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Pricing collateralised options in the presence of counterparty credit risk: An extension of the Heston–Nandi model

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In this paper, a closed-form expression for a collateralised European option in the presence of counterparty credit risk and stochastic volatility is derived. The model is applied to Standard and Poor's 500 index options. The model prices obtained are consistent with expectations.

Keywords: Collateral, Counterparty credit risk, GARCH, Option pricing.

1. Introduction

The Global Financial Crisis (GFC) of 2008 has changed financial markets permanently, especially with regard to the pricing of derivatives. Before the GFC, derivative pricing was typically done using the Black–Scholes model (Black and Scholes, 1973). The basic Black–Scholes theory presumes, in essence, that transactions have no balance sheet impact for a derivative market maker. In a pre-GFC environment participants ignored most of their balance sheet costs. However, post-GFC derivatives pricing has evolved to account for numerous frictional costs; examples include differential rate spreads, collateral, as well as default intensity of derivative counterparts.

An important assumption required for the Black–Scholes framework is the existence of a unique risk-free rate. This is not necessarily a reasonable assumption. Piterbarg (2010) developed a framework that specifically relaxes this assumption. The purpose of the model derived by Piterbarg (2010) is to price derivatives in the presence of collateral.

When using the Black–Scholes model for the pricing of derivatives, it is also assumed that volatility remains constant over time. Heston (1993) extended the model to incorporate stochastic volatility. Later, Heston and Nandi (2000) derived a model in which volatility follows a generalised autoregressive conditional heteroskedasticity (GARCH) process that allows for a closed-form solution for European call options. Note that this model assumes the existence of a risk-free rate and no counterparty credit risk.

As a solution to the impractical assumption of no counterparty credit risk in the Heston–Nandi model, Wang (2017) extended this model to incorporate counterparty credit risk for vanilla options. This has also been extended to exotic options by Wang (2020). However, this model also assumes

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the existence of a risk-free rate. Furthermore, Von Boetticher (2017) extended the work by Heston and Nandi (2000) to incorporate the Piterbarg framework.

Quantitative finance has developed into a multidisciplinary field of study incorporating methods of mathematical statistics, physics, and numerical analysis, for example. In this paper, we show a rigorous and pedagogical extension of the well-known Heston–Nandi model to the pricing of default risky collateralised options in a mathematical statistics setting. Essentially, the work by Von Boetticher (2017) is extended to the pricing of collateralised options using the Heston–Nandi model in the Piterbarg framework, in the presence of counterparty credit risk. This is especially relevant after the GFC given regulatory changes in the industry.

The rest of this paper is structured as follows. Section 2 reviews the recent and relevant literature, Section 3 focuses on the theoretical framework and derivation of the pricing model, Section 4 focuses on the empirical results of a numerical example, and finally concluding remarks are considered in Section 5.

2. Literature review

Research focusing on the pricing of derivatives using GARCH models to model volatility is well documented in the literature. Duan (1995) initially considered a risk-neutral pricing framework based on GARCH volatility modelling. This allowed for the pricing of derivatives with time-varying volatility in the Black and Scholes (1973) framework. A shortcoming of the model by Duan (1995) (based on a non-linear asymmetric GARCH model) is that it does not have a closed-form solution, and therefore numerical methods are required.

Heston and Nandi (2000) extended the work by Duan (1995) and derived a closed-form solution for vanilla options when volatility is modelled using a GARCH process. This model addresses the assumption of constant volatility in the Black–Scholes framework. However, it still requires the assumption of a unique risk-free rate and no credit risk; these are not necessarily reasonable assumptions.

The GFC of 2008 has changed the way in which derivative instrument trades are conducted. An important factor that needs to be considered in modern pricing frameworks is the presence of collateral (Hunzinger and Labuschagne, 2015). Piterbarg (2010) extended the Black–Scholes framework to the pricing of derivatives in the presence of collateral. This framework allows for three different interest rates; the discount rate is dependent on the amount of collateral that is posted. Von Boetticher (2017) extended the work by Piterbarg (2010) to incorporate the Heston–Nandi methodology to address the constant volatility assumption.

The Heston–Nandi model in the Black–Scholes framework requires the assumption of no counterparty credit risk. The GFC of 2008 has clearly shown that counterparty credit risk is an important factor to consider when pricing derivative instruments. To address this problem, Wang (2017) extended the Heston–Nandi model to incorporate counterparty credit risk. This model allows for correlation between the conditional variance of the underlying asset and the default intensity process.

Wang (2017) shows numerically that vanilla options are cheaper in the presence of counterparty credit risk. Intuitively, this makes sense because the option holder suffers losses if the counterpary defaults. This model was later extended to the pricing of executive stock options (Wang, 2018) and Asian options (Wang, 2020). However, this model still relies on a unique risk-free rate (single-curve framework), and does not take collateral into account. Therefore, this paper contributes to the existing literature by extending the Heston–Nandi model to the pricing of derivatives in the presence of collateral and counterparty credit risk.

3. Theoretical framework

In this section, the theory applied in this paper is outlined. This section is divided into four subsections. The first focuses on the Heston–Nandi model (in the Black–Scholes framework). The second subsection briefly discusses the Piterbarg framework (Black–Scholes with collateral). Thereafter, the Heston–Nandi model with collateral is briefly outlined. Finally, the Heston–Nandi model with collateral and counterparty credit risk is considered.

