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A Lot-Sizing Model for a Multi-State System with Deteriorating Items, Variable Production Rate, and Imperfect Quality

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Abstract
Conventional production systems assume that during the manufacturing processes, machines operate without breakdown over an infinite planning horizon and manufacture only products of good quality. Imperfect production processes as a result of machine degradation are common in manufacturing. This paper deals with a problem that concerns the modelling and evaluation of the performance of a multi-state production system that is subject to degradation and its effect on lot sizing. Here, we consider that the cycle starts with a particular production rate until a point when the inventory reaches a certain level after which the failure mode is activated due to the deterioration of certain components, leading to a reduction in the production rate in order to ensure the continuity of supply until the maximum inventory level is reached. Production then stops to restore the machine and the cycle starts again. We have assumed that the rate at which inventory deteriorates is exponential and that demand is constant. A numerical example is used to illustrate the model application, followed by sensitivity analysis. This paper contributes to lot sizing in the area of machine reliability by considering a production system in a degraded state with a non-increasing production rate for deteriorating items with imperfect quality and partial backlogging.

Keywords- Multi-state systems, Inventory; Shortages, Deteriorating process, Variable production rate.

1. Introduction

Manufacturing organizations face significant challenges such as availability of equipment and other resources, flexibility of production systems, reliability of the processes, the quality of the output products, deterioration of goods, and integration of new products and services into the existing production process. The management and control of inventory has become a core part of operations management, and it plays a significant role through achieving efficient and profitable operations for many organizations. Hence, considerable efforts have been made to develop models that can be implemented to optimize inventory systems without compromising customer needs. The classic Economic Order/Production Quantity (EPQ/EPQ) model is the most widely used of these models. However, the traditional EPQ model made a number of simplifying assumptions that might be unrealistic in real-world situations. Ever since the Economic Production Quantity (EPQ) model was first introduced in the early decades of the 21st century, researchers have extended it in many ways through the relaxation of key assumptions, including considerations of shortages, degradation of equipment, deterioration of goods, variable demand, imperfect quality of the outputs, and some combinations of these. The objective of this paper is to develop a model based on the EPQ concept to optimize a flexible manufacturing system that is subject to breakdown of machines, to produce both perfect and imperfect items that deteriorate over time and allows partial backorders.

2. Literature Review

Many inventory models assume that the item has an infinite shelf life while it is in storage. In many real-life situations, this assumption may not be true. The management of deteriorating inventories has received much attention because deterioration of items is one of the important factors in inventory control problems. In many real-life situations, decay or deterioration of items is a natural phenomenon. Chemicals, fruits, vegetables, fertilizers, perfumes, pharmaceutical products, radioactive substances, gasoline, and different types of oils are examples of deteriorating items. The classic production model of Taft (1918) assumes that the depletion of inventory is due only to the constant demand rate, while in many inventory systems, the effect of deterioration cannot be ignored. Whitin (1957) was the first to consider the effect of deterioration on fashion items after a prescribed date. Ghare and Schrader (1963) proposed a replenishment policy for an exponentially decaying inventory. Datta and Pal (1990) proposed a deterministic inventory system for deteriorating items with constant deterioration rate and demand rate that is a linear function of stock level. Panda et al. (2009) developed an inventory model for perishable products with time varying demand. Guchhait et al. (2013) presented an EPQ model for damageable items with variable demand rate in which both the inventory carrying cost and the production rate are assumed to be time-dependent. Pandey and Vaish (2017) formulated an inventory policy for deteriorating goods with seasonal demand under the effect of price discounting on the unit selling price. Agi and Soni (2020) presented a deterministic inventory policy for a perishable product subject to both physical deterioration and degradation subject to freshness condition. Çalışkan (2022) developed an EOQ model for exponentially deteriorating items with planned backorders without using differential calculus. Rahaman et al. (2022) presented an inventory model for deteriorating inventory in which preservation technology to recover substantial loss of items during production is implemented.

Also, many researchers have studied production systems where perfect quality items are always produced, but in actual situations, manufactured products may include a number of imperfect items. This defect in the quality of products, which may be the result of many factors such as human errors, wide tolerance, equipment failure, mishandling and incorrect specifications of raw materials (Al-Salamah, 2019), is now being studied. Zhang and Gerchak (1990) extended the classic EOQ model to systems with imperfect quality by including an inspection policy with random yield on lot sizing while assuming that defective units are replaced by non-defective ones. Cheng (1991) presented an EOQ model with imperfect production processes and price dependent demand. Chang (2004) presented a model in which a proportion of items is considered imperfect and the demand rate is assumed to be fuzzy variables. Eroglu and Ozdemir (2007) examined an EOQ model with defective items and backordering. Elyasi et al. (2014) presented three distinct game theoretical approaches to solve a decentralized EOQ model which considers the defective items in a two-echelon decentralized supply chain. Sajjad et al. (2022) discussed a shipping policy that considers the imperfections of production processes and some related factors such as transportation cost, actual production inventory, defective items and backorders.

Furthermore, typical models of production systems do not consider processes with speed losses or breakdown of machines. However, considering the manufacturing system as a complex sequence with several unit processes, each with its characteristics, the issues of resource reliability has become an issue that producers have to address because the performance of a system depends on the availability of machinery. Process degradation is a natural phenomenon in a production process that runs for long time, hence, the problem of degradation of processes has been addressed by several authors. Hall (1983) studied the effect of malfunctions of equipment on the quality of products. Sana et al. (2007) extended Krasa and Pal (1990)'s model by considering a system with imperfect production (due to staff impatience), constant demand, loss of sale, and price discounting based on the quality of items. Sethi et al. (2002) developed a stochastic production planning model subject to random failures to control and optimize a flexible manufacturing system and used the dynamic programming approach to solve the problem. Ben-Daya et al. (2008) studied an EPQ model with a shifting production rate under stoppages due to speed losses. They demonstrated that process deterioration could be the result of minor stoppages and speed losses, which in practice may affect the efficiency of the process. Kenne´ and Nkeungoue (2008) proposed homogenous Markov Processes using the Hedging point policy with failures and repairs of machines. The authors assumed that the machine failures were age-dependent. Emami-Mehrgani et al. (2014) studied production systems with machines subjected to random breakdowns and repairs under preventive maintenance with human error.

The study of a system's reliability has traditionally been based on binary modelling (using two states), namely the operational state and the complete failure state. However, growing literature now considers numerous scenarios that may occur during the lifetime of some systems. Systems may be Multi-Production Systems (MPS) or Multi-State Systems (MSS). MPS usually start with low production rate and then ramp up later in order to lower the average holding cost as smaller stock level is held for longer time, while large stocks are held for shorter period. MSS, however, may serve both the purpose of holding cost reduction and management of a degraded state. If, from the production point of view, a system is conceived in a way that at the occurrence of any failure, a reconfiguration is undertaken automatically allowing the degraded machine or any other equipment to be functional but with a decrease of the service delivered, we refer to this as a multi-state system (MSS) or degraded system. MSS may be subject to multiple failure modes which may have different effects on their performance. Degradation, being one of these failure modes, allows a machine to continue to perform its function after a breakdown has occurred, but resulting in a partial reduction of its nominal performance However, the continuity of a service depends mainly on the state of the manufacturing system. For such systems, the breakdown of any component only minimally or at least partially disrupts their performance. This approach differs from the conventional methods that normally imply the complete shutdown of the system.

