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Differential chain of algebras of generalized functions

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Abstract. In this paper, it is shown how the spaces of generalized functions associated with the construction of the generalized solution for nonlinear partial differential equations through the order completion method using convergence spaces, may be interpreted as a chain of algebras of generalized functions. In particular, we showed that the spaces of normal lower semi-continuous functions that contain the generalized solution of the nonlinear partial differential equation under consideration is a differential chain of algebras of generalized functions. Consequently, this generalized solution is shown to be a chain generalized solution. The relationships between the chain of normal lower semi-continuous functions and the chain of nowhere dense algebras, as well as the chain of almost everywhere algebras of generalized functions are shown. We further show that the chain generalized solutions of nonlinear partial differential equations obtained in the chain of normal lower semi-continuous functions corresponds to the chain generalized solution for nonlinear partial differential equation obtained in the chain of nowhere dense algebras of generalized functions as well as the chain of almost everywhere algebra of generalized functions.

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1. Introduction

In the late 1960s, differential algebras of generalized functions were introduced as an alternative approach to dealing with the difficulties associated with the application of distributions to nonlinear partial differential equations (PDEs). One of the most notable of these difficulties is the Schwartz impossibility problem [16] which is summarized as the inability to extend the multiplication of C^{∞}-smooth function with distribution to all of the distributions in such a way that the class of distributions, together with the usual vector space operations, is an algebra.

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This alternative approach, known as the ‘algebra first’ approach, is to construct suitable algebras of generalized functions that contain the $\mathcal{D}'(\Omega)$-distributions as a linear subspace. In this way, multiplication of distributions can be done consistently and in a meaningful way, although the result of multiplying two distributions will, in general, not be a distribution. The ‘algebra first’ approach has been developed extensively and has been useful in the study of generalized solutions to linear and nonlinear PDEs in mathematical physics and related fields, see for instance [7, 8, 9, 12, 13, 14, 15].

Let Ω be a nonempty open subset of $\mathbb{R}^n$. An algebra $\mathcal{A}(\Omega)$ is called a differential algebra of generalized functions if it is equipped with the generalized partial derivative operators

$$D^p : \mathcal{A}(\Omega) \longrightarrow \mathcal{A}(\Omega), \quad p \in \mathbb{N}^n$$

that satisfy the Leibnitz rule for derivative of product of functions, given as

$$D^p(uv) = \sum_{q \leq p} \left(\begin{array}{c} p \\ q \end{array} \right) D^{p-q}uD^q v \tag{1.1}$$

for all $u, v \in \mathcal{A}(\Omega)$. Here $D^p = D_1^{p_1} \cdots D_n^{p_n}$, $D_j = \frac{\partial}{\partial x_j}$ with $|p| = p_1 + \cdots + p_n$.

In [14], Rosinger gave sufficient conditions for the construction of an algebra of generalized functions $\mathcal{A}(\Omega)$ that guaranteeis the commutativity of the following diagram

$$\begin{tikzcd}
C^\infty(\Omega) \arrow[rr, "\subset"] \arrow[dr, "\mathcal{E}_\infty"'] & & \mathcal{D}'(\Omega) \arrow[dl, "E"] \\
& \mathcal{A}(\Omega) &
\end{tikzcd} \tag{1.2}$$

where $E : \mathcal{D}'(\Omega) \to \mathcal{A}(\Omega)$ is a linear injection, and $\mathcal{E}_\infty : C^\infty(\Omega) \longrightarrow \mathcal{A}(\Omega)$ is the canonical injective algebra homomorphism. However, there is an essential limitation on the way in which distributions are embedded into a differential algebra. Indeed, an embedding of $\mathcal{D}'(\Omega)$ into a differential algebra cannot, at the same time, preserve both the algebraic structure of $C(\Omega)$ and the differential structure of $\mathcal{D}'(\Omega)$, see [12, 13, 15]. Furthermore, different embeddings of $\mathcal{D}'(\Omega)$ into an algebra $\mathcal{A}(\Omega)$ may not determine the same differential structure on $\mathcal{D}'(\Omega)$. This limitation is due to a basic conflict between the trio of insufficient smoothness, multiplication and differentiability, see [15].

In order to overcome the above mentioned limitation of the embedding of $\mathcal{D}'(\Omega)$ into a differential algebra $\mathcal{A}(\Omega)$, the concept of the *chain of algebras of generalized functions* was introduced, see [13, 14, 15]. In [15, Chapter 7], chains of algebras of generalized functions were applied to resolve the closed, nowhere dense singularities occurring in the solutions of certain polynomial partial differential equations. In particular, weak solutions of the nonlinear hyperbolic conservation laws

$$u_t(t,x) + cu(t,x)u_x(t,x) = 0 \quad t > 0, \quad x \in \mathbb{R}$$

with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R},$$

where c is an arbitrary polynomial, were interpreted within the framework of chains of algebras of generalized solutions, as chain weak solution.

In this paper, the space of generalized functions underlying the recent development of Order Completion Method [10], presented in [21, 22], are interpreted as a *differential chain of algebras of generalized functions.* Any generalized solution in the underlying space is interpreted as a chain generalized solution. The mentioned chain of algebras of generalized functions are shown to be related to the chain of closed nowhere dense algebras of generalized functions introduced by Rosinger [13, 14, 15], and the chain of almost everywhere algebras of generalized functions introduced and studied in [1] which was constructed based on techniques introduced by Vernaeve for constructing the almost everywhere algebra of generalized functions [26]. The existence results for chain generalized solutions for nonlinear PDEs lead to the corresponding existence results in the chain of closed nowhere dense algebras of generalized functions and the chain of almost everywhere algebras of generalized functions, respectively.

The following notation are used throughout this paper. Ω denotes an open subset of $\mathbb{R}^n$. For $x \in \Omega$, $\mathcal{V}_x$ is the set of open neighbourhoods of x. The extended real line is denoted by $\overline{\mathbb{R}}$, and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, with the natural order.

2. Spaces of generalized functions

In this section we recall the main points in the construction of the space $\{\mathcal{NL}^l(\Omega) \ : \ l \in \overline{\mathbb{N}}\}$, see [19, 22]. We denote by $\mathcal{NL}(\Omega)$ the set of all nearly finite normal lower semi continuous functions $u : \Omega \longrightarrow \overline{\mathbb{R}}$. A function $u : \Omega \longrightarrow \overline{\mathbb{R}}$ is normal lower semi-continuous if and only if

$$I(S(u)) = u$$

where I and S are the Lower and Upper Baire operators, [3, 5], defined by

$$I(u)(x) = \sup\{\inf\{u(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}_x\} \tag{2.1}$$

and

$$S(u)(x) = \inf\{\sup\{u(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}(x)\}, \tag{2.2}$$

respectively. A normal lower semi-continuous function is nearly finite whenever the set $\{x \in \Omega : u(x) \in \mathbb{R}\}$ is finite. We define the following algebraic operations on $\mathcal{NL}(X)$. For $u, v \in \mathcal{NL}(X)$ and $\alpha \in \mathbb{R}$ we set

$$u + v = I(S(u \oplus v)), \quad \alpha u = I(S(\alpha \odot u)), \quad uv = I(S(u \otimes v)) \tag{2.3}$$

where the algebraic operations $\oplus$, $\odot$ and $\otimes$ are taken as the usual pointwise operations on real functions, with understanding that the result of any operation involving $\pm\infty$ is again $\pm\infty$, with the appropriate sign determined as usual [23]. We note that, for $u, v \in \mathcal{NL}(X)$, the function $u \oplus v$ may fail to be normal lower semi-continuous. Indeed, if

$$u(x) = \left\{ \begin{array}{ll} 1 & if x > 0 \\ 0 & if x \leq 0 \end{array} \right.$$

and

$$v(x) = \left\{ \begin{array}{ll} -1 & if x \geq 0 \\ 0 & if x < 0 \end{array} \right.$$

then

$$(u \oplus v)(x) = \left\{ \begin{array}{ll} -1 & if x = 0 \\ 0 & if x \neq 0 \end{array} \right.$$

so that $u \oplus v \notin \mathcal{NL}(\mathbb{R})$.

Let

$$\mathcal{ML}^l(\Omega) := \left\{ u \in \mathcal{NL} : \left| \begin{array}{ll} \exists & \Gamma_u \subset \Omega \text{ closed, nowhere dense :} \\ & u \in C^l(\Omega \backslash \Gamma_u) \end{array} \right. \right\}. \tag{2.4}$$

and

$$\mathcal{ML}^0(\Omega) := \left\{ u \in \mathcal{NL} : \left| \begin{array}{ll} \exists & \Gamma_u \subset \Omega \text{ closed, nowhere dense :} \\ & u \in C^0(\Omega \backslash \Gamma_u) \end{array} \right. \right\}. \tag{2.5}$$

The space $\mathcal{ML}^0(\Omega)$ is a σ-order dense subalgebra of $\mathcal{NL}(\Omega)$, see [20]. On $\mathcal{ML}^0(\Omega)$ we introduced the following uniform convergence structure [20]. Let Λ consists of all nonempty order intervals in $\mathcal{ML}^0(\Omega)$. Let $\mathcal{J}_0$ denote the family of filters on $\mathcal{ML}^0(\Omega) \times \mathcal{ML}^0(\Omega)$.

Definition 2.1. A filter $\mathcal{U}$ belongs to $\mathcal{J}_0$ if there exists $k \in \mathbb{N}$ such that

$$\begin{array}{ll} \forall & j = 1, \cdots, k : \\ \exists & \Lambda_j = \{I_n^j\} \subseteq \Lambda : \\ \exists & u_j \in \mathcal{NL}(\Omega) : \\ & \quad (i) \quad I_{n+1}^j \subseteq I_n^j, \quad n \in \mathbb{N} \\ & \quad (ii) \quad \sup\limits_n \{\inf\{I_n^j\}\} = u_j = \inf\limits_n \{\sup\{I_n^j\}\} \\ & \quad (iii) \quad ([\Lambda_1] \times [\Lambda_1]) \cap \cdots \cap ([\Lambda_k] \times [\Lambda_k]) \subseteq \mathcal{U}. \end{array} \tag{2.6}$$

where, for $\Lambda' \subset \Lambda$, $[\Lambda']$ denotes the filter generated by Λ' if this filter exists.

The uniform convergence structure $\mathcal{J}_0$ is Hausdorff, first countable and induces the order convergence structure, see [4, 24, 20]. A filter $\mathcal{F}$ on $\mathcal{ML}^0(\Omega)$ order convergence to $u \in \mathcal{ML}^0(\Omega)$ if and only if

$$\begin{array}{ll} \exists & (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ & \quad (i) \quad \lambda_n \leq \lambda_{n+1} \leq u \leq \mu_{n+1} \leq \mu_n \quad n \in \mathbb{N}, \\ & \quad (ii) \quad \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\} \\ & \quad (iii) \quad [\{[\lambda_n, \mu_n] \ : \ n \in \mathbb{N}\}] \subseteq \mathcal{F}. \end{array} \tag{2.7}$$

In particular, a sequence $\{u_n\}$ converges to $u \in \mathcal{ML}^0(\Omega)$ if it order converges to u. Cauchy sequences on $\mathcal{ML}^0(\Omega)$ are characterized in the following way, see [23]

Proposition 2.2. *A sequence in $\mathcal{ML}^0(\Omega)$ is Cauchy with respect to $\mathcal{J}_0$ if and only if there exists a set $B \subset \Omega$ of first Baire category such that $(u_n(x))$ is convergent in $\mathbb{R}$ for all $x \in \Omega\backslash B$.*

The space $\mathcal{NL}(\Omega)$ equipped with a suitable uniform convergence structure, see [20] is the (Wyler) completion [11, 28] of $\mathcal{ML}^0(\Omega)$.