3.1 Heston–Nandi model

The main assumption of the model derived by Heston and Nandi (2000) is that the asset price dynamics under the real-world measure, P, are given by

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t + \sqrt{h_t} z_t,$$

where S_t is the asset price at time t, r is the constant risk-free rate, λ is the unit risk premium, and z_t is a standard normal random variable. Furthermore, the conditional variance is modelled using the following GARCH process:

$$h_t = \alpha_0 + \beta_1 h_{t-1} + \alpha_1 (z_{t-1} - \delta_1 \sqrt{h_{t-1}})^2.$$

The conditional generating function (given filtration $\mathcal{F}_t$) of the asset price under the measure P is given by

$$f_{BS}(t, \phi) = \mathbb{E}^P \left[S_T^{\phi} | \mathcal{F}_t \right].$$

The conditional generating function is dependent on the parameters and state variables, however, this is suppressed for notational convenience.

Heston and Nandi (2000) show that the asset price dynamics under the risk-neutral measure Q in the Black–Scholes framework are given by

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2} h_t + \sqrt{h_t} z_t^*,$$

where

$$\begin{aligned} h_t &= \alpha_0 + \beta_1 h_{t-1} + \alpha_1 (z_{t-1}^* - \delta_1^* \sqrt{h_{t-1}})^2, \qquad (1) \\ z_t^* &= z_t + \left(\lambda + \frac{1}{2}\right) \sqrt{h_t}, \\ \delta_1^* &= \delta_1 + \lambda + \frac{1}{2}. \end{aligned}$$

The risk-neutral generating function (ensures that the risk-neutral expected price at time $T > t$ is $S_t e^{r(T-t)}$) takes the following log-linear form:

$$f_{BS}^*(t, \phi) = S_t^{\phi} \exp\left\{ A_{BS}(t, \phi) + B_{BS}(t, \phi) h_{t+1} \right\},$$

where

$$A_{BS}(t,\phi) = \phi r + A_{BS}(t+1,\phi) + \alpha_0 B_{BS}(t+1,\phi) - \frac{1}{2}\ln\left(1 - 2\alpha_1 B_{BS}(t+1,\phi)\right), \quad (2)$$

$$B_{BS}(t,\phi) = \beta_1 B_{BS}(t+1,\phi) - \frac{1}{2}\delta_1^2 + \phi(\lambda + \delta_1) + \frac{(\phi - \delta_1)^2}{2\left(1 - 2\left(\alpha_1 B_{BS}(t+1,\phi)\right)\right)}. \quad (3)$$

These coefficients can be calculated recursively using the terminal conditions

$$A_{BS}(T,\phi) = B_{BS}(T,\phi) = 0.$$

Heston and Nandi (2000) explain that the generating function of the spot price is the moment generating function of the logarithm of the spot price. Hence, $f^*_{BS}(t, i\phi)$ is the characteristic function of the logarithm of the spot price (where $i = \sqrt{-1}$). Using the risk-neutral dynamics, it is possible to derive a closed form formula for a European call option. The Heston–Nandi price of a European call option is stated in the theorem below.

Theorem 1 (Heston–Nandi call option in the Black–Scholes framework)**.** *The price of a European call option at time t is given by*

$$V_t = \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi} f^*_{BS}(t, i\phi + 1)}{i\phi} d\phi\right] - Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi} f^*_{BS}(t, i\phi)}{i\phi} d\phi\right]\right), \quad (4)$$

where $Re[\cdot]$ *denotes the real portion of a complex number.*

Proof. See Heston and Nandi (2000). ■

3.2 The Piterbarg framework

An important assumption required for the Heston–Nandi model in the Black–Scholes framework is the existence of a unique risk-free rate, which is not necessarily a practical assumption. The purpose of the model derived by Piterbarg (2010) is the pricing of derivatives in the presence of collateral. The Piterbarg framework is an extension of the Black–Scholes framework, which relaxes the assumption of a unique risk-free rate.

The framework comprises three interest rates, the collateral rate (r_C), the repurchase agreement rate (r_R), and the funding rate (r_F). In general, we have that

$$r_C \leq r_R \leq r_F,$$

the collateral rate is associated with least risk. It is assumed that collateral is posted in the form of cash. The funding rate is associated with the most risk (unsecured lending), and finally a repurchase agreement (collateralised loan) is less risky than unsecured lending; however, there is more risk associated with the underlying asset than there is with cash.

Piterbarg (2010) assumes the following asset price dynamics under the real world measure, P:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ is the drift, σ is the implied volatility, and W_t is a standard Brownian Motion under P. In the Piterbarg framework, pricing is done under the Q_{r_R} measure. Hence, the dynamics under the risk-neutral measure Q_{r_R} are given by

$$dS_t = r_R S_t dt + \sigma S_t d\tilde{W}_t,$$

where $\tilde{W}_t$ is a standard Brownian Motion under Q_{r_R}. Using a replicating portfolio argument and an application of the Feynman–Kac theorem (see Shreve, 2004), it is possible to derive an expression for the price of a call option in the Piterbarg framework (essentially a derivation of the Black–Scholes model with different interest rates); this is given in the theorem below:

Theorem 2 (Piterbarg call option)**.** *The price of a European call option at time t is given by*

$$V_t = \mathbb{E}^{Q_{r_R}}\left[e^{-\int_t^T r_C(u)du}(S_T - K)^+|\mathcal{F}_t\right] - \mathbb{E}^{Q_{r_R}}\left[\int_t^T e^{-\int_t^T r_C(u)du}(r_F(s) - r_C(s))(V(s) - \gamma_C(s))ds|\mathcal{F}_t\right],$$

where γ_C denotes the collateral account.

