

THE NUMBER OF CONFIGURATIONS OF RADII THAT CAN OCCUR IN COMPACT PACKINGS OF THE PLANE WITH DISCS OF n SIZES IS FINITE

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ABSTRACT. By a compact packing of the plane by discs, P , we mean a collection of closed discs in the plane with pairwise disjoint interior so that, for every disc $C \in P$, there exists a sequence of discs $D_0, \dots, D_{m-1} \in P$ so that each D_i is tangent to both C and $D_{i+1 \pmod m}$.

We prove, for every $n \in \mathbb{N}$, that there exist only finitely many tuples $(r_0, r_1, \dots, r_{n-1}) \in \mathbb{R}^n$ with $0 < r_0 < r_1 < \dots < r_{n-1} = 1$ that can occur as the radii of the discs in any compact packing of the plane with n distinct sizes of disc.

1. INTRODUCTION

For any set of discs Q in the plane, with $D \in Q$ and $r_D \in \mathbb{R}$ denoting the radius of a disc $D \in Q$, we define $\text{radii}(Q) := \{r_D : D \in Q\}$. With $m \in \mathbb{N}$ by a *corona* we mean a collection of closed discs in the plane $C, D_0, D_1, \dots, D_{m-1}$ that have pairwise disjoint interiors and are labeled in such a way so that for every $i \in \{0, \dots, m-1\}$ the disc D_i is tangent to both the discs C and $D_{i+1 \pmod m}$. We call C *the center* of the corona and D_0, D_1, \dots, D_{m-1} *the petals* of the corona.

By a *compact packing (of the plane by discs)* P we mean a collection of closed discs in the plane with pairwise disjoint interiors, so that every disc $C \in P$ is the center of a corona with petals from P (e.g., Figure 1.1). The set of all maximal (with respect to inclusion) coronas that occur in a compact packing P is denoted by $\text{coronas}(P)$.

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2010 *Mathematics Subject Classification.* Primary: 05B40, Secondary: 52C26, 52C15.

Key words and phrases. Circle packing, Compact packing.

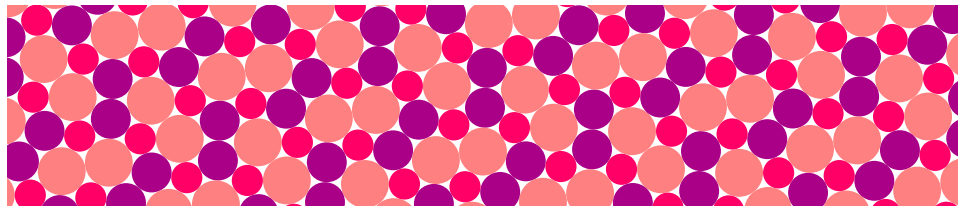


FIGURE 1.1. An example of a compact 3-packing [FHS21, Appendix A. Example (53)].

With P a compact packing, in this paper, we restrict ourselves always to the case $|\text{radii}(P)| < \infty$. For $n \in \mathbb{N}$, we will say P is a *compact n -packing* if $|\text{radii}(P)| = n$. We define the set

$$\Pi_n := \left\{ \text{radii}(P) \mid \begin{array}{l} P \text{ a compact } n\text{-packing with} \\ \max \text{ radii}(P) = 1 \end{array} \right\}.$$

In this paper we prove the following result:

Theorem 1.1. *For all $n \in \mathbb{N}$, the set Π_n is finite.*

Before discussing our approach in this paper toward proving Theorem 1.1, we first make some brief historical remarks.

Since the hexagonal packing is the only compact packing with one size of disc we have $|\Pi_1| = 1$. This was likely known to the ancients. The equality $|\Pi_2| = 9$ was only established in 2006 by Kennedy in [Ken06]. Of the nine elements in Π_2 , some were known earlier, with seven appearing in Fejes Toth's 1964 book [FT64, p.185–187] and a further one appearing in 1993 in [LH93].

Determining whether or not $|\Pi_3|$ is finite and computing its value took a number of years after publication of [Ken06] and proved to be a tantalizing problem in its own right. The bound $|\Pi_3| \leq 13617$ was established in [Mes20]. Working roughly at the same time, but independently from the author, a better approach was developed by Fernique, Hashemi and Sizova who in [FHS21] showed that $|\Pi_3| = 164$. Based on the values of $|\Pi_1|$, $|\Pi_2|$ and $|\Pi_3|$, Connelly and Pierre made the conjecture in a previous version of the preprint [CP19] that the sequence $(|\Pi_n|)$ is the OEIS sequence A086759 [SI21].

Analogous work, studying compact packings of three dimensional space with spheres of two and three sizes, was performed by Fernique respectively in [Fer21] and [Fer19]. Surprisingly, the situation in three dimensions is not quite as rich as the two dimensional case (at least for two and three sizes of spheres), in that all the compact packings of three dimensional space with two and three sizes of spheres arise through merely filling the interstitial holes in a close-packing of spheres with unit radius. This is not the case for the compact packings of discs of the plane with two and three sizes of discs as is evident in [Ken06] and [FHS21].

As the ideas in the current paper are partially informed by those employed in analyzing Π_3 , we briefly describe the process involved in establishing $|\Pi_3| < \infty$ along with some of the difficulties that arise when analyzing Π_3 that do not exist in determining Π_2 .

Define

$$\mathcal{P}_3 := \{P \mid P \text{ a compact 3-packing with } \max \text{ radii}(P) = 1\}.$$

As there occur exactly three sizes of discs in any packing $P \in \mathcal{P}_3$, for convenience we refer to the disc sizes as *small*, *medium* and *large*. To any corona one may associate an abstract *coronal code* (see Sections 2.1 and 2.4 for exact definitions). The length of the code determined by a corona is exactly equal to the length of the sequence of petals in the corona. Mimicking Kennedy's argument from [Ken06], it is not difficult to see that for any compact 3-packing $P \in \mathcal{P}_3$, all codes determined by coronas in P with small centers have a universal bound of six on their length.