The partial derivatives

$$D^p : C^l(\Omega) \longrightarrow C^0(\Omega), \quad p \in \mathbb{N}^n, \ |p| \leq l$$

extends to the mappings

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \ni u \mapsto (I \circ S)(D^p u) \in \mathcal{ML}^0(\Omega), \quad p \in \mathbb{N}^n, \ |p| \leq l.$$

On the space $\mathcal{ML}^l(\Omega)$ we consider the initial uniform convergence structure, denoted by $\mathcal{J}_l$, with respect to the mappings

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^0(\Omega), \ |p| \leq l \tag{2.8}$$

Definition 2.3. A filter on $\mathcal{ML}^l(\Omega)$ belongs to $\mathcal{J}_l$ if and only if

$$\begin{array}{ll} \forall & p \in \mathbb{N}^n, \ |p| \leq l : \\ & (\mathcal{D}^p \times \mathcal{D}^p)(\mathcal{U}) \in \mathcal{J}_0. \end{array}$$

Proposition 2.4. *A filter $\mathcal{F}$ on $\mathcal{ML}^l(\Omega)$ converges to $u \in \mathcal{ML}^l(\Omega)$ with respect to the induced convergence structure λ_l if and only if $\mathcal{D}^p(\mathcal{F})$ converges to $\mathcal{D}^p u$ in $\mathcal{ML}^0(\Omega)$ for every $p \in \mathbb{N}$, $|p| \leq l$. In particular, a sequence (u_n) converges to $u \in \mathcal{ML}^l(\Omega)$ if and only if*

$$\begin{array}{ll} \forall & p \in \mathbb{N}^n, \ |p| \leq l : \\ & \mathcal{D}^p(u_n) \text{ order converges to } \mathcal{D}^p(u) \in \mathcal{ML}^0(\Omega). \end{array}$$

From Definition 2.3, it is clear that each of the mappings in (2.8) is uniformly continuous with respect to the uniform convergence structure, $\mathcal{J}_l$ and $\mathcal{J}_0$ of $\mathcal{ML}^l(\Omega)$ and $\mathcal{ML}^0(\Omega)$, respectively. In fact, see [21, 22], the mapping

$$\mathbf{D} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)^M$$

defined through

$$\mathbf{D}(u) = (\mathcal{D}^p u)_{|p| \leq l}.$$

is a uniformly continuous embedding. Therefore, see [22], the mapping $\mathbf{D}$ extends uniquely to an injective, uniformly continuous mapping

$$\mathbf{D}^\sharp : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^0(\Omega)^M. \tag{2.9}$$

where $\mathcal{NL}^l(\Omega)$ denotes the completion of $\mathcal{ML}^l(\Omega)$. This gives a first and basic regularity result: The generalized functions in $\mathcal{NL}^0(\Omega)$ may be represented, through their generalized partial derivatives, as normal lower semi-continuous functions. Indeed, the mapping (2.9) may be represented as

$$\mathbf{D}^\sharp(u) = (\mathcal{D}^{p^\sharp} u^\sharp)_{|p| \leq l}$$

where, for $|p| \leq l$, $(\mathcal{D}^{p^\sharp})$ denotes the unique uniformly continuous extension of $\mathcal{D}^p$ to $\mathcal{NL}^l(\Omega)$.

We now discuss,briefly, the concept of generalized solutions of nonlinear PDEs in the space $\mathcal{NL}^l(\Omega)$. Consider a nonlinear PDE

$$T(x, D)u(x) = h(x), x \in \Omega \tag{2.10}$$

of order l, where $h : \Omega \to \mathbb{R}$ is continuous, and the differential operator $T(x, D)$ is defined through a jointly continuous function

$$F : \Omega \times \mathbb{R}^M \to \mathbb{R}$$

by the expression

$$T(x, D)u(x) = F(x, u(x), ..., D^p u(x), ...), \quad |p| \leq l, \tag{2.11}$$

where M is the cardinality of the set $\{p \in \mathbb{N}^n : |p| \leq l\}$. In [18, 22], it was shown that the partial differential operator (2.11) induces a uniformly continuous mapping

$$T : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^0(\Omega) \tag{2.12}$$

defined as follows

$$Tu = (I \circ S)(F(\cdot, u, \cdots \mathcal{D}^p u \cdots)). \tag{2.13}$$

Therefore, the mapping (2.13) extends uniquely to a uniformly continuous mapping

$$T^\sharp : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}(\Omega)$$

where $\mathcal{NL}^l(\Omega)$ and $\mathcal{NL}(\Omega)$ are the completion of $\mathcal{ML}^l(\Omega)$ and $\mathcal{ML}^0(\Omega)$, respectively. The main existence result for generalized solutions of (2.10) in $\mathcal{NL}^l(\Omega)$ is the following.

Theorem 2.5. *[22, Theorem 7] If for each $x \in \Omega$ there is some $\zeta \in \mathbb{R}^M$ and neighborhoods V and W of x and ζ so that*

$$F(x, \zeta) = h(x)$$

and

$$F : V \times W :\longrightarrow \mathbb{R}$$

is open, then there exists $u^\sharp \in \mathcal{NL}^l(\Omega)$ such that

$$T^\sharp u^\sharp = h.$$

We also discuss in brief the existence of C^∞-smooth generalized solution of nonlinear PDEs in the space $\mathcal{NL}^\infty(\Omega)$, detailed exposition of the result is found in [25]. In this regard, consider the PDE

$$T(x, D)u(x) = h(x), \; x \in \Omega, \tag{2.14}$$

where the differential operator $T(x, D)$ is defined by a C^∞-smooth mapping

$$F : \Omega \times \mathbb{R}^M \longrightarrow \mathbb{R} \tag{2.15}$$

through

$$T(x, D)u(x) = F(x, u(x), \cdots, D^p u(x), \cdots),\ x \in \Omega,\ |p| \le l \tag{2.16}$$

for sufficiently smooth $u : \Omega \longrightarrow \mathbb{R}$. The right-hand term $h \in \mathcal{C}^\infty(\Omega)$. Assume that the PDE (2.14) satisfies

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & \xi(x) \in \mathbb{R}^{\mathbb{N}^n} : \\ \exists & V \in \mathcal{V}_x,\ W \in \mathcal{V}_{\xi(x)} : \\ & 1)\ F^\infty : V \times W \longrightarrow \mathbb{R}^{\mathbb{N}^n} \text{ open,} \\ & 2)\ F^\infty(x, \xi(x)) = (D^\beta f(x))_{\beta \in \mathbb{N}^n} \end{array} \tag{2.17}$$

where $\mathbb{R}^{\mathbb{N}^n}$ is equipped with the product topology τ, the mapping

$$F^\infty : \Omega \times \mathbb{R}^{\mathbb{N}^n} \longrightarrow \mathbb{R}^{\mathbb{N}^n}$$

is defined by setting

$$F^\infty(x, (\xi_\alpha)_{\alpha \in \mathbb{N}^n}) = (F^\beta(x, \cdots, \xi_\alpha, \cdots,))_{\beta \in \mathbb{N}^n}, \tag{2.18}$$

where, for each $\beta \in \mathbb{N}^n$, the mapping

$$F^\beta : \Omega \times \mathbb{R}^{\mathbb{N}^n} \longrightarrow \mathbb{R}$$

is defined by setting

$$D^\beta(T(x, D)u(x)) = F^\beta(x, \cdots, D^\alpha u(x), \cdots), \qquad |\alpha| \le l + |\beta| \tag{2.19}$$

for all $u \in C^\infty(\Omega)$.

The nonlinear operator $T(x, D)$, which is a mapping

$$T : C^\infty(\Omega) \longrightarrow C^\infty(\Omega) \tag{2.20}$$

may be extended to the mapping

$$T : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{ML}^\infty(\Omega)$$

defined by setting

$$Tu = (I \circ S)(F(., u, \cdots, \mathcal{D}^p u, \cdots)),\ |p| \le l. \tag{2.21}$$

Theorem 2.6. *The mapping* $T : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{ML}^\infty(\Omega)$ *defined through* (2.21) *is uniformly continuous.*

As a consequence of Theorem 2.6, there exists a unique uniformly continuous extension

$$T^{\sharp} : \mathcal{NL}^{\infty}(\Omega) \longrightarrow \mathcal{NL}^{\infty}(\Omega)$$

of T. This give rise to the concept of generalized solution of (2.14) as a solution $u^{\sharp} \in \mathcal{NL}^{\infty}(\Omega)$ of the extended equation

$$T^{\sharp}u^{\sharp} = h. \tag{2.22}$$

The main existence result for the C^{∞}-smooth PDE (2.14) is the following, see [25].

Theorem 2.7. *Consider the nonlinear PDE of the form* (2.14). *If the nonlinear operator T satisfies* (2.17), *then there exists some $u^{\sharp} \in \mathcal{NL}^{\infty}(\Omega)$ that satisfies* (2.22).

3. Differential algebra of generalized functions

In this section we briefly outline the main points in showing that the space $\mathcal{NL}^{l}(\Omega)$, for each $l \in \overline{\mathbb{N}}$, is an algebra of generalized functions referred to as *order convergence algebra of generalized functions*, admitting an embedding of $\mathcal{C}^{l}(\Omega)$ as a subalgebra, details of the results presented here can be found in [1], see also [2].

Proposition 3.1. *The space $\mathcal{ML}^{l}(\Omega)$ is a subalgebra of $\mathcal{NL}(\Omega)$. Furthermore, the differential operators*

$$\mathcal{D}^{p} : \mathcal{ML}^{l}(\Omega) \longrightarrow \mathcal{NL}, \quad |p| \leq l$$

are linear and satisfy the Leibnitz rule

$$\mathcal{D}^{p}(uv) = \sum_{q \leq p} \left(\begin{array}{c} p \\ q \end{array} \right) \mathcal{D}^{p-q}u\mathcal{D}^{q}v$$

Proposition 3.2. *The induced convergence structure λ_l on $\mathcal{ML}^{l}(\Omega)$ is a Hausdorff and first countable algebra convergence structure.*

We remark that the uniform convergence structure $\mathcal{J}_l$ on $\mathcal{ML}^{l}(\Omega)$ is the uniform convergence structure induced by the convergence structure λ_l, see [6]. Based on the abstract construction of the completion of a uniform convergence space, see [20], the set $\mathcal{NL}^{l}(\Omega)$ may be represented as

$$\mathcal{NL}^{l}(\Omega) = C[\mathcal{ML}^{l}(\Omega)]/ \sim_{C}, \tag{3.1}$$

where,

$$\mathcal{F} \sim_{C} \mathcal{G} \Leftrightarrow \mathcal{F} - \mathcal{G} \in \lambda_l(0).$$

The representation of $\mathcal{NL}^l(\Omega)$ can be further particularised. By Propostition 3.2, λ_l is first countable. Hence for $\mathcal{F} \in C[\mathcal{ML}^l(\Omega)]$ there exists $\mathcal{G} = [\{G_n | n \in \mathbb{N}\}] \longrightarrow 0$ in $\mathcal{ML}^l(\Omega)$ so that $\mathcal{G} \subseteq \mathcal{F} - \mathcal{F}$. Thus

$$(3.2) \qquad \begin{array}{ll} \forall & n \in \mathbb{N} : \\ \exists & F_n \in \mathcal{F} : \\ & F_n - F_n \subseteq G_n. \end{array}$$

For each $n \in \mathbb{N}$, select $u \in F_1 \cap \cdots \cap F_n$. Then $\langle u_n \rangle - \langle u_n \rangle \supseteq \mathcal{G}$ so that (u_n) is a Cauchy sequence in $\mathcal{ML}^l(\Omega)$. Furthermore, $\langle u_n \rangle \sim_C \mathcal{F}$ so that each $\sim_C$-equivalence class contains a Cauchy sequence. Therefore we may represent $\mathcal{NL}^l(\Omega)$ as

$$\mathcal{NL}^l(\Omega) = C_s[\mathcal{ML}^l(\Omega)] / \sim_{C_s}$$

where $C_s[\mathcal{ML}^l(\Omega)]$ denotes the set of Cauchy sequences in $\mathcal{ML}^l(\Omega)$, and for $(u_n), (v_n) \in C_s[\mathcal{ML}^l(\Omega)]$,

$$(u_n) \sim_{C_s} (v_n) \Longleftrightarrow \langle u_n - v_n \rangle \in \lambda_l(0).$$

In view of (3.1), the structure of $\mathcal{NL}^l(\Omega)$ depends only on the properties of the Cauchy sequences in $\mathcal{ML}^l(\Omega)$. Thus we have the following

Proposition 3.3. *A sequence (u_n) in $\mathcal{ML}^l(\Omega)$ is Cauchy sequence with respect to the uniform convergence structure on $\mathcal{J}_l$ if and only if there exists a residual set $R \subset \Omega$ such that $(\mathcal{D}^p u_n(x))$ is a convergent sequence in $\mathbb{R}$ for each $x \in R$ and $p \in \mathbb{N}^n$ with $|p| \leq l$.*

By Proposition 3.3, we have that

$$(3.3) \quad (u_n) \sim_{C_s} (v_n) \Longleftrightarrow \left(\begin{array}{ll} \exists & R \subseteq \Omega, \text{ a residual set} : \\ \forall & p \in \mathbb{N}^n, \ |p| \leq l, \ x \in R : \\ & \mathcal{D}^p u_n(x) - \mathcal{D}^p v_n(x) \to 0 \text{ in } \mathbb{R}. \end{array} \right)$$

In order to represent the space $\mathcal{NL}^l(\Omega)$ as an algebra of generalized functions, we show that each $\sim_{C_s}$-equivalence class contains a sequence of C^l-smooth functions. To do this, we make use of the Principle of Partition of Unity, see [17].

