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Best Proximity Point Results in Generalized Metric Spaces

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Abstract In this paper we define a new class of mappings called $(\theta, \alpha^+)-$ proximal admissible contractionse and obtain a unique best proximity point for such mappings in the setting of complete generalized metric space. Our result is an extension of comparable results in the existing literature. Some examples are presented to support the results proved herein.

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1. Introduction and Preliminaries

Let (X, d) be a complete metric space. A point x in X is called a fixed point of a mapping $T : X \to X$ if $d(x, Tx) = 0$.

Banach contraction principle [1] states that *a selfmapping T on a complete metric space X has a unique fixed point in X provided that there exists $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \le \alpha d(x, y),$$

is satisfied for all $x, y \in X$.

Several authors have extended and generalized this principle in various directions. Recently, Branciari [2] defined a generalized metric space, where the axiom of triangular inequality in the definition of ordinary metric space was replaced with so called rectangular inequality by choosing four points instead of three points. He also proved a fixed point result in such spaces.

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Definition 1.1. [2] Let (X,d) be a nonempty set. A mapping $d : X \times X \to [0,\infty)$ is said to be a generalized metric on X if for any $x, y \in X$, $u, v \in X \setminus \{x, y\}$, the following conditions hold:

(1) $d(x,y) = 0 \Leftrightarrow x = y$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

The pair (X,d) is called a generalized metric space.

Definition 1.2. [2] Let (X,d) be a generalized metric space. A sequence $\{x_n\}$ in X is called:

(a) convergent to $x \in X$ if and only if $x_n \to x$ as $n \to \infty$. In this case, we write
$$\lim_{n\to\infty} x_n = x.$$
(b) Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.3. [2] A generalized metric space (X,d) is said to be complete if every Cauchy sequence in X converges to some element in X.

Several authors have obtained fixed point results in such spaces (see, [20, 21] and [3]) references are mentioned therein. Sarama [21] presented an example which shows that there exist a generalized metric space that does not satisfy any of the following conditions:

(1) The metric d is continuous in both variables.
(2) Each open ball is an open set.
(3) The topology induced by d is Hausdorff.
(4) Each convergent sequence is a Cauchy sequence.
(5) In particular, a sequence may converge to at the most one point.

Recently, Jleli and Samet [8] introduced a new class of mappings called $\theta-$contraction and established an interesting fixed point theorem for such mappings in the setting of generalized metric spaces.

Definition 1.4. [8] Let Δ_θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1): θ is increasing,
(Θ_2): for any sequence $\{t_n\}$ in $(0,\infty)$, $\lim_{n\to\infty} t_n = 0^+$ if and only if
$$\lim_{n\to\infty} \theta(t_n) = 1,$$
(Θ_3): there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t)-1}{t^r} = l$.

Example 1.5. [8] Let $i \in \{1,2,3\}$. Define $\theta_i : (0,\infty) \to (1,\infty)$ as follows:
$$\theta_1(t) = e^{\sqrt{te^t}},\ \theta_2(t) = 4^{\sqrt{t}},\ \theta_3(t) = e^{\sqrt{t}}.$$
Note that θ_1, θ_2, $\theta_3 \in \Delta_\theta$.

Jleli and Samet presented the following definition and fixed point theorem in [8].

Definition 1.6. [8] Let (X,d) be a generalized metric space and $\theta \in \Delta_\theta$. A mapping $T : X \to X$ is called $\theta-$contraction if
$$\theta\left(d\left(T(x), T(y)\right)\right) \leq \left[\theta\left(d\left(x,y\right)\right)\right]^k, \tag{1.1}$$
for any x, $y \in X$, where $d(T(x), T(y)) > 0$ and $0 \leq k < 1$.

Theorem 1.7. [8] *Let (X, d) be a generalized metric space and a selfmapping T on X a $\theta-$contraction. Then T has a unique fixed point in X.*

Let A and B be two nonempty subsets of a generalized metric space X such that $A \cap B = \phi$ and $T : A \to B$. In this case $d(x, Tx) > 0$ for any $x \in A$, and hence the solution of a functional equation $x = Tx$ does not exist. It is then an interesting problem to find an element $x^* \in A$ which solves the following optimization problem:

$$\min \{d(x, Tx) : x \in A\}.$$

A point x^* in A which not only solves the above optimization problem but also satisfies $d(x^*, Tx^*) = d(A, B)$ is known as best proximity point of mapping T, where $A \cap B = \phi$ and

$$d(A, B) = \inf\{d(x, y), \text{ where } x \in A \text{ and } y \in B\}.$$

If $A \cap B \neq \phi$ then $d(A, B) = 0$, the best proximity point of T becomes a fixed point of T.