A first difficulty arises: That of determining a universal bound on the length of codes associated to any coronas with medium centers in any compact 3-packing $P \in \mathcal{P}_3$. Failing this, there *could* exist infinitely many coronas with medium centers

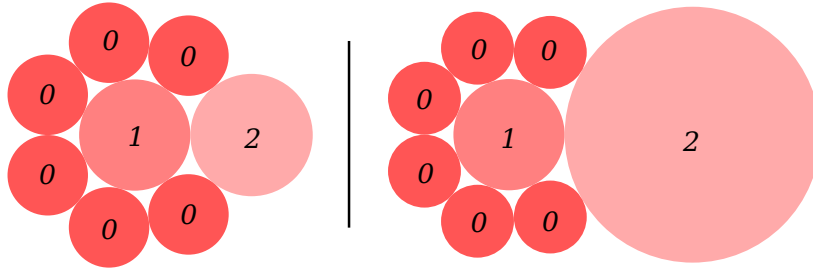


FIGURE 1.2. Two coronas with three sizes of discs that are not congruent (modulo rigid motions and scaling) that determine the same coronal code 1:2000000.

that occur across compact 3-packings in \mathcal{P}_3 , all with distinct associated coronal codes. This would imply $|\Pi_3| = \infty$. A crucial observation in showing that this is not the case is the result [FHS21, Lemma 6.1] paraphrased here:

Lemma 1.2. *There exists a universal constant $K > 0$ so that, for any compact 3-packing P with $\text{radii}(P) = \{r_0, r_1, r_2\}$ satisfying $0 < r_0 < r_1 < r_2 = 1$, we have*

$$0 < K \leq \frac{r_0}{r_1}.$$

The largest value of K can take on is $\min\{r : \{r, 1\} \in \Pi_2\}$.

The proof of Lemma 1.2 proceeds through leveraging knowledge of Π_2 to determine a universal lower bound K for all possible ratios of small-to-medium radii that can occur in any compact 3-packing from \mathcal{P}_3 . This is crucial in placing a universal bound on the length of the coronal codes associated to coronas with medium center that can occur across all compact 3-packings. This universal bound allows one to enumerate all of the finitely many pairs of codes that can possibly be associated to coronas with small and medium centers occurring in any compact 3-packing from \mathcal{P}_3 . This idea returns in Section 5 of the current paper where we prove a more general version of Lemma 1.2 in Lemma 5.1.

A second difficulty now arises: There exist infinitely many coronas containing discs of three different radii that are not congruent (modulo rigid motions and scaling), but who nevertheless determine the same coronal code (see Figure 1.2). By inflating and deflating discs in a compact packing, one could thus conceivably obtain different compact packings yet with the same combinatorics (see for example [CG21] where such techniques are used to continuously deform finite compact packings whose contact graphs are related by edge flips into one another). For this reason, there could exist infinitely many more elements in Π_3 than the finite set all possible pairs of coronal codes that can be associated to coronas with small and medium centers in any compact 3-packing from \mathcal{P}_3 . One then needs to show that this is not the case: In any compact 3-packing $P \in \mathcal{P}_3$ there exists a pair of coronas with small and medium centers so that their associated coronal codes necessarily satisfy conditions (which we do not state here) that uniquely determine the values in $\text{radii}(P)$. This shows that Π_3 has at most as many elements as pairs of codes that satisfy the mentioned necessary conditions, of which there are only finitely many. This idea also returns in the current paper (cf. Sections 3 and 4).

The third and last difficulty arises in computing the exact values of elements of Π_3 . Elements of Π_3 are determined as solutions to systems of trigonometric equations in two variables which are obtained from the pairs of codes associated with coronas with small and medium radii in some compact 3-packing $P \in \mathcal{P}_3$. Such systems of equations can be manipulated into systems of equations containing radicals of rational expressions, and further into systems of polynomial equations in two variables (cf. [Mes20, Section 8] and [FHS21, Sections 5 and 6]). The solutions of these systems of polynomial equations can be determined as the roots of single variable polynomials through computing appropriate Gröbner bases of the occurring polynomials. Some of these computations can become expensive (cf. [FHS21, Sections 3.5 and 3.6]). As our methods in this paper do not require explicit computation of elements in any of the sets Π_n in order to prove Theorem 1.1, the computational difficulties mentioned in this paragraph are not encountered.

We now turn to the approach taken in this paper to prove Theorem 1.1. The proof follows through strong induction and is somewhat informed by the arguments and computations from [Mes20] and [FHS21], but does not at all rely on any computer calculations.

In Section 2 we define a simple symbolic language so that we may analyze compact packings entirely abstractly. This is done to facilitate the combinatorial arguments without the need to refer to actual compact packings. In Sections 2.1 and 2.2, we define what we call *coronal codes* and *angle symbols*. Coronal codes are abstract representations of coronas, and angle symbols are abstract representations of the angle formed at the center of a specific disc through connecting the centers of three mutually tangent discs with disjoint interior (cf. Figure 2.1). For a coronal code we define what we call its *angle sum* which is the formal sum of the angle symbols that represent the sum of the angles formed by connecting the center of the central disc of a corona to the centers of its petals.

In Sections 2.3 and 2.4, we describe what we call *realizations* and what we call the *canonical labeling of a compact packing*. By a realization we merely mean assigning concrete positive values to indeterminates in a formal algebraic expression. For a compact packing, we describe what we call the *canonical labeling of the packing* (labeling all discs with non-negative integers in increasing order of size), along with its *canonical realization* (mapping the label of a disc to the radius of the disc). As such, all coronal codes from coronas obtained from a canonically labeled compact packing are necessarily such that their angle sums, when realized by the canonical realization, evaluate to 2π .

In Section 3 we define what we call a *fundamental set* of coronal codes. We continue to prove in Theorem 3.2 that every canonically labeled compact packing is such that it necessarily contains a set of coronas so that their associated coronal codes determines a fundamental set.

In Section 4 we prove Theorem 4.2. This is an entirely abstract result which shows that any fundamental set of coronal codes is such that a certain map from a set of certain possible realizations to the tuple of realized angle sums is injective. This allows us to prove Theorem 4.3, which states that the coronal codes from a canonically labeled compact packing determine a unique realization ρ (i.e., the canonical realization) for which these coronal codes' angle sums, when realized by ρ , evaluate to 2π .

Section 5 sees proof of what we will call The Bootstrapping Lemma (Lemma 5.1). This lemma is a generalization of Lemma 1.2. It allows us to relate ratios of radii that occur in certain different realizations for the same fundamental set of coronal codes. This lemma will be a crucial ingredient in a strong induction argument that forms part of the main result.