Proof. See Piterbarg (2010). ∎

It is clear from the above that the price of a fully collateralised call option is given by

$$V_t^{(FC)} = \mathbb{E}^{Q_{r_R}}\left[e^{-\int_t^T r_C(u)du}(S_T - K)^+|\mathcal{F}_t\right], \tag{5}$$

and the price of a zero collateral call option is given by

$$V_t^{(ZC)} = \mathbb{E}^{Q_{r_R}}\left[e^{-\int_t^T r_F(u)du}(S_T - K)^+|\mathcal{F}_t\right].$$

The extended Heston–Nandi model is considered in the next subsection.

3.3 Heston–Nandi model with collateral

The Heston–Nandi model in the Black–Scholes framework relies on the existence of a unique risk-free rate. Therefore, Von Boetticher (2017) extended the model by Heston and Nandi (2000) to price collateralised options; this was done in the Piterbarg framework. Von Boetticher (2017) shows that the Heston–Nandi asset price dynamics under measure Q_{r_R} take the following form:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r_R - \frac{1}{2}h_t + \sqrt{h_t}z_t^*, \tag{6}$$

where h_t is the same as (1). Furthermore, the log-linear form of the risk-neutral generating function is given by

$$f_P^*(t,\phi) = S_t^\phi \exp\left\{A_P(t,\phi) + B_P(t,\phi)h_{t+1}\right\},$$

where

$$A_P(t,\phi) = \phi r_R + A_P(t+1,\phi) + \alpha_0 B_P(t+1,\phi) - \frac{1}{2}\ln\left(1 - 2\alpha_1 B_P(t+1,\phi)\right),$$

$$B_P(t,\phi) = \beta_1 B_P(t+1,\phi) - \frac{1}{2}\delta_1^2 + \phi(\lambda + \delta_1) + \frac{(\phi - \delta_1)^2}{2\left(1 - 2\left(\alpha_1 B_P(t+1,\phi)\right)\right)}.$$

Clearly, $A_P(t,\phi)$ and $B_P(t,\phi)$ are the same as equations (2) and (3) respectively, the only difference is that the risk-free rate r is replaced by the repurchase agreement rate r_R. Using the boundary conditions

$$A_P(T,\phi) = B_P(T,\phi) = 0,$$

$A_P(t,\phi)$ and $B_P(t,\phi)$ are calculated recursively.

The Heston–Nandi price of a European call option in the Piterbarg framework is given in the following theorem.

Theorem 3 (Heston–Nandi call option in the Piterbarg framework)**.** *The Heston–Nandi price of a fully collateralised call option at time t is given by*

$$V_t^{(FC)} = \frac{1}{2}\left(S(t)e^{\int_t^T (r_R(u)-r_C(u))du} - Ke^{\int_t^T r_C(u)du}\right) + \frac{K}{\pi}\psi(r_C),$$

where the function

$$\psi(r_C) = e^{-\int_t^T r_C(s)ds}\frac{e^{\int_t^T r_R(s)ds}}{K}\int_0^\infty Re\left[\frac{K^{-i\phi f_P^*(t,i\phi+1)}}{i\phi}\right]d\phi - e^{-\int_t^T r_C(s)ds}\int_0^\infty Re\left[\frac{K^{-i\phi f_P^*(t,i\phi)}}{i\phi}\right]d\phi,$$

and $f_P^*(t,\phi)$ ensures that $\mathbb{E}^{Q_{r_R}}[S_T|\mathcal{F}_t] = S_t e^{r_R(T-t)}$. The price of a zero collateral call option is,

$$V_t^{(ZC)} = \frac{1}{2}\left(S(t)e^{\int_t^T (r_R(u)-r_F(u))du} - Ke^{\int_t^T r_F(u)du}\right) + \frac{K}{\pi}\psi(r_F).$$

Proof. See Von Boetticher (2017), Section 3.2. ∎

It is clear from the above that if $r_C = r_R = r_F = r$, the Heston–Nandi price of a call option in the Black–Scholes framework (4) is obtained. The use of the above model addresses the unreasonable assumption of a unique risk-free rate. However, it does not take the effect of counterparty credit risk into account; this is the focus of the next subsection.

3.4 Heston–Nandi model with collateral and counterparty credit risk

The focus of this subsection is the derivation of the Heston–Nandi price of a default risky European call option (in the presence of collateral) in the Piterbarg framework. The overall objective is to extend the model derived by Von Boetticher (2017) to incorporate counterparty credit risk by making use of an approach similar to Wang (2017).