Ky Fan [6] proved the following result:

Theorem 1.8. [6] *If A is nonempty convex subset of a Housdroff locally convex topologically vector space X and $T : A \to X$ is a continuous mapping, then there exist an element x^* in A such that*

$$d(x^*, Tx^*) = d(A, B).$$

Several authors have extended and generalized Fan's theorem in different directions (see, [4, 5, 9–11, 13–19, 22, 23, 25]).

Following definitions are also needed in the sequel.

Definition 1.9. [5] If $x \in X$ and A is any nonempty subset of X, then

$$d(x, A) = \inf \{d(x, a) : a \in A\},$$

is the *distance of a point x from a set A*. Also,

$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\}, \text{ and}$$
$$B_0 = \{x \in B : d(x, y) = d(A, B), \text{ for some } y \in A\}.$$

Throughout this paper, we assume that A and B are nonempty disjoint subsets of a complete generalized metric space (X, d).

We now state "the $p-$ property" and "the weak $p-$property" in the setting of generalized metric space.

Definition 1.10. [12, 24] Let $A_0 \neq \phi$. The pair (A, B) is said to satisfy:

(a) the $p-$property, if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$ with

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \text{implies } d(x_1, x_2) = d(y_1, y_2).$$

(b) the weak $p-$property, if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ with

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \text{implies } d(x_1, x_2) \leq d(y_1, y_2).$$

Definition 1.11. [7] Let $\alpha : A\times A \to [-\infty,\infty)$. A mapping $T : A \to B$ is called proximal α^+- admissible, if for any $x,\ y,\ u,\ v \in A$,

$$\left.\begin{array}{r}\alpha(x,y)\geq 0\\ d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow \alpha(u,v)\geq 0. \tag{1.2}$$

In this paper, we establish a best proximity point result for $(\theta,\alpha^+)-$proximal admissible contraction in the set up of generalized metric spaces.

2. Main Results

We start the section by introducing the notion of $(\theta,\alpha^+)-$proximal admissible contraction of first kind and second kind as follows.

Definition 2.1. Let $\alpha : A\times A \to [-\infty,\infty)$ and $\theta \in \triangle_\theta$. A mapping $T : A \to B$ is said to be $(\theta,\alpha^+)-$proximal admissible contraction of

(i) first kind, if

$$\left.\begin{array}{r}\alpha(x,y)\geq 0\\ d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow 0\leq \alpha\left(u,v\right)+\theta\left[d\left(u,v\right)\right]\leq\left[\theta\left(d\left(x,y\right)\right)\right]^{k}, \tag{2.1}$$

(ii) second kind, if

$$\left.\begin{array}{r}\alpha(x,y)\geq 0\\ d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow 0\leq \alpha\left(u,v\right)+\theta\left[d\left(Tx,Ty\right)\right]\leq\left[\theta\left(d\left(x,y\right)\right)\right]^{k},$$

where $x,y,u,v \in A$ and $k \in (0,1)$.

If we choose $\alpha(x,y) = 0$, for all $x,y \in A$. Then we obtain the following definitions.

Definition 2.2. Let $\alpha : A\times A \to [-\infty,\infty)$ and $\theta \in \triangle_\theta$. A mapping $T : A \to B$ is said to be $\theta-$ proximal admissible contraction of:

(i) first kind, if

$$\left.\begin{array}{r}d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow \theta\left[d\left(u,v\right)\right]\leq\left[\theta\left(d\left(x,y\right)\right)\right]^{k}, \tag{2.2}$$

(ii) second kind, if

$$\left.\begin{array}{r}d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow \theta\left[d\left(Tx,Ty\right)\right]\leq\left[\theta\left(d\left(x,y\right)\right)\right]^{k}, \tag{2.3}$$

where $x,y,u,v \in A$ and $k \in (0,1)$.

Now we prove our main result of the article.