In Section 6, for $n \in \mathbb{N}$ with $n \geq 2$, we define what we call *n-essential sets*. An *n-essential set* is a fundamental set of coronal codes for which there exist two monotone realizations so that the angle sum of all coronal codes in the set, when realized by these two maps, the results respectively lie in the intervals $(0, 2\pi]$ and $[2\pi, \infty)$ (cf. Definition 6.1). This definition allows for leveraging The Bootstrapping Lemma in a strong induction argument to prove that, for all $n \in \mathbb{N}$ with $n \geq 2$, there exist at most finitely many *n-essential sets* (Corollary 6.6). We finally argue that every compact *n-packing* determines at least one of the finitely many *n-essential sets* and that every *n-essential set* determines at most one (perhaps no) element of Π_n . Therefore, for every $n \in \mathbb{N}$ with $n \geq 2$, cardinality of Π_n is bounded by the cardinality of the set of all *n-essential sets* (cf. Corollary 6.8).

We do remark that our method for showing $|\Pi_n| < \infty$ is non-constructive and therefore does not determine quantitative bounds for $|\Pi_n|$. As it stands, determining quantitative bounds would require some significant computational resources (cf. Remark 6.9). For this reason, we opt to use bounds for the size of certain combinatorial objects that are obvious, but definitely not optimal, as using tighter bounds or even the exact values will not provide any advantage.

2. PRELIMINARIES

2.1. Coronal codes. Let S be any set of symbols and let $m \in \mathbb{N}$. With symbols $c, p_0, p_1, \dots, p_{m-1} \in S$, by a *coronal code (over S of length m)*, we will mean a formal string of the form

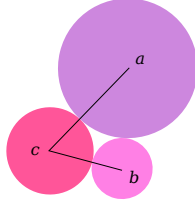
$$c : p_0 p_1 \dots p_{m-1}.$$

In the coronal code $x := c : p_0 p_1 \dots p_{m-1}$, we call the symbol, c , *the center* of the coronal code x , and the symbols, p_0, p_1, \dots, p_{m-1} , the *petals* of the coronal code x . We define $\text{center}(x) := c$, $\text{petals}(x) := \{p_0, p_1, \dots, p_{m-1}\}$ and $\text{length}(x) := m$. We will say two coronal codes $c : p_0 p_1 \dots p_{m-1}$ and $d : q_0 q_1 \dots q_{k-1}$ are equivalent if they have the same length, $m = k$, have the same centers, $c = d$, and the formal string $p_0 p_1 \dots p_{m-1}$ equals some rotation and/or reflection of the formal string $q_0 q_1 \dots q_{m-1}$. We denote the set of all equivalence classes of coronal codes over S by $\mathfrak{C}(S)$. We will not make explicit distinction between an equivalence class of coronal codes in $\mathfrak{C}(S)$ and the members of the equivalence class.

2.2. Angle symbols. Let S be any set of symbols. For symbols $a, b, c \in S$ we define the formal symbol

$$c_b^a := \arccos \left(\frac{(c+a)^2 + (c+b)^2 - (a+b)^2}{2(c+a)(c+b)} \right)$$

and call the symbol c_b^a an *angle symbol (over S)*. In an arrangement of three mutually tangent discs, labeled a, b and c , an angle symbol c_b^a is defined to abstractly represent the angle formed at the center of the disc c by the line segments connecting the center of the disc labeled c to the centers of the discs labeled a and b (cf. Figure 2.1).

FIGURE 2.1. The angle represented by the angle symbol c_b^a .

We explicitly note that that elements in S are *always* to be considered as indeterminate symbols in an angle symbol over S . For an angle symbol c_b^a we call c the *vertex* of the angle symbol c_b^a and we call a, b the *petals* of the angle symbol c_b^a . We regard c_b^a and c_a^b as identical. We denote the set of all angle symbols over S by $\mathfrak{A}(S)$. For distinct symbols $a, b, c, d \in S$ we define the following formal symbols:

$$\begin{aligned} \partial_a c_b^a = \partial_a c_b^a &:= \frac{\sqrt{bc}}{(c+a)\sqrt{a}\sqrt{a+b+c}}, & \partial_a c_b^a &:= 0, \\ \partial_a c_c^a = \partial_a c_c^a &:= \frac{c}{(c+a)\sqrt{a^2+2ac}}, & \partial_a c_c^a &:= 0, \\ \partial_a c_a^a &:= \frac{2\sqrt{c}}{(c+a)\sqrt{2a+c}}, & \partial_a c_a^a &:= 0, \\ \partial_c c_a^b = \partial_c c_a^a &:= -\frac{(a+b+2c)\sqrt{ab}}{(c^2+ab+ac+bc)\sqrt{c}\sqrt{a+b+c}}, & \partial_c c_c^c &:= 0, \\ \partial_c c_c^c = \partial_c c_c^a &:= -\frac{a}{(c+a)\sqrt{a^2+2ac}}, & & \end{aligned}$$

The symbols defined above are exactly the partial derivatives of the involved angle symbols toward the stated indeterminate. For our purpose, the exact values of the expressions above are not as important as their signs, as will be seen in the proof of Lemma 4.1. For $a \in S$, we will regard ∂_a as an additive operator on sums of angle symbols.

Let $x := c : p_0 p_1 \dots p_{m-1} \in \mathfrak{C}(S)$ be any coronal code. We define the formal sum

$$\alpha(x) := c_{p_0}^{p_1} + c_{p_1}^{p_2} + \dots + c_{p_{m-2}}^{p_{m-1}} + c_{p_{m-1}}^{p_0}.$$

The expression $\alpha(x)$ is called an *angle sum* for the coronal code x . We assume in this formal sum that addition commutes, and therefore as a map, α is well defined on equivalence classes in $\mathfrak{C}(S)$. The set of angle symbols that occur in coronal code $x := c : p_0 p_1 \dots p_{m-1} \in \mathfrak{C}(S)$ is defined and denoted as

$$\text{angles}(x) := \{c_{p_0}^{p_1}, c_{p_1}^{p_2}, \dots, c_{p_{m-2}}^{p_{m-1}}, c_{p_{m-1}}^{p_0}\} \subseteq \mathfrak{A}(S).$$

2.3. Realizations. Let S be any set of symbols. By a *realizer* of S we mean a map $\rho : S \rightarrow (0, \infty)$. With a realizer $\rho : S \rightarrow (0, \infty)$ and any (formal) arithmetic expression E containing symbols from S , by $E|_\rho$ we mean the expression E with every symbol $s \in S$ occurring in E replaced by $\rho(s)$ and we will call $E|_\rho$ the *realization of E by ρ* . E.g.,

$$c_b^a|_\rho := \arccos \left(\frac{(\rho(c) + \rho(a))^2 + (\rho(c) + \rho(b))^2 - (\rho(a) + \rho(b))^2}{2(\rho(c) + \rho(a))(\rho(c) + \rho(b))} \right) \in \mathbb{R}.$$

We will say that $x \in \mathfrak{C}(S)$ is *tightly realized by $\rho : S \rightarrow (0, \infty)$* if $\alpha(x)|_\rho = 2\pi$. With $C \subseteq \mathfrak{C}(S)$, we will say that C is *tightly realized by ρ* if all $t \in C$ are tightly realized by ρ .