The asset price dynamics under measure Q_{r_R} are consistent with equation (6). The random default time τ is modelled as the first jump time of a Cox process with intensity κ_t (Wang, 2017). The following Q_{r_R} dynamics are assumed for the default intensity:

$$\kappa_t = \omega + b\kappa_{t-1} + az_{t-1}^{(\kappa)}.$$

The mean arrival rate of default in $(t, t+1]$ is given by

$$Q_{r_R}(\tau > t+1|\mathcal{F}_t) = \mathbb{E}^{Q_{r_R}}[e^{-\kappa_t}] = e^{-\kappa_{t+1}}.$$

We also assume that z_t^* and $z_t^{(\kappa)}$ have correlation coefficient ρ.

Wang (2017) explains that when pricing a default risky call option, two parts need to be considered. The first part is if no default event during the life of the trade, in this case the payoff, is equal to the payoff of a vanilla call option (full collateral or zero collateral in this case). The second part considers that a default event occurs during the life of the trade, in this case only a portion of the option value can be recovered.

By making use of (5) and the argument above, the price of a default risky European call option, which is fully collateralised, is formulated as follows (assuming constant interest rates):

$$\begin{aligned}\tilde{V}_t^{(FC)} &= e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^+\middle|\mathcal{F}_t\right]\\ &\quad+\mathbb{E}^{Q_{r_R}}\left[1_{\{t\le\tau\le T\}}\theta e^{r_C(\tau-t)}\mathbb{E}^{Q_{r_R}}\left[e^{r_C(T-\tau)}(S_T-K)^+\middle|\mathcal{F}_\tau\right]\middle|\mathcal{F}_t\right]\\ &= e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^+\middle|\mathcal{F}_t\right]+\theta e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{t\le\tau\le T\}}(S_T-K)^+\middle|\mathcal{F}_t\right]\\ &= e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^++\theta 1_{\{t\le\tau\le T\}}(S_T-K)^+\middle|\mathcal{F}_t\right],\end{aligned}$$

where θ is the recovery rate, and $1_{\{\tau>T\}}$ is an indicator function that takes a value of one if a default event occurs after the expiry of the option ($\tau > T$) and zero otherwise. The value of a zero collateral option is expressed as

$$\tilde{V}_t^{(ZC)} = e^{r_F(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^++\theta 1_{\{t\le\tau\le T\}}(S_T-K)^+\middle|\mathcal{F}_t\right].$$

It is important to note that $1_{\{t\le\tau\le T\}} = 1 - 1_{\{\tau>T\}}$. Hence, the fully collateralised price can be simplified as follows:

$$\begin{aligned}\tilde{V}_t^{(FC)} &= e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^++\theta 1_{\{t\le\tau\le T\}}(S_T-K)^+\middle|\mathcal{F}_t\right]\\ &= (1-\theta)e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}(S_T-K)^+\middle|\mathcal{F}_t\right]+\theta e^{r_C(T-t)}\mathbb{E}^{Q_{r_R}}\left[(S_T-K)^+\middle|\mathcal{F}_t\right]\\ &= (1-\theta)e^{r_C(T-t)}(I_1-KI_2)+\theta e^{r_C(T-t)}(I_3-KI_4),\end{aligned}$$

where

$$I_1 = \mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau>T,S_T\ge K\}}\middle|\mathcal{F}_t\right], \tag{7}$$

$$I_2 = \mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T,S_T\ge K\}}\middle|\mathcal{F}_t\right], \tag{8}$$

$$I_3 = \mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{S_T\ge K\}}\middle|\mathcal{F}_t\right], \tag{9}$$

$$I_4 = \mathbb{E}^{Q_{r_R}}\left[1_{\{S_T\ge K\}}\middle|\mathcal{F}_t\right]. \tag{10}$$

I_2 is the probability of the counterparty surviving up to time T and the option expiring in the money (I_1 scales this value by the expected value of the underlying price at expiry). I_4 is the probability of the option expiring in the money (default risk is not taken into account), I_4 scales this value by the expected value of the underlying price at expiry. The zero collateral price of a default risky call option takes a similar form:

$$\tilde{V}_t^{(ZC)} = (1-\theta)e^{r_F(T-t)}(I_1-KI_2)+\theta e^{r_F(T-t)}(I_3-KI_4).$$

Deriving closed form expressions for (7) to (10) will allow for the derivation of a closed form expression for the Heston–Nandi price of an option in the Piterbarg framework in the presence of counterparty credit risk. Wang (2017) derived an expression for the characteristic function under the Q measure. Under the Q_{r_R} measure the derivation is the same, the only difference is the drift of the underlying asset is equal to r_R. In this case, the risk-neutral generating (log-linear) function is given by

$$f^*_{PD}(t,\phi_1,\phi_2) = \exp\left\{\phi_1 x_t + \phi_2 \sum_{s=1}^{t} \kappa_s + A_{PD}(t,\phi_1,\phi_2)\right\} \times \exp\left\{B^{(1)}_{PD}(t,\phi_1,\phi_2)h_{t+1} + B^{(2)}_{PD}(t,\phi_1,\phi_2)\kappa_{t+1}\right\},$$

