+} d_\lambda(x) = 1. \tag{III.3.26}$$

Hence, $C = \int_0^\infty \beta(x)u(x)dx$ and

$$u(x) = R(\lambda, Z^-_{0,m})f(x) + \frac{d_\lambda(x)}{r(x)}\langle \beta, u\rangle, \tag{III.3.27}$$

where $\langle \beta, u\rangle$ is defined in (III.3.8). We multiply by $\beta(x)$ both sides of (III.3.27), integrate from

$0$ to $\infty$ and then solve for $\langle \beta, u \rangle$, to obtain

$$\langle \beta, u \rangle = \frac{\langle \beta, R(\lambda, Z^{-}_{0,m}) f \rangle}{1 - \langle \beta, \frac{d_\lambda}{r} \rangle}, \tag{III.3.28}$$

provided $1 \neq \langle \beta, \dfrac{d_\lambda}{r} \rangle$. We have

$$\begin{aligned} \langle \beta, \tfrac{d_\lambda}{r} \rangle &= \int_0^\infty \frac{\beta(x) e^{-\lambda R(x) - Q(x)}}{r(x)} dx \\ &\leq \int_0^\infty \frac{\beta(x)(1+x^m) e^{-\lambda R(x) - Q(x)}}{(1+x^m) r(x)} dx \quad \leq \beta_\infty \int_0^\infty \frac{(1+x^m) e^{-\lambda R(x)}}{r(x)} dx \leq \frac{\beta_\infty}{\lambda - w_{r,m}}, \end{aligned} \tag{III.3.29}$$

where we used the assumption on function $\beta$ and Lemma III.3.1. The right hand side of (III.3.29) approaches 0 as $\lambda \to \infty$. In particular, $\frac{\beta_\infty}{\lambda - w_{r,m}} < 1$, provided $\lambda > w_{r,m} + \beta_\infty$. Hence, for $\lambda > w_{r,m} + \beta_\infty$,

$$u(x) = R(\lambda, Z^{-}_{0,m}) f(x) + \frac{d_\lambda(x)}{r(x)} \frac{\langle \beta, R(\lambda, Z^{-}_{0,m}) f \rangle}{1 - \langle \beta, \frac{d_\lambda}{r} \rangle}. \tag{III.3.30}$$

For $f \in X_m$, taking the norm of (III.3.27), we obtain

$$\begin{aligned} \| u \|_m &= \int_0^\infty (1+x^m) u(x) dx \\ &= \int_0^\infty (1+x^m) R(\lambda, Z^{-}_{0,m}) f(x) dx + \int_0^\infty (1+x^m) \frac{d_\lambda(x)}{r(x)} \langle \beta, u \rangle dx \\ &= A_1 + A_2. \end{aligned} \tag{III.3.31}$$

On the account of (III.3.17), $A_1 \leq \frac{1}{\lambda - w_{r,m}}$, while for $A_2$, we have

$$A_2 = \frac{\langle \beta, R(\lambda, Z^{-}_{0,m}) f \rangle}{1 - \langle \beta, \frac{d_\lambda}{r} \rangle} \int_0^\infty \frac{d_\lambda(x)}{r(x)} (1+x^m) dx. \tag{III.3.32}$$

We consider the integral first

$$\int_0^\infty \frac{d_\lambda(x)}{r(x)} (1+x^m) dx \leq \frac{1}{\lambda - w_{r,m}}, \tag{III.3.33}$$

by Lemma III.3.1. The constant coefficient satisfies

$$\begin{aligned}\langle \beta, R(\lambda, Z_{0,m}^{-})f\rangle &= \int_0^\infty \beta(x)\frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}f(y)dydx \\ &= \int_0^\infty (1+x^m)\frac{\beta(x)}{1+x^m}\frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}f(y)dydx \\ &\leq \beta_\infty \int_0^\infty f(y)e^{\lambda R(y)+Q(y)}\int_y^\infty (1+x^m)\frac{e^{-\lambda R(x)-Q(x)}}{r(x)}dxdy \\ &\leq \frac{\beta_\infty \parallel f \parallel_m}{\lambda - w_{r,m}},\end{aligned} \tag{III.3.34}$$

where we used the assumption on function $\beta$, changed the order of integration and applied Lemma III.3.1. For the remaining term, we have

$$\frac{1}{1-\langle \beta, \frac{d_\lambda}{r}\rangle} \leq \frac{1}{1-\frac{\beta_\infty}{\lambda - w_{r,m}}} = \frac{\lambda - w_{r,m}}{\lambda - w_{r,m} - \beta_\infty}, \tag{III.3.35}$$

by (III.3.29). Hence,

$$A_2 \leq \parallel f \parallel_m \frac{\beta_\infty}{(\lambda - w_{r,m} - \beta_\infty)(\lambda - w_{r,m})} \tag{III.3.36}$$

and

$$\parallel u \parallel_m \leq \frac{\parallel f \parallel_m}{\lambda - w_{r,m} - \beta_\infty}. \tag{III.3.37}$$

Since the resolvent candidate is a sum of the resolvent $R(\lambda, Z_{0,m}^{-})$ and the term matching the boundary condition, the direct substitution yields

$$\lambda[R(\lambda)f](x) + \partial_x(r(x)[R(\lambda)f](x)) + a(x)[R(\lambda)f] = f(x).$$

Hence, $R(\lambda)$ is the right inverse of $\lambda I - Z_{\beta,m}^{-}$.

It remains to show that $R(\lambda)$ is the left inverse of $\lambda I - Z_{\beta,m}^{-}$. Consider $R(\lambda)(\lambda I - Z_{\beta,m}^{-})g$, with

$g \in D(Z^{-}_{\beta,m})$. We have

$$\Big[R(\lambda)(\lambda I - Z^{-}_{\beta,m})g\Big](x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}(\lambda I - Z^{-}_{\beta,m})g(y)dy$$
$$+ \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \frac{\int_0^\infty \frac{\beta(x)}{r(x)} e^{-\lambda R(x)-Q(x)} \int_0^x e^{\lambda R(y)+Q(y)}(\lambda I - Z^{-}_{\beta,m})g(y)dydx}{1-\int_0^\infty \frac{\beta(x)}{r(x)} e^{-\lambda R(x)-Q(x)}dx}$$
$$= C_1 + C_2. \tag{III.3.38}$$

Considering $C_1$ separately, we obtain

$$\begin{aligned} C_1 &= \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}\Big(\lambda g(y) + \frac{d}{dy}[r(y)g(y)] + a(y)g(y)\Big)dy \\ &= \frac{\lambda e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x g(y)e^{\lambda R(y)+Q(y)}dy + \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}a(y)g(y)dy \\ &+ \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}\frac{d}{dy}[r(y)g(y)]dy. \end{aligned} \tag{III.3.39}$$

Considering the integral in the last term and integrating by parts, we have

$$\begin{aligned} \int_0^x e^{\lambda R(y)+Q(y)}\frac{d}{dy}[r(y)g(y)]dy &= e^{\lambda R(y)+Q(y)}r(y)g(y)\Big|_0^x - \int_0^x e^{\lambda R(y)+Q(y)}r(y)g(y)\frac{(\lambda + a(y))}{r(y)}dy \\ &= e^{\lambda R(x)+Q(x)}r(x)g(x) - \int_0^\infty \beta(x)g(x)dx \\ &- \int_0^x e^{\lambda R(y)+Q(y)}g(y)(\lambda + a(y))dy. \end{aligned} \tag{III.3.40}$$

Finally, substituting (III.3.40) into (III.3.39), we obtain

$$C_1 = g(x) - \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^\infty \beta(x)g(x)dx. \tag{III.3.41}$$

In a similar way, we calculate $C_2$ to find

$$\begin{aligned}C_2 &= \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\left[\frac{\int_0^\infty \beta(x)g(x)dx - \int_0^\infty \frac{\beta(y)}{r(y)}e^{-\lambda R(y)-Q(y)}\int_0^\infty \beta(x)g(x)dxdy}{1-\int_0^\infty \frac{\beta(x)}{r(x)}e^{-\lambda R(x)-Q(x)}dx}\right]\\ &= \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\left[\frac{\int_0^\infty \beta(x)g(x)dx\big(1-\int_0^\infty \frac{\beta(y)}{r(y)}e^{-\lambda R(y)-Q(y)}dy\big)}{1-\int_0^\infty \frac{\beta(x)}{r(x)}e^{-\lambda R(x)-Q(x)}dx}\right] \\ &= \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^\infty \beta(x)g(x)dx.\end{aligned} \tag{III.3.42}$$

Combining (III.3.41) and (III.3.42) then simplifying, we get $R(\lambda)(\lambda I - Z^-_{\beta,m})g(x) = g(x)$. Hence, $R(\lambda, Z^-_{\beta,m}) = R(\lambda)$ is the resolvent. □

**Theorem III.3.3.** *The operator $(Z^-_{\beta,m}, D(Z^-_{\beta,m}))$ generates a strongly continuous positive semigroup, say $\left(G_{Z^-_{\beta,m}}(t)\right)_{t\geq 0}$, on $X_m$ for any given $m \geq 1$.*

*Proof.* The proof follows from the positivity of the resolvent, Lemma III.3.2 and Theorem II.2.6.

□

## 3.2 Growth fragmentation semigroup

The existence of a strongly continuous positive transport semigroup $(G_{Z^-_{\beta,m}}(t))_{t\geq 0}$ for the associated problem (III.3.9) has been established. Next, we consider the complete equation (III.3.1), that is

$$u_t = T^- u(x) + Fu(x) = Z^-_{\beta,m}u(x) + B_m u(x), \quad t \geq 0, \tag{III.3.43}$$

where $Z^-_{\beta,m}u(x) = T^- u(x) - Au(x)$.

**Theorem III.3.4.** *Let $m > \max\{1, l\}$ and (III.1.2), (III.1.9) and (III.3.21) hold. Then $B_m$ is a Miyadera perturbation of $Z^-_{\beta,m}$, and hence $(K^-_{\beta,m}, D(Z^-_{\beta,m})) := (Z^-_{\beta,m} + B_m, D(Z^-_{\beta,m}))$ generates a positive semigroup, $(S_{K^-_{\beta,m}}(t))_{t\geq 0}$ in $X_m$.*

*Proof.* Theorem II.4.2 will be used to establish that the operator $(Z^-_{\beta,m} + B_m, D(Z^-_{\beta,m}))$ generates a semigroup. The theorem requires that $(Z^-_{\beta,m}, D(Z^-_{\beta,m}))$ generates a positive semigroup, $B_m$ is a positive operator that satisfies the condition

$$\| B_m R(\lambda, Z^-_{\beta,m}) \|_m < 1. \tag{III.3.44}$$

Note that $B_m$ is an integral operator with positive kernel and hence is positive. We further consider,

$$\begin{aligned}
\| B_m R(\lambda, Z^-_{\beta,m}) g \|_m &= \int_0^\infty (1+x^m) \int_x^\infty a(y) b(x,y) \left[R(\lambda, Z^-_{\beta,m})g\right](y) dy dx \\
&= \int_0^\infty \left[R(\lambda, Z^-_{\beta,m})g\right](y) \int_0^y (1+x^m) b(x,y) dx dy \\
&= \int_0^\infty a(y) \left[R(\lambda, Z^-_{\beta,m})g\right](y)(n_0(y)+n_m(y)) dy,
\end{aligned} \tag{III.3.45}$$

where to change the order of integration we used the Fubini-Tonelli theorem, and also (III.1.10), (III.1.13) to write

$$\int_0^y (1+x^m) b(x,y) dx = n_0(y) + n_m(y). \tag{III.3.46}$$

As in the proof of Theorem III.2.3, we split the last integral in (III.3.45) into two

$$\begin{aligned}
&\int_0^s a(y) \left[R(\lambda, Z^-_{\beta,m})g\right](y)(n_0(y)+n_m(y)) dy \\
&\qquad + \int_s^\infty a(y) \left[R(\lambda, Z^-_{\beta,m})g\right](y)(n_0(y)+n_m(y)) dy = I_1 + I_2,
\end{aligned} \tag{III.3.47}$$

for some $s > y_m$, where $y_m$ is defined in (III.1.15) .

Consider the first integral $I_1$. We know that for a finite $s$, $a(y)$ is essentially bounded on $[0,s]$.

As a result,

$$\begin{aligned}
I_1 &= \int_0^s a(y)\left[R(\lambda, Z^-_{\beta,m})g\right](y)(n_0(y)+n_m(y))dy \\
&\le a_s \int_0^s \left[R(\lambda, Z^-_{\beta,m})g\right](y)(b_0(1+y^l)+y^m)dy \\
&\le a_s D_{m,l} \int_0^s \left[R(\lambda, Z^-_{\beta,m})g\right](y)(1+y^m)dy \\
&\le a_s D_{m,l}\Bigg( \int_0^s \frac{(1+y^m)e^{-\lambda R(y)-Q(y)}}{r(y)} \int_0^y e^{\lambda R(x)+Q(x)} g(x)dxdy \\
&+ \frac{\langle \beta, R(\lambda, Z^-_{0,m})g\rangle}{1-\langle \beta, \frac{d_\lambda}{r}\rangle} \int_0^s \frac{(1+y^m)}{r(y)} d_\lambda(y)dy \Bigg) \\
&= a_s D_{m,l}\left(A_0 + A_1\right),
\end{aligned} \tag{III.3.48}$$

where we used (III.1.13) and (III.1.15), $a_s = \operatorname{ess\,sup}_{y\in(0,s)} a(y)$ and

$$D_{m,l} = \sup_{y\in(0,s)} b_0 \frac{1+y^l}{1+y^m} + \frac{y^m}{1+y^m} \le 2b_0 + 1. \tag{III.3.49}$$

The estimates of $A_0$ and $A_1$ are identical to the resolvent $R(\lambda, Z^-_{\beta,m})$ estimates (III.3.17) and (III.3.36) for $s \to \infty$ and hence we have

$$A_0 \le \frac{\| g \|_m}{\lambda - w_{r,m}} \tag{III.3.50}$$

and

$$A_1 = \frac{\langle \beta, u_\lambda\rangle}{1-\langle \beta, d_\lambda r^{-1}\rangle} \int_0^s \frac{(1+y^m)}{r(y)} d_\lambda(y)dy \le \frac{\beta_\infty \| g \|_m}{(\lambda - w_{r,m})(\lambda - w_{r,m} - \beta_\infty)}. \tag{III.3.51}$$

Combining the last two bounds, we arrive at

$$I_1 \le a_s D_{m,l}\left(A_0 + A_1\right) \le \frac{a_s D_{m,l} \| g \|_m}{\lambda - w_{r,m} - \beta_\infty}. \tag{III.3.52}$$

Since $l < m$,

$$b_0 \frac{1+y^l}{1+y^m} \to 0, \quad y \to \infty, \tag{III.3.53}$$

there is $s$ such that $b_0 \frac{1+y^l}{1+y^m} < 1 - c_m$ for $0 < y_m < s < y$. Then it follows that

$$b_0 \frac{1+y^l}{1+y^m} + c_m < 1, \quad \text{for } y > s. \tag{III.3.54}$$

Hence, we can choose $s$ so that $\alpha + c_m < 1$, for $\alpha = \operatorname{ess\,sup}_{y>s} b_0 \frac{1+y^l}{1+y^m}$.

For the second integral $I_2$ in (III.3.47), we have

$$\begin{aligned}
I_2 &\leq (\alpha + c_m) \int_s^\infty a(y) \left[R(\lambda, Z^-_{\beta,m})f\right](y)(1+y^m)dy) \\
&\leq (\alpha + c_m) \Bigg( \int_s^\infty a(y) \frac{(1+y^m)e^{-\lambda R(y) - Q(y)}}{r(y)} \int_0^y e^{\lambda R(x) + Q(x)} f(x) dx dy \\
&+ \frac{\langle \beta, R(\lambda, Z^-_{0,m})g \rangle}{1 - \langle \beta, \frac{d_\lambda}{r} \rangle} \int_s^\infty a(y) \frac{(1+y^m)}{r(y)} d_\lambda(y) dy \Bigg) \\
&= (\alpha + c_m)(B_0 + B_1).
\end{aligned} \tag{III.3.55}$$

Further,

$$\begin{aligned}
B_0 &= \int_0^\infty a(y) \frac{(1+y^m)e^{-\lambda R(y) - Q(y)}}{r(y)} \int_0^y e^{\lambda R(x) + Q(x)} f(x) dx dy \\
&= \int_0^\infty e^{\lambda R(x) + Q(x)} f(x) \int_x^\infty a(y) \frac{(1+y^m)e^{-\lambda R(y) - Q(y)}}{r(y)} dy dx \\
&\leq \int_0^\infty e^{\lambda R(x) + Q(x)} f(x) \int_x^\infty (a(y) + \lambda) \frac{(1+y^m)e^{-\lambda R(y) - Q(y)}}{r(y)} dy dx \\
&\leq \int_0^\infty e^{\lambda R(x) + Q(x)} f(x) \left( \frac{\lambda e^{-\lambda R(x) - Q(x)}(1+x^m)}{\lambda - w_{r,m}} \right) dx \\
&\leq \frac{\lambda \parallel f \parallel_m}{\lambda - w_{r,m}},
\end{aligned} \tag{III.3.56}$$

where we used Lemma III.3.1 on the third line. Considering the last estimate, we use Lemma III.3.1 and (III.3.34) to obtain

$$B_1 \leq \frac{\parallel f \parallel_m \lambda \beta_\infty}{(\lambda - w_{r,m} - \beta_\infty)(\lambda - w_{r,m})}. \tag{III.3.57}$$

Finally,

$$I_2 \leq \frac{\lambda(\alpha + c_m) \parallel f \parallel_m}{\lambda - w_{r,m} - \beta_\infty} \tag{III.3.58}$$

and

$$\| B_m R(\lambda, Z^-_{\beta,m}) f \|_m \leq \left( \frac{a_s D_{m,l}}{\lambda - w_{r,m} - \beta_\infty} + \frac{\lambda(\alpha + c_m)}{\lambda - w_{r,m} - \beta_\infty} \right) \| f \|_m . \qquad \text{(III.3.59)}$$

Since $\frac{a_s D_{m,l}}{\lambda - w_{r,m} - \beta_\infty} \to 0$ and $\frac{\lambda(\alpha+c_m)}{\lambda - w_{r,m} - \beta_\infty} \to \alpha + c_m$ as $\lambda \to \infty$, with $\alpha + c_m < 1$, there exist $\lambda_0$ such that for $\lambda > \lambda_0$

$$\left( \frac{a_s D_{m,l}}{\lambda - w_{r,m} - \beta_\infty} + \frac{\lambda(\alpha + c_m)}{\lambda - w_{r,m} - \beta_\infty} \right) < 1. \qquad \text{(III.3.60)}$$

Hence $B_m$ is a Miyadera perturbation of $Z^-_{\beta,m}$. We conclude that the sum $(Z^-_{\beta,m} + B_m, D(Z^-_{\beta,m})) =: (K^-_{\beta,m}, D(Z^-_{\beta,m}))$ generates a positive semigroup in $X_m$, say $(S_{K^-_{\beta,m}}(t))_{t\geq 0}$ by Theorem II.4.2. □

# 4 Smoothing properties of the transport-fragmentation semigroups

In this section, we study moments of solutions to the transport-fragmentation equation (III.2.16) and (III.3.43). In particular, we have three cases to consider, that is, decay with no boundary conditions and growth with the homogenous Dirichlet and the McKendrick-von Foerster boundary conditions. In the first scenario the semigroup solutions are generated by $(T^+ + F, D(Z^+_m))$, while in the second and third cases the solutions are generated by $(T^+ + F, D(Z^+_{0,m}))$ and $(T^+ + F, D(Z^+_{\beta,m}))$, respectively. As we shall see shortly, the proof of the moment regularisation property is similar in all three scenarios. To avoid repetition, we introduce the following uniform notation, i.e., we let

$$(K^\pm_m, D(Z^\pm_m)) := \begin{cases} (K^\pm_m, D(Z^+_m)) & \text{for decay with no boundary condition,} \\ (K^\pm_m, D(Z^-_{0,m})) & \text{for growth with homogenous boundary conditions,} \\ (K^\pm_m, D(Z^-_{\beta,m})) & \text{for growth with McKendrick-von Foerster boundary conditions.} \end{cases}$$

Further, we denote the associated semigroups by

$$S_{K_m^{\pm}}(t) = \begin{cases} S_{K_m^{+}}(t) & \text{semigroup for the decay case,} \\ S_{K_{0,m}^{-}}(t) & \text{for growth with homogenous boundary conditions,} \\ S_{K_{\beta,m}^{-}}(t) & \text{for growth with McKendrick-von Foerster boundary conditions.} \end{cases}$$

Using this notation, for nonnegative initial data $u_0 \in D(Z_m^{\pm})$ and $u(t) = S_{K_m^{\pm}}(t)u_0$, we have

$$\frac{d \parallel u(t) \parallel_m}{dt} = \int_0^{\infty} (T^{\pm} + F)u(x,t)(1 + x^m)dx. \tag{III.4.1}$$

The proof of the smoothing property relies on accurate evaluation of the right-hand side of (III.4.1). The following technical result simplifies our calculations.

**Lemma III.4.1.** *Under the adopted assumptions of Sections III.2 and III.3, for $u \in D(Z_m^{\pm})$, we have*

$$\begin{aligned} \int_0^{\infty} (T^{\pm} + F)u(x)w_m(x)dx = \mathcal{K}^{\pm} \mp m \int_0^{\infty} r(x)u(x)x^{m-1}dx \\ - \int_0^{\infty} a(x)u(x)(N_0(x) + N_m(x))dx, \end{aligned} \tag{III.4.2}$$

*where $\mathcal{K}^- = \int_0^{\infty} \beta(x)u(x)dx$, $\mathcal{K}^+ = -\lim_{x\to 0} r(x)u(x)$ and $w_m(x) = 1 + x^m$.*

*Proof.* a) We follow the technique used in [18, Lemma 5.2.13 & 14]. We note that the left-hand side of (III.4.2) is linear, while $X_m$ is a Banach lattice. Consequently, any $u \in D(Z_m^{\pm})$ can be written as $u = u^+ - u^-$, where $u^{\pm} \in D(Z_m^{\pm})$, $u^{\pm} \in X_m^+$ and $u^{\pm} = R(\lambda, Z_m^{\pm})g^{\pm}$, with some $g^{\pm} \in X_m^+$. It turns out that it is sufficient to prove (III.4.2) for $u \in D(Z_m^{\pm})$ of the form $u = R(\lambda, Z_m^{\pm})g$, with $g \in X_m^+$.

b) For $u$ as above, $B_m u \in X_m^+$ and thus we can write

$$\begin{aligned}\int_0^\infty (T^\pm + B_m)u(x)w_m(x)dx &= \pm \int_0^\infty \partial_x(u(x)r(x))w_m(x)dx - \int_0^\infty a(x)u(x)w_m(x)dx \\ &\quad + \int_0^\infty B_m u(x)w_m(x)dx \\ &= \pm \int_0^\infty \partial_x(u(x)r(x))w_m(x)dx \\ &\quad - \int_0^\infty a(x)u(x)(N_0(x) + N_m(x))dx,\end{aligned} \tag{III.4.3}$$

where we used (III.1.10) and (III.1.11). By the definition of the domain $D(Z_m^\pm)$, both integrals appearing on the right-hand side of (III.4.3) exist. We note that the second integral cannot be simplified any further and it remains to compute the first integral only. The calculations depend on assumptions on the coefficient of $r(x)$, which are different in the decay and the growth scenarios. For that reason, we do the calculations separately.

c) First, we consider the decay case. For $0 < a_0 < a_1 < \infty$, using integration by parts, we obtain

$$\begin{aligned}\int_{a_0}^{a_1} (u(x)r(x))_x w_m(x)dx &= u(a_1)r(a_1)w_m(a_1) - u(a_0)r(a_0)w_m(a_0) \\ &\quad - m\int_{a_0}^{a_1} x^{m-1}u(x)r(x)dx.\end{aligned} \tag{III.4.4}$$

By the definition of $D(Z_m^+)$, the limit of the integral in the left-hand side of (III.4.4), as $a_0 \to 0$ and $a_1 \to \infty$, exists. We show that the limit of each term on the right-hand side of this formula also exist. Indeed, by our assumption $u = R(\lambda, Z_m^+)g$, with $g \in X_m^+$. Using this fact for the integral term on the right-hand side of (III.4.4) and allowing $a_0 \to 0$ and $a_1 \to \infty$ for nonnegative

$u(x)$, we have

$$
\begin{aligned}
m\int_0^\infty x^{m-1}u(x)r(x)dx &= m\int_0^\infty x^{m-1}e^{\lambda R(x)+Q(x)}\int_x^\infty g(y)e^{-\lambda R(y)-Q(y)}dydx \\
&= m\int_0^\infty e^{-\lambda R(y)-Q(y)}g(y)\int_0^y x^{m-1}e^{\lambda R(x)+Q(x)}dxdy \\
&= m\int_0^\infty e^{-\lambda R(y)-Q(y)}g(y)\int_0^y x^{m-1}e^{\lambda R(x)+Q(x)}dxdy \\
&= m\int_0^\infty g(y)(1+y^m)\frac{e^{-\lambda R(y)-Q(y)}}{1+y^m}\int_0^y x^{m-1}e^{\lambda R(x)+Q(x)}dxdy \\
&= \int_0^\infty g(y)(1+y^m)\frac{y^m}{1+y^m}dy \\
&\leq \|g\|_m,
\end{aligned}
\tag{III.4.5}
$$

where we used $\frac{y^m}{1+y^m} \leq 1$, for all $y \geq 0$, the Fubini-Tonelli theorem to change the order of integration and the monotonicity of $R(x)$ and $Q(x)$. Since the integral term in (III.4.5) is monotone in $a_0$ and $a_1$ and is uniformly bounded, it follows that

$$
\lim_{\substack{a_1\to\infty \\ a_0\to 0}}\int_{a_0}^{a_1}(u(x)r(x))_x w_m(x)dx = \int_0^\infty x^{m-1}u(x)r(x)(x)dx \tag{III.4.6}
$$

does exist and is finite.