Theorem 2.3. *Let $\alpha : A\times A \to [-\infty,\infty)$, $T : A \to B$ be a $(\theta,\alpha^+)-$proximal admissible contraction of first kind. Moreover, if A_0 is nonempty closed set in X and $T(A_0) \subseteq B_0$, then there exist a best proximity point of a mapping T provided that there exists $x_0, x_1 \in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B)\ \textit{and}\ \alpha\left(x_0,x_1\right)\geq 0.$$

Moreover, if $\alpha\left(x,y\right)\geq 0$, for every $x,y \in A$ satisfying $d(x,Tx) = d(A,B) = d(y,Ty)$, then x^* is a unique best proximity point of a mapping T.

Proof. Let $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 0$. As $Tx_0 \in T(A_0) \subseteq B_0$, there exists x_2 in A_0 such that $d(x_2, Tx_1) = d(A,B)$. Since T is proximal α^+- admissible, we have $\alpha(x_1, x_2) \geq 0$. Similarly, by $T(A_0) \subseteq B_0$, we obtain a point $x_3 \in A_0$ such that $d(x_3, Tx_2) = d(A,B)$ which further implies that $\alpha(x_2, x_3) \geq 0$. Continuing this way, we can obtain a sequence $\{x_n\}$ in A_0 such that

$$\begin{aligned} &d(x_n, Tx_{n-1}) = d(A,B), \\ &d(x_{n+1}, Tx_n) = d(A,B),\ \alpha(x_n, x_{n+1}) \geq 0, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{2.4}$$

Since T is $(\theta, \alpha^+)-$ proximal admissible contraction of first kind, we have

$$\alpha(x_n, x_{n+1}) + \theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^k.$$

Since $\alpha(u,v) \geq 0$ for all $x, y \in A$, we obtain that

$$\theta(d(x_n, x_{n+1})) \leq \alpha(x_n, x_{n+1}) + \theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^k.$$

Thus

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^k, \text{ for all } n \in \mathbb{N}.$$

Now,

$$\begin{aligned} 1 &< \theta(d(x_n, x_{n+1})) \\ &\leq (\theta(d(x_{n-1}, x_n)))^k \\ &\leq (\theta(d(x_{n-2}, x_{n-1})))^{k^2} \\ &\ \vdots \\ &\leq \theta(d(x_0, x_1))^{k^n}, \end{aligned} \tag{2.5}$$

gives that

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1^+,$$

and hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

From definition of θ, there exist $r \in (0,1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[\theta(d(x_n, x_{n+1}))]^r} = l.$$

Suppose that $l < \infty$. In this case, for $C = \frac{l}{2} > 0$, there exist some $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{[\theta(d(x_n, x_{n+1}))]^r} - l \right| \leq C, \text{ for all } n \geq n_0 \in \mathbb{N}.$$

This implies that

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[\theta(d(x_n, x_{n+1}))]^r} \geq l - C = l - \frac{l}{2} = \frac{l}{2} = C, \text{ for all } n \geq n_0 \in \mathbb{N}.$$

Thus

$$n[\theta(d(x_n, x_{n+1}))]^r \leq nD[\theta(d(x_n, x_{n+1})) - 1],$$

holds for all $n \geq n_0 \in \mathbb{N}$, where $D = \frac{1}{C}$. Now suppose that $l = \infty$ and $C > 0$ is any arbitrary positive real number. Then there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1}{\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r} \geq C, \text{ for all } n \geq n_0.$$

Thus

$$n\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r \leq nD\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right],$$

holds for all $n \geq n_0 \in \mathbb{N}$. So, in all cases, there exist $D > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r \leq nD\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right], \text{ for all } n \geq n_0 \in \mathbb{N}.$$

From (2.5), we have

$$n\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r \leq nD\left[\theta\left(d\left(x_0, x_1\right)\right)^{k^n} - 1\right], \text{ for all } n \geq n_0 \in \mathbb{N}.$$

On taking limit as $n \to \infty$ on both side of the above inequality, we obtain that

$$\lim_{n\to\infty} n\left[d\left(x_n, x_{n+1}\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$, such that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{1/r}}, \text{ for all } n \geq n_1 \in \mathbb{N}. \tag{2.6}$$