2.4. Canonical labeling and canonical realizers. Let $n \in \mathbb{N}$ and let Q be any collection of discs in the plane with $|\text{radii}(Q)| = n$. With $0 < r_0 < \dots < r_{n-1}$ such that $\text{radii}(Q) = \{r_0, \dots, r_{n-1}\}$ we label every disc $D \in Q$ with the unique index $i \in \{0, \dots, n-1\}$ attached to its radius r_i and we denote this label by L_D . We call this labeling of discs *the canonical labeling of Q* . The map $\rho : S \rightarrow (0, \infty)$ defined, for all $i \in S$, by $\rho(i) := r_i$ is called the *canonical realizer of Q* .

Let P be a canonically labeled compact n -packing with $0 < r_0 < \dots < r_{n-1}$ so that $\text{radii}(P) = \{r_0, \dots, r_{n-1}\}$. With $S := \{0, 1, \dots, n-1\}$, we define the function $\text{code} : \text{coronas}(P) \rightarrow \mathfrak{C}(S)$ for any maximal corona $\mathcal{C} \in \text{coronas}(P)$ with center $C \in P$ and petals $D_0, \dots, D_{m-1} \in P$, as

$$\text{code}(\mathcal{C}) := L_C : L_{D_0} L_{D_1} \dots L_{D_{m-1}} \in \mathfrak{C}(S).$$

It is clear, by definition of a compact packing, that the set $(\text{code} \circ \text{coronas})(P) \subseteq \mathfrak{C}(S)$ is necessarily tightly realized by the canonical realizer of P .

3. FUNDAMENTAL SETS AND FUNDAMENTAL SETS FROM COMPACT PACKINGS

In this brief section we introduce define the following structure of coronal codes, and subsequently show that every compact n -packing determines such a structure.

Definition 3.1. Let $n \in \mathbb{N}$ and $S := \{0, 1, \dots, n-1\}$. We say a non-empty set $C \subseteq \mathfrak{C}(S)$ is a *fundamental set (of coronal codes)* if $\{\text{center}(x) : x \in C\} = \{0, \dots, n-2\}$ and for every non-empty set $K \subseteq \{0, 1, \dots, n-2\}$ there exists a subset $D \subseteq C$ satisfying $\{\text{center}(x) : x \in D\} = K$ so that

$$\left(\bigcup_{x \in D} \text{petals}(x) \right) \setminus K \neq \emptyset.$$

Theorem 3.2. Let $n \in \mathbb{N}$ with $n > 2$ and set $S := \{0, \dots, n-1\}$. Let P be a canonically labeled compact n -packing. The set

$$C := \{x \in (\text{code} \circ \text{coronas})(P) : \text{center}(x) \leq n-2\} \subseteq \mathfrak{C}(S)$$

is a fundamental set.

Proof. Suppose C is not a fundamental set. As P is a compact n -packing, we have $\{\text{center}(x) : x \in C\} = \{0, \dots, n-2\}$. For C to not be fundamental there then exists a non-empty set $K \subseteq \{0, 1, \dots, n-2\}$ so that, for every subset $D \subseteq C$ with $\{\text{center}(x) : x \in D\} = K$, we have that $(\bigcup_{x \in D} \text{petals}(x)) \setminus K = \emptyset$. Therefore every corona in P with center labeled by an element from K , only has petals labeled by K as well. Therefore only discs labeled by elements of K occur in P , and hence $|\text{radii}(P)| \leq |K| < n$, contradicting $|\text{radii}(P)| = n$. \square

Figure 3.1 displays an example of a fundamental set determined from a canonically labeled 3-packing.

4. UNIQUE TIGHT REALIZATIONS FOR CORONAL CODES FROM CANONICALLY LABELED COMPACT PACKINGS

The purpose of this section is to prove Theorem 4.2. The main idea is that fundamental sets C (cf. Definition 3.1) are such that a map $\mathbf{G} : (0, \infty)^S \rightarrow \mathbb{R}^C$, as defined in Theorem 4.2, is injective when restricted to a certain subset of $(0, \infty)^S$. Combining Theorems 3.2 and 4.2 then allows us to show, for any compact n -packing P , that there exists *exactly* one element Π_n that tightly realizes $(\text{code} \circ \text{coronas})(P)$.

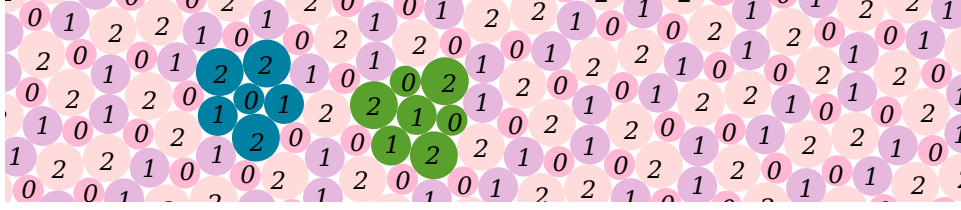


FIGURE 3.1. An example of a canonically labeled compact 3-packing P with fundamental set $\{0 : 22121, 1 : 212020\} \subseteq (\text{code} \circ \text{coronas})(P)$ determined from the blue and green coronas (cf. Definition 3.1). The set of coronal codes $\{0 : 22121, 1 : 212020\}$ determined from the blue and green coronas also form a 3-essential set (cf. Definition 6.1).