A manufacturing system's overall effectiveness is usually determined by combining the system's efficiency, the process' availability, and the product's quality. Over the past few years, studies have started to address the effects of process availability, however, MSS and its influence on both the efficiency of equipment and the quality of the items resulting from such a system have not been fully addressed. Some researchers have tried to address this problem by simply inflating the quantity produced in the lot size by an amount needed to take care of the quantity that needs to be discarded without considering the implications of degraded state of the machine and other factors such as the quality of items produced and the shelf life of items stored (Hu et al., 1994; Martinelli, 2007; Nodem et al., 2011). While this assumption seems reasonable and may lead to simpler and more direct mathematical solutions, these authors did not discuss the issue of having the machine performing continuously at a higher rate of production, which may lead to more damage to the system, and may increase the level of defective items produced beyond what would have happened by shifting to a lower production rate. Silver (1990) considered the case of a manufacturing system in which the production is deliberately slowed down to permit the production rates to be treated as a controllable variable without taking into account the degraded mode. Khouja and Mehrez (1994) developed an EPQ model by extending the traditional EPQ to cases of flexible production rates with variable production cost function. Eiamkanchanalai and Banerjee (1999) extended the work of Khouja and Mehrez (1994) by including a linear penalty function due to unused capacity. Sana et al. (2007) considered a production system with adjustable rate, with demand for both perfect and imperfect quality items. Ben-Daya et al. (2008) developed a two-state production system, demonstrating the effect of varying production rate on batch sizing due to speed loss. Bhowmick and Samanta (2011) investigated a production model for deteriorating items with shifting production rates and shortages. Uthayakumar and Sekar (2017) developed an imperfect EPQ model for deteriorating items, including both the rework and salvage value. Shaikh et al. (2020) proposed a replenishment policy for deteriorating items under the effect of reliability and price-dependent demand. Manna et al. (2021) developed a multi-item non-perfect production-inventory model over a finite time horizon. Ritha and Saarumathi (2021) discussed the implementation of EPQ model to consider the optimal production time over a finite planning horizon in which the production rate is dependent on the demand rate. We now discuss the identified gap in literature filled by this work.

Table 1. Analysis of related literature.

Characteristics of the EPQ system									
Year	**Authors**	**Single Production**	**MPS**	**MSS**	**Imperfect quality**	**Deterioration**	**Rework**	**Partial backorder**	**Complete backorder**
1918	Taft (1918)	Yes	no	no	no	no	no	no	no
1994	Khouja & Mehrez (1994)	Yes	no	yes	yes	no	yes	no	no
2008	Ben-Daya et al. (2008)	No	yes	no	no	no	no	no	no
2012	Bhowmick & Samanta (2012)	No	yes	no	no	no	no	no	yes
2017	Gothi et al. (2017)	Yes	yes	no	no	yes	no	yes	no
2019	Al-Salamah (2019)	Yes	no	no	yes	no	yes	no	yes
2022	This paper	No	yes	yes	yes	yes	no	yes	no

An analysis of the published EPQ models with MSS/MPS systems previously studied is provided in Table 1, which illustrates the various factors considered with the MSS/MPS models by different research articles in the extant literature, and what this paper adds to the research on production system with machines in degraded mode. A review of current literature seems to suggest that there are no extensions of inventory model for deteriorating items which considered a multi-state production system with non-increasing production rate, imperfect quality, and partial backlogging. This paper, hence, considers an MSS for a deteriorating item with imperfect items and degrading production rates, while also allowing partial backlogging of demand with lost sales. The model extends the work of Bhowmick and Samanta (2012) and Al-Salamah (2019). The goal is to examine deterioration of both products and processes and their impacts on the economic production quantity decisions.

The rest of this article is structured as follows: the development of the mathematical model is outlined in section 3, while a numerical example is presented in section 4. Finally, conclusions are given in Section 5.

3. Formulation of the Proposed MSS

3.1 Notations and Assumptions

3.1.1 Assumptions

The following assumptions are made for development of the model:

- The production-inventory system produces a single item.
- Shortages are partially backlogged and partially lost.

- The changeover cost and time from k_1 to k_2 is assumed to be negligible
- All imperfect items are scrapped and disposed as a batch, and a disposal cost is incurred per item scrapped.
- The production shift occurs during the production run, and is consequent to the optimized inventory parameters.
- It is assumed that the cost of repair can be ignored since it is handled by using in house capacity.

The deterioration of an item produced follows the exponential function $\theta e^{-\theta t}$ for ≥ 0 , where θ is the deterioration rate $(0 \leq \theta \ll 1)$.

3.1.2 Notations

The following notations are adopted to develop the model given in Table 2.

Table 2. Summary of notations used.

Symbol	Description
c_a	The deterioration cost per item
c_b	The disposal cost per unit item
c_s	The shortage cost per item per time
c_p	The penalty cost per unit lost sale
$D(t)$	Demand at time t which is assumed constant and equal to A
d_1, d_2	The proportion of defective units produced during the interval $[0, t_1]$and time interval $[t_1, t_2]$ respectively $0 \leq d_1, d_2 < 1$
G	The production setup cost
h	The inventory carrying cost per item per time
H_M	The Hessian Matrix
$I(t)$	The instantaneous state of the inventory level at any time t $(0 \leq t \leq T)$
k_1, k_2	The Constant production rates during the time intervals $[0, t_1]$and time interval $[t_1, t_2]$ respectively
I_1	The inventory level at the end of time $t = t_1$
I_2	The inventory level at the end of time $t = t_2$
r	The fraction of demand lost due to inventory stock out $(0 < r < 1)$
p_{c1}, p_{c2}	The constant unit production costs when the rate of productions are k_1 and k_2 respectively
S	The maximum shortage, occurring at $t = t_4$
T	The total cycle time
TC	The average total cost for the time period $[0, T]$
$\theta(t)$	The deterioration rate per (units/unit time).
$\rho_1, \rho_2, \beta, \tau, \Psi, B, C, E, F$	Aggregation parameters for some known variables