Next, using the representation $u = R(\lambda, Z_m^+)g$, $g \in X_m^+$, and employing the ideas of [18, Lemma 5.2.14], on account of the monotonicity of $R(x)$, $Q(x)$ and $w_m(x)$, we obtain

$$
\begin{aligned}
r(a_1)u(a_1)w_m(a_1) &\leq w_m(a_1)e^{\lambda R(a_1)+Q(a_1)}\int_{a_1}^\infty e^{-\lambda R(x)-Q(x)}g(x)dx \\
&\leq w_m(a_1)e^{\lambda R(a_1)}\int_{a_1}^\infty \frac{e^{-\lambda R(x)}}{w_m(x)}w_m(x)g(x)dx \\
&\leq \int_{a_1}^\infty u(x)w_m(x)dx.
\end{aligned}
\tag{III.4.7}
$$

This leads to the conclusion that

$$
\lim_{a_1\to\infty} r(a_1)u(a_1)w_m(a_1) = 0. \tag{III.4.8}
$$

Finally, from formula (III.4.5), it follows that the limit $\lim_{a_0\to 0} r(a_0)u(a_0)$ does exist and is finite.

Combining all calculations, we arrive at (III.4.2) in the decay case.

d) Now we turn to the growth scenario. As in part c) above, it follows that the limit of the the left-hand side of (III.4.5) as $a_0 \to 0$ and $a_1 \to \infty$ does exist and is finite. In the growth settings, $r(x)$ satisfies (III.3.4). Therefore, direct calculations give

$$m\int_{a_0}^{a_1} x^{m-1}u(x)r(x)dx \leq mr_0\int_{a_0}^{a_1} x^{m-1}(1+x)u(x)dx \leq 2mr_0\int_{a_0}^{a_1} w_m(x)u(x)dx \\ \leq 2mr_0 \parallel u \parallel_m \leq \infty, \quad 0 < a_0 < a_1 < \infty. \tag{III.4.9}$$

The bound shows that $\lim_{\substack{a_1\to\infty \\ a_0\to 0}} \int_{a_0}^{a_1} x^{m-1}u(x)r(x)dx = \int_0^\infty x^{m-1}u(x)r(x)dx$ exist and is finite. Further, since $u \in D(Z_m^-)$, it follows that

$$\lim_{a_0\to 0} r(a_0)u(a_0)w_m(a_0) = \begin{cases} 0 & \text{for the homogeneous Dirichlet boundary conditions,} \\ \int_0^\infty \beta(x)u(x)dx & \text{for the McKendrick-von Foerster boundary conditions.} \end{cases}$$

The fact that formula (III.4.5) is bounded and the limit as $a_0 \to 0$ exists, indicate that $\lim_{a_1\to\infty} r(a_1)u(a_1)w_m(a_1)$ does exist and is finite.

To show that this limit is zero, we argue by contradiction. That is, we assume that $\lim_{a_1\to\infty} r(a_1)u(a_1)w_m(a_1) = L > 0$. If this is the case,

$$r(x)u(x)w_m(x) \geq C > 0, \tag{III.4.10}$$

for all sufficiently large values of $x$. In view of (III.3.4), for this values of $x$ we have

$$u(x)w_m(x) \geq \frac{C}{r(x)} \geq \frac{C'}{1+x}, \quad \text{for } C' > 0. \tag{III.4.11}$$

Since the latter is impossible as $u \in X_m$, we conclude that

$$\lim_{a_1\to\infty} r(a_1)u(a_1)w_m(a_1) = 0 \tag{III.4.12}$$

and (III.4.2) is completely settled. □

## 4.1 Regularisation property of the semigroups

It is at this point that we will show that both decay and growth semigroups have a moment regularisation property. The following theorem is a generalisation of a result in [22].

**Theorem III.4.2.** *Assume that there is $x_1 > 0$, such that for $x_1 \leq x$ we have*

$$a(x) \geq a_0 x^{\xi_0}, \quad \xi_0 > 0. \tag{III.4.13}$$

*Then, for any $l < n < p < m$, there are constants $\delta = \delta(m,n,p) > 0$ and $\rho = \rho(m,n) > 0$ such that for all $u_0 \in X_p$,*

$$\| S_{K_p^\pm}(t)u \|_m \leq \delta e^{\rho t} t^{\frac{n-m}{\xi_0}} \| u_0 \|_p . \tag{III.4.14}$$

*Proof.* Let $u_0 \in D(Z_m^\pm)_+$ and $u(\cdot) := [S_{K_m^\pm}(t)u_0](\cdot) = [S_{K_p^\pm}(t)u_0](\cdot) \in D(Z_m^\pm)$, for $t > 0$. Then, multiplying equation (III.1.1) without the coagulation by $w_m(x)$, integrating term by term in the interval $(0, \infty)$ and using Lemma III.4.1, we have

$$\begin{aligned} \frac{d\|u(t)\|_m}{dt} &= \mathcal{K}^\pm \mp m \int_0^\infty r(x)u(x)x^{m-1}dx - \int_0^\infty a(x)u(x)(N_0(x) + N_m(x))dx \\ &\leq \int_0^\infty \Big[ B^\pm(x) \mp m x^{m-1} r(x) - [N_0(x) + N_m(x)]a(x) \Big] u(x)dx \end{aligned} \tag{III.4.15}$$

where $\mathcal{K}^\pm$ are defined in Lemma III.4.1, and dropped $-\lim_{x\to 0} r(x)u(x)$ for the decay case so that $B^-(x) = \beta(x)$ and $B^+(x) = 0$. We choose $R_m > \max\{1, x_1, y_m\}$, where $y_m$ is defined in (III.1.15), so that

$$b_0(1 + x^l) - 1 - (1 - c_m)x^m \leq 0, \tag{III.4.16}$$

for $x > R_m$. Then using (III.1.12), (III.1.13), (III.1.15) and (III.4.13), for any $R_m \leq R \leq x$, we have

$$
\begin{aligned}
\Gamma^{\pm}(x) :&= B^{\pm}(x) \mp m x^{m-1} r(x) - (N_0(x) + N_m(x)) a(x) \\
&\leq \mathbb{B}^{\pm} w_m(x) + ((b_0(1 + x^l) - 1) - (1 - c_m) x^m) a_0 R^{\xi_0} \\
&= \mathbb{B}^{\pm} w_m(x) + (b_0(1 + x^l) - (1 - c_m) - c_m - (1 - c_m) x^m) a_0 R^{\xi_0} \\
&= \mathbb{B}^{\pm} w_m(x) - (1 - c_m) w_m(x) a_0 R^{\xi_0} + (b_0(1 + x^l) - c_m) a_0 R^{\xi_0} \\
&\leq (\mathbb{B}^{\pm} - (1 - c_m) a_0 R^{\xi_0}) w_m(x) + b_0(1 + x^l) a_0 R^{\xi_0},
\end{aligned}
\tag{III.4.17}
$$

where $\mathbb{B}^+ = 0$ and $\mathbb{B}^- = \beta_\infty + 2mr_0$. We choose $R_m$ so that

$$
\mathbb{B}^{\pm} - (1 - c_m) a_0 R^{\xi_0} \leq -\phi_m^{\pm} R^{\xi_0}, \tag{III.4.18}
$$

holds for all $R \geq R_m$ and some constant $\phi_m^{\pm} > 0$. We let $l \leq n \leq m$. Then for any $x \geq R \geq R_m$,

$$
\Gamma^{\pm}(x) \leq -\phi_m^{\pm} R^{\xi_0} w_m(x) + b_0 a_0 R^{\xi_0} w_n(x). \tag{III.4.19}
$$

For the case of $x \leq R$, we observe that (III.4.16) still holds for $R_m \leq x \leq R$, while $-N_m(x) \leq 0$ for any $x$. For any $R \geq R_m$ fixed, then there exists constants $P_m^{\pm}$ such that

$$
\begin{aligned}
\Gamma^{\pm}(x) &\leq \mathbb{B}^{\pm} w_m(x) + (b_0(1 + R_m^l) - 1) a_{R_m} \\
&\leq -\phi_m^{\pm} w_m(x) R^{\xi_0} + \left( \phi_m w_m(x) R^{\xi_0} + \mathbb{B}^{\pm} w_n(x) + (b_0(1 + R_m^l) - 1) a_{R_m} \right) \\
&\leq -\phi_m^{\pm} w_m(x) R^{\xi_0} + \left( \phi_m R^{\xi_0} \frac{w_m(x)}{w_n(x)} + \frac{(b_0(1 + R_m^l) - 1) a_{R_m}}{w_n(x)} \right) w_n(x) \\
&\leq -\phi_m^{\pm} w_m(x) R^{\xi_0} + P_m^{\pm} R^{\xi_0 + m - n} w_n(x),
\end{aligned}
\tag{III.4.20}
$$

where we applied (III.1.13) and the inequality,

$$
\frac{1 + x^m}{1 + x^n} = \frac{1}{1 + x^n} + \frac{x^m}{1 + x^n} \leq 1 + x^{m-n}, \tag{III.4.21}
$$

and $a_{R_m} = \operatorname{ess\,sup}_{x \in [0, R_m]} a(x) < \infty$. The inequalities obtained above are similar to the ones in the paper [16], with $2m\tilde{r}$ replaced by $\mathbb{B}^{\pm}$. Then for $R \geq R_m \geq 1$, we can find $D_m$ such that for all $x \in \mathbb{R}_+$ the following holds

$$\frac{d\|u(t)\|_m}{dt} \leq -\phi_m^{\pm} R^{\xi_0} \|u(t)\|_m + D_m R^{\xi_0+m-n} \|u(t)\|_n. \tag{III.4.22}$$

Recall that $X_m \subset X_n$ for $n \leq m$. Then

$$\|u(t)\|_n = \| S_{K_m^{\pm}}(t) u_0 \|_n = \| S_{K_n^{\pm}}(t) u_0 \|_n = F_n e^{\omega_n t} \| u_0 \|_n =: g_n(t) \| u_0 \|_n,$$

for some constants $F_n$, $\omega_n$, and $g_n(t)$ is exponentially bounded as $t \to \infty$. Then, from (III.4.22) we obtain

$$\frac{d\|u(t)\|_m}{dt} \leq -\phi_m^{\pm} R^{\xi_0} \|u(t)\|_m + D_m R^{\xi_0+m-n} g_n(t) \| u_0 \|_n, \tag{III.4.23}$$

which is equivalent to

$$\frac{d}{dt} \left( e^{\phi_m^{\pm} R^{\xi_0} t} \|u(t)\|_m \right) \leq D_m F_n R^{\xi_0+m-n} e^{(\phi_m^{\pm} R^{\xi_0} + \omega_n) t} \| u_0 \|_n . \tag{III.4.24}$$

Therefore, for any $R > R_m$,

$$\begin{aligned} \|u(t)\|_m &= \int_0^\infty (S_{K_m^{\pm}} u_0)(x) w_m(x) dx = \int_0^\infty u_0(x) \left( S^*_{K_m^{\pm}}(t) w_m(x) \right) dx = \int_0^\infty u_0(x) \Theta(x,t) dx \\ &\leq \| u_0 \|_m e^{-\phi_m^{\pm} R^{\xi_0} t} + \frac{F_n D_m R^{\xi_0}}{\phi_m^{\pm} R^{\xi_0} + \omega_n} R^{m-n} (g_n(t) - e^{-\phi_m^{\pm} R^{\xi_0} t}) \| u_0 \|_n \\ &\leq \| u_0 \|_m e^{-\phi_m^{\pm} R^{\xi_0} t} + H_m R^{m-n} g_n(t) \| u_0 \|_n \\ &= \int_0^\infty \left( e^{-\phi_m^{\pm} R^{\xi_0} t} w_m(x) + H_m R^{m-n} g_n(t) w_n(x) \right) u_0(x) dx. \end{aligned} \tag{III.4.25}$$

We note that the integrand in the brackets is continuous. The domain $D(Z_m^{\pm})_+$ contains the space of positive $C_0^\infty(\mathbb{R}_+)$ functions which are dense in $X_m^+$, hence, the functions can be bounded,

$$\Theta(x,t) \leq e^{-\phi_m^{\pm} R^{\xi_0} t} w_m(x) + H_m R^{m-n} g_n(t) w_n(x), \tag{III.4.26}$$

for $x > 0$ and $R \geq R_m$. Since $t$, $x$ and $R$ are independent, we can define $R$ using the two other variables. Considering $R^{\xi_0} = \frac{(m-n)\log x}{d_m t}$, we get for $x \geq e^{\frac{\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}$,

$$\begin{aligned} \Theta(x,t) &\leq x^{n-m} w_m(x) + H_m \left(\frac{m-n}{\phi_m^{\pm}}\right)^{\frac{m-n}{\xi_0}} t^{\frac{n-m}{\xi_0}} (\log x)^{\frac{m-n}{\xi_0}} g_n(t) w_n(x) \\ &\leq H'_{m,n,p} g_n(t) t^{\frac{n-m}{\xi_0}} w_p(x), \end{aligned} \tag{III.4.27}$$

where $H'_{m,n,p}$ is positive and independent of $x$ and $t$ for any $p > n$.

To establish this, we consider

$$f(x) = \log x - x^{\xi} \quad \text{for } x > 0 \text{ and } \xi > 0. \tag{III.4.28}$$

Letting $y = x^{\xi}$ leads to $\hat{f}(y) = \frac{1}{\xi}\log y - y$. We take the first derivative and equate it to zero to obtain the critical point $y = \frac{1}{\xi}$. The second derivative is $\hat{f}''(y) = -\frac{1}{\xi y^2}$, which shows that the critical point is a maximum. Therefore, $\hat{f}(y) \leq \hat{f}(\frac{1}{\xi})$ for $y > 0$. This leads to

$$\begin{aligned} &\tfrac{1}{\xi}\log y < y + \left[\tfrac{1}{\xi}\log(\tfrac{1}{\xi}) - \tfrac{1}{\xi}\right], \\ &\log y < c\xi + \xi y \\ &\log x \leq \max\{1, c_{\xi}\}(1 + x^{\xi}). \end{aligned} \tag{III.4.29}$$

From (III.4.27),

$$(\log x)^{\frac{m-n}{\xi_0}} \leq [\max\{1, c_{\xi}\}(1 + x^{\xi})]^{\frac{m-n}{\xi_0}} \leq c_1(1 + x^{\xi^*}), \quad \text{for any } \xi^* > 0.$$

Further,

$$(\log x)^{\frac{m-n}{\xi_0}} w_n(x) \leq c_1(1 + x^{\xi^*})(1 + x^n) \leq C(1 + x^{n+\xi^*}) = C(1 + x^p), \tag{III.4.30}$$

for some $C > 0$, and $p := n + \xi^*$, which leads to the last statement of (III.4.27). Now, for

$x \leq e^{\frac{\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}$, we use $R = R_m$ in (III.4.26) to get

$$\begin{aligned}
\Theta(x,t) &\leq e^{-\phi_m^{\pm} R_m^{\xi_0} t} w_m(x) + H_m g_n(t) R_m^{m-n} w_n(x) \\
&= e^{-\phi_m^{\pm} R_m^{\xi_0} t} w_m\left(e^{\frac{\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) + H_m g_n(t) R_m^{m-n} w_n\left(e^{\frac{\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) \\
&= e^{-\phi_m^{\pm} R_m^{\xi_0} t}\left(1 + e^{\frac{m\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) + H_m g_n(t) R_m^{m-n}\left(1 + e^{\frac{n\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) \qquad \text{(III.4.31)} \\
&\leq e^{-\phi_m^{\pm} R_m^{\xi_0} t}\left(1 + e^{\frac{m\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) + H_m g_n(t) R_m^{m-n}\left(1 + e^{\frac{m\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right) \\
&= \left(1 + e^{\frac{m\phi_m^{\pm} R_m^{\xi_0} t}{m-n}}\right)\left(1 + H_m g_n(t) R_m^{m-n}\right).
\end{aligned}$$

Finally, considering (III.4.27) and (III.4.31), there exist positive constants $\delta = \delta(m,n,p)$ and $\rho = \rho(m,n)$ such that

$$\Theta(x,t) \leq \delta e^{\rho t} t^{\frac{n-m}{\xi_0}} w_p(x) \qquad \text{(III.4.32)}$$

and by (III.4.25), we have

$$\| S_{K_p^{\pm}}(t) u_0 \|_m \leq \delta e^{\rho t} t^{\frac{n-m}{\xi_0}} \int_0^{\infty} u_0(x) w_p(x) dx. \qquad \text{(III.4.33)}$$

This concludes the proof. □

# 5 The complete transport-coagulation-fragmentation model

In this section, we consider the complete nonlinear model (III.1.3). We use the unified notation of Section III.4 for the semigroups and generators associated to the decay- and growth-fragmentation equations and treat both abstract scenarios simultaneously. Any differences, if they occur, will be stated explicitly.

## 5.1 Local solutions

To begin, we discuss the local solvability of (III.1.3). Our presentation follows closely [16]. As in [16], we assume

$$k(x,y) \le c_0(1+x^\eta)(1+y^\eta), \tag{III.5.1}$$

where $c_0 > 0$, $0 < \eta < \xi_0$ and $m := p + \eta$. Again, as in [16], instead of $C[u,u]$, we consider its modification

$$C_\mho[u,u](x) := C[u,u](x) + \mho(1+x^\eta)u(x), \tag{III.5.2}$$

for an appropriately chosen $\mho > 0$. Further, instead of $(K_m^\pm, D(Z_m^\pm))$, we employ $(L_m^\pm, D(Z_m^\pm))$ given by

$$\left[L^\pm u\right](x) := [K_m^\pm u](x) - \mho(1+x^\eta)u(x), \quad u \in D(Z_m^\pm). \tag{III.5.3}$$

With this notation, the original problem (III.1.3) can be written as a semilinear ACP

$$\frac{du(t)}{dt} = L_m^\pm u(t) + C_\mho[u(t), u(t)], \quad u(0) = u_0. \tag{III.5.4}$$

We remark that for every $\mho > 0$, the semigroup $(S_{L_m^\pm}(t))_{t\ge 0}$ generated by $(L_m^\pm, D(Z_m^\pm))$ has exactly the same regularisation properties as $(S_{K_m^\pm}(t))_{t\ge 0}$, provided all assumptions of Theorem III.4.2 are satisfied. Furthermore, choosing $\mho$ as in the proof of [16, Theorem 3.1], it follows that the nonlinear part of (III.5.4) has exactly the same properties as the map $K_{0,m}^{(\beta)}$ defined in [16, equation (3.4)]. As a consequence, the local wellposedness analysis of [16, equation (3.4)] can be repeated verbatim for problem (III.5.4). In particular, all conclusions of [16, Theorems 3.1 and 3.5] hold in the case of (III.5.4) and we have

**Theorem III.5.1.** *[16, Theorems 3.1 and 3.5] Assume $u_0 \in X_m$. Then there exists a unique local mild solution of* (III.5.4)*, $u \in C([0,t^*), X_m)$. The mild solution is nonnegative if $u_0 \in X_m^+$. Furthermore, if $u_0 \in X_m \cap D(Z_p^\pm)$, where $p = m - \eta$. Then the mild solution $u$, defined on its maximal interval of existence $[0,t^*)$, satisfies $u \in C([0,t^*), X_m) \cap C^1((0,t^*), X_m) \cap C((0,t^*), D(Z_p^\pm))$ and is classical in $X_p$.*

## 6 Global solutions

The solutions of Theorem III.5.1 are local in time and are defined on the maximal interval of existence $0 \le t < t^*$ only. We demonstrate global solvability of (III.1.1) for positive input data under additional restrictions on the model coefficients. In the sequel, we assume that

$$k(x,y) \le c_0(1 + x^\eta + y^\eta), \quad 0 < \eta < \xi_0. \tag{III.6.1}$$

To show that (III.5.4) is globally solvable, we will look at the evolution of the $i$-th moments

$$\| \, u(t) \, \|_{[i]} := \int_0^\infty u(x,t) x^i dx, \quad i \ge 0. \tag{III.6.2}$$

Global solvability of (III.1.3) is guaranteed if none of the moments $\|u(t)\|_{[i]}, 0 \le i \le m$, blow up in finite time.

In the sequel, we make use of the following two technical lemmas. The first one is used to control the coagulation operator $C[u,u]$, while the second one controls the term associated to the fragmentation process.

**Lemma III.6.1.** *Assume $u \in D(Z_m^\pm)_+$ and the coagulation kernel and the fragmentation rate satisfy (III.6.1) and (III.4.13), respectively. Then for $m \ge 2$ and any $\epsilon > 0$, we have*

$$\int_0^\infty C[u,u](x)dx \le \epsilon \|u(t)\|_{[m+\xi_0]} + c\epsilon^{-\frac{m+\eta-2}{\xi_0-\eta+1}} \left[ \|u(t)\|_{[1]}^{\frac{m+2\xi_0}{\xi_0+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0+1}{\xi_0-\eta+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0-1}{\xi_0-\eta+1}} \right], \tag{III.6.3}$$

*where the constant $c > 0$ depends on $m$, $\xi_0$ and $\eta$ only.*

*Proof.* Assume $u \in D(Z_m^\pm)_+$. Then in view of our definition of $D(Z_m^\pm)$ and of the assumption (III.4.13), the moments $\|u\|_{[\alpha]}$ are well defined for all $0 \le \alpha \le m + \xi_0$. Using this fact and changing the order of integration, for $m \geq 1$ we obtain

$$\begin{aligned}
\int_0^\infty x^m C[u,u](x)dx &= \frac{1}{2}\int_0^\infty \int_0^\infty (x+y)^m k(x,y)u(x)u(y)dxdy \\
&- \int_0^\infty x^m \int_0^\infty k(x,y)u(x)u(y)dydx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty (x+y)^m k(x,y)u(x)u(y)dxdy \\
&- \frac{1}{2}\int_0^\infty x^m \int_0^\infty k(x,y)u(x)u(y)dydx \\
&- \frac{1}{2}\int_0^\infty x^m \int_0^\infty k(x,y)u(x)u(y)dydx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \big((x+y)^m - x^m - y^m\big)k(x,y)u(x)u(y)dxdy \\
&\leq \frac{\Gamma_m c_0}{2}\int_0^\infty \int_0^\infty \big(xy^{m-1} + x^{m-1}y\big)(1+x^\eta+y^\eta)u(x)u(y)dxdy \\
&\leq P_m\Bigg[\parallel u(t) \parallel_{[1]}\parallel u(t) \parallel_{[m-1]} + \parallel u(t) \parallel_{[\eta+1]}\parallel u(t) \parallel_{[m-1]} \\
&+ \parallel u(t) \parallel_{[1]}\parallel u(t) \parallel_{[\eta+m-1]}\Bigg],
\end{aligned} \tag{III.6.4}$$

where $P_m = \Gamma_m c_0$ and we used the symmetry of the coagulation kernel and Lemma II.5.3. To obtain formula (III.6.3), we estimate each moment appearing in the last line of (III.6.4) separately.

If $m = 2$, then $\parallel u(t) \parallel_{[m-1]} = \|u(t)\|_{[1]}$ and the estimate is trivial. If $m > 2$, we let $\theta = \frac{\xi_0+1}{(m-1)(m+\xi_0-1)}$, $p = \frac{m+\xi_0-1}{m-2}$ and $q = \frac{m+\xi_0-1}{\xi_0+1}$. Since $\xi_0 > \eta > 0$, it follows that $1 < p, q$ and $0 < \theta < 1$. Therefore, using Hölder's inequality with exponents $p$ and $q$, we infer

$$\begin{aligned}
\parallel u(t) \parallel_{[m-1]} &= \int_0^\infty u^{\frac{1}{p}}(x,t)x^{(1-\theta)(m-1)}u^{\frac{1}{q}}(x,t)x^{\theta(m-1)}dx \\
&\leq \|u(t)\|_{[m+\xi_0]}^{\frac{m-2}{m+\xi_0-1}}\|u(t)\|_{[1]}^{\frac{\xi_0+1}{m+\xi_0-1}}.
\end{aligned} \tag{III.6.5}$$

Then using Young's inequality with the same exponents $p$ and $q$, we obtain

$$\|u(t)\|_{[1]} \parallel u(t) \parallel_{[m-1]} \leq \frac{m-2}{m+\xi_0-1}\epsilon^{\frac{m+\xi_0-1}{m-2}}\|u(t)\|_{[m+\xi_0]} + \frac{\xi_0+1}{m+\xi_0-1}\epsilon^{\frac{1-m-\xi_0}{\xi_0+1}}\|u(t)\|_{[1]}^{\frac{m+2\xi_0}{\xi_0+1}}. \tag{III.6.6}$$

It is not difficult to verify that (III.6.6) holds for all $m \geq 2$ and $\xi_0 > \eta > 0$.