Theorem 3.4. *Let O be a locally finite open cover of an open subset Ω of $\mathbb{R}^n$. Then there is a collection*

$$\{\phi_U : \Omega \longrightarrow [0,1] : U \in O\}$$

of C^l-smooth functions ϕ_U such that the following hold:

(i) For each $U \in O$, the support of ϕ_U is contained in U.

(ii) $\sum_{U \in O} \phi_U(x) = 1$, for each $x \in M$.

A consequence of Theorem 3.4 is that disjoint, closed sets in Ω are separated by C^l-smooth, real valued functions. In this regard, let A and B be disjoint, nonempty, closed subsets of Ω. Then it follows from Theorem 3.4 that

$$\begin{array}{ll} \exists & \phi \in C^l(\Omega,[0,1]): \\ & (1)\ x \in A \Longrightarrow \phi(x) = 1 \\ & (2)\ x \in B \Longrightarrow \phi(x) = 0 \end{array} \tag{3.4}$$

Lemma 3.5. *Let (u_n) be a Cauchy sequence in $\mathcal{ML}^l(\Omega)$ with respect to $\mathcal{J}_l$. Then $C^l(\Omega)^{\mathbb{N}} \cap [(u_n)]_{C_s} \neq \emptyset$, where $[(u_n)]_{C_s}$ denotes the $\sim_{C_s}$-equivalence class generated by (u_n).*

The main result of this section is the following.

Theorem 3.6. *Let $\mathcal{S}^l_{cs} = C_s[\mathcal{ML}^l(\Omega)] \cap C^l(\Omega)^{\mathbb{N}}$ and $\mathcal{I}^l_{cs} = \lambda_l(0) \cap C^l(\Omega)^{\mathbb{N}}$. Then*

(i) $\mathcal{S}^l_{cs}$ is a subalgebra of $C^l(\Omega)^{\mathbb{N}}$ and $\mathcal{I}^l_{cs}$ is an ideal in $\mathcal{S}^l_{cs}$.

(ii) $\Delta(C^l(\Omega)) \subseteq \mathcal{S}^l_{cs}$ and $\Delta(C^l(\Omega)) \cap \mathcal{I}^l_{cs} = \{0\}$.

(iii) There exists a bijective mapping $E^l_{cs} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$ such that the diagram

$$\begin{tikzcd} \mathcal{S}^l_{cs} \arrow[r, "L"] \arrow[dr, "q_{\mathcal{S}^l_{cs}}"'] & \mathcal{NL}^l(\Omega) \arrow[d, "E^l_{cs}"] \\ & \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \end{tikzcd} \tag{3.5}$$

commutes. Here, $q_{\mathcal{S}^l_{cs}}$ is the canonical mapping associated with the quotient algebra $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$, and the mapping L is defined as

$$L : \mathcal{S}^l_{cs} \ni u = (u_n) \mapsto u^\sharp \in \mathcal{NL}^l(\Omega), \tag{3.6}$$

where $u^\sharp$ is the limit of (u_n) in $\mathcal{NL}^l(\Omega)$.

4. The chain of order convergence algebras of generalized functions

In this section we study the chain structure

$$\mathcal{NL}^\infty(\Omega) \to \cdots \to \mathcal{NL}^l(\Omega) \to \mathcal{NL}^{l-1}(\Omega) \to \cdots \to \mathcal{NL}^0(\Omega).$$

we show how the spaces of generalized functions $\mathcal{NL}^l(\Omega)$, $l \in \overline{\mathbb{N}}$, may be represented as a chain of algebras of generalized functions, referred to as chain of order convergence algebras of generalized functions. We further show how the existence result for generalized solutions of $\mathcal{C}^\infty$-smooth PDEs may be interpreted in the differential-algebraic frame work. This chain is denoted with

$$\mathbf{A}_{oc} = \{(\mathcal{NL}^l(\Omega), \mathcal{NL}^k(\Omega), \gamma_k^l) \mid k, l \in \overline{\mathbb{N}},\ k \leq l\}.$$

Our method for constructing the chain $\mathbf{A}_{oc}$ is based on Rosinger's technique for constructing an abstract chain of algebras of generalized functions, see [13, 14].

Definition 4.1. Let $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \mid k, l \in \overline{\mathbb{N}},\ k \leq l\}$, where $\mathcal{A}^l(\Omega)$ is a unital, commutative algebra for each $l \in \overline{\mathbb{N}}$ and
$\gamma_k^l : \mathcal{A}^l(\Omega) \to \mathcal{A}^k(\Omega)$
is an algebra homomorphism for $k \leq l$. $\mathbf{A}$ is a chain of algebras of generalized functions if the following hold.

(i) The diagram

$$\begin{tikzcd} \mathcal{A}^l(\Omega) \arrow[rr, "\gamma_h^l"] \arrow[dr, "\gamma_k^l"'] & & \mathcal{A}^h(\Omega) \\ & \mathcal{A}^k(\Omega) \arrow[ur, "\gamma_h^k"'] & \end{tikzcd} \tag{4.1}$$

commutes for all $h, k, l \in \overline{\mathbb{N}}$ with $h \leq k \leq l$.

(ii) For $l \geq k > 0$ and $p \in \mathbb{N}^n$, with $|p| + k \leq l$, there exists a linear differential operator $D^p : \mathcal{A}^l(\Omega) \to \mathcal{A}^k(\Omega)$ that satisfies the Leibnitz rule.

(iii) If, in addition, the diagram

$$\begin{tikzcd} \mathcal{A}^k(\Omega) \arrow[r, "D^p"] & \mathcal{A}^{k-|p|}(\Omega) \\ \mathcal{A}^l(\Omega) \arrow[u, "\gamma_k^l"] \arrow[r, "D^p"] & \mathcal{A}^{l-|p|}(\Omega) \arrow[u, "\gamma_{k-|p|}^{l-|p|}"'] \end{tikzcd} \tag{4.2}$$

commutes for all $l \geq k$ and $p \in \mathbb{N}^n$, $|p| \leq k \leq l$, we call the chain $\mathbf{A}$ differential.

We outline, briefly, how such chain of algebras of generalized functions may be constructed, see [14]. Let $l \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be given. Let Λ be an infinite index set, and define

$$\mathcal{C}^l(\Omega)^\Lambda = \left\{ u = (u_\lambda)_{\lambda \in \Lambda} \;\middle|\; \begin{array}{l} \forall \quad \lambda \in \Lambda \ : \\ \quad u_\lambda \in \mathcal{C}^l(\Omega) \end{array} \right\}. \tag{4.3}$$

It is easy to see that the set $\mathcal{C}^l(\Omega)^\Lambda$ is a commutative algebra with unit element, when considered with the termwise operations on sequences of functions. For a subalgebra $\mathcal{S}^l$ of $\mathcal{C}^l(\Omega)^\Lambda$, and a proper ideal $\mathcal{I}^l$ in $\mathcal{S}^l$, the quotient algebra

$$\mathcal{A}^l(\Omega) = \mathcal{S}^l / \mathcal{I}^l \tag{4.4}$$

is a unital and commutative algebra of generalized functions on Ω.

If for $k \leq l$, the inclusions

$$\mathcal{S}^l \subseteq \mathcal{S}^k, \;\; \mathcal{I}^l \subseteq \mathcal{I}^k \tag{4.5}$$

hold, then

$$\gamma^l_k : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto u + \mathcal{I}^k \in \mathcal{A}^k(\Omega) \tag{4.6}$$

defines an algebra homomorphism. Clearly, in this case the diagram (4.1) commutes for $h \leq k \leq l$.

Suppose further that, for $l > 0$ and $p \in \mathbb{N}^n$ with $|p| \leq l$ and $k \leq |p| \leq l$ we have

$$D^p(\mathcal{S}^l) \subseteq \mathcal{S}^k, \;\; D^p(\mathcal{I}^l) \subseteq \mathcal{I}^k. \tag{4.7}$$

Then

$$D^p : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto D^p(u) + \mathcal{I}^k \in \mathcal{A}^k(\Omega)$$

defines a linear differential operator that satisfies the Leibnitz rule (1.1). For $k \leq l$ and $p \in \mathbb{N}^n$ such that $|p| \leq k$, the diagram (4.2) commutes. Hence we have the following.

Theorem 4.2. *For each $l \in \overline{\mathbb{N}}$, let $\mathcal{S}^l$ be a subalgebra of $\mathcal{C}^l(\Omega)^\Lambda$ and $\mathcal{I}^l$ an ideal in $\mathcal{S}^l$. If (4.5) and (4.7) are satisfied, then $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma^l_k) \; : \; k, l \in \overline{\mathbb{N}}, \; k \leq l\}$, with $\mathcal{A}^l(\Omega) = \mathcal{S}^l / \mathcal{I}^l$ and γ^l_k defined by (4.6), is a differential chain of algebras of generalized functions.*

The embedding of $\mathcal{C}^l$ - smooth function into the chain $\mathbf{A}$ follows directly from the embedding of $\mathcal{C}^l(\Omega)$ into $\mathcal{A}^l(\Omega)$, for each $l \in \overline{\mathbb{N}}$. The existence of an algebra embedding

$$\mathcal{C}^l(\Omega) \hookrightarrow \mathcal{A}^l(\Omega), \;\; l \in \overline{\mathbb{N}} \tag{4.8}$$

is determined by the neutrix condition

$$\mathcal{U}^l_\Lambda(\Omega) \subseteq \mathcal{S}^l, \qquad \mathcal{U}^l_\Lambda(\Omega) \cap \mathcal{I}^l = \{0\} \tag{4.9}$$

where

$$\mathcal{U}^l_\Lambda(\Omega) = \left\{ u = (u_\lambda)_{\lambda \in \Lambda} \;\middle|\; \begin{array}{ll} \exists & v \in \mathcal{C}^l(\Omega) : \\ \forall & \lambda \in \Lambda \;\; : \\ & u_\lambda = v \end{array} \right\}.$$

Theorem 4.3. *Suppose that (4.9) is satisfied for each $l \in \overline{\mathbb{N}}$. Then*

$$\mathcal{C}^l(\Omega) \ni u \mapsto \Delta(u) + \mathcal{I}^l \in \mathcal{A}^l(\Omega) \tag{4.10}$$

defines an injective algebra homomorphism for each $l \in \overline{\mathbb{N}}$. Furthermore, the diagrams

$$\begin{tikzcd}
\mathcal{A}^l(\Omega) \arrow[rr, "\gamma^l_k"] & & \mathcal{A}^k(\Omega) \\
\mathcal{C}^l(\Omega) \arrow[u, "\hookrightarrow"] \arrow[rr, "\subseteq"] & & \mathcal{C}^k(\Omega) \arrow[u, "\hookrightarrow"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}^l(\Omega) \arrow[r, "D^p"] & \mathcal{A}^{l-|p|}(\Omega) \\
\mathcal{C}^l(\Omega) \arrow[r, "D^p"] \arrow[u] & \mathcal{C}^{l-|p|}(\Omega) \arrow[u]
\end{tikzcd}$$

commute for all $l, k \in \overline{\mathbb{N}}$ *and* $p \in \mathbb{N}^n$, $|p| \leq l$.

We now show how the spaces $\mathcal{NL}^l(\Omega)$, $l \in \overline{\mathbb{N}}$, form a chain of algebras of generalized functions. By virtue of the definition of the uniform convergence structure on $\mathcal{ML}^l(\Omega)$, the partial derivative operators

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega), \quad k + |p| \leq l \tag{4.11}$$

are uniformly continuous. Hence there exist unique uniformly continuous extensions

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + |p| \leq l \tag{4.12}$$

of the mappings in (4.11). On the other hand, since $\mathcal{S}^l_{cs} \subset C_s[\mathcal{ML}^l(\Omega)]$ and $\mathcal{I}^l_{cs}$ consists of null sequences in $\mathcal{ML}^l(\Omega)$, it follows by uniform continuity of the mapping in (4.11) that

$$D^p(\mathcal{I}^l_{cs}) \subseteq \mathcal{I}^k_{cs}, \quad \text{and} \quad D^p(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad p \in \mathbb{N}^n, \quad |p| \leq l - k, \tag{4.13}$$

so that

$$D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \ni (u) + \mathcal{I}^l_{cs} \mapsto D^p(u) + \mathcal{I}^k_{cs} \in \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs} \tag{4.14}$$

define linear mappings that satisfy the Leibnitz rule.