3.2 Mathematical Formulation

In the following system, a company produces a certain item which deteriorates over time, for which the behavior of the inventory profile and the state of the machine that produces the items through a degradable process are presented in Figures 1 and 2. The system is designed to start operating at a production rate of k_1 (Figure 2), and an inventory of goods accumulates during the first part of the cycle at a rate $(1 - d_1)k_1 - A$ while the imperfect quality items accumulate at the rate $d_1 k_1$ and are disposed as a single batch at the end of the cycle. We assume the time until when the production switches to a degraded state to be t_1, by which time the stock of good items has reached the level I_1 . It is assumed that when breakdown occurs, the system is automatically reconfigured to continue to be operational but at a lower production rate, k_2 (Figure 2). The same concept was used by Khouja (2005) in dealing with shifts in production. Therefore, the production rate switches over to k_2 and inventory of perfect items accumulates at the rate $(1-d_2)k_2 - A$ until a level I_2 is reached. The quantity of imperfect quality item continues to accumulate at the rate d_2k_2 in the second production consumption cycle which ends at time t_2 and is also disposed at the end of the cycle. We assume the failure state to be static. This implies that no further deterioration occurs over time after the system shifts to a failure state. Once the level I_2 is reached, production is then stopped for maintenance, and the inventory decreases until the stock level reaches zero (Figure 1). There are unit costs of production associated with each of the two states of production. These costs are assumed constant in each state of the production. After the stock is drawn down to zero, the system goes into a state of backlog of demand up to S (the maximum backorder level), and thereafter production starts to clear the backlog in $[t_4, T]$. Gothi et al. (2017) assumed that demand during the time interval $[t_4, T]$ is satisfied as the production has already started at time $t = t_4$, and so lost sales cost during this interval is not taken into account (Figure 1). However, in reality some customers are not willing to wait as the company is still not able to meet all the outstanding demand (i.e. backlog) at the time instance. It should be noted that stock cannot build up in the period $[t_4, T]$ because there isn't enough extra capacity to meet the instantaneous demand and also clear backlog during the period $[t_4, T]$, hence, it continues to lose some sales until there is no more backlog at time T.

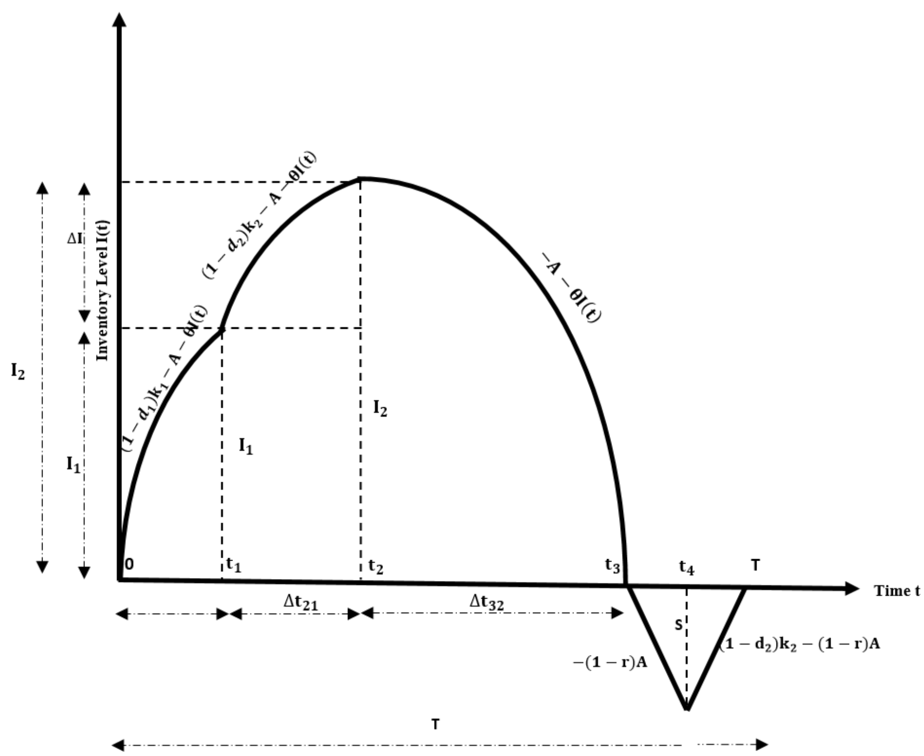

Figure 1. Inventory profile with alternating production rates of the MSS.

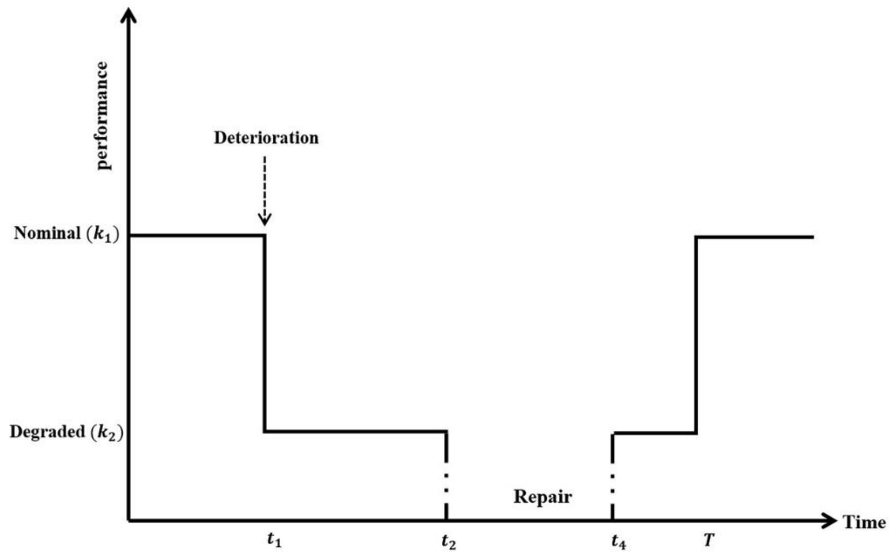

Figure 2. State of a MSS in terms of equipment deterioration.

The differential equations that represent the problem statement of the EPQ model in the interval $[0, T]$ are given by:

$$\frac{dI(t)}{dt} + \theta I(t) = (1 - d_1)k_1 - a \qquad 0 \le t \le t_1 \tag{1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = (1 - d_2)k_2 - a \qquad t_1 \le t \le t_2 \tag{2}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -a \qquad t_2 \le t \le t_3 \tag{3}$$

$$\frac{dI(t)}{dt} = -(1 - r)a \qquad t_3 \le t \le t_4 \tag{4}$$

$$\frac{dI(t)}{dt} = (1 - d_2)k_2 - (1 - r)a \qquad t_4 \le t \le T \tag{5}$$

Under the boundary conditions $(0) = 0$, $I(t_1) = I_1$, $I(t_2) = I_2$, $I(t_3) = 0$, $I(t_4) = S$, and $I(T) = 0$, we obtain the following:

$$I(t) = \left[\frac{(1-d_1))k_1}{\theta} - \frac{a}{\theta}\right]\left(1 - e^{-\theta t}\right) \qquad 0 \le t \le t_1 \tag{6}$$

$$I(t) = \left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t-t_1)} \qquad t_1 \le t \le t_2 \tag{7}$$

$$I(t) = -\frac{a}{\theta} + \left(I_2 + \frac{a}{\theta}\right) e^{-\theta(t-t_2)} \qquad t_2 \le t \le t_3 \tag{8}$$

$$I(t) = -(1 - r)a(t - t_3) \qquad t_3 \le t \le t_4 \tag{9}$$

$$I(t) = -S + (1 - d_2)(t - t_4)k_2 - (1 - r)a(t - t_4) \qquad t_4 \le t \le T \tag{10}$$

From (6) we have:

$$I_1 = \left[\frac{(1-d_1)k_1}{\theta} - \frac{a}{\theta}\right]\left(1 - e^{-\theta t_1}\right) \tag{11}$$

$$t_1 = -\frac{1}{\theta} ln \left[1 - \frac{\theta I_1}{(1-d_1)k_1 - a}\right] \tag{12}$$