where $x_T = \ln S_T$ and

$$\begin{aligned} A_{PD}(t,\phi_1,\phi_2) &= \phi_1 r_R + A_{PD}(t+1,\phi_1,\phi_2) + \alpha_0 B^{(1)}_{PD}(t+1,\phi_1,\phi_2) \\ &\quad + \omega B^{(2)}_{PD}(t+1,\phi_1,\phi_2) - \frac{1}{2}\ln\left(1 - 2aB^{(2)}_{PD}(t+1,\phi_1,\phi_2)(1-\rho^2)\right) \\ &\quad - \frac{1}{2}\ln\left(1 - 2\left(\alpha_1 B^{(1)}_{PD}(t+1,\phi_1,\phi_2) + \frac{aB^{(2)}_{PD}(t+1,\phi_1,\phi_2)\rho^2}{1-2aB^{(2)}_{PD}(t+1,\phi_1,\phi_2)(1-\rho^2)}\right)\right), \\ B^{(1)}_{PD}(t,\phi_1,\phi_2) &= \beta_1 B^{(1)}_{PD}(t+1,\phi_1,\phi_2) - \frac{1}{2}\phi_1 + \alpha_1(\delta_1+\lambda)^2 + B^{(1)}_{PD}(t+1,\phi_1,\phi_2) \\ &\quad + \frac{(\phi_1 - 2\alpha_1(\delta_1+\lambda)B^{(1)}_{PD}(t+1,\phi_1,\phi_2))^2}{2\left(1 - 2\left(\alpha_1 B^{(1)}_{PD}(t+1,\phi_1,\phi_2) + \frac{aB^{(2)}_{PD}(t+1,\phi_1,\phi_2)\rho^2}{1-2aB^{(2)}_{PD}(t+1,\phi_1,\phi_2)(1-\rho^2)}\right)\right)}, \\ B^{(2)}_{PD}(t,\phi_1,\phi_2) &= bB^{(1)}_{PD}(t+1,\phi_1,\phi_2) + \phi_2, \end{aligned}$$

with boundary conditions

$$A_{PD}(T,\phi_1,\phi_2) = B^{(1)}_{PD}(T,\phi_1,\phi_2) = B^{(2)}_{PD}(T,\phi_1,\phi_2) = 0.$$

Given the boundary conditions, $A_{PD}(t,\phi_1,\phi_2)$, $B^{(1)}_{PD}(t,\phi_1,\phi_2)$ and $B^{(2)}_{PD}(t,\phi_1,\phi_2)$ are calculated recursively. The closed form expression is reported in the theorem below.

Theorem 4 (Heston–Nandi call option in the Piterbarg framework in the presence of counterparty credit risk)**.** *In the presence of counterparty credit risk, the price of a fully collateralised European call option is given by*

$$\tilde{V}^{(FC)}_t = e^{r_C(T-t)}(1-\theta)\left(\Pi_1(t,T) + \frac{1}{2}f^*_{PD}(t,1,-1) - K\Pi_2(t,T) - \frac{1}{2}Kf^*_{PD}(t,0,-1)\right) + e^{r_C(T-t)}\theta\left(\Pi_3(t,T) + \frac{1}{2}f^*_{PD}(t,1,0) - K\Pi_4(t,T) - \frac{1}{2}K\right),$$

where $f^*_{PD}(\cdot)$ is the risk-neutral characteristic function and

$$\Pi_1(t,T) = \frac{1}{\pi}\int_0^{-\infty} Re\left[\frac{e^{i\phi_1 \ln K} f^*_{PD}(t, i\phi_1 + 1, -1)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_2(t,T) = \frac{1}{\pi}\int_0^{-\infty} Re\left[\frac{e^{i\phi_1 \ln K} f^*_{PD}(t, i\phi_1, -1)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_3(t,T) = \frac{1}{\pi}\int_0^{-\infty} Re\left[\frac{e^{i\phi_1 \ln K} f^*_{PD}(t, i\phi_1 + 1, 0)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_4(t,T) = \frac{1}{\pi}\int_0^{-\infty} Re\left[\frac{e^{i\phi_1 \ln K} f^*_{PD}(t, i\phi_1, 0)}{i\phi_1}\right] d\phi_1.$$

Similarly, the zero collateral price in the presence of counterparty credit risk is

$$\tilde{V}_t^{(ZC)} = e^{r_F(T-t)}(1-\theta)\left(\Pi_1(t,T) + \frac{1}{2} f^*_{PD}(t,T,1,-1) - K\Pi_2(t,T) - \frac{1}{2} K f^*_{PD}(t,T,0,-1)\right) + e^{r_C(T-t)}\theta\left(\Pi_3(t,T) + \frac{1}{2} f^*_{PD}(t,T,1,0) - K\Pi_4(t,T) - \frac{1}{2}K\right).$$

Proof. See the Appendix. ■

4. Empirical results

In this study, the Heston–Nandi option pricing model in the Piterbarg framework is applied to the pricing of three-year S&P500 index call options in the presence of counterparty credit risk. Daily data from 4-Jan-2010 to 31-Jan-21 were obtained from the Thomson–Reuters Datastream databank. The Heston–Nandi parameters are calibrated to historical returns in the Black–Scholes framework ($r_C = r_R = r_F = r$) using the maximum likelihood method. Furthermore, for the default intensity, parameters consistent with a Ba rating for corporate bonds are assumed (consistent with Wang, 2020). According to Hull (2017), the average cumulative issuer-weighted default rate (based on 1970 to 2009) of a Ba rated bond with a three-year term is 4.492%. The risk-free rate is assumed to be equal to the three-year US treasury yield. The parameters are given in Tables 1 and 2.