Next, we consider $\| \ u \ \|_{[\eta+1]}$. To bound this norm, we let $\theta = \frac{\eta(m+\xi_0)}{(1+\eta)(m+\xi_0-1)}$ and use Hölder's inequality with exponents $p = \frac{m+\xi_0-1}{\eta}$, $q = \frac{m+\xi_0-1}{m+\xi_0-1-\eta}$. If $m \geq 2$ and $\xi_0 > \eta > 0$, then $1 < p, q$ and $0 < \theta < 1$, consequently

$$\begin{aligned} \| \ u(t) \ \|_{[\eta+1]} &= \int_0^\infty u^{\frac{1}{p}}(x,t) x^{\theta(1+\eta)} u^{\frac{1}{q}}(x,t) x^{1-\theta(1+\eta)} dx \\ &\leq \|u(t)\|_{[m+\xi_0]}^{\frac{\eta}{m+\xi_0-1}} \|u(t)\|_{[1]}^{\frac{m+\xi_0-1-\eta}{m+\xi_0-1}}. \end{aligned} \tag{III.6.7}$$

Now, combining (III.6.5) and (III.6.7), we have

$$\begin{aligned} \|u(t)\|_{[1+\eta]} \|u(t)\|_{[m-1]} &\leq \|u(t)\|_{[m+\xi_0]}^{\frac{m-2}{m+\xi_0-1}+\frac{\eta}{m+\xi_0-1}} \|u(t)\|_{[1]}^{\frac{\xi_0+1}{m+\xi_0-1}+\frac{m+\xi_0-1-\eta}{m+\xi_0-1}} \\ &\leq \|u(t)\|_{[m+\xi_0]}^{\frac{m+\eta-2}{m+\xi_0-1}} \|u(t)\|_{[1]}^{\frac{m+2\xi_0+\eta}{m+\xi_0-1}} \end{aligned} \tag{III.6.8}$$

and, using Young's inequality with exponents $p = \frac{m+\xi_0-1}{m+\eta-2}$ and $q = \frac{m+\xi_0-1}{\xi_0-\eta+1}$ (which satisfy $1 \leq p, q$, when $m \geq 2$ and $\xi_0 > \eta > 0$),

$$\|u(t)\|_{[1+\eta]} \|u(t)\|_{[m-1]} \leq \frac{m+\eta-2}{m+\xi_0-1} \epsilon^{\frac{m+\xi_0-1}{m+\eta-2}} \|u(t)\|_{[m+\xi_0]} + \frac{\xi_0-\eta+1}{m+\xi_0-1} \epsilon^{\frac{1-m-\xi_0}{\xi_0-\eta+1}} \|u(t)\|_{[1]}^{\frac{m+2\xi_0+\eta}{\xi_0-\eta+1}}. \tag{III.6.9}$$

Finally, to bound $\|u(t)\|_{[\eta+m-1]}$, we employ Hölder's inequality with exponents $p = \frac{m+\xi_0-1}{m+\eta-2}$, $q = \frac{m+\xi_0-1}{\xi_0-\eta+1}$ and let $\theta = \frac{(m+\xi_0)(m+\eta-2)}{(m+\eta-1)(m+\xi_0-1)}$. Since $1 < p, q$ and $0 < \theta < 1$ when $m \geq 2$ and $\xi_0 > \eta > 0$, this gives

$$\begin{aligned} \|u(t)\|_{[m+\eta-1]} &= \int_0^\infty u^{\frac{1}{p}}(x,t) x^{\theta(m+\eta-1)} u^{\frac{1}{q}}(x,t) x^{(1-\theta)(m+\eta-1)} dx \\ &\leq \|u(t)\|_{[m+\xi_0]}^{\frac{m+\eta-2}{m+\xi_0-1}} \|u(t)\|_{[1]}^{\frac{\xi_0-\eta+1}{m+\xi_0-1}}. \end{aligned} \tag{III.6.10}$$

From (III.6.10), using Young's inequality with the same exponents as above, we infer

$$\begin{aligned} \|u(t)\|_{[1]} \|u(t)\|_{[m+\eta-1]} &\leq \|u(t)\|_{[m+\xi_0]}^{\frac{m+\eta-2}{m+\xi_0-1}} \|u(t)\|_{[1]}^{\frac{m+2\xi_0-\eta}{m+\xi_0-1}} \\ &\leq \frac{m+\eta-2}{m+\xi_0-1} \epsilon^{\frac{m+\xi_0-1}{m+\eta-2}} \|u(t)\|_{[m+\xi_0]} + \frac{\xi_0-\eta+1}{m+\xi_0-1} \epsilon^{\frac{1-m-\xi_0}{\xi_0-\eta+1}} \|u(t)\|_{[1]}^{\frac{m+2\xi_0-1}{\xi_0-\eta+1}}. \end{aligned} \tag{III.6.11}$$

Formula (III.6.3) follows from (III.6.4) by adjusting the free parameter $\epsilon > 0$ in (III.6.6), (III.6.9) and (III.6.11) and by observing that $\frac{m-2}{\xi_0+1} \leq \frac{m+\eta-2}{\xi_0-\eta+1}$, for $m$, $\xi_0$ and $\eta$ satisfying assumption of the Lemma. $\square$

**Lemma III.6.2.** *Let $u_0 \in D(Z_m^{\pm})$ and (III.1.14) and (III.4.13) hold. Then for $N_m(y)$, defined in (III.1.11), we have the estimate*

$$-\int_0^\infty N_m(x)a(x)u(x)dx \leq -\frac{\mathfrak{d}}{3}\int_{x_0}^\infty a(x)x^m u(x)dx - \frac{2\mathfrak{d}}{3} \parallel u(t) \parallel_{[m+\xi_0]} + \mathfrak{d}_\mathfrak{m} \parallel u(t) \parallel_{[m]}, \tag{III.6.12}$$

*where $\mathfrak{d}_\mathfrak{m} = d_m \operatorname{ess\,sup}_{0\leq x\leq x_0} a(x)$ and $\mathfrak{d} = d_m a_0$.*

*Proof.* Taking into account (III.1.14) and results in [11], if for some $m_0 > 1$ and $x_0 \geq 0$, $N_{m_0}(x)/x^{m_0} \geq d_{m_0}$, $x \geq x_0$, then for any $m > 1$ there exists $d_m > 0$ such that $N_m(x)/x^m \geq d_{m_0} > 0$ for all $x \geq x_0$. Then, similarly to [16], we have

$$\begin{aligned}
-\int_0^\infty N_m(x)a(x)u(x,t)dx &\leq -\frac{1}{3}\int_{x_0}^\infty N_m(x)a(x)u(x,t)dx - \frac{2}{3}\int_{x_0}^\infty N_m(x)a(x)u(x,t)dx \\
&\quad - \int_0^{x_0} N_m(x)a(x)u(x,t)dx \\
&< -\frac{d_m}{3}\int_{x_0}^\infty a(x)x^m u(x,t)dx - \frac{2a_0 d_m}{3}\int_0^\infty x^{\xi_0}x^m u(x,t)dx \\
&\quad + \operatorname*{ess\,sup}_{0\leq x\leq x_0} a(x)d_m \int_0^{x_0} x^m u(x,t)dx \\
&\leq -\frac{\mathfrak{d}}{3}\int_{x_0}^\infty a(x)x^m u(x,t)dx - \frac{2\mathfrak{d}}{3} \parallel u(t) \parallel_{[m+\xi_0]} + \mathfrak{d}_\mathfrak{m} \parallel u(t) \parallel_{[m]},
\end{aligned} \tag{III.6.13}$$

and the conclusion follows. $\square$

Now we are in the position to prove global solvability of (III.1.3) for nonnegative input data. Due to technical reasons, we study the decay- and growth-fragmentation-coagulation cases independently. First, we consider the decay scenario.

**Theorem III.6.3.** *Let the assumptions of Theorem III.4.2 hold. If the coagulation kernel satisfy (III.6.1) and if $m \geq 2$, then the mild solutions to the decay-fragmentation-coagulation equation, associated to positive initial data $u_0 \in X_m^+$, are global in time.*

*Proof.* Assume initially that $u_0 \in C^0_\infty(\mathbb{R}_+)_+$, then the local mild solution satisfies $u(t) \in C([0,t^*), D(Z^+_m))$ for any $m$, in its maximal interval of existence $[0,t^*)$. For such a solution the differential equation (III.5.4) holds in the classical sense. Substituting $u(t)$ into the equation, multiplying by $1$, $x$ and $x^m$, respectively, and integrating over $\mathbb{R}_+$, we obtain

$$\begin{aligned}
\frac{d\|u(t)\|_{[0]}}{dt} &\le \int_0^\infty (n_0(x)-1)a(x)u(x,t)dx,\\
\frac{d\|u(t)\|_{[1]}}{dt} &\le -\int_0^\infty r(x)u(x,t)dx \le 0,\\
\frac{d\|u(t)\|_{[m]}}{dt} &\le -\int_0^\infty \left[mx^{m-1}r(x)+N_m(x)a(x)\right]u(x,t)dx\\
&\quad+\frac{1}{2}\int_0^\infty\int_0^\infty \big((x+y)^m-x^m-y^m\big)k(x,y)u(x,t)u(y,t)dxdy,
\end{aligned} \tag{III.6.14}$$

where (III.1.13) and (III.1.11) are used.

The second inequality in (III.6.14) implies that $\|u(t)\|_{[1]} \le \|u_0\|_{[1]}$ for any $t \ge 0$. From the second inequality in (III.6.14) and from Lemmas III.6.1 and III.6.2, we infer

$$\begin{aligned}
\frac{d\|u(t)\|_{[m]}}{dt} &\le -\frac{\mathfrak{d}}{3}\int_{x_0}^\infty a(x)x^m u(x,t)dx - \frac{2\mathfrak{d}}{3}\parallel u(t)\parallel_{[m+\xi_0]} + \mathfrak{d}_{\mathfrak{m}}\parallel u(t)\parallel_{[m]}\\
&\quad+\epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{-\frac{m+\eta-2}{\xi_0-\eta+1}}\left[\|u(t)\|_{[1]}^{\frac{m+2\xi_0}{\xi_0+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0+1}{\xi_0-\eta+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0-1}{\xi_0-\eta+1}}\right].
\end{aligned} \tag{III.6.15}$$

Using Hölder's and Young's inequalities with exponents $p=\frac{m+\xi_0-1}{m-1}$, $q=\frac{m+\xi_0-1}{\xi_0}$ and letting $\theta=\frac{(m-1)(m+\xi_0)}{m(m+\xi_0-1)}$, we obtain

$$\begin{aligned}
\|u(t)\|_{[m]} &= \int_0^\infty u^{\frac{1}{p}}(x,t)x^{\theta m}u^{\frac{1}{q}}(x,t)x^{(1-\theta)m}dx\\
&\le \|u(t)\|_{[m+\xi_0]}^{\frac{m-1}{m+\xi_0-1}}\|u(t)\|_{[1]}^{\frac{\xi_0}{m+\xi_0-1}}\\
&\le \epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{\frac{1-m}{\xi_0}}\|u(t)\|_{[1]},
\end{aligned} \tag{III.6.16}$$

for all $\epsilon > 0$ and some $c > 0$ that depends on $m$ and $\xi_0$ only.

Substituting (III.6.16) into (III.6.15) and choosing $\epsilon > 0$ such that $\epsilon(1+\mathfrak{d}_{\mathfrak{m}}) \le \frac{2\mathfrak{d}}{3}$, we deduce that

$$\frac{d\|u(t)\|_{[m]}}{dt} \leq -\frac{\mathfrak{d}}{3}\int_{x_0}^{\infty} a(x)x^m u(x,t)dx + Z_{m,1}, \tag{III.6.17}$$

where $Z_{m,1} \geq 0$ and depends on $m$, $\eta$, $\xi_0$ and $\|u_0\|_{[1]}$ only.

From (III.6.17) it follows that

$$\|u(t)\|_{[m]} + \frac{\mathfrak{d}}{3}\int_0^t \int_{x_0}^{\infty} a(x)x^m u(x,t)dx \leq \|u_0\|_{[m]} + tZ_{m,1}, \tag{III.6.18}$$

for any $t \in [0, t^*)$.

To bound the zeroth-order moment $\|u(t)\|_{[0]}$, we employ (III.1.13) to get

$$\begin{aligned} \frac{d\|u(t)\|_{[0]}}{dt} &\leq b_0 \int_0^{\infty} u(x,t)(1+x^l)a(x)dx \leq 2b_0 \int_0^{\infty} (1+x^m)a(x)u(x,t)dx \\ &\leq 2b_0 \int_0^{x_0} a(x)(1+x^m)u(x,t)dx + 2b_0 \int_{x_0}^{\infty} a(x)(1+x^m)u(x,t)dx \\ &\leq B_0\|u(t)\|_{[0]} + 2b_0 \int_{x_0}^{\infty} a(x)(1+x^m)u(x,t)dx, \end{aligned} \tag{III.6.19}$$

where $B_0 = 2b_0 ess\sup_{y\in[0,x_0]} a(x)(1+x^m)$. Consequently, from (III.6.19) we deduce

$$\begin{aligned} \|u(t)\|_{[0]} &\leq e^{B_0 t}\left[\|u_0\|_{[0]} + 2b_0(1+x_0^{-m})\int_0^t e^{-B_0 s}\int_{x_0}^{\infty} a(x)u(x,s)x^m dxds\right] \\ &\leq e^{B_0 t}\left[\|u_0\|_{[0]} + 2b_0(1+x_0^{-m})\int_0^t \int_{x_0}^{\infty} a(x)u(x,s)x^m dxds\right] \\ &\leq e^{B_0 t}\left[\|u_0\|_{[0]} + \frac{6b_0}{\mathfrak{d}}(1+x_0^{-m})\big(\|u_0\|_{[m]} + tZ_{m,1}\big)\right], \end{aligned} \tag{III.6.20}$$

for any $t \in [0, t^*)$.

From (III.6.18) and(III.6.20) it follows that $\|u(t)\|_{[m]}$ does not blow up for any positive value of $t$. Therefore, positive solutions emanating from $C_\infty^0(\mathbb{R}_+)_+$ are globally defined.

To conclude the proof, consider $u_0 \in X_m^+$. By Theorem III.5.1 every such $u_0$ gives rise to a mild solution of (III.1.3), which is defined in a maximal interval of existence $[0, t^*)$. Assume that $t^*$ is finite. Then the $X_m$-norm $\|u(t)\|_m$ of the solutions blows up at $t^*$. On the other hand, every $u_0 \in X_m^+$ is a strong $X_m$ limit of a sequence $(u_{0,k})_{t\geq 0} \in C_\infty^0(\mathbb{R}_+)_+$. Furthermore, the elements of the sequence can be chosen so that $\sup_{k\geq 0}\|u_{0,k}\|_m \leq 2\|u_0\|_m$.

Let the classical solutions emanating from $u_{0,k}$ be denoted by $u_k(t)$. By what was proven above, all these solutions are global and satisfy $\|u_k(t)\|_m \leq e^{B_0 t}[A + Ct]$ uniformly in $k \geq 0$, where the constants $A \geq 0$ and $C \geq 0$ depend on $m$, the model coefficients and $\|u_0\|_m$ only, while $B_0$ is defined (III.6.20). Assume now that $(t_n)_{n\geq 0}$ is a monotone increasing sequence, with $\lim_{n\to\infty} t_n = t^*$. Since $t^*$ is finite, we have $\lim_{n\to\infty}\|u(t_n)\|_m = \infty$. On another hand, classical solutions continuously depend on the initial data, consequently for each $n \geq 0$ there exists $k_n \geq 0$ such that $\|u(t_n) - u_{k_n}(t_n)\|_m \leq e^{B_0(t^*)}[A + Ct^*]$.

Let $M \geq 3e^{B_0 t^*}[A + Ct^*]$. Since $\lim_{n\to\infty}\|u(t_n)\|_m = \infty$, for $n$ sufficiently large we have

$$\begin{aligned} 3e^{B_0 t^*}[A + Ct^*] \leq M < \|u(t_n)\|_m &\leq \|u_{k_n}(t_n)\|_m + \|u(t_n) - u_{k_n}(t_n)\|_m \\ &\leq 2e^{B_0 t^*}[A + Ct^*]. \end{aligned} \tag{III.6.21}$$

The latter is clearly impossible. Hence, $t^* = \infty$ and for every $u_0 \in X_m^+$ the associated mild solution is globally defined. □

Now, we turn to the growth case.

**Theorem III.6.4.** *Let the assumptions of Theorem III.4.2 hold and the coagulation kernel satisfy* (III.6.1)*. If $m \geq 2$ and either*

*a) $r(x) \leq rx$ for all all $x > 0$, or*

*b) there are constants $\alpha_0$ and $\alpha_1$ satisfying $\beta(x) + (n_0(x) - 1)a(x) \leq \alpha_0 + \alpha_1 x$ for all $x > 0$,*

*holds, then the mild solutions to the growth-fragmentation-coagulation equation, associated to positive initial data $u_0 \in X_m^+$, are global in time.*

*Proof.* As in the previous theorem, we follow the ideas of [16]. Assume initially that $u_0 \in C^0_\infty(\mathbb{R}_+)_+$, then the local mild solution satisfies $u(t) \in C([0, t^*), D(Z_m^-))$ in its maximal interval of existence $[0, t^*)$. For such a solution the differential equation (III.5.4) holds. Substituting $u(t)$

into the equation, multiplying by $1$, $x$ and $x^m$ and integrating over $\mathbb{R}_+$, we obtain

$$\begin{aligned}
\frac{d\|u(t)\|_{[0]}}{dt} &= \int_0^\infty \beta(x)u(x,t)dx + \int_0^\infty (n_0(x)-1)a(x)u(x,t)dx \\
&\quad - \frac{1}{2}\int_0^\infty\int_0^\infty k(x,y)u(x,t)u(y,t)dxdy \\
\frac{d\|u(t)\|_{[1]}}{dt} &= \int_0^\infty r(x)u(x,t)dx \\
\frac{d\|u(t)\|_{[m]}}{dt} &= \int_0^\infty \big[mx^{m-1}r(x) - N_m(x)a(x)\big]u(x,t)dx \\
&\quad + \frac{1}{2}\int_0^\infty\int_0^\infty \big((x+y)^m - x^m - y^m\big)k(x,y)u(x,t)u(y,t)dxdy,
\end{aligned} \tag{III.6.22}$$

where (III.1.13) and (III.1.11) are used. First, we assume that condition a) of the theorem is satisfied. In this scenario, the first two inequalities in (III.6.22) reduce to

$$\frac{d\|u(t)\|_{[0]}}{dt} \leq \alpha_0\|u(t)\|_{[0]} + \alpha_1\|u(t)\|_{[1]}, \tag{III.6.23a}$$

$$\frac{d\|u(t)\|_{[1]}}{dt} \leq b_0\|u(t)\|_{[0]} + b_1\|u(t)\|_{[1]}. \tag{III.6.23b}$$

Adding (III.6.23a) and (III.6.23b), we obtain

$$\begin{aligned}
\frac{d\big(\|u(t)\|_{[0]} + \|u(t)\|_{[1]}\big)}{dt} &\leq (\alpha_0 + b_0)\|u(t)\|_{[0]} + (\alpha_1 + b_1)\|u(t)\|_{[1]} \\
&\leq B_0\big(\|u(t)\|_{[0]} + \|u(t)\|_{[1]}\big),
\end{aligned} \tag{III.6.24}$$

with $B_0 = \max\{(\alpha_0 + b_0), (\alpha_1 + b_1)\}$. Hence, from Gronwall's inequality, we infer

$$\|u(t)\|_{[0]} + \|u(t)\|_{[1]} \leq (\|u_0\|_{[0]} + \|u_0\|_{[1]})e^{B_0 t}. \tag{III.6.25}$$

Applying Young's inequality to the right-hand side of (III.6.5), we obtain

$$\| u \|_{[m-1]} \leq \begin{cases} \|u(t)\|_{[1]}, & m = 2, \\ \epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{-\frac{\xi_0+1}{m-2}}\|u(t)\|_{[1]}, & m > 2. \end{cases} \tag{III.6.26}$$

Using the inequalities (III.6.16), (III.6.26) and Lemmas III.6.1 and III.6.2 (with the first term dropped), we have

$$\frac{d\|u(t)\|_{[m]}}{dt} \leq b_0 m \begin{cases} \|u(t)\|_{[1]}, & m=2, \\ \epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{-\frac{\xi_0+1}{m-2}}\|u(t)\|_{[1]}, & m>2, \end{cases}$$
$$+ (b_1 m + \mathfrak{d}_{\mathfrak{m}})\left(\epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{\frac{1-m}{\xi_0}}\|u(t)\|_{[1]}\right) - \frac{2\mathfrak{d}}{3} \parallel u(t) \parallel_{[m+\xi_0]}$$
$$+ \epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{-\frac{m+\eta-2}{\xi_0-\eta+1}}\left[\|u(t)\|_{[1]}^{\frac{m+2\xi_0}{\xi_0+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0+1}{\xi_0-\eta+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0-1}{\xi_0-\eta+1}}\right]. \tag{III.6.27}$$

If we choose $\epsilon > 0$ so that $\epsilon[m(b_0+b_1) + \mathfrak{d}_{\mathfrak{m}} + 1] \leq \frac{2\mathfrak{d}}{3}$, then

$$\frac{d\|u(t)\|_{[m]}}{dt} \leq F(\|u(t)\|_{[1]}),$$

where $F(\cdot)$ is a linear combination of powers of $\|u(t)\|_{[1]}$, whose coefficients depend on $m$, and the model's parameters only. By (III.6.25), $\|u(t)\|_{[1]}$ is exponentially bounded, hence $F(\|u(t)\|_{[1]})$ is exponentially bounded and, on the account of Gronwall's inequality,

$$\|u(t)\|_{[m]} \leq Ae^{Bt}, \tag{III.6.28}$$

for any $t > 0$, with $A > 0$ that depends on $m$, the model parameters and $\|u_0\|_1$, while $B > 0$ depends on the model's parameters but is independent of the size of the initial data $u_0$. Bound (III.6.28) shows that $\|u(t)\|_{[m]}$ grows at most exponentially and therefore cannot blow up in finite time. Hence, under assumption a) of the Theorem, solutions with initial data from $C_0^\infty(\mathbb{R}_+)_+$ are globally defined.

Assume that condition b) of the theorem is satisfied. Then the second inequality in (III.6.22) decouples from the system and we have the exponential bound

$$\|u(t)\|_{[1]} \leq e^{rt}\|u_0\|_{[1]}. \tag{III.6.29}$$

Using the inequality (III.6.26) and Lemmas III.6.1 and III.6.2, we have

$$\begin{aligned}\frac{d\|u(t)\|_m}{dt} &\leq (r+\mathfrak{d}_{\mathfrak{m}})\left(\epsilon\|u(t)\|_{[m+\xi_0]} + c\epsilon^{\frac{1-m}{\xi_0}}\|u(t)\|_{[1]}\right) - \frac{2\mathfrak{d}}{3}\parallel u(t)\parallel_{[m+\xi_0]} \\ &\quad - \frac{\mathfrak{d}}{3}\int_{x_0}^{\infty} a(x)x^m u(x,t)dx + \epsilon\|u(t)\|_{[m+\xi_0]} \\ &\quad + c\epsilon^{-\frac{m+\eta-2}{\xi_0-\eta+1}}\left[\|u(t)\|_{[1]}^{\frac{m+2\xi_0}{\xi_0+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0+1}{\xi_0-\eta+1}} + \|u(t)\|_{[1]}^{\frac{m+2\xi_0-1}{\xi_0-\eta+1}}\right].\end{aligned} \tag{III.6.30}$$

If we choose $\epsilon > 0$ in such a way that $\epsilon(1+r+\mathfrak{d}_{\mathfrak{m}}) \leq \frac{\mathfrak{d}}{3}$, then

$$\frac{d\|u(t)\|_{[m]}}{dt} \leq -\frac{\mathfrak{d}}{3}\int_{x_0}^{\infty} a(x)x^m u(x,t)dx + Ae^{Bt}, \tag{III.6.31}$$

where $A > 0$ depends on $m$, the model parameters and on $\|u_0\|_{[1]}$, while $B > 0$ depends on $m$, the model's parameters but is independent of the size of the input data $u_0$. As in the proof of Theorem III.6.3, we integrate (III.6.31) to obtain

$$\|u(t)\|_{[m]} + \frac{\mathfrak{d}}{3}\int_0^t \int_{x_0}^{\infty} a(x)x^m u(x,t)dx \leq \|u_0\|_{[m]} + \frac{A}{B}e^{Bt}, \tag{III.6.32}$$

for any $t \in [0, t^*)$.