Suppose that $x_n \neq x_m$ for every $n, m \in \mathbb{N}$ with $n \neq m$. By (2.1), we have

$$\theta\left(d\left(x_n, x_{n+2}\right)\right) \leq \alpha\left(x_n, x_{n+2}\right) + \theta\left[d\left(x_n, x_{n+2}\right)\right] \leq \left[\theta(d\left(x_{n-1}, x_{n+1}\right))\right]^k.$$

That is,

$$\theta\left(d\left(x_n, x_{n+2}\right)\right) \leq \left[\theta(d\left(x_{n-1}, x_{n+1}\right))\right]^k, \text{ for all } n \in \mathbb{N}.$$

Therefore
$1 < \theta\left[d\left(x_n, x_{n+2}\right)\right] \leq \left[\theta\left(d\left(x_{n-1}, x_{n+1}\right)\right)\right]^k \leq \left[\theta\left(d\left(x_{n-2}, x_n\right)\right)\right]^{k^2} \leq \ldots \leq \left[\theta\left(d\left(x_0, x_2\right)\right)\right]^{k^n}$,
for all $n \in \mathbb{N}$, gives that

$$\lim_{n\to\infty} \theta[d(x_n, x_{n+2})] = 1^+,$$

and hence,

$$\lim_{n\to\infty} d\left(x_n, x_{n+2}\right) = 0.$$

Now, from the definition of θ and using the similar arguments as above, we can choose $n_2 \in \mathbb{N}$, such that

$$d\left(x_n, x_{n+2}\right) \leq \frac{1}{n^{1/r}}, \text{ for all } n \geq n_2 \in \mathbb{N}.$$

Let $N = \max\{n_0, n_1\}$. We consider two cases.

Case 1: If $m > 2$ and m is odd, then $m = 2L + 1$, where $L \geq 1$. Note that

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d\left(x_{n+2}, x_{n+m}\right) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+m}) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d\left(x_{n+2}, x_{n+3}\right) + \dots \\
&\quad + d(x_{n+m-1}, x_{n+m}) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \frac{1}{(n+2)^{1/r}} + \dots + \frac{1}{(n+m-1)^{1/r}} \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}},
\end{aligned}$$

for all $m \geq n \geq N \in \mathbb{N}$.

Case 2: If $m > 2$ and m is even, then $m = 2L$, where $L \geq 2$. Note that

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d\left(x_{n+3}, x_{n+m}\right) \\
&\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+5}) \\
&\quad + d(x_{n+5}, x_{n+m}) \\
&\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + \\
&\quad + d(x_{n+m-1}, x_{n+m}) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \dots + \frac{1}{(n+2L-1)^{1/r}} \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}},
\end{aligned}$$

for all $m \geq n \geq N$. Thus in all the cases, we have

$$d(x_n, x_{n+m}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}, \text{ for all } m, n \geq N \in \mathbb{N}.$$

On taking limit as $n \to \infty$, we have

$$d(x_n, x_{n+m}) \to 0, \text{ for all } m, n \geq N \in \mathbb{N},$$

which shows that $\{x_n\}$ is a Cauchy sequence in $A_0 \subseteq A \subset X$. As X is complete and A_0 is closed, there exists $x^* \in A_0$ such that $x_n \to x^* \in A_0$. Note that $T(A_0) \subseteq B_0$ gives that $Tx^* \in B_0$ and hence we may choose a point $z \in A_0$ such that $z \neq x^*$ and

$$d(z, Tx^*) = d(A, B). \tag{2.7}$$

By (2.4) and (2.7), we have

$$\begin{aligned}
\theta(d(z, x_{n+1})) &\leq \alpha(z, x_{n+1}) + \theta(d(z, x_{n+1})) \\
&\leq \left[\theta(d(x^*, x_n))\right]^k \\
&< \theta(d(x^*, x_n)).
\end{aligned}$$

which implies that

$$d(z, x_{n+1}) < d(x^*, x_n). \tag{2.8}$$

Now, by the rectangular property, (2.6) and (2.8), we obtain that

$$\begin{aligned} d(x^*, z) &\leq d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, z) \\ &\leq d(x^*, x_n) + \frac{1}{n^{1/r}} + d(x^*, x_n). \end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, we conclude that $z = x^*$. Hence

$$d(A, B) = d(x^*, Tx^*),$$

and x^* is a best proximity point of the mapping T.