Proposition 4.4. *The diagram*

$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "\mathcal{D}^{p\sharp}"] \arrow[d, "E^l_{cs}"'] & \mathcal{NL}^k(\Omega) \arrow[d, "E^l_{cs}"] \\
\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \arrow[r, "D^p"] & \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs}
\end{tikzcd} \tag{4.15}$$

commutes for all $p \in \mathbb{N}^n$, $l, k \in \overline{\mathbb{N}}$, *so that* $k + |p| \leq l$ *with*

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + |p| \leq l.$$

given by (4.12) *and*

$$D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \longrightarrow \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs} \tag{4.16}$$

given by (4.14).

Proof. Fix $u^\sharp \in \mathcal{NL}^l(\Omega)$. According to Theorem 3.6, $u^\sharp \in L(u)$ for some $u \in \mathcal{S}^l_{cs}$, and $E^l_{cs}(u^\sharp) = u + \mathcal{I}^l_{cs}$. So $\mathcal{D}^p(E^l_{cs}(u^\sharp)) = \mathcal{D}^p(u) + \mathcal{I}^k_{cs}$. But $\mathcal{D}^{p\sharp}u^\sharp = L(\mathcal{D}^p u)$ so that, by Theorem 3.6, $E^k_{cs}(\mathcal{D}^{p\sharp}u^\sharp) = \mathcal{D}^p u + \mathcal{I}^k_{cs} = \mathcal{D}^p(E^l_{cs}(u^\sharp)$. Thus the diagram (4.15) commutes. ☐

Observe that

$$\mathcal{S}^l_{cs} \subseteq \mathcal{S}^k_{cs} \quad \text{and} \quad \mathcal{I}^l_{cs} \subseteq \mathcal{I}^k_{cs} \tag{4.17}$$

for all $l, k \in \overline{\mathbb{N}}$ such that $k \leq l$. Indeed, it follows directly from the definition of the uniform convergence structure on $\mathcal{ML}^l(\Omega)$ and $\mathcal{ML}^k(\Omega)$, respectively, that the inclusion map

$$\mathcal{ML}^l(\Omega) \ni u \mapsto u \in \mathcal{ML}^k(\Omega)$$

is uniformly continuous. Thus (4.17) follows immediately from the definition of $\mathcal{I}^l_{cs}$ and $\mathcal{S}^l_{cs}$. Thus

$$\gamma^l_k : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \ni (u) + \mathcal{I}^l_{cs} \mapsto u + \mathcal{I}^k_{cs} \in \mathcal{S}^k_{cs})/\mathcal{I}^k_{cs} \tag{4.18}$$

defines an algebra homomorphism, see (4.5) and (4.6).

In view of Theorem 3.6 and Proposition 4.4, the spaces $\mathcal{NL}^l(\Omega)$, and the differential operators

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega)$$

with $|p| + k \leq l$, may be identified with the algebras $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$, with differential operators $D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \longrightarrow \mathcal{S}^k/\mathcal{I}^k_{cs}$ defined in (4.14). Therefore we denote the algebras $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$ by $\mathcal{NL}^l(\Omega)$.

As a direct application of Theorem 4.2 we now have the following

Theorem 4.5. *With the algebra homomorphism*

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega)$$

define as in (4.18) *and the differential operator*

$$\mathcal{D}^p : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega),$$

with $k + |p| \leq l$ define as in (4.16),

$$\mathbf{A}_{oc} = \{(\mathcal{NL}^l(\Omega), \mathcal{NL}^k(\Omega), \gamma^l_k) \mid k, l \in \overline{\mathbb{N}},\ k \leq l\}$$

is a differential chain of algebras of generalized functions.

Proof. The result follows from (4.13), (4.17) and Theorem 4.2. ☐

Next we address the issue of embedding smooth functions into the chain $\mathbf{A}_{oc}$ of algebras of generalized functions.

Theorem 4.6. *For each $l \in \mathbb{N}$, there exists an injective algebra homomorphism*

$$\mathcal{E}^l_{cs} : C^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

so that the diagram

$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "\gamma^l_k"'] \arrow[rr, bend left, "\gamma^l_h"] & \mathcal{NL}^k(\Omega) \arrow[r, "\gamma^k_h"'] & \mathcal{NL}^h \\
C^l(\Omega) \arrow[r, "\subseteq"] \arrow[u, "\mathcal{E}^l_{cs}"] & C^k(\Omega) \arrow[r, "\subseteq"] \arrow[u, "\mathcal{E}^k_{cs}"] & C^h(\Omega) \arrow[u, "\mathcal{E}^h_{cs}"]
\end{tikzcd} \tag{4.19}$$

commutes. Here γ^l_k, γ^k_h, γ^l_h, are injective algebra homomorphisms defined by (4.18)*, while $\mathcal{E}^l_{cs}$, $\mathcal{E}^h_{cs}$, $\mathcal{E}^k_{cs}$ are linear injective algebra homomorphisms defined as in* (4.10)*.*

Proof. Since $\mathcal{S}^l_{cs}$ is contained in the set $C_s[\mathcal{ML}^l(\Omega)]$ of $\mathcal{J}_l$ - Cauchy sequences in $\mathcal{ML}^l(\Omega)$, it follows that $\mathcal{U}^l_{\mathbb{N}}(\Omega) \subseteq \mathcal{S}^l_{cs}$. Furthermore, $\mathcal{I}^l_{cs} \subset \lambda_l(0)$, so that, since λ_l is Hausdorff $\mathcal{I}^l_{cs} \cap \mathcal{U}^l_{\mathbb{N}}(\Omega) = \{0\}$. The result now follows from Theorem 4.3. □

5. Embedding spaces of normal lower semi-continuous functions

The embedding $\mathcal{C}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$ extends in a natural way to an embedding

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega). \tag{5.1}$$

Theorem 5.1. *For each $l \in \overline{\mathbb{N}}$ there exists an injective homomorphism*

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega).$$

so that the following hold.

(i) The diagram

$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "\gamma^l_k"] & \mathcal{NL}^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[r, "\subset"] \arrow[u, "H^l_{oc}"] & \mathcal{ML}^k \arrow[u, "H^k_{oc}"']
\end{tikzcd} \tag{5.2}$$

commutes whenever $k \leq l$.

ii The diagram

(5.3)
$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "D^p"] & \mathcal{NL}^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "H^l_{oc}"] \arrow[r, "\mathcal{D}^p"] & \mathcal{ML}^k \arrow[u, "H^k_{oc}"']
\end{tikzcd}$$

commutes whenever $k + |p| \leq l$.

(iii) The diagram

(5.4)
$$\begin{tikzcd}
\mathcal{ML}^l(\Omega) \arrow[rr, "H^l_{oc}"] & & \mathcal{NL}^l(\Omega) \\
& \mathcal{C}^l(\Omega) \arrow[ul, "\subset"] \arrow[ur, "\mathcal{E}^l_{cs}"'] &
\end{tikzcd}$$

commutes for all $l \in \overline{\mathbb{N}}$.

Proof. Consider the map

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \ni u \mapsto (u_n) + \mathcal{I}^l_{cs} \in \mathcal{NL}^l(\Omega) \tag{5.5}$$

where $(u_n) \in \mathcal{S}^l_{cs}$ converges to u with respect to λ_l. The existence of such a sequence follows from Lemma 3.5. To see that H^1_{oc} is well defined, let $(u_n), (v_n)$ be two sequences in $\mathcal{S}^l_{cs}$ converging to u with respect to λ_l. Based on Proposition 3.2 we conclude that $(u_n - v_n)$ converges to 0 with respect to λ_l, so that $(u_n - v_n) \in \mathcal{I}^l_{cs}$. It follows from Proposition 3.2 that H^l_{oc} is an injective algebra homomorphism. Indeed, if $H^l_{oc}(u) = H^l_{oc}(v)$ for some $u, v \in \mathcal{ML}(\Omega)$, then there exists $(u_n) \in \mathcal{S}^l_{cs}$ that converges to u and v with respect to λ_l. Since λ_l is Hausdorff, it follows that $u = v$. If $(u_n), (v_n) \in \mathcal{S}^l_{cs}$ converge to $u, v \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, respectively, then $(u_n v_n)$ converges to uv with respect to λ_l. Hence

$$\begin{aligned}
H^l_{oc}(u)H^l_{oc}(v) &= ((u_n) + \mathcal{I}^l_{cs})((v_n) + \mathcal{I}^l_{cs}) \\
&= (u_n v_n) + \mathcal{I}^l_{cs} \\
&= H^l_{oc}(uv).
\end{aligned}$$

Linearity of H^l_{oc} follows the same way.

(i) The commutativity of the diagram (5.2), follows immediately from th definitions of the homomorphisms H^l_{oc}, H^k_{oc} and γ^l_k.

(ii) Recall that, for $k + |p| \leq l$, the partial differential operator

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega)$$

is uniformly continuous, thus continuous with respect to the convergence structure λ_l and λ_k. Thus if

$$H^l_{oc}(u) = (u_n) + \mathcal{I}^l_{oc}$$

for some $u \in \mathcal{ML}^l(\Omega)$, then $D^p(u_n) = (\mathcal{D}^p u_n)$ converges to $\mathcal{D}^p u$ in $\mathcal{ML}^k(\Omega)$ with respect to λ_k. Hence

$$H^k_{oc}(\mathcal{D}^p u) = D^p(u_n) + \mathcal{I}^k_{cs}.$$

By definition,

$$D^p(H^l_{oc} u) = D^p(u_n) + \mathcal{I}^k_{cs}.$$

Thus (5.3) is commutative.

(iii) The embedding $\mathcal{E}^l_{cs} : \mathcal{C}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$ is given by

$$\mathcal{E}^l_{cs}(u) = \Delta(u) + \mathcal{I}^l_{cs}$$

where $\Delta : \mathcal{C}^l(\Omega) \longrightarrow \mathcal{S}^l_{cs}$ maps each $u \in \mathcal{C}^l(\Omega)$ to the constant sequence with all terms equal to u. Since this sequence converges to u with respect to λ_l, the result follows immediately from the definition of the map H^l_{oc}.

□

6. Existence of chain generalized solutions

In this section, we give an interpretation of the existence result for smooth PDEs, Theorem 2.7, in the context of the chain

$$\mathbf{A}_{oc} = \{(\mathcal{NL}^l, \mathcal{NL}^k, \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

of algebra of generalized functions. In particular, we show that the generalized solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ obtained through the Theorem 2.7 is a chain generalized solution.

Definition 6.1. Let $\mathbf{A}$ be a chain. A generalized function $u + \mathcal{I}^\infty \in \mathcal{A}^\infty(\Omega)$ is a chain generalized solution of (2.10) in the chain $\mathbf{A}$ if

$$T(\gamma^\infty_l(u + \mathcal{I}^\infty)) = \gamma^\infty_k(f + \mathcal{I}^\infty)$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k + m \leq l$.

In order to show the generalized solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ is a chain generalized solution we consider the nonlinear partial differential operator

$$T : C^l(\Omega) \longrightarrow C^k(\Omega),\quad k + m \leq l \tag{6.1}$$

of order at most m, defined through a C^∞-smooth mapping

$$F : \Omega \times \mathbb{R}^M \longrightarrow \mathbb{R}$$

by setting

$$Tu(x) = F(x, u(x), \cdots, D^p u(x), \cdots),\ |p| \leq m \tag{6.2}$$

for each $x \in \Omega$. Since

$$T(C^l(\Omega)) \subseteq C^k(\Omega),$$

it follows that

$$T(C^l(\Omega)^{\mathbb{N}}) \subseteq C^k(\Omega)^{\mathbb{N}}.$$

By the definition of $\mathcal{ML}^l(\Omega)$, see (2.4), and owing to F being C^∞-smooth, the mapping (6.1) may be extended to a map

$$T : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega) \quad k + m \leq l. \tag{6.3}$$

It follows from the uniform continuity of the mapping T, see [22, Theorem 6] and the uniform continuity of the embedding

$$\mathcal{ML}^l(\Omega) \ni u \mapsto u \in \mathcal{ML}^k(\Omega) \quad k \leq l.$$

that (6.3) is uniformly continuous for all $l, k \in \overline{\mathbb{N}}$ such that $k + m \leq l$. Hence there exists unique uniformly continuous extensions

$$T^\sharp : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + m \leq l. \tag{6.4}$$

of (6.3).