It remains to show that the zeroth-order moment $\|u(t)\|_{[0]}$ is bounded. We use (III.3.7) and (III.1.13) to obtain

$$\begin{aligned}\frac{d\|u(t)\|_{[0]}}{dt} &\leq \int_0^{\infty}[\beta(x) + n_0(x)a(x)]u(x,t)dx \\ &\leq \int_0^{\infty}\left[\beta(x)\frac{(1+x^m)}{(1+x^m)} + b_0(1+x^l)a(x)\right]u(x,t)dx \\ &\leq \beta_\infty\int_0^{\infty}(1+x^m)u(x,t)dx + 2b_0\int_0^{\infty}(1+x^m)a(x)u(x,t)dx \\ &\leq \mathfrak{s}\int_0^{x_0}u(x,t)dx + \left(\frac{\beta_\infty}{a_0x_0^{\xi_0}} + 2b_0\right)\int_{x_0}^{\infty}a(x)u(x,t)(1+x^m)dx \\ &\leq \mathfrak{s}\|u(t)\|_{[0]} + \left(\frac{\beta_\infty}{a_0x_0^{\xi_0}} + 2b_0\right)\int_{x_0}^{\infty}a(x)u(x,t)(1+x^m)dx,\end{aligned} \tag{III.6.33}$$

where $\mathfrak{s} = 2b_0 ess\sup_{x\in[0,x_0]}(a(x) + \beta_\infty)(1 + x^m)$. By Gronwall's lemma and (III.6.32), we have

$$\begin{aligned}\|u(t)\|_{[0]} &\le e^{\mathfrak{s}t}\left(\|u_0\|_{[0]} + \left(\frac{\beta_\infty}{a_0 x_0^{\xi_0}} + 2b_0\right)(1 + x_0^{-m}) \int_0^t \int_{x_0}^\infty a(x)u(x,s)x^m dxds\right) \\ &\le e^{\mathfrak{s}t}\left[\|u_0\|_{[0]} + \frac{3}{\mathfrak{d}}\left(\frac{\beta_\infty}{a_0 x_0^{\xi_0}} + 2b_0\right)(1 + x_0^{-m})\left(\|u_0\|_{[m]} + \frac{A}{B}e^{Bt}\right)\right],\end{aligned} \tag{III.6.34}$$

for $t \in [0, t^*)$.

On the account of (III.6.32) and (III.6.34), it follows that $\|u(t)\|_{[m]}$ does not blow up for any positive value of $t$. Therefore, positive solutions emanating from $C_0^\infty(\mathbb{R}_+)_+$ are globally defined.

Using the density argument employed in the proof of Theorem III.6.3, we conclude that for every $u_0 \in X_m^+$ the associated mild solution is globally defined. $\square$

Chapter IV

# Explicit solutions of the transport-fragmentation equation

## 1 Introduction

In this chapter, we provide a systematic way of finding exact and physically meaningful solutions to the continuous transport-fragmentation models. The constant and linear decay/growth rates are considered. In the growth case, we first deal with the homogeneous boundary conditions and then show that the technique is applicable to some cases of fragmentation equation with the McKendrick-von Foerster boundary condition.

The search for exact solutions in the continuous case can be traced back to Ziff and McGrady [55, 56] for the pure fragmentation model while Huang et al [35, 36] considered fragmentation with decay for constant and linear decay rates. The work was further continued by Banasiak et al., [20], where a systematic approach for both linear decay/growth and constant rates was considered. It is worth re-emphasizing that exact solutions play a crucial role in understanding the model dynamics, grasping features that are not easily captured by general theoretical analysis. Furthermore, such solutions can benchmark numerical methods and aid in improving the computational procedures.

## 2 The model equation

Consider the transport-fragmentation equation

$$\partial_t u(x,t) \pm \partial_x(r(x)u(x,t)) = -a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)dy, \quad x,t \in \mathbb{R}_+,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \tag{IV.2.1}$$

where $-$ and $+$ refer to the decay and the growth scenarios, respectively, and the coefficients are defined in Section III.1.1. In the case of growth, if

$$\int_0^1 \frac{dx}{r(x)} < \infty, \tag{IV.2.2}$$

then (IV.2.1) must be coupled with boundary conditions and we assume that

$$\lim_{x \to 0^+} r(x)u(x) = 0. \tag{IV.2.3}$$

The work presented here aims to extend the results of [35, 36] where only the decay case was considered and explicit solutions were found only for a constant decay rate. As in [35, 36], we work with the power law coefficients and extend the results to constant and linear rates and to the growth case. The power law coefficients are defined by

$$r(x) = kx^\gamma, \quad a(x) = ax^\alpha, \quad b(x,y) = \frac{\nu+2}{y}\left(\frac{x}{y}\right)^\nu, \quad -2 < \nu \leq 0, \tag{IV.2.4}$$

where $k, a > 0$ and $\alpha \neq 0$. The lower bound of $\nu$ guarantees existence of the integral

$$\int_0^y xb(x,y)dx = y, \quad y > 0. \tag{IV.2.5}$$

Due to the physical interpretation (see [18, Lemma 2.2.3]), the fragmentation kernel must satisfy

$$\int_0^z xb(x,y)dx \geq \int_{y-z}^y (y-x)b(x,y)dx, \quad 0 \leq z < \frac{y}{2}, \tag{IV.2.6}$$

which is possible only if $\nu \leq 0$. We substitute (IV.2.4) into equation (IV.2.1) to get:

$$\partial_t u^{\pm}(x,t) \pm \partial_x(kx^{\gamma}u^{\pm}(x,t)) = -ax^{\alpha}u^{\pm}(x,t) + \int_x^{\infty} ay^{\alpha}\frac{\nu+2}{y}\left(\frac{x}{y}\right)^{\nu}u^{\pm}(y,t)dy, \quad x,t \in \mathbb{R}_+,$$
$$u^{\pm}(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \tag{IV.2.7}$$

We start our analysis by transforming (IV.2.7) as

$$z = ax^{\alpha}, \qquad v^{\pm}(z,t) = x^{-\nu}u^{\pm}(x,t), \tag{IV.2.8}$$

which leads to

$$x = \left(\frac{z}{a}\right)^{\frac{1}{\alpha}}, \qquad v^{\pm}(z,t) = \left(\frac{z}{a}\right)^{-\frac{\nu}{\alpha}}u^{\pm}\left(\left(\frac{z}{a}\right)^{\frac{1}{\alpha}},t\right). \tag{IV.2.9}$$

The substitution transforms the partial derivative with respect to $t$ as follows

$$\partial_t u^{\pm}(x,t) = x^{\nu}\partial_t v^{\pm}(z,t) = \left(\frac{z}{a}\right)^{\nu/\alpha}\partial_t v^{\pm}(z,t). \tag{IV.2.10}$$

Further, the partial derivative with respect to $x$ is given by

$$\begin{aligned}\partial_x[r(x)u^{\pm}(x,t)] &= \partial_x\Big[kx^{\gamma+\nu}v^{\pm}(z,t)\Big]\\ &= k\Big((\nu+\gamma)x^{\nu+\gamma-1}v^{\pm}(z,t) + x^{\nu+\gamma}\partial_z v^{\pm}(z,t)\frac{dz}{dx}\Big)\\ &= k\left((\nu+\gamma)\left(\frac{z}{a}\right)^{(\nu+\gamma-1)/\alpha}v^{\pm}(z,t) + \alpha a\left(\frac{z}{a}\right)^{(\nu+\gamma+\alpha-1)/\alpha}\partial_z v^{\pm}(z,t)\right).\end{aligned} \tag{IV.2.11}$$

Considering the integral, we note that $\alpha$ is either negative or positive, but not equal to zero. Limits of integration are obtained by replacing $x$ and $y$ with $z = ax^{\alpha}$ and $y = as^{x}$. If $\alpha > 0$, the inequality $0 \leq x \leq y$ implies $0 \leq z \leq s$, while when $\alpha < 0$, we have $0 \leq s \leq z$. Combining both cases together, we obtain

$$\int_x^{\infty} a(y)b(x,y)u(y,t)dy = \left(\frac{z}{a}\right)^{\frac{\nu}{\alpha}}\frac{\nu+2}{|\alpha|}\begin{cases}\int_z^{\infty} v^{\pm}(s,t)ds & \alpha > 0\\ \int_0^z v^{\pm}(s,t)ds & \alpha < 0.\end{cases}$$

Formally, the substitution of (IV.2.10), (IV.2.11) and the last integral into (IV.2.7) gives

$$\partial_t v^{\pm}(z,t) \pm \beta z^{\mu}\partial_z v^{\pm}(z,t) = -\big[\pm\theta z^{\mu-1} + z\big] v^{\pm}(z,t) + m \begin{cases} \int_z^{\infty} v^{\pm}(s,t)ds & \text{if} \quad \alpha > 0, \\ \int_0^z v^{\pm}(s,t)ds & \text{if} \quad \alpha < 0, \end{cases} \tag{IV.2.12}$$

$$v^{\pm}(z,0) = v_0(z) := \left(\frac{z}{a}\right)^{-\frac{\nu}{\alpha}} u_0\left(\left(\frac{z}{a}\right)^{\frac{1}{\alpha}}\right), \tag{IV.2.13}$$

for $t, z \in \mathbb{R}_+$, with

$$\beta = a^{\frac{1-\gamma}{\alpha}} k\alpha, \qquad \theta = a^{\frac{1-\gamma}{\alpha}} k(\gamma+\nu),$$
$$m = \frac{(\nu+2)}{|\alpha|}, \qquad \mu = \frac{\gamma+\alpha-1}{\alpha}. \tag{IV.2.14}$$

If $\gamma > 1$ in the growth case, then (IV.2.7) is not well-posed in $X_m$, with $m \geq 1$, see [18]. Consequently, we assume that $0 \leq \gamma \leq 1$ in this situation. It is worth pointing out that the sign of $\alpha$ in (IV.2.12) transforms the growth problem to the decay one and vice versa. We consider the decay and the growth scenarios with both constant and linear rates, including the constant decay rate that was solved in [35, 36]. Here, however, we provide a comprehensive approach covering in a unified way all cases.

We introduce the following notation

$$\mathcal{G}^-[v](x) = \int_0^x v(s)ds \qquad \text{and} \qquad \mathcal{G}^+[v](x) = \int_x^{\infty} v(s)ds. \tag{IV.2.15}$$

Symbol $\mathcal{G}^{\mathrm{sg}\,\alpha}$, where $\mathrm{sg}\,\alpha = +$ and $\mathrm{sg}\,\alpha = -$ for $\alpha < 0$, allows us to treat two integrals terms in (IV.2.12) simultaneously.

We start with the decay case.

## 2.1 Constant decay and growth rates

In this section, we consider (IV.2.12) with $\mu = 0$ in both the decay and the growth cases. To avoid singularities at zero in (IV.2.12), we assume also that $\theta = 0$. If $\mu = \theta = 0$, then $\gamma = -\nu$ and $\alpha = 1 - \gamma$. Under the assumptions on $\mu$, $\theta$, $\gamma$ and $\alpha$, we have the following possibilities:

(i) The decay case with $\alpha < 0$ implies $\gamma > 1$. The decay case is transformed into the growth case with $\mathcal{G}^-$, which requires boundary conditions for for $v^-(z,t)$ as $z \to 0^+$. This corresponds to boundary conditions $x^\gamma u(x,t)$ as $x \to +\infty$. Using (IV.2.3) we conclude that $\lim_{x\to\infty} x^\gamma u(x,t) = 0$. Accordingly, in this case we use $\lim_{z\to 0^+} v^-(z,t) = 0$.

(ii) The growth case with $\alpha < 0$ implies $\gamma > 1$. The growth case is transformed into a decay case with $\mathcal{G}^-$ and no boundary conditions are needed.

(iii) The decay case with $\alpha > 0$ implies $\gamma < 1$. The decay is transformed into decay with $\mathcal{G}^+$, which does not require boundary conditions as $z \to 0^+$.

(iv) The growth case with $\alpha > 0$ leads to $\gamma < 1$. The growth remains the growth case with $\mathcal{G}^+$, requiring boundary conditions for $v^+(z,t)$ as $z \to 0^+$. The boundary conditions match with $x^\gamma u(x,t)$ as $x \to 0^+$. Similarly to i), $\lim_{x\to 0^+} x^\gamma u(x,t) = 0$ and $\lim_{z\to 0^+} v^+(z,t) = 0$.

With the assumptions $\mu = \theta = 0$, equation (IV.2.14) is transformed into

$$\partial_t v^\pm(z,t) \pm \beta \partial_z v^\pm(z,t) = -z v^\pm(z,t) + m\mathcal{G}^{\operatorname{sg}\alpha}[v^\pm(\cdot,t)](z), \tag{IV.2.16}$$

with the initial condition as in (IV.2.13) and, in the growth case whenever $\pm\beta > 0$, we supplement (IV.2.16) with the homogeneous boundary condition, that is,

$$\lim_{z\to 0^+} v^\pm(z,t) = 0. \tag{IV.2.17}$$

The characteristics of (IV.2.16) are given by

$$z(\xi,t) = \pm\beta t + \xi, \quad \xi > 0. \tag{IV.2.18}$$

We set $w^\pm(\xi,t) = v^\pm(z(\xi,t),t)$, hence (IV.2.16) becomes

$$\begin{aligned} \partial_t w^\pm(\xi,t) &= -(\xi \pm \beta t) w^\pm(\xi,t) + m\mathcal{G}^{\operatorname{sg}\alpha}[w^\pm(\cdot,t)](\xi) \\ w^\pm(\xi,0) &= w_0^\pm(\xi) = v_0^\pm(\xi), \end{aligned} \tag{IV.2.19}$$

for $\xi > 0$. The characteristics given by (IV.2.18) fill half of the first quadrant if $\beta$ is positive for $\pm\alpha$. We need to further address solutions that emanate from the upper half of the quadrant.

To be able to account for any characteristics crossing the first quadrant, we extend (IV.2.19) to $\xi < 0$ by assuming that $w_0^\pm(\xi) = 0$ for $\xi < 0$ and then we redefine the initial condition as $\phi(\xi) = w_0^\pm(\xi) + \psi(\xi)$, where $\psi(\xi) = 0$ for $\xi > 0$ and must be determined for $\xi < 0$. For the extended solution $w^\pm$, we have

$$w^\pm(\xi, \mp\beta^- \xi) = 0, \tag{IV.2.20}$$

for $\xi < 0$ whenever $\pm\beta > 0$.

To integrate (IV.2.16) explicitly, we set

$$f^\pm(\xi, t) = e^{\pm\frac{\beta t^2}{2} + \xi t} w^\pm(\xi, t). \tag{IV.2.21}$$

This transforms (IV.2.16) into the following equation

$$\begin{aligned} \partial_t f^\pm(\xi, t) &= m\mathcal{G}^{\mathrm{sg}\,\alpha}[e^{-t(\cdot - \xi)} f^\pm(\cdot, t)](\xi) \\ f^\pm(\xi, 0) &= \phi(\xi), \end{aligned} \tag{IV.2.22}$$

with $\xi > 0$ if $\pm\beta < 0$ and $\xi \in \mathbb{R}$ if $\pm\beta > 0$. In the second case, we further require

$$f^\pm(\xi, \mp\beta^{-1}\xi) = 0, \quad \xi < 0. \tag{IV.2.23}$$

## 2.2 Linear decay and growth rates

In this section, we consider the cases of linear growth and decay, that is, we assume that $\mu = 1$. We keep in mind that, depending on the sign of $\alpha$, the growth and the decay can swap. The transformed equation (IV.2.12) reads

$$\partial_t v^\pm(z, t) \pm \beta z \partial_z v^\pm(z, t) = -\big[\pm\theta z^{\mu-1} + z\big] v^\pm(z, t) + m\mathcal{G}^{\mathrm{sg}\,\alpha}[v^\pm(\cdot, t)](z), \tag{IV.2.24}$$

supplemented with the initial conditions. We apply the method of characteristics to solve the problem. The characteristic equation associated to (IV.2.24) is given by

$$\frac{dz}{dt} = \pm\beta z, \qquad z(0) = \xi, \qquad \xi, t \in \mathbb{R}_+. \tag{IV.2.25}$$

Solving, we obtain

$$z(\xi, t) = \xi e^{\pm\beta t}, \qquad \xi, t \in \mathbb{R}_+. \tag{IV.2.26}$$

It should be noted that for all admissible values of $\alpha$ the characteristics lines fill up the whole region, where the problem is defined, that is, the first quadrant and hence no boundary conditions are required. Setting $w^{\pm}(\xi, t) = v^{\pm}(z, t)$, the partial derivatives are transformed into

$$\begin{aligned} \partial_t v^{\pm}(z, t) &= \partial_t w^{\pm}(\xi, t) \pm \beta e^{\pm\beta t} \partial_\xi w^{\pm}(\xi, t) \\ \partial_z v^{\pm}(z, t) &= \pm e^{\pm\beta t} w_\xi^{\pm}(\xi, t). \end{aligned} \tag{IV.2.27}$$

The integral terms in (IV.2.24) are given by

$$\begin{aligned} \mathcal{G}^{+}[v^{\pm}(\cdot, t)](z) &= \int_z^\infty v^{\pm}(z(t, s), t) \frac{dz(t, s)}{ds} ds \\ &= \int_\xi^\infty w^{\pm}(s, t) ds = \mathcal{G}^{+}[w^{\pm}(\cdot, t)](\xi) \end{aligned} \tag{IV.2.28}$$

and

$$\begin{aligned} \mathcal{G}^{-}[v^{\pm}(\cdot, t)](z) &= \int_0^z v^{\pm}(z(t, s), t) \frac{dz(t, s)}{ds} ds \\ &= \int_0^\xi w^{\pm}(s, t) ds = \mathcal{G}^{-}[w^{\pm}(\cdot, t)](\xi). \end{aligned} \tag{IV.2.29}$$

Hence, equation (IV.2.24) reduces to

$$w_t^{\pm}(\xi, t) = -[\pm\theta + \xi e^{\pm\beta t}] w^{\pm}(\xi, t) + m \mathcal{G}^{\mathrm{sg}\,\alpha}[w^{\pm}(\cdot, t)](\xi). \tag{IV.2.30}$$

Integrating factor of (IV.2.30) is given by

$$I.F = e^{\int_0^t (\pm\theta + e^{\pm\beta s}) ds} = e^{\pm\theta t \pm \frac{\xi}{\beta}(e^{\pm\beta t} - 1)}. \tag{IV.2.31}$$

Setting $\tau^{\pm} = \pm\frac{1}{\beta}(e^{\pm\beta t} - 1)$ leads to $\pm t = \frac{1}{\beta}\ln(1 + \beta\tau^{\pm})$. Multiplying by the integrating factor and using the substitution, we have

$$\left((1 + \beta\tau^{\pm})^{\frac{\theta}{\beta}} e^{\tau^{\pm}\xi} w^{\pm}(\xi, \tau^{\pm})\right)_{\tau^{\pm}} = (1 + \beta\tau^{\pm})^{\frac{\theta}{\beta}} e^{\tau^{\pm}\xi} m\mathcal{G}^{\mathrm{sg}\,\alpha}[w^{\pm}(\cdot, t)](\xi). \tag{IV.2.32}$$

Setting $f^{\pm}(\xi, \tau^{\pm}) = (1 + \beta\tau^{\pm})^{\frac{\theta}{\beta}} e^{\tau^{\pm}\xi} w^{\pm}(\xi, \tau^{\pm})$ results in

$$\begin{aligned} f_{\tau^{\pm}}(\xi, \tau^{\pm}) &= m\mathcal{G}^{\mathrm{sg}\,\alpha}\left[e^{-\tau^{\pm}(\cdot - \xi)} f^{\pm}(\cdot, t)\right](\xi) \\ f_{\tau^{\pm}}(\xi, 0) &= f_0^{\pm}(\xi) := \xi^{-\frac{\nu}{\alpha}} u_0\left(\left(\frac{\xi}{a}\right)^{\frac{1}{\alpha}}\right), \quad \xi \in \mathbb{R}_+. \end{aligned} \tag{IV.2.33}$$

## 2.3 The model analysis

The considerations of Section IV.2 show that a large class of fragmentation equations with decay or growth can be transformed into the linear integro-differential equation

$$\begin{aligned} &f_t^{\pm}(x, t) = m\mathcal{G}^{\pm}[\varphi(\pm t(\cdot - x) f^{\pm}(\cdot, t)](x), \\ &f^{\pm}(x, 0) = f_0^{\pm}(x), \end{aligned} \tag{IV.2.34}$$

where $x \in \mathbb{R}_+$ or $x \in \mathbb{R}$, and the integral kernel is given by an entire function

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi_n}{n!} z^n, \tag{IV.2.35}$$

which we assume to be of finite exponential type $l > 0$, [33]. We establish classical solvability of (IV.2.34) for input data $f_0^{\pm}$ in an abstract setting of a suitable Banach spaces $X^{\pm}$. We assume that

$$\left\|\mathcal{G}^{\pm}\right\|_{X^{\pm} \to X^{\pm}} \leq c, \tag{IV.2.36}$$

for some $c > 0$. For the analysis, we write (IV.2.34) as an abstract linear non-autonomous ODE in $X^{\pm}$.

Firstly, for both the decay and the growth, the operator $\mathcal{G}^{\pm}$ satisfies the equality

$$\left(\mathcal{G}^{\pm}\right)^{n+1}[u](x) = \frac{(\pm 1)^n}{n!}\mathcal{G}^{\pm}[(\cdot - x)^n u(\cdot)](x), \quad n \geq 1, \tag{IV.2.37}$$

and on account of the definition (IV.2.15), to obtain the right hand side we recursively integrated the left hand side. With the assumption on (IV.2.35), we define

$$\begin{aligned}\Phi(z) :&= \int_0^\infty \varphi(sz)e^{-s}ds = \sum_{n=0}^{\infty} \frac{\varphi_n}{n!} z^n \int_0^\infty s^n e^{-s} ds \\ &= \sum_{n=0}^{\infty} \varphi_n z^n, \\ \tilde{\Phi}(z) &= \sum_{n=0}^{\infty} |\varphi_n| z^n.\end{aligned} \tag{IV.2.38}$$

By the assumptions, $\Phi$ and $\tilde{\Phi}$ are analytic in the disk $\mathbb{D}_{\frac{1}{l}} = \{|z| < \frac{1}{l}\}$. We define the map $z \mapsto \Phi(z\mathcal{G}^{\pm})$ as

$$\Phi(z\mathcal{G}^{\pm}) := \sum_{n=0}^{\infty} \varphi_n (z\mathcal{G}^{\pm})^n, \quad 0 \leq |z| < \frac{1}{cl} =: r. \tag{IV.2.39}$$

By construction $\Phi(\cdot\mathcal{G}^{\pm}) : \mathbb{D}_r \to \mathcal{L}(X^{\pm})$ is analytic operator-valued function of $z$ satisfying

$$\left\|\Phi\big(z\mathcal{G}^{\pm}\big)\right\|_{X^{\pm}\to X^{\pm}} \leq \tilde{\Phi}(c|z|), \tag{IV.2.40}$$

where $\mathcal{L}(X^{\pm})$ is the space of bounded linear operators in $X^{\pm}$. As a result, locally in time, (IV.2.34) can be written as an abstract Cauchy problem

$$f_t^{\pm} = m\Phi(\pm t\mathcal{G}^{\pm})\mathcal{G}^{\pm} f^{\pm}, \quad f^{\pm}(0) = f_0^{\pm}, \quad f^{\pm} \in C^1\big((0,r), X^{\pm}\big). \tag{IV.2.41}$$

Now we are ready to state a useful lemma.

**Lemma IV.2.1.** *Problem (IV.2.41) is classically solvable. That is, for any $f_0^{\pm} \in X^{\pm}$, there exists a unique classical solution $f^{\pm} \in C([0,r), X^{\pm}) \cap C^1((0,r), X^{\pm})$ satisfying (IV.2.41). Furthermore, the solutions are given explicitly by*

$$f^{\pm}(x,t) = \phi_0 f_0^{\pm}(x) + t\mathcal{G}^{\pm}[F(\pm t(\cdot - x)) f_0^{\pm}(\cdot)](x), \tag{IV.2.42a}$$

*where the kernel $F$, defined by*

$$F(z) = \sum_{n\geq 0} \frac{\phi_{n+1}}{n!(n+1)!} z^n, \qquad \phi_n = \frac{d^n}{dz^n} \exp\Big\{ m \int_0^z \Phi(\xi) d\xi \Big\}\Big|_{z=0},$$

*is of finite exponential type $l > 0$. Equivalently,*

$$f^{\pm}(t) = \phi(t\mathcal{G}^{\pm}) f_0^{\pm}, \qquad t \in [0, r), \tag{IV.2.42b}$$

*where*

$$\phi(z) = \exp\Big\{ m \int_0^z \Phi(\xi) d\xi \Big\} = \sum_{n\geq 0} \frac{\phi_n}{n!} z^n. \tag{IV.2.42c}$$

*Proof.* a) First, we rewrite (IV.2.41) as Volterra integral equation of the first kind

$$f^{\pm}(t) = f_0^{\pm} + m \int_0^t \Phi(\tau\mathcal{G}^{\pm}) \mathcal{G}^{\pm} f^{\pm}(\tau) d\tau, \qquad 0 < t < r. \tag{IV.2.43}$$

Using (IV.2.36) and (IV.2.40), we have $\|\Phi(t\mathcal{G}^{\pm})\mathcal{G}^{\pm}\|_{X^{\pm}\to X^{\pm}} \leq c\widetilde{\Phi}(ct)$ and it follows that the integral equation is uniquely solvable in $C([0, t_0^{\pm}], X^{\pm})$, for some $0 < t_0^{\pm} < r$. Furthermore, since the right-hand side of the equation is in $C^1((0, t_0^{\pm}), X^{\pm})$, we conclude that the solutions $f^{\pm} \in C([0, t_0^{\pm}], X^{\pm})$ to (IV.2.43) are in fact of class $C^1((0, t_0^{\pm}), X^{\pm})$ and hence satisfy (IV.2.41) in the classical sense.

b) To obtain (IV.2.42a), we note that the function $\phi$, defined by (IV.2.42c), is analytic in $\mathbb{D}_{\frac{1}{l}}$ as a composition of an entire and an analytic function. Moreover, the reciprocal is given by

$$\phi^{-1}(z) = \exp\Big\{ -m \int_0^z \Phi(\xi) d\xi \Big\}.$$

Thus, on account of the commutativity of $t\mathcal{G}^{\pm}$ and $s\mathcal{G}^{\pm}$ for scalar $t, s$,

$$\frac{d}{dt}\big[\phi^{-1}(t\mathcal{G}^{\pm}) f^{\pm}(t)\big] = \phi^{-1}(t\mathcal{G}^{\pm}) \partial_f^{\pm}(t) - \phi^{-1}(t\mathcal{G}^{\pm}) m\Phi(t\mathcal{G}^{\pm}) \mathcal{G}^{\pm} f^{\pm}(t) = 0,$$

in $C((0, t_0^{\pm}), X^{\pm})$. Integrating with respect to time and observing that $\phi^{-1}(t\mathcal{G}^{\pm}) = (\phi(t\mathcal{G}^{\pm}))^{-1}$, we infer

$$f^{\pm}(t) = \phi(t\mathcal{G}^{\pm})f_0^{\pm}, \qquad t \in [0, t_0^{\pm}).$$

In connection with the last formula, we note that the right-hand side of (IV.2.42b) is analytic when $t \in \mathbb{D}_r$, with values in $X^{\pm}$. Using this fact, $f^{\pm}$ defined in (IV.2.42b), satisfies (IV.2.41) in the classical sense for $t \in (0, r)$. Finally, using the analyticity of $\phi(z)$ and the standard Cauchy estimates [48], we obtain (IV.2.42a) with $F(z)$ of finite exponential type $l > 0$.