Uniqueness: We now show that x^* is a unique best proximity point of the mapping T. Suppose that x^* and w are two best proximity points of a mapping T, that is,

$$d(w, Tw) = d(A, B) = d(x^*, Tx^*).$$

Since $\alpha(x, y) \geq 0$, for every $x, y \in A$ and T is $(\theta, \alpha^+)-$ proximal admissible contraction of first kind, we obtain that

$$\begin{aligned} \theta\left(d(x^*, w)\right) &\leq \alpha\left(x^*, w\right) + \theta\left(d\left(x^*, w\right)\right) \\ &\leq \left[\theta\left(d\left(x^*, w\right)\right)\right]^k \\ &< \theta\left(d\left(x^*, w\right)\right), \end{aligned}$$

a contradiction. Therefore, $x^* = w$. Thus, $x^*$is a unique best proximity point of the mapping T. ∎

Corollary 2.4. *Let $\alpha : A \times A \to [-\infty, \infty)$, $T : A \to B$ be a $\theta-$ proximal admissible contraction of first kind. Suppose that A_0 is nonempty closed set and $T(A_0) \subseteq B_0$. Then there exist a unique best proximity point of T provided that there exists $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B).$$

Proof. The result follows from theorem (2.3) by choosing $\alpha(x, y) = 0$, for all $x, y \in A$. ∎

To support the Theorem (2.3), following example is provided.

Example 2.5. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = d(y, x) \text{ and } d(x, x) = 0 \text{ for all } x, y \in X,$$

and

$$\begin{aligned} &d(4, 6) = d(5, 7) = 1, \\ &d(2, 7) = d(4, 7) = d(6, 1) = d(6, 7) = 7, \\ &d(3, 7) = d(2, 6) = 2, \ d(0, 2) = d(1, 3) = 3, \\ &d(2, 4) = d(3, 5) = 4, \ d(0, 4) = d(1, 5) = 5, \\ &d\,(0, 1) = d(2, 3) = d(4, 5) = d(6, 7) = d(0, 5) = d(0, 7) = 7, \\ &d(0, 8) = d(2, 8) = d(4, 8) = d(6, 8) = d(1, 8) = d(6, 5) = d(4, 1) = d(2, 1) = 8. \\ &d(0, 6) = d(1, 7) = d(6, 3) = d(4, 3) = d(0, 3) = d(3, 8) = d(5, 8) = d(2, 5) \\ &\qquad = d(7, 8) = 8. \end{aligned}$$

Note that (X, d) is a generalized metric space but not a metric space. Indeed,

$$d\,(0, 6) \not\leq d(0, 2) + d(2, 6).$$

If $A = \{2, 4, 6\}$ and $B = \{1, 5, 7\}$, then $d(A, B) = 7$, $A = A_0$ and $B = B_0$. Define mapping $T : A \to B$ by

$$T(6) = 1,\ T(2) = 5,\ T(4) = 1.$$

Note that $T(A_0) \subseteq B_0$. Define a mapping $\alpha : A \times A \to [-\infty, \infty)$ by

$$\alpha(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in \{(2, 4), (4, 2), (4, 6), (6, 4)\} \\ 0, & \text{if } x, y \in \{(2, 6), (6, 2)\}. \end{cases}$$

Define a function $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(t) = e^{\sqrt{t}}.$$

Note that $u = 4, v = 6, x = 2, y = 4 \in A$ with $\alpha(2, 4) = \dfrac{1}{2} \geq 0$ and

$$d(4, T(2)) = d(6, T(4)) = 7,$$

gives that

$$\alpha(u, v) + \theta[d(u, v)] = 0.584 \leq [\theta(d(x, y))]^k = k.$$

The inequality (2.2) is satisfied when $k \in [0.584, 1)$. Thus, all conditions of Theorem (2.3) holds. Moreover, $x = 6$ is a unique best proximity point of T.

Theorem 2.6. *Let $\alpha : A \times A \to [-\infty, \infty)$, $T : A \to B$ be a $(\theta, \alpha^+)-$proximal admissible contraction of second kind and A_0 a nonempty closed set such that $T(A_0) \subseteq B_0$. Then there exist a best proximity point of a mapping T provided that (A, B) satisfies weak $p-$property and there exists $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 0.$$

Moreover, if $\alpha(x, y) \geq 0$, for every $x, y \in A$ satisfying $d(x, Tx) = d(A, B) = d(y, Ty)$, then x^* is a unique best proximity point of a mapping T.