Lemma 4.1. *Let S be any set of symbols and let $x := c : p_0 \dots p_{m-1} \in \mathfrak{C}(S)$. For any $\rho \in (0, \infty)^S$ and any $v \in [0, \infty)^S$ satisfying both $v(c) = 0$ and $v(p) > 0$ for at least one petal $p \in \{p_0, \dots, p_{m-1}\}$, the map $[0, \infty) \ni t \mapsto \alpha(x)|_{\rho+tv}$ is strictly increasing.*

Proof. Let $t \in [0, \infty)$ be arbitrary. Let $\rho \in (0, \infty)^S$ and let $v \in [0, \infty)^S$ satisfy $v(c) = 0$ and $v(p) > 0$ for at least one petal $p \in \{p_0, \dots, p_{m-1}\}$. Since $v(p) > 0$ and $v(c) = 0$ we have $p \neq c$. Keeping the definitions of Section 2.2 in mind, for any $d \in S \setminus \{c\}$ and any $a, b \in S$ we have $(\partial_d c_b^a)|_{\rho+tv} \geq 0$ and therefore $(\partial_d \alpha(x))|_{\rho+tv} \geq 0$. Further, there exists some $s \in S$ so that the angle symbol c_s^c occurs at least once as a term in the angle sum $\alpha(x)$. Therefore, since $\rho + tv \in (0, \infty)^S$, we have $(\partial_p \alpha(x))|_{\rho+tv} > 0$. Remembering that $v(c) = 0$, obtain

$$\begin{aligned} \sum_{s \in S} (\partial_s \alpha(x))|_{\rho+tv} v(s) &= \sum_{s \in S \setminus \{c\}} (\partial_s \alpha(x))|_{\rho+tv} v(s) \\ &\geq (\partial_p \alpha(x))|_{\rho+tv} v(p) \\ &> 0. \end{aligned}$$

Now notice that the directional derivative of the map $(0, \infty)^S \ni \sigma \mapsto \alpha(x)|_\sigma$ at $\rho + tv$ in the direction of v is a positive scalar multiple of the left of the above inequality and hence this directional derivative is strictly positive. Therefore the map $[0, \infty) \ni t \mapsto \alpha(x)|_{\rho+tv}$ is strictly increasing. \square

Theorem 4.2. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $S := \{0, \dots, n-1\}$. Let $C \subseteq \mathfrak{C}(S)$ be a fundamental set. The map $\mathbf{G} : (0, \infty)^S \rightarrow \mathbb{R}^C$ defined as*

$$\mathbf{G}(\rho) := (\alpha(x)|_\rho)_{x \in C} \quad (\rho \in (0, \infty)^S).$$

is injective when restricted to the set $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$.

Proof. Take $\rho, \sigma \in \{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ and assume that $\mathbf{G}(\rho) = \mathbf{G}(\sigma)$, but suppose that $\rho \neq \sigma$. Hence, by exchanging the roles of ρ and σ if necessary, we are assured that there exists some index $j \in \{0, \dots, n-2\}$ so that $\rho(j) < \sigma(j)$.

Define $t_0 := \sup\{t \in (0, 1) : \forall j \in S, t\sigma(j) < \rho(j)\} < 1$. As the values of realized angle symbols are invariant under positive scaling of the realizer, it is readily seen for all $\lambda \in (0, \infty)$ that $\mathbf{G}(\sigma) = \mathbf{G}(\lambda\sigma)$, and in particular, we have $\mathbf{G}(\sigma) = \mathbf{G}(t_0\sigma)$. The set $J := \{j \in S : t_0\sigma(j) = \rho(j)\}$ is non-empty, otherwise t_0 cannot be the

supremum of the set $\{t \in (0, 1) : \forall j \in S, t\sigma(j) < \rho(j)\}$. Further, since $t_0 < 1$ we have $n - 1 \notin J$ and hence $S \setminus J$ is also non-empty. Since C is a fundamental set, there exists a code $x \in C$, which we fix, so that $c := \text{center}(x) \in J$ and $\text{petals}(x) \setminus J$ is not empty. Fix a symbol $p \in \text{petals}(x) \setminus J$.

Define $v := \rho - t_0\sigma \in [0, \infty)^S$. The support of v is exactly the set $S \setminus J$ and v takes on positive values on $S \setminus J$ so that $v(p) > 0$ and $v(c) = 0$. By Lemma 4.1, the map $[0, \infty) \ni t \mapsto \alpha(x)|_{\rho+tv}$ is strictly increasing. Therefore, since $\mathbf{G}(\sigma) = \mathbf{G}(t_0\sigma)$, we have

$$\alpha(x)|_{\sigma} = \alpha(x)|_{t_0\sigma} = \alpha(x)|_{t_0\sigma+0v} < \alpha(x)|_{t_0\sigma+1v} = \alpha(x)|_{\rho}.$$

But this contradicts the equality $\alpha(x)|_{\sigma} = \alpha(x)|_{\rho}$ obtained from the assumption $\mathbf{G}(\rho) = \mathbf{G}(\sigma)$. Therefore the supposition $\rho \neq \sigma$ is false, and therefore we obtain $\rho = \sigma$. We conclude that \mathbf{G} is injective when restricted to $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$. \square

Theorem 4.3. *Let $n \in \mathbb{N}$ with $n \geq 2$ and set $S := \{0, \dots, n-1\}$. Let P be any canonically labeled compact packing with $|\text{radii}(P)| = n$ and $\max \text{radii}(P) = 1$. The canonical realizer of P is the unique element from $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ that tightly realizes the set $(\text{code} \circ \text{coronas})(P) \subseteq \mathfrak{C}(S)$.*

Proof. By Theorem 3.2, the set $C := \{x \in \text{code} \circ \text{coronas}(P) : \text{center}(x) \leq n-2\}$ is a fundamental set. By Theorem 4.2, the map $\mathbf{G} : (0, \infty)^S \rightarrow \mathbb{R}^C$ defined as

$$\mathbf{G}(\rho) := (\alpha(x)|_{\rho})_{x \in C}, \quad (\rho \in (0, \infty)^S)$$

is injective when restricted to $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$. The canonical realizer ρ of P is an element of $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ and satisfies $\mathbf{G}(\rho)_x = \alpha(x)|_{\rho} = 2\pi$ for all $x \in C$, so ρ is the unique element in $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ that tightly realizes C . Since every realizer that tightly realizes $(\text{code} \circ \text{coronas})(P)$ also tightly realizes C , the canonical realizer ρ of P is the unique element of $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ that tightly realizes $(\text{code} \circ \text{coronas})(P)$. \square

5. THE BOOTSTRAPPING LEMMA

In this section we prove what we call The Bootstrapping Lemma (Lemma 5.1) which should be read as a generalization of Lemma 1.2. The proof of Lemma 5.1 leverages the definition of a fundamental set C to allow for the comparison of ratios of values from two different realizations ρ and σ satisfying $\alpha(x)|_{\rho} \leq 2\pi \leq \alpha(x)|_{\sigma}$ for all $x \in C$.