From Taylor's series expansion, and the assumption that $\theta^2 \ll 1$ (neglecting higher powers of θ) and the boundary conditions, the expansion of the logarithmic function in (12) is given by:

$$ln\left[1 - \frac{\theta I_1}{[\,(1-d_1)k_1 - a]}\right] = -\frac{\theta I_1}{(1-d_1)k_1 - a} - \frac{\theta^2 I_1^2}{2[\,(1-d_1)k_1 - a]^2} - \frac{2\theta^3}{6[(1-d_1)k_1 - a]^3} I_1^3 \tag{13}$$

Therefore:

$$t_1 = -\frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_1)k_1 - a} - \frac{\theta^2 I_1^2}{2[\,(1-d_1)k_1 - a]^2} - \frac{2\theta^3}{6[(1-d_1)k_1 - a]^3} I_1^3\right] \tag{14}$$

$$t_1 \approx \frac{I_1}{(1-d_1)k_1 - a} + \frac{\theta I_1^2}{2[\,(1-d_1)k_1 - a]^2} \tag{15}$$

Thus, t_1 can be written in terms of I_1 and so, t_1 is not a decision variable.

From (7), under the boundary condition $(t_2) = I_2$, we have:

$$I_2 = \left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right]\right\} e^{-\theta(t_2 - t_1)} \tag{16}$$

$$t_2 - t_1 = \frac{1}{\theta} ln\left[1 - \frac{\theta I_1}{(1-d_2)k_2 + a}\right] - \frac{1}{\theta} ln\left[1 - \frac{\theta I_2}{(1-d_2)k_2 + a}\right] \tag{17}$$

From Taylor's series expansion, and the assumption that $\theta^2 \ll 1$ (neglecting higher powers of θ), the expansion of the logarithmic functions in (17) is given by:

$$\begin{aligned}&\frac{1}{\theta} ln\left[1 - \frac{\theta I_1}{(1-d_2)k_2 + a}\right] - \frac{1}{\theta} ln\left[1 - \frac{\theta I_2}{(1-d_2)k_2 + a}\right] \\ &= \frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_2)k_2 - a} - \frac{\theta^2 I_1^2}{2[\,(1-d_2)k_2 - a]^2} - \frac{2\theta^3 I_1^3}{6[(1-d_2)k_2 - a]^3}\right] - \frac{1}{\theta}\left[-\frac{\theta I_2}{(1-d_2)k_2 - a} - \frac{\theta^2 I_2^2}{2[\,(1-d_2)k_2 - a]^2} - \right. \\ &\quad \left. \frac{2\theta^3 I_2^3}{6[(1-d_2)k_2 - a]^3}\right]\end{aligned} \tag{18}$$

Therefore:

$$\begin{aligned}t_2 - t_1 = &\frac{1}{\theta}\left[-\frac{\theta I_1}{(1-d_2)k_2 - a} - \frac{\theta^2 I_1^2}{2[\,(1-d_2)k_2 - a]^2} - \frac{2\theta^3 I_1^3}{6[(1-d_2)k_2 - a]^3}\right] - \frac{1}{\theta}\left[-\frac{\theta I_2}{(1-d_2)k_2 - a} - \frac{\theta^2 I_2^2}{2[\,(1-d_2)k_2 - a]^2} - \right. \\ &\left. \frac{2\theta^3 I_2^3}{6[(1-d_2)k_2 - a]^3}\right]\end{aligned} \tag{19}$$

That is,

$$t_2 = \frac{I_2 - I_1}{(1-d_2)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2[\,(1-d_2)k_2 - a]^2} + \frac{I_1}{(1-d_1)k_1 - a} + \frac{\theta I_1^2}{2[\,(1-d_1)k_1 - a]^2} \tag{20}$$

Thus, t_2 can be written in terms of I_1 and I_2 . Therefore, t_2 is not a decision variable.

From (8), under the boundary condition $I(t_3) = 0$, we get:

$$0 = -\frac{a}{\theta} + \left[I_2 + \frac{a}{\theta}\right] e^{-\theta(t_3 - t_2)} \tag{21}$$

$$t_3 - t_2 = \frac{1}{\theta} ln\left(\frac{a+\theta I_2}{a}\right) \tag{22}$$

For small values of θ and using Taylor series approximation, we expand the logarithmic function in (22) as follow:

$$ln\left(\frac{a+\theta I_2}{a}\right) = \frac{\theta I_2}{a} - \frac{\theta^2 I_2^2}{2a^2} + \frac{2\theta^3 I_2^3}{6a^3} \tag{24}$$

Therefore:

$$t_3 - t_2 = \frac{1}{\theta}\left[\frac{\theta}{a} I_2 - \frac{\theta^2}{2a^2} I_2^2 + \frac{2\theta^3}{6a^3} I_2^3\right] \tag{25}$$

That is,

$$t_3 = \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} + \frac{I_2 - I_1}{(1-d_2)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2[\,(1-d_2)k_2 - a]^2} + \frac{I_1}{(1-d_1)k_1 - a} + \frac{\theta I_1^2}{2[\,(1-d_1)k_1 - a]^2} \tag{26}$$

Thus, t_3 can be written in terms of I_1 and I_2 . Therefore, t_3 is not a decision variable.

From (9) and the boundary condition $I(t_4) = -S$, we get:

$$-(1-r)a(t_4 - t_3) = -S \tag{27}$$

$$t_4 = \frac{S}{(1-r)a} + \frac{I_1}{(1-d_1)k_1 - a} + \frac{\theta I_1^2}{2[\,(1-d_1)k_1 - a]^2} + \frac{I_2 - I_1}{(1-d_2)k_2 - a} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2[\,(1-d_2)k_2 - a]^2} + \frac{I_2}{a} - \frac{\theta I_2^2}{2a^2} \tag{28}$$

Thus, t_4 can be written in terms of I_1 and I_2. Therefore, t_4 is not a decision variable.

From (10) and the boundary condition $I(T) = 0$, we get:

$$-S + (1-d_2)(T - t_4)k_2 - (1-r)a(T - t_4) = 0 \tag{29}$$

$$T - t_4 = \frac{S}{[(1-d_2)k_2 - (1-r)a]} \tag{30}$$

Substituting (27) into (30), we get:

$$T - t_3 - \frac{S}{(1-r)a} = \frac{S}{[(1-d_2)k_2 - (1-r)a]} \tag{31}$$

$$S = \frac{(1-r)a[(1-d_2)k_2 - (1-r)a]}{(1-d_2)k_2}\left[T - \frac{I_2}{a} + \frac{\theta I_2^2}{2a^2} - \frac{I_2 - I_1}{\rho_2} - \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2} - \frac{I_1}{\rho_1} - \frac{\theta I_1^2}{2\rho_1^2}\right]$$

$$= \frac{(1-r)a[(1-d_2)k_2 - (1-r)a]}{(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2} C I_2^2 + E I_1 + \frac{\theta}{2} F I_1^2\right] \tag{32}$$

With

$$(1-d_1)k_1 - a = \rho_1 \tag{33}$$

$$(1-d_2)k_2 - a = \rho_2 \tag{34}$$
$$\frac{1}{a}+\frac{1}{\rho_2} = B \tag{35}$$
$$\frac{1}{a^2}-\frac{1}{\rho_2^2} = C \tag{36}$$
$$\frac{1}{\rho_2}-\frac{1}{\rho_1} = E \tag{37}$$
$$\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2} = F \tag{38}$$

Thus, S can be written in terms of I_1 ,I_2 and T. Therefore, S is not a decision variable.

