SOME PRODUCTS OF SUBGROUPS AND VANISHING CONJUGACY CLASS SIZES

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Abstract. In this note we investigate some products of subgroups and vanishing conjugacy class sizes of finite groups. We prove some supersolubility criteria for groups with restrictions on the vanishing conjugacy class sizes of their subgroups.

1. Introduction

All groups considered in this note will be finite.

The study of the structural influence of the conjugacy classes of a group is a question of interest in the Theory of Groups. In particular, the set of vanishing conjugacy class sizes has acquired special significance (see [3], [4] and [8]). An element x of a group G is called a *vanishing element* if there exists some irreducible character χ of G such that $\chi(x) = 0$; the conjugacy class x^G is called a *vanishing conjugacy class.* We denote $\mathrm{Van}(G)$ the set of all elements of G which are vanishing in G.

During the past three decades, factorised groups whose factors are connected by means of certain permutability properties, namely mutually permutable products, have been extensively studied (see [1]). Recall that two subgroups A and B of a group G are called *mutually permutable* if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that a group G is a *mutually permutable product* of the subgroups A and B if $G = AB$ and A and B are mutually permutable. Note that normal products are typical examples of mutually permutable products. However, the symmetric group of degree 3 is an easy example of a group which is a non-normal mutually permutable product.

Kong and Chen [10] studied normal products of groups in which every element of prime power order of each factor is vanishing. They proved:

Theorem 1.1. [10, Theorem A] *Let $G = AB$ be the product of two normal subgroups A and B of G. If $|x^G|$ is square-free for every $x \in \mathrm{Van}(A) \cup \mathrm{Van}(B)$ of prime power order, then G is supersoluble.*

On the other hand, Felipe, Martínez-Pastor and Ortiz-Sotomayor in [7] proved, as a consequence of a more general result, the following:

Theorem 1.2. *Let $G = AB$ be the mutually permutable product of the subgroups A and B. If $|x^G|$ is square-free for every $x \in A \cup B$ of prime power order that is vanishing in G, then G is supersoluble.*

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We prove an analogous result but instead of assuming that all elements of $A \cup B$ of prime power order that are vanishing in G have square-free conjugacy class size, we assume that all elements of prime power order in $\mathrm{Van}(A) \cup \mathrm{Van}(B)$ have square-free conjugacy class size, and it extends Theorem 1.1.

Theorem A. *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. If $|x^G|$ is square-free for every $x \in \mathrm{Van}(A) \cup \mathrm{Van}(B)$ of prime power order, then G is supersoluble.*

Theorem 1.1 is not true if only one of the factors is normal: the alternating group G of degree 4 is a product of a normal subgroup A of order 4 and a subgroup B of order 3 which is not supersoluble and $\mathrm{Van}(A) = \mathrm{Van}(B) = \emptyset$.

However, an additional condition allows us to obtain the following extension:

Theorem B. *Let $G = AB$ be a product of two subgroups A and B. Assume that A is normal in G and B permutes with every maximal subgroup of A. If $|x^G|$ is square-free for every $x \in \mathrm{Van}(A) \cup \mathrm{Van}(B)$ of prime power order, then G is supersoluble.*

2. Proofs

Given a normal subgroup N of a group G and $x \in G$, then both conjugacy class sizes $|x^N|$ and $|x^{G/N}|$ divide $|x^G|$. Moreover, there is a natural bijection between the set of irreducible characters of G/N and the set of all irreducible characters of G with N in their kernel. These facts, together with [1, Lemmas 1.2.7 and 4.1.10], show that the hypotheses of our main theorems are inherited by factor groups.

The proofs of our main results also depend on the following lemmas.

Lemma 2.1. [4, Theorem 1.5] *Let G be a group. If $|x^G|$ is square-free for every vanishing element of prime power order $x \in G$, then G is supersoluble.*

Lemma 2.2 ([8, Theorem 1.1])**.** *If G is nilpotent-by-supersoluble, then every non-vanishing element of G belongs to $F(G)$, the Fitting subgroup of G.*

Lemma 2.3. *Let N be a non-abelian minimal normal subgroup of G. Then there exists a non-trivial element x of N such that*

(i) *x of prime power order;*
(ii) *x is a vanishing element of H for any subgroup H such that $N \leq H \leq G$;*
(iii) *4 divides $|x^G|$.*

Proof. We may write $N = S_1 \times \cdots \times S_n$, where $S_i \cong S$ for some non-abelian simple group. Suppose that S has an irreducible character of p-defect zero for every prime p dividing $|S|$. Applying [3, Lemma 2.2], we conclude that every element of N is a vanishing element of any subgroup H such that $N \leq H \leq G$. Since S is not soluble, there exists a non-trivial p-element of S such that 4 divides $|x^S|$ using [5, Corollary]. Hence 4 divides $|x^G|$ as required.