□

We will see in the coming sections that the integral kernels $F(z)$ of the growth/decay problems are entire functions of $z$ with moderate growth in $\mathbb{R}_+$. This allows for global extensions of formulae (IV.2.42b) and (IV.2.42c) beyond the initial data classes in $X^{\pm}$ and the time interval $(0, r)$ after some minor modifications.

# 3 Solutions to growth/decay fragmentation equation with linear rates

The theory presented in Section IV.2.3 is applicable to the model (IV.2.33), which is in the form (IV.2.34) with $\varphi(z) := e^{-(\operatorname{sg}\alpha)z}$. We define

$$X_{\pm\rho}^{\pm\sigma} := L_1(\mathbb{R}_+, x^{\pm\sigma}e^{\pm\rho x}dx), \quad \|u\|_{X_{\pm\rho}^{\pm\sigma}} = \int_{\mathbb{R}_+} |u|(x)x^{\pm\sigma}e^{\pm\rho x}dx, \tag{IV.3.1}$$

for $\sigma \geq 0, \rho > 0$. Then, by (IV.2.15), (IV.2.36) and integration by parts, we have

$$\|\mathcal{G}^{\pm}\|_{X_{\pm\rho}^{\pm\sigma} \to X_{\pm\rho}^{\pm\sigma}} \leq \frac{1}{\rho}, \qquad \rho > 0, \qquad \sigma \geq 0.$$

We observe that the choice of spaces with exponential weights for the solution of (IV.2.33) is a natural option as the later problem is obtained from the original ones via an exponential scalings such as (IV.2.8) and (IV.2.21). On the contrary, the solutions to the original problems are considered in spaces with no exponential weights, as we shall see in a section concerned with moments of solutions.

Consequently, from Lemma IV.2.1, the local in time classical solutions to (IV.2.33), with the initial data $w_0^{\pm} \in X^{\mathrm{sg}\,\alpha\,\sigma}_{\mathrm{sg}\,\alpha\,\rho}$, $\rho > 0$, $\sigma \geq 0$, are given explicitly by

$$w^{\pm}(\xi,\tau^{\pm}) = w_0^{\pm}(\xi) + m\tau^{\pm}\mathcal{G}^{\mathrm{sg}\,\alpha}\Big[{}_1F_1\Big(1-(\mathrm{sg}\,\alpha)m;2;-\tau^{\pm}(\cdot-\xi)\Big)w_0^{\pm}\Big](\xi), \tag{IV.3.2}$$

where

$${}_1F_1(a;b;z) = \sum_{n\geq 0}\frac{(a)_n z^n}{(b)_n n!},$$

is the Kummer confluent hypergeometric function of the first kind, [1, Formula 13.1.2]. We note that, using Kummer transformation, see [1, Formula 13.1.27] the confluent hypergeometric function can be expressed as

$${}_1F_1(a;b;z) = e^z{}_1F_1(b-a;b;-z). \tag{IV.3.3}$$

For $\alpha < 0$, we have the kernel in (IV.3.2) as ${}_1F_1\Big(1+m;2;\tau^{\pm}(\xi-\cdot)\Big)$ and it is positive and the solution is positive for non-negative input data $w_0^{\pm}$. On the other hand, for $\alpha > 0$, it is not obvious that the kernel is positive. Hence, using the Kummer transformation of the confluent hypergeometric function, we have the kernel as ${}_1F_1\Big(1+m;2;\tau^{\pm}(\cdot-\xi)\Big)$ and integrating from $\xi$ to $\infty$. Similarly, the kernel is positive and clearly the solution will remain positive for positive initial data.

We use (IV.2.42b), and note that, since $\varphi(z) = e^{-(\mathrm{sg}\,\alpha)z}$,

$$\Phi(z) = \frac{1}{1+(\mathrm{sg}\,\alpha)z}.$$

While the series expansion defining $\Phi$ in (IV.2.38) converges only for $|z| < 1$, $\Phi$ is analytic everywhere except for $z = -(\mathrm{sg}\,\alpha)1$. Thus, $\phi(z) = (1+(\mathrm{sg}\,\alpha)z)^{(\mathrm{sg}\,\alpha)m}$ where, for non-integer $m > 0$, we assume that the functions are defined in $\mathbb{C}$, cut along the ray $(-\infty,-1]$, when $\alpha > 0$, or along $[1,+\infty)$, when $\alpha < 0$. Then, as long as the spectrum of $\tau^{\pm}\mathcal{G}^{\mathrm{sg}\,\alpha}$ does not intersect the respective line, we have

$$w^{\pm}(\xi,\tau^{\pm}) = (I + (\operatorname{sg}\alpha)\tau^{\pm}\mathcal{G}^{\operatorname{sg}\alpha})^{(\operatorname{sg}\alpha)m}[w_0^{\pm}](\xi). \tag{IV.3.4}$$

For the case of $\alpha > 0$, this point of view yields global in $\tau^{\pm} \in \mathbb{R}_+$ solutions $w^{\pm}$. To validate this fact, we prove the following proposition.

**Proposition IV.3.1.** *Assume $\alpha > 0$, then the unique classical solution to (IV.3.3) in $X^{\pm\sigma}_{\pm\rho}$, $\rho, \sigma > 0$, is given by*

$$w^{\pm}(\xi,t) = (I + t\mathcal{G}^{\pm})^m[w_0](\xi), \qquad t \in [0,T], \qquad \xi \in \mathbb{R}_+, \tag{IV.3.5}$$

*for any finite $T > 0$.*

*Proof.* By Lemma IV.2.1 we know that (IV.3.5) holds at least for $0 < T < \rho$. By solving the resolvent equation

$$\lambda f - t\mathcal{G}^{\pm} f = g, \tag{IV.3.6}$$

we obtain that a candidate for the resolvent of $t\mathcal{G}^{\pm}$ is given by

$$(\lambda I - t\mathcal{G}^{\pm})^{-1}[g](\xi) = \frac{t}{\lambda^2}\mathcal{G}^{\pm}[e^{\pm\frac{t}{\lambda}(\cdot-\xi)}g(\cdot)](\xi) + \frac{1}{\lambda}g(\xi), \quad \xi \in \mathbb{R}_+. \tag{IV.3.7}$$

To show this is a resolvent, we first evaluate the norm. Changing the order of integration, we obtain

$$\begin{aligned} &\int_0^{\infty} \xi^{\pm\sigma} e^{\pm\rho\xi} \left|\mathcal{G}^{\pm}[e^{\pm\frac{t}{\lambda}(\cdot-\xi)}g(\cdot)](\xi)\right| d\xi \\ &\le \begin{cases} \int_0^{\infty}\left(e^{\frac{t\Re\lambda}{|\lambda|^2}\eta}|g(\eta)|\int_0^{\eta}\xi^{\sigma}e^{\left(\rho-\frac{t\Re\lambda}{|\lambda|^2}\right)\xi}d\xi\right)d\eta, & \text{if} \quad \alpha > 0, \\ \int_0^{\infty}\left(e^{-\frac{t\Re\lambda}{|\lambda|^2}\eta}|g(\eta)|\int_{\eta}^{\infty}\xi^{-\sigma}e^{-\left(\rho-\frac{t\Re\lambda}{|\lambda|^2}\right)\xi}d\xi\right)d\eta, & \text{if} \quad \alpha < 0. \end{cases} \end{aligned}$$

Using the monotonicity of $x^{\pm\sigma}$ on the respective intervals we factor it out from the inner integrals. Furthermore, we observe that if $\rho - \frac{t\Re\lambda}{|\lambda|^2} > 0$, then both integrals can be computed explicitly, leading to

$$\left\|\mathcal{G}^{\pm}[e^{\pm\frac{t}{\lambda}(\cdot-\xi)}g(\cdot)]\right\|_{X^{\pm\sigma}_{\pm\rho}} \leq \frac{|\lambda|^2}{\rho|\lambda|^2 - t\Re\lambda}\|g\|_{X^{\pm\sigma}_{\pm\rho}}. \tag{IV.3.8}$$

Hence, the resolvent $(I - t\mathcal{G}^{\pm})^{-1}$ exists in $X^{\pm\sigma}_{\pm\rho}$ as long as the left hand side is finite. Thus, the spectrum of $t\mathcal{G}^{\pm}$ is contained in $\left\{\lambda \in \mathbb{C};\ \left(\Re\lambda - \frac{t}{2\rho}\right)^2 + (\Im\lambda)^2 \leq \frac{t^2}{4\rho^2}\right\}$, which belongs to the closed right complex half-plane. Also, the spectrum of $0\mathcal{G}^{\pm}$ is 0. Thus, for any function $\mathcal{G}$ that is analytic in an open set containing the spectrum of $t\mathcal{G}^{\pm}$ (for a fixed $t \geq 0$), we can evaluate $\Phi(t\mathcal{G}^{\pm})$ by means of the Dunford integral

$$\mathcal{F}(t\mathcal{G}^{\pm}) = \frac{1}{2\pi i}\int_{\mathcal{Z}} \mathcal{F}(z)(I - t\mathcal{G}^{\pm})^{-1}dz,$$

where $\mathcal{Z}$ is a curve surrounding the spectrum of $t\mathcal{G}^{\pm}$ in a positive direction. We note that if we change $t$ from $0$ to $T$, then the spectra of $t\mathcal{G}^{\pm}$ will continuously change from 0 to the disc centred at $\left(\frac{T}{2\rho}, 0\right)$ with radius $\frac{T}{2\rho}$, so each one will be contained in the latter. By the analyticity of the resolvent, we can define a smooth function $[0,T] \ni t \mapsto \mathcal{F}(t\mathcal{G}^{\pm})$ for any $0 < T < \infty$, provided the analyticity domain of $\mathcal{F}$ includes the largest spectral disc of $t\mathcal{G}^{\pm}$. Since the functions $z \mapsto (1+tz)^m, m > 0, t \in [0,T]$ are analytic in $\mathbb{C}$ with the cut along the negative ray $\{z \in \mathbb{C};\ \Re z > -1/T\}$, the solution (IV.3.5) can be extended to $[0,T]$ for any $T > 0$. □

We point out that the appearance of the weight $\xi^{-\sigma}$ in the case $\alpha < 0$ are natural if one has in mind transformation (IV.2.9) for $z$.

## 3.1 Explicit solutions

Backward substitution to the original coordinates shows that the solutions to (IV.2.1) are given explicitly by

$$\begin{aligned} u^{\pm}(x,t) = \exp\Big\{\mp kt \mp \frac{ax^{\alpha}}{k\alpha}(1 - e^{\mp k\alpha t})\Big\}\Big[u_0^{\pm}(xe^{\mp kt}) \pm \frac{a(\nu+2)}{k\alpha}(1 - e^{\mp k\alpha t}) \\ \times \int_x^{\infty} {}_1F_1\Big(1 - \frac{\nu+2}{\alpha}; 2; \frac{\mp a}{k\alpha}(1 - e^{\mp k\alpha t})(y^{\alpha} - x^{\alpha})\Big)x^{\nu}y^{\alpha-\nu-1}u_0^{\pm}(ye^{\mp kt})dy\Big]. \end{aligned} \tag{IV.3.9}$$

We remark at this point that (IV.3.9) follows from (IV.3.2) in a purely formal manner, hence it requires proper justification, which is done in Appendix A. We just state that (IV.3.9) are solutions in the sense of distributions. That is, these solutions satisfy (IV.2.1) in the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}_+^2)$, see [23], for initial data $u_0^\pm$ in $\mathcal{D}'(\mathbb{R}_+)$. As an immediate consequence, for the monodisperse initial data $u_0^\pm = \delta(x - x_0)$, $x_0 \in \mathbb{R}_+$, we have

$$\begin{aligned} u^\pm(x,t) = \exp\Big\{\mp\frac{ax_0^\alpha}{k\alpha}(e^{\pm k\alpha t}-1)\Big\}\Big[&\delta(x - x_0e^{\pm kt}) \\ &\pm \chi_{[0,x_0e^{\pm kt}]}(x)\frac{a(\nu+2)}{k\alpha}(e^{\pm k\alpha t}-1)x_0^{\alpha-\nu-1}x^\nu \\ &{}_1F_1\Big(1+\frac{\nu+2}{\alpha};2;\frac{\mp a}{k\alpha}(1-e^{\mp k\alpha t})(x^\alpha - x_0^\alpha e^{\pm k\alpha t})\Big)\Big]. \end{aligned} \tag{IV.3.10}$$

Expanding the right-hand side of (IV.3.10), we see that the term containing $\delta$-distribution describes the evolution and mass loss of the original particle of size $x_0$, while the term with the Kummer function provides a continuous mass distribution of daughter particles generated by multiple fragmentation and transport events.

**Remark.** Usually, stronger properties of solutions are obtained for regular initial data $u_0^\pm$. For example, if $(u_0^\pm)_x \in X_0^{p+1}$, $u_0^\pm \in X_0^{p+\alpha}$ and

$$\alpha > 0, \qquad p \geq \alpha - \nu - 1, \tag{IV.3.11a}$$

$$\alpha < 0, \qquad p \geq 0, \qquad p > 1 + \alpha, \tag{IV.3.11b}$$

then

$$u_t^\pm, (ru^\pm)_x, au \in C\big((0,T), X_0^p\big), \tag{IV.3.12}$$

and (IV.3.9) satisfies (IV.2.1) in the classical sense of $X_0^p$, where $X_0^p$ are the spaces introduced in (IV.3.1) with $\sigma = 0$ and $p = \pm\delta > 0$. For the proof, see Corollary A.0.2 of Appendix A.

## 3.2 Moments

The moments $M_p^{\pm}(t) := \|u^{\pm}(\cdot,t)\|_{X_0^p}$ of the solutions are of physical importance as they provide information about the global state of the system during its evolution. We note that functions $M_p^{\pm}$ coincide with norms $\|\cdot\|_{[p]}$ defined in (III.6.2). The zeroth moment $M_0^{\pm}(t)$ gives the number of particles in the system at time $t$, the first moment $M_1^{\pm}(t)$ describes the total mass of the system, while the higher order moments are related to the distribution of mass between small and large clusters. Moreover, the behaviour of the first moment is related to the occurrence of a phase transition phenomenon, known as "shattering", [41, 30], that describes an unaccounted for loss of mass from the system. In the context of fragmentation with growth or decay, shattering refers to the fact that the evolution of the total mass of the system is not determined solely by the mass growth/decay terms $\pm(r(x)u^{\pm}(x,t))_x$ built into the model, [24, 26] and [18, Section 5.2.7&8].

As mentioned in Section IV.3.1, for integrable input data $u_0^{\pm} \in X_0^p$, the nonnegative $p$-th order moments $M_p^{\pm}(t)$ are well defined and remain finite at each instance of time $t \geq 0$, only if the inequalities (IV.3.11a)-(IV.3.11b) are satisfied. The interesting difference in the behaviour of higher and lower order moments of $u^{\pm}$, $t > 0$, occurs when $\alpha > 0$ and when $\alpha < 0$. It follows from (A.0.9a) of Corollary A.0.2 that in the first case all higher order moments $p \leq q \leq p+\nu+2$ become finite instantly at $t > 0$. In contrast, for $\alpha < 0$ only the lower order moments $1+\alpha < q \leq p$ and $0 \leq q$, remain well defined. The first scenario is related to the moment regularization found recently in [22, 16] (see also Section III.4 of the thesis) induced by the fragmentation rate $a$, with $\alpha > 0$.

**Remark.** The explicit formulae (IV.3.9) and (IV.3.10) allow for direct calculations of nonnegative-order moments for integrable and monodisperse input data $u_0^{\pm}$, respectively. Nevertheless, in view of the linearity of the model (IV.2.1) and of the moment functionals $M_p^{\pm}(t)$, $p \geq 0$, we present computations only for the latter case. For general integrable data $u_0^{\pm} \in X_0^p$, the dynamics of $M_p(t)$ can be read off the monodisperse case via the standard superposition principle.

To emphasize the dependence on the initial data, in the monodisperse case, we denote $M_p^{\pm}(t,x) :=$

$\|u^{\pm}(\cdot,t)\|_{X_0^p}$. Then, integrating (IV.3.10) using the identities (A.0.4b) and (A.0.5), for the general value of $p \geq 0$ satisfying (IV.3.11a)–(IV.3.11b), we obtain

$$\begin{aligned} M_p^{\pm}(t,x_0) &= \exp\Big\{\pm pkt \mp \frac{ax_0^{\alpha}}{k\alpha}(e^{\pm k\alpha t}-1)\Big\}x_0^p \\ &\quad \times {}_1F_1\Big(\frac{\nu+2}{\alpha};\frac{p+\nu+1}{\alpha};\frac{\pm ax_0^{\alpha}}{k\alpha}(e^{\pm k\alpha t}-1)\Big), \quad \alpha>0 \end{aligned} \quad \text{(IV.3.13a)}$$

and

$$\begin{aligned} M_p^{\pm}(t,x_0) &= \frac{\Gamma\big(\frac{\alpha-p+1}{\alpha}\big)}{\Gamma\big(\frac{\alpha-p-\nu-1}{\alpha}\big)}\exp\Big\{\pm pkt \mp \frac{ax_0^{\alpha}}{k\alpha}(e^{\pm k\alpha t}-1)\Big\}x_0^p \\ &\quad \times \Psi\Big(\frac{\nu+2}{\alpha};\frac{p+\nu+1}{\alpha};\frac{\pm ax_0^{\alpha}}{k\alpha}(e^{\pm k\alpha t}-1)\Big), \qquad \alpha<0, \end{aligned} \quad \text{(IV.3.13b)}$$

where $\Psi(a;b;z)$ is the Kummer hypergeometric function of the second kind, see [1, Formulae 13.2.5 and 13.1.29]. In particular, letting $p=1$ in (IV.3.13a)-(IV.3.13b), we see that the total mass evolution of the particle system associated to a monodisperse initial distribution is given by the formulae

$$M_1^{\pm}(t,x_0) = e^{\pm kt}x_0, \qquad \alpha>0, \quad \text{(IV.3.14a)}$$

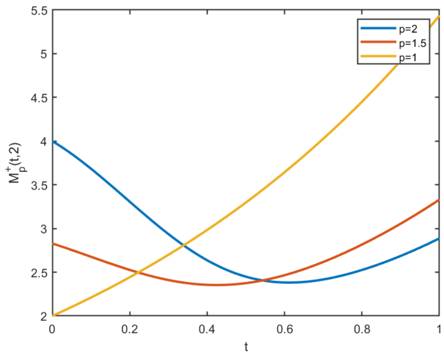
$$M_1^{\pm}(t,x_0) = \frac{e^{\pm kt}x_0}{\Gamma\big(1-\frac{\nu+2}{\alpha}\big)}\Gamma\Big(1-\tfrac{\nu+2}{\alpha};\tfrac{\pm ax_0^{\alpha}}{k\alpha}(e^{\pm k\alpha t}-1)\Big), \qquad \alpha<0, \quad \text{(IV.3.14b)}$$

where $\Gamma(a;z)=\int_z^{\infty} e^{-s}s^{a-1}ds$, $a>0$, is the incomplete gamma function (see [1, Formula 6.5.3, p. 260]). Since (IV.3.14a) describes also the the total mass evolution due to the transport terms $\pm k(xu(x,t))_x$, we see that there is no shattering if $\alpha>0$. In contrast, (IV.3.14b) shows that shattering occurs in both decay and the growth scenarios for $\alpha<0$. However, in the growth case, on account of the asymptotic identity

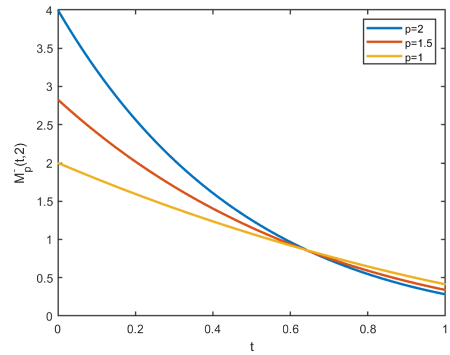
$$\Gamma(a;x) = \mathcal{O}(x^{a-1}e^{-x}), \quad x\to+\infty,$$

[1, Formula 6.5.32], for large values of $t$ shattering is dominated by the linear growth and in this case we have

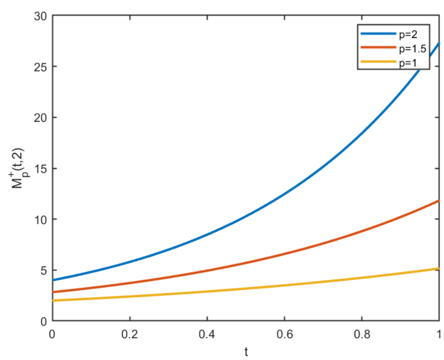
$$\lim_{t\to\infty} M_1^{+}(t,x_0) = \infty, \qquad x_0\in\mathbb{R}_+, \qquad \alpha<0.$$



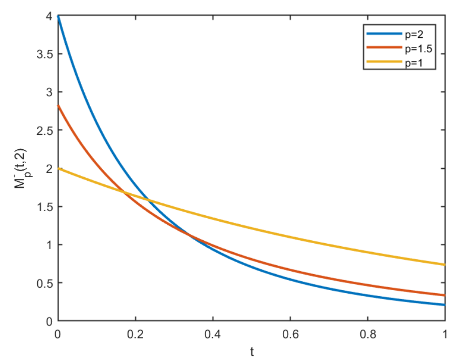

(a) Moments given by $k = a = 1$, $\nu = \frac{-3}{2}$, $\alpha = 2$.




(b) Moments given by $k = a = 1$, $\nu = \frac{-3}{2}$, $\alpha = 2$.




(c) Moments given by $k = a = 1$, $\nu = \frac{-3}{2}$, $\alpha = -2$.




(d) Moments given by $k = a = 1$, $\nu = \frac{-3}{2}$, $\alpha = -2$.

Figure 3.1: Evolution of moments

The behaviour of moments in all four cases covered by (IV.2.1), with the monodisperse initial data $u_0^{\pm}(x) = \delta_2(x)$, is shown in Fig 3.1.

## 3.3 Non-uniqueness

It is a known fact that pure fragmentation ($k = 0$) equation (IV.2.1), with $\alpha > 0$, has multiple solutions satisfying the same initial data. This aspect was first observed in [3], and it was explained in [9, 10], where the author also shown that it is related to the non-maximality of the generator of the semigroup associated with the fragmentation equation.