On the other hand, in view of the construction of the extension of a uniformly continuous map to the completion of its domain, the map

$$T : C^l(\Omega)^{\mathbb{N}} \ni (u_n) \mapsto (Tu_n) \in C^k(\Omega)^{\mathbb{N}} \quad k + m \leq l.$$

satisfies

$$T(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad k + m \leq l$$

and

$$(u_n) - (v_n) \in \mathcal{I}^l_{cs} \Longrightarrow T(u_n) - T(v_n) \in \mathcal{I}^k_{cs}, \quad k + m \leq l$$

Thus in view of (2.14) - (2.16), and since $\mathcal{S}^l_{cs}$ and $\mathcal{I}^l_{cs}$ satisfy the neutrix condition (4.9), it follows that

$$T : \mathcal{NL}^l(\Omega) \ni u + \mathcal{I}^l_{cs} \mapsto Tu + \mathcal{I}^k_{cs} \in \mathcal{NL}^k(\Omega) \quad k + m \leq l. \tag{6.5}$$

defines an extension of (6.1). Using the same argument as in the proof of Theorem 4.5, it follows that (6.4) and (6.5) are equal, in the sense that the diagram

$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "T^\sharp"] \arrow[d, "E^l_{cs}"] & \mathcal{NL}^k(\Omega) \arrow[d, "E^l_{cs}"] \\
\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \arrow[r, "T"] & \mathcal{S}^k_{cs}/\mathcal{I}^l_{cs}
\end{tikzcd} \tag{6.6}$$

commutes for all $l, k \in \overline{\mathbb{N}}, \quad k + m \leq l$

Our main result is the following

Theorem 6.2. *Assume that the PDE*

$$T(x, D)u(x) = f(x), \quad x \in \Omega \tag{6.7}$$

with $f \in C^\infty(\Omega)$ *and* T *defined as in* (6.2) *satisfies* (2.17). *Then* (6.7) *admits a chain generalized solution* $u + \mathcal{I}^\infty_{cs} \in \mathcal{NL}^\infty(\Omega)$.

Proof. According to Theorem 2.7, there exists a generalized solution

$$u \in \mathcal{NL}^\infty(\Omega)$$

of the PDE (6.7). Thus there exists a sequence $(u_n) \in \mathcal{S}^l_{cs}$ so that

$$u = (u_n) + \mathcal{I}^\infty_{cs}$$

satisfies $Tu = f$ in $\mathcal{NL}^\infty(\Omega)$. That is,

$$(Tu_n) - f \in \mathcal{I}^\infty_{cs} \subseteq \mathcal{I}^k_{cs}, \quad k \in \overline{\mathbb{N}}. \tag{6.8}$$

By definition of the algebra homomorphism

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{6.9}$$

we have

$$T(\gamma^l_k(u)) = T(u_n) + \mathcal{I}^k_{cs}, \quad k + m \leq l \tag{6.10}$$

and

$$\gamma^\infty_k(f) = f + \mathcal{I}^l_{cs} \tag{6.11}$$

Thus (6.8), (6.10) and (6.11) imply that

$$T(\gamma^l_k(u)) = \gamma^\infty_{cs}(f + \mathcal{I}^\infty), \quad k + m \leq l$$

□

7. Differential chains of nowhere dense algebras

In this section we briefly outline the main points in the construction of the so-called differential chain of nowhere dense algebras of generalized functions

$$\mathbf{A}_{nd} = \{(\mathcal{A}^l_{nd}(\Omega), \mathcal{A}^k_{nd}(\Omega), \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

introduced by Rosinger [13, 14, 15], as well as the differential chain of almost everywhere algebras of generalized functions

$$\mathbf{A}_{ae} = \{(\mathcal{A}^l_{ae}(\Omega), \mathcal{A}^k_{ae}(\Omega), \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

which was obtained using Verneave's construction of the almost everywhere algebra $\mathcal{A}^\infty_{ae}(\Omega)$, see [26, 27]. In this regard, let $l \in \overline{\mathbb{N}}$ and denote by $\mathcal{I}^l_{nd}$ the set of all sequences of functions in $C^l(\Omega)$ satisfying the following asymptotic vanishing condition:

$$u = (u_n)_{n\in\mathbb{N}} \in \mathcal{I}^l_{nd} \Longleftrightarrow \left\{ \begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \forall & x \in \Omega\backslash\Gamma : \\ \exists & V \subset \Omega\backslash\Gamma, \text{ neighbourhood of } x,\ N_V \in \mathbb{N} : \\ \forall & y \in V,\ n \geq N_V : \\ & u_n(y) = 0 \end{array} \right.$$

In other words, the terms of the sequence (u_n) vanish at each point of the open and dense subset $\Omega\backslash\Gamma$, provided $n \in \mathbb{N}$ is sufficiently large. The set $\mathcal{I}^l_{nd}$ is an ideal in $C^l(\Omega)^{\mathbb{N}}$, see [13, Chapter 1 Sction 7]. The ideal $\mathcal{I}^l_{nd} \subseteq C^l(\Omega)^{\mathbb{N}}$ is called the *nowhere dense ideal* on Ω and satisfies the neutrix condition, 4.9 so that

$$\mathcal{A}^l_{nd}(\Omega) = C^l(\Omega)^{\mathbb{N}}/\mathcal{I}^l_{nd}$$

is an algebra of generalized functions. Furthermore the inclusions

$$\mathcal{I}^l_{nd} \subset \mathcal{I}^k_{nd}, \quad k \leq l$$

and

$$D^p(\mathcal{I}^l_{nd}) \subset \mathcal{I}^k_{nd}, \quad |p| + k \leq l$$

hold. Thus we have the following

Theorem 7.1.

$$\mathbf{A}_{nd} = \{(\mathcal{A}^l_{nd}(\Omega), \mathcal{A}^k_{nd}(\Omega), \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

is a differential chain of algebras of generalized functions.

The algebra homomorphism

$$\gamma^l_k : \mathcal{A}^l_{nd}(\Omega) \longrightarrow \mathcal{A}^k_{nd}(\Omega), \quad k \leq l$$

is defined as (4.6). That is,

$$\gamma^l_k : \mathcal{A}^l_{nd}(\Omega) \ni u + \mathcal{I}^l_{nd} \mapsto u + \mathcal{I}^k_{nd} \in \mathcal{A}^k_{nd}(\Omega), \quad k \leq l \tag{7.1}$$

Since the neutrix condition (4.9) is satisfied, it follows from Theorem 4.3 that

$$\mathcal{E}_l : C^l(\Omega) \ni u \mapsto \Delta(u) + \mathcal{I}^l_{nd} \in \mathcal{A}^k_{nd}(\Omega).$$

defines an injective algebra homomorphism for each $l \in \overline{\mathbb{N}}$. Furthermore, the diagrams

$$\begin{tikzcd}
\mathcal{A}^l_{nd} \arrow[r, "\gamma_{lk}"'] \arrow[rr, bend left, "\gamma_{lh}"] & \mathcal{A}^k_{nd} \arrow[r, "\gamma_{kh}"'] & \mathcal{A}^h_{nd} \\
C^l(\Omega) \arrow[u, "\mathcal{E}_l"] \arrow[r, "\subseteq"] & C^k(\Omega) \arrow[u, "\mathcal{E}_h"] \arrow[r, "\subseteq"] & C^h(\Omega) \arrow[u, "\mathcal{E}_k"]
\end{tikzcd} \tag{7.2}$$

and

$$\begin{tikzcd}
\mathcal{A}^{l'}_{nd}(\Omega) \arrow[r, "D^p"] & \mathcal{A}^{k'}_{nd}(\Omega) \\
\mathcal{A}^{l}_{nd}(\Omega) \arrow[u, "\gamma^l_{l'}"] \arrow[r, "D^p"] & \mathcal{A}^{k}_{nd}(\Omega) \arrow[u, "\gamma^k_{k'}"']
\end{tikzcd} \tag{7.3}$$

commute, for all $h \leq k \leq l$ and all $k + |p| \leq l$.

We next discuss the construction of the chain of almost everywhere algebras of generalized functions, $\mathbf{A}_{ae}$. Let $\mathcal{M}_0$ be a set of closed nowhere dense subset of Ω that is closed under the formation of finite unions of its elements. For $l \in \overline{\mathbb{N}}$, let

$$\mathcal{E}^l_{ae}(\Omega) = \left\{ (u_n) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & (1)\ u_n : \Omega \longrightarrow \mathbb{R} : \\ & (2)\ u_n \in C^l(\Omega \backslash \Gamma) \end{array} \right\} \tag{7.4}$$

It is easy to see that $\mathcal{E}^l_{ae}(\Omega)$ is an algebra over $\mathbb{R}$ with respect to the termwise operations on sequences of functions. Consider the ideals

$$\mathcal{I}^l_E := \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \end{array} \right\} \tag{7.5}$$

and

$$\mathcal{I}^l_{ae} := \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & u_n(x) = 0, \quad x \in \Omega \setminus \Gamma \end{array} \right\} \tag{7.6}$$

Since $\mathcal{I}^l_E$ and $\mathcal{I}^l_{ae}$ are ideals, so is

$$(7.7)\quad \mathcal{I}^l_E + \mathcal{I}^l_{ae} = \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ \ n \geq N : \\ & u_n(y) = 0, \ \ y \in V \setminus \Gamma \end{array} \right\}.$$

The algebra $\mathcal{A}^l_{ae}(\Omega)$ is defined as

$$(7.8)\qquad \mathcal{A}^l_{ae}(\Omega) = \mathcal{E}^l_{ae}/(\mathcal{I}^l_E + \mathcal{I}^l_{ae}).$$

Since

$$(7.9)\qquad \mathcal{E}^l_{ae}(\Omega) \subseteq \mathcal{E}^k_{ae}(\Omega) \ \text{ and } \ \mathcal{I}^l_E + \mathcal{I}^l_{ae} \subseteq \mathcal{I}^k_E + \mathcal{I}^k_{ae}$$

whenever $l \geq k$, it follows that

$$(7.10)\quad \gamma^l_k : \mathcal{A}^l_{ae}(\Omega) \ni u + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \mapsto u + (\mathcal{I}^k_E + \mathcal{I}^k_{ae}) \in \mathcal{A}^k_{ae}(\Omega)$$

defines an algebra homomorphism.

Theorem 7.2.

$$\mathbf{A}_{ae} = \{\mathcal{A}^l_{ae}(\Omega), \mathcal{A}^k_{ae}(\Omega), \gamma^l_k\} \mid l,\ k \in \overline{\mathbb{N}},\ k \leq l\}$$

is a differential chain of algebras of generalized functions with γ^l_k *defined in* (7.10).

8. Relationship between the chain of order convergence Algebras and the chain of nowhere Dense Algebras

In this section we show how the chain $\mathbf{A}_{oc}$ is related to the chain of nowhere dense algebra of generalized functions, denoted as $\mathbf{A}_{nd}$, which was introduced by Rosinger [14] In order to establish the mentioned relationship between the chains $\mathbf{A}_{oc}$ and $\mathbf{A}_{nd}$, we introduce an auxiliary chain $\mathbf{A}^0_{nd}$. In this regard, we note that

$$(8.1)\qquad \mathcal{I}^l_{nd} \subset \mathcal{I}^l_{cs} \subset \mathcal{S}^l_{cs}, \quad l \in \overline{\mathbb{N}}.$$

Indeed, for each $(u_n) \in \mathcal{I}^l_{nd}$ there exists $\Gamma \subset \Omega$ closed and nowhere dense such that

$$\begin{array}{ll} \forall & x \in \Omega \setminus \Gamma : \\ \exists & N \in \mathbb{N} : \\ \forall & n \geq N, \ \ |p| \leq l : \\ & D^p u_n(x) = 0 \end{array}$$

Thus (u_n) converges to 0 pointwise on an open and dense, hence residual, subset of Ω. It follows from Proposition 3.3 that $(u_n) \in \mathcal{I}^l_{cs}$. Since $\mathcal{I}^l_{nd}$ is an ideal in $C^l(\Omega)^{\mathbb{N}}$, it is also an ideal in $\mathcal{S}^l_{cs}$. Furthermore, the inclusions,

$$\mathcal{I}^l_{nd} \subseteq \mathcal{I}^k_{nd}, \ \ \mathcal{S}^l_{cs} \subseteq \mathcal{S}^k_{cs}, \ \ k \leq l$$

and

$$D^p(\mathcal{I}^l_{nd}) \subseteq \mathcal{I}^k_{nd}, \quad D^p(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad |p| + k \leq l$$

imply that

$$\mathbf{A}^0_{nd} = \{\mathcal{A}^l_0(\Omega), \mathcal{A}^k_0(\Omega), \gamma^l_k)|l, k \in \mathbb{N}, \quad k \leq l\}$$

with $\mathcal{A}^l_0(\Omega) = \mathcal{S}^l_{cs}/\mathcal{I}^l_{nd}$ and γ^l_k defined as

$$\gamma^l_k : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto (u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_0(\Omega) \;\; k \leq l \tag{8.2}$$

is a differential chain of algebras of generalized functions. The way in which $\mathbf{A}_{oc}$ is related to $\mathbf{A}_{nd}$ is given in the following

Theorem 8.1. *For each $l \in \overline{\mathbb{N}}$ then there exists an injective algebra homomorphism*

$$H^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega)$$

and a surjective algebra homomorphism

$$G^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

such that the following hold.