We assume a correlation of $\rho = 0.5$. The three-year US treasury yield on 31-Jan-20 is $r = 1.3\%$. We assume that $r_R = r$, $r_C = 1\%$ (collateralised and therefore less risky, which implies a lower rate),

Table 1. Underlying process parameters.

Parameter	Value
λ	4.6429
α_0	0.0000
α_1	5.2800×10^{-6}
β_1	0.7557
δ_1	183.7511

Table 2. Default intensity parameters.

Parameter	Value
ω	1.540×10^{-7}
a	2.600×10^{-11}
b	0.977

and $r_F = 1.6\%$ (higher rate because there is no collateral). The call option prices (relative to the spot price) assuming no default risk in a single-curve framework (Heston and Nandi, 2000), full collateral options, and zero collateral (Von Boetticher, 2017) options are plotted in Figure 1. In addition, the default risky (Wang, 2017, for the single-curve case) prices are also included. Moneyness is defined as the spot price over the strike price.

It is clear from the above that default risky options are cheaper than options that are not default risky. This is consistent with expectations (if a default occurs, the payoff is less than that of a vanilla option). Furthermore, fully collateralised options are more expensive (less risk) when compared to zero collateral options. To illustrate the effect of correlation, at-the-money (ATM) option prices assuming different correlation values are illustrated in Figure 2. It is clear from Figure 2 that the correlation between the underlying volatility process and default intensity does not have a significant impact on ATM option prices using the parameters outlined above.

5. Conclusion

In this paper, a closed-form expression for the Heston–Nandi price of a collateralised European call option in the presence of counterparty credit risk was derived. This is an extension of the work by Von Boetticher (2017), who derived an expression for the Heston–Nandi price of a collateralised European call option in the absence of counterparty credit risk in the Piterbarg framework. Using an approach similar to Wang (2017), the work by Von Boetticher (2017) is extended to incorporate counterparty credit risk.

As a numerical example, the model was applied to three-year S&P500 index options. The underlying process parameters were calibrated in a single-curve framework, assuming no default risk. The assumed default risk parameters are consistent with a Ba rated corporate bond. The prices obtained are consistent with expectations, default risky bonds are cheaper than options with no counterparty credit risk, and fully collateralised options are more expensive when compared to zero collateral options.

The effect of correlation is tested by plotting the default risky ATM option price for different levels of correlation. The results indicate that correlation has an insignificant impact when pricing using the calibrated parameters. Areas for future research include the extension of the model to exotic options and also a comparison of different default intensity assumptions.

Appendix A: Proof of Theorem 4

The proof of Theorem 4 is outlined below. The proof follows Wang (2017) closely. However, Wang (2017) derived expressions for the required probabilities in the Black–Scholes framework (measure Q). This relies on the existence of a unique risk-free rate. To extend the model to the Piterbarg framework (multiple interest rates to account for the presence of collateral), it is necessary to derive expressions for the probabilities under the Q_{r_R} measure.

Proof. To evaluate the integral I_1, it is necessay to define a new probability measure

$$Q^{(1)}_{r_R}(y) = \frac{\mathbb{E}^{Q_{r_R}}\left[1_{\{y\}} S_T 1_{\{\tau>T\}} \middle| \mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau>T\}} \middle| \mathcal{F}_t\right]},$$

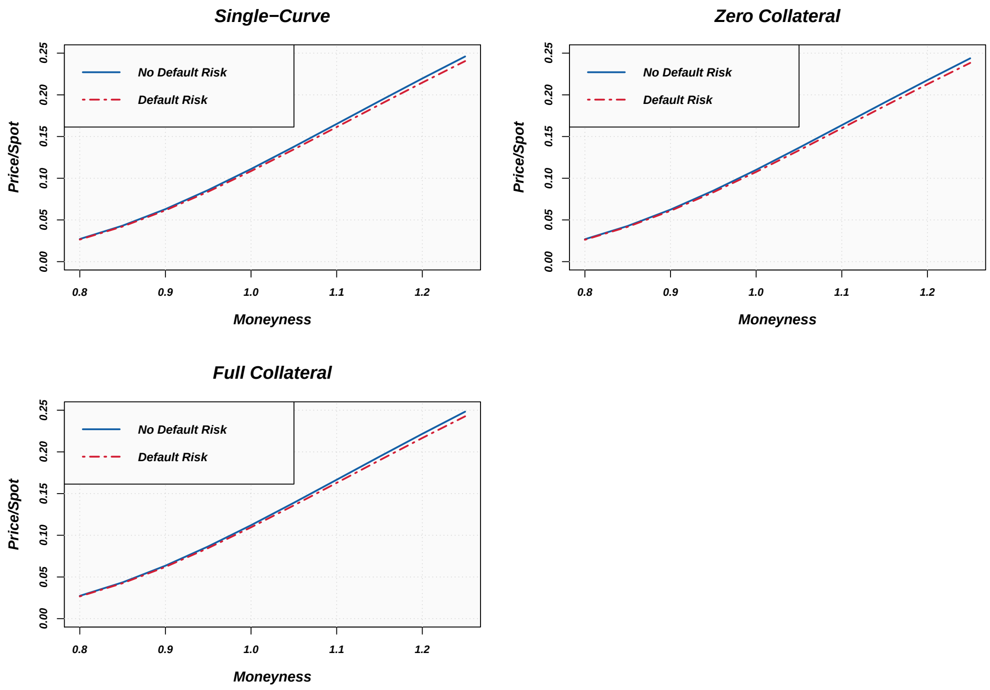

Figure 1. S&P500 index option prices.