A similar phenomenon occurs in the context of growth/decay-fragmentation model (IV.2.1), for general $k > 0$. Indeed, separating variables in (IV.2.33) and using change of variables (A.0.10a)-

(A.0.10b), we infer that the functions

$$\hat{u}^{\pm}(x,t) = x^{\nu} \int_0^{\infty} \frac{\exp\{\mp k(\nu+1)t \pm \frac{\mu}{k\alpha}(e^{\pm k\alpha t}-1)\}}{\left(\frac{\mu}{a} + x^{\alpha} e^{\mp k\alpha t}\right)^{\frac{\alpha+\nu+2}{\alpha}}} \hat{u}_0^{\pm}(\mu) d\mu, \tag{IV.3.15}$$

with $\alpha > 0$, satisfying (IV.2.1) pointwise, for any $\hat{u}_0^{\pm} \in L_1(\mathbb{R}_+)$. It is an exercise to verify that these solutions are $p$-integrable, provided $-(1+\nu) < p < 1+\alpha$; and are classical in $X_0^p$, provided $-(1+\nu) < p < 1$. In the former case, the $p$-th order moments $\hat{M}_p^{\pm}(t) := \|\hat{u}^{\pm}(\cdot,t)\|_{X_0^p}$ are well defined and are given by the formula

$$\hat{M}_p^{\pm}(t) = \frac{a^{\frac{2-p}{\alpha}}}{\alpha} B\left(\frac{p+\nu+1}{\alpha}, \frac{\alpha-p+1}{\alpha}\right) \int_0^{\infty} \exp\{\pm pkt \pm \frac{\mu}{k\alpha}(e^{\pm k\alpha t}-1)\} \mu^{\frac{p-2}{\alpha}} \hat{u}_0^{\pm}(\mu) d\mu.$$

Hence, we see that in both the growth and the decay cases, the total mass of the system described by (IV.3.15) is magnified by the spurious factor $\exp\{\pm \frac{\mu}{k\alpha}(e^{\pm k\alpha t}-1)\}$, rendering these solutions physically infeasible.

# 4 Constant growth and decay fragmentation solutions

As in (A.0.10a)–(A.0.10b), in this section our focal point is only on deriving the formulae for solutions to (IV.2.22).

We have four cases stated in paragraph *Constant growth or decay* of Section IV.2.1. However, the cases (ii) and (iii) do not require boundary conditions and both are confined to the first quadrant, as in Section IV.3.

## 4.1 Cases of $\pm\beta < 0$.

This scenario covers items (ii) and (iii), that is, the decay case with $\alpha > 0$ and the growth case with $\alpha < 0$ in (IV.2.1) and (IV.2.4). Here, problem (IV.2.22) takes the form

$$\begin{aligned} w_t^{\pm}(\xi,t) &= m\mathcal{G}^{\operatorname{sg}\alpha}\big[e^{-t(\cdot-\xi)}w^{\pm}(\cdot,t)\big](\xi), && \text{(IV.4.1a)} \\ w^{\pm}(\xi,0) &= w_0(\xi), \qquad \xi \in \mathbb{R}_+. && \text{(IV.4.1b)} \end{aligned}$$

In this case, as in (IV.3.4), the local solutions are given by

$$w^{\pm}(\xi,t) = (I + (\operatorname{sg}\alpha)t\mathcal{G}^{\operatorname{sg}\alpha})^{(\operatorname{sg}\alpha)m}[w_0](\xi). \tag{IV.4.2}$$

The scenario is completely identical to that considered in detail in Section IV.3. We mention that, in consideration of Lemma IV.2.1, the solutions (IV.4.2) are classical for small values of $t > 0$, in the sense of spaces $X^{(\operatorname{sg}\alpha)\sigma}_{(\operatorname{sg}\alpha)\rho}$ from (IV.3.1). Further, on the account of Proposition IV.3.1, (IV.4.2) with $\alpha > 0$ holds for any finite value of $t > 0$. The explicit solutions to (IV.2.1) and (IV.2.4) follow from (IV.4.2) by backward substitution from the characteristic $(\xi, t)$ back to the physical $(x, t)$ variables. We have

$$\begin{aligned} &u^{-}(x,t) = 0, \quad x^{\alpha} < -k\alpha t, \qquad \alpha < 0; \\ &u^{\pm}(x,t) = e^{\pm\frac{k\alpha a t^2}{2} - a x^{\alpha} t}\Bigg[\left(1 \mp \frac{k\alpha t}{x^{\alpha}}\right)^{\frac{1-\alpha}{\alpha}} u_0\left(x\left[1 \mp \frac{k\alpha t}{x^{\alpha}}\right]^{\frac{1}{\alpha}}\right) + a(1+\alpha)t x^{\alpha-1} \\ &\qquad \times \int_{(x^{\alpha} \mp k\alpha t)^{\frac{1}{\alpha}}}^{\infty} {}_1F_1\left(-\frac{1}{\alpha}; 2; at[x^{\alpha} \mp k\alpha t - y^{\alpha}]\right) u_0(y)\,dy\Bigg], \quad x^{\alpha} \geq \pm k\alpha t. \end{aligned} \tag{IV.4.3}$$

The formulae in (IV.4.3) give solution in the decay scenario $(-)$ and in the growth $(+)$, with $\alpha < 0$.

## 4.2 The case of $\pm\beta > 0$ and $\alpha < 0$.

This is only possible if $\beta < 0$, i.e., we deal with (i) – the decay with $\alpha < 0$ in (IV.2.1) and (IV.2.4). Then (IV.2.22) takes the form

$$w_t^-(\xi,t) = m\mathcal{G}^-\left[e^{-t(\cdot-\xi)}w^-(\cdot,t)\right](\xi) = m\int_0^\xi e^{-t(\eta-\xi)}w^-(\eta,t)d\eta, \tag{IV.4.4a}$$

$$w^-(\xi,0) = \phi(\xi), \tag{IV.4.4b}$$

with $\xi \in \mathbb{R}$, $\phi(\xi) = v_0(\xi) + \psi(\xi)$, where $v_0$ is assumed to be extended by 0 to $\mathbb{R}_-$, $\psi(\xi) = 0$ for $\mathbb{R}_+$ and must be computed on $\mathbb{R}_-$, so that

$$w^-\left(\xi, \beta^{-1}\xi\right) = 0, \qquad \xi \in \mathbb{R}_-. \tag{IV.4.4c}$$

Having in mind the analogous representation of the solution, given by (IV.2.42a), we see that $(I - t\mathcal{G}^-)^{-m}[v_0](\xi)$ converges to zero as $\xi \to 0^+$ if so does $v_0$. This can be ascertained, at least for small $t > 0$, by taking the series expansion of $(I - t\mathcal{G}^-)^{-m}[v_0](\xi)$ and noting that its terms are $v_0$ and integrals from $0$ to $\xi$. Hence, $(I - t\mathcal{G}^-)^{-m}[v_0](\xi)$ can be continuously extended by $0$ to $\mathbb{R}_-$ for any $t \geq 0$ and a quick reflection leads us to the conclusion that the function

$$w^-(\xi,t) = \begin{cases} 0 & \text{for} \quad \xi \in \mathbb{R}_-, \\ (I - t\mathcal{G}^-)^{-m}[v_0](\xi) & \text{for} \quad \xi \in \mathbb{R}_+, \end{cases} \tag{IV.4.5}$$

is a solution to (IV.4.4a).

**Remark.** This result can be better understood if we take into account [12, Theorem 9.4] and comment (i) in Section IV.2. Indeed, according to the former, if $\gamma > 1$, then the characteristics of the transport term fill only the region in $\mathbb{R}_+ \times \mathbb{R}_+$ bounded by the limit characteristic $t = \frac{x^{1-\gamma}}{k(\gamma-1)}$ and the solutions vanish identically outside of it. Since for $\mu = 0$ we have $\alpha = 1 - \gamma$, on account of (IV.2.9) and (IV.2.14), this characteristic is transformed into

$$t = -\frac{1}{k\alpha a}ax^\alpha = -\frac{1}{\beta}z,$$

which is precisely the limiting characteristic (IV.2.18), separating the region of influence of the initial condition from the region of influence of the boundary condition in (IV.2.17).

We illustrate the above with an example for $\alpha < 0$, with $\alpha = 1 - \gamma$. For simplicity, we set $a = k = 1$ and we set $\alpha = -\frac{1}{2}$. Then $\gamma = \frac{3}{2} = -\nu$, $m = 1$ and $\beta = -\frac{1}{2}$.

In original coordinates, we consider

$$u_t^-(x,t) - (x^{\frac{3}{2}} u^-(x,t))_x = -x^{-\frac{1}{2}} u^-(x,t) + \frac{1}{2x^{\frac{3}{2}}} \int_x^\infty u^-(y,t)dy$$
$$u^-(x,0) = u_0^-(x), \quad x \in \mathbb{R}_+.$$

Passing to $v^-(z,t)$, as explained in Section IV.2, we obtain

$$v_t^-(z,t) + \frac{1}{2} v_z^-(z,t) = -z v^-(z,t) + \int_0^z v^-(s,t)ds$$
$$v^-(z,0) = v_0^-(z) = z^{-3} u_0^-(z^{-2}).$$

The characteristics equation is

$$z(\xi,t) = \frac{1}{2}t + \xi,$$

so that setting $g^-(\xi,t) = v^-(z(\xi,t),t)$, we have

$$g_t^-(\xi,t) = -\left(\xi + \frac{1}{2}t\right) g^-(\xi,t) + m\mathcal{G}^-[g^-(\cdot,t)](\xi),$$
$$g^-(\xi,0) = g_0^-(\xi) = v_0^-(\xi).$$

We further set $w^-(\xi,t) = e^{\frac{t^2}{4} + \xi t} g^-(\xi,t)$, which leads to

$$w_t^-(\xi,t) = m\mathcal{G}^-[e^{-t(\cdot-\xi)} w^-(\cdot,t)](\xi).$$

The solution is given by

$$w^-(\xi,t) = (I - t\mathcal{J}^-)^{-1}[w_0^-](\xi),$$

which, using (IV.3.7) with $\lambda = 1$, can be rewritten as

$$w^-(\xi,t) = w_0^-(\xi) + t\int_0^{\xi} e^{-t(s-\xi)} w_0^-(s)ds. \tag{IV.4.6}$$

For the exact solution, we use the monodisperse initial condition $u_0(x) = \delta(x - x_0)$. For any test function $\phi$ we have

$$\begin{aligned}\int_0^{\infty} u_0(z)\phi(z)dz &= \int_0^{\infty} z^{-3}\delta(z^{-2} - x_0)\phi(z)dz \\ &= \frac{1}{2}\int_0^{\infty} \delta(p - x_0^{-\frac{1}{2}})\phi(p)dp.\end{aligned} \tag{IV.4.7}$$

Using (IV.4.7), we have

$$w^-(\xi,t) = \frac{1}{2}\delta(\xi - x_0^{-\frac{1}{2}}) + \left[\frac{t}{2}e^{t\left(\xi - x_0^{-\frac{1}{2}}\right)}\right]\mathcal{X}_{[x_0^{-\frac{1}{2}},\infty)}(\xi)$$

and then

$$g^-(\xi,t) = e^{-\frac{t^2}{4}-\xi t}\left(\frac{1}{2}\delta(\xi - x_0^{-\frac{1}{2}}) + \left[\frac{t}{2}e^{t\left(\xi - x_0^{-\frac{1}{2}}\right)}\right]\mathcal{X}_{[x_0^{-\frac{1}{2}},\infty)}(\xi)\right).$$

Then,

$$\begin{aligned}v^-(z,t) &= e^{-\frac{t^2}{4}-(z-\frac{t}{2})t}\left(\frac{1}{2}\delta(z - \tfrac{t}{2} - x_0^{-\frac{1}{2}}) + \left[\frac{t}{2}e^{t\left(z-\frac{t}{2} - x_0^{-\frac{1}{2}}\right)}\right]\mathcal{X}_{[x_0^{-\frac{1}{2}},\infty)}(z - \tfrac{t}{2})\right) \\ &= e^{-\frac{t^2}{4}-(z-\frac{t}{2})t}\frac{1}{2}\delta(z - \tfrac{t}{2} - x_0^{-\frac{1}{2}}) + \left[\frac{t}{2}e^{-t\left(\frac{t}{4} + x_0^{-\frac{1}{2}}\right)}\right]\mathcal{X}_{[x_0^{-\frac{1}{2}}+\frac{t}{2},\infty)}(z).\end{aligned}$$

Finally, backward substitutions to the original coordinates yields

$$\begin{aligned}u^-(x,t) &= x^{-\frac{3}{2}}\left[e^{-\frac{t^2}{4}-(x^{-\frac{1}{2}}-\frac{t}{2})t}\frac{1}{2}\delta(x^{-\frac{1}{2}} - \tfrac{t}{2} - x_0^{-\frac{1}{2}}) + \left[\tfrac{t}{2}e^{-t\left(\frac{t}{4}+x_0^{-\frac{1}{2}}\right)}\right]\mathcal{X}_{[x_0^{-\frac{1}{2}}+\frac{t}{2},\infty)}(x^{-\frac{1}{2}})\right] \\ &= e^{-t\left(\frac{t}{4}+\frac{1}{\sqrt{x_0}}\right)}\left(\delta\left(x - \frac{4x_0}{(\sqrt{x_0}t+2)^2}\right) + \frac{tx^{-\frac{3}{2}}}{2}\mathcal{X}_{[0,\frac{4x_0}{(t\sqrt{x_0}+2)^2}]}(x)\right),\end{aligned} \tag{IV.4.8}$$

where $\mathcal{X}$ is the characteristic function of the indicated set.

## 4.3 The case of $\pm\beta > 0$ and $\alpha > 0$.

Here, $\beta > 0$ and we have the growth case with $\alpha > 0$ in (IV.2.1), that is, case (iv). Then the problem (IV.2.22) takes the form

$$w_t^+(\xi,t) = m\mathcal{G}^+\big[e^{-t(\cdot-\xi)}w^+(\cdot,t)\big](\xi) = m\int_\xi^\infty e^{-t(\eta-\xi)}w^+(\eta,t)d\eta, \qquad \text{(IV.4.9a)}$$

$$w^+(\xi,0) = \phi(\xi), \qquad \text{(IV.4.9b)}$$

with $\xi \in \mathbb{R}$, $\phi(\xi) = v_0(\xi) + \psi(\xi)$, where, as before, $v_0$ is extended by $0$ to $\xi < 0$, $\psi(\xi) = 0$ for $\xi > 0$, but must be computed for $\xi < 0$, so that

$$w^+\left(\xi, -\beta^{-1}\xi\right) = 0, \qquad \xi < 0, \qquad \text{(IV.4.9c)}$$

is satisfied. The operator $\mathcal{G}^+$ is bounded on the space $X^{+\rho} := L_1(\mathbb{R}, e^{\rho x}dx)$ for any $\rho > 0$. The solution to (IV.4.4) is given by

$$w^+(\xi,t) = (I + t\mathcal{G}^+)^m[\phi](\xi), \qquad \xi \in \mathbb{R}, \qquad t \in [0,T], \qquad \text{(IV.4.10)}$$

for any $T < \infty$. Thus, we have

$$w^+(\xi,t) = (I + t\mathcal{G}^+)^m[v_0](\xi), \quad \xi > 0. \qquad \text{(IV.4.11)}$$

For $\xi < 0$, we note that the formal resolvent of $\mathcal{G}^+$ is given by (IV.3.7), with $\xi \in \mathbb{R}$ and the spectrum of $t\mathcal{G}^+$ is contained in $\left\{\lambda \in \mathbb{C};\ \left(\Re\lambda - \frac{t}{2\rho}\right)^2 + (\Im\lambda)^2 \le \frac{t^2}{4\rho^2}\right\}$. Thus, the solution to (IV.4.9) is given by the formula (IV.3.5)

$$w^+(\xi,t) = (I + t\mathcal{G}^+)^m[\phi](\xi), \qquad t \in [0,T],$$

for any $T < \infty$ but extended to $\xi \in \mathbb{R}$.

Later, we shall need the solution to

$$\lambda f(\xi) - t\int_{\xi}^{0} f(\eta)d\eta = \lambda f(\xi) - t\mathcal{G}[f](\xi) = g(\xi), \qquad \xi \in \mathbb{R}_-, \tag{IV.4.12}$$

which is given by

$$f(\xi) = \frac{t}{\lambda^2} e^{-\frac{t}{\lambda}\xi} \int_{\xi}^{0} e^{\frac{t}{\lambda}\eta} g(\eta)d\eta + \frac{1}{\lambda} g(\xi). \tag{IV.4.13}$$

Using the Dunford integral representation for (IV.4.10), we obtain

$$\begin{aligned} w^+(\xi,t) &= (I+t\mathcal{G}^+)^m[\phi](\xi) = \frac{1}{2\pi i}\int_{\mathcal{C}} (1+z)^m (zI - t\mathcal{G}^+)^{-1}[\phi](\xi)dz \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} (1+z)^m \left( \frac{t}{z^2} e^{-\frac{t}{z}\xi} \int_{\xi}^{\infty} e^{\frac{t}{z}\eta}\phi(\eta)d\eta + \frac{1}{z}\phi(\xi)\right) dz. \end{aligned} \tag{IV.4.14}$$

Recall that for $\xi > 0$, we have $\phi(\xi) = v_0(\xi)$, with known $v_0$. Hence, for such $\xi$, (IV.4.14) provides a complete solution, given by (IV.4.11). Next, using the definition of $\phi$, for $\xi < 0$, we write

$$\begin{aligned} w^+(\xi,t) &= \frac{1}{2\pi i}\int_{\mathcal{C}} (1+z)^m \left( \frac{t}{z^2} e^{-\frac{t}{z}\xi} \int_{0}^{\infty} e^{\frac{t}{z}\eta} v_0(\eta)d\eta \right) dz \\ &\quad + \frac{1}{2\pi i}\int_{\mathcal{C}} (1+z)^m \left( \frac{t}{z^2} e^{-\frac{t}{z}\xi} \int_{\xi}^{0} e^{\frac{t}{z}\eta}\psi(\eta)d\eta + \frac{1}{z}\psi(\xi)\right) dz \\ &=: F(\xi,t) + (I+t\mathcal{G})^m[\psi](\xi), \end{aligned} \tag{IV.4.15}$$

where

$$\mathcal{G}[f](\xi) := \int_{\xi}^{0} f(\eta)d\eta. \tag{IV.4.16}$$

Thus, in view of (IV.4.9c), we get

$$\begin{aligned} 0 &= (I+t\mathcal{G}^+)^m[v_0](\xi)|_{t=-\beta^{-1}\xi} + (I+t\mathcal{G})^m[\psi](\xi)|_{t=-\beta^{-1}\xi} \\ &= F(\xi, -\beta^{-1}\xi) + (I+t\mathcal{G})^m[\psi](\xi)|_{t=-\beta^{-1}\xi}. \end{aligned} \tag{IV.4.17}$$

The problem is that the operators $(I+t\mathcal{G}^+)^m$ and $(I+t\mathcal{G})^m$ are nonlocal and their evaluation at a given $t$ for general $m$ is quite involved. Thus we shall proceed under a simplifying assumption that $m \in \mathbb{N}$.

Let us introduce the (probabilistic) Hermite polynomials, see [1, Section 20.3],

$$He_m(\zeta) = m! \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^i}{i!(m-2i)!} \frac{\zeta^{m-2i}}{2^i}. \tag{IV.4.18}$$

Then, as shown in Lemma A.0.3, the unique solution $\psi \in X^{+\rho}$ to (IV.4.17) reads

$$\begin{aligned} \psi(\xi) &= (-1)^m \frac{d^m}{d\xi^m} \left( e^{-\frac{\xi^2}{2\beta}} y\left(\frac{\xi}{\sqrt{\beta}}\right)\right) \\ &= (-1)^m \beta^{\frac{m}{2}} \frac{d^m}{d\zeta^m} \left( e^{-\frac{\zeta^2}{2}} y(\zeta)\right)\Big|_{\zeta = \frac{\xi}{\sqrt{\beta}}}, \end{aligned} \tag{IV.4.19a}$$

where $y$ is given by

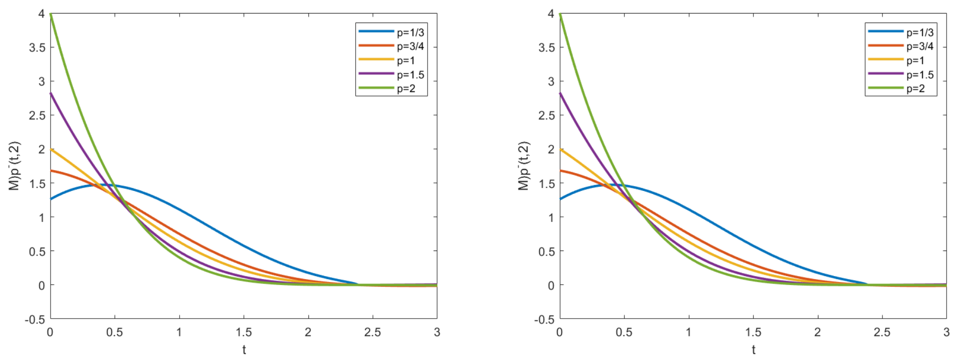
$$y(\zeta) = \int_0^\zeta \left( \sum_{i=1}^m \frac{1}{(He_m)'(\lambda_{m,i})} e^{\lambda_{m,i}(\zeta-\sigma)} \right) e^{\frac{\sigma^2}{2}} g(\sigma) d\sigma, \tag{IV.4.19b}$$

$\lambda_{m,1}, \ldots, \lambda_{m,m}$ are simple real roots of $He_m$ and $g(\zeta) = -\beta^{\frac{m}{2}} F\left(\zeta\beta^{\frac{1}{2}}, -\beta^{-\frac{1}{2}}\zeta\right)$.

## 4.4 Solutions

As mentioned in Section IV.4.1, the formulae for solutions in the constant growth/decay case are based on the same representation (IV.4.2) as in Section IV.3 and, as a result, their properties are established as in Appendix A.

In this section, we provide formulae for the solutions in the decay case of (IV.2.1)–(IV.2.4). Note that since we cover both positive and negative $\alpha$, some restrictions below are redundant. The




(a) Moments given by $k = a = 1$, $\nu = \frac{1}{3}$, $\alpha = \frac{2}{3}$

(b) Moments given by $k = a = 1$, $\nu = \frac{15}{8}$, $\alpha = \frac{-7}{8}$

Figure 4.1: The dynamics of the moments

explicit solutions to (IV.2.1)–(IV.2.4) in the decay case are given by

$$
\begin{aligned}
u(x,t) &= 0, \quad x^\alpha < -k\alpha t, \quad x,t \in \mathbb{R}_+, \\
u(x,t) &= e^{-\frac{1}{2}ka\alpha t^2 - ax^\alpha t}\Bigg[\Big(1 + \frac{k\alpha t}{x^\alpha}\Big)^{\frac{1-\alpha}{\alpha}} u_0\Bigg(x\Big(1+\frac{k\alpha t}{x^\alpha}\Big)^{\frac{1}{\alpha}}\Bigg) \\
&\qquad + a(\alpha+1)x^{\alpha-1}t \int_{(x^\alpha + k\alpha t)^{\frac{1}{\alpha}}}^{\infty} {}_1F_1\Bigg(-\frac{1}{\alpha}; 2; at(x^\alpha + k\alpha t - y^\alpha)\Bigg) \\
&\qquad\qquad u_0(y)dy\Bigg], \quad x^\alpha \geq -k\alpha t, \quad x,t \in \mathbb{R}_+.
\end{aligned}
$$

Here, since $\mu = \theta = 0$, we have $\gamma = 1 - \alpha$ and $\nu = \alpha - 1$. In particular, for the monodisperse initial data $u_0(x) = \delta(x - x_0)$, we obtain,

$$
\begin{aligned}
u(x,t) &= 0, \quad x_0^\alpha < k\alpha t, \quad x,t \in \mathbb{R}_+, \\
u(x,t) &= e^{-\frac{1}{2}ka\alpha t^2 - ax^\alpha t}\Bigg[\delta\Bigg(x - x_0\Big(1 - \frac{k\alpha t}{x_0^\alpha}\Big)^{\frac{1}{\alpha}}\Bigg) \\
&\qquad + \chi_{[0,(x_0^\alpha - k\alpha t)^{\frac{1}{\alpha}}]} a(\nu+2)x^{\alpha-1}t\, {}_1F_1\Bigg(-\frac{1}{\alpha}; 2; at(x^\alpha + k\alpha t - x_0^\alpha)\Bigg)\Bigg], \\
&\qquad x_0^\alpha \geq k\alpha t, \quad x,t \in \mathbb{R}_+.
\end{aligned}
$$

Similarly to Section IV.3, we also provide formulae for the moments of solutions, focusing on the monodisperse initial data only. First, we consider $\alpha > 0$ and $p \geq 0$, in which case we have

$$
\begin{aligned}
&M_p^-(t, x_0) = 0, \quad x_0^\alpha < k\alpha t, \quad x_0, t \in \mathbb{R}_+, \\
&M_p^-(t, x_0) = e^{-\frac{1}{2}ka\alpha t^2} x_0^p \Big(1 - \frac{k\alpha t}{x_0^\alpha}\Big)^{\frac{p}{\alpha}} {}_1F_1\left(\frac{p-1}{\alpha}; \frac{p+\alpha}{\alpha}; -at(x_0^\alpha - k\alpha t)\right), \\
&\qquad\qquad x_0^\alpha \geq k\alpha t, \quad x_0, t \in \mathbb{R}_+.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&M_1^-(t, x_0) = 0, \quad x_0^\alpha < k\alpha t, \quad x_0, t \in \mathbb{R}_+, \\
&M_1^-(t, x_0) = e^{-\frac{1}{2}ka\alpha t^2} x_0 \Big(1 - \frac{k\alpha t}{x_0^\alpha}\Big)^{\frac{1}{\alpha}}, \quad x_0^\alpha \geq k\alpha t, \qquad x_0, t \in \mathbb{R}_+,
\end{aligned}
$$

and no shattering occurs.