(i) The diagrams

$$\begin{tikzcd} \mathcal{A}^l_0(\Omega) \arrow[r, "H^l"] \arrow[d, "\gamma^l_k"'] & \mathcal{A}^l_{nd}(\Omega) \arrow[d, "\gamma^l_k"] \\ \mathcal{A}^k_0(\Omega) \arrow[r, "H^k"] & \mathcal{A}^k_{nd}(\Omega) \end{tikzcd} \tag{8.3}$$

and

$$\begin{tikzcd} \mathcal{A}^l_0(\Omega) \arrow[r, "G^l"] \arrow[d, "\gamma^l_k"'] & \mathcal{NL}^l(\Omega) \arrow[d, "\gamma^l_k"] \\ \mathcal{A}^k_0(\Omega) \arrow[r, "G^k"] & \mathcal{NL}^k(\Omega) \end{tikzcd} \tag{8.4}$$

commute for all $k \leq l$.

(ii) The diagrams

$$\begin{tikzcd} \mathcal{A}^l_0(\Omega) \arrow[r, "D^p"] \arrow[d, "H^l"'] & \mathcal{A}^k_0(\Omega) \arrow[d, "H^k"] \\ \mathcal{A}^l_{nd}(\Omega) \arrow[r, "D^p"] & \mathcal{A}^k_{nd}(\Omega) \end{tikzcd} \tag{8.5}$$

and

(8.6)
$$\begin{tikzcd}
\mathcal{A}^l_0(\Omega) \arrow[r, "D^p"] \arrow[d, "G^l"'] & \mathcal{A}^k_0(\Omega) \arrow[d, "G^k"] \\
\mathcal{NL}^l(\Omega) \arrow[r, "D^p"] & \mathcal{NL}^k(\Omega)
\end{tikzcd}$$

commute whenever $k + |p| \leq l$.

Proof. For each $l \in \overline{\mathbb{N}}$ define H^l and G^l as

$$H^l : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto (u_n) + \mathcal{I}^l_{nd} \in \mathcal{A}^l_{nd}(\Omega) \tag{8.7}$$

and

$$G^l : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto (u_n) + \mathcal{I}^l_{cs} \in \mathcal{NL}^l(\Omega) \tag{8.8}$$

H^l is well defined since $\mathcal{S}^l_{cs} \subseteq C^l(\Omega)^{\mathbb{N}}$, while G^l also well defined since $\mathcal{I}^l_{nd} \subseteq \mathcal{I}_{cs}$. Clearly H^l is injective, and G^l is surjective.

The commutativity of the diagrams in (i) follows immediately from (8.2), (8.7) and (8.8) as well as the definition of the algebra homomorphisms

$$\gamma^l_k : \mathcal{A}^l_{nd}(\Omega) \longrightarrow \mathcal{A}^k_{nd}(\Omega) \quad k \leq l \tag{8.9}$$

and

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{8.10}$$

see (7.1) and (4.18).

The commutativity of the diagrams in (ii) follows in a similar way taking into account the definitions of the differential operators in the algebras $\mathcal{A}^l_{nd}(\Omega)$, $\mathcal{A}^l_0(\Omega)$ and $\mathcal{NL}^l(\Omega)$, respectively. □

Each of the algebras $\mathcal{A}^l_{nd}$ and $\mathcal{NL}^l(\Omega)$ contain $\mathcal{ML}^l(\Omega)$ as a subalgebra. Indeed, there exist injective algebra homomorphisms

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega) \tag{8.11}$$

and

$$H^l_{nd} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega) \tag{8.12}$$

so that the diagrams

(8.13)
$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "\gamma_k^l"] & \mathcal{NL}^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "H_{oc}^l"] \arrow[r, "\subset"] \arrow[d, "H_{nd}^l"'] & \mathcal{ML}^k(\Omega) \arrow[u, "H_{oc}^k"'] \arrow[d, "H_{nd}^k"] \\
\mathcal{A}_{nd}^l(\Omega) \arrow[r, "\gamma_k^l"] & \mathcal{A}_{nd}^k(\Omega)
\end{tikzcd}$$

and

(8.14)
$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[r, "D^p"] & \mathcal{NL}^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "H_{oc}^l"] \arrow[r, "\mathcal{D}^p"] \arrow[d, "H_{nd}^l"'] & \mathcal{ML}^k(\Omega) \arrow[u, "H_{oc}^k"'] \arrow[d, "H_{nd}^k"] \\
\mathcal{A}_{nd}^l(\Omega) \arrow[r, "D^p"] & \mathcal{A}_{nd}^k(\Omega)
\end{tikzcd}$$

commute whenever $k \leq l$ and $k + |p| \leq l$, respectively.

Therefore we have the following

Proposition 8.2. *For each $l \in \overline{\mathbb{N}}$, there exists an injective algebra homomorphism*

$$\Gamma_0^l : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}_0^l(\Omega)$$

such that the diagrams

(8.15)
$$\begin{tikzcd}
\mathcal{A}_0^l(\Omega) \arrow[r, "\gamma_k^l"] & \mathcal{A}_0^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "\Gamma_0^l"] \arrow[r, "\subset"] & \mathcal{ML}^k(\Omega) \arrow[u, "\Gamma_0^k"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_0^l(\Omega) \arrow[r, "D^p"] & \mathcal{A}_0^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "\Gamma_0^l"] \arrow[r, "\mathcal{D}^p"] & \mathcal{ML}^k(\Omega) \arrow[u, "\Gamma_0^k"']
\end{tikzcd} \tag{8.16}$$

commute, whenever $k \leq l$ and $|p| + k \leq l$, respectively.

As we show next, the homomorphisms

$$H^l : \mathcal{A}_0^l(\Omega) \longrightarrow \mathcal{A}_{nd}^l(\Omega)$$

and

$$G^l : \mathcal{A}_0^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

leave $\mathcal{ML}^l(\Omega)$ invariant.

Theorem 8.3. *The following diagrams*

$$\begin{tikzcd}
\mathcal{A}_0^l(\Omega) \arrow[rr, "H^l"] & & \mathcal{A}_{nd}^l(\Omega) \\
& \mathcal{ML}^l(\Omega) \arrow[ul, "\Gamma_0^l"] \arrow[ur, "H_{nd}^l"'] &
\end{tikzcd} \tag{8.17}$$

and

$$\begin{tikzcd}
\mathcal{A}_0^l(\Omega) \arrow[rr, "G^l"] & & \mathcal{NL}^l(\Omega) \\
& \mathcal{ML}^l(\Omega) \arrow[ul, "\Gamma_0^l"] \arrow[ur, "H_{oc}^l"'] &
\end{tikzcd} \tag{8.18}$$

commute for all $l \in \overline{\mathbb{N}}$.

Proof. For each $u \in \mathcal{ML}^l(\Omega)$,

$$\Gamma_0^l(u) = (u_n) + \mathcal{I}_{nd}^l$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}} \subset \mathcal{S}_{cs}^l$ satisfies

$$\begin{array}{ll} \forall & x \in \Omega\backslash\Gamma : \\ \exists & V \in \mathcal{V}_X,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq \mathbb{N} : \\ & u_n(y) = u(y),\ y \in V. \end{array} \tag{8.19}$$

where $\Gamma \subset \Omega$ is closed, nowhere dense set so that $u \in \mathcal{C}^l(\Omega \backslash \Gamma)$. Likewise, the map $H^l_{nd}(u)$ may be expressed as

$$H^l_{nd}(u) = (u_n) + \mathcal{I}^l_{nd}$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ satisfies (8.19). Clearly, $(u_n) \in \mathcal{S}^l_{cs}$ for any $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (8.19). Thus the commutativity of (8.17) follows from Definition 8.7 of H^l.

Since any sequence $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (8.19) converges to $u \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, the commutativity of (8.18) follows the same way as that of (8.17), taking into account the definition (5.5) of H^l_{oc}. $\square$

9. Relationship between the chain of order convergence algebras and the chain of almost everywhere algebras

In this section we consider the relationship between the chain $\mathbf{A}_{oc}$ and $\mathbf{A}_{ae}$. In this regards, we note that

$$\mathcal{I}^l_0 = (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \cap C^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{I}^l_{cs} \subset \mathcal{S}^l_{cs}, \;\; l \in \overline{\mathbb{N}}$$

Indeed, for $(u_n) \in \mathcal{I}^l_0$ there exists, by (7.4), a closed nowhere dense set $\Gamma \in \mathcal{M}_0$ so that

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x, \; N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \; n \geq N : \\ & u_n(y) = 0, \;\; y \in V \setminus \Gamma. \end{array}$$

Hence (u_n) converges pointwise to 0 on the open and dense set $\Omega \setminus \Gamma$ so that $(u_n) \in \mathcal{I}^l_{cs}$.

Furthermore, the inclusions

$$\mathcal{I}^l_0 \subset \mathcal{I}^k_0, \quad \mathcal{S}^l_{cs} \subset \mathcal{S}^k_{cs}, \quad l, \; k \in \mathbb{N} \tag{9.1}$$

and

$$D^p(\mathcal{I}^l_0) \subset \mathcal{I}^k_0, \quad D^p(\mathcal{S}^l_{cs}) \subset \mathcal{S}^k_{cs}, \quad k + |p| \leq l \tag{9.2}$$

hold. Therefore

$$\mathbf{A}^0_{ae} = \{(\mathcal{B}^l_{ae}, \mathcal{B}^k_{ae}, \gamma^l_k) | l, k \in \overline{\mathbb{N}}, k \leq l\}$$

is a differential chain of algebra of generalized functions, where

$$\mathcal{B}^l_{ae}(\Omega) = \mathcal{S}^l_{cs} / \mathcal{I}^l_0$$

and

$$\gamma^l_k : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto (u_n) + \mathcal{I}^k_0 \in \mathcal{B}^k_{ae}(\Omega) \tag{9.3}$$

for all $l, k \in \overline{\mathbb{N}}$ with $k \leq l$. The differential operators are defined in the usual way, that is

$$D^p : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto D^p(u_n) + \mathcal{I}^k_0 \in \mathcal{B}^k_{ae}(\Omega), \;\; |p| + k \leq l.$$

Theorem 9.1. *For every $l \in \overline{\mathbb{N}}$ there exists an injective algebra homomorphism*

$$F^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$$

and a surjective algebra homomorphism

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

so that the following hold.

(i) The diagrams

(9.4)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "F^l_{ae}"] \arrow[d, "\gamma^l_k"'] & \mathcal{A}^l_{ae}(\Omega) \arrow[d, "\gamma^l_k"] \\
\mathcal{B}^k_{ae}(\Omega) \arrow[r, "F^k_{ae}"] & \mathcal{A}^k_{ae}(\Omega)
\end{tikzcd}$$

and

(9.5)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "G^l_{ae}"] \arrow[d, "\gamma^l_k"'] & \mathcal{NL}^l(\Omega) \arrow[d, "\gamma^l_k"] \\
\mathcal{B}^k_{ae}(\Omega) \arrow[r, "G^k_{ae}"] & \mathcal{NL}^k(\Omega)
\end{tikzcd}$$

commute for all $k \leq l$.

(ii) The diagrams

(9.6)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "D^p"] \arrow[d, "F^l_{ae}"'] & \mathcal{B}^k_{ae}(\Omega) \arrow[d, "F^k_{ae}"] \\
\mathcal{A}^l_{ae}(\Omega) \arrow[r, "D^p"] & \mathcal{A}^k_{ae}(\Omega)
\end{tikzcd}$$

and

(9.7)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "D^p"] \arrow[d, "G^l_{ae}"'] & \mathcal{B}^k_{ae}(\Omega) \arrow[d, "G^k_{ae}"] \\
\mathcal{NL}^l(\Omega) \arrow[r, "D^p"] & \mathcal{NL}^k(\Omega)
\end{tikzcd}$$

commute whenever $k + |p| \leq l$.