Lemma 5.1 (The Bootstrapping Lemma). *Let $n \in \mathbb{N}$ with $n \geq 2$ and $S := \{0, \dots, n-1\}$. Let $C \subseteq \mathfrak{C}(S)$ be a fundamental set. Let $\rho, \sigma \in (0, \infty)^S$ be maps such that for all $x \in C$ we have $\alpha(x)|_{\rho} \leq 2\pi \leq \alpha(x)|_{\sigma}$. Then*

$$\frac{\sigma(n-2)}{\sigma(n-1)} \leq \frac{\rho(n-2)}{\rho(n-1)}.$$

Proof. If, for some $t \in (0, \infty)$, it holds that $\rho = t\sigma$, then we immediately obtain

$$\frac{\sigma(n-2)}{\sigma(n-1)} = \frac{t\sigma(n-2)}{t\sigma(n-1)} = \frac{\rho(n-2)}{\rho(n-1)}.$$

We therefore assume, for all $t \in (0, \infty)$, that $\rho \neq t\sigma$.

Define $t_0 := \sup\{t \in (0, \infty) : \forall j \in S, t\sigma(j) < \rho(j)\}$ and $J := \{j \in S : t_0\sigma(j) = \rho(j)\}$. The set J is non-empty, else t_0 cannot be the supremum of the set

$\{t \in (0, \infty) : \forall j \in S, t\sigma(j) < \rho(j)\}$. Furthermore, since $\rho \neq t\sigma$ for all $t \in (0, \infty)$, the set J does not equal the whole of S .

We consider the two cases $n-1 \in J$ and $n-1 \notin J$.

If $n-1 \in J$, then by definition of t_0 and J , we have $t_0\sigma(n-1) = \rho(n-1)$ and $t_0\sigma(n-2) \leq \rho(n-2)$. Hence

$$\frac{\sigma(n-2)}{\sigma(n-1)} = \frac{t_0\sigma(n-2)}{t_0\sigma(n-1)} = \frac{t_0\sigma(n-2)}{\rho(n-1)} \leq \frac{\rho(n-2)}{\rho(n-1)}.$$

The case $n-1 \notin J$ remains. We show that this case cannot occur. Suppose $n-1 \notin J$, so that $J \subseteq \{0, \dots, n-2\}$. Since C is fundamental, there exists a code $x \in C$ so that $c := \text{center}(x) \in J$ but with a petal $p \in \text{petals}(x)$ so that $p \notin J$. Since realized angle sums are invariant under positive scaling of the realizer, we have $\alpha(x)|_\sigma = \alpha(x)|_{t_0\sigma}$. Define $v := \rho - t_0\sigma$. The support of v is exactly the set $S \setminus J$ and v takes on positive values on $S \setminus J$ so that $v(c) = 0$ and $v(p) > 0$. By Lemma 4.1, the map $[0, \infty) \ni t \mapsto \alpha(x)|_{\rho+tv}$ is strictly increasing. However this yields the absurdity

$$2\pi \leq \alpha(x)|_\sigma = \alpha(x)|_{t_0\sigma} = \alpha(x)|_{t_0\sigma+0v} < \alpha(x)|_{t_0\sigma+1v} = \alpha(x)|_\rho \leq 2\pi,$$

from which we conclude that the supposition $n-1 \notin J$ is false. \square

We briefly show how Lemma 5.1 implies Lemma 1.2. For every canonically labeled corona Q with $|\text{radii}(Q)| = 2$ with smaller disc as center, let $\sigma_Q \in (0, \infty)^{\{0,1\}}$ denote the canonical realizer of Q . Furthermore, there are only finitely many such coronas (modulo rigid motion and scaling). Take any canonically labeled compact 3-packing P with canonical realizer ρ . Obviously $\alpha(x)|_\rho = 2\pi$ for all $x \in (\text{code} \circ \text{coronas})(P)$. Take any coronal code $x \in (\text{code} \circ \text{coronas})(P) \subseteq \mathfrak{C}(\{0, 1, 2\})$ that has center 0 and that has a petal that is not equal to 0. Construct the code $y \in \mathfrak{C}(\{0, 1\})$ from x by replacing every symbol 2 occurring in x with a 1. The code y is noticed to coincide with a code determined by a canonically labeled corona Q' with $|\text{radii}(Q')| = 2$ with smaller disc as center. But now $\alpha(y)|_\rho \leq 2\pi = \alpha(y)|_{\sigma_{Q'}}$. Therefore, with fundamental set $C := \{y\}$, by Lemma 5.1,

$$\frac{\sigma_{Q'}(0)}{\sigma_{Q'}(1)} \leq \frac{\rho(0)}{\rho(1)}.$$

As there exist only finitely many canonically labeled coronas Q with $|\text{radii}(Q)| = 2$ having small disc as center (modulo rigid motion and scaling), taking $K := \min_Q \{\sigma_Q(0)/\sigma_Q(1)\} > 0$, we obtain

$$0 < K \leq \frac{\sigma_{Q'}(0)}{\sigma_{Q'}(1)} \leq \frac{\rho(0)}{\rho(1)}$$

with K independent of the compact 3-packing P , establishing Lemma 1.2.

6. ESSENTIAL SETS OF CORONAL CODES AND PROOF OF THE MAIN RESULT

We now turn to proving the main result of this paper. We introduce what we call n -essential sets of coronal codes (Definition 6.1). We firstly show, for every $n \in \mathbb{N}$ with $n \geq 2$, that the set of all n -essential sets, \mathcal{E}_n , is finite (cf. Corollary 6.6). This proceeds through a strong induction argument in Lemma 6.4 which utilizes the The Bootstrapping Lemma (Lemma 5.1) to place a universal bound on the length of all coronal codes that can occur as elements of any n -essential set. Finally we show every compact n -packing determines one of the finitely many n -essential sets in \mathcal{E}_n

(cf. Theorem 6.7) and that every element in \mathcal{E}_n determines at most one element of Π_n so that $|\Pi_n| \leq |\mathcal{E}_n|$ (Corollary 6.8). This will then prove the main result of this paper, Theorem 1.1.

Definition 6.1. Let $n \in \mathbb{N}$ with $n \geq 2$ and $S := \{0, \dots, n-1\}$. We will say that a set $C \subseteq \mathfrak{C}(S)$ is an n -essential set if C satisfies both of the following conditions:

- (1) The set C is a fundamental set.
- (2) There exist monotone maps $\rho, \sigma \in (0, \infty)^S$ so that, for all $x \in C$, we have $\alpha(x)|_\rho \leq 2\pi \leq \alpha(x)|_\sigma$.