Furthermore, when $-1 < \alpha < 0$ and $0 < 1 + \alpha < p$, the moments are given by

$$
\begin{aligned}
M_p^-(t, x_0) = &\frac{\Gamma\big(\frac{\alpha - p + 1}{\alpha}\big)}{\Gamma\big(\frac{p}{|\alpha|}\big)} e^{\frac{1}{2}ka\alpha t^2 - a x_0^\alpha t} x_0^p \Big(1 - \frac{k\alpha t}{x_0^\alpha}\Big)^{\frac{p}{\alpha}} \\
&\times \Psi\left(\frac{\alpha+1}{\alpha}; \frac{\alpha+p}{\alpha}; at(x_0^\alpha - k\alpha t)\right), \qquad x_0, t \in \mathbb{R}_+,
\end{aligned}
$$

and, in particular, the formula

$$
M_1^-(t, x_0) \;\; = \frac{e^{-\frac{1}{2}ka\alpha t^2}}{\Gamma\big(\frac{1}{|\alpha|}\big)} x_0 \Big(1 - \tfrac{k\alpha t}{x_0^\alpha}\Big)^{\frac{1}{\alpha}} \Gamma\Big(\tfrac{1}{|\alpha|}; at(x_0^\alpha - k\alpha t)\Big), \quad -1 < \alpha < 0,
$$

indicates that there exists spurious loss of mass not connected with the transport processes, i.e., we have shattering solutions for $-1 < \alpha < 0$.

The typical behaviour of moments, with the monodisperse initial data $u_0^-(x) = \delta(x - x_0)$, is shown in Fig 4.1.

# 5 Explicit solution with McKendrick-von Foerster boundary conditions

In this section, we consider the growth case (IV.2.1), equipped with the McKendrick-von Foerster boundary condition,

$$\lim_{x\to 0^+} r(x)u(x,t) = \int_0^\infty \beta(y)u(y,t)dy. \tag{IV.5.1}$$

As in Sections IV.2.1 and IV.4.3, we focus on the linear rate. However, this time, in order to obtain explicit solutions we assume that $\gamma = \nu = 0$, $\alpha = 1$ and the reproduction rate $\beta(x)$ is linear, i.e., $\beta(x) = \beta_0 + \beta_1 x$, for some $\beta_0, \beta_1 > 0$. Since $k$ is a positive constant, we set it to $1$ for convenience. Then, equation (IV.2.1) becomes

$$\begin{aligned}
\partial_t u(x,t) &= -\partial_x u(x,t) - axu(x,t) + 2a\int_x^\infty u(y,t)dy, \quad x,t \in (0,\infty),\\
u(x,0) &= u_0(x),\\
\lim_{x\to 0^+} u(x,t) &= \beta_0 M_0(t) + \beta_1 M_1(t).
\end{aligned} \tag{IV.5.2}$$

According to the theory presented in Chapter III, for $u_0 \in X_m^+ \cap D(Z_m^-)_+$, (IV.5.2) is globally well-posed in $\mathbb{R}_+$ and, furthermore, the associated solutions are classical. Using this fact, our assumption on the reproduction rate $\beta(x)$ in the McKendrick-von Foerster boundary condition allows for finding the zeroth and the first order moments explicitly. Since we assumed that $u_0 \in D(Z_{\beta,m})$, then $u \in C^1\big((0,\infty), D(Z_{\beta,m})\big)$, which enables changing the order of integration and differentiation leading to

$$\frac{d}{dt}\int_0^\infty u(x,t)dx = \frac{dM_0(t)}{dt} \quad \text{and} \quad \frac{d}{dt}\int_0^\infty xu(x,t)dx = \frac{dM_1(t)}{dt}.$$

Applying Lemma III.4.1, $\lim_{x\to\infty} x^i u(x,t) = 0$, $i = 0,1$. Hence, multiplying (IV.5.2) by $1$ and $x$ and using Lemma III.4.1, we obtain a system of two linear ordinary differential equations,

$$\begin{aligned}
\frac{dM_0(t)}{dt} &= \beta_0 M_0(t) + (\beta_1 + a)M_1(t),\\
\frac{dM_1(t)}{dt} &= M_0(t).
\end{aligned} \tag{IV.5.3}$$

To solve (IV.5.3), we rewrite the system in the matrix-vector form

$$\underline{M'} = \begin{pmatrix} M_0' \\ M_1' \end{pmatrix} = A\underline{M} = \begin{pmatrix} \beta_0 & \beta_1 + a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix}. \tag{IV.5.4}$$

Let $\alpha_0 = \beta_0 \neq 0$ and $\alpha_1 = \beta_1 + a \neq 0$. With this notation, the characteristics equation for $A$ reads

$$\begin{vmatrix} \alpha_0 - \lambda & \alpha_1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda - \alpha_0) - \alpha_1 = \lambda^2 - \alpha_0\lambda - \alpha_1, \tag{IV.5.5}$$

and the eigenvalues are given by

$$\lambda_\pm = \frac{\alpha_0 \pm \sqrt{\alpha_0^2 + 4\alpha_1}}{2}. \tag{IV.5.6}$$

The solution of (IV.5.3) is given by the formulae

$$\begin{aligned} M_0(t) &= M_0(0)\left[\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right] + M_1(0)\left[\frac{\lambda_+\lambda_-(e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-}\right] \\ &= M_0(0)K_1(t) + M_1(0)K_2(t). \\ M_1(t) &= M_0(0)\left[\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right] + M_1(0)\left[\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right] \\ &= M_0(0)K_3(t) + M_1(0)K_4(t). \end{aligned} \tag{IV.5.7}$$

Since, $\lambda_- < 0 < \lambda_+$, we have $K_i \geq 0$, $i = 1, 2, 3, 4$ for $t \in \mathbb{R}_+$. Hence, the moments $M_0(t)$ and $M_1(t)$ remain positive for a regular data $u_0$ from $D(Z_{\beta,m})_+$ for $t \geq 0$. We note that under our assumptions the results of Section IV.4 apply. In particular, the characteristic lines are given by $x = t + \xi$, and letting $w(\xi, t) = e^{at(\xi + \frac{t}{2})}u(t + \xi, t)$ as in Section IV.4.2, we obtain

$$w(\xi, t) = (I + at\mathcal{G}^+)^2[u_0](\xi) + (I + at\mathcal{G})^2[\psi](\xi), \tag{IV.5.8}$$

where, instead of (IV.4.9c), the unknown function $\psi$ is chosen so that

$$w(\xi, -\xi) = \beta_0 M_0(-\xi) + \beta_1 M_1(-\xi) \tag{IV.5.9}$$

is satisfied. We mention that in the relation of Section IV.2, we have $u(x,t) = v(ax,t)$, hence we maintain the use of $u$.

Thus, solution for $\xi > 0$ is

$$\begin{aligned} w(\xi,t) &= (I + at\mathcal{G}^+)^2[u_0](\xi) \\ &= u_0(\xi) + at\int_\xi^\infty [2 + at(s-\xi)]u_0(s)ds. \end{aligned} \tag{IV.5.10}$$

Since the characteristic lines do not cover all the first quadrant, for the extended solution, i.e., for $\xi < 0$, we have

$$\begin{aligned} w(\xi,t) &= (I + at\mathcal{G}^+)^2[u_0](\xi) + (I + at\mathcal{G})^2[\psi](\xi) \\ &= u_0(\xi) + 2at\int_\xi^\infty u_0(s)ds + (at)^2\int_\xi^\infty (s-\xi)u_0(s)ds \\ &+ \psi(\xi) + 2at\int_\xi^0 \psi(s)ds + (at)^2\int_\xi^0\int_s^0 \psi(p)dpds \\ &= at(2 - at\xi)M_0(0) + (at)^2M_1(0) + \psi(\xi) + 2at\int_\xi^0 \psi(s)ds + (at)^2\int_\xi^0\int_s^0 \psi(p)dpds. \end{aligned} \tag{IV.5.11}$$

To determine $\psi(\xi)$, for $\xi < 0$, we write

$$w(\xi,-\xi) = (I + at\mathcal{G}^+)^2[u_0](\xi)\Big|_{t=-\xi} + (I + at\mathcal{G})^2[\psi](\xi)\Big|_{t=-\xi},$$

which leads to

$$\begin{aligned} e^{\frac{a\xi^2}{2}}\Big[\beta_0M_0(-\xi) + \beta_1M_1(-\xi)\Big] &= -a\xi\big(2 + a\xi^2\big)M_0(0) + a^2\xi^2M_1(0) \\ &+ \psi(\xi) + 2at\int_\xi^0 \psi(s)ds + (at)^2\int_\xi^0\int_s^0 \psi(r)drds. \end{aligned}$$

The last formula can be re-written as

$$F(\xi) = \psi(\xi) - 2a\xi\int_\xi^0 \psi(s)ds + a^2\xi^2\int_\xi^0\int_s^0 \psi(r)drds, \tag{IV.5.12}$$

where

$$F(\xi) = e^{\frac{a\xi^2}{2}}[\beta_0 M_0(-\xi) + \beta_1 M_1(-\xi)] + a\xi\big(2 + \frac{a\xi^2}{k}\big)M_0(0) - a^2\xi^2 M_1(0), \quad \xi < 0,$$

is known explicitly. We set $Y(\xi) = \int_\xi^0 \int_s^0 \psi(r)drds$. With this notation, (IV.5.12) reads

$$F(\xi) = \frac{d^2Y(\xi)}{d\xi^2} + 2a\xi\frac{dY(\xi)}{d\xi} + a^2\xi^2 Y(\xi), \qquad Y(0) = Y'(0) = 0. \tag{IV.5.13}$$

The last differential equation can be recast as

$$e^{\frac{a\xi^2}{2}}F(\xi) = \frac{d^2}{d\xi^2}\Big(e^{\frac{a\xi^2}{2}}Y(\xi)\Big) - ae^{\frac{a\xi^2}{2}}Y(\xi). \tag{IV.5.14}$$

We set $G(\xi) := e^{\frac{a\xi^2}{2}}Y(\xi)$, then (IV.5.13) is equivalent to

$$e^{\frac{a\xi^2}{2}}F(\xi) := \hat{F}(\xi) = \frac{d^2G(\xi)}{d\xi^2} - aG(\xi), \qquad G(0) = G'(0) = 0. \tag{IV.5.15}$$

The general solution of the homogeneous part of (IV.5.15) is given by

$$G_h(\xi) = C_1 e^{\sqrt{a}\xi} + C_2 e^{-\sqrt{a}\xi}. \tag{IV.5.16}$$

To find a particular solution, we choose $v_1(\xi)$ and $v_2(\xi)$ so that $v_1(\xi)e^{\sqrt{a}\xi} + v_2(\xi)e^{-\sqrt{a}\xi}$ satisfies (IV.5.15). It follows from (IV.5.15) that

$$v_1'(\xi) = -\frac{y_2\hat{F}(\xi)}{W(\xi)}, \qquad v_2'(\xi) = \frac{y_1\hat{F}(\xi)}{W(\xi)}, \tag{IV.5.17}$$

where the Wronskian $W$ of $\{e^{\sqrt{a}\xi}, e^{-\sqrt{a}\xi}\}$ is given by,

$$W(\xi) = \begin{vmatrix} e^{\sqrt{a}\xi} & e^{-\sqrt{a}\xi} \\ \sqrt{a}e^{\sqrt{a}\xi} & -\sqrt{a}e^{-\sqrt{a}\xi} \end{vmatrix} = -2\sqrt{a} \neq 0. \tag{IV.5.18}$$

Using the last identity from (IV.5.17) we infer

$$v_1(\xi) = \frac{1}{2\sqrt{a}}\int_0^\xi e^{-s}\hat{F}(s)ds, \qquad v_2(\xi) = -\frac{1}{2\sqrt{a}}\int_0^\xi e^{s}\hat{F}(s)ds \tag{IV.5.19}$$

and the general solution to (IV.5.15) is

$$\begin{aligned} e^{\frac{a\xi^2}{2}}Y(\xi) = G(\xi) = \Big(C_1 e^{\sqrt{a}\xi} + C_2 e^{-\sqrt{a}\xi} + \sqrt{\frac{1}{a}}\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)\hat{F}(\xi - s)ds\Big),\\ Y(\xi) = e^{-\frac{a\xi^2}{2}}\Big(C_1 e^{\sqrt{a}\xi} + C_2 e^{-\sqrt{a}\xi} + \sqrt{\frac{1}{a}}\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}F(\xi - s)ds\Big). \end{aligned} \tag{IV.5.20}$$

From the initial conditions, $Y(0) = \frac{dY}{d\xi}\big|_{\xi=0} = 0$, it follows $C_1 = C_2 = 0$. Hence, the exact solution $Y(\xi)$ of (IV.5.13) and its first and second order derivatives are given by

$$\begin{aligned} Y(\xi) &= e^{-\frac{a\xi^2}{2}}\sqrt{\frac{1}{a}}\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{21}}F(\xi - s)ds\Big),\\ \frac{dY(\xi)}{d\xi} &= e^{-\frac{\xi^2}{2a}}\sqrt{\frac{1}{a}}\Big[-a\xi\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}F(\xi - s)ds - \sinh\Big(\sqrt{\tfrac{1}{a}}\xi\Big)F(0)\\ &\quad + \int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}\big[F'(\xi - s) - a(\xi - s)F(\xi - s)\big]ds\Big],\\ \frac{d^2Y(\xi)}{d\xi^2} &= \psi(\xi) = F(\xi) - 2a\xi Y'(\xi) - a^2\xi^2 Y(\xi)\\ &= F(\xi) - 2\xi\sqrt{a}\Big[-a\xi e^{-\frac{\xi^2}{2a}}\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}F(\xi - s)ds - \sinh\Big(\sqrt{\tfrac{1}{a}}\xi\Big)F(0)\\ &\quad + \int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}\big[F'(\xi - s) - a(\xi - s)F(\xi - s)\big]ds\Big]\\ &\quad - a^2\xi^2 e^{-\frac{a\xi^2}{2}}\Big[\sqrt{\frac{1}{a}}\int_\xi^0 \sinh\Big(\sqrt{\tfrac{1}{a}}s\Big)e^{-\frac{a(\xi-s)^2}{2}}F(\xi - s)ds\Big)\Big]. \end{aligned} \tag{IV.5.21}$$

The solution of (IV.5.11) is given by

$$w(\xi, t) = \begin{cases} (1 + t\mathcal{G}^+)^2[u_0](\xi) & \text{if } \xi > 0;\\ (1 + t\mathcal{G}^+)^2[u_0](\xi) + (1 + t\mathcal{G}^+)^2[\psi](\xi) & \text{if } \xi < 0. \end{cases} \tag{IV.5.22}$$

Backward substitution leads to

$$u(x,t) = e^{\frac{at^2}{2} - axt} \begin{cases} u_0(x-t) + at \displaystyle\int_{x-t}^{\infty} [2 + at(s - x + t)] u_0(s) ds, & \text{if } x \geq t; \\ at(2 - at(x-t)) M_0(0) + (at)^2 M_1(0) + \psi(x-t) & \\ \quad + at \displaystyle\int_{x-t}^{0} [2 + at(s - x + t)] \psi(s) ds, & \text{if } x < t. \end{cases} \tag{IV.5.23}$$

The zeroth and the first moments are given by (IV.5.7).

Chapter V

# Conclusion

## 1 Summary

In this thesis, we established the wellposedness of the transport-fragmentation-coagulations, where transport refers to either decay or growth of particles; in the latter case the equation is coupled with the McKendrick-von Foerster boundary condition. In addition, we provided a systematic method of finding explicit solutions for the transport-fragmentation equation with the homogenous boundary data and power-law coefficients. Furthermore, we have shown that for specific coefficients, the method is applicable to the growth-fragmentation equation equipped with the McKendrick-von Foerster boundary conditions.

In the preliminary chapter, Chapter II, we collected classical results of the theory of semigroups and some elementary inequalities, which were employed in subsequent chapters. In Chapter III, we applied the semigroup theory to the transport-fragmentation-coagulation equation in $X_m = L_1([0,\infty),(1+x^m)dx)$, for $m \geq 1$. First, we considered the transport-fragmentation part, which is given as the sum of three operators, that is transport, loss and gain operators. We treated the sum of the transport and the loss operators as the leading operator and the gain operator as its perturbation. We established the existence of a semigroup generated by the leading operator in Theorems III.2.2 & III.3.3, and then used the Miyadera-Desch perturbation theorem to obtain the transport-fragmentation semigroup in Theorems III.2.3 & III.3.4. Further, we proved that the constructed semigroups have the moment improving property, which was first derived in [22] in a more restrictive setting. Next, we considered the full transport-fragmentation-coagulation problem and, as in [16], we established the existence of local positive solutions using a fixed point argument. We concluded Chapter III by proving that the local solutions obtained in Theorem III.5.1 are global in time.

Other main results are presented in Chapter IV, where we provided explicit solutions for the transport-fragmentation equation subject to the homogenous boundary condition. We assumed power-law for fragmentation and for transport rates and the fragmentation kernel. We provided a systematic method of obtaining explicit solutions for the constant and linear transport rates of the equation. It follows that the solutions can be written in terms of the confluent hypergeometric functions. In addition, we found the moments of the solutions and, in particular, confirmed the existence of shattering solutions. Additionally, we demonstrated that the developed method is applicable to the growth-fragmentation equation with the McKendrick-von Foerster boundary condition, constant growth rate and binary fragmentation.

## 2 Future work

In Chapter III, we were able to establish a global semigroup solution for the transport-fragmentation-coagulation equation under restrictive assumptions. Further, in Chapter IV, we only considered the power-law form of the coefficients. Then, the following ideas can be investigated in future:

- proving global stability of the growth-fragmentation-coagulation equation with no additional assumptions of Section III.6;
- extending techniques of finding explicit solutions of the growth-fragmentation equation to a larger class of coefficients;
- finding stationary solution to the growth-fragmentation-coagulation equation and establishing their stability.

**Extension of** (IV.3.2). It can be verified by direct calculations that $w^{\pm}$, with $\alpha < 0$ are local, i.e., these solutions blow-up in a finite time in the sense of $X_{-\rho}^{-\sigma}$, $\sigma, \rho \in \mathbb{R}_+$. For the forthcoming analysis, it is convenient to replace $w^{\pm}$ with

$$f^{\pm}(\xi, \tau^{\pm}) := e^{-\tau^{\pm}\xi} w_0^{\pm}(\xi, \tau^{\pm}), \qquad f_0^{\pm}(\xi) := w_0^{\pm}(\xi). \tag{A.0.1}$$

It turns out that the latter functions are defined globally for $\tau^{\pm} \in \mathbb{R}_+$. Furthermore, we have

**Lemma A.0.1.** *Assume that $f_0^{\pm} \in X_0^{(\mathrm{sg}\,\alpha)\sigma}$ and either $\alpha > 0$ and $\sigma \geq 0$, $0 \leq \delta \leq m$ or $\alpha < 0$, $\sigma > m$ and $0 \leq \delta < \sigma - m$. Then*

$$(\tau^{\pm})^{\delta} f^{\pm} \in C\big([0, T], X_0^{(\mathrm{sg}\,\alpha)\sigma+\delta}\big),$$

*for every finite value of $T > 0$. In addition, if $f_0^{\pm} \in X_0^{(\mathrm{sg}\,\alpha)\sigma+1}$, then*

$$f^{\pm} \in C^1\big((0, T), X_0^{(\mathrm{sg}\,\alpha)\sigma+\delta}\big) \tag{A.0.2a}$$

*and $f^{\pm}$ is the global classical solution to*

$$f^{\pm}_{\tau^{\pm}}(\xi, \tau^{\pm}) + \xi f^{\pm}(\xi, \tau^{\pm}) = m\mathcal{G}^{\mathrm{sg}\,\alpha}\big[f^{\pm}(\cdot, \tau^{\pm})\big](\xi), \qquad \xi, \tau^{\pm} \in \mathbb{R}_+, \tag{A.0.3a}$$

$$f^{\pm}(\xi, 0) = f_0^{\pm}(\xi), \qquad \xi \in \mathbb{R}_+, \tag{A.0.3b}$$

*in the sense of $X_0^{(\mathrm{sg}\,\alpha)\sigma}$.*

*Proof.* (a) We let

$$\begin{aligned}\mathcal{A}^\pm(\tau^\pm)[w_0^\pm](\xi) &:= e^{-\tau^\pm\xi} w_0^\pm(\xi,\tau^\pm),\\ \mathcal{B}^\pm(\tau^\pm)[w_0^\pm](\xi) &:= m\tau^\pm e^{-\tau^\pm\xi}\\ &\mathcal{G}^{\operatorname{sg}\alpha}\Big[{}_1F_1\Big(1-(\operatorname{sgn}\alpha)m;2;-\tau^\pm(y-\xi)\Big)w_0^\pm\Big](\xi).\end{aligned}$$

Since $\xi^\delta e^{-\xi} \le c_\delta$, $\delta \in \mathbb{R}_+$, for some $c_\delta > 0$, uniformly in $\xi \in \mathbb{R}^+$, it follows that

$$\|\mathcal{A}^\pm(\tau^\pm)\|_{X^{(\operatorname{sg}\alpha)\sigma}\to X^{(\operatorname{sg}\alpha)\sigma+\delta}} \le c_\delta(\tau^\pm)^{-\delta}, \qquad \sigma,\delta \ge 0,$$

uniformly for $\tau^\pm \in \mathbb{R}_+$. To estimate the norm of $\mathcal{B}^\pm$, we consider separately the cases of $\alpha > 0$ and $\alpha < 0$.

(b) Assume initially that $\alpha > 0$. Then, using (IV.3.3), we have

$$\|\mathcal{B}^\pm(\tau^\pm)\|_{X^\sigma\to X^{\sigma+\delta}} \le m(\tau^\pm)^{-\delta}\sup_{\xi\in\mathbb{R}_+}\xi^{-\sigma}e^{-\xi}\int_0^\xi y^{\sigma+\delta}{}_1F_1\Big(1+m;2;\xi-y\Big)dy.$$

Next, by the asymptotic identity (see, e.g., [1, Formulae 13.4.1 and 13.5.5])

$${}_1F_1\Big(a;b;-z\Big) = \begin{cases}\mathcal{O}(1), & z\to 0,\\ \mathcal{O}(z^{-a}), & z\to\infty, \quad \operatorname{Re} z > 0,\end{cases} \tag{A.0.4a}$$

and the formula

$$\int_0^x y^p{}_1F_1(a+1;2;x-y)dy = \frac{x^p}{a}\Big[{}_1F_1\Big(a;p+1;x\Big)-1\Big], \tag{A.0.4b}$$

which holds for all $a \ne 0$ and $p > -1$, we have

$$\|\mathcal{B}^\pm(\tau^\pm)\|_{X^\sigma\to X^{\sigma+\delta}} \le c(\tau^\pm)^{-\delta}, \qquad 0 \le \delta \le m,$$

for some $c > 0$. Hence, (A.0.2a), with $\alpha > 0$, is settled.

(c) Let now $\alpha < 0$. In this case,

$$\|\mathcal{B}^{\pm}(\tau^{\pm})\|_{X^{-\sigma}\to X^{-\sigma+\delta}} \le m(\tau^{\pm})^{-\delta} \sup_{\xi\in\mathbb{R}_+} \xi^{\sigma} e^{-\xi} \int_{\mathbb{R}_+} (\xi+y)^{\delta-\sigma} e^{-y}{}_1F_1\Big(1+m;2;y\Big)dy.$$

It is not difficult to verify that for $0 < a < p$,

$$\begin{aligned}
\textstyle\int_{\mathbb{R}_+}(x+y)^{-p}e^{-y}{}_1F_1(a+1;2;y)dy &= \tfrac{x^{-p}}{a\Gamma(p)}\textstyle\int_{\mathbb{R}_+} e^{-t}t^{p-a-1}\Big[(x+t)^a - t^a\Big]dt \\
&= \tfrac{x^{-p}}{a\Gamma(p)}\Big[\Gamma(p-a)\Psi(-a;1-p;x) - \Gamma(p)\Big] \\
&\le \begin{cases} \frac{\Gamma(p-a)}{a\Gamma(p)}x^{a-p}, & 0 < a \le 1, \\ \frac{2^{a-1}\Gamma(p-a)}{a\Gamma(p)}x^{a-p} + \frac{2^{a-1}-1}{a}x^{-p}, & a > 1, \end{cases}
\end{aligned} \tag{A.0.5}$$

where $\Psi(a;b;z)$ is the Kummer hypergeometric function of the second kind, see [1, Formulae 13.2.5 and 13.1.29]. Hence,

$$\|\mathcal{B}^{\pm}(\tau^{\pm})\|_{X^{-\sigma}\to X^{-\sigma+\delta}} \le c_\delta(\tau^{\pm})^{-\delta}, \qquad 0 \le \delta \le \sigma - m$$

and the proof of (A.0.2a) is complete.