Proof. For each $l \in \overline{\mathbb{N}}$ define algebra homomorphisms F^l_{ae} and G^l_{ae} as

$$F^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto (u_n) + (\mathcal{I}^l_E + \mathcal{I}_{ae}) \in \mathcal{A}^l_{ae}(\Omega) \tag{9.8}$$

and

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto (u_n) + \mathcal{I}^l_{cs} \in \mathcal{NL}^l(\Omega). \tag{9.9}$$

Since $\mathcal{S}^l_{cs} \subseteq C^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{E}^l_{ae}$ and $\mathcal{I}^l_0 \subseteq (\mathcal{I}^l_E + \mathcal{I}_{ae})$ it follows that F^l_{ae} is well defined. Also G^l_{ae} is well defined since $\mathcal{I}^l_0 \subseteq \mathcal{I}_{cs}$. The mapping F^l_{ae} is injective since $\mathcal{I}^l_0 = (\mathcal{I}^l_E + \mathcal{I}^l_{oc}) \cap \mathcal{S}^l_{cs}$ which implies that $\{(u_n) + \mathcal{I}^l_0 \in \mathcal{B}^l_{ae} \mid F^l_{ae}((u_n) + \mathcal{I}^l_0) = 0\} = \{0\}$. G^l_{ae} is surjective since

$$\mathcal{I}^l_0 \subseteq \mathcal{I}^l_{cs}.$$

The commutativity of the diagrams in (i) follows immediately from (9.3), (9.8) and (9.9) as well as the definition of the algebra homomorphisms

$$\gamma^l_k : \mathcal{A}^l_{ae}(\Omega) \longrightarrow \mathcal{A}^k_{ae}(\Omega) \quad k \leq l \tag{9.10}$$

and

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{9.11}$$

given by (7.10) and (4.18) respectively.

The commutativity of the diagrams in (ii) follows in a similar way taking into account the definitions of the differential operators in the algebras $\mathcal{A}^l_{ae}(\Omega)$, $\mathcal{B}^l_{ae}(\Omega)$ and $\mathcal{NL}^l(\Omega)$ respectively. □

If $\mathcal{M}_0$ consists of all closed nowhere dense subsets of Ω, then each of the algebras $\mathcal{A}^l_{ae}(\Omega)$ contain $\mathcal{ML}^l(\Omega)$ as a subalgebra. In particular, there exists for each $l \in \overline{\mathbb{N}}$ an injective algebra homomorphism

$$H^l_{ae} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega) \tag{9.12}$$

so that the diagrams

$$\begin{tikzcd}
\mathcal{NL}^l(\Omega) \arrow[rr, "\gamma^l_k"] & & \mathcal{NL}^k(\Omega) \\
\mathcal{ML}^l(\Omega) \arrow[u, "H^l_{oc}"] \arrow[d, "H^l_{ae}"'] \arrow[rr, "\subset"] & & \mathcal{ML}^k(\Omega) \arrow[u, "H^k_{oc}"'] \arrow[d, "H^k_{ae}"] \\
\mathcal{A}^l_{ae}(\Omega) \arrow[rr, "\gamma^l_k"] & & \mathcal{A}^k_{ae}(\Omega)
\end{tikzcd} \tag{9.13}$$

and

$$\begin{tikzcd}
\mathcal{NL}^{l}(\Omega) \arrow[r, "D^{p}"] & \mathcal{NL}^{k}(\Omega) \\
\mathcal{ML}^{l}(\Omega) \arrow[u, "H^{l}_{oc}"] \arrow[r, "\mathcal{D}^{p}"] \arrow[d, "H^{l}_{ae}"'] & \mathcal{ML}^{k}(\Omega) \arrow[u, "H^{k}_{oc}"'] \arrow[d, "H^{k}_{ae}"] \\
\mathcal{A}^{l}_{ae}(\Omega) \arrow[r, "D^{p}"] & \mathcal{A}^{k}_{ae}(\Omega)
\end{tikzcd} \tag{9.14}$$

commute whenever $k \leq l$ and $k + |p| \leq l$, respectively, where H^{l}_{oc} is defined by (8.11).

Thus we have the following

Proposition 9.2. *Assume that* $\mathcal{M}_0 = \{\Gamma \subset \Omega | \Gamma \text{ is closed nowhere dense}\}$. *Then for each* $l \in \overline{\mathbb{N}}$, *there exists an injective algebra homomorphism*

$$H^{l}_{ae} : \mathcal{ML}^{l}(\Omega) \longrightarrow \mathcal{B}^{l}_{ae}(\Omega)$$

such that the diagrams

$$\begin{tikzcd}
\mathcal{B}^{l}_{ae}(\Omega) \arrow[r, "\gamma^{l}_{k}"] & \mathcal{B}^{k}_{ae}(\Omega) \\
\mathcal{ML}^{l}(\Omega) \arrow[u, "H^{l}_{ae}"] \arrow[r, "\subset"] & \mathcal{ML}^{k}(\Omega) \arrow[u, "H^{k}_{ae}"']
\end{tikzcd} \tag{9.15}$$

and

$$\begin{tikzcd}
\mathcal{B}^{l}_{ae}(\Omega) \arrow[r, "D^{p}"] & \mathcal{B}^{k}_{ae}(\Omega) \\
\mathcal{ML}^{l}(\Omega) \arrow[u, "H^{l}_{ae}"] \arrow[r, "\mathcal{D}^{p}"] & \mathcal{ML}^{k}(\Omega) \arrow[u, "H^{k}_{ae}"']
\end{tikzcd} \tag{9.16}$$

commute, whenever $k \leq l$ *and* $|p| + k \leq l$, *respectively.*

We note that the algebra homomorphism

$$H^{l}_{oc}(u) : \mathcal{ML}^{l}(\Omega) \longrightarrow \mathcal{B}^{l}_{ae}(\Omega)$$

is obtained by setting

$$H^l_{oc}(u)(u) = (u_n) + \mathcal{I}^l_0$$

where $(u_n) \in \mathcal{S}^l_{cs}$. The existence of such a sequence is guaranteed by Lemma 3.5.

The homomorphism

$$F^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$$

and

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

leave the subalgebra $\mathcal{ML}^l(\Omega)$ of $\mathcal{B}^l_{ae}(\Omega)$ invariant as shows in the following.

Theorem 9.3. *Assume that* $\mathcal{M}_0 = \{\Gamma \subset \Omega | \Gamma$ is closed nowhere dense . *Then the diagrams*

(9.17)

$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[rr, "F^l_{ae}"] & & \mathcal{A}^l_{ae}(\Omega) \\
& \mathcal{ML}^l(\Omega) \arrow[ul, "H^l_{ae}"] \arrow[ur, "H^l_{ae}"'] &
\end{tikzcd}$$

and

(9.18)

$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[rr, "G^l_{ae}"] & & \mathcal{NL}^l(\Omega) \\
& \mathcal{ML}^l(\Omega) \arrow[ul, "H^l_{ae}"] \arrow[ur, "H^l_{oc}"'] &
\end{tikzcd}$$

commute for all $l \in \overline{\mathbb{N}}$.

Proof. The proof is similar to that of Theorem 8.3 which we outline below.

For each $u \in \mathcal{ML}^l(\Omega)$,

$$\Gamma^l_0(u) = (u_n) + \mathcal{I}^l_{nd}$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}} \subset \mathcal{S}^l_{cs}$ satisfies

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \in \mathcal{V}_X, \ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \ n \geq \mathbb{N} : \\ & u_n(y) = u(y), \ y \in V. \end{array} \tag{9.19}$$

where $\Gamma \subset \Omega$ is closed, nowhere dense set so that $u \in \mathcal{C}^l(\Omega \backslash \Gamma)$. Likewise, the map $H^l_{ae}(u)$ may be expressed as

$$H^l_{ae}(u) = (u_n) + \mathcal{I}^l_{nd}$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ satisfies (9.19). Clearly, $(u_n) \in \mathcal{S}^l_{cs}$ for any $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (9.19). Thus the commutativity of (9.17) follows from the definition (9.8) of F^l_{ae}.

Since any sequence $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (9.19) converges to $u \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, the commutativity of (9.18) follows the same way as that of (9.17), taking into account the definition (5.5) of H^l_{oc}. □

10. Chain generalized solutions in nowhere dense algebras and almost everywhere algebras

In this section we show how the existence result for chain generalized solutions of nonlinear PDEs in $\mathbf{A}_{oc}$ given in Theorem 6.2 leads to corresponding existence results in the chains $\mathbf{A}_{ae}$ and $\mathbf{A}_{nd}$, respectively. In this regard, consider a polynomial nonlinear differential operator

$$T = \sum_{1 \leq i \leq h} c_i(x) \prod_{1 \leq j \leq k_i} D^{p_{ij}}, \quad x \in \Omega \tag{10.1}$$

where $h, k_i \in \mathbb{N}$, $c_i \in C^\infty(\Omega)$ and $p_{ij} \in \mathbb{N}^n$ satisfies $|p_{ij}| \leq m$ for all $i = 1, \cdots, h$ and $j = 1, \cdots, k_i$. For $f \in C^\infty(\Omega)$ we show that, under a mild assumption on the operator T, the polynomial PDE,

$$Tu = f. \tag{10.2}$$

admits a chain generalized solutions in $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ respectively.

We deal first with the case of solutions in $\mathbf{A}_{nd}$. In this regard, it is clear that

$$T(\mathcal{I}^l_{nd}) \subset \mathcal{I}^k_{nd}$$

whenever $k + m \leq l$ and, obviously,

$$T(C^l(\Omega)^{\mathbb{N}}) \subset C^k(\Omega)^{\mathbb{N}}, \quad k + m \leq l$$

Therefore, since $\mathcal{I}^l_{nd}$ is off diagonal, $(u_n) - (v_n) \in \mathcal{I}^l_{nd}$ which implies $(Tu_n) - (tv_n) \in \mathcal{I}^k_{nd}$, so that

$$T_{nd} : \mathcal{A}^l_{nd}(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto T(u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_{nd}(\Omega) \quad k + m \leq l$$

defines an extension of

$$T : C^l(\Omega) \longrightarrow C^k(\Omega),$$

for $k + m \leq l$. In the same way,

$$T_{oc} : \mathcal{NL}^l(\Omega) \ni (u_n) + \mathcal{I}^l_{cs} \mapsto T(u_n) + \mathcal{I}^k_{cs} \in \mathcal{NL}^k(\Omega), \quad k + m \leq l$$

and

$$T_0 : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto T(u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_0(\Omega) \quad k + m \leq l$$

defines an extension of $T : C^l(\Omega) \longrightarrow C^k(\Omega)$.

Proposition 10.1. *The diagrams*

(10.3)
$$\begin{tikzcd}\mathcal{A}_0^l(\Omega) \arrow[r, "T_0"] \arrow[d, "F_{nd}^l"'] & \mathcal{A}_0^k(\Omega) \arrow[d, "F_{nd}^k"] \\ \mathcal{A}_{nd}^l(\Omega) \arrow[r, "T_{nd}"] & \mathcal{A}_{nd}^k(\Omega)\end{tikzcd}$$

and

(10.4)
$$\begin{tikzcd}\mathcal{A}_0^l(\Omega) \arrow[r, "T_0"] \arrow[d, "G_{nd}^l"'] & \mathcal{A}_0^k(\Omega) \arrow[d, "G_{nd}^k"] \\ \mathcal{NL}^l(\Omega) \arrow[r, "T_{oc}"] & \mathcal{NL}^k(\Omega)\end{tikzcd}$$

commute whenever $k+m \leq l$, with F_{nd}^l and G_{nd}^l algebra homomorphisms.

Proof. For $u = (u_n) + \mathcal{I}_{nd}^l \in \mathcal{A}_0^l$ with $k+m \leq l$,

$$\begin{aligned} T_{nd}(F_{nd}^l(u)) &= T_{nd}((u_n) + \mathcal{I}_{nd}^l) \\ &= T(u_n) + \mathcal{I}_{nd}^k \end{aligned}$$

and

$$\begin{aligned} F_{nd}^k(T_0(u)) &= F_{nd}^k(T(u_n) + \mathcal{I}_{nd}^k) \\ &= T(u_n) + \mathcal{I}_{nd}^k \end{aligned}$$

Hence (10.3) commutes. The commutativity of diagram (10.4) follows in the same way. ☐

Theorem 10.2. *If $f \in C^\infty(\Omega)$, and the operator T defined in* (10.1) *satisfies* (2.16) *to* (2.17) *then the PDE*

$$Tu = f \tag{10.5}$$

admits a chain generalized solution in $\mathbf{A}_{nd}$.