3.3 Cost Components Involved in the Mathematical Formulation

To find the optimum quantities, we first calculate the total cycle cost, which is the sum of setup cost, deteriorating cost, inventory holding cost, shortage costs, loss of sales cost, production cost, and cost of disposing defective items. The cost components are derived as follows:

3.3.1 Setup Cost (SUC)

The setup cost is considered fixed and is represented by:

$$SUC = G \tag{39}$$

3.3.2 Deteriorated Items' Cost (DC)

The total number of deteriorated items over the time interval $[0, T]$ is obtained by integrating the deterioration function over the interval $[0, \mathrm{t}_3]$. It should be noted that the product can only deteriorate in the interval $[0, \mathrm{t}_3]$ when there is stock. There is no stock in the interval $[\mathrm{t}_3, \mathrm{T}]$. Hence,

$$\int_0^{t_1}[(1-d_1)k_1 - a]\,dt + \int_{t_1}^{t_2}[(1-d_2)k_2 - a]\,dt - \int_{t_2}^{t_3} a\,dt \tag{40}$$
$$= [(1-d_1)k_1 - a]t_1 + [(1-d_2)k_2 - a](t_2 - t_1) - a(t_3 - t_2) \tag{41}$$

Substituting (15), (20), and (26) appropriately, the total cost of deteriorated items over $[0, T]$ is given by:

$$DC = C_a\left[\frac{\theta I_1^2}{2\rho_1}+\frac{\theta(I_2^2-I_1^2)}{2\rho_2}+\frac{\theta I_2^2}{2a}\right] \tag{42}$$

3.3.3 Inventory Carrying Cost (ICC)

According to Figure 1, it can be concluded that the total holding cost over $[0, T]$ can be summarized as follow:

$$ICC = h \times \left\{\int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)\,dt + \int_{t_2}^{t_3} I(t)dt\right\} \tag{43}$$
$$\int_0^{t_1} I(t)dt = \int_0^{t_1}\left\{\left[\frac{(1-d_1)k_1}{\theta} - \frac{a}{\theta}\right]\left(1-e^{-\theta t}\right)\right\}dt = \frac{I_1^2}{2[(1-d_1)k_1 - a]} + \frac{\theta I_1^3}{3[(1-d_1)k_1 - a]^2} \tag{44}$$
$$\int_{t_1}^{t_2} I(t)\,dt = \int_{t_1}^{t_2}\left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right] + \left\{I_1 - \left[\frac{(1-d_2)k_2}{\theta} - \frac{a}{\theta}\right]\right\}e^{-\theta(t-t_1)}\,dt$$
$$= \frac{(I_2^2-I_1^2)}{2[(1-d_2)k_2 - a]} + \frac{\theta(I_2^3-I_1^3)}{3[(1-d_2)k_2 - a]^2} \tag{45}$$
$$\int_{t_2}^{t_3} I(t)dt = \int_{t_2}^{t_3}\left[-\frac{a}{\theta} + \left(I_2 + \frac{a}{\theta}\right)e^{-\theta(t-t_2)}\right]dt = \frac{I_2^2}{2a} - \frac{\theta I_2^3}{3a^2} \tag{46}$$

Therefore, the total inventory carrying cost over the period $[0, \mathrm{T}]$ is given by:

$$ICC = h \times \left[\frac{I_1^2}{2\rho_1} + \frac{\theta I_1^3}{3\rho_1^2} + \frac{(I_2^2 - I_1^2)}{2\rho_2} + \frac{\theta(I_2^3 - I_1^3)}{3\rho_2^2} + \frac{I_2^2}{2a} - \frac{\theta I_2^3}{3a^2}\right] \quad (47)$$

3.3.4 Shortage Cost (SC)

The shortage cost over the period $[t_3, T]$ can be obtained as follow:

$$SC = C_s \times \left\{\int_{t_3}^{t_4} -a(1-r)(t-t_3)dt + \int_{t_4}^{T} [-S + (1-d)k_2 - a(1-r)](t-t_4)dt\right\} \quad (48)$$

$$\int_{t_3}^{t_4} -a(1-r)(t-t_3)dt = \left[-a(1-r)(\frac{1}{2}t^2 - tt_3)\right]_{t_3}^{t_4} = -\frac{1}{2}\frac{S^2}{(1-r)a} \quad (49)$$

With:

$$t_4 - t_3 = \frac{S}{a(1-r)} \quad (50)$$

$$\int_{t_4}^{T} \{-S + [(1-d_2)k_2 - a(1-r)](t-t_4)\}dt = -\frac{S^2}{2[(1-d_2)k_2 - a(1-r)]} \quad (51)$$

Hence the total quantity backordered over the period $[t_3, T]$ is given by:

$$= \frac{-S^2(1-d_2)k_2}{2a(1-r)[(1-d_2)k_2 - a(1-r)]} \quad (52)$$

Therefore, the cost of shortage over $[\mathrm{t}_3, \mathrm{T}]$ is:

$$SC = C_s \frac{a(1-r)[(1-d_2)k_2 - a(1-r)]}{2(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 \quad (53)$$

3.3.5 Disposal Cost (DIC)

During each production cycle, a proportion d_i of manufactured items is defective and a cost is incurred by the company to dispose those imperfect items, and thus the disposal cost per cycle is given by:

$$\begin{aligned} DIC &= C_b \times [d_1 k_1 t_1 + d_2 k_2 (t_2 - t_1) + d_2 k_2 (T - t_4)] \\ &= C_b \times \left\{d_1 k_1 \left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right] + d_2 k_2 \left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta(I_2^2 - I_1^2)}{2\rho_2^2}\right] + \frac{(1-r)a}{(1-d_2)} d_2 \left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]\right\} \end{aligned} \quad (54)$$

3.3.6 Lost Sale Cost (LC)

The expression for the loss of sale per cycle is determined by:

$$LC = C_p \times r\left[\int_{t_3}^{T} a \times dt\right] = C_p \times r \times a\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] \quad (55)$$

3.3.7 Production Cost (PC)

The production cost over the period [0, T] is:

$$\begin{aligned} PC &= p_{c1}k_1 t_1 + p_{c2}k_2(t_2 - t_1) + p_{c2}k_2(T - t_4) \\ &= p_{c1}k_1\left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right] + p_{c2}k_2\left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta(I_2^2 - I_1^2)}{2\rho_2^2}\right] + p_{c2}\frac{(1-r)a}{(1-d_2)}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] \end{aligned} \quad (56)$$