Proof. Let $x_0, x_1 \in A_0$ be such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 0$. Following arguments similar to those given in the proof of Theorem (2.3), we obtain a sequence $\{x_n\}$ in A_0 such that

$$\begin{aligned} &d(x_n, Tx_{n-1}) = d(A, B), \\ &d(x_{n+1}, Tx_n) = d(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq 0, \text{ for all } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{2.9}$$

Since T is $(\theta, \alpha^+)-$proximal admissible contraction of second kind, we have

$$\alpha(x_n, x_{n+1}) + \theta(d(Tx_{n-1}, Tx_n)) \leq (\theta(d(x_{n-1}, x_n)))^k.$$

As $\alpha(u, v) \geq 0$ for all $x, y \in A$, we obtain that

$$\theta(d(Tx_{n-1}, Tx_n)) \leq \alpha(x_n, x_{n+1}) + \theta(d(Tx_{n-1}, Tx_n)) \leq (\theta(d(x_{n-1}, x_n)))^k.$$

Using weak $p-$property of the pair (A, B) given by (2.9), we have

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n), \text{ for all } n \in \mathbb{N}.$$

As θ is increasing,

$$\theta(d(x_n, x_{n+1})) \leq \theta(d(Tx_{n-1}, Tx_n)).$$

Thus

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^k, \text{ for all } n \in \mathbb{N}.$$

Again by using the arguments as in the proof of Theorem (2.3), we obtain

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{1/r}}, \text{ for all } n \geq n_1 \in \mathbb{N}. \tag{2.10}$$

Suppose that $x_n \neq x_m$ for every $n, m \in \mathbb{N}$ with $n \neq m$. By (2.9), we have

$$\theta\left(d\left(Tx_{n-1}, Tx_{n+1}\right)\right) \leq \alpha\left(x_n, x_{n+2}\right)+\theta\left[d\left(Tx_{n-1}, Tx_{n+1}\right)\right] \leq \left[\theta(d\left(x_{n-1}, x_{n+1}\right))\right]^k.$$

As (A, B) satisfies weak $p-$property, from (2.9), we have

$$d(x_n, x_{n+2}) \leq d\left(Tx_{n-1}, Tx_{n+1}\right), \text{ for all } n \geq n_1,$$

which further implies that

$$\theta\left(d\left(x_n, x_{n+2}\right)\right) \leq \left[\theta(d\left(x_{n-1}, x_{n+1}\right))\right]^k, \text{ for all } n \in \mathbb{N}.$$

By using the similar arguments as in the proof of Theorem (2.3) we obtain that $\{x_n\}$ is a Cauchy sequence in $A_0 \subseteq A \subset X$ and

$$d(z, Tx^*) = d(A, B). \tag{2.11}$$

By (2.9) and (2.11), we have

$$\begin{aligned}
\theta(d(Tx^*, Tx_n)) &\leq \alpha(z, x_n) + \theta(d(Tx^*, Tx_n)) \\
&\leq \left(\theta\left(d\left(x^*, x_n\right)\right)\right)^k \\
&< \theta(d(x^*, x_n)).
\end{aligned}$$

which implies that

$$d(Tx^*, Tx_n) < d(x^*, x_n).$$

By the weak $p-$property, we have

$$d(z, x_{n+1}) \leq d(Tx^*, Tx_n) < d(x^*, x_n). \tag{2.12}$$

Then, by the rectangular property, (2.10) and (2.12), we have

$$\begin{aligned}
d(x^*, z) &\leq d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, z) \\
&\leq d(x^*, x_n) + \frac{1}{n^{1/r}} + d(x^*, x_n),
\end{aligned}$$

which on taking limit $n \to \infty$ implies that $z = x^*$. Hence

$$d(A, B) = d(x^*, Tx^*).$$

and x^* is a best proximity point of the mapping T.