Proof. According to Theorem 6.2, there exists a chain generalized solution of (10.5) in $\mathbf{A}_{oc}$. That is, there exists $(u_n) \in \mathcal{S}_{cs}^\infty$ so that $u = (u_n) + \mathcal{I}_{cs}^l$ satisfies

$$Tu = T(u_n) + \mathcal{I}_{cs}^k = f + \mathcal{I}_{cs}^k$$

for all $l, k \in \overline{\mathbb{N}}$ with $k+m \leq l$. Since

$$G_{nd}^l : \mathcal{A}_0^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

is surjective for each $l \in \overline{\mathbb{N}}$, it follows that there exists $v = (v_n) + \mathcal{I}^l_{nd} \in \mathcal{A}^\infty_0(\Omega)$ so that $G^\infty_{nd}(v) = u$. It follows from Proposition 10.1 that

$$T_o((v_n) + \mathcal{I}^l_{nd}) = f + \mathcal{I}^k_{nd}$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k + m \leq l$. In the same way, it follows that

$$F^\infty_{nd}(v) = (v_n) + \mathcal{I}^\infty_{nd} \in \mathcal{A}^\infty_{nd}(\Omega)$$

is a chain generalized solution of (10.5) in $\mathbf{A}_{nd}$. $\square$

Let us now consider the existence of chain generalized solutions of the PDE (10.5) in the chain $\mathbf{A}_{ae}$. It is clear that

$$(u_n) - (v_n) \in \mathcal{I}^l_0 \Longrightarrow T(u_n) - T(v_n) \in \mathcal{I}^k_0$$

for all $(u_n), (v_n) \in C^\infty(\Omega)^{\mathbb{N}}$ and $k + m \leq l$. Thus

$$T_{\mathcal{B}} : \mathcal{B}^l_{ae}(\Omega) \in (u_n) + \mathcal{I}^l_0 \mapsto T(u_n) + \mathcal{I}^k_0 \in \mathcal{B}^k_{ae}(\Omega)$$

is a well-defined extension of $T : C^l(\Omega) \longrightarrow C^k(\Omega)$, for all $l, k \in \overline{\mathbb{N}}$ such that $m + k \leq l$. With each $(u_n) \in \mathcal{E}^l_{ae}(\Omega)$ and $k \in \overline{\mathbb{N}}$, we associate the set

$$(10.6)\quad \overline{T}_{ae}(u_n) = \left\{ (v_n) \in \mathcal{E}^k_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma_0 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & v_n(y) = Tu_n(y), \quad y \in V \setminus \Gamma \end{array} \right\}.$$

This gives rise to a relation

$$\mathcal{E}^l_{ae}(\Omega) \ni (u_n) \mapsto \overline{T}_{ae}(u_n) \subset \mathcal{E}^k_{ae}(\Omega).$$

It follows that

$$\overline{T}_{ae}(u_n) - \overline{T}_{ae}(u_n) \subseteq \mathcal{I}^k_E + \mathcal{I}^k_{ae}$$

and

$$(v_n) \in \overline{T}_{ae}(u_n), \quad ((v_n) - (w_n)) \in \mathcal{I}^k_E + \mathcal{I}^k_{ae} \quad \Longrightarrow (w_n) \in \overline{T}_{ae}(u_n)$$

for all $(u_n) \in \mathcal{E}^l_{ae}(\Omega)$ and $l, k \in \overline{\mathbb{N}}$ such that $k + m \leq l$. Therefore,

$$(10.7)\qquad T_{ae} : \mathcal{A}^l_{ae}(\Omega) \ni (u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \mapsto \overline{T}_{ae}(u_n) \in \mathcal{A}^k_{ae}(\Omega)$$

is well-defined for all $k, l \in \overline{\mathbb{N}}$ such that $m + k \leq l$. Note that

$$T_{ae}((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) = (v_n) + (\mathcal{I}^k_E + \mathcal{I}^l_{ae})$$

where (v_n) is any member of the set $\overline{T}_{ae}(u_n)$. Since the ideal $\mathcal{I}^l_E + \mathcal{I}^l_{ae}$ is off diagonal, it follows that (10.7) is an extension of

$$T : C^l(\Omega) \longrightarrow C^k(\Omega), \quad k + m \leq l.$$

Proposition 10.3. *For all $k, l \in \overline{\mathbb{N}}$ so that $m + k \leq l$, the diagrams*

(10.8)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "T_{\mathcal{B}}"] \arrow[d, "G^l_{ae}"'] & \mathcal{B}^k_{ae}(\Omega) \arrow[d, "G^k_{ae}"] \\
\mathcal{NL}^l(\Omega) \arrow[r, "T_{oc}"] & \mathcal{NL}^k(\Omega)
\end{tikzcd}$$

and

(10.9)
$$\begin{tikzcd}
\mathcal{B}^l_{ae}(\Omega) \arrow[r, "T_{\mathcal{B}}"] \arrow[d, "F^l_{ae}"'] & \mathcal{B}^k_{ae}(\Omega) \arrow[d, "F^k_{ae}"] \\
\mathcal{A}^l_{ae}(\Omega) \arrow[r, "T_{ae}"] & \mathcal{A}^k_{ae}(\Omega)
\end{tikzcd}$$

commute.

Proof. For $u = (u_n) + \mathcal{I}^l_0 \in \mathcal{B}^l_{ae}$ with $k + m \leq l$,

$$\begin{aligned} T_{oc}(G^l_{ae}(u)) &= T_{oc}((u_n) + \mathcal{I}^l_{cs}) \\ &= T(u_n) + \mathcal{I}^k_{cs} \end{aligned}$$

and

$$\begin{aligned} G^k_{ae}(T_{\mathcal{B}}(u)) &= G^k_{ae}(T(u_n) + \mathcal{I}^k_0) \\ &= T(u_n) + (\mathcal{I}^k_{cs}) \end{aligned}$$

Hence (10.8) commutes. The commutativity of diagram (10.9) follows in the same way. □

Theorem 10.4. *If $f \in C^\infty(\Omega)$, and the operator T defined in* (10.1) *satisfies* (2.16) *to* (2.17) *then the PDE*

$$Tu = f \qquad (10.10)$$

admits a chain generalized solution in $\mathbf{A}_{ae}$.

Proof. According to Theorem 6.2, there exists a chain generalized solution of (10.5) in $\mathbf{A}_{oc}$. That is, there exists $(u_n) \in \mathcal{S}^\infty_{cs}$ so that $u = (u_n) + \mathcal{I}^l_{cs}$ satisfies

$$Tu = T(u_n) + \mathcal{I}^k_{cs} = f + \mathcal{I}^k_{cs}$$

for all $l, k \in \overline{\mathbb{N}}$ with $k + m \leq l$. Since

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

is surjective for each $l \in \overline{\mathbb{N}}$, there exists $v = (v_n) + \mathcal{I}_0^l \in \mathcal{B}_{ae}^{\infty}(\Omega)$ so that $G_{ae}^{\infty}(v) = (u_n) + \mathcal{I}_0^l$. It follows from Proposition 10.3 that

$$T_{\mathcal{B}}((v_n) + \mathcal{I}_0^l) = f + \mathcal{I}_0^k$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k + m \leq l$. In the same way, it follows that

$$F_{ae}^{\infty}(v) = (v_n) + \mathcal{I}_0^{\infty} \in \mathcal{A}_{nd}^{\infty}(\Omega)$$

is a chain generalized solution of (10.5) in $\mathbf{A}_{ae}$. □

Theorem 10.4 establishes the existence of a chain generalized solution in $\mathbf{A}_{ae}$ for a large class of PDEs, as demonstrated in the following

Example 10.5. Consider the PDE

$$D_t u(x,t) = \sum_{1 \leq i \leq h} c_i(x) \prod_{1 \leq j \leq k_i} D_x^{p_{ij}} u(x,t), \quad (x,t) \in \Omega = \Omega' \times \mathbb{R} \tag{10.11}$$

where $\Omega' \subset \mathbb{R}^{n-1}$ is open, $h, k_i \in \mathbb{N}$, $c_i \in C^{\infty}(\Omega)$ and $p_{ij} \in \mathbb{N}^n$ satisfies $|p_{ij}| \leq m$ for all $i = 1, \cdots, h$ and $j = 1, \cdots, k_i$. The PDE (10.11) can be written in the form

$$T(x,t,D)u(x,t) = 0, \quad (x,t) \in \Omega$$

where 0 denotes the zero function on Ω. The operator $T(x,t,D)$ is defined through a jointly continuous, C^{∞}-smooth mapping

$$F : \Omega \times \mathbb{R}^{M+1} \longrightarrow \mathbb{R} \tag{10.12}$$

as

$$T(x,t,D) = F(x,t,u(x,t), \cdots, D_x^{p_{ij}} u(x,t), \cdots, D_t u(x,t)).$$

where M is the cardinality of $\{p_{ij} \mid i = 1 \cdots h,\ j = 1 \cdots k_i\}$. In particular,

$$F(x,t,\xi_1 \cdots, \xi_{M+1}) = \xi_{M+1} - \sum_{1 \leq i \leq h} c_i(x) \prod_{1 \leq j \leq k_i} \xi_{p_{ij}}, \quad (x,t) \in \Omega = \Omega' \times \mathbb{R}.$$

Since the PDE in (10.11) is linear in ξ_{M+1}, it follows that the range of F in $\mathbb{R}$ is given by

$$R_F = \{F(x,t,\xi_1 \cdots, \xi_{M+1}) | (x,t) \in \Omega, \ \ (x,t,\xi_1 \cdots, \xi_{M+1}) \in \mathbb{R}^{M+1}\} = \mathbb{R}.$$

Hence R_F is open and F is surjective. Furthermore, $R_F = intR_F = \mathbb{R}$ so that $0 \in intR_F$.

Now define the mapping

$$F^{\infty} : \Omega \times \mathbb{R}^{\mathbb{N}^{n+1}} \longrightarrow \mathbb{R}^{\mathbb{N}^{n+1}}$$

by setting

$$F^{\infty}(x,t,(\xi_{M+1})_{M \in \mathbb{N}^n}) = (F^{\beta}(x,t,\cdots,\xi_M,\xi_{M+1})), \ \beta \in \mathbb{N}^{n+1}$$

where, for each $\beta \in \mathbb{N}^{n+1}$, the mapping

$$F^\beta : \Omega \times \mathbb{R}^{\mathbb{N}^{n+1}} \longrightarrow \mathbb{R}^{\mathbb{N}^{N+1}}$$

is defined by setting

$$D^\beta(T(x,t,D)u(x,t)) = F^\beta(x,t,\cdots,D^{p_{ij}}u(x,t),\cdots,D_tu(x,t)),\ |p_{ij}| \leq m + |\beta|$$

for all $u \in C^\infty(\Omega)$. Note that for each $\beta \in \mathbb{N}^{n+1}$, F^β is linear in at least one factor of $\mathbb{R}^{\mathbb{N}^{n+1}}$, so that, for $\beta' \neq \beta$, F^β is independent of this factor. Hence

$$\begin{array}{ll} \forall & (x,t) \in \Omega \\ \exists & \xi(x,t) \in \mathbb{R}^{\mathbb{N}^{n+1}}, \quad F^\infty(x,t,\xi(x,t)) = \mathbf{0} \\ \exists & V \in \mathcal{V}_{(x,t)}, W \in \mathcal{V}_{\xi(x,t)} : \\ & F^\infty : V \times W \in \mathbb{R}^{\mathbb{N}^{n+1}} \text{ is open} \end{array}$$

Thus the PDE (10.11) satisfies (2.17). Therefore by Theorems 10.4, the PDE (10.11) has a chain generalized solution in $\mathbf{A}_{ae}$.

11. Conclusion

The underlying spaces of generalized functions, $\mathcal{NL}^l(\Omega)$, involve in the Order Completion Method as formulated in the setting of convergence spaces, has been shown to form a differential chain $\mathbf{A}_{oc}$ of algebras of generalized functions. Any generalized solution in the underlying space may be interpreted as a chain generalized solution.

We also established a relationship between the differential chains $\mathbf{A}_{oc}$ and $\mathbf{A}_{nd}$, as well as a relationship between the differential chain $\mathbf{A}_{oc}$ and $\mathbf{A}_{ae}$ of almost-everywhere algebras was introduced. It was shown that the existence results for chain generalized solution of nonlinear PDEs lead to corresponding existence results in $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$, respectively. It was shown that chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ admits embeddings of the spaces $\mathcal{ML}^l(\Omega)$ which preserve both the algebraic and differential structure of $\mathcal{NL}^l(\Omega)$. These results demonstrate the extent to which these chains of are able to handle singularities occurring on a closed, nowhere dense set.

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