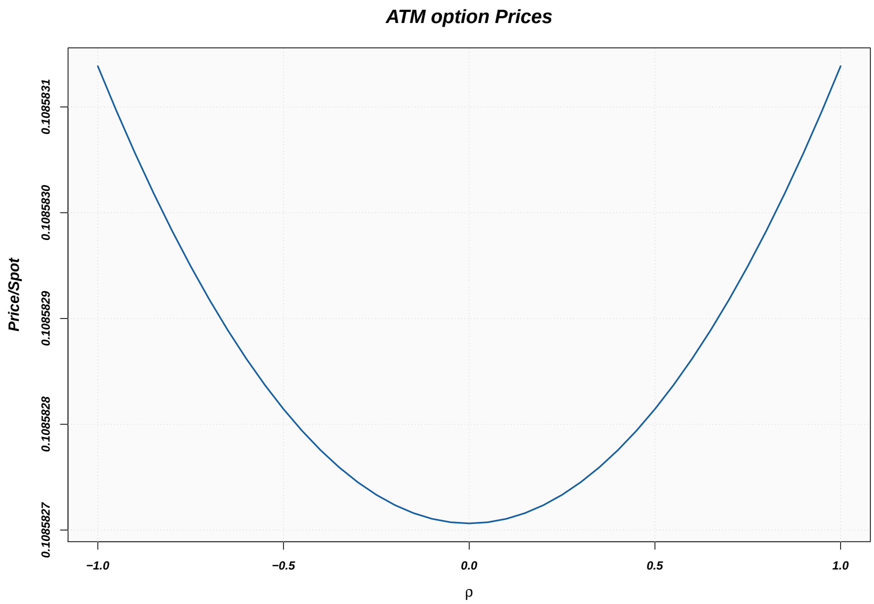

Figure 2. Effect of correlation.

for any event $y \in \mathcal{F}_T$. The characteristic function of x_T under $Q_{r_R}^{(1)}$ is given by

$$\begin{aligned} f^{(1)}(t, i\phi_1) &= \mathbb{E}^{Q_{r_R}^{(1)}}\left[e^{i\phi_1 x_T}\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{i\phi_1 x_T} S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]} \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{(i\phi_1+1) x_T} 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]} \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{(i\phi_1+1) x_T} e^{-\sum_{s=t}^{T}\kappa_s}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[e^{x_T - \sum_{s=t}^{T}\kappa_s}\middle|\mathcal{F}_t\right]} \\ &= \frac{f_{PD}^*(t, i\phi_1 + 1, -1)}{f_{PD}^*(t, 1, -1)}. \end{aligned}$$

In this case, standard probability theory applies (Kendall and Stuart, 1977) and the distribution function corresponding to $f^{(1)}$ is given by

$$F^{(1)}(x_T; x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 x} f^{(1)}(t, i\phi_1)}{i\phi_1}\right] d\phi_1.$$

This implies that

$$\begin{aligned} Q_{r_R}^{(1)}(x_T \geq \ln K) &= 1 - F^{(1)}(x_T; \ln K) \\ &= \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 \ln K} f^{(1)}(t, i\phi_1)}{i\phi_1}\right] d\phi_1 + \frac{1}{2}. \end{aligned}$$

By using the definition of $Q_{r_R}^{(1)}$, we have that

$$Q_{r_R}^{(1)}(x_T \geq \ln K) = \frac{\mathbb{E}^{Q_{r_R}}\left[1_{\{x_T \geq \ln K\}} S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]},$$

which implies

$$\begin{aligned} \mathbb{E}^{Q_{r_R}}\left[1_{\{x_T \geq \ln K\}} S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right] &= Q_{r_R}^{(1)}(x_T \geq \ln K) \times \mathbb{E}^{Q_{r_R}}\left[S_T 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right] \\ &= \left(\frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1} f^{(1)}(t, i\phi_1)}{i\phi_1}\right] d\phi_1 + \frac{1}{2}\right) f_{PD}^*(t, 1, -1) \\ &= \Pi_1(t, T) + \frac{1}{2} f_{PD}^*(t, 1, -1), \end{aligned}$$

which is an explicit expression for I_1.

For I_2 a similar process is required. Define the probability measure

$$Q_{r_R}^{(2)}(y) = \frac{\mathbb{E}^{Q_{r_R}}\left[1_{\{y\}} 1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau > T\}}\middle|\mathcal{F}_t\right]}.$$

The characteristic function of x_T takes the form

$$\begin{aligned} f^{(2)}(t, i\phi_1) &= \mathbb{E}^{Q^{(2)}_{r_R}}\left[e^{i\phi_1 x_T}\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{i\phi_1 x_T}1_{\{\tau>T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}\middle|\mathcal{F}_t\right]} \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{i\phi_1 x_T}1_{\{\tau>T\}}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T\}}\middle|\mathcal{F}_t\right]} \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{i\phi_1 x_T}e^{-\sum_{s=t}^{T}\kappa_s}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[e^{\sum_{s=t}^{T}\kappa_s}\middle|\mathcal{F}_t\right]} \\ &= \frac{f^*_{PD}(t, i\phi_1, -1)}{f^*_{PD}(t, 0, -1)}. \end{aligned}$$