Uniqueness: Now, we show that x^* is a unique best proximity point of the mapping T. Let x^* and w be two best proximity points of a mapping T, that is,

$$d(w, Tw) = d(A, B) = d(x^*, Tx^*).$$

Since $\alpha(x, y) \geq 0$, for every $x, y \in A$ and by using properties of T and reasoning as above, we obtain that

$$\begin{aligned}
\theta\left(d(x^*, w)\right) &\leq \theta\left(d(Tx^*, Tw)\right) \\
&\leq \alpha\left(x^*, w\right) + \theta\left(d\left(Tx^*, Tw\right)\right) \\
&\leq \left[\theta\left(d\left(x^*, w\right)\right)\right]^k \\
&< \theta\left(d\left(x^*, w\right)\right),
\end{aligned}$$

a contradiction. Therefore, $x^* = w$. Thus x^* is a unique best proximity point of the mapping T. ■

Corollary 2.7. *Let $\alpha : A \times A \to [-\infty, \infty)$, $T : A \to B$ be a $\theta-$ proximal admissible contraction of second kind and A_0 a nonempty closed set with $T(A_0) \subseteq B_0$. Then there exist a unique best proximity point of T provided that (A, B) satisfies weak $p-$property and there exists $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B).$$

Proof. The result follows from Theorem (2.6) by choosing $\alpha(x, y) = 0$, for all $x, y \in A$. ■

We now provide the following example to support the Theorem (2.6).

Example 2.8. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = d(y, x) \text{ and } d(x, x) = 0 \text{ for all } x, y \in X,$$

and

$$\begin{aligned}
&d(4,6) = d(5,7) = 1,\\
&d(3,7) = d(2,6) = 2,\ d(0,2) = d(1,3) = 3,\\
&d(2,4) = d(3,5) = 4,\ d(0,4) = d(1,5) = 5,\\
&d(0,8) = d(2,8) = d(4,8) = d(6,8) = d(1,8) = d(4,1) = 8,\\
&d(0,1) = d(2,3) = d(4,5) = d(6,7) = d(0,5) = d(0,7) = d(2,1) = 7,\\
&d(2,7) = d(4,7) = d(6,1) = d(6,5) = d(6,7) = d(2,5) = 7,\\
&d(0,6) = d(1,7) = d(6,3) = d(4,3) = d(0,3) = d(3,8) = d(5,8) = d(7,8) = 8.
\end{aligned}$$

Note that (X, d) is a generalized metric space but not a metric space. Indeed,

$$d(0,6) \nleq d(0,2) + d(2,6).$$

If $A = \{0, 2, 4\}$ and $B = \{1, 3, 5\}$, then $d(A, B) = 7$, $A = A_0$, $B = B_0$ and (A, B) satisfies the weak $p-$property. Define a mapping $T : A \to B$ by

$$T(0) = 3,\ T(4) = 1,\ T(2) = 3.$$

Note that $T(A_0) \subseteq B_0$. Define a mapping $\alpha : A \times A \to [-\infty, \infty)$ by

$$\alpha(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in \{(0,2), (2,0), (2,4), (4,2)\} \\ 0, & \text{if } x, y \in \{(0,4), (4,0)\}. \end{cases}$$

Define a function $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(t) = e^{\sqrt{t}}.$$

Note that $u = 0, v = 2, x = 4, y = 0 \in A$ with $\alpha(2, 4) = \dfrac{1}{2} \geq 0$ and

$$d(0, T(4)) = d(2, T(0)) = 7,$$

gives that

$$\alpha(u, v) + \theta[d(T(x), T(y))] = 0.8125 \leq [\theta(d(x, y))]^k = k.$$

The inequality (2.3) is satisfied when $k = [0.8125, 1)$. Thus all conditions of Theorem (2.6) holds. Moreover, $x = 2$ is a unique best proximity point of T.

3. Application to Fixed Point Theory

If we take $A = B = X$, then from Definition (1.11), proximal $\alpha^{+}-$ admissible mapping implies

$$d(u, Tx) = d(A, B) = 0 \Rightarrow u = Tx,$$

and

$$d(v, Ty) = d(A, B) = 0 \Rightarrow v = Ty.$$

Since $u = Tx$ and $v = Ty$, the inequality (1.2) becomes

$$\alpha(x, y) \geq 0 \text{ implies } \alpha(Tx, Ty) \geq 0,$$

for all $x, y \in X$.

Remark 3.1. Note that, every proximal $\alpha^{+}-$ admissible self mapping is $\alpha^{+}-$ admissible mapping.