(d) From (A.0.4a)–(A.0.4b), (A.0.5), the inclusion $f_0^{\pm} \in X_0^{(\mathrm{sgn}\,\alpha)\sigma+1}$ and the standard identity (see [1, Formula 13.4.8, p. 505])

$$\frac{d}{dz}{}_1F_1(a;b;z) = \frac{a}{b}{}_1F_1(a+1;b+1;z), \tag{A.0.6}$$

it follows (as in parts (b) and (c) above) that $f^{\pm}$, defined by the explicit formulae (IV.3.2) and (A.0.1), satisfies

$$f^{\pm}_{\tau^{\pm}}, \xi f^{\pm}, \mathcal{G}^{\mathrm{sg}\,\alpha}[f^{\pm}] \in C\big((0,T), X_0^{(\mathrm{sg}\,\alpha)\sigma+\delta}\big),$$

for any finite value of $T > 0$. Using this fact and the direct substitution of $f^{\pm}$ into (A.0.3a)–(A.0.3b), it is not difficult to verify that (A.0.3a) and (A.0.3b) hold in $X_0^{(\mathrm{sg}\,\alpha)\sigma}$ globally for $\tau^{\pm} \in \mathbb{R}_+$. □

**Distributional solutions.** In connection with Lemma A.0.1, we note that $f^{\pm}$, being integrable, satisfies (A.0.3a)–(A.0.3b) in the sense of Schwartz distributions. Moreover, from (IV.3.3) and (A.0.6), it follows that $f^{\pm}$, given by (IV.3.2) and (A.0.1), satisfies

$$0 = \int_{\mathbb{R}_+} f_0^{\pm}(\xi)v(\xi,0) + \int_{\mathbb{R}_+} \bigg( \int_{\mathbb{R}_+} f^{\pm}(\xi,\tau^{\pm}) \\ \bigg[ v_{\tau^{\pm}}(\xi,\tau^{\pm})(\xi,\tau^{\pm}) - \xi v(\xi,\tau^{\pm}) + m\mathcal{G}^{-(\operatorname{sgn}\alpha)}[v(\cdot,\tau^{\pm})](\xi) \bigg] d\xi \bigg) d\tau^{\pm}, \tag{A.0.7}$$

for any $v \in \mathcal{D}(\mathbb{R}_+^2)$ and $f_0^{\pm} \in \mathcal{D}'(\mathbb{R}_+)$. That is, formulae (IV.3.2), (A.0.1) solve (A.0.3a)–(A.0.3b) in the sense of Schwartz distributions for any distributional initial data in $\mathcal{D}'(\mathbb{R}_+)$. In particular, for the monodisperse initial condition $f_0^{\pm}(\cdot) = \delta_{\xi_0}(\cdot) = \delta(\cdot - \xi_0)$, supported at $\xi_0 \in \mathbb{R}_+$, we have

$$f^{\pm}(\xi,\tau^{\pm}) = e^{-\tau^{\pm}\xi} \Big[ \delta_{\xi_0}(\xi) + \chi_{[0,\xi_0]}(\xi) m\tau^{\pm} {}_1F_1\Big(1-m; 2; \tau^{\pm}(\xi-\xi_0)\Big) \Big], \qquad \alpha > 0, \tag{A.0.8a}$$

$$f^{\pm}(\xi,\tau^{\pm}) = e^{-\tau^{\pm}\xi} \Big[ \delta_{\xi_0}(\xi) + \chi_{[\xi_0,\infty)}(\xi) m\tau^{\pm} {}_1F_1\Big(1+m; 2; \tau^{\pm}(\xi-\xi_0)\Big) \Big], \qquad \alpha < 0. \tag{A.0.8b}$$

As an immediate consequence of the preliminary calculations, presented above, we have

**Corollary A.0.2.** *For $u_0^{\pm} \in \mathcal{D}'(\mathbb{R}_+)$, the distributional solutions to (IV.2.1) are given explicitly by (IV.3.9). In particular, for the monodisperse initial data $u_0^{\pm} = \delta_{x_0}$, $x_0 \in \mathbb{R}_+$, formula (IV.3.10) holds.*

*Further, for any finite value of $T > 0$ and input data $u_0^{\pm} \in X^p$, with either $\alpha > 0$, $p \geq \alpha - \nu - 1$ and $0 \leq \delta \leq \frac{\nu+2}{\alpha}$, or $\alpha < 0$, $p > 1 + \alpha$ and $0 \leq \delta < \frac{\alpha - p + 1}{\alpha}$, the solutions (IV.3.9) satisfy*

$$t^{\delta} u^{\pm} \in C\big([0,T], X_0^{p+\alpha\delta}\big), \tag{A.0.9a}$$

*In addition, if the initial datum is regular, i.e., if $(u_0^{\pm})_x \in X_0^{p+1}$ and $u_0^{\pm} \in X_0^{p+\alpha}$, then*

$$u_t^{\pm}, (ru^{\pm})_x, au \in C\big((0,T), X_0^p\big), \tag{A.0.9b}$$

*and (IV.3.9) satisfies (IV.2.1) in the classical sense of $X_0^p$.*

*Proof.* (a) By virtue of our definitions of $v^{\pm}$, $w^{\pm}$, $f^{\pm}$, $f_0^{\pm}$ and $\xi$, $z^{\pm}$, $\tau^{\pm}$, the solution to (IV.2.1) is formally given by

$$u(x,t) = (1 \pm \beta\tau^{\pm}(t))^{\frac{\nu}{\alpha}-\frac{\theta}{\beta}}\xi^{\frac{\nu}{\alpha}}(x,t)f^{\pm}(\xi(x,t),\tau^{\pm}(t)), \tag{A.0.10a}$$

$$\xi(x,t) = x^{\alpha}e^{\mp\beta t}, \qquad \tau^{\pm}(t) = \pm\tfrac{1}{\beta}\big(e^{\pm\beta t}-1\big), \qquad x,t \in \mathbb{R}_+. \tag{A.0.10b}$$

Since the coordinate transformation $(x,t) \mapsto (\xi,\tau^{\pm})$, defined in (A.0.10b), is a diffeomorphism from $\mathbb{R}^2_+$ to $\mathbb{R}_+ \times I^{\pm}$ and since $f^{\pm} \in \mathcal{D}'(\mathbb{R}^2_+)$ satisfies (A.0.3a)–(A.0.3b) in the sense of distributions for any $f_0^{\pm} \in \mathcal{D}'(\mathbb{R}_+)$, it follows (after changing variables in (A.0.7)) that $u^{\pm}$, defined in (IV.3.9), satisfies

$$0 = \int_{\mathbb{R}_+} u_0^{\pm}(x)v(x,0)dx \quad + \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} u^{\pm}(x,t)\Big[v_t(x,t) \pm r(x)v_x(x,t) \right.$$
$$\left. -a(x)v(x,t) + a(x)\int_0^x b(y,x)v(y,t)dy\Big]dt\right)dx,$$

for any $u_0^{\pm} \in \mathcal{D}'(\mathbb{R}^+)$ and hence our first claim is settled.

(b) The right hand side of formula (A.0.10a) defines one-to-one linear maps $\mathcal{T}^{\pm} : f^{\pm} \mapsto u^{\pm}$. Elementary calculations shows that these maps satisfy

$$\mathcal{T}^{\pm} \in \mathcal{L}\big(C([0,T^{\pm}],X_0^{\sigma}),C([0,T],X_0^{p})\big),$$
$$\Big(\mathcal{T}^{\pm}\Big)^{-1} \in \mathcal{L}\big(C([0,T^{\pm}],X_0^{p}),C([0,T],X_0^{\sigma})\big),$$
$$p = \alpha(\sigma+1)-\nu-1, \qquad \sigma \in \mathbb{R}, \qquad T = \pm\tfrac{1}{\beta}(1 \pm \beta T^{\pm}),$$

for any finite $T^{\pm} \in I^{\pm}$. These inclusions, together with (A.0.2a)–(A.0.2a) and the identity $\tau^{\pm}(t) = \mathcal{O}(t)$, $t \to 0$, yield (A.0.9a). In addition, if $(u_0^{\pm})_x \in X_0^{p+1}$ and $u_0^{\pm} \in X_0^{p+\alpha}$, direct calculations, using (A.0.6), (A.0.10a)–(A.0.10b) and (A.0.2a)–(A.0.2a), show that $u^{\pm}$, defined in (IV.3.9), satisfy (A.0.9b). Using this fact and the direct substitution, it is not difficult to verify that $u^{\pm}$, defined in (IV.3.9), satisfy (IV.2.1) in the classical sense of $X_0^{p}$. The proof is complete. □

## Proof of (IV.4.19).

The proof of (IV.4.19) is quite involved, hence we split it into a series of lemmas.

**Lemma A.0.3.** *The substitution* $z = e^{-\frac{\zeta^2}{2}} y$ *transforms the differential equation*

$$\sum_{k=0}^{m} \binom{m}{k} \zeta^{m-k} z^{(k)}(\zeta) = g(\zeta), \qquad \zeta \in \mathbb{R}, \tag{A.0.11}$$

*into to the constant coefficient equation*

$$He_m \left( \frac{d}{d\zeta} \right) [y](\zeta) = e^{\frac{\zeta^2}{2}} g(\zeta), \tag{A.0.12}$$

*where* $He_m$ *is the probabilist's Hermite polynomial of order* $m$*, [1, Section 20.3].*

*Proof.* Consider the substitution $z = hy$ for some unknown differentiable function $h$. Then, using the Leibniz product formula and changing the order of summation,

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} \zeta^{m-k} (hy)^{(k)}(\zeta) = \zeta^m \sum_{r=0}^{m} y^{(r)}(\zeta) \left( \sum_{k=r}^{m} \binom{m}{k} \binom{k}{r} \zeta^{-k} h^{(k-r)}(\zeta) \right) \\
&= \zeta^m \sum_{r=0}^{m} y^{(r)}(\zeta) \left( \sum_{l=0}^{m-r} \binom{m}{l+r} \binom{l+r}{r} \zeta^{-(l+r)} h^{(l)}(\zeta) \right) \\
&=: \sum_{r=0}^{m} a_{m,r}(\zeta) y^{(r)}(\zeta) =: L_m[y](\zeta).
\end{aligned} \tag{A.0.13}$$

We shall prove that for $r \geq 1$, we have

$$a_{m+1,r}(\zeta) = \frac{m+1}{r} a_{m,r-1}. \tag{A.0.14}$$

Indeed,

$$a_{m+1,r}(\zeta) = \zeta^{m+1} \sum_{l=0}^{m+1-r} \binom{m+1}{l+r} \binom{l+r}{r} \zeta^{-(l+r)} h^{(l)}(\zeta) \tag{A.0.15}$$

and

$$\binom{m+1}{l+r} \binom{l+r}{r} = \frac{m+1}{r} \binom{m}{l+r-1} \binom{l+r-1}{r-1}.$$

Thus, taking into account $\zeta^{m+1}\zeta^{-(l+r)} = \zeta^m\zeta^{-(l+(r-1))}$ in (A.0.15), we obtain (A.0.14). Then, by iterations, (A.0.14) yields

$$a_{m,r}(\zeta) = \binom{m}{r} a_{m-r,0}(\zeta). \tag{A.0.16}$$

Hence, to specify all $a_{m,r}$, it suffices to know $a_{m,0}$ for any $m \in \mathbb{N}_0$, with $a_{0,0} = 1$. In what follows, we specify $h(\zeta) = e^{-\frac{\zeta^2}{2}}$ and note that

$$h^{(k)}(\zeta) = h^{-1}(\zeta)(-1)^k He_k(\zeta),$$

where $He_k$ is the Hermite polynomial, defined in (IV.4.18). Hence,

$$a_{m,0}(\zeta) \;=\; h^{-1}(\zeta)\zeta^m \sum_{l=0}^{m} \binom{m}{l} (-1)^l \left( \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(-1)^i}{i!(l-2i)!} \frac{\zeta^{-2i}}{2^i} \right).$$

Considering the sum in the formula above, there are only even powers of $\zeta^{-1}$, running from $1$ to $2\lfloor \frac{m}{2} \rfloor$. The power $\zeta^{-2i}$ appears in the expansion only for $l \geq 2i$, with the coefficient

$$\begin{aligned} b_{m,i} \;&:=\; \sum_{l=2i}^{m} \binom{m}{l} \frac{(-1)^{l+i} l!}{i!(l-2i)!2^i} = \frac{(-1)^i}{i!2^i} \sum_{p=0}^{m-2i} (-1)^{p+2i} \frac{m!}{(m-p-2i)!p!} \\ &= \frac{(-1)^i}{i!2^i} \prod_{k=0}^{2i-1} (m-k) \sum_{p=0}^{m-2i} (-1)^p \binom{m-2i}{p}. \end{aligned}$$

Thus, we have

$$b_{m,i} = \begin{cases} 0 & \text{for} \quad 2i < m, \\ (-1)^{\frac{m}{2}}(m-1)!! & \text{for} \quad 2i = m. \end{cases}$$

Hence

$$a_{m,0}(\zeta) = e^{-\frac{\zeta^2}{2}} c_m := e^{-\frac{\zeta^2}{2}} \begin{cases} 0 & \text{for} \quad m \text{ odd}, \\ (-1)^{\frac{m}{2}}(m-1)!! & \text{for} \quad m \text{ even}, \end{cases} \tag{A.0.17}$$

with the convention $(0-1)!!=1$. Therefore, by (A.0.16),

$$L_m[y](\zeta) = e^{-\frac{\zeta^2}{2}} \sum_{r=0}^{m} \binom{m}{r} c_{m-r} y^{(r)}(\zeta)$$
$$= \begin{cases} e^{-\frac{\zeta^2}{2}} \sum\limits_{i=0}^{k} \binom{2k}{2i} (-1)^{k-i} (2(k-i)-1)!! y^{(2i)}(\zeta) & \text{for } m = 2k, \\ e^{-\frac{\zeta^2}{2}} \sum\limits_{i=0}^{k} \binom{2k+1}{2i+1} (-1)^{k-i} (2(k-i)-1)!! y^{(2i+1)}(\zeta) & \text{for } m = 2k+1. \end{cases}$$

Using the change of variable $l = k-i$ and the definition of the double factorial, the coefficients in both equations can simplified to $(-1)^l \frac{m!}{(m-2l)!} \frac{1}{2^l l!}$ and, using (IV.4.18), both differential operators can be combined into

$$L_m[y](\zeta) = e^{-\frac{\zeta^2}{2}} m! \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^l}{l!(m-2l)!2^l} y^{(m-2l)}(\zeta) = e^{-\frac{\zeta^2}{2}} He_m \left( \frac{d}{d\zeta} \right) [y](\zeta).$$

Hence, (A.0.12) is proved. □

**Lemma A.0.4.** *The solution to* (A.0.12) *satisfying*

$$y(0) = y'(0) = \cdots = y^{(m-1)}(0) = 0, \tag{A.0.18}$$

*is given by*

$$y(\zeta) = \int_0^\zeta \left( \sum_{i=1}^{m} \frac{1}{(He_m)'(\lambda_{m,i})} e^{\lambda_{m,i}(\zeta-\sigma)} \right) e^{\frac{\sigma^2}{2}} g(\sigma) d\sigma, \tag{A.0.19}$$

*where* $\lambda_{m,1}, \ldots, \lambda_{m,m}$ *are simple real roots of* $He_m$.

*Proof.* The solution to (A.0.12) can be found by the variation of constants formula. Clearly, the characteristic polynomial for (A.0.12) is $He_m(\lambda) = 0$. By [1, Section 22.16], all zeroes of Hermite polynomials (as orthogonal polynomials) are real and simple. Then the functions $\theta_{m,i}(\zeta) = e^{\lambda_{m,i}\zeta}$ form a basis of the solution space of the homogeneous equation (A.0.12), hence we seek a particular solution to the inhomogeneous equation (A.0.12) as

$$y(\zeta) = C_1(\zeta) e^{\lambda_{m,1}\zeta} + \cdots + C_m(\zeta) e^{\lambda_{m,m}\zeta}. \tag{A.0.20}$$

In the general setting of the variation of constants method, $C_i'$s are given by

$$C_i' = (-1)^{m-i} e^{\frac{\zeta^2}{2}} g(\zeta) \frac{W_i(\zeta)}{W(\zeta)},$$

where $W$ is the Wronskian of $\{\theta_{m,1}, \dots, \theta_{m,m}\}$ and $W_i$ is the minor of the element $(m, i)$ of $W$. When the roots of the characteristic polynomial are simple, this formula can be made more explicit. Indeed,

$$W(\zeta) = \prod_{i=1}^{m} e^{\lambda_{m,i}\zeta} \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_{m,1} & \lambda_{m,2} & \dots & \lambda_{m,m} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{m,1}^{m-1} & \lambda_{m,2}^{m-1} & \dots & \lambda_{m,m}^{m-1} \end{vmatrix} = V(\lambda_{m,1}, \dots, \lambda_{m,m}) \prod_{i=1}^{m} e^{\lambda_{m,i}\zeta},$$

where $V$ is the Vandermonde determinant, whose value is

$$V(\lambda_{m,1}, \dots \lambda_{m,m}) = \prod_{1 \le i < j \le m} (\lambda_{m,j} - \lambda_{m,i}).$$

Now, for a given $r$,

$$W_r = \prod_{i=1, i \ne r}^{m} e^{\lambda_{m,i}\zeta} V(\lambda_{m,1}, \dots, \lambda_{r-1}, \lambda_{r+1}, \dots, \lambda_{m,m}).$$

To relate these two Vandermonde determinants, we write

$$\begin{aligned} & V(\lambda_{m,1}, \dots \lambda_{m,m}) \\ & \quad = \prod_{1<j\le m} (\lambda_{m,j} - \lambda_{m,1}) \cdot \ldots \cdot \prod_{r<j\le m} (\lambda_{m,j} - \lambda_{m,r}) \cdot \ldots \cdot (\lambda_{m,m} - \lambda_{m,m-1}) \\ & = \prod_{1\le i<r} (\lambda_{m,r} - \lambda_{m,i}) \cdot \prod_{r<j\le m} (\lambda_{m,j} - \lambda_{m,r}) \cdot \prod_{\substack{1\le i<j\le m \\ i,j\ne r}} (\lambda_{m,j} - \lambda_{m,i}) \\ & = (-1)^{m-r} \prod_{\substack{1\le i\le m \\ i\ne r}} (\lambda_{m,r} - \lambda_{m,i}) \cdot V(\lambda_{m,1}, \dots, \lambda_{r-1}, \lambda_{r+1}, \dots, \lambda_{m,m}) \\ & = (-1)^{m-r} (He_m)'(\lambda_{m,r}) V(\lambda_{m,1}, \dots, \lambda_{r-1}, \lambda_{r+1}, \dots, \lambda_{m,m}). \end{aligned}$$

Hence,

$$C_i'(\zeta) = \frac{e^{\frac{\zeta^2}{2}} g_m(\zeta)}{e^{\lambda_{m,i}\zeta}(He_m)'(\lambda_{m,i})}$$

and (A.0.19) follows from (A.0.20). □

**Theorem A.0.5.** *Let $m \in \mathbb{N}$. Then the unique solution $\psi \in X^{+\rho}$ to (IV.4.15) satisfying (IV.4.9c) is given by*

$$\begin{aligned} \psi(\xi) &= (-1)^m \frac{d^m}{d\xi^m}\left(e^{-\frac{\xi^2}{2\beta}} y\left(\frac{\xi}{\sqrt{\beta}}\right)\right) \\ &= (-1)^m \beta^{\frac{m}{2}} \frac{d^m}{d\zeta^m}\left(e^{-\frac{\zeta^2}{2}} y(\zeta)\right)\Big|_{\zeta=\frac{\xi}{\sqrt{\beta}}}, \end{aligned} \tag{A.0.21}$$

*where $y$ is given by (A.0.19) with $g(\zeta) = -\beta^{\frac{m}{2}} F\left(\zeta\beta^{\frac{1}{2}}, -\beta^{-\frac{1}{2}}\zeta\right)$.*

*Proof.* In this particular case, $F(\xi,t)$ is a known function, for $\xi < 0$ given by

$$F(\xi,t) = \sum_{r=1}^{m} \binom{m}{r} \frac{1}{(r-1)!} t^r \int_0^\infty (\eta-\xi)^{r-1} w_0(\eta) d\eta. \tag{A.0.22}$$

Hence, we can re-write (IV.4.15) as

$$(I + t\mathcal{J})^m[\psi](\xi) = \sum_{k=0}^{m} \binom{m}{k} t^k \mathcal{G}^k[\psi](\xi) = w^+(\xi,t) - F(\xi,t). \tag{A.0.23}$$

Now, we use (IV.4.9c) and arrive at

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^k \beta^{-k} \xi^k \mathcal{G}^k[\psi](\xi) = -F(\xi, -\beta^{-1}\xi) =: G(\xi), \tag{A.0.24}$$

where $G$ is a known function. If we denote $Z(\xi) = (-1)^m \mathcal{G}^m[\psi](\xi)$, then $Z^{(l)}(\xi) = (-1)^{m+l}\mathcal{G}^{m-l}[\psi](\xi)$ and (A.0.24) becomes

$$\sum_{k=0}^{m} \binom{m}{k} \left(\frac{\xi}{\beta}\right)^{m-k} Z^{(k)}(\xi) = G(\xi). \tag{A.0.25}$$

Next, with $\zeta = \frac{\xi}{\sqrt{\beta}}$ and $z(\zeta) = Z(\xi)$, (A.0.25) takes the form

$$\sum_{k=0}^{m} \binom{m}{k} \zeta^{m-k} z^{(k)}(\zeta) = \beta^{\frac{m}{2}} G(\zeta\sqrt{\beta}) =: g(\zeta), \tag{A.0.26}$$

and hence, by Lemma A.0.3, using $z(\zeta) = e^{-\frac{\zeta^2}{2}} y(\zeta)$, we transform (A.0.11) into

$$He_m\left(\frac{d}{d\zeta}\right)[y](\zeta) = e^{\frac{\zeta^2}{2}} g(\zeta). \tag{A.0.27}$$

Further, we observe that, by definition, $Z^k(0) = 0$ for $k = 0, \dots, m-1$ and, since $y(\zeta) = e^{-\frac{\zeta^2}{2}} z(\zeta)$, the Leibniz formula shows that $y^k(0) = 0$, provided $y^{(l)}(0) = 0$ for $l = 0, \dots, k-1$, thus, by induction, the initial (or terminal) conditions for (A.0.27) are given by (A.0.18). Thus, the solution to (A.0.12) satisfying these conditions is given by (A.0.19) and we recover formula (A.0.21) for $\psi$ by backward substitution.

To estimate $\psi$, we observe that, by (A.0.22), $e^{-\frac{\zeta^2}{2}} y(\zeta)$ is a linear combination of terms of the form

$$I_\lambda := e^{\lambda\zeta - \frac{\zeta^2}{2}} \int_0^\zeta p_{2m-1}(\sigma) e^{-\lambda\sigma + \frac{\sigma^2}{2}} d\sigma = e^{-\frac{\upsilon^2}{2}} \int_{-\lambda}^{\upsilon} q_{2m-1}(\varsigma) e^{\frac{\varsigma^2}{2}} d\varsigma,$$

where $p_{2m-1}(\sigma)$ and $q_{2m-1}(\varsigma) = p_{2m-1}(\varsigma + \lambda)$ are polynomials of degree $2m-1$ with coefficients depending on the moments of $w_0$ of order from $0$ to $m-1$. Now, by the Leibniz rule, the $m$th derivative of $I_\lambda$ is a linear combination of product of $k$th derivatives of $e^{-\frac{\upsilon^2}{2}}$ (which are the Hermite polynomials of degree $k$), $e^{-\frac{\upsilon^2}{2}}$ and the $(m-k)$th derivative of $\int_{-\lambda}^{\upsilon} q_{2m-1}(\varsigma) e^{\frac{\varsigma^2}{2}} d\varsigma$, $0 \le k \le m$, which, apart from the case $k = m$, is the derivative of order $m-k-1$ of $q_{2m-1}(\varsigma) e^{\frac{\varsigma^2}{2}}$. Using again the Leibniz rule, we see that the $r$th derivative of the latter is given by $\bar{q}_{2m-1+r}(\varsigma) e^{\frac{\varsigma^2}{2}}$, where $\bar{q}_{2m-1+r}$ is a polynomial of degree $2m-1+r$. Thus, the terms of $I_\lambda^{(m)}$ are products of polynomials of degrees $k$ and $3m-k-2$, and hence polynomials of degree $3m-2$, except for $k = m$, which is given by

$$He_m(\upsilon) e^{-\frac{\upsilon^2}{2}} \int_{-\lambda}^{\upsilon} q_{2m-1}(\varsigma) e^{\frac{\varsigma^2}{2}} d\varsigma.$$

By the l'Hôspital rule,

$$\lim_{\upsilon\to-\infty}\frac{He_m(\upsilon)e^{-\frac{\upsilon^2}{2}}\int_{-\lambda}^{\upsilon}q_{2m-1}(\varsigma)e^{\frac{\varsigma^2}{2}}d\varsigma}{He_m(\upsilon)\bar{p}_{2m-2}(\upsilon)}=\lim_{\upsilon\to-\infty}\frac{\int_{-\lambda}^{\upsilon}q_{2m-1}(\varsigma)e^{\frac{\varsigma^2}{2}}d\varsigma}{e^{\frac{\upsilon^2}{2}}\bar{p}_{2m-2}(\upsilon)}=\lim_{\upsilon\to-\infty}\frac{q_{2m-1}(\upsilon)}{\tilde{p}_{2m-1}(\upsilon)}=l,$$

for some finite $l$, where $\bar{p}_{2m-2}$ and $\tilde{p}_{2m-1}$ are polynomials of respective degrees. Thus,

$$\psi(\xi)=\mathcal{O}(\xi^{3m-2})\quad\text{as }\xi\to-\infty$$

and hence $\phi\in X^{+\rho}$. □

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