We denote the set of all n -essential sets by $\mathcal{E}_n \subseteq 2^{\mathfrak{C}(S)}$.

Definition 6.2. Let S be a totally ordered set. Let E be any formal expression containing symbols from S . For $k \in S$, by the formal expression $E \downarrow_k$ we mean the formal expression constructed from E with every symbol that occurs in E that is strictly larger than k being replaced by k .

Lemma 6.3. Let $n \in \mathbb{N}$ with $n \geq 2$ and $S := \{0, \dots, n-1\}$. If $C \subseteq \mathfrak{C}(S)$ is n -essential, then for every $m \in \{2, \dots, n-1\}$ with $S_m := \{0, \dots, m-1\}$, the set $C' := \{x \downarrow_{m-1} : x \in C, \text{center}(x) \leq m-2\} \subseteq \mathfrak{C}(S_m)$ is m -essential.

Proof. Assume that C is n -essential. By definition, C is fundamental and there exist monotone maps $\rho, \sigma \in (0, \infty)^S$ so that, for all $x \in C$, we have $\alpha(x)|_\rho \leq 2\pi \leq \alpha(x)|_\sigma$.

Let $m \in \{2, \dots, n-1\}$, and define $C' := \{x \downarrow_{m-1} : x \in C, \text{center}(x) \leq m-2\} \subseteq \mathfrak{C}(S_m)$. We firstly claim that C' is a fundamental set. That the set of centers of codes from C' is $\{0, \dots, m-2\}$ follows from construction. Let

$$K \subseteq \{0, \dots, m-2\} \subseteq \{0, \dots, n-2\}$$

be any non-empty set. As C is fundamental, there exists a subset $D \subseteq C$ so that $\{\text{center}(x) : x \in D\} = K$ with $(\bigcup_{x \in D} \text{petals}(x)) \setminus K \neq \emptyset$. We define $D' := \{x \downarrow_{m-1} : x \in D\} \subseteq C'$. Since, for all $k \in K$ we have $k < m-1$, then

$$\{\text{center}(x) : x \in D'\} = \{\text{center}(x) : x \in D\} = K.$$

Further for any $p \in (\bigcup_{x \in D} \text{petals}(x)) \setminus K \neq \emptyset$ we either have that $p \geq m-1$ or that $p < m-1$ with $p \notin K$. In the case that $p \geq m-1$, we have that $p \downarrow_{m-1} = m-1 \notin K$ so that $p \downarrow_{m-1} = m-1 \in (\bigcup_{x \in D'} \text{petals}(x)) \setminus K$. In the case that $p < m-1$ with $p \notin K$, we have $p \downarrow_{m-1} = p \notin K$ so that $p \downarrow_{m-1} = p \in (\bigcup_{x \in D'} \text{petals}(x)) \setminus K$. In both cases $(\bigcup_{x \in D'} \text{petals}(x)) \setminus K \neq \emptyset$. Therefore C' is fundamental.

We define $\rho', \sigma' \in (0, \infty)^{S_m}$ as

$$\rho'(s) := \begin{cases} \rho(s) & s < m-1 \\ \rho(m-1) & s = m-1 \end{cases} \quad (s \in S_m)$$

and

$$\sigma'(s) := \begin{cases} \sigma(s) & s < m-1 \\ \sigma(n-1) & s = m-1 \end{cases} \quad (s \in S_m).$$

Since ρ and σ are both monotone, so are ρ' and σ' . Further, for all $y \in C'$ take $x \in C$ so that $y = x \downarrow_{m-1}$, we have

$$\alpha(y)|_{\rho'} = \alpha(x \downarrow_{m-1})|_{\rho'} \leq \alpha(x)|_\rho \leq 2\pi \leq \alpha(x)|_\sigma \leq \alpha(x \downarrow_{m-1})|_{\sigma'} = \alpha(y)|_{\sigma'}.$$

We therefore conclude that C' is m -essential. \square

Lemma 6.4. *Let $n \in \mathbb{N}$ with $n \geq 3$. If, for all $m \in \{2, \dots, n-1\}$, the set \mathcal{E}_m is finite, then the set \mathcal{E}_n is finite.*

Proof. Set $S := \{0, \dots, n-1\}$. Assume for all $m \in \{2, \dots, n-1\}$, that $|\mathcal{E}_m| < \infty$. For every $m \in \{2, \dots, n-1\}$ define $S_m := \{0, \dots, m-1\}$ and for every $D \in \mathcal{E}_m$ fix a monotone map $\sigma_D \in (0, \infty)^{S_m}$ so that, for all $x \in D$, we have $2\pi \leq \alpha(x)|_{\sigma_D}$. For $m \in \{2, \dots, n-1\}$ define

$$K_{m-2} := \min_{D \in \mathcal{E}_m} \frac{\sigma_D(m-2)}{\sigma_D(m-1)} \in (0, 1].$$

Now, take any n -essential set $C \in \mathcal{E}_n$. Let $\rho \in (0, \infty)^S$ be any monotone map such that, for all $x \in C$, we have $\alpha(x)|_\rho \leq 2\pi$. By Lemma 6.3, for every $m \in \{2, \dots, n-1\}$, the set $C_m := \{x \downarrow_{m-1} : x \in C, \text{center}(x) \leq m-2\}$ is m -essential so that $C_m \in \mathcal{E}_m$. For every $m \in \{2, \dots, n-1\}$, define ρ_m as the restriction of ρ to S_m . Then, for every $y \in C_m$ taking $x \in C$ so that $y = x \downarrow_{m-1}$, since ρ_m is monotone, we obtain

$$\alpha(y)|_{\rho_m} \leq \alpha(x)|_\rho \leq 2\pi \leq \alpha(y)|_{\sigma_{C_m}}.$$

Hence, by Lemma 5.1, for every $m \in \{2, \dots, n-1\}$, we have

$$0 < K_{m-2} = \min_{D \in \mathcal{E}_m} \frac{\sigma_D(m-2)}{\sigma_D(m-1)} \leq \frac{\sigma_{C_m}(m-2)}{\sigma_{C_m}(m-1)} \leq \frac{\rho_m(m-2)}{\rho_m(m-1)} = \frac{\rho(m-2)}{\rho(m-1)}.$$