The corresponding distribution function is given by

$$F^{(2)}(x_T; x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 x} f^{(2)}(0, i\phi_1)}{i\phi_1}\right] d\phi_1,$$

which implies

$$\begin{aligned} Q^{(2)}_{r_R}(x_T \geq \ln K) &= 1 - F^{(2)}(x_T; \ln K) \\ &= \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 \ln K} f^{(2)}(t, i\phi_1)}{i\phi_1}\right] d\phi_1 + \frac{1}{2}. \end{aligned} \quad (11)$$

By using (11) above and the definition of $Q^{(2)}_{r_R}$ it is possible to derive an expression for I_2,

$$\begin{aligned} I_2 &= \mathbb{E}^{Q_{r_R}}\left[1_{\{\tau>T, S_T\geq K\}}\middle|\mathcal{F}_t\right] \\ &= Q^{(2)}_{r_R}(x_T \geq \ln K) \times Q_{r_R}(\tau > T) \\ &= Q^{(2)}_{r_R}(x_T \geq \ln K) \times \mathbb{E}^{Q_{r_R}}\left[e^{\sum_{s=1}^{T}\kappa_s}\middle|\mathcal{F}_t\right] \\ &= Q^{(2)}_{r_R}(x_T \geq \ln K) \times f^*_{PD}(t, 0, -1) \\ &= \Pi_2(t, T) + \frac{1}{2} f^*_{PD}(t, 0, -1). \end{aligned}$$

For the derivation of I_3, the following probability measure is defined:

$$Q^{(3)}_{r_R}(y) = \frac{\mathbb{E}^{Q_{r_R}}\left[1_{\{y\}}S_T\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T\middle|\mathcal{F}_t\right]},$$

and the characteristic function of x_T under $Q^{(3)}_{r_R}$,

$$\begin{aligned} f^{(3)}(t, i\phi_1) &= \mathbb{E}^{Q^{(3)}_{r_R}}\left[e^{i\phi_1 x_T}\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbb{E}^{Q_{r_R}}\left[e^{i\phi_1 x_T}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{Q_{r_R}}\left[S_T\middle|\mathcal{F}_t\right]} \\ &= \frac{f^*_{PD}(t, i\phi_1 + 1, 0)}{f^*_{PD}(t, 1, 0)}. \end{aligned}$$

It is possible to show that

$$Q_{r_R}^{(3)}(x_T \geq \ln K) = \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 \ln K} f^{(3)}(t, i\phi_1)}{i\phi_1}\right] d\phi_1 + \frac{1}{2},$$

which implies

$$\begin{aligned} I_3 &= Q_{r_R}^{(3)}(x_T \geq \ln K) \times \mathbb{E}^{Q_{r_R}}\left[S_T \middle| \mathcal{F}_t\right] \\ &= Q_{r_R}^{(3)}(x_T \geq \ln K) \times f_{PD}^*(t, 1, 0) \\ &= \Pi_3(t, T) + \frac{1}{2} f_{PD}^*(t, 1, 0). \end{aligned}$$

Finally,

$$\begin{aligned} I_4 &= \mathbb{E}^{Q_{r_R}}\left[1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] \\ &= Q_{r_R}(x_T \geq \ln K) \\ &= 1 - Q_{r_R}(x_T \leq \ln K) \\ &= \frac{1}{\pi}\int_0^{-\infty} \mathrm{Re}\left[\frac{e^{i\phi_1 \ln K} f_{PD}^*(t, i\phi_1, 0)}{i\phi_1}\right] d\phi_1 + \frac{1}{2} \\ &= \Pi_4(t, T) + \frac{1}{2}. \end{aligned}$$

This implies

$$\begin{aligned} \tilde{V}_t^{(FC)} &= (1-\theta)e^{r_C(T-t)}(I_1 - KI_2) + \theta e^{r_C(T-t)}(I_3 - KI_4) \\ &= (1-\theta)e^{r_C(T-t)}\left(\Pi_1(t, T) + \frac{1}{2} f_{PD}^*(t, 1, -1) - K\left(\Pi_2(t, T) + \frac{1}{2} f_{PD}^*(t, 0, -1)\right)\right) \\ &+ \theta e^{r_C(T-t)}\left(\Pi_3(t, T) + \frac{1}{2} f_{PD}^*(t, 1, 0) - K\left(\Pi_4(t, T) + \frac{1}{2}\right)\right), \end{aligned}$$

and

$$\begin{aligned} \tilde{V}_t^{(ZC)} &= (1-\theta)e^{r_F(T-t)}(I_1 - KI_2) + \theta e^{r_F(T-t)}(I_3 - KI_4) \\ &= (1-\theta)e^{r_F(T-t)}\left(\Pi_1(t, T) + \frac{1}{2} f_{PD}^*(t, 1, -1) - K\left(\Pi_2(t, T) + \frac{1}{2} f_{PD}^*(t, 0, -1)\right)\right) \\ &+ \theta e^{r_F(T-t)}\left(\Pi_3(t, T) + \frac{1}{2} f_{PD}^*(t, 1, 0) - K\left(\Pi_4(t, T) + \frac{1}{2}\right)\right), \end{aligned}$$

which completes the proof. ■

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