We may assume that there exists a prime p such that S does not have an irreducible character of p-defect zero. By [4, Corollary 2.1], $S \cong \mathrm{A}_n$, $n \geq 7$, or S is isomorphic to an sporadic simple group. If $S \cong \mathrm{A}_n$, $n \geq 7$, then there exists an q-element x of S for some prime $q \geq 5$ such that 4 divides $|x^S|$ by [4, Lemma 2.4]. Note that every non-abelian simple group S has an irreducible character of q-defect zero for every prime $q \geq 5$ dividing $|S|$ (see [4, Corollary 2.1]). Hence the result follows by the above argument.

Suppose that S is isomorphic to one of the sporadic simple groups given in [4, Corollary 2.1]. Using [4, Lemma 2.4], we have that there exists a non-trivial element x of S

of prime power order such that 4 divides $|x^S|$ and there exists $\chi \in \mathrm{Irr}(S)$ with the following properties: χ extends to $\mathrm{Aut}(S)$ and χ vanishes on x^S. By [3, Proposition 2.3], $\chi \times \cdots \times \chi \in \mathrm{Irr}(N)$ extends to any subgroup H such that $N \leq H \leq G$. Hence x is a vanishing element of H and also 4 divides $|x^G|$. This concludes our proof. □

Proof of Theorem A. Assume the result is false and let G be a minimal counterexample. Applying Lemma 2.1, we obtain that A and B are proper subgroups of G. Since the hypotheses are inherited by factor groups and the class of all supersoluble groups is a saturated formation, it follows that G is a primitive group. Let N be the unique minimal normal subgroup of G. Then, G/N is supersoluble. By [1, Theorem 4.3.11], $N \leq A$ or $N \leq B$, suppose without loss of generality that $N \leq A$. Assume N is non-abelian. By Lemma 2.3, there exists a non-trivial element $x \in N$ of prime power order such that 4 divides $|x^G|$ and x is vanishing in A. This contradiction implies that N is a q-elementary abelian group of order q^n for some prime q, $n > 1$. In particular, G is nilpotent by supersoluble. By [1, Lemma 4.3.3, 4 and 5], $N \leq B$ or $B \cap N = 1$. Assume that $B \cap N = 1$. Since N is not cyclic, it follows that $N \leq C_G(B)$. Hence $B \leq C_G(N) = N \leq A$, and $G = A$ which is not possible. Consequently, $N \leq A \cap B$.

Since $G/N = (A/N)(B/N)$ is the mutually permutable product of A/N and B/N, there exists a minimal normal subgroup L/N of G/N contained in A/N or in B/N. Suppose $L/N \leq A/N$. Since G/N is supersoluble, it follows that L/N is a cyclic group of prime order p. Moreover $p \neq q$. Therefore $L = N\langle x\rangle$, where $|\langle x\rangle| = p$. By Lemma 2.2, x is a vanishing element of A. Since G is a primitive group, there exists a complement of N, M say, such that $MN = G$ and $M \cap N = 1$ and M is maximal in G. Hence $L/N \cong \langle x\rangle$ is a minimal normal subgroup of M. Since $\langle x\rangle$ is normal in M, $C_G(\langle x\rangle) \leqslant N_G(\langle x\rangle) = M$ or G. Since $N \cap M = 1$, $N_G(\langle x\rangle) = M$ and so $|N|$ divides $|x^G|$, our final contradiction. □

Proof of Theorem B. Suppose that the theorem is not true and let G be a minimal counterexample. Then G has a unique minimal normal subgroup N, N is not cyclic, G is a primitive group and A and B are proper subgroups of G. Moreover N is contained in A. Since A is a normal subgroup of G, we have that $|x^A|$ divides $|x^G|$. Applying Lemma 2.1, we obtain A is supersoluble and N is abelian. By [6, Theorem A.15.2], $C_G(N) = N$ and so $F(G) = N$. Note that $F(A)$ is a normal nilpotent subgroup of G contained in $F(G) = N$. Therefore $F(A) = N$.

Assume that N is a proper subgroup of A. Then there exists a minimal normal subgroup L/N of G/N such that $L/N \subseteq A/N$. We can argue as in Theorem A to reach a contradiction. Consequently, $A = N$, B is a maximal subgroup of G and $B \cap N = 1$. Since N is not cyclic, N has a maximal subgroup $D \neq 1$. By hypothesis, DB is a subgroup of G. Hence $D = N \cap DB$ is normalised by N and B. Thus D is a normal subgroup of G, contradicting the fact that N is minimal normal in G.

The proof of the theorem is now complete.

□

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