Remark 3.2. If $T : X \to X$, then $(\theta, \alpha^{+})-$proximal admissible contraction of first kind and second kind satisfies:

$$\alpha\left(Tx, Ty\right) + \theta\left[d\left(Tx, Ty\right)\right] \leq \left[\theta\left(d\left(x, y\right)\right)\right]^{k}, \tag{3.1}$$

where $\theta \in \triangle_{\theta}$, $k \in (0, 1)$, $u = Tx$ and $v = Ty$, for all $x, y \in X$.

Definition 3.3. A self mapping T and $\alpha : X \times X \to [-\infty, \infty)$ satisfying (3.1) is known as $(\theta, \alpha)-$contraction.

Corollary 3.4. *Let X be a complete generalized metric space, $\alpha : X \times X \to [-\infty, \infty)$ and $T : X \to X$ a $(\theta, \alpha)-$ contraction. If $\{x_n\}$ is a sequence in X such that $\alpha\left(x_n, x_{n+1}\right) \geq 0$ and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha\left(x_n, x\right) \geq 0$ for all $n \in N$. Then there exists a fixed point of T provided that $x_0 \in X$ such that $\alpha\left(x_0, Tx_0\right) \geq 0$. Moreover, if $\alpha\left(x, y\right) \geq 0$, for every $x, y \in X$, then x^{*} is a unique fixed point of a mapping T.*

Proof. Let $A = B = X$. We now show that T satisfies $(\theta, \alpha^{+})-$proximal admissible contraction of first kind and second kind. Note that

$$\begin{aligned} \alpha\left(x, y\right) &\geq 0 \\ d\left(u, Tx\right) &= d\left(A, B\right) \\ d\left(v, Ty\right) &= d\left(A, B\right). \end{aligned}$$

for all $x, y, u, v \in A$. As $d\left(A, B\right) = 0$, $u = Tx$ and $v = Ty$. Since T satisfies condition (3.1). Therefore

$$\alpha\left(u, v\right) + \left[\theta\left(d\left(u, v\right)\right)\right] = \alpha\left(Tx, Ty\right) + \theta\left(d\left(Tx, Ty\right)\right) \leq \left[\theta\left(d\left(x, y\right)\right)\right]^{k},$$

implies that

$$\alpha\left(u, v\right) + \left[\theta\left(d\left(u, v\right)\right)\right] \leq \left[\theta\left(d\left(x, y\right)\right)\right]^{k},$$

which further implies that T is $(\theta, \alpha^{+})-$ proximal admissible contraction of first kind. Also,

$$\alpha\left(u, v\right) + \left[\theta\left(d\left(Tx, Ty\right)\right)\right] = \alpha\left(Tx, Ty\right) + \theta\left(d\left(Tx, Ty\right)\right) \leq \left[\theta\left(d\left(x, y\right)\right)\right]^{k},$$

implies that

$$\alpha\left(u, v\right) + \left[\theta\left(d\left(Tx, Ty\right)\right)\right] \leq \left[\theta\left(d\left(x, y\right)\right)\right]^{k},$$

which further implies that T is $(\theta, \alpha^+)-$ proximal admissible contraction of second kind. Since T is proximal α^+- admissible, by Remark (3.1), T is α^+- admissible mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 0$ and $d(x, Tx) = d(A, B) = 0$, then x^* is a fixed point of T.

Uniqueness: Let $\alpha(x, y) \geq 0$ for all $x, y \in X$. Now we will show that x^* is a unique fixed point of a mapping T. On the contrary, suppose that w^* is another fixed point of mapping T. Hence

$$d(w^*, Tw^*) = 0.$$

Then by using the properties of T, we obtain

$$\begin{aligned}\theta(d(x^*, w^*)) &= \theta(d(Tx^*, Tw^*))\\ &\leq \alpha(Tx^*, Tw^*) + \theta(d(Tx^*, Tw^*))\\ &\leq (\theta(d(x^*, w^*)))^k\\ &< \theta(d(x^*, w^*)),\end{aligned}$$

a contradiction. Hence x^* is a unique fixed point of mapping T. ■

Remark 3.5. By taking $\alpha(x, y) = 0$ for all $x, y \in X$ in the above Corollary (3.4), we obtain the main result (1.7) of Jleli [8].

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