3.4 Total Cost (TC)
The total cost per cycle $[0, T]$ is, therefore:

$$TC = \frac{1}{T}\Big\{[p_{c1}k_1 + C_b d_1 k_1]\left[\frac{I_1}{\rho_1} + \frac{\theta I_1^2}{2\rho_1^2}\right] + [p_{c2}k_2 + C_b d_2 k_2]\left[\frac{I_2 - I_1}{\rho_2} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2^2}\right] + \Big[p_{c2}\frac{(1-r)a}{(1-d_2)} + C_b d_2 \frac{(1-r)a}{(1-d_2)} + C_p \times r \times a\Big]\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right] + C_s \frac{a(1-r)[(1-d_2)k_2 - a(1-r)]}{2(1-d_2)k_2}\left[T - BI_2 + \frac{\theta}{2}CI_2^2 + EI_1 + \frac{\theta}{2}FI_1^2\right]^2 + h \times \left[\frac{I_1^2}{2\rho_1} + \frac{\theta I_1^3}{3\rho_1^2} + \frac{\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta\left(I_2^3 - I_1^3\right)}{3\rho_2^2} + \frac{I_2^2}{2a} - \frac{\theta I_2^3}{3a^2}\right] + Ca\left[\frac{\theta I_1^2}{2\rho_1} + \frac{\theta\left(I_2^2 - I_1^2\right)}{2\rho_2} + \frac{\theta I_2^2}{2a}\right] + G\Big\} \tag{57}$$

3.4.1 Proof of Convexity
From (57), the MSS may be classified as an unconstrained multivariate optimization problem with a differentiable objective function at I_1,I_2 and T. The optimum values could be obtained by solving the following three equations concurrently:

$$\frac{\partial CT}{\partial I_1} = 0, \frac{\partial CT}{\partial I_2} = 0, \frac{\partial TC}{\partial T} = 0 \tag{58}$$

Due to highly non-linearity of the functions in (58), a closed form analytical proof is difficult to obtain. However, the convexity of the cost function can be proven numerically by showing that it is positive (semi)definite through the matrix minor relations in (59), (60) and (61):

$$H_1 = \frac{\partial^2 TC}{\partial I_1^2} \geq 0 \tag{59}$$

$$H_2 = \left[\frac{\partial^2 \mathrm{TC}}{\partial I_1^2}\frac{\partial^2 \mathrm{TC}}{\partial I_2^2}\right] - \left[\frac{\partial^2 \mathrm{TC}}{\partial I_1\, \partial I_2}\right]^2 \geq 0 \tag{60}$$

$$H_M = \frac{\partial^2 \mathrm{TC}}{\partial I_1^2}\left[\frac{\partial^2 \mathrm{TC}}{\partial I_2^2} \cdot \frac{\partial^2 \mathrm{TC}}{\partial T^2} - \frac{\partial^2 \mathrm{TC}}{\partial I_2\, \partial T} \cdot \frac{\partial^2 \mathrm{TC}}{\partial T\, \partial I_2}\right] - \frac{\partial^2 \mathrm{TC}}{\partial I_2\, \partial I_1}\left[\frac{\partial^2 \mathrm{TC}}{\partial I_1\, \partial I_2} \cdot \frac{\partial^2 \mathrm{TC}}{\partial T^2} - \frac{\partial^2 \mathrm{TC}}{\partial I_1\, \partial T} \cdot \frac{\partial^2 \mathrm{TC}}{\partial T\, \partial I_2}\right] + \frac{\partial^2 \mathrm{TC}}{\partial T\, \partial I_1}\left[\frac{\partial^2 \mathrm{TC}}{\partial I_1\, \partial I_2} \cdot \frac{\partial^2 \mathrm{TC}}{\partial I_2\, \partial T} - \frac{\partial^2 \mathrm{TC}}{\partial I_2^2} \cdot \frac{\partial^2 \mathrm{TC}}{\partial I_1\, \partial T}\right] \geq 0 \tag{61}$$

Many researchers such as Al-Salamah (2019), Uthayakumar and Sekar (2017) and Sana (2007) used this approach in solving their inventory problems. This proof is presented with the data used in the numerical example.

4. Numerical Examples and Sensitivity Analysis
4.1 Numerical Examples
To demonstrate the effectiveness of the proposed model, an example with various data ranges has been studied. A sensitivity analysis is also performed to demonstrate the reliability of the model. The Newton Raphson method is used to solve the problem since it is difficult to find analytical solution.

Consider a production process for making a single item where the demand rate for the items is 25 units per time. The first and second production rates are 80 units and 55 units per time respectively (per time here can be per year, month or day as appropriate). Defective rates in the two production-consumptions cycles are 7% and 14% respectively. The setup cost is \$2700. The holding cost of one unit of the finished product is \$0.5 per time. The production costs of one unit of the finished item during the two production phases are \$21 and \$20 respectively. The cost of deterioration is charged at \$18 per item. The imperfect quality items that are screened may be disposed of at a unit cost of \$3 per item. The proportion of stock out demand sale lost is 20%, the rate at which items deteriorate is 0.002, the shortage cost is \$5 per unit per time and the lost sales cost is \$11 per unit.

To verify whether or not the solution obtained from the total cost function (57) is actually optimal, we first establish the sufficiency conditions of optimality by substituting the data provided into (59), (60) and (61). We confirmed the three conditions to be greater than or equal to zero ($|H_1| = 0, |H_2| = 0, |H_3| = 0$). This result proves that the cost function is positive (semi)definite.

Table 3. Results from the numerical example.

t_1	t_2	t_3	t_4	T^*	I_1^*	I_2^*	S	G	DC	ICC	SC	DIC	LC	PC	TC^*
4.6	8.95	21.6	24.1	25.92	224.18	319.88	50.06	2700	134	1860	543	220	239	14511	20207

The optimum values from the numerical example summarised in Table 3 show that the company reaches a maximum inventory of 224.18 units in the first production-consumption cycle and stops production once a stock level of 319.88 units is reached. A maximum backlog of 50.06 units is allowed. This maximum backlog is reached at $t_4 = 24.1$ time units, and it takes 1.82 time units for the company to clear this backlog in the system. The total cycle time is 25.92 time units, after which the cycle starts again. Deterioration, inventory holding, shortage, disposal, loss of profit, setup, and production costs per unit time are \$2700, \$134, \$1860, \$543, \$220, \$239 and \$14511 respectively, with a total cost \$20207 per cycle.

The graphical illustration is shown in Figure 3, which shows the total cost against two decision variables.

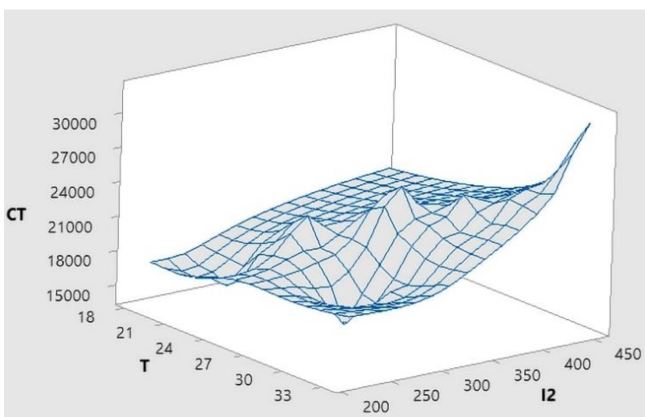

Figure 3. Total cost per cycle vs. inventories levels and cycle time.

4.2 Sensitivity Analysis

In this section, sensitivity analysis was performed to analyse the effects of parameter changes on the total cost. Results from the previous numerical example is considered for this sensitivity analysis. The effects of parameter changes on T, I_1, I_2, S and TC are shown on Tables 4 to 8, and graphical representations of how these changes affect the cycle time and the total cost are presented in Figures 4 and 5.

(i) T decreases with increase in holding cost rate, h, deteriorating cost rate, C_a, shortage cost rate, C_s, production rates, k_1 and k_2, fraction of demand lost (r) and deterioration rate (θ), whereas it increases with increase in setup cost G, disposal cost c_b, penalty cost for lost sales, C_p, and proportion, d_2, of defective units produced. As demand rate increases, the cycle time, T, automatically decreases, resulting in lower average inventory holding cost.

(ii) The optimum inventory levels I_1 and I_2 decrease with an increase in the deteriorating cost, holding cost rate, disposal cost, deterioration rate, production rate k_2, fraction of demand lost r, and proportion d_2 of defective items produced. It is seen that higher production rate k_1, setup cost G, proportion d_1 of defective items produced during the first production cycle and shortage costs increase the optimum level, of I_1 and I_2. However, higher demand rate, a, results in higher optimal level of I_1 and lower level of I_2.

(iii) The backorder level, S, increases with an increase in the unit deterioration cost, holding cost rate, item deterioration rate, production rates k_1 and k_2, fraction of demand lost r, and proportions d_1 and d_2 of defectives items produced. Consequently, higher shortage cost rate, unit disposal cost and lost sale cost rate lead to lower optimal backorder level.

(iv) The total cost TC, increases considerably with the increase in set up cost G and demand rate a.

The following is observed about the level of sensitivity of the decision variables to the parameters:

(i) T is highly sensitive to changes in h, G and a. T is slightly sensitive to changes in values of c_s, k_1, θ, k_2, d_1 and d_2. However, T remains fairly insensitive to changes c_a, c_b, c_p and r.

(ii) $I_1$is insensitive to changes in c_a, however, it is considerably sensitive to c_s, c_p, h, c_b and r. Moreover, I_1 increases drastically with the increase in G, k_1 and d_1 due to its high sensitivity, whereas it decreases with the increase in d_2 and k_2.

(iii) I_2 is not very sensitive to changes in c_b, moderately sensitive to changes in c_s, c_a, c_p, p_{c1}, p_{c2}, a, r, d_1, and d_2, and highly sensitive to changes in h and G.

(iv) The backorder level increases drastically with an increase in h, G and a, and decreases with the increase in C_s. The backorder level increases moderately with the increase in k_1, k_2, d_1, and d_2 and decreases with the increase in C_p.

(v) The cost TC is insensitive to changes in c_a, c_b, c_p, r and θ, and slightly sensitive to changes in h, k_1 , k_2, d_1 and d_2 . Moreover, it has been observed that TC is highly sensitive to changes in G, and a.

4.3 Managerial Insights

(i) It can be seen that the set up costs and demand rate have significant impact on the rate of increase of the cost rate, hence the manager should pay attention to these cost parameters. These are seen to also negatively affect the optimum inventory levels I_1 and I_2 significantly. Larger batch sizes mean higher holding costs, which escalates the total costs. The reverse is also true, and this gives manager the lever to reduce total costs through carrying less inventory.

(ii) Managers should also be concerned about production rates k_1 and k_2. It may be tempting to increase production capacity, but it comes at a cost because higher values of $k_1$and k_2 imply higher holding and disposal costs, which increase the total cost rate. In order to keep total costs down, managers should consider producing items at lower production rates because they can reduce the total costs.

Table 4. Variation of T with respect to parameter changes.

Change in T (%)													
Change in parameter	c_a	h	c_b	c_s	G	k_1	c_p	θ	k_2	a	d_1	r	d_2
-35	0.99	17.76	-0.33	3.70	-18.81	*	-0.26	2.34	3.58	13.59	2.73	0.05	*
-20	0.56	8.87	-0.79	1.73	-10.21	7.66	-0.14	1.32	1.74	6.14	1.43	0.04	-2.09
-5	0.14	1.98	-0.05	0.37	-2.45	1.38	-0.03	0.32	0.39	1.22	0.32	-0.28	-0.77
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	-0.28	-3.57	0.10	-0.64	4.73	-2.19	0.06	-0.64	0.14	-1.94	-0.55	-0.02	2.03
25	-0.69	-8.14	0.25	-1.41	11.45	-4.56	0.13	-1.26	1.19	-3.82	-1.14	-0.07	6.14
30	-0.82	-9.49	0.31	-1.63	13.60	-5.19	0.13	-1.88	1.94	-4.08	-1.27	-0.09	7.77

Table 5. Variation of I_1 with respect to parameter changes.

Change in I_1 (%)													
Change in parameter	c_a	h	c_b	c_s	G	k_1	c_p	θ	k_2	a	d_1	r	d_2
-35	0.75	12.32	5.39	-8.31	-31.7	*	-2.61	1.09	26.5	-28.8	-32.7	3.1	*
-20	0.43	6.73	3.07	-3.99	-17.22	-41.2	-1.47	0.62	24.6	-14.7	-19.4	1.8	50.65
-5	0.11	1.61	0.77	-0.86	-4.14	-6.0	-0.36	0.15	15.9	-3.27	-5.1	0.21	11.65
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	-0.21	-3.06	-1.53	1.51	8.00	8.72	0.72	-0.3	-18.7	5.79	10.6	-0.97	-21.4
25	-0.53	-7.29	-3.82	3.37	19.38	17.29	1.77	-0.6	-70.3	13.06	27.5	-2.51	-49.3
30	-0.64	-8.61	-4.58	3.91	23.03	19.44	1.92	-0.9	-100	14.5	33.52	-3.05	-57.5

Table 6. Variation of I_2 with respect to parameter changes.

Change in I_2 (%)													
Change in parameter	c_a	h	c_b	c_s	G	k_1	c_p	θ	k_2	a	d_1	r	d_2
-35	1.22	21.27	0.78	-5.74	-21.9	*	-1.8	2.7	-7.35	-8.07	-3.36	2.12	*
-20	0.69	10.83	0.44	-2.75	-11.89	-4.92	-1.02	1.52	-4.68	-3.37	-2.13	1.25	8.43
-5	0.17	2.45	0.11	-0.59	-2.86	-1.09	-0.25	0.37	-1.94	-0.56	-0.59	-0.34	1.87
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	-0.34	-4.48	-0.22	1.04	5.52	1.90	0.49	-0.74	1.33	0.6	1.31	-0.67	-3.3
25	-0.85	-10.33	-0.53	2.32	13.36	4.17	1.22	-1.46	3.85	0.24	3.62	-1.73	-7.34
30	-1.01	-12.08	-0.64	2.69	15.88	4.80	1.32	-2.17	5.08	-0.21	4.49	-2.11	-8.47

Table 7. Variation of S with respect to parameter changes.

	Change in S (%)												
Change in parameter	c_a	h	c_b	c_s	G	k_1	c_p	θ	k_2	a	d_1	r	d_2
-35	-0.72	-11.28	0.2	47.27	-16.32	*	6.35	-1.81	-6.21	-14.7	-2.5	-7.33	*
-20	-0.41	-6.1	0.11	22.4	-8.86	-3.66	3.64	-1.02	-3.85	-7.49	-159	-4.16	-3.66
-5	-0.1	-1.45	0.03	4.8	-2.13	-0.82	0.91	-0.25	-1.66	-1.66	-0.4	-0.20	-0.8
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.2	2.77	-0.05	-8.4	4.11	1.42	-1.83	0.5	1.05	2.93	0.97	2.05	1.42
25	0.5	6.63	-0.12	-18.6	9.96	3.11	-4.59	1	2.98	6.56	2.7	5.07	3.11
30	0.6	7.85	-0.15	-21.53	11.83	3.58	-5.01	1.5	3.91	7.27	3.34	6.07	3.58

Table 8. Variation of TC with respect to parameter changes.

	Change in TC (%)												
Change in parameter	c_a	h	c_b	c_s	G	k_1	c_p	θ	k_2	a	d_1	r	d_2
-35	-0.23	-3.62	-0.38	-1.37	-5.23	*	-0.43	-0.58	-1.99	-28.5	-0.8	0.51	*
-20	-0.13	-1.96	-0.22	-0.66	-2.84	-1.18	-0.24	-0.33	-1.24	-16	-0.5	0.3	-075
-5	-0.03	-0.47	-0.05	-0.14	-0.68	-0.26	-0.06	-0.08	-0.50	-3.93	-014	0.07	-0.26
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.07	0.89	0.11	0.11	1.32	0.45	0.12	0.16	0.34	7.73	0.31	-0.16	0.66
25	0.16	2.13	0.27	0.27	3.19	1.00	0.29	0.32	0.96	19.8	0.86	-0.41	1.96
30	0.19	2.52	0.33	0.33	3.79	1.15	0.32	0.48	1.26	22.7	1.07	-0.5	2.47

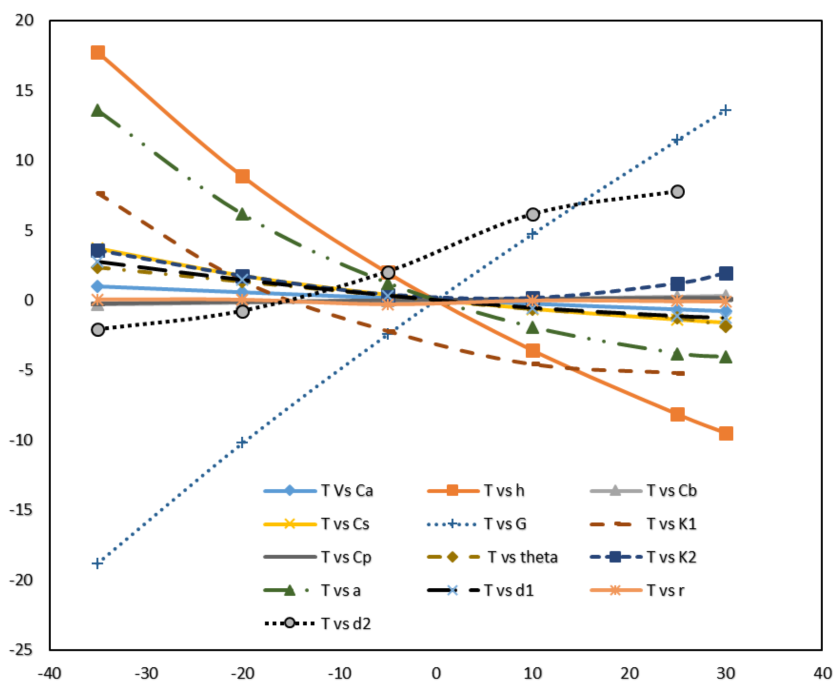

Figure 4. How the cycle time changes with the change of key parameters.

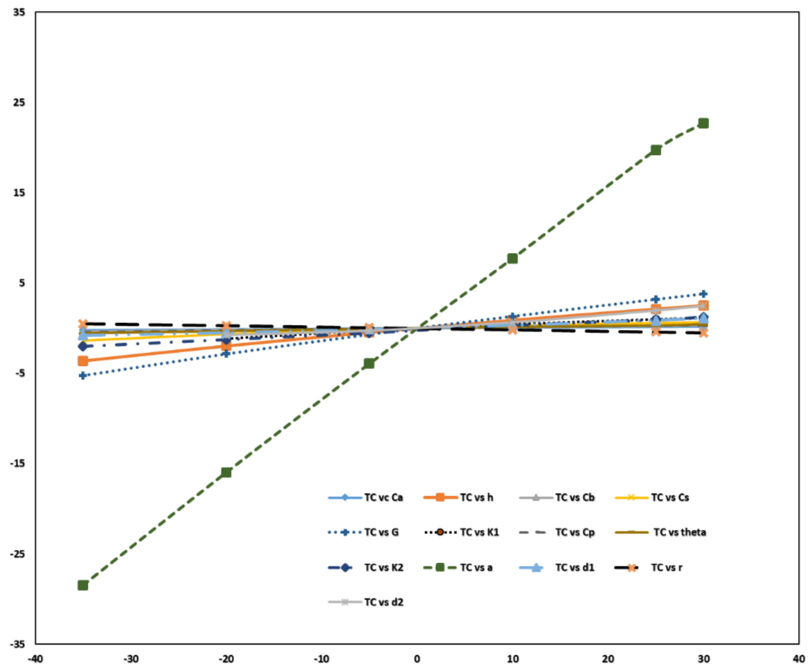

Figure 5. How the total cost changes with the change of key parameters.

5. Conclusion

This study presents an imperfect EPQ model for a multi-state system with deteriorating items and alternating production rates, while allowing shortages leading to partial backlogging and lost sale. In this model, the equipment's deterioration affects the quantity and quality of the outputs, i.e. the system operates in a degraded state. In such systems, the productivity is assumed to be dependent on the equipment's speed. In addition, it is assumed that the system produces both good and bad items. After screening, the imperfect items are disposed as a batch after the production process is completed, whilst the perfect quality items are used to meet customer demand. A portion of stock-out demand is allowed in the model formulation. The demand for the item is considered constant, and the deterioration rate follows an exponential function. It is assumed that items start to deteriorate right from when finished inventory begins to accumulate after production. Numerical examples were presented to demonstrate the use of the model. The model can be under different areas such as assembly lines for automotive parts subject to non-essential equipment failures, manufacturing plants for mechanical parts, in hydrometallurgical plants for production of metals such as copper, cobalt, zinc and other production systems configured in line with the model proposed in this paper. Sensitivity analysis indicated the parameters to which the cycle time, stock level and total cost are most sensitive. The model can be extended in many ways, for example, it may be extended to multiple machines problem and that of multiple items with rework. Also, the model can be extended to cases considering time dependent deterioration and different demand conditions like power demand and price dependent demand.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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