Therefore, from

$$0 < K_0 \leq \frac{\rho(0)}{\rho(1)}, \quad 0 < K_1 \leq \frac{\rho(1)}{\rho(2)}, \quad \dots, \quad 0 < K_{n-3} \leq \frac{\rho(n-3)}{\rho(n-2)},$$

we surmise that

$$0 < \left(\prod_{j=0}^{n-3} K_j \right) \leq \frac{\rho(0)}{\rho(n-2)},$$

with $\left(\prod_{j=0}^{n-3} K_j \right) \in (0, 1]$. We define

$$\kappa(s) := \begin{cases} \left(\prod_{j=0}^{n-3} K_j \right) & s = 0 \\ 1 & s = n-2, \end{cases} \quad (s \in \{0, n-2\}).$$

Since ρ is monotone, we have $(n-2)_0^0|_\rho \leq \min\{\text{angles}(x)|_\rho : x \in C\}$, and since $0 < \left(\prod_{j=0}^{n-3} K_j \right) \leq \rho(0)/\rho(n-2)$, we obtain

$$0 < (n-2)_0^0|_\kappa \leq (n-2)_0^0|_\rho \leq \min\{\text{angles}(x)|_\rho : x \in C\}.$$

Therefore the length of any of the codes in C can be at most

$$N := \lfloor 2\pi / ((n-2)_0^0|_\kappa) \rfloor.$$

It is crucial to observe that N is independent of the choice of $C \in \mathcal{E}_n$. The cardinality of \mathcal{E}_n is thus bounded by the cardinality of the powerset of the finite set

$$\{x \in \mathfrak{C}(S) : \text{length}(x) \leq N, \text{center}(x) \leq n-2\}.$$

In particular $|\mathcal{E}_n| < \infty$. □

Lemma 6.5. *The set \mathcal{E}_2 is finite.*

Proof. Let $C \in \mathcal{E}_2$. Since C is fundamental, we have that $C \subseteq \{x \in \mathfrak{C}(\{0,1\}) : \text{center}(x) = 0\}$ and there exist monotone maps $\rho, \sigma \in (0, \infty)^{\{0,1\}}$ so that for all $x \in C$

$$\alpha(x)|_\rho \leq 2\pi \leq \alpha(x)|_\sigma.$$

For all $a, b \in \{0,1\}$ we have $\pi/3 \leq 0_b^a|_\rho$ which implies that the angle sum $\alpha(x)$ can have at most six terms and hence the length of any code $x \in C$ is at most six. Therefore the cardinality of \mathcal{E}_2 is bounded by the cardinality of powerset of the finite set $\{x \in \mathfrak{C}(\{0,1\}) : \text{length}(x) \leq 6, \text{center}(x) = 0\}$. \square

Corollary 6.6. *For every $n \in \mathbb{N}$ with $n \geq 2$, the set \mathcal{E}_n is finite.*

Proof. By strong induction and Lemmas 6.5 and 6.4, for every $n \in \mathbb{N}$ with $n \geq 2$, the set \mathcal{E}_n is finite. \square

Theorem 6.7. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $S := \{0, \dots, n-1\}$. Let P be a canonically labeled compact n -packing. Then there exists a subset of $C \subseteq (\text{code} \circ \text{coronas})(P)$ that is n -essential, i.e., $C \in \mathcal{E}_n$.*

Proof. By Theorem 3.2, there exists a fundamental set $C \subseteq (\text{code} \circ \text{coronas})(P)$. With ρ the canonical realizer of P , for all $x \in C$ we have $\alpha(x)|_\rho = 2\pi$, so taking $\sigma := \rho$, we obtain $\alpha(x)|_\rho = 2\pi = \alpha(x)|_\sigma$ for all $x \in C$. Therefore $C \in \mathcal{E}_n$. \square

Figure 3.1 displays an example of a 3-essential set determined from a canonically labeled 3-packing.

Corollary 6.8. *For every $n \in \mathbb{N}$ with $n \geq 2$, the cardinality of the set*

$$\Pi_n := \left\{ \text{radii}(P) \mid \begin{array}{l} P \text{ a compact } n\text{-packing with} \\ \max \text{ radii}(P) = 1 \end{array} \right\}$$

is at most the cardinality of \mathcal{E}_n , and therefore Π_n is finite.

Proof. Let $n \in \mathbb{N}$ with $n \geq 2$ and set $S := \{0, \dots, n-1\}$. By Theorem 4.2, for every element C of the finite set \mathcal{E}_n , there is at most one (but perhaps no) element $\rho \in \{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ so that, for all $x \in C$, we have $\alpha(x)|_\rho = 2\pi$.

On the other hand, let P be any compact n -packing with $\max \text{ radii}(P) = 1$. By Theorem 6.7, there exists a subset $C \subseteq (\text{code} \circ \text{coronas})(P)$ that is an element of \mathcal{E}_n and, by Theorem 4.3, the canonical realizer $\rho : S \rightarrow (0, 1]$ of P is an element of $\{\tau \in (0, \infty)^S : \tau(n-1) = 1\}$ so that, for all $x \in C$, we have $\alpha(x)|_\rho = 2\pi$.

We therefore obtain $|\Pi_n| \leq |\mathcal{E}_n| < \infty$. \square

The previous corollary proves Theorem 1.1, as stated in the introduction.

Remark 6.9. For $n \in \mathbb{N}$, from Corollary 6.8 a quantitative bound for $|\Pi_n|$ is the finite number $|\mathcal{E}_n|$. However, for a fixed $n \in \mathbb{N}$, given exact knowledge of the sets $\mathcal{E}_2, \dots, \mathcal{E}_{n-1}$, Lemma 6.4 only places a bound on the size of the set \mathcal{E}_n , but gives no information on the elements of \mathcal{E}_n . Specifically, Lemma 6.4 and its proof does not give information on the monotone maps ρ and σ that exist for every element of \mathcal{E}_n (Definition 6.1). But as seen in the proof of Lemma 6.4 this information is required to be able to determine a bound on the size of the next set \mathcal{E}_{n+1} in the sequence and, by extension, a bound on the size of Π_{n+1} . For this reason, determining a quantitative bound for $|\Pi_n|$ for $n \in \mathbb{N}$ using the ideas presented in this paper will at least require computing monotone maps ρ and σ , as in Definition 6.1, that exist for each element in each of the sets $\mathcal{E}_2, \dots, \mathcal{E}_{n-